

École polytechnique de Louvain

Simulation of optical wave propagation through atmospheric turbulence using phase screens

A study of the turbulence impact on metrics

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Abstract

Phase screen propagation is a method used to simulate turbulent atmosphere via computation. It performs free propagation between diffractive layers, called phase screens, intended to symbolise the turbulent atmosphere. The main objective of this master thesis is to use metrics to describe the impact of the atmospheric turbulence on the Earth to satellite propagation of an optical wave. Therefore, a *Python* simulation tool has been developed to execute phase screens propagation. It applies Fresnel diffraction formula for free propagating and subharmonic generation of phase screen for representing the turbulence. The impact of turbulence on the propagated optical wave is analysed at constant turbulence. On top of that, physical turbulence profiles are also studied to investigate the impact of the wind and ground turbulence on telecommunications metrics.

La propagation par écrans de phase est une méthode utilisée pour simuler un atmosphère turbulent à l'aide de programmes informatiques. Le principe repose sur des propagations libres entre écrans de phase conçus pour représenter la turbulence sur la couche atmosphérique considérée. L'objectif principal de ce mémoire est d'étudier l'impact de cette turbulence sur une transmission optique entre la Terre et un satellite au travers de mesures. Pour ce faire, un programme de simulation *Python* a été développé pour exécuter la propagation par écrans de phase. Ce programme applique la formule de diffraction de Fresnel pour propager l'onde ainsi que des sous harmoniques pour la création des écrans de phase. L'impact de la propagation turbulente est d'abord analysée à turbulence constante. Par après, des profils réels sont employés afin d'étudier les effets du vent et de la turbulence au sol sur les mesures.

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List of abbreviations

AO	Adaptive Optics
BER	Bit Error Rate
CDF	Cumulative Density Function
CLT	Central Limit Theorem
DFT	Discrete Fourier Transform
DFT-SH	Discrete Fourier Transform with Subharmonics
EM	Electromagnetic
FSO	Free Space Optic
FT	Fourier Transform
FWHM	Full Width at Half Maximum
GSM	Global System for Mobile communications
HV	Hufnagel Valley
HWHM	Half Width at Half Maximum
IDFT	Inverse Discrete Fourier Transform
IFT	Inverse Fourier Transform
K	Kolmogorov
LE	Long Exposure
LEO	Low Earth Orbit
LoS	Line of Sight
MIMO	Multiples Inputs Multiples Outputs
MLE	Maximum Likelihood Estimator
mvK	modified von Karman
NASA	National Aeronautics and Space Administration
NEP	Noise equivalent Power

PDF	Probability Density Function
PS	Phase Screen
PSD	Power Spectral Density
QQ	Quantile - Quantile
RF	Radio Frequency
SH	Subharmonic
SNR	Signal to Noise Ratio
SS	Sparse Spectrum
SSBPM	Split Step Beam Propagation Method
SSUD	Sparse Spectrum with Uniform Distribution
SSW	Split-Step Wavelet
vK	von Karman

Nomenclature

Dimensions of the problem

α_{edges}	Angle between limit of mD_1 and perpendicular
α_k	Angle between limit of D_2 and perpendicular
α_{max}	Maximal opening angle
μ	Center of the image vector
f	Frequency vector
R	Position vector between two analysed points
r	Position vector
κ	Wavenumber for frequencies
Δh_i	Thickness of the atmospheric slab between z_{i-1} and z_i
δ_i	Spatial spacing at z_i
δ_{f_i}	Frequency spacing at z_i
κ	Norm of the wavenumber for frequencies (κ)
λ	Wavelength
$\phi_i(\mathbf{r})$	Phase of optical field at position $\mathbf{r}$ for a distance z_i
ψ	Gouy phase
σ^2	Variance of the image
ξ	Zenith angle
$A_i(\mathbf{r})$	Amplitude of optical field at position $\mathbf{r}$ for a distance z_i
D_i	Maximal extend at position z_i
D_{grid}	Dimension of the grid
D_{illum}	Illuminated region
f	Norm of frequency vector

f_{loc}	Local frequency
f_{max}	Maximal authorised frequency
$H(\mathbf{f})$	Transfer function
$h(\mathbf{r})$	Free-space amplitude spread function
$I_i(\mathbf{r})$	Intensity of optical field at position $\mathbf{r}$ for a distance z_i
k	Wavenumber
m	Ratio of the spacing between two plane
N	Number of pixels in one direction
n	Number of phase screens
n_{LE}	Number of outputs for long exposure
R	Norm of position vector between two analysed points
r	Norm of position vector
R_c	Radius of curvature
S_i	Aperture size at z_i
$U_i(\mathbf{r})$	Optical field at position $\mathbf{r}$ for a distance z_i
w_i	Intensity radius at z_i
z_i	Distance to phase screen i
z_R	Rayleigh distance

Random process

τ	Short amount of time
$m_x(t)$	Mean at time t
t	Time
$x(t)$	Random variable at time t
$x_1(t)$	Varying part of random variable at time t

Turbulence

$(\alpha + i\beta)_{m,n}$	Random complex Gaussian noise for coefficient m, n
η	Zenith angle
$\hat{\mu}$	Mean estimator
$\hat{\sigma}^2$	Mean variance

κ_0	Lower wavenumber boundary of power spectrum
κ_l	Variable associated with modified atmospheric power spectrum
κ_m	Upper wavenumber boundary of power spectrum
ν	Kinematic viscosity
Φ_x	Power spectrum of variable x
ϕ_{SH}	Subharmonic phase
ϕ_{tot}	Total phase
ρ_0	Coherence length
σ_I^2	Scintillation index
σ_R^2	Rytov variance
A	Ground turbulence
C_x^2	Structure constant of variable x
$c_{m,n}$	Fourier coefficient
D_x	Structure function of variable x
f_0	Lower spatial frequency boundary of power spectrum
f_m	Upper spatial frequency boundary of power spectrum
f_X	probability density function of variable X
I	Received intensity array
I_T	Threshold intensity
I_{free}	Received intensity value under free propagation
$L(x_i)$	Likelihood estimator
l	Dimension of the turbulence flow
L_0	Outer scale
l_0	Inner scale
$n(\mathbf{r})$	Refractive index at position $\mathbf{r}$
$P(\mathbf{r})$	Pressure at position $\mathbf{r}$
p	Order of the subharmonic
r_0	Fried parameter
Re	Reynolds number

SR Strehl ratio

$T(\boldsymbol{r})$ Temperature at position $\boldsymbol{r}$

w Wind velocity

w_{LE} Long exposure intensity radius

Chapter 1

Introduction

The spatial conquest during the cold war marked the start of the race for the launch of a satellite. The USSR was able to place Spoutnik 1 in orbit in 1957 [1]. Since then, the number of satellites in orbit has grown fast. At the end of 2021, there were 4852 operational satellites around the Earth and in 2022, less than four months were necessary to launch almost 900 satellites. The increasing trend will not stop because of project like Starlink which intends to launch 42 000 satellites [2].

Therefore, more than ever there is a need in improving the Earth to satellite and satellite to satellite communication links. Optical transmission is an efficient alternative to radio frequency (RF) communication because it ensures faster and more secured transmission of data [3, 4]. Moreover, there is no need for frequency coordination like in RF communications [5].

In an Earth to satellite optical communication link, the wavefront of the optical wave is distorted due to scattering, absorption and atmospheric turbulence. Atmospheric turbulence is the most impacting cause of the degradation and is therefore studied in this report [6, 7]. The three factors analysed in this report responsible for the apparition of this turbulence are the fluctuations in the refractive index (n), the variation of wind velocity (w) and the turbulence caused by the ground (A).

Free space optic (FSO) communication presents some challenges and must still be investigated. Indeed, characterising the behaviour of the atmosphere is primordial to improve the transmission of data. In this context, various scenarios are currently explored to correct the degradation. Several techniques such as aperture averaging, partially coherent source, different optical beam profiles and adaptive optics (AO) are potential solutions [8].

Using computer simulations is an efficient method to collect information over the degradation of the transmission. In this report, the phase screens (PS) propagation method is studied and implemented to allow an analysis of the impact of the turbulence on telecommunications metrics. The computational part of this report mainly gets its inspiration from the book written by Schmidt [9] and the associated *Python* package *AOtools* [10].

Chapter 2 lists previous studies on the subject with the different techniques to perform turbulent propagation. Afterwards, a comparison between FSO and RF communications is done, giving the pros and cons of each transmission method. Finally, a theoretical background on turbulence is developed starting from the Earth atmosphere to the different existing turbulence models.

Chapter 3 clearly states the general problem. It details the input and output fields and explains the PS propagation method. It introduces all the variables related to the geometry of the problem and states the different necessary hypotheses. The diffraction of the beam and the implementation of angular resolution are explained as well.

The propagation procedure can be divided in two main topics : the free propagation and the generation of PS. Chapter 4 is approaching the free propagation performed between two PS. The Rayleigh-Sommerfeld formula is used under Kirchoff and Fresnel assumptions to obtain a discrete equation for computer implementation. Then come Fourier transforms (FT) and conditions restricting the propagation variables.

Chapter 5 considers the turbulence through some characteristic variables and then develops the PS generation procedure with and without subharmonics. After a validation of the PS, the distribution of the received intensity is discussed and confirmed using a maximum likelihood estimator (MLE). The last point of the chapter is an example of a turbulent propagation.

Chapter 6 uses the working simulation tool to describe, via different metrics, the atmospheric channel at constant turbulence. Note that a constant turbulence has no physical meaning. It first studies the evolution of the probability density function (PDF) and the cumulative density function (CDF) with the turbulence. Then, the bit error rate (BER) is computed depending on the signal to noise ratio (SNR). Afterwards, the long exposure (LE) is explained and the impact of turbulence on the growth of the intensity radius is analysed. Finally, the scintillation index and the Strehl ratio are developed.

Chapter 7 analyses the same metrics as in chapter 6 but for physical profiles with the impact caused by the wind velocity (w) and the ground turbulence (A).

The last chapter sums up the contribution of this work and give ideas for future works. It also explains the current trend in terms of phase screen propagation.

Chapter 2

State of the art and background

This chapter is devoted to establishing the state of the art. it also gives a theoretical background to the reader on atmospheric turbulence. The first part of the chapter is the state of the art and is composed of three sections :

- The history, explaining the evolution of the theory from the first approach of Kolmogorov to the framework established by Phillips and Andrews. Contributions of certain authors such as Obukhov or Tatarskii are enumerated.
- A summary on phase screens generation and propagation methods. 4 generation methods are enunciated : discrete Fourier transform with and without subharmonics and sparse spectrum with or without uniform distributions. Studies on the number of phase screens and their location are also developed.
- A comparison between free space optics and radio frequency communications. The pros and cons of each method are listed and developed.

Afterwards, a second part about the theoretical background is necessary to understand this report. It starts with an analysis of the temperature evolution across the atmosphere followed by the introduction of the continuum of turbulence and the definition of the inertial subrange. Afterwards, a reminder on random process and associated structure function is done. Structure function is given for velocity, temperature and refractive index, defining each time the structure constant. Finally, Kolmogorov, Tatarskii, modified von Karman and the modified atmospheric power spectrum of the refractive index are computed.

2.1 State of the art

2.1.1 History

The propagation of electromagnetic (EM) wave has been studied for a few decades and was first introduced by Kolmogorov in 1941 [11]. He found a power spectrum expression for the refractive-index following a $-11/3$ power law according to the frequencies. His work inspired others and in 1947, Obukhov published an article on the structure function of temperature in turbulent flows [12]. Afterwards, Tatarskii [13], von Karman and many others published articles on the topic. Tatarskii is notably responsible for the unification of Kolmogorov's theory with the work of Keller on classical wave propagation through random medium [14]. Andrews

and Phillips created a framework by synthesizing the theory and making it comprehensive [15]. It is currently considered as the reference in free space optic (FSO).

With the appearance of computers, many scientists implemented programs to simulate the propagation of an EM wave through a random medium applying the aforementioned theory. Those simulations usually are performed with the split step beam propagation method (SSBPM). SSBPM is the discretisation of the turbulence into n phase screens (PS). Free propagation is executed between those PS. The next section focuses on the generation, number and location of the phase screens.

2.1.2 Study of the phase screens

Phase screens are thin layers of turbulence generated following a power spectrum (e.g. : Kolmogorov) and are applied onto the phase of the optical field. They represent slabs of atmosphere and distort the wavefront of the optical wave. To generate PS, one needs to color white noise using the power spectrum of atmospheric turbulence as explained by the university of Dayton [16]. There are multiples ways to generate PS. The most common one is using discrete Fourier transform (DFT). To enhance the low frequency spectrum, one can add subharmonics to the DFT generation (DFT-SH). The sparse spectrum (SS) method can also be used to generate PS if one phase sample is assumed to only contain a handful of spectral components and if the amplitude and wave vector are random. In 2019, Daniel Paulson et al. combined both the DFT and SS method improving the efficiency [17]. Finally, the sparse spectrum with uniform distributions (SSUD) is developed by Charnotskii as the continuity of SS [18].

PS can be correlated or independent. The situation of independent screens is equivalent to the Markov approximation used in analytical studies and allows for comparison between theory and simulation [19]. The idea is the generation of one PS is not impacting the generation of another PS. Moreover, the PS are generated by pairs (one is found in the real part and the other in the imaginary part) which are also independent [20]. Correlated phase screens have also been investigated. It considers the medium to move slowly compared to the transmission of information and thus the two following propagations have very similar PS. Correlated PS have been investigated by Naeh and Katzir through a sparse spectrum harmonic augmentation (SSHA) method which produces exact correlated PS [21].

There are multiple ways of performing SSBPM. The number of phase screens n to represent a given distance z is a parameter to take into account in a simulation. Indeed, increasing n improves the performances and gives a closer result to the physical one. This is possible for small grids. In the current case, $N = 256$ pixels, implying a larger computational time. Therefore, there is a trade-off between time and accuracy as explained by the NASA [22]. Troxel et al. used only 4 PS for a simulation placing them at 200 m, 2 km, 10 km and 18 km matching the atmospheric turbulence with around 1% error margin as calculated in an ad hoc study [23].

An university in China recently published a document on a split-step wavelet (SSW) method with the objective of reducing the computational time. This efficient method propagates each wavelet and then perform a linear combination of the results [24].

The position of the PS along the path hugely impacts the performances. In 2016, a Chinese institute minimised an error function due to the approximation of the PS and obtained an optimal location for each PS [25]. Another approach found by Martin and Flatté assumed that the contribution of each slab of atmosphere should not be too important to the global scintillation index (measure of the quality of the transmission) [26]. Finally, a good technique is to match the discrete and continuous moments of the turbulence (C_n^2) using weights. Those weights determine where to place the PS within the slab [9, 22].

In this work, the DFT-SH is implemented with $n = 14$ independent PS in order to propagate over a distance of 24 km. The placement of those PS is studied in section 4.4. The DFT generation is also explained.

2.1.3 Comparison with radio frequency

Propagation of information through the atmosphere is of interest since the apparition of the telecommunications. Optical transmission is one of the multiple ways for communication. In this section, focus is laid on radio frequency (RF) communication by comparing it to FSO.

RF appeared at the end of the 19th with its discovery by H.Hertz but G.Marconi understood the concept of a carrier years later [27]. This was the beginning of wireless communications. RF evolved a lot since then with the emergence of the internet or the global system for mobile communications (GSM). Nowadays, elaborate systems are implemented such as multiple inputs multiple outputs (MIMO) antennas array for the 4G/5G.

	RF	FSO
Transmission rate	Less than 100 Mbps	100 Mbps to 10 Gbps
Source of signal degradation	Multipath fading Rain User interferences	Atmospheric turbulence Obscuration Pointing errors Geometric losses
Line of sight (LoS)	No	Yes
Side lobes	Yes	No
Channel security	Weak	Strong
Component dimensions	Large	Small
Power consumption	High (23.1 mJ/Mb)	Low (2 mJ/Mb)
Bandwidth	3 kHz - 300 GHz	0.8 THz - 1.5 THz

Table 2.1: Comparison between radio frequency (RF) and free space optics (FSO) communications.

Both RF and FSO presents advantages and drawbacks. Table 2.1 displays the different parameters of interest when communicating. The values and information for this section were found in the literature [4, 28, 29].

The first element of comparison analysed is the transmission rate. It is one of the biggest advantage of FSO over RF. In 2018, a data rate of 10 Gbps was achieved between two satellite using FSO while RF transmissions transmits at a rate lower than 100 Mbps. With the increasing need of communication, FSO is a good solution.

Another point to take into account when comparing the two transmission technologies is the degradation of the signal. The sources of degradation are not the same between RF and FSO. The signal is degraded when crossing the atmosphere whether it is optical or radio frequency communication. When using RF, rain is a major cause of attenuation. Indeed, for certain wavelengths in the range of RF, there are high peaks of absorption by H_2O . This also occurs for O_2 . For FSO communication, the path must be clear as the propagation occurs with line of sight (LoS). The impact of fog, haze, rain, ... on FSO is studied in [28] outlining the attenuation. On top of that, Rayleigh scattering influence the propagation as well. This report studies the impact of atmospheric turbulence but other non-atmospheric parameters also influence the communication. For RF, the presence of sidelobes at the antenna is causing multipath fading with a loss of power. User employing the same frequencies will also face interference problems. For FSO, there is a need for the sender and receiver to be align otherwise the information would be lost.

Security of the information transmitted is primordial. The security for RF is poor as anyone could intercept the signal using an antenna. Therefore, the signal needs encryption. For FSO, it is more difficult as the intruder should be in the LoS to detect the signal.

With the increasing number of satellite, technical improvement have to be considered. One of them is the weight and dimensions of the components. FSO transmitter and receiver are lighter and smaller than for RF therefore facilitating the implementation. On the same topic, the power consumption needed for communication is a parameter to optimise. Once again, using FSO is more interesting as it consumes 2 mJ/M which is less than RF (23.1 mJ/Mbb) [29].

The last element of comparison is the bandwidth. FSO allows a bandwidth of 0.8 THz - 1.5 THz which is significantly higher than for RF (3 kHz - 300 GHz). When looking at the signal to noise ratio (SNR) and the noise equivalent power (NEP) needed for transmission. Assuming identical bandwidth, RF perform better than FSO by tens of dB according to [3]. However, when taking into account the bandwidth available in RF and optical transmission, FSO performs better as more bandwidth is allocated. The frequencies allocated to RF are often limited due to the transmitter or receiver [3].

To sum up, FSO seems to be a good solution to compensate for the lack of performances of the RF communication. In the next section, the theoretical background necessary to perform FSO simulations is developed starting with the structure of the earth atmosphere.

2.2 The Earth atmosphere

In this section, the Earth turbulent atmosphere is studied starting with the different layers of atmosphere. Afterwards, the turbulence is analysed to go from the refractive index fluctuations to an associated power spectrum density (PSD). This PSD will be necessary to implement the PS in chapter 5.

2.2.1 Layers of atmosphere and impact on temperature

The Earth atmosphere extends from ground level to about 10 000 km in the sky [30] but most of the air mass is concentrated between 0 km and 24 km (roughly 75% of the air mass between

0 km and 11 km [15]). Above this height, there is not enough air and thus turbulence to impact the optical waves [22, 31]. This statement will also be observe in section 5.1 with the turbulence profiles. The atmosphere is composed of different layers, each having different temperatures and thickness. The variation of temperature in the atmosphere is mainly responsible for the change in refractive index [15]. This variation is causing turbulence and will be used to create turbulence profiles. Figure 2.1 represents the different layers at their respective heights and the associated variation of temperature [32]. The fluctuations in temperatures are the result of several phenomena. For example, absorption of ultraviolet rays in the stratosphere induces a temperature increase [15, 31, 33].

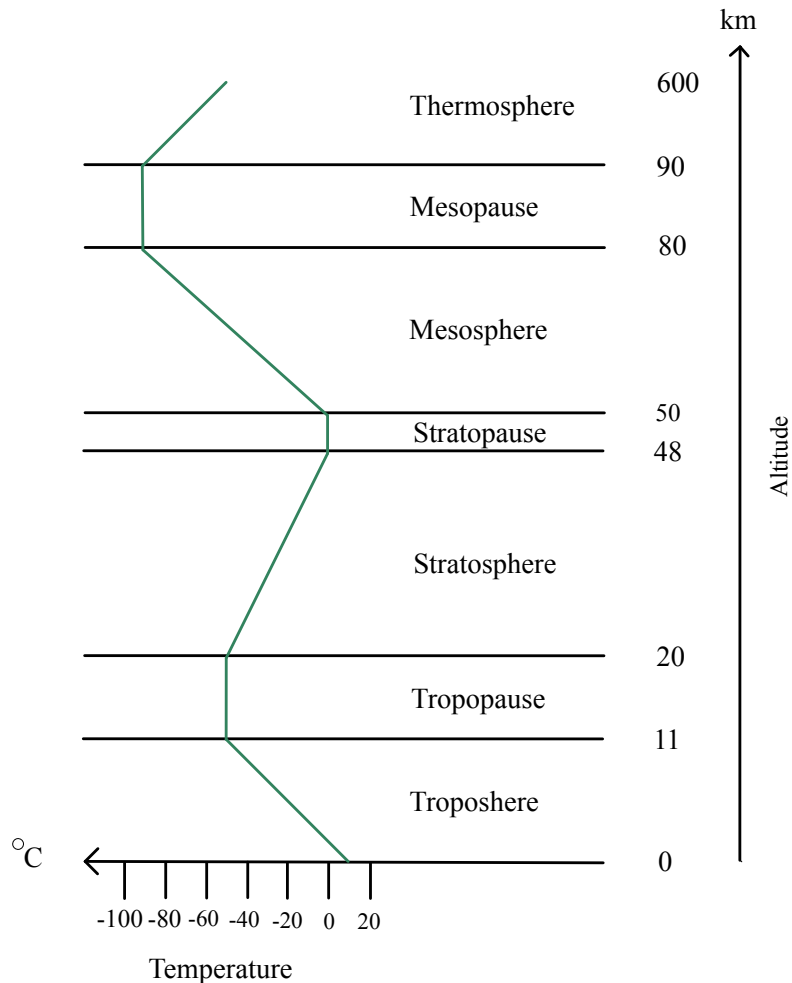


Figure 2.1: Evolution of the temperature (green) in the Earth atmosphere with the altitude. The names of the layers and their range are stated on the right.

2.2.2 Continuum of turbulence

To model the turbulence it is essential to understand its behaviour across the atmosphere. To do so, one must know that fluids are classified between two states of motion : laminar or turbulent. The distinction between both is made through the Reynolds number (Re) depending on the wind velocity (w), the dimension of the flow (l) and the kinematic viscosity (ν) (in units of m/s, m and m^2/s). A large Reynolds number represents a turbulent motion [15, 32].

$$Re = \frac{wl}{\nu} \quad (2.1)$$

With a tool to distinguish the two states, the continuum of turbulence also known as energy cascade theory can be explained. At high altitude, there is an energy injection due to strong winds or convection as represented on Figure 2.2. The wind increases and so does the Reynolds number until it reach a critical level : the boundary between laminar and turbulent motions. At this point, the air mass becomes unstable and eddies begin to appear. To reduce the Reynolds number, the large eddies are broken into smaller eddies due to large inertial forces, decreasing therefore the dimension of the flow l (see Figure 2.2). This reduction increases the amount of energy dissipated by viscous forces. The scheme is repeated creating the continuum between flow dimensions L_0 and l_0 , respectively called the outer and inner scale. They represent the maximal and minimal flow dimensions and are the boundaries of the inertial subrange. When reaching l_0 the Reynolds number is around the unity [15, 34].

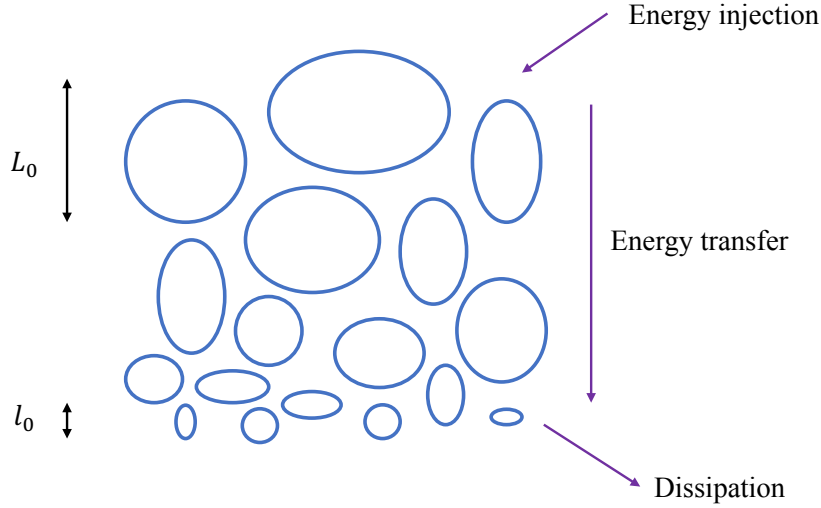


Figure 2.2: Eddies continuum across the atmosphere from large scale flows of size L_0 to small scale of size l_0 .

Outside the inertial subrange, various phenomena occur. For dimension flow larger than L_0 , the eddies are considered non-isotropic as the horizontal dimensions become much larger than the vertical one. For l smaller than l_0 , the viscous forces are strong enough to dissipate the turbulence into heat.

2.2.3 Random process

Study of random process is important when treating a subject such as atmospheric turbulence. Therefore, a resume is done in this section to introduce the necessary theoretical background.

To start, one must know a random variable ($x(t)$) can be separated in two parts, as shown in equation (2.2). The stationary part is the mean ($m_x(t)$) and the varying part is $x_1(t)$, with the peculiarity : $\langle x_1(t) \rangle = 0$. This statement impose a distribution with zero mean for the fluctuations.

$$x(t) = m_x(t) + x_1(t) \quad (2.2)$$

A random process is usually seen as stationary but it is not the case for atmospheric parameters such as the wind velocity, temperature or index of refraction fluctuations. Indeed, their mean are constant only over a short time period. Such functions have slow-varying means requiring some important hypothesis used later in this work. To alleviate the problem, instead of working with the random variable $x(t)$, the work is performed with $x(t + \tau) - x(t)$ hence taking into account a short time period (τ). With this idea in mind, one can study the structure function D_x of a field x . The definition of the structure function is given in equation (2.3).

$$\begin{aligned} D_x(t, t + \tau) &= \langle [x(t) - x(t + \tau)]^2 \rangle \\ &= [m_x(t) - m_x(t + \tau)]^2 + \langle [x_1(t) - x_1(t + \tau)]^2 \rangle \\ &\approx \langle [x_1(t) - x_1(t + \tau)]^2 \rangle \end{aligned} \quad (2.3)$$

Using the property of slow-varying mean, $m_x(t) \approx m_x(t + \tau)$, thus the stationary part is equal to 0 and the expression is approximated to only the varying part as illustrated in equation (2.3).

2.2.3.1 Velocity fluctuations

Historically in turbulence theory, the first atmospheric parameter described by a structure function is the fluctuation in the wind velocity. Therefore, it is used to establish the theoretical background for the temperature and refractive index fluctuations which have the same behaviour.

With the same idea as section 2.2.3, the wind fluctuation can be expressed as equation (2.4). The analysis performed in the scope of this work is a spatial variation and not a temporal one thus w is expressed in terms of $\mathbf{r}$. The variable $\mathbf{r}$ is the location in a three dimensional environment ($\mathbf{r} = (x, y, z)$). The temporal variable is not studied as the speed of light is much greater than the speed changes of the atmospheric parameters. This hypothesis is called frozen atmosphere.

$$w(\mathbf{r}) = m_w(\mathbf{r}) + w_1(\mathbf{r}) \quad (2.4)$$

Introducing a vector $\mathbf{R}$ as the distance between the two studied locations allows to work on structure functions. The hypothesis of statistically homogeneous and isotropic atmosphere as well as slow-varying mean must still be valid. The wind structure function D_w is given in equation (2.5).

$$\begin{aligned} D_w(\mathbf{r}, \mathbf{r} + \mathbf{R}) &= \langle [w(\mathbf{r}) - w(\mathbf{r} + \mathbf{R})]^2 \rangle \\ &= [m_w(\mathbf{r}) - m_w(\mathbf{r} + \mathbf{R})]^2 + \langle [w_1(\mathbf{r}) - w_1(\mathbf{r} + \mathbf{R})]^2 \rangle \\ &\approx \langle [w_1(\mathbf{r}) - w_1(\mathbf{r} + \mathbf{R})]^2 \rangle \end{aligned} \quad (2.5)$$

In early works, Kolmogorov proved that the fluctuation in velocities between two points is only depending on the distance between those two points (R) to the power of $2/3$ and to a structure constant (C_w^2). R is the magnitude of the vector $\mathbf{R}$ ($R = |\mathbf{R}|$). Kolmogorov's formula is only valid in the inertial subrange. For $R \leq l_0$, another formula depending on the square of the distance was found and contains a $l_0^{-4/3}$ factor to match the expression of Kolmogorov when $R = l_0$.

$$D_w(R) = \begin{cases} C_w^2 l_0^{-4/3} R^2 & 0 \leq R \ll l_0 \\ C_w^2 R^{2/3} & l_0 \ll R \ll L_0 \end{cases} \quad (2.6)$$

2.2.3.2 Temperature fluctuations

Turbulence is mainly caused by fluctuations in the refractive index. Those fluctuations are very sensitive to the temperature variations and therefore it is primordial to understand their behaviour. The definition of structure function is used as foundation for equation (2.7) expressing the variation in temperature between two points.

$$D_T(\mathbf{r}, \mathbf{r} + \mathbf{R}) = \langle (T(\mathbf{r}) - T(\mathbf{r} + \mathbf{R}))^2 \rangle \quad (2.7)$$

The theory of Kolmogorov for structure functions on wind fluctuations can be extended to conservative passive scalars like the potential temperature giving equation (2.8). Under the same hypothesis, D_T is expressed as a function of the structure constant (C_T^2) the inner scale (l_0) and the magnitude of the distance between the two analysed points (R). The 2/3 power law and the quadratic behaviour are the same as in equation (2.6).

$$D_T(R) = \begin{cases} C_T^2 l_0^{-4/3} R^2 & 0 \leq R \ll l_0 \\ C_T^2 R^{2/3} & l_0 \ll R \ll L_0 \end{cases} \quad (2.8)$$

2.2.3.3 Refractive index fluctuations

As stated previously, the refractive index has a strong correlation to the temperature variation. To analyse the evolution of n when T is fluctuating, one must start back from a statistical point of view (see section 2.2.3). For a random location at a random time, the index of refraction is given by equation (2.9), where n_0 is the mean and n_1 is the varying part. In the case of the refractive index of the Earth atmosphere, $n_0(\mathbf{r}, t) = 1$. One must also recall from section 2.2.3 that $\langle n_1(\mathbf{r}, t) \rangle = 0$. Finally, in equation (2.9), the time dependence is removed because of the hypothesis of frozen atmosphere

$$\begin{aligned} n(\mathbf{r}, t) &= n_0(\mathbf{r}, t) + n_1(\mathbf{r}, t) \iff \\ n(\mathbf{r}) &= 1 + n_1(\mathbf{r}) \end{aligned} \quad (2.9)$$

The varying part of equation (2.9) can be expressed in terms of the optical wavelength (λ) in micro meters, the pressure ($P(\mathbf{r})$) in millibars and the temperature ($T(\mathbf{r})$) in kelvin as done in equation (2.10). For a classical wavelength in optics of around 500 nm, the equation is approximated hereunder. Note that the pressure variations are neglectable in comparison to temperature fluctuations [34].

$$\begin{aligned} n(\mathbf{r}) &= 1 + 77.6 \times 10^{-6} (1 + 7.52 \times 10^{-3} \lambda^{-2}) \frac{P(\mathbf{r})}{T(\mathbf{r})} \\ &\approx 1 + 79 \times 10^{-6} \frac{P(\mathbf{r})}{T(\mathbf{r})} \end{aligned} \quad (2.10)$$

As performed previously, the definition of the structure function allows to express the variation of refractive index between two locations separated by a distance $R = |\mathbf{R}|$. D_n is shown in equation (2.11).

$$D_n(\mathbf{r}, \mathbf{r} + \mathbf{R}) = \langle (n(\mathbf{r}) - n(\mathbf{r} + \mathbf{R}))^2 \rangle \quad (2.11)$$

Under the same assumptions, the structure function of the refractive index follows the same evolution as the ones of the wind velocity and temperature. C_n^2 is in $\text{m}^{-2/3}$ as the structure function for the refractive index has no unit.

$$D_n(R) = \begin{cases} C_n^2 l_0^{-4/3} R^2 & 0 \leq R \ll l_0 \\ C_n^2 R^{2/3} & l_0 \ll R \ll L_0 \end{cases} \quad (2.12)$$

On Figure 2.3, the structure function of the refractive index is plotted as a function of the distance between the two analysed points to illustrate the $2/3$ (orange line) and quadratic (blue line) power laws. The intersection of these two lines is found when the distance R is equal to the inner scale (l_0). In real conditions, a smooth transition between the two regimes appears.

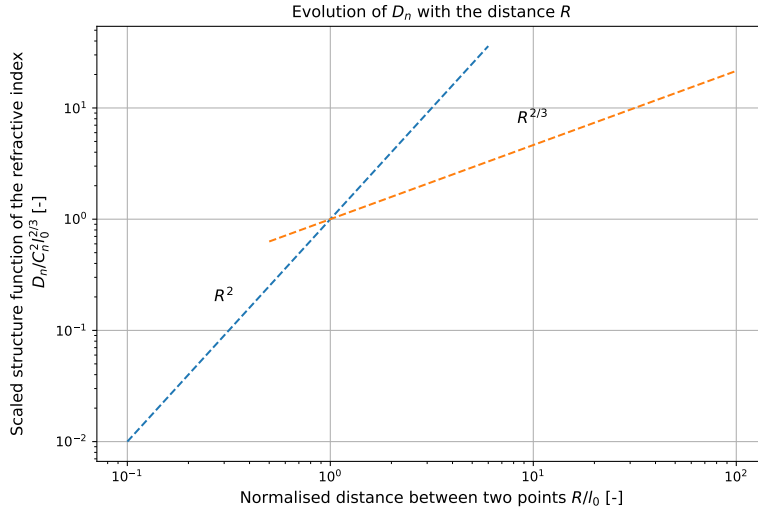


Figure 2.3: 2 regimes of the structure function for the refractive index. R^2 evolution for $R < l_0$ (blue) and $R^{2/3}$ for $R > l_0$ (orange).

Substituting equation (2.10) into equation (2.11) and combining with equation (2.7) gives equation (2.13).

$$\begin{aligned} D_n(\mathbf{r}, \mathbf{r} + \mathbf{R}) &= \left\langle \left(1 + 79 \times 10^{-6} \frac{P(\mathbf{r})}{T(\mathbf{r})} - \left(1 + 79 \times 10^{-6} \frac{P(\mathbf{r} + \mathbf{R})}{T(\mathbf{r} + \mathbf{R})} \right) \right)^2 \right\rangle \\ &= (79 \times 10^{-6})^2 \left\langle \left(\frac{P(\mathbf{r})}{T(\mathbf{r})} - \frac{P(\mathbf{r} + \mathbf{R})}{T(\mathbf{r} + \mathbf{R})} \right)^2 \right\rangle \\ &\stackrel{(h_1)}{=} (79 \times 10^{-6} P)^2 \left\langle \left(\frac{1}{T(\mathbf{r})} - \frac{1}{T(\mathbf{r} + \mathbf{R})} \right)^2 \right\rangle \\ &= (79 \times 10^{-6} P)^2 \left\langle \left(\frac{T(\mathbf{r} + \mathbf{R}) - T(\mathbf{r})}{T(\mathbf{r})T(\mathbf{r} + \mathbf{R})} \right)^2 \right\rangle \\ &\stackrel{(h_2)}{=} \left(79 \times 10^{-6} \frac{P}{T^2} \right)^2 D_T(\mathbf{r}, \mathbf{r} + \mathbf{R}) \end{aligned} \quad (2.13)$$

Two hypothesis are made in the mathematical development :

- h_1 : The pressure is constant : $P(\mathbf{r}) = P(\mathbf{r} + \mathbf{R}) = P$
- h_2 : if $\mathbf{R}$ is not too large, the temperature variation can be approximated by : $T(\mathbf{r})T(\mathbf{r} + \mathbf{R}) \approx T^2(\mathbf{r}) = T^2$

From the last expression in equation (2.13) and because D_n and D_T both evolve according to $R^{2/3}$ in the inertial subrange as shown in equations (2.8) and (2.12), the structure functions are replaced by their structure constant. The result is displayed in equation (2.14) [35].

$$C_n^2 = \left(79 \times 10^{-6} \frac{P}{T^2} \right)^2 C_T^2 \quad (2.14)$$

Nowadays, balloons are sent into space with precise thermometers picking up temperature measurements each couple of meters allowing for C_T^2 values to be computed using equation (2.8). With the last relationship developed, C_n^2 can be computed even though values of refractive index are not measurable [15].

C_n^2 values are varying with the altitude and different profiles are representing the Earth turbulence. One of the most used is the Hufnagel-Valley 5/7 profile. C_n^2 values are usually found between $1 \times 10^{-17} \text{ m}^{-2/3}$ for weak turbulence and $1 \times 10^{-13} \text{ m}^{-2/3}$ for strong turbulence.

The next section will transform the structure function into power spectrum using the freshly acquired knowledge of structure functions dependency on R .

2.2.4 Power spectrum

This section aims at deriving the Kolmogorov power spectrum using the structure functions. Afterwards, different power spectra are defined to improve the behaviour of the Kolmogorov power spectrum at low and high frequencies. The Tatarskii spectrum impact the high frequencies at a threshold depending on l_0 while the modified von Karman spectrum add a threshold to the Tatarskii spectrum on the low frequencies depending on L_0 .

In section 2.2.3, the random fluctuations of the wind, temperature and refractive index were considered using two measurements as they are not stationary process. Under this hypothesis and with the assumption of slow-varying mean, the structure function was approximated by equation (2.3). According to [36, 37], in the situation of a locally homogeneous random field x , the structure function D_x is defined by equation (2.15) where $\boldsymbol{\kappa}$ is the wavenumber vector in three dimensions : $\boldsymbol{\kappa} = (\kappa_x, \kappa_y, \kappa_z)$.

$$D_x(\mathbf{r}) = 2 \int_{-\infty}^{\infty} \Phi_x(\boldsymbol{\kappa}) [1 - \cos(\boldsymbol{\kappa} \cdot \mathbf{r})] d\boldsymbol{\kappa} \quad (2.15)$$

Under the assumption of a locally homogeneous and isotropic field, the previous integral is simplified and can be seen as a 1 dimensional integral of the magnitude of the wavenumber vector $\kappa = |\boldsymbol{\kappa}|$ related to spatial frequencies.

$$D_x(R) = 8\pi \int_0^{\infty} \kappa^2 \Phi_x(\kappa) \left(1 - \frac{\sin(\kappa R)}{\kappa R} \right) d\kappa \quad (2.16)$$

Finally an expression of the power spectrum of the field x depending on the structure function is found in equation (2.17). With the knowledge of the evolution of D_x with R , one can find the power spectrum of the field x .

$$\Phi_x(\kappa) = \frac{1}{4\pi^2\kappa^2} \int_0^\infty \frac{\sin(\kappa R)}{\kappa R} \frac{d}{dR} \left[R^2 \frac{d}{dR} D_x(R) \right] dR \quad (2.17)$$

For the next part, the structure function of the refractive index found previously in equation (2.12) is used to find the Kolmogorov spectrum.

2.2.4.1 Kolmogorov

Plugging the structure function of the refractive index (D_n) from equation (2.12) in equation (2.17) results in the Kolmogorov's spectrum described by equation (2.18). The evolution is linear in log-scale with a slope of $-\frac{11}{3}$ and not bounded at the inner and outer scale. The validity of the spectrum is thus limited between l_0 and L_0 . Note that in certain cases, to have validity over all frequencies, l_0 is set to 0 and L_0 to ∞ but one needs to be careful.

$$\Phi_n(\kappa) = 0.033C_n^2\kappa^{-11/3} \quad (2.18)$$

2.2.4.2 Tatarskii

To improve the performances at high frequencies, Tatarskii slightly modified the work of Kolmogorov. In order to take the inner scale into account, he added a decreasing exponential factor. For values of $\kappa \geq \kappa_m$, the exponential tends to 0 and creates an upper boundary. For values of $\kappa < \kappa_m$, the exponential is close to 1 and doesn't affect the $-11/3$ power law.

$$\Phi_n(\kappa) = 0.033C_n^2\kappa^{-11/3}e^{-\left(\frac{\kappa}{\kappa_m}\right)^2} \quad \kappa \gg \frac{1}{L_0} \quad \kappa_m = \frac{5.92}{l_0} \quad (2.19)$$

In the current case, l_0 is set to 1 mm and κ_m equal 5920 m^{-1} using the formula in equation (2.19).

2.2.4.3 von Karman

The last upgrade to the Kolmogorov spectrum is the creation of a boundary at low frequencies depending on the outer scale. von Karman added to the $-11/3$ power law a denominator containing the variable κ_0 , usually defined as $\frac{2\pi}{L_0}$ but sometimes as $\frac{1}{L_0}$. The expression is the first in equation (2.20). For κ lower than κ_0 , the denominator tends to $\kappa_0^{11/3}$ and therefore the normalised spectrum (see y -axis on Figure 2.4) tends to $\kappa_0^{-11/3}$. For higher frequencies ($\kappa \geq \kappa_0$), the spectrum follows the power law of Kolmogorov.

To extend the validity of the formula to all the frequencies, he used Tatarskii's work and multiplied his formula with the decreasing exponential. This spectrum is known as the modified von Karman spectrum.

$$\Phi_n(\kappa) = \begin{cases} \frac{0.033C_n^2}{(\kappa^2 + \kappa_0^2)^{11/6}} & 0 \leq \kappa \ll \frac{1}{l_0} \\ \frac{0.033C_n^2}{(\kappa^2 + \kappa_0^2)^{11/6}} e^{-\left(\frac{\kappa}{\kappa_m}\right)^2} & 0 \leq \kappa < \infty \end{cases} \quad (2.20)$$

With a L_0 of 100 m, the variable κ_0 is set to 0.063 m^{-1} . For this value of κ_0 , the modified von Karman spectrum tends to around $25\,250 \text{ m}^{11/3}$ when $\kappa \rightarrow 0$.

Figure 2.4 shows the asymptotic behaviour of the modified von Karman (green) and Tatarskii (orange) models compared to the one of Kolmogorov (blue). The important values such as the inner and outer scale are reminded and the inertial subrange is limited by the values κ_0 ($= 0.063 \text{ m}^{-1}$) and κ_m ($= 5920 \text{ m}^{-1}$) [35].

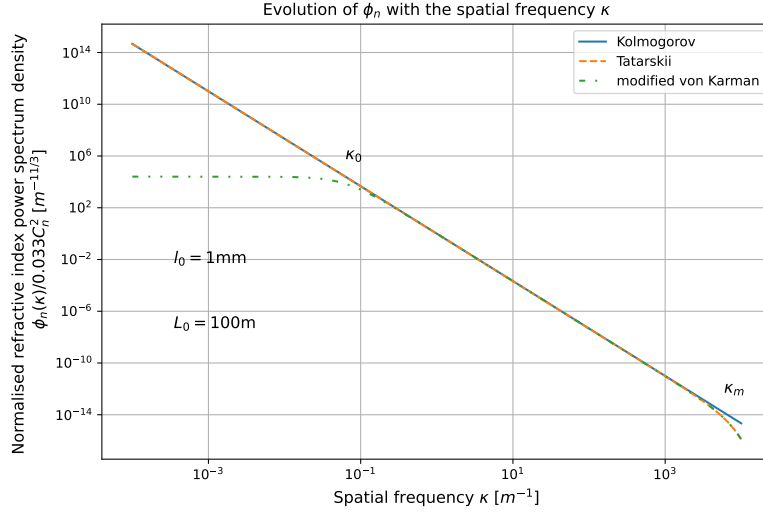


Figure 2.4: Evolution of the 3 power spectrum (Kolmogorov, Tatarskii and modified von Karman) of the refractive index (n) as a function of the frequencies κ . The outer and inner scale are set to $L_0 = 100 \text{ m}$ and $l_0 = 0.001 \text{ m}$.

2.2.4.4 Modified atmospheric

A last spectrum, called the modified atmospheric spectrum, is considered as it takes the physical behaviour of the atmosphere into account outside the inertial subrange. Indeed, the three previous models are physically valid between κ_0 and κ_m but outside are described for mathematical convenience and theoretical simulation. However, they are enough for the study conducted here.

$$\Phi_n(\kappa) = 0.033C_n^2 \left[1 + 1.802 \left(\frac{\kappa}{\kappa_l} \right) - 0.254 \left(\frac{\kappa}{\kappa_l} \right)^{7/6} \right] \frac{e^{-\left(\frac{\kappa}{\kappa_l} \right)^2}}{(\kappa^2 + \kappa_0^2)^{11/6}} \quad (2.21)$$

The modified atmosphere model has the power spectrum defined in equation (2.21) where $\kappa_l = \frac{3.3}{l_0}$. The formula has a physical meaning for all frequencies. To confirm, a plot of the spectrum at the upper limit is plotted in Figure 2.5.

When looking at Figure 2.5, one can observe that near $\kappa = \frac{1}{l_0}$, the modified atmosphere shows a bump whereas the 2 other spectra do not have this feature. From temperature measurements, a bump should be visible in the temperature spectrum and thus in the refractive-index spectrum.

In this report, the power spectrum used is the modified von Karman. It is the most complete spectrum compared to Kolmogorov and Tatarskii with the upper and lower boundaries. Its

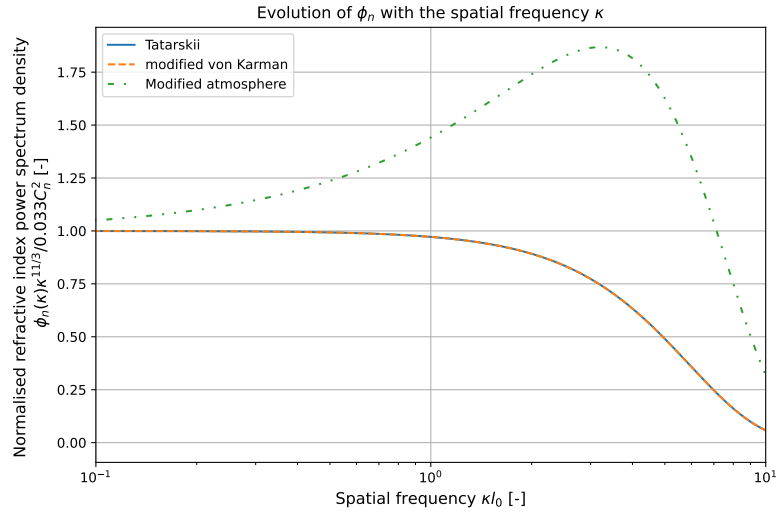


Figure 2.5: Zoom on the evolution of the 3 power spectrum (Tatarskii, modified von Karman and modified atmosphere) of the refractive index (n) as a function of the frequencies multiplied by the l_0 . The outer and inner scale are set to $L_0 = 100$ m and $l_0 = 0.001$ m.

validity in the subrange is sufficient for the work performed and there is no need to used the modified atmospheric spectrum which is much more complicated.

Chapter 3

Problem statement

The purpose of the master thesis is to develop a tool in *Python* allowing the simulation of turbulent propagation of optical waves through the atmosphere. It is a complex subject involving many new concepts. The problem is clearly stated to ensure a good comprehension of the global work.

First, the input grids in the spatial and frequency domains are defined. The dimensions of each window as well as the number of pixels and spacings associated are also determined. Then the amplitude and intensity of a typical input (Gaussian beam) are displayed on a spatial grid.

The second section dives into the Gaussian beam theory by linking the classical definition to the one defined in this report. Then the notion of radius and full width at half maximum are approached and explained in terms of the variance of the image.

The third section is the global process to achieve phase screen propagation. It develops the entire optical chain and introduces several concepts such as the growth of the beam as a result of diffraction and the angular resolution implemented to cope with this growth. The attenuation due to free space propagation is also presented.

Finally, after turbulent propagation of the optical field, a typical output of its amplitude and intensity is illustrated.

3.1 Input image

This section aims at defining the optical field at the transmitter. The input image is a squared pixel image defined by two quantities : N and S . N is the number of pixels per direction, hence there will be N^2 pixels in the image and S is the physical dimension of the image. Therefore, the aperture of the transmitter is of size S^2 . Using these two variables, the spatial size of one pixel can be computed with equation (3.1) and is denoted by δ . The spatial grid for the input image is shown on the left in Figure 3.1.

$$\delta = \frac{S}{N} \tag{3.1}$$

During the procedure of phase screen propagation, the optical field is transformed to the Fourier plane and thus a frequency grid has to be defined. Quantities related to turbulence

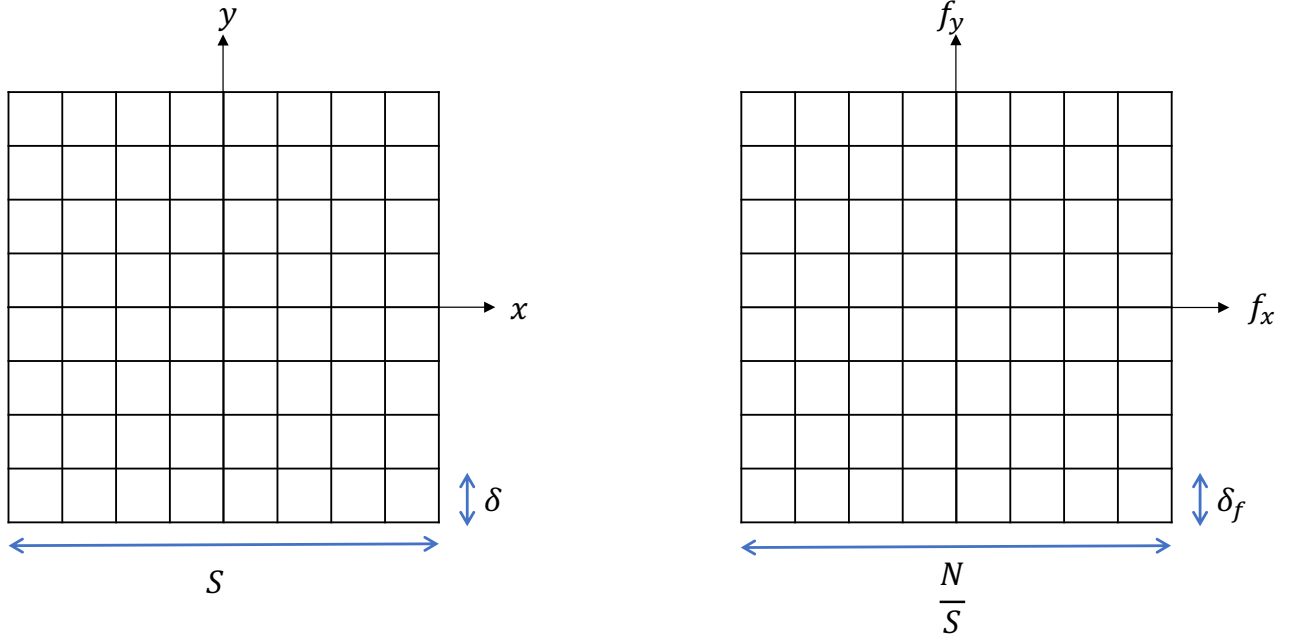


Figure 3.1: Spatial (left) and frequency (right) grids used for the optical fields. There are N pixels of size δ for the spatial grid and δ_f for the frequency grid.

are also defined in the frequency domain. The pixel size in the frequency domain is δ_f and its definition is given in equation (3.2). It only depends on the size of the window. Based on δ_f and the number of pixels N , the size of the window in frequency is $N\delta_f = \frac{N}{S}$ as represented on the right in Figure 3.1.

$$\delta_f = \frac{1}{N\delta} = \frac{1}{S} \quad (3.2)$$

For both cases, the $(0,0)$ coordinate is at the centre of the image. The axes defined around 0 have spacing of δ for the spatial grid and δ_f for the frequency grid. Moreover, in this report, the number N is always even and thus a symmetrical array around 0 is not possible. By convention, the axes range from $-\frac{N}{2}$ to $\frac{N}{2} - 1$. The axes x , y , f_x and f_y are created following equations (3.3) and (3.4).

$$x = y = \left[-\frac{N}{2}, -\frac{N}{2} + 1, \dots, -1, 0, 1, \dots, \frac{N}{2} - 2, \frac{N}{2} - 1 \right] \times \delta \quad (3.3)$$

In the frequency domain, having an even number of pixels is needed to avoid getting twice the same frequency. Indeed, the frequency domain is circular implying $-\frac{N}{2}\delta_f = \frac{N}{2}\delta_f$.

$$f_x = f_y = \left[-\frac{N}{2}, -\frac{N}{2} + 1, \dots, -1, 0, 1, \dots, \frac{N}{2} - 2, \frac{N}{2} - 1 \right] \times \delta_f \quad (3.4)$$

To improve the readability of the formulas in the report, the norm of a point $\mathbf{r} = (x, y)$ will be denoted by $r = \sqrt{x^2 + y^2}$ and the norm of $\mathbf{f} = (f_x, f_y)$ by $f = \sqrt{f_x^2 + f_y^2}$.

In the scope of this project, the transmission of a Gaussian beam is considered. The Gaussian has a radius w_0 at the sender and will be detailed in section 3.2. Moreover, the Gaussian is located in the middle of the frame so that all the information is contained within the plot.

The generic form of the optical field can be seen as the combination of an amplitude and a phase. Its form is described in equation (3.5) for the input field (index = 0) where $A_0(\mathbf{r})$ is the initial amplitude and $\phi_0(\mathbf{r})$ the initial phase. For the input, the phase is set to 0 and the amplitude is a Gaussian beam with maximum of 1 at the centre of the image ($A_0(\mathbf{r} = \mathbf{0}) = 1$). Under those conditions, $U_0(\mathbf{r}) = A_0(\mathbf{r})$ via equation (3.5).

$$U_0(\mathbf{r}) = A_0(\mathbf{r})e^{j\phi_0(\mathbf{r})} \quad (3.5)$$

The notion of intensity for an optical field needs to be introduced as it is what can be observed through a pupil (such as the human eye). Multiple measurements are also based on intensity. At the pixel $\mathbf{r}$, the intensity ($I(\mathbf{r})$) is found by multiplying the optical field and its complex conjugate at $\mathbf{r}$. Given one specific property of the complex conjugate, it is simply the square of the amplitude as written in equation (3.6). This relationship is valid during the entire propagation and not only at the input plane.

$$I(\mathbf{r}) = U(\mathbf{r})U^*(\mathbf{r}) = |U(\mathbf{r})|^2 = A^2(\mathbf{r}) \quad (3.6)$$

The input optical field is displayed using two representations : the amplitude of the optical field ($A_0(\mathbf{r})$) and its intensity ($I_0(\mathbf{r})$). Figure 3.2 shows the optical field's amplitude (left) and intensity (right) on a grid composed of 256^2 pixels and of dimensions $S_0 = 20$ cm with $w_0 = 2.12$ cm.

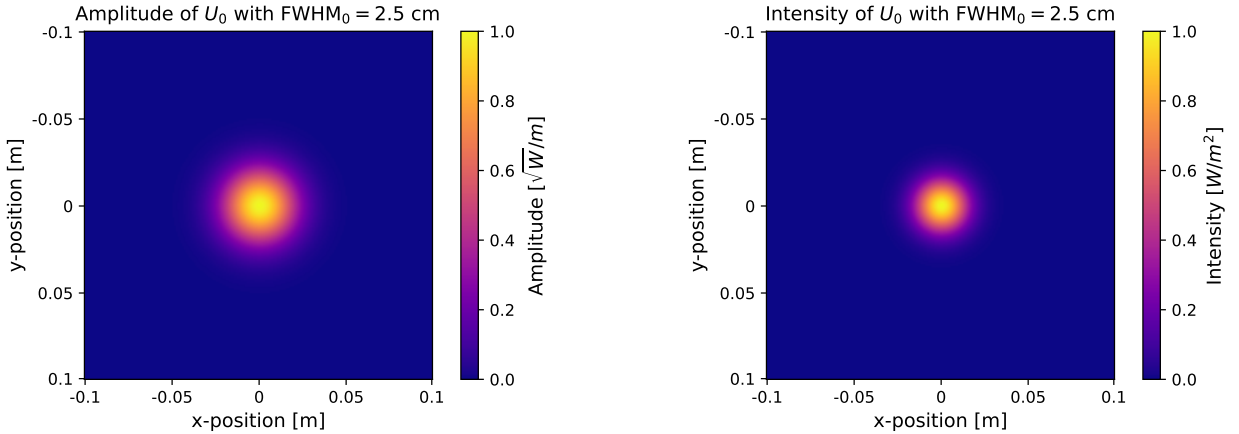


Figure 3.2: Amplitude (left) and intensity (right) of the input optical field. The aperture window is $S_0 = 20 \text{ cm} \times 20 \text{ cm}$, the number of pixel is $N^2 = 256^2$ and the intensity radius is $w_0 = 2.12 \text{ cm}$.

3.2 Gaussian beam

Before going any further, one must recall the definition of the intensity of a Gaussian beam in two dimensions. It is described in equation (3.7), where $\mathbf{r} = (x, y)$ and $\boldsymbol{\mu} = (\mu_x, \mu_y)$ are 2 dimensional vectors containing respectively the position on the grid and the mean in x and y directions. σ^2 is the variance, a measure of spreading of the beam.

$$I(\mathbf{r}|\boldsymbol{\mu}, \sigma) = e^{-\frac{|\mathbf{r}-\boldsymbol{\mu}|^2}{2\sigma^2}} = e^{-\frac{(x-\mu_x)^2 + (y-\mu_y)^2}{2\sigma^2}} \quad (3.7)$$

3.2. GAUSSIAN BEAM

For the purpose of this work, the Gaussian beam is considered centered, meaning $\boldsymbol{\mu} = (0, 0)$ and the variance (σ^2) depends on the parameter w_0 already mentioned in the previous section. w_0 is the radius of the intensity of the optical field at the input plane. This radius circles the area where the intensity has decreased by a factor $1/e^2$ (see purple line on the right graph of Figure 3.3) and is represented by the green line/circle on Figure 3.3. An equivalent explanation is a decrease of a factor $1/e$ for the amplitude $A_0(\mathbf{r})$.

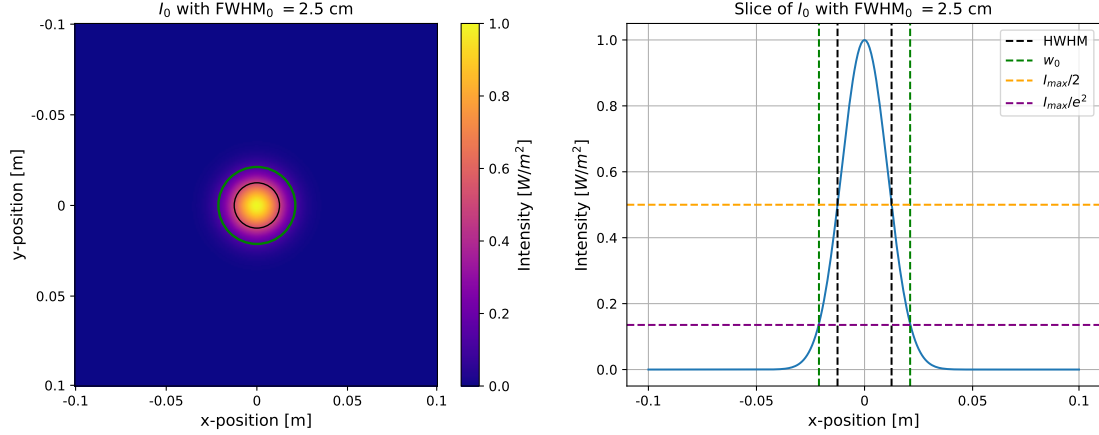


Figure 3.3: Initial Gaussian beam with $N = 256$ pixels, $S_0 = 20$ cm and $w_0 = 2.12$ cm. The left graph is the top view of the Gaussian beam and the right graph is a slice taken at the middle. The green and black lines/circles indicate respectively w_0 and FWHM_0 . The yellow and purple lines indicate a decrease of $1/2$ and $1/e^2$ in intensity.

In this report, the amplitude and intensity of a Gaussian optical field are defined following equation (3.8). The factor 2 in the second expression comes from the definition of the intensity (3.6).

$$A_0(\mathbf{r}) = U_0(\mathbf{r}) = e^{-\frac{|\mathbf{r}-\boldsymbol{\mu}|^2}{w_0^2}} \quad I_0(\mathbf{r}) = e^{-\frac{2|\mathbf{r}-\boldsymbol{\mu}|^2}{w_0^2}} \quad (3.8)$$

In the simulation, the field $U_0(\mathbf{r})$ is the one defined for the propagation but the expression of $I_0(\mathbf{r})$ helps to find a link between the radius w_0 and the standard deviation σ . Equation (3.9) is found by matching the arguments of the exponential of $I_0(\mathbf{r})$ in equations (3.7) and (3.8) setting the radius to twice the standard deviation.

$$\frac{w_0^2}{2} = 2\sigma^2 \iff w_0 = 2\sigma \quad (3.9)$$

The intensity of Gaussian beams is also characterised by their full width at half maximum (FWHM). It is the diameter of the circle created by the points reaching half the maximal intensity. In the example on Figure 3.3 the half maximal intensity is $A_0(\mathbf{r} = \mathbf{0})^2/2 = 0.5 \text{ W/m}^2$ (see yellow line on Figure 3.3). In terms of radius like it is done for w_0 , the half width at half maximum is introduced (HWHM) and is simply the FWHM divided by 2. The new parameter can be related to the radius w_0 and to the standard deviation σ according to equation (3.10). For the input image, the FWHM is written as FWHM_0 . A $\text{FWHM}_0 = 2.5$ cm is represented on Figure 3.3 by the black line/circle.

$$w_0 = \frac{\text{FWHM}_0}{\sqrt{2 \ln 2}} \quad \text{FWHM}_0 = 2\sqrt{2 \ln 2} \sigma \quad (3.10)$$

3.3 Phase screen propagation

Phase screen propagation is the method used in this work to simulate the propagation of an optical wave through a turbulent medium. The principle is to represent slabs of atmosphere by thin diffractive screens called phase screens (PS) and then perform free propagation over the slab width until reaching the following PS [38]. The diffractive screens are generated according to the altitude using a constant $C_{n,i}^2$ and are added to the phase of the optical wave therefore creating a distortion of the wavefront. The situation is represented on Figure 3.4 where the input, phase screen and output field are shown.

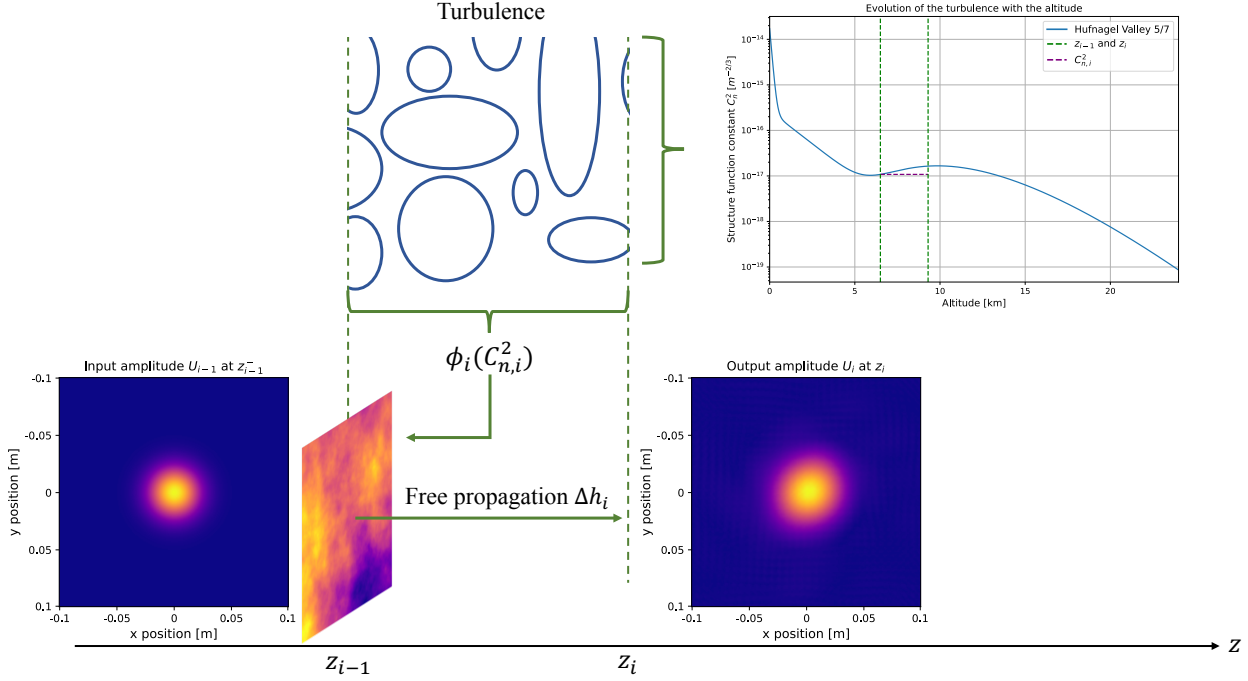


Figure 3.4: Representation of turbulence by associated PS with constant C_n^2 . The free propagation occurs over a distance Δh_i between z_{i-1} and z_i with the PS ϕ_i .

The geometry of the problem is also an important factor. The turbulent layer of width Δh_i between z_{i-1} and z_i is associated with a turbulent parameter $C_{n,i}^2$. By convention, the $C_{n,i}^2$ is taken at an altitude z_{i-1} . It is shown on Figure 3.4 with the dotted purple line. The $C_{n,i}^2$ is the value of the refractive index structure constant used to generated the PS ϕ_i . To perform a propagation over the 24 km of turbulent atmosphere, the i ranges from 1 to n where n is the number of free propagation/PS used.

Phase screens generation will be approached in section 5.1 but for the moment, it can be assumed to be a $C_{n,i}^2$ dependent two dimensional screen where each pixel represents the local phase variation to apply to the propagating optical field.

To apply the PS onto the wave, some notations are introduced. The optical fields just before and just after the application of the PS are written $U_i(x, y, z_i^-)$ and $U_i(x, y, z_i^+)$. The PS generated for the altitude z_i is $\phi_i(x, y)$ and by putting it all together the PS is applied to the field according to equation (3.11) [39].

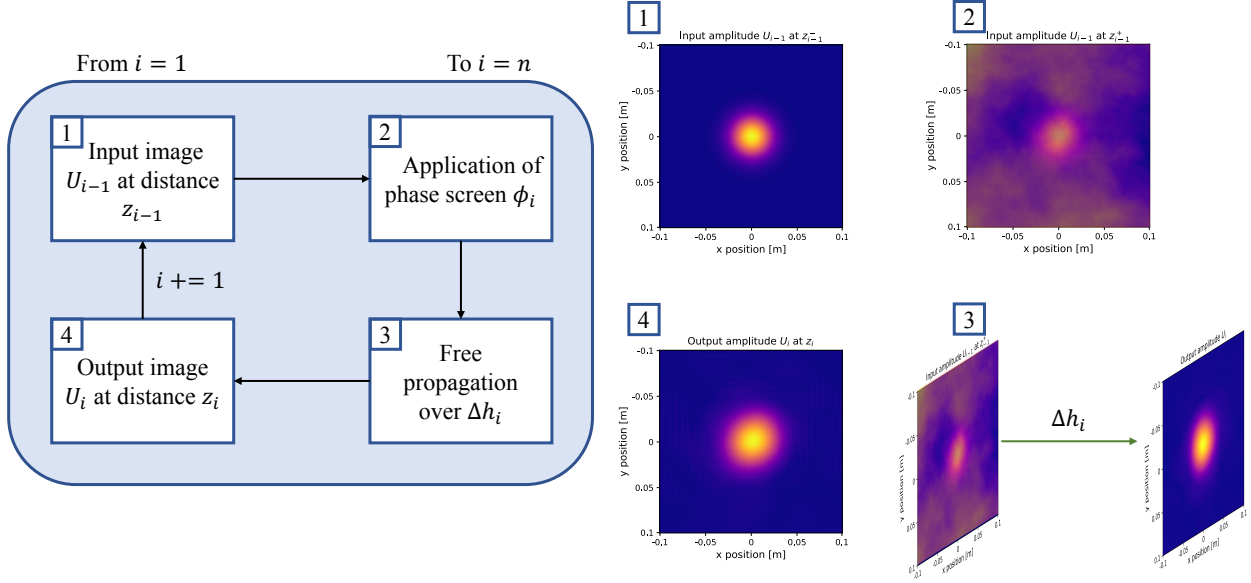


Figure 3.5: Phase screen propagation scheme. Propagation over a distance z_n with n PS. Δh_i is the separation between PS. The optical field propagated is the yellow circle and each purple arrow is a free propagation.

Only the phase of the optical field is impacted thus the amplitude before and after the application of the PS are equivalent ($A_i(x, y, z_i^-) = A_i(x, y, z_i^+)$). To observe modifications in the amplitude, one needs to free-propagate $U_i(x, y, z_i^+)$ further.

$$U_{i-1}(x, y, z_{i-1}^+) = U_{i-1}(x, y, z_{i-1}^-) e^{j\phi_i(x, y)} \quad (3.11)$$

The process of adding a PS and then propagating is repeated n times from the input to the output planes where n defines the number of phase screens generated. Figure 3.5 shows the principle with the growth and distortion of the Gaussian beam.

Finding a good n is important. On the one hand, using not enough layers is not representative of the real behaviour of the wave. On the other hand, applying modification in the phase of the wave too many times is time consuming and requires high computation forces while the improvements in precision are not significant. As a reference, James Osborn published a paper with 7 layers separated by unequal distances [40] for a propagation of 24 km. In this work, propagation with $n = 14$ layers are performed. Those layers are presented in chapter 4.

The presence of diffraction is visible on Figures 3.4 and 3.5. Diffraction is a phenomenon that always occurs when propagating and needs to be clarified. The next section approaches the subject.

3.3.1 Growth of the radius during propagation

When propagating, the radius of the intensity of the Gaussian optical field increases with the distance independently of the turbulence. This phenomenon is called diffraction and the behaviour of the intensity radius w_i is described by equation (3.12) where w_0 is the initial radius, z_i is the propagated distance and z_R is the Rayleigh distance [32]. The square root term is denoted m_i and symbolises the growth factor of the radius after a propagation of z_i .

This situation is represented on Figure 3.6.

$$w_i = w_0 \underbrace{\sqrt{1 + \left(\frac{z_i}{z_R}\right)^2}}_{m_i} \quad \text{where } z_R = \frac{\pi w_0^2 n}{\lambda} = \frac{\pi w_0^2}{\lambda} \quad (3.12)$$

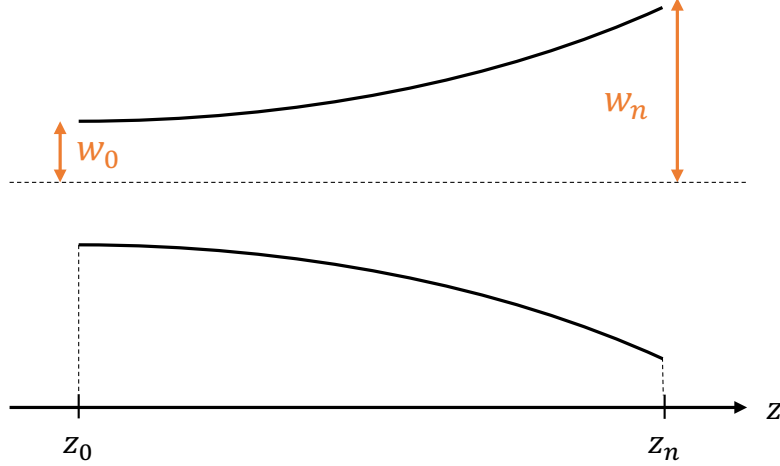


Figure 3.6: Evolution of the intensity radius (w_i) with the propagation distance (z_i). S_i is the window of propagation size at location i .

The weak turbulence is assumed to have little impact on the growth of the radius. This statement will be demonstrated in section 6.4. Under this assumption, turbulence is not considered in the growth of the radius. Therefore, the refractive index n is constant and equal to 1 hence z_R only depends on the initial radius and the wavelength (λ). The wavelength in this report is always 550 nm, a typical value in optical propagation [15].

Observations over the radius evolution and the spreading of the Gaussian beam are possible based on the initial radius. A narrow w_0 diffracts faster (with a larger angle) leading to a reduction of the received peak intensity. Indeed, the energy contained in the wave remains the same between the initial and final states meaning an equivalent area under the blue and orange curves on Figure 3.7. If the support at the output is wide, the Gaussian is flat and the intensity low.

To illustrate, a propagation of 8 km with 2 initial radius of $w_0 = 2.12$ cm (FWHM₀ of 2.5 cm) and $w_0 = 4.24$ cm (FWHM₀ of 5 cm) is simulated on Figure 3.7. The left graph represents the case of the small radius and the right graph the large radius. After computation, the output radius for the small and large cases are respectively $w_1 = 6.93$ cm and $w_1 = 5.38$ cm implying factors of $m_1 = 3.26$ and $m_1 = 1.27$. The factor of growth is the square root term in equation (3.12). In terms of peak intensities, for a FWHM₀ of 5 cm, the received peak intensity (orange curve) is 0.62 W/m². For a FWHM₀ twice smaller, the intensity drops to 0.094 W/m².

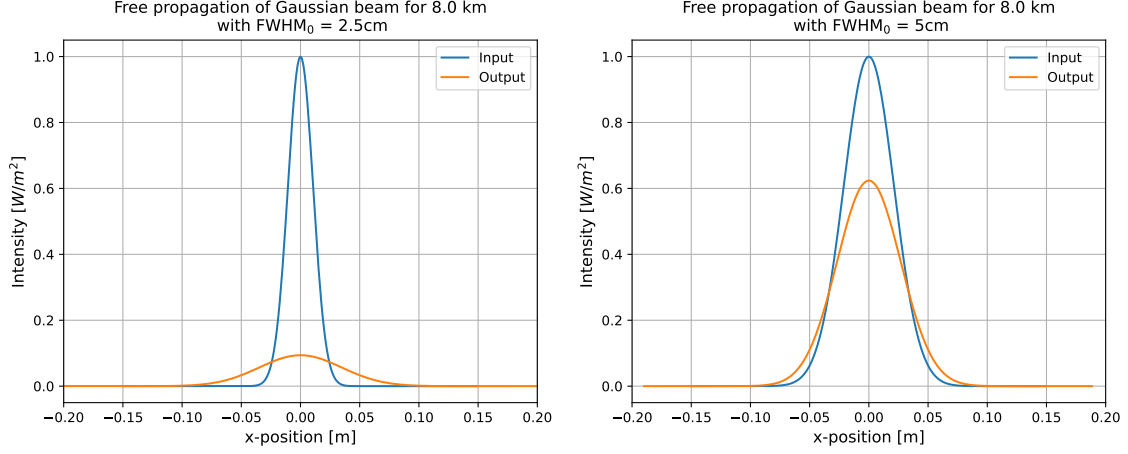


Figure 3.7: Input (blue) and output (orange) intensities for a 8 km free propagation with on the left a FWHM_0 of 2.5 cm and on the right a FWHM_0 of 5 cm.

3.3.2 Angular resolution : evolution of the δ

Angular resolution is implemented in the *Python* script implying the number of pixels (N^2) per image remains constant when propagating. To propagate, 2 dimensional Fourier transforms (FT) are used. It is important to keep N fixed because the complexity of an FT is $(N \log N)^2$ [41]. To keep N constant, the size of one pixel (δ) must be adapted as the size of the aperture (S) grows due to diffraction. More on this topic will be explained in section 4.2.1. From this point onwards, for a plane i , the pixel size will be referred to as δ_i , the size of the aperture as S_i and its radius as w_i . The input plane has a δ_0 whereas the output plane has a δ_n .

$$\delta_i = m_i \delta_0 \quad (3.13)$$

The objective is to find the value of the parameter m_i based on the propagation distance so that the same amount of pixels is dedicated to the representation of the Gaussian between each screen. In terms of equation, it is summarized by (3.13). This statement implies that the Gaussian beam has always the same size on the representations even though it is in fact growing. A way to find the parameter is to impose the spatial spacing to follow the same growing pattern as the intensity radius (see equation (3.12)).

Section 3.3.1 explains how the intensity of the propagated wave evolves. For the intensity of the wave to be represented with the same size whatever the distance, the initial spatial spacing (δ_0) is multiplied by the factor m_i so that δ_i grows like w_i .

It is also possible to prove this using the condition of a constant number of pixels N . With this basis, the parameters of the input and output planes can be linked through equation (3.14). The output plane spatial spacing (δ_i) is isolated.

$$N = \frac{S_0}{\delta_0} = \frac{S_i}{\delta_i} \iff \delta_i = \frac{S_i}{S_0} \delta_0 \quad (3.14)$$

To avoid having a growing Gaussian beam when plotting it, the ratio of aperture dimensions (S_i/S_0) must behave like the ratio of intensity diameters and thus the ratio of radius (w_i/w_0). This is shown in equation (3.15).

$$\delta_i(z_i) = \frac{w_i}{w_0} \delta_0 = \sqrt{1 + \left(\frac{z_i}{z_R}\right)^2} \delta_0 \quad (3.15)$$

3.3.3 Evolution of the optical field

The propagation modifies the input optical field ($U_0(\mathbf{r})$) described in section 3.2. The amplitude decreases with the propagation as shown by Figure 3.7. Comparing equation (3.16) with the definition of the input field (3.8), the drop in amplitude is equal to the ratio of radius intensity w_0/w_i multiplied by an exponential : $e^{(w_i/w_0)^2}$. The phase (ϕ_i) is becoming non-zero due to propagation. There are three contributions to the phase term : the first is the propagation of the wave itself, the second is the curvature of the wavefront and the last is the Gouy phase [32].

$$U_i(\mathbf{r}) = \underbrace{\frac{w_0}{w_i} e^{-\frac{|\mathbf{r}-\boldsymbol{\mu}_i|^2}{w_i^2}}}_{A_i(\mathbf{r})} e^{\overbrace{i(-kz_i - \frac{k|\mathbf{r}|^2}{2R_{c,i}} + \psi_i)}^{\phi_i(\mathbf{r})}} \quad (3.16)$$

The variable $R_{c,i}$ is called the radius of curvature at the location z_i . It is the radius of the propagated wavefront. At the input plane, $R_{c,0}$ tends to infinity and at $z_n \rightarrow \infty$, $R_{c,n}$ tends to 0. At the Rayleigh distance $z_i = z_R$, the radius of curvature is equal to $2z_R$. Its behaviour as a function of the distance is described by equation (3.17) [32].

$$R_{c,i} = z_i \left[1 + \left(\frac{z_i}{z_R}\right)^2 \right] \quad (3.17)$$

k is simply the wavenumber depending on the wavelength as follows : $k = \frac{2\pi}{\lambda}$. Finally, ψ_i is the value of the Gouy phase at distance z_i as shown in equation (3.18) [32].

$$\psi_i = \arctan\left(\frac{z_i}{z_R}\right) \quad (3.18)$$

Finally, if the amplitude of the optical field changes during the propagation, so will its intensity. This variation is displayed in equation (3.19) by using the definition of the intensity in (3.6).

$$I_i(\mathbf{r}) = A_i^2(\mathbf{r}) = \left(\frac{w_0}{w_i}\right)^2 e^{-\frac{2|\mathbf{r}-\boldsymbol{\mu}_i|^2}{w_i^2}} \quad (3.19)$$

3.4 Output image

Figure 3.8 is a typical output after a propagation of $z = 24$ km using $n = 14$ PS. As always, the amplitude (left) and intensity (right) are represented. The image parameters at the input are the same as in section 3.1 : $S_0 = 20$ cm, $N = 256$ pixels and $\text{FWHM}_0 = 2.5$ cm. Under these conditions, the output dimensions are $S_{14} = 1.88$ m, $N = 256$ pixels and $\text{FWHM}_{14} = 24.8$ cm.

The distortion of the wavefront caused by the turbulence is visible on Figure 3.8. The turbulence profile across the atmosphere is weak and the received amplitude and intensity are not completely changed.

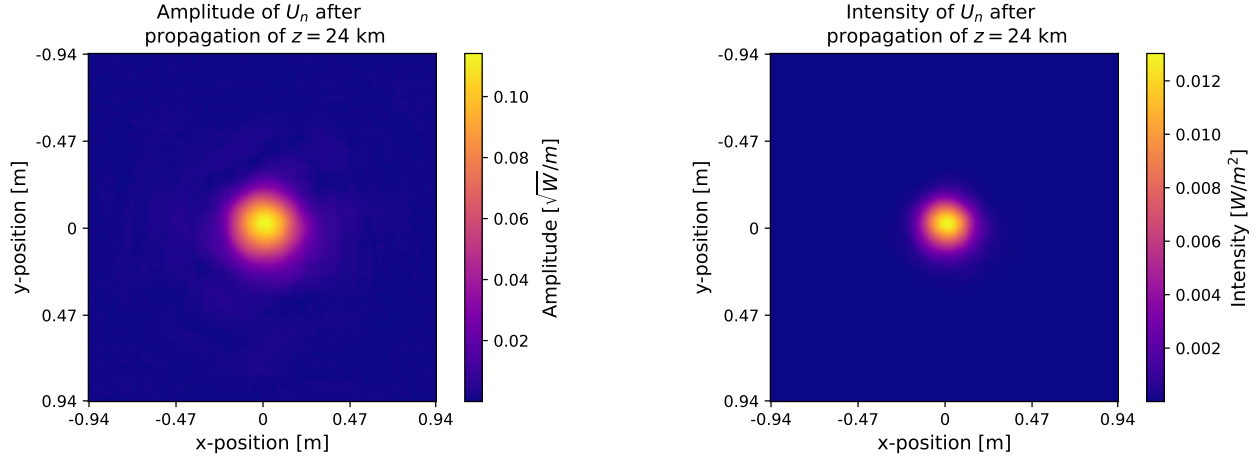


Figure 3.8: Output amplitude (left) and intensity (right) of the optical field ($U_n(\mathbf{r})$) after a propagation of 24 km.

The last point of this chapter is to establish the perfect tracking hypothesis for a real application where the output plane is large. Above 24 km, the optical field propagates in free space. A communication between the ground and a low Earth orbit (LEO) satellite covers a propagation distance of 400 km. At the output plane the aperture reaches the large size of 31.76 m $\times$ 31.76 m and thus the pixel size is 12.4 cm $\times$ 12.4 cm. It is impossible for the satellite to capture all the information contained in such a large aperture. The hypothesis of perfect tracking is assuming only one pixel is captured and it is always the same implying a perfect alignment between the base station and the satellite. The pixel considered in this report is the central pixel $(\frac{N}{2}, \frac{N}{2})$.

Chapter 4

Free Propagation

Free propagation is used to describe the evolution of the optical wave between two phase screens. The simulated propagation must reproduce the physical behaviour of light in free space. To obtain this, the mathematical development is started with the wave equation governing all EM waves propagation.

The first part starts from the exact solution of the wave equation developed by Rayleigh-Sommerfeld's and applied Kirchoff's approximation. Afterwards, under the assumption of a paraxial propagation and using mathematical tricks, one can find the Fresnel free propagation formula to implement, using computer simulations. The use of 2 dimensional Fourier transforms is essential in this chapter and therefore those transforms are investigated.

The second part approaches the sampling conditions due to the discretisation. Those conditions are due to the geometry of the problem and to the Fourier transforms.

The mathematical development in this chapter mainly comes from the book written by J.Schmidt [9].

4.1 Fresnel approximation of Rayleigh-Sommerfeld

The behaviour of EM wave is governed by the four Maxwell equations. The wave equation can be derived from these four equations and the result is displayed in the left expression of equation (4.1) where k is the wavenumber and $\mathbf{E}(\mathbf{r}, t)$ is the electric field. The electric field is a vector with three components, one for each direction ($\mathbf{E} = (E_x, E_y, E_z)$). The optical field U is one of the component of $\mathbf{E}$ as written in the right expression in equation (4.1) [42].

$$(\nabla^2 + k^2)\mathbf{E}(\mathbf{r}, t) = 0 \longrightarrow (\nabla^2 + k^2)U(\mathbf{r}, t) = 0 \quad (4.1)$$

Starting from the exact formulation of free propagation derived by Rayleigh-Sommerfeld which is a solution of equation (4.1) [43], the objective is to make a convolution form appear. Having a convolution in the spatial domain implies a multiplication in the frequency domain which is possible using Fourier transforms. To meet this objective, the Kirchoff and Fresnel approximations of Rayleigh-Sommerfeld are studied.

Firstly, the situation must be properly introduced. The propagation considered in this section is between two planes separated by a distance Δh . The coordinates of the first plane are $(x_1, y_1, 0)$

and of the second plane are (x_2, y_2, z) , therefore, $\Delta h = z$. The wavelength of the propagated field is noted by λ and the associated wavenumber is $k = \frac{2\pi}{\lambda}$. Figure 4.1 illustrates the situation.

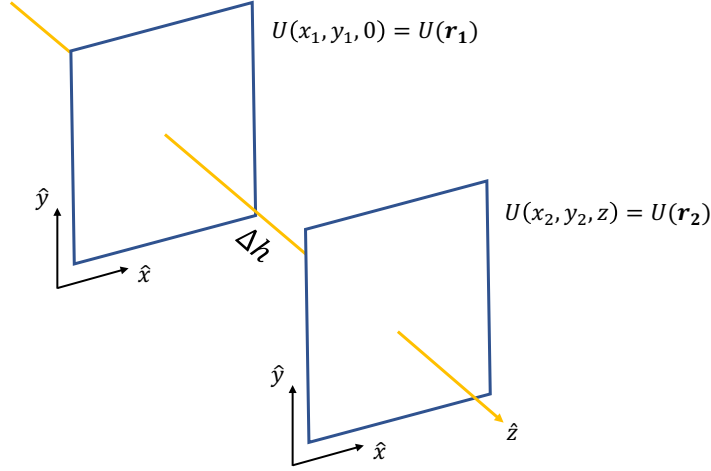


Figure 4.1: Input and output planes for a propagation of Δh . The input field is characterised by $\mathbf{r}_1 = (x_1, y_1, 0)$ and the output field by $\mathbf{r}_2 = (x_2, y_2, z)$.

Starting from the Rayleigh-Sommerfeld diffraction formula (4.2) [44, 45], the optical field at the output plane $U(x_2, y_2, z)$ after a propagation of length z is expressed as an integral of the known source field $U(x_1, y_1, 0)$ [46]. The expression is considering each pixel of the input plane to emit a spherical wave. To determine the value of a pixel at the output plane, every contribution has to be taken into account. This is Huygens principle [47]. To simplify the expression, the variable R is introduced as $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + z^2}$. It is the distance between one point in the input plane and another in the output plane.

$$\begin{aligned}
 U(x_2, y_2, z) &= \frac{1}{\lambda} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U(x_1, y_1, 0) \left[\frac{1}{k\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + z^2}} - i \right] \\
 &\quad \frac{ze^{ik\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + z^2}}}{(x_2 - x_1)^2 + (y_2 - y_1)^2 + z^2} dx_1 dy_1 \\
 &= \frac{1}{\lambda} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U(x_1, y_1, 0) \left[\frac{1}{kR} - i \right] \frac{ze^{ikR}}{R^2} dx_1 dy_1
 \end{aligned} \tag{4.2}$$

Kirchoff's approximation is intended to simplify the term in the bracket. and is valid for a propagation distance larger than a few wavelengths ($z \gg \lambda$) [45]. By definition R is always greater or equal to z thus $R \gg \lambda$. Writing the wavenumber k in terms of the wavelength λ as in equation (4.3) shows that under the assumption the term in brackets tends to $-i$.

$$\left[\frac{1}{kR} - i \right] = \left[\frac{\lambda}{2\pi R} - i \right] \longrightarrow -i \quad \text{if } R \gg \lambda \tag{4.3}$$

Substituting equation (4.3) in equation (4.2) and using the identity $-i = \frac{1}{i}$ gives equation (4.4). This formulation is Rayleigh-Sommerfeld approximated by Kirchoff.

$$U(x_2, y_2, z) = \frac{1}{i\lambda} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U(x_1, y_1, 0) \frac{ze^{ikR}}{R^2} dx_1 dy_1 \quad (4.4)$$

The next step is to make a convolution form appear from Kirchoff's formulation. This will allow to avoid computing the integral by going in the frequency domain. To do so, the Fresnel approximation is developed.

A Taylor serie development is used in order to approach R at the first order after it has been modified so that the Taylor expansion of $\sqrt{1 + \gamma}$ is possible. The variable R can be modified to obtain a small term in the square root as done in equation (4.5).

$$\begin{aligned} R &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + z^2} \\ &= z \sqrt{\underbrace{\frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{z^2}}_{=\gamma} + 1} \end{aligned} \quad (4.5)$$

Fresnel came up with the paraxial approximation which states that the propagation distance is much larger than the dimensions of the optical field ($z \gg x, y$). Looking at the modified form of R in equation (4.5), the first term in the square root (γ) is very small. The square root can be approximated by its Taylor serie at the first order as demonstrated in equation (4.6) [48].

$$\begin{aligned} R &= z\sqrt{1 + \gamma} \approx z \left(1 + \frac{\gamma}{2} \right) \quad \text{if } \gamma \text{ is small} \quad \text{where } \gamma = \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{z^2} \\ &\approx z + \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{2z} \end{aligned} \quad (4.6)$$

With the paraxial approximation, $R \approx z$ and equation (4.4) can be simplified into equation (4.7). Note the R in the exponential is not replaced by z as an exponential is very sensitive to small variations.

$$U(x_2, y_2, z) \approx \frac{1}{i\lambda z} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U(x_1, y_1, 0) e^{ikR} dx_1 dy_1 \quad (4.7)$$

Substituting R by its Taylor first order approximation into equation (4.7) leads to the Fresnel diffraction form shown in equation (4.8).

$$U(x_2, y_2, z) \approx \frac{e^{ikz}}{i\lambda z} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U(x_1, y_1, 0) e^{\frac{ik}{2z} [(x_2 - x_1)^2 + (y_2 - y_1)^2]} dx_1 dy_1 \quad (4.8)$$

Before going any further into the mathematical development, position and frequency vectors are defined. The z axis is not taken into account in the vectors in equation (4.9). As a convention, the initial plane is at location $z_1 = 0$ and the final plane at $z_2 = z$ [45].

$$\begin{cases} \mathbf{r}_1 = (x_1, y_1) \\ \mathbf{r}_2 = (x_2, y_2) \\ \mathbf{f}_1 = (f_{x1}, f_{y1}) \end{cases} \quad (4.9)$$

Equation (4.8) can be written as (4.10) with the new vectors in (4.9). In this expression, a phase shift is applied to the entire field via the term e^{ikz} . This shift can be ignored for the rest of the development as it is constant for the entire field.

$$U(\mathbf{r}_2) \approx \frac{1}{i\lambda z} \int_{-\infty}^{\infty} U(\mathbf{r}_1) e^{\frac{ik}{2z} |\mathbf{r}_2 - \mathbf{r}_1|^2} d\mathbf{r}_1 \quad (4.10)$$

Finally, equation (4.10) can be written as an integral between the field $U(\mathbf{r}_1)$ and a free-space amplitude spread function ($h(\mathbf{r}_2 - \mathbf{r}_1)$). This expression is a convolution in the spatial domain and thus a simple product in the frequency domain [43, 44, 41]. With $h(\mathbf{r})$ is associated a transfer function $H(\mathbf{f})$ after performing a Fourier transform ($\mathcal{F}$).

$$\begin{aligned} U(\mathbf{r}_2) &\approx \frac{1}{i\lambda z} \int_{-\infty}^{\infty} U(\mathbf{r}_1) e^{\frac{ik}{2z} |\mathbf{r}_2 - \mathbf{r}_1|^2} d\mathbf{r}_1 \\ &\approx \int_{-\infty}^{\infty} U(\mathbf{r}_1) h(\mathbf{r}_2 - \mathbf{r}_1) d\mathbf{r}_1 \quad \text{with } h(\mathbf{r}_2 - \mathbf{r}_1) = \frac{1}{i\lambda z} e^{\frac{ik}{2z} |\mathbf{r}_2 - \mathbf{r}_1|^2} \\ &\approx U(\mathbf{r}_1) * h(\mathbf{r}_1) \xrightarrow{\mathcal{F}} U(\mathbf{f}_1) H(\mathbf{f}_1) \end{aligned} \quad (4.11)$$

Figure 4.2 shows how to obtain $U(\mathbf{r}_2)$ from $U(\mathbf{r}_1)$ and $h(\mathbf{r}_1)$. The blue rectangles indicate the spatial domain and the green rectangles the frequency domain.

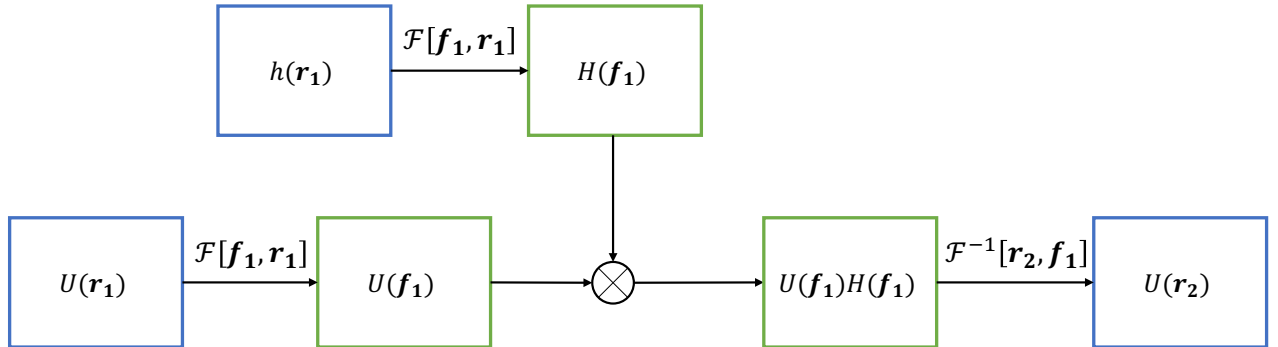


Figure 4.2: Block scheme of the steps to compute $U(\mathbf{r}_2)$ starting with $U(\mathbf{r}_1)$ and $h(\mathbf{r}_1)$. The rectangles are in blue and green for the spatial and frequency domain.

The use of Fourier transform ($\mathcal{F}$ and $\mathcal{F}^{-1}$) is needed to go in the frequency domain and obtain a simple product between the input field $U(\mathbf{f}_1)$ and the transfer function $H(\mathbf{f}_1)$ associated with $h(\mathbf{r}_1)$. The next section introduces the Fourier transforms and the discretisation used in this work. Afterwards, the Nyquist condition induced by going from continuous to discrete form is derived.

4.2 Two dimensional Fourier transforms

As mentioned in the previous section and shown in equation (4.11), Fourier transforms are needed to go in the frequency domain and perform a simple product instead of a convolution. This section first develops the continuous form of the Fourier transforms and then the discrete version. The direct and inverse Fourier transforms (FT and IFT) in the continuous domain can be expressed as equation (4.12).

$$\begin{aligned}\mathcal{F}[\mathbf{f}, \mathbf{r}] \{U(\mathbf{r})\} &\equiv \int_{-\infty}^{\infty} U(\mathbf{r}) e^{-i2\pi \mathbf{f} \cdot \mathbf{r}} d\mathbf{r} \\ \mathcal{F}^{-1}[\mathbf{r}, \mathbf{f}] \{U(\mathbf{f})\} &\equiv \int_{-\infty}^{\infty} U(\mathbf{f}) e^{i2\pi \mathbf{f} \cdot \mathbf{r}} d\mathbf{f}\end{aligned}\tag{4.12}$$

For this work, the Fourier transforms have to be discretized as the spatial and frequency grids implemented are finite. The spatial step is δ and the frequency step is δ_f as defined in section 3.1. *Python* is the programming language used for this work as it offers various powerful libraries. One of them is *Numpy* which allows for discrete Fourier transforms with built-in functions.

The discrete Fourier transform (DFT) and the inverse discrete Fourier transform (**IDFT**) used to propagate are displayed in equations (4.13) and (4.14). The couple $\mathbf{r} = (x, y)$ is associated to the spatial domain and $\mathbf{f} = (f_x, f_y)$ to the frequency domain. To obtain one pixel value in a domain, all the pixels in the other domain are weighted and added up. It is computationally heavy for large images as its complexity is $\mathcal{O}(N^2 \log(N)^2)$ [49].

For notation purposes in this report, the DFT will be written $\mathcal{F}[\mathbf{f}, \mathbf{r}] \{U(\mathbf{r}), \delta\}$ and the **IDFT** $\mathcal{F}^{-1}[\mathbf{r}, \mathbf{f}] \{U(\mathbf{f}), \delta_f\}$ where U is the field to transform and δ, δ_f are the discretisation steps. By convention in this work, the values of δ and δ_f are computed on the input plane.

$$\mathcal{F}[\mathbf{f}, \mathbf{r}] \{U[x, y], \delta\} = U[f_x, f_y] = \delta^2 \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} U[x, y] e^{-\frac{i2\pi(f_x x + f_y y)}{N^2}}\tag{4.13}$$

$$\begin{aligned}\mathcal{F}^{-1}[\mathbf{r}, \mathbf{f}] \{U[f_x, f_y], \delta_f\} &= U[x, y] = \frac{1}{(N\delta)^2} \sum_{f_x=0}^{N-1} \sum_{f_y=0}^{N-1} U[f_x, f_y] e^{\frac{i2\pi(f_x x + f_y y)}{N^2}} \\ &= \delta_f^2 \sum_{f_x=0}^{N-1} \sum_{f_y=0}^{N-1} U[f_x, f_y] e^{\frac{i2\pi(f_x x + f_y y)}{N^2}}\end{aligned}\tag{4.14}$$

The formulas applied by built-in functions of *Numpy* are described in equation (4.15) and (4.16). To establish a comparison with the formulas needed for this work, the Fourier transforms need to be normalised by δ^2 for the DFT (to go from (4.15) to (4.13)) and by $(N\delta_f)^2$ for the **IDFT** (to go from (4.16) to (4.14)). To avoid any confusion, the two functions have been re-implemented with the normalisation in the *Python* code and therefore, each time a Fourier transform is used, the value of δ (for the DFT) or δ_f (for the **IDFT**) must be indicated.

$$U[f_x, f_y] = \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} U[x, y] e^{-\frac{i2\pi(f_x x + f_y y)}{N^2}}\tag{4.15}$$

$$U[x, y] = \frac{1}{N^2} \sum_{f_x=0}^{N-1} \sum_{f_y=0}^{N-1} U[f_x, f_y] e^{\frac{i2\pi(f_x x + f_y y)}{N^2}}\tag{4.16}$$

The following section approaches the sampling conditions on the image dimensions induced by the discretisation performed to implement the DFT and **IDFT**.

4.2.1 Sampling conditions on image dimensions

The parameters of the image need to be adjusted to avoid the occurrence of aliasing and rippling. There are 3 parameters that can be tuned : N , the number of samples per direction on the square image, S , the physical size of the image and δ , the spacing between two pixels. Because $\delta = \frac{S}{N}$, it is only possible to act on two parameters at the same time.

According to the Nyquist-Shannon sampling theorem, a bandlimited signal, with maximal frequency being f_{max} , must have a spatial spacing (δ) respecting the condition in equation (4.17). In terms of frequency, the condition is given in equation (4.18), where f is the sampling frequency [50].

$$\delta \leq \frac{1}{2f_{max}} \quad (4.17)$$

$$f = \frac{1}{\delta} \geq 2f_{max} \quad (4.18)$$

Figure 4.3 shows three cases when analysing one dimension of the signal (f_x) [50]. The first case respects the condition in equation (4.18). The second one is the limiting case and there is still no spectral coverage. Finally, the third case shows spectral coverage in blue and there is a corruption of information. The result is an aliased image in the spatial domain.

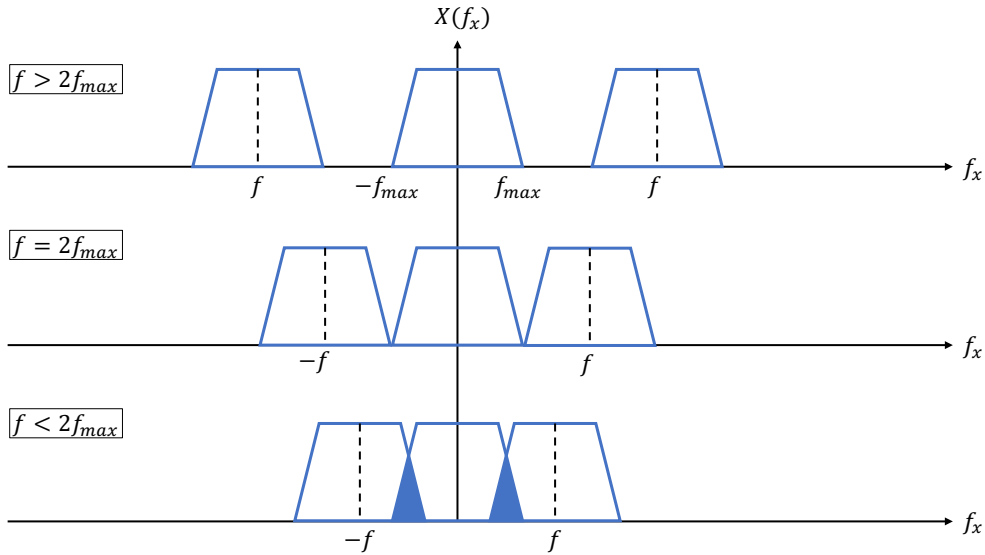


Figure 4.3: Representation of Nyquist's principle in 3 cases : the first one respects the condition, the second one is the limiting case and the last one contains aliasing (colored in blue).

Changing the input field parameters N and S enables to tune the δ so that there is no aliasing. There are three cases to analyse [9] :

1. Increasing δ and keeping N constant implies that the physical window size S is also increased. This will reduce the rippling (S higher) but increase the aliasing (δ higher).
2. Increasing N and keeping S constant implies the reduction of δ . This leads to a reduction of the aliasing but the rippling can't be changed.

3. Increasing S and decreasing δ allow for both the improvement of the aliasing and rippling but requires a lot of samples N and thus is computationally longer and uses more memory.

In chapter 3, choices were made to keep N constant to have a low computational time and to increase the δ during the propagation. The case selected is therefore the first one above and thus aliasing might appear due to the growth of δ (decrease of f). For each Fourier transform performed, the maximal frequency of the signal (f_{max}) must be compared to the sampling frequency (f) as done in section 4.4.

4.3 Propagation with varying spatial spacing

Looking back at equation (4.11), it could be possible to directly use built-in function of *Python* to perform the convolution. However, there would be no control over the spatial spacing in the output plane (δ_2) yet the δ_2 must evolve according to equation (3.15) (see section 3.3.2) to be a propagation at fixed angular resolution. This section investigates the possibility to obtain a convolution form taking into account the variation of δ .

To consider this variation, the argument of the exponential in equation (4.10) is developed and the variable m appears. To ease the reading, the square of the norm of position vectors in the input and output planes are indicated as $r_1^2 = |\mathbf{r}_1|^2$ and $r_2^2 = |\mathbf{r}_2|^2$. The goal is to extract a term independent of $\mathbf{r}_1$ out of the integral as well as to isolate a difference of position vectors. To do so, the square is distributed and a trick of adding and subtracting the same value is used. Those terms are indicated in green and red. The terms indicated in purple can be smartly regrouped to factorise the expression. Using calculus, a term depending only on r_2^2 is also isolated.

$$\begin{aligned}
 |\mathbf{r}_2 - \mathbf{r}_1|^2 &= r_2^2 - 2\mathbf{r}_2 \cdot \mathbf{r}_1 + r_1^2 \\
 &= r_2^2 + \frac{r_2^2}{m} - \frac{r_2^2}{m} - 2\mathbf{r}_2 \cdot \mathbf{r}_1 + r_1^2 + mr_1^2 - mr_1^2 \\
 &= r_2^2 \left(1 - \frac{1}{m}\right) + \frac{r_2^2}{m} - 2\mathbf{r}_2 \cdot \mathbf{r}_1 + r_1^2(1 - m) + mr_1^2 \\
 &= m \left[\left(\frac{r_2}{m}\right)^2 - 2\left(\frac{\mathbf{r}_2}{m}\right) \cdot \mathbf{r}_1 + r_1^2 \right] + r_2^2 \left(1 - \frac{1}{m}\right) + r_1^2(1 - m) \\
 &= m \left| \frac{\mathbf{r}_2}{m} - \mathbf{r}_1 \right|^2 - r_2^2 \left(\frac{1 - m}{m}\right) + r_1^2(1 - m)
 \end{aligned} \tag{4.19}$$

The parameter m is the scaling defined in section 3.3.2. In the current case, $m = \delta_2/\delta_1$. Plugging the result of equation (4.19) into the exponential of equation (4.10) gives equation (4.20).

$$U(\mathbf{r}_2) \approx \frac{1}{i\lambda z} \int_{-\infty}^{\infty} U(\mathbf{r}_1) e^{\frac{ik}{2z} \left[m \left| \frac{\mathbf{r}_2}{m} - \mathbf{r}_1 \right|^2 - \left(\frac{1-m}{m}\right) r_2^2 + (1-m)r_1^2 \right]} d\mathbf{r}_1 \tag{4.20}$$

The exponential is splitted into its three terms and the one containing only r_2 is placed outside the integral in equation (4.21) as there is no dependence on $\mathbf{r}_1$.

$$\begin{aligned}
 &\approx \frac{1}{i\lambda z} \int_{-\infty}^{\infty} U(\mathbf{r}_1) e^{\frac{ik}{2z} m |\frac{\mathbf{r}_2}{m} - \mathbf{r}_1|^2} e^{-\frac{ik}{2z} (\frac{1-m}{m}) r_2^2} e^{\frac{ik}{2z} (1-m) r_1^2} d\mathbf{r}_1 \\
 &\approx \frac{1}{i\lambda z} e^{-\frac{ik}{2z} (\frac{1-m}{m}) r_2^2} \int_{-\infty}^{\infty} U(\mathbf{r}_1) e^{\frac{ik}{2z} m |\frac{\mathbf{r}_2}{m} - \mathbf{r}_1|^2} e^{\frac{ik}{2z} (1-m) r_1^2} d\mathbf{r}_1
 \end{aligned} \tag{4.21}$$

The goal is to transform the term inside the integral in equation (4.21) into a convolution of a field depending only on $\mathbf{r}_1$ and a term with an exponential of the difference between two positions such as $\mathbf{r}_2 - \mathbf{r}_1$. To aim for this apparition, a new field, denoted $U''(\mathbf{r}_1)$, is defined in equation (4.22). Another way of expressing it is by multiplying both sides by m . This last formula is used for the next step.

$$U''(\mathbf{r}_1) \equiv \frac{1}{m} U(\mathbf{r}_1) e^{\frac{ik}{2z} (1-m) r_1^2} \longrightarrow mU''(\mathbf{r}_1) \equiv U(\mathbf{r}_1) e^{\frac{ik}{2z} (1-m) r_1^2} \tag{4.22}$$

Figure 4.4 shows how to obtain $U''(\mathbf{r}_1)$ by multiplying $U(\mathbf{r}_1)$ to a quadratic term and introduce the normalisation $\frac{1}{m}$.

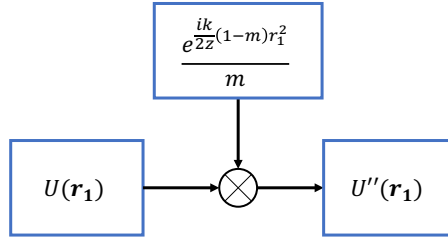


Figure 4.4: Block scheme representing equation (4.22). It is a multiplication of the optical field with a quadratic term.

Equation (4.23) is obtained by substituting equation (4.22) in equation (4.21). The scaling parameter m appears in the integral as a result of the definition of U'' .

$$U(\mathbf{r}_2) \approx \frac{1}{i\lambda z} e^{-\frac{ik}{2z} (\frac{1-m}{m}) r_2^2} \int_{-\infty}^{\infty} mU''(\mathbf{r}_1) e^{\frac{ik}{2z} m |\frac{\mathbf{r}_2}{m} - \mathbf{r}_1|^2} d\mathbf{r}_1 \tag{4.23}$$

Equation (4.23) is already a convolution form but by introducing the scaled position vector $\mathbf{r}'_2$ and the scaled propagated distance z' , a better convolution expression can be found. The scaling is done with the factor m . The variable change is expressed in equation (4.24).

$$\begin{cases} \mathbf{r}'_2 = \frac{\mathbf{r}_2}{m} \\ z' = \frac{z}{m} \end{cases} \tag{4.24}$$

Writing expression (4.23) with the new set of coordinates gives equation (4.25). In this formula, a difference of position vectors appears with $|\mathbf{r}'_2 - \mathbf{r}_1|$ and the convolution is found by considering the exponential term in the integral in equation (4.25) as an impulse response, also known as free-space amplitude spread function (h). The definition of the impulse response is exactly the same as in equation (4.11).

$$\begin{aligned}
 U(m\mathbf{r}'_2) &\approx \frac{1}{i\lambda z'} e^{-\frac{ik}{2z'}(1-m)r_2'^2} \int_{-\infty}^{\infty} U''(\mathbf{r}_1) e^{\frac{ik}{2z'}|\mathbf{r}'_2 - \mathbf{r}_1|^2} d\mathbf{r}_1 \\
 &\approx e^{-\frac{ik}{2z'}(1-m)r_2'^2} \underbrace{\int_{-\infty}^{\infty} U''(\mathbf{r}_1) h(\mathbf{r}'_2 - \mathbf{r}_1) d\mathbf{r}_1}_{U'(m\mathbf{r}'_2)}
 \end{aligned} \tag{4.25}$$

The impulse response and its Fourier transform are written in equation (4.26). The function $H(\mathbf{f}_1)$ is called the amplitude transfer function.

$$h(\mathbf{r}_1) = \frac{1}{i\lambda z'} e^{\frac{ik}{2z'}|\mathbf{r}_1|^2} \longrightarrow H(\mathbf{f}_1) = \mathcal{F}[\mathbf{f}_1, \mathbf{r}_1] \{h(\mathbf{r}_1), \delta_1\} = e^{-i\pi\lambda z'|\mathbf{f}_1|^2} \tag{4.26}$$

A new function ($U'(m\mathbf{r}'_2)$) is defined to take only the integral of equation (4.25) into account. It is expressed as a convolution in equation (4.27) performing the same procedure as displayed on Figure 4.2.

$$\begin{aligned}
 U'(m\mathbf{r}'_2) &= \int_{-\infty}^{\infty} U''(\mathbf{r}_1) h(\mathbf{r}'_2 - \mathbf{r}_1) d\mathbf{r}_1 \\
 &= U''(\mathbf{r}_1) * h(\mathbf{r}_1) \\
 &= \mathcal{F}^{-1}[\mathbf{r}'_2, \mathbf{f}_1] \{U''(\mathbf{f}_1) H(\mathbf{f}_1), \delta_{f_1}\}
 \end{aligned} \tag{4.27}$$

Putting it all together, equation (4.28) shows the procedure to apply on $U''(\mathbf{r}_1)$ to obtain $U'(m\mathbf{r}'_2)$. The field $U''(\mathbf{r}_1)$ once placed in the frequency domain is multiplied by the transfer function $H(\mathbf{f}_1)$ and placed back into the spatial domain [39]. For notation purposes and as done for $\mathbf{r}_1$ and $\mathbf{r}_2$, the norm of the frequency vector $|\mathbf{f}_1|^2$ is noted f_1^2 .

$$U'(m\mathbf{r}'_2) = \mathcal{F}^{-1}[\mathbf{r}'_2, \mathbf{f}_1] \left\{ \mathcal{F}[\mathbf{f}_1, \mathbf{r}_1] \{U''(\mathbf{r}_1), \delta_1\} e^{-i\pi\lambda z' f_1^2}, \delta_{f_1} \right\} \tag{4.28}$$

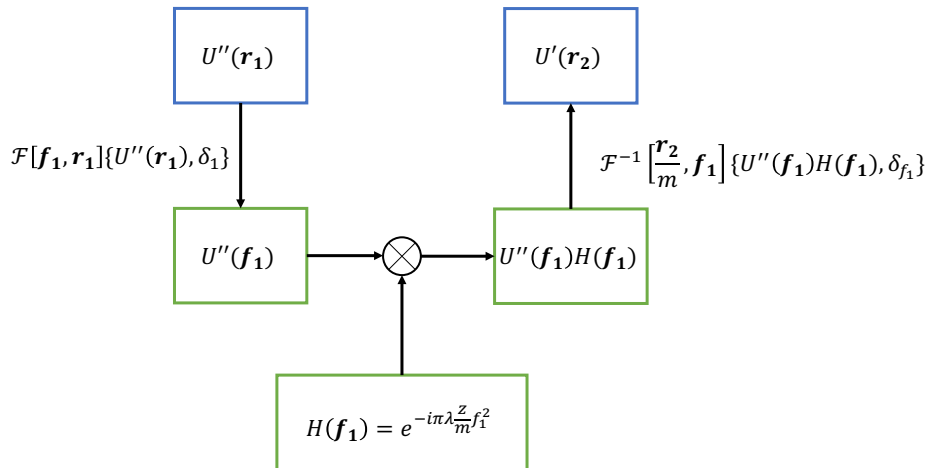


Figure 4.5: Block scheme representing equation (4.29) with a switch in the frequency domain.

The expression in equation (4.28) is retranscribed back in terms of $\mathbf{r}_2$ and z using the variable change previously introduced. The result is displayed in equation (4.29). Figure 4.5 displays the procedure to obtain $U'(\mathbf{r}_2)$ by starting from $U''(\mathbf{r}_1)$.

$$U'(\mathbf{r}_2) = \mathcal{F}^{-1} \left[\frac{\mathbf{r}_2}{m}, \mathbf{f}_1 \right] \left\{ \mathcal{F}[\mathbf{f}_1, \mathbf{r}_1] \{U''(\mathbf{r}_1), \delta_1\} e^{-i\pi\lambda \frac{z}{m} f_1^2}, \delta_{f_1} \right\} \quad (4.29)$$

Equation (4.25) can also be expressed in terms of $\mathbf{r}_2$ and z . The result is shown in equation (4.30). This last step is represented by a simple multiplication by a quadratic term on Figure 4.6.

$$U(\mathbf{r}_2) \approx e^{-i\frac{k}{2z} \frac{(1-m)}{m} r_2^2} U'(\mathbf{r}_2) \quad (4.30)$$

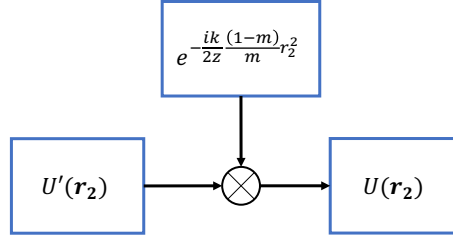


Figure 4.6: Block scheme of equation 4.30. It is a multiplication of the optical field with a quadratic term.

To obtain the complete propagation expression as shown in equation (4.31), the two last equations must be combined.

$$\begin{aligned} U(\mathbf{r}_2) &\approx e^{-i\frac{k}{2z} \frac{(1-m)}{m} r_2^2} \mathcal{F}^{-1} \left[\frac{\mathbf{r}_2}{m}, \mathbf{f}_1 \right] \left\{ \mathcal{F}[\mathbf{f}_1, \mathbf{r}_1] \{U''(\mathbf{r}_1), \delta_1\} e^{-i\pi\lambda \frac{z}{m} f_1^2}, \delta_{f_1} \right\} \\ &\approx e^{-i\frac{k}{2z} \frac{(1-m)}{m} r_2^2} \mathcal{F}^{-1} \left[\frac{\mathbf{r}_2}{m}, \mathbf{f}_1 \right] \left\{ \mathcal{F}[\mathbf{f}_1, \mathbf{r}_1] \left\{ \frac{1}{m} U(\mathbf{r}_1) e^{i\frac{k}{2z} (1-m) r_1^2}, \delta_1 \right\} e^{-i\pi\lambda \frac{z}{m} f_1^2}, \delta_{f_1} \right\} \end{aligned} \quad (4.31)$$

Introducing operators for the quadratic terms improves the readability of the formula. The operators are defined according to equation (4.32)

$$\begin{cases} \mathcal{Q}[c, \mathbf{r}] \{U(\mathbf{r})\} = e^{i\frac{k}{2} c |\mathbf{r}|^2} \{U(\mathbf{r})\} \\ \mathcal{Q}_2[c, \mathbf{f}] \{U(\mathbf{f})\} = e^{i\pi^2 \frac{2c}{k} |\mathbf{f}|^2} \{U(\mathbf{f})\} \end{cases} \quad (4.32)$$

Finally, using the freshly defined operators in equation (4.32) on equation (4.31) gives equation (4.33). Combining the three previous block schemes on Figures 4.4, 4.5 and 4.6 enables to draw a block scheme for the entire propagation as shown on Figure 4.7. The scheme is describing equation (4.33).

$$\begin{aligned} U(\mathbf{r}_2) &\approx \mathcal{Q} \left[\frac{m-1}{mz}, \mathbf{r}_2 \right] \\ &\quad \mathcal{F}^{-1} \left[\frac{\mathbf{r}_2}{m}, \mathbf{f}_1 \right] \left\{ \mathcal{F}[\mathbf{f}_1, \mathbf{r}_1] \left\{ \mathcal{Q} \left[\frac{1-m}{z}, \mathbf{r}_1 \right] \frac{1}{m} U(\mathbf{r}_1), \delta_1 \right\} \mathcal{Q}_2 \left[\frac{-z}{m}, \mathbf{f}_1 \right], \delta_{f_1} \right\} \end{aligned} \quad (4.33)$$

This last expression is the one implemented in *Python* to simulate the free propagation. It needs the input image and its dimensions (N and S_1) to compute δ_1 as well as the pixel dimension at the output plane (δ_2) in order to compute the variable m and create the grid of $U(\mathbf{r}_2)$. Applying the theoretical background acquired in section 3.3.2, δ_2 is found using the diffraction formula in free space.

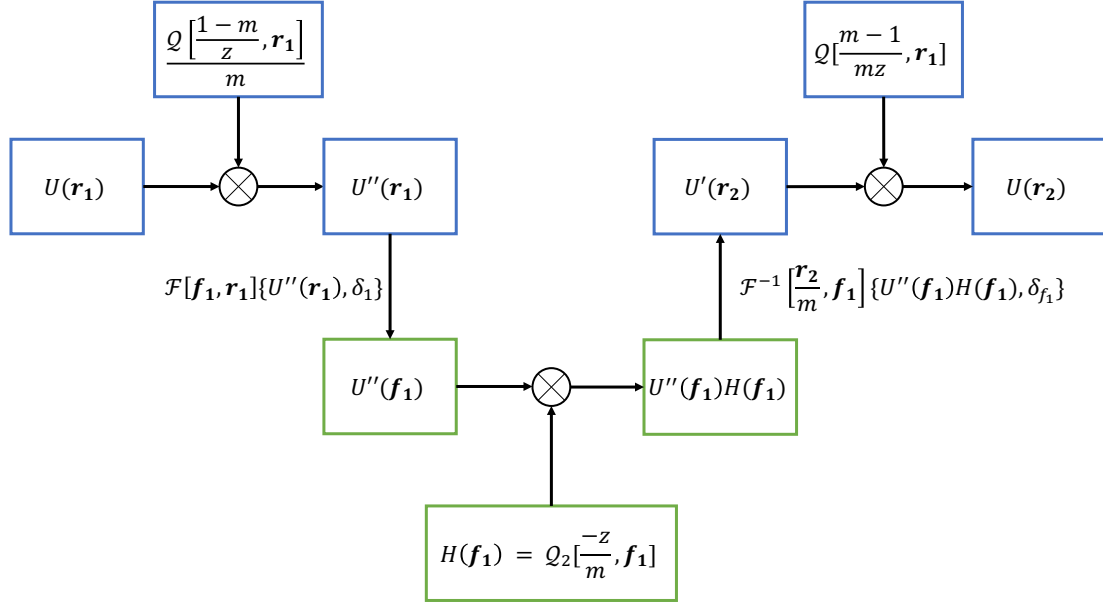


Figure 4.7: Global block scheme representing equation (4.33). It is the combination of Figures 4.4, 4.5 and 4.6.

4.4 Sampling Conditions

An expression for the free propagation has been derived but the conditions under which it is valid must still be investigated. Indeed, there are sampling considerations to analyse because of the geometry of the problem and the Fourier transforms. There are four sampling conditions developed in this section.

4.4.1 Geometrical conditions

The two first conditions analysed are based on the geometry of the propagation. They make use of the Nyquist theorem and creates boundaries for δ_2 and z .

4.4.1.1 Maximal opening condition

The first geometrical consideration is the diffraction of a point source resulting in a maximal opening angle α_{max} . Each pixel in the input plane is emitting a spherical wave with a maximal opening denoted α_{max} . Figure 4.8 shows the angle is dependent on the size δ_1 . A small δ_1 creates a wide α_{max} .

The maximal opening angle can be related to the maximal frequency f_{max} using the wavelength through equation (4.34). Afterwards, the Nyquist theorem defined in equation (4.17) can be written as a function of α_{max} .

$$\alpha_{max} = \lambda f_{max} \longrightarrow \delta_1 \leq \frac{\lambda}{2\alpha_{max}} \quad (4.34)$$

The objective is to find an expression for the maximal opening angle α_{max} depending on the dimensions of the input and output planes as well as on the propagation parameters.

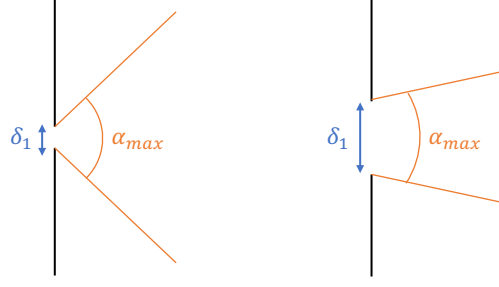


Figure 4.8: Representation of the maximal opening angle depending on the pixel size δ .

The current situation is represented on Figure 4.9 where D_1 and D_2 are the maximal extend of the input and output optical fields. D_1 can be seen as the size of the transmitter lens and D_2 can be seen as the lens dimension of the receiver. The situation is represented on Figure 4.9.

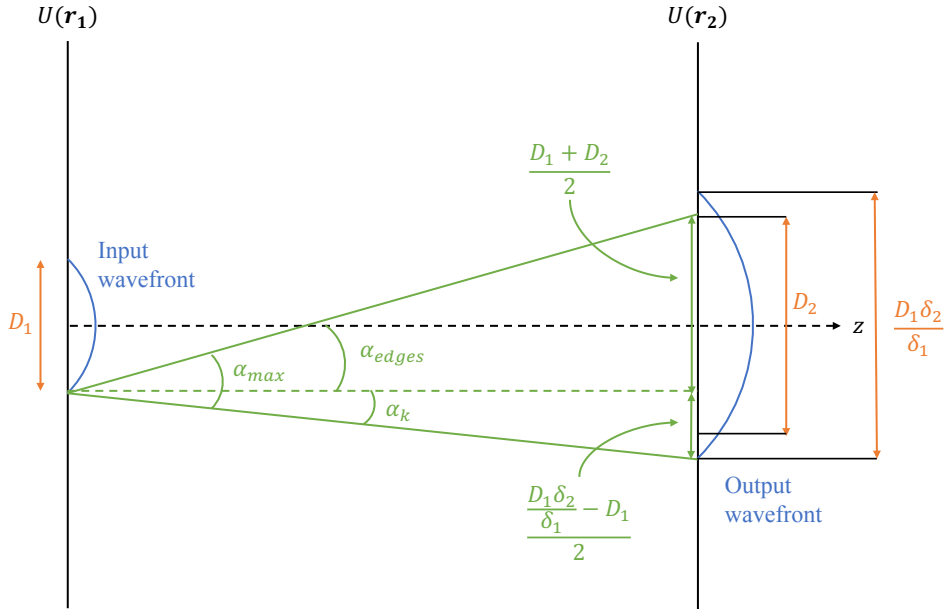


Figure 4.9: Situation of a propagation between two planes with the necessary dimensions for the maximal opening condition being represented.

The requirement to fulfil is to ensure the spatial sampling (δ_1) is fine enough so that the angle α_{max} is wide and each point source of the input plane illuminates all D_2 . The limiting case where $x_1 = -D_1/2$ (the last non-zero pixel of the input plane) has its illumination pattern opening over an angle α_{max} and including all D_2 . The pixel $x_1 = -D_1/2$ (or $D_1/2$) defines the upper (or lower) boundary of D_2 .

In order to compute α_{max} , the angle is divided in two components (α_{edges} and α_k) represented on Figure 4.9. As the propagation distance z is much larger than the image dimensions, the use of the paraxial approximation allows to assume the angle are close to their tangents : $\tan(\alpha_{edges}) \approx \alpha_{edges}$ and $\tan(\alpha_k) \approx \alpha_k$. Using the definition of the tangent (opposite side divided by adjacent side), equations (4.35) and (4.36) are found and thus an expression for the opening angle is derived in equation (4.37).

$$\alpha_{edges} \approx \tan(\alpha_{edges}) = \frac{D_1 + D_2}{2z} \quad (4.35)$$

$$\alpha_k \approx \tan(\alpha_k) = \frac{D_1 \delta_2}{2\delta_1 z} - \frac{D_1}{2z} = \frac{D_1}{2z} \left(\frac{\delta_2}{\delta_1} - 1 \right) \quad (4.36)$$

$$\alpha_{max} = \alpha_{edges} + \alpha_k = \frac{D_1 \delta_2 / \delta_1 + D_2}{2z} \quad (4.37)$$

The last step towards the first condition is to apply Nyquist's criterion in equation (4.34). After isolating δ_2 , equation (4.38) is deduced.

$$\delta_2 \leq -\frac{D_2}{D_1} \delta_1 + \frac{\lambda z}{D_1} \quad (4.38)$$

4.4.1.2 Illuminated region condition

The second condition focuses on the grid dimension needed at the output to avoid the appearance of aliasing. In the same situation as for the maximal opening condition, the maximal opening angle is found based on the δ_1 . The second condition considers the illumination of pixels outside D_2 . Those pixels receive information from some but not all of the input pixels because they are located outside D_2 .

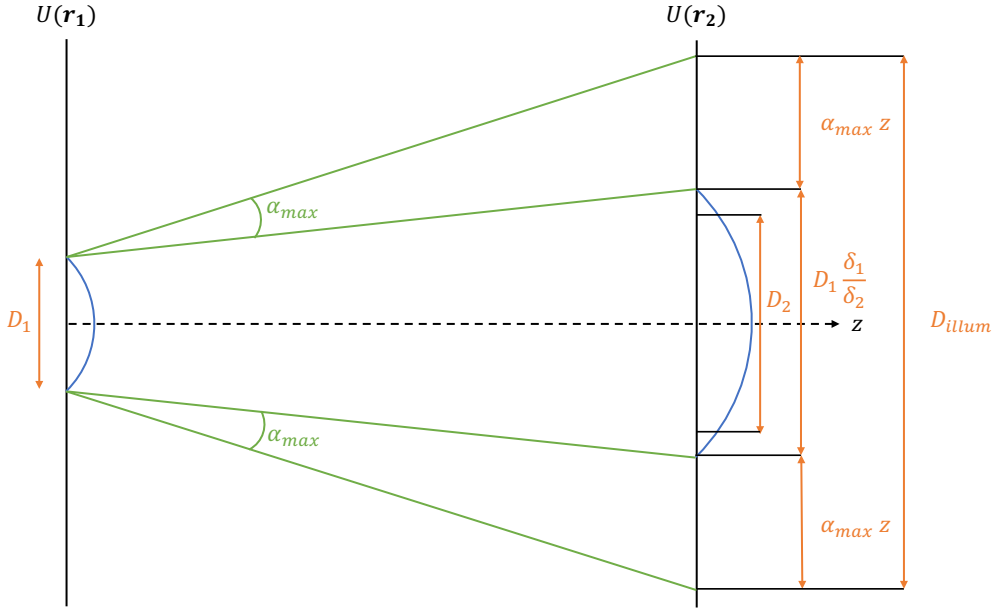


Figure 4.10: Representation of all the characteristic dimensions needed for the illuminated region condition.

$$D_{illum} = D_1 \frac{\delta_2}{\delta_1} + 2\alpha_{max} z = D_1 \frac{\delta_2}{\delta_1} + \frac{\lambda z}{\delta_1} \quad (4.39)$$

The illuminated area (D_{illum}) can be computed using the growth of the Gaussian $m = \delta_2/\delta_1$ and α_{max} . D_{illum} is defined based on the two extreme points of D_1 ($x_1 = \pm D_1/2$). Considering those points propagate so that they have no contribution to the output wavefront (blue curve)

results in the case presented on Figure 4.10.

Under the paraxial approximation the distance x illuminated by α_{max} can be found : $\alpha_{max} \approx \tan(\alpha_{max}) = \frac{x}{z} \iff x \approx \alpha_{max}z$. Therefore, the total illuminated region is given by equation (4.39).

When propagating from $U(\mathbf{r}_1)$ to $U(\mathbf{r}_2)$, D_{illum} becomes quite large. The output grid is fixed to $D_{grid} = S_2 = N\delta_2$ as explained in chapter 3 and if it is smaller than D_{illum} , aliasing appears. This apparition is characterised by the content located at the upper section of D_{illum} found back on the lower section of D_{grid} as shown in purple on Figure 4.11 [51]. To avoid corrupting the useful information located in D_2 , the aliased information must be contained in a section smaller or equal to $\frac{D_{illum}-D_2}{4}$. The example shown on Figure 4.11 is a version where the aliasing does not impact the information in D_2 . To ensure no corruption, the condition on the grid in equation (4.40) must be respected.

$$D_{grid} = N\delta_2 \geq D_2 + 2\left(\frac{D_{illum} - D_2}{4}\right) = \frac{D_{illum} + D_2}{2} = \frac{D_1\delta_2/\delta_1 + \lambda z/\delta_1 + D_2}{2} \quad (4.40)$$

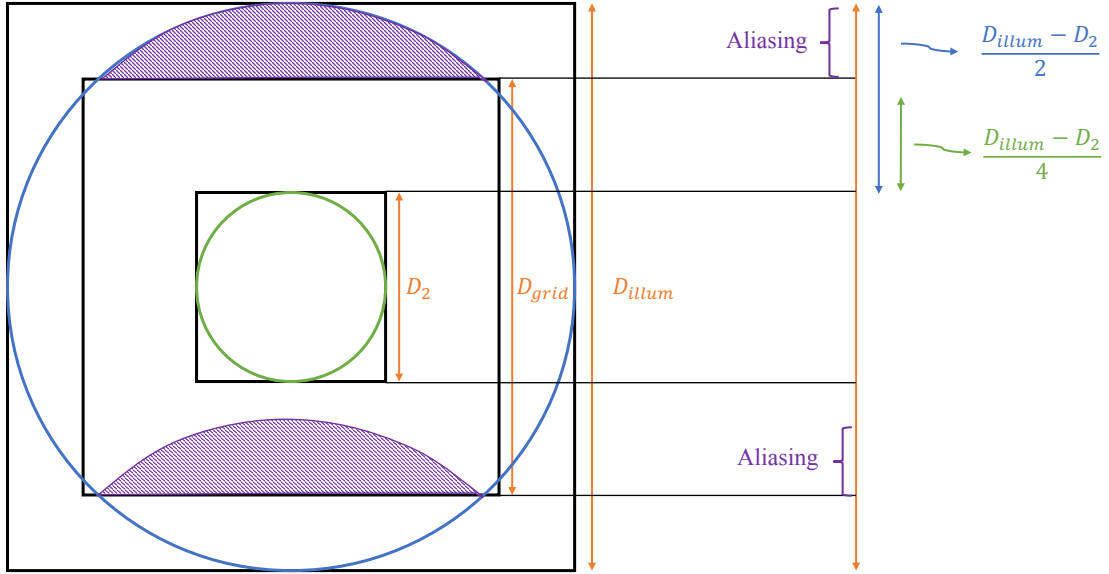


Figure 4.11: Representation of the apparition of aliasing due to the size of D_{grid} being smaller than D_{illum} . The aliasing at the top appears at the bottom as shown in purple.

Solving the inequality by isolating z results in the second condition of propagation shown in equation (4.41). This restriction implies the propagation distance has an upper boundary.

$$z \leq \frac{2N\delta_1\delta_2 - D_1\delta_2 - D_2\delta_1}{\lambda} \quad (4.41)$$

4.4.2 Fourier transform conditions

Section 4.2.1 established that a Fourier transform imposes a condition on the size of the pixel δ . The maximal frequency of the signal can not be larger than $\frac{1}{2\delta}$ otherwise aliasing appears

due to frequency coverage and corrupts the propagated information.

The purpose of this section is to find the maximal frequency in the quadratic terms inside the Fourier transforms to obtain conditions on the parameter of the problem.

The Gaussian beam considered in this report is defined in section 3.3.3. For the purpose of this section, only the term depending on $\mathbf{r}_1$ is considered as the derivative according to this variable is taken. The resulting Gaussian beam is the field presented in equation (4.42) [52]. Note that the apodized beam presented for this development in the book written by Schmidt is defining $e^{\frac{ik}{2R_c}r_1^2}$ with $R_c < 0$ for a diverging beam whereas in this report, the R_c is positive and thus the negative sign disappears [9].

$$U(\mathbf{r}_1) = e^{-\frac{|\mathbf{r}_1 - \boldsymbol{\mu}|^2}{w_1^2}} e^{-\frac{ik}{2R_c}r_1^2} = A(\mathbf{r}_1)e^{-\frac{ik}{2R_c}r_1^2} \quad (4.42)$$

Substituting the beam definition above in equation (4.43) and after combining the exponential terms on the second line, a new expression can be implemented numerically considering only the FT of the amplitude ($A(\mathbf{r}_1)$).

$$\begin{aligned} U(\mathbf{r}_2) &\approx \mathcal{Q}\left[\frac{m-1}{mz}, \mathbf{r}_2\right] \\ \mathcal{F}^{-1}\left[\frac{\mathbf{r}_2}{m}, \mathbf{f}_1\right] &\left\{ \mathcal{F}[\mathbf{f}_1, \mathbf{r}_1] \left\{ \mathcal{Q}\left[\frac{1-m}{z}, \mathbf{r}_1\right] \frac{1}{m} U(\mathbf{r}_1), \delta_1 \right\} \mathcal{Q}_2\left[\frac{-z}{m}, \mathbf{f}_1\right], \delta_{f_1} \right\} \\ U(\mathbf{r}_2) &\approx \mathcal{Q}\left[\frac{m-1}{mz}, \mathbf{r}_2\right] \\ \mathcal{F}^{-1}\left[\frac{\mathbf{r}_2}{m}, \mathbf{f}_1\right] &\left\{ \mathcal{F}[\mathbf{f}_1, \mathbf{r}_1] \left\{ \mathcal{Q}\left[\frac{1-m}{z}, \mathbf{r}_1\right] \frac{1}{m} A(\mathbf{r}_1) e^{-\frac{ik}{2R_c}r_1^2}, \delta_1 \right\} \mathcal{Q}_2\left[\frac{-z}{m}, \mathbf{f}_1\right], \delta_{f_1} \right\} \\ U(\mathbf{r}_2) &\approx \mathcal{Q}\left[\frac{m-1}{mz}, \mathbf{r}_2\right] \\ \mathcal{F}^{-1}\left[\frac{\mathbf{r}_2}{m}, \mathbf{f}_1\right] &\left\{ \mathcal{F}[\mathbf{f}_1, \mathbf{r}_1] \left\{ \mathcal{Q}\left[\frac{1-m}{z} - \frac{1}{R_c}, \mathbf{r}_1\right] \frac{1}{m} A(\mathbf{r}_1), \delta_1 \right\} \mathcal{Q}_2\left[\frac{-z}{m}, \mathbf{f}_1\right], \delta_{f_1} \right\} \end{aligned} \quad (4.43)$$

Previously in this chapter the Nyquist theorem was approached for the Fourier transform (see section 4.2.1) and imposes conditions on the frequencies. To avoid the apparition of aliasing, the two quadratic terms inside the DFT and IDFT must be analysed. They are represented in pink in equation (4.43).

The quadratic terms give information on the optical phase (ϕ in radians). To go from phase to frequency, where the Nyquist condition is applied, the expression must be derived and normalised by 2π . The computation is simplified thanks to the phase variations being the same in x and y directions. Only one dimension (x) is analysed as shown in equation (4.44) [44, 53].

$$\mathbf{f} = \frac{1}{2\pi} \boldsymbol{\nabla} \phi \longrightarrow f_x = \frac{1}{2\pi} \frac{\partial \phi(x)}{\partial x} \quad (4.44)$$

Once the frequency expression is derived, the extreme case where the frequency is maximal is analysed to finally obtain an inequality thanks to Nyquist.

4.4.2.1 IDFT condition on quadratic phase term

The phase of quadratic pink term associated to the IDFT is extracted. Recalling the definition of $\mathcal{Q}_2$, the full expression can be written and the phase ϕ_a can be extracted. Note that the quadratic term is defined according to f_{x_1} as it is in the frequency domain.

$$\mathcal{Q}_2\left[\frac{-z}{m}, f_{x_1}\right] = e^{-i\pi^2 \frac{2z}{mk} f_{x_1}^2} = e^{-i\phi_a} \longrightarrow \phi_a = \pi^2 \frac{2z}{mk} f_{x_1}^2 \quad (4.45)$$

Applying the definition of the frequency in equation (4.44) but with the derivative of the phase computed according to the variable f_{x_1} and recalling $k = \frac{2\pi}{\lambda}$, the local frequency ($f_{loc,a}$) is found. Nyquist theorem is restricting the value of the maximal frequency. Therefore, the variable f_{x_1} has to be set to its maximal value in order to analyse the limiting situation. In the current case it is the last pixel of the frequency grid : $f_{x_1} = \frac{N}{2S_1} = \frac{1}{2\delta_1}$. The expression of $f_{max,a}$ is then found in equation (4.46)

$$f_{loc,a} = \frac{1}{2\pi} \frac{\partial \phi_a}{\partial f_{x_1}} = \frac{\lambda z}{m} f_{x_1} \longrightarrow f_{max,a} = \frac{\lambda z}{m} \frac{1}{2\delta_1} \quad (4.46)$$

Finally, the Nyquist theorem defined in equation (4.17) is applied. Note that the theorem is reversed as an IDFT is performed. The result is thus a condition over the spatial domain as one can see by computing the units of equation (4.46). In this first case, the spacing considered between frequencies is thus δ_{f_1} (and not δ_1 as in equation (4.17)).

$$f_{max,a} \leq \frac{1}{2\delta_{f_1}} \iff \frac{\lambda z}{m} \frac{1}{2\delta_1} \leq \frac{N\delta_1}{2} \quad (4.47)$$

One of the conditions of the propagation is found in equation (4.48). It restricts the propagated distance to a value depending on the number of pixels (N), the input and output spatial pixel dimensions (δ_1 and δ_2) and the wavelength of the propagated wave (λ).

$$\boxed{z \leq \frac{N\delta_1\delta_2}{\lambda}} \quad (4.48)$$

4.4.2.2 DFT condition on quadratic phase term

Considering the second quadratic phase term, the same procedure can be applied. In this case, the quadratic term is defined in the spatial domain and in the x direction. Once again, by developing the term $\mathcal{Q}$ using its definition, the phase variation can be extracted. Equation (4.49) displays the resulting phase ϕ_b .

$$\mathcal{Q}\left[\frac{1-m}{z} - \frac{1}{R_c}, x_1\right] = e^{\frac{ik}{2} \left(\frac{1-m}{z} - \frac{1}{R_c}\right) x_1^2} = e^{-i\phi_b} \longrightarrow \phi_b = -\frac{k}{2} \left(\frac{1-m}{z} - \frac{1}{R_c}\right) x_1^2 \quad (4.49)$$

Taking the derivative of the phase ϕ_b , the local frequency $f_{loc,b}$ in the second case is found. This frequency evolves in function of the variable x_1 . To apply Nyquist the extreme values of x_1 must be found. At the defined input aperture, it happens at $x_1 = -D_1/2$. After substitution, the maximal frequency ($f_{max,b}$) is found in equation (4.50).

$$f_{loc,b} = \frac{1}{2\pi} \frac{\partial \phi_b}{\partial x_1} = -\frac{1}{\lambda} \left(\frac{1-m}{z} - \frac{1}{R_c}\right) x_1 \longrightarrow f_{max,b} = \frac{1}{\lambda} \left(\frac{1-m}{z} - \frac{1}{R_c}\right) \frac{D_1}{2} \quad (4.50)$$

i	Simulation parameters		Maximum/minimum authorised values				
	z_i	$\delta_{i,simu}$	$\delta_{i,max} =$	$z_{i,max} =$	$z_{i,max} =$	$\delta_{i,min}/\delta_{i,max} =$	
1	280	0.00079	0.00647	510.9	285.8	-0.00647	0.00803
2	290	0.0008	0.00667	523.1	292.7	-0.00664	0.00829
3	300	0.00082	0.00673	548.3	307.1	-0.00673	0.00841
4	330	0.00086	0.00706	589.4	330.8	-0.00706	0.00882
5	360	0.00091	0.0072	651.0	366.4	-0.00721	0.00907
6	400	0.00098	0.0073	739.0	417.4	-0.00731	0.00931
7	490	0.00108	0.00811	869.4	492.8	-0.00811	0.01031
8	600	0.00121	0.00878	1069.1	607.8	-0.00878	0.01125
9	780	0.0014	0.00995	1385.0	789.2	-0.00995	0.0128
10	1070	0.00168	0.01162	1917.7	1094.4	-0.0116	0.01502
11	1600	0.00212	0.01433	2902.8	1657.9	-0.0143	0.01861
12	2800	0.00293	0.01983	5060.8	2890.2	-0.01978	0.02571
13	6000	0.00471	0.03076	11241.3	6413.8	-0.03065	0.04018
14	8700	0.00732	0.02505	28162.3	16043.1	-0.02493	0.03964

Table 4.1: Simulation parameters for the 14 free propagations with their minimal/maximal values authorised computed based on the four conditions. The red values are irrelevant because another is more restrictive.

With the knowledge of the maximal frequency, Nyquist's theorem is applied. The spacing considered is spatial thus the sampling frequency is $1/\delta_1$ and the condition is given in equation (4.51).

$$f_{max,b} \leq \frac{1}{2\delta_1} \iff \frac{1}{\lambda} \left| \frac{1-m}{z} - \frac{1}{R_c} \right| \frac{D_1}{2} \leq \frac{1}{2\delta_1} \quad (4.51)$$

Finally, after developing equation (4.51), the last propagation condition is found in equation (4.52).

$$\left(1 - \frac{z}{R_c} \right) \delta_1 - \frac{\lambda z}{D_1} \leq \delta_2 \leq \left(1 - \frac{z}{R_c} \right) \delta_1 + \frac{\lambda z}{D_1} \quad (4.52)$$

4.4.3 Discussion

The four conditions are displayed from inequalities (4.53) to (4.56). Those inequalities must remain valid after each propagation otherwise aliasing appears.

$$\delta_2 \leq -\frac{D_2}{D_1} \delta_1 + \frac{\lambda z}{D_1} \quad (4.53)$$

$$z \leq \frac{2N\delta_1\delta_2 - D_1\delta_2 - D_2\delta_1}{\lambda} \quad (4.54)$$

$$z \leq \frac{N\delta_1\delta_2}{\lambda} \quad (4.55)$$

$$\left(1 - \frac{z}{R_c} \right) \delta_1 - \frac{\lambda z}{D_1} \leq \delta_2 \leq \left(1 - \frac{z}{R_c} \right) \delta_1 + \frac{\lambda z}{D_1} \quad (4.56)$$

To find a set of variables fulfilling the requirements, one can try to solve the system but it is really difficult. Another way of setting the grid parameters is thus explained. When having defined the input dimensions (δ_1 , D_1) for a constant value of N , the output dimensions (δ_2 , D_2) are dependent only on the propagated distance (see chapter 3). Thus, for each propagation, the parameter z must be tuned so that each inequality is verified.

For each propagation, the four values restricting the δ_i and z_i are computed based on the conditions. From the four inequalities, two are applied on δ_i and the two others on z_i . Therefore, there will always be an irrelevant condition as the other one is more restrictive. Table 4.1 clearly shows the minimal/maximal values with the irrelevant values coloured in red. It appears that the second and fourth conditions are never used. Indeed, the first and the third are more restrictive. Moreover, a negative value for the δ_i makes no physical sense and thus the lower boundary is in fact 0.

4.5 Analysis of aliasing

The analysis of the four conditions should serve as a guideline as mentioned by J.Schmidt [9]. They are not strict rules but gives a general intuition on optical wave propagation. The most important point is to ensure the FT are sampled at an appropriate rate. This section analyses 2 cases presenting aliasing.

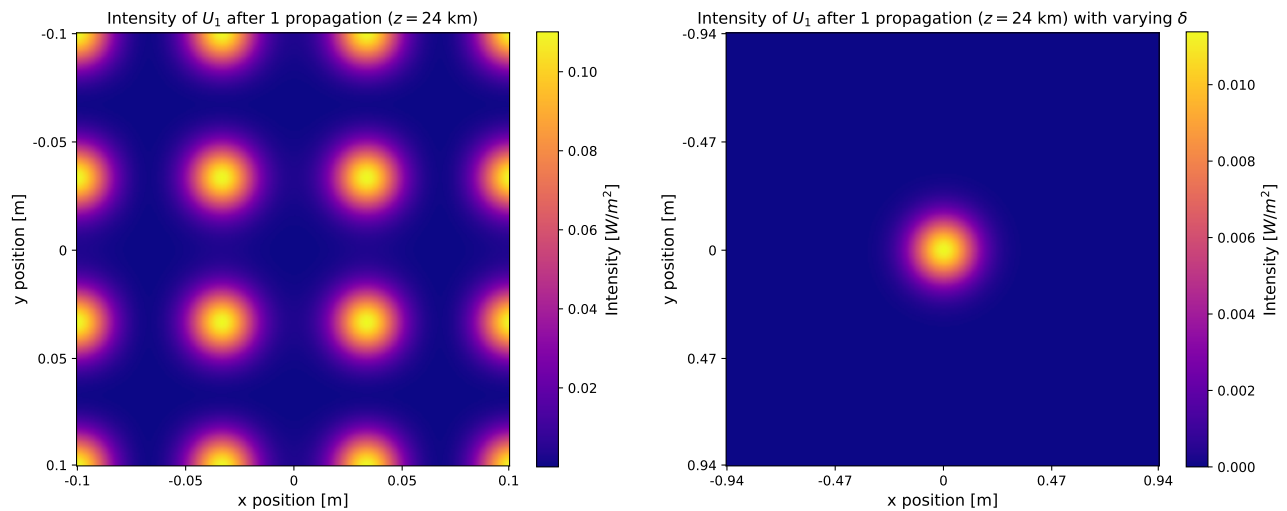


Figure 4.12: 24 km propagation with (right) and without (left) growth of the radius. The left graph presents aliasing visible by the repeated pattern.

An investigation of the number of free propagation showed that one propagation of 24 km gives exactly the same result as the 14 propagations of Table 4.1. This similarity really shows that the above conditions are more guidelines. Even though the results are the same, every time a propagation is performed, it will be with the parameters respecting the four conditions.

To go further in the aliasing analysis, the impact of the growth of the spatial spacing is studied. Figure 4.12 displays two graphs. The left graph is a propagation of a Gaussian beam with no

adaptation of the δ while the δ of the right graph follows the growth of the radius intensity. The left output presents strong aliasing visible by the repetition of a pattern. On the other hand, the right graph is a perfect propagation and the important information is contained at the center of the image.

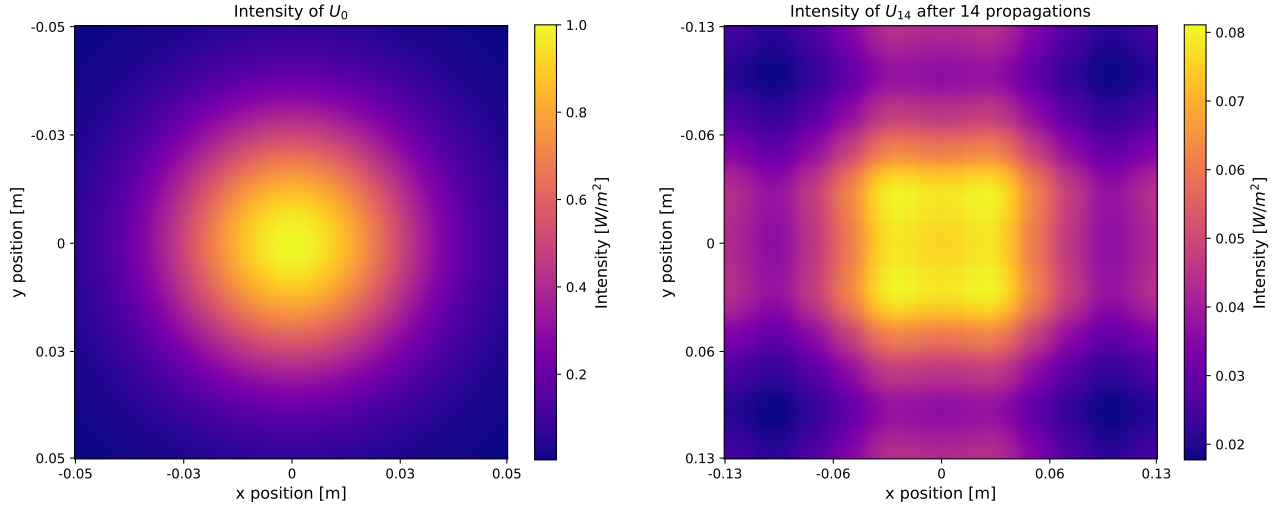


Figure 4.13: Input and output intensities for a large $\text{FWHM}_0 = 5$ cm compared to the initial aperture $S_0 = 10$ cm. The result (right) presents aliasing.

The second analysis conducted here approaches a propagation with an input presenting non zero values at the edges. This is for example the case if the input is characterised by FWHM_0 of 5 cm and an initial aperture of $S_0 = 10$ cm. This input is represented on the left plot on Figure 4.13. After propagation, aliasing appears on the output as visible on the same Figure.

To sum up, for this report, ensuring a growth of the δ with the intensity radius and avoiding the presence of non zero values near the edges at the input plane seems sufficient conditions to propagate freely without error. To go further in the analysis, it could be interesting to study the presence of aliasing outside the scope of this work (e.g. longer propagation).

Chapter 5

Turbulence

Knowledge on the power spectrum of the index of refraction was studied in chapter 2 and a method to propagate freely with an adaptation of the pixel size depending on the distance was explained in chapter 4. The last step towards a functional simulation as described in chapter 3 is the creation of PS matching real turbulent profiles.

The first section tackles the turbulent profile under hypothesis and introduces several variables such as the phase structure function and the Fried parameter. The values of the variables used in the simulations are specified.

Afterwards, the creation of PS is explained, starting from the modified von Karman spectrum, through a mathematical development. Then, the generation of PS with the coloring noise procedure is described. Subharmonics PS are also developed followed by a validation of the generated screens.

Finally, the appropriate distribution to represent the received intensity fluctuations after turbulence is analysed through high order moment matching. The log-normal distribution is more specifically explained and maximum likelihood estimators (MLE) are computed. An example of a turbulent propagation is displayed to conclude this chapter.

5.1 Profiles of turbulence

Turbulence profiles exist to correctly represent the Earth turbulent atmosphere and to help generate accurate phase screens. Those profiles are a representation of the refractive-index structure constant (C_n^2), presented in chapter 2, as a function of the altitude. When placing a phase screen at an altitude z_i , one must take the corresponding $C_{n,i}^2$ value where i refers to the index of the phase screen. The definition of C_n^2 is written in equation (2.14) showing the link with the C_T^2 . Profiles of C_n^2 are drawn following the temperature measurements.

Various profiles exist for representing the turbulence. This report uses the Hufnagel-Valley 5/7. This profile takes into account the wind speed (w) in m/s and the turbulence due to the ground (A) in $\text{m}^{-2/3}$. Its evolution with the altitude z is given by equation (5.1) where z is in meters [44].

$$C_n^2(z) = 5.94 \times 10^{-3} \left(\frac{w}{27} \right)^2 \left(\frac{z}{10^5} \right)^{10} e^{-\frac{z}{1000}} + 2.7 \times 10^{-16} e^{-\frac{z}{1500}} + A e^{-\frac{z}{100}} \quad (5.1)$$

The C_n^2 profile considered in this report is one of the most used in the literature. It is plotted in blue on Figure 5.1 for a wind speed of 21 m/s and a ground turbulence of $1.7 \times 10^{-14} \text{ m}^{-2/3}$. The parameters of the profile are the ones used in a paper written by James Osborn [40].

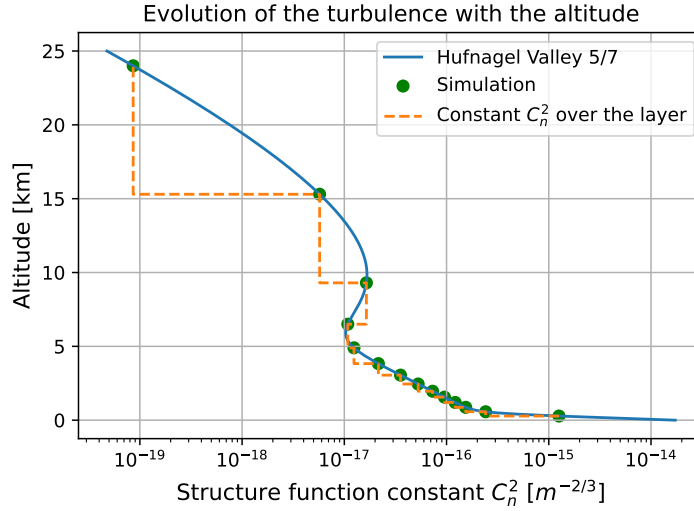


Figure 5.1: Hufnagel-Valley 5/7 turbulence profile for a wind speed of 21 m/s and a ground turbulence of $1.7 \times 10^{-14} \text{ m}^{-2/3}$. The green dots are the location of the PS and the orange dotted line the value of C_n^2 taken for each slab of atmosphere.

A bump is shown on the graph at an altitude of around 10 km implying a more turbulent layer. Note that this layer corresponds to the limit of the troposphere according to Figure 2.1.

In turbulence theory, the Fried parameter (r_0) is often used instead of C_n^2 and will, in this report, generate the phase screens in the *Python* code. To understand its definition, one needs to start back from structure function theory. A structure function can be built for the phase (D_ϕ) by using the same methodology as in section 2.2.3.3 for D_n . Considering two opposite points on the beam spot ($\mathbf{r}$ and $-\mathbf{r}$), the structure function can be expressed with a dependence only on R , the norm of $\mathbf{R}$, as shown by equation (5.2).

$$D_\phi(\mathbf{r}, \mathbf{r} + \mathbf{R}) = D_\phi(\mathbf{r}, -\mathbf{r}) = D_\phi(R) \quad (5.2)$$

The structure function can be expressed for a Kolmogorov turbulence according to the Fried parameter through equation (5.3) [9, 54].

$$D_\phi(R) = 6.88 \left(\frac{R}{r_0} \right)^{5/3} \quad (5.3)$$

The structure function of the phase can also be written in terms of the index of refraction structure constant (C_n^2) as shown in equation (5.4) where ξ is the zenith angle, k is the wavenumber and μ_0 is an integral of the structure constant over the slab of analysed atmosphere. Starting from those two definitions, the r_0 can be linked to the C_n^2 [15].

$$D_\phi(R) = 2.91k^2 R^{5/3} \mu_0 \sec(\xi) \quad \text{and} \quad \mu_0 = \int_{h_i}^{h_{i+1}} C_n^2(h) dh \quad (5.4)$$

To advance further in the development, the spatial coherent radius (ρ_0) has to be defined. This radius is the value R must take so that the structure function $D_\phi(R = \rho_0) = 2 \text{ rad}^2$ [9]. From this result, the Fried parameter can be related to ρ_0 by equation (5.5) and ρ_0 can be related to C_n^2 by equation (5.6).

$$r_0 = \rho_0 \left(\frac{6.88}{D_\phi(\rho_0)} \right)^{3/5} = \rho_0 \left(\frac{6.88}{2} \right)^{3/5} = 2.1 \rho_0 \quad (5.5)$$

$$\rho_0 = \left(\frac{D_\phi(\rho_0) \cos(\xi)}{2.91 \mu_0 k^2} \right)^{3/5} = \left(\frac{2 \cos(\xi)}{2.91 \mu_0 k^2} \right)^{3/5} = \left(\frac{\cos(\xi)}{1.45 \mu_0 k^2} \right)^{3/5} \quad (5.6)$$

After computation, the Fried parameter is 2.1 times greater than the spatial coherence radius. ρ_0 is then expressed in terms of ξ , k and μ_0 .

i	Δh_i [m]	z_i [m]	$C_{n,i}^2$ [$\text{m}^{-2/3}$]	$\rho_{0,i}$ [m]	$r_{0,i}$ [m]
1	280	280	1.258×10^{-15}	0.08	0.169
2	290	570	2.415×10^{-16}	0.212	0.444
3	300	870	1.540×10^{-16}	0.272	0.57
4	330	1200	1.214×10^{-16}	0.296	0.621
5	360	1560	9.544×10^{-17}	0.324	0.681
6	400	1960	7.310×10^{-17}	0.357	0.75
7	490	2450	5.275×10^{-17}	0.385	0.808
8	600	3050	3.546×10^{-17}	0.432	0.908
9	780	3830	2.154×10^{-17}	0.498	1.046
10	1070	4900	1.243×10^{-17}	0.573	1.204
11	1600	6500	1.082×10^{-17}	0.489	1.028
12	2800	9300	1.645×10^{-17}	0.272	0.571
13	6000	15300	5.734×10^{-18}	0.324	0.681
14	8700	24000	8.604×10^{-20}	3.222	6.765

Table 5.1: Computed values of $C_{n,i}^2$, $\rho_{0,i}$ and $r_{0,i}$ for each phase screens i . The thickness of the slab of atmosphere is variable and given by Δh_i and the output altitude is z_i .

Bringing it all together, equations (5.5) and (5.6) become equation (5.7). The Fried parameter is approximated as a constant $C_{n,i}^2$ is considered for the entire atmosphere slab of thickness Δh_i . Therefore, the integral becomes a simple multiplication. The orange dotted line in Figure 5.1 represents the constant value of the $C_{n,i}^2$ for each slab of atmosphere i . Based on equation (5.7), the Fried parameter and the C_n^2 are inversely proportional meaning a strong turbulence (large C_n^2) results in a small Fried number.

$$\begin{aligned}
 r_{0,i} &= 2.1\rho_{0,i} \quad \text{with } \rho_{0,i} = \left(\frac{\cos(\xi)}{1.45\mu_{0,i}k^2} \right)^{3/5} \quad \text{and } \mu_{0,i} = \int_{z_{i-1}}^{z_i} C_n^2(z) dz \\
 &= \left[0.423 \sec(\xi) k^2 \int_{z_{i-1}}^{z_i} C_n^2(z) dz \right]^{-3/5} \approx \left[0.423 k^2 C_{n,i}^2 \Delta h_i \right]^{-3/5}
 \end{aligned} \tag{5.7}$$

For example, one simulation with $n = 14$ PS resulted in r_0 displayed in Table 5.1. The 14 points considered are represented in green on Figure 5.1 and are the same than the ones respecting the free propagation conditions in section 4.4.

This report does not study the impact of the zenith angle. A perpendicular to the ground propagation is considered and therefore, ξ is set to 0° .

To sum up, this section approached the profiles of turbulence by describing the evolution of the C_n^2 with the altitude. Then a link with the Fried parameter was found through two phase structure function definitions. Finally, for each PS, the constant $C_{n,i}^2/r_{0,i}$ value chosen to represent the turbulence of the i^{th} slab of atmosphere at altitude z_i and of thickness Δh_i is computed. With the known values $r_{0,i}$, the phase screen can be generated. The next section explain this generation.

5.2 Creation of phase screens

5.2.1 Mathematical description

Obtaining the power spectral density (PSD) of the phase is essential to generate the PS with the correct turbulent statistics. At the moment, the PSD of the refractive-index is known (see section 2.2.4) and must be transformed according to equation (5.8) [22]. k is the wave number and Δh the thickness of the atmosphere slab.

$$\Phi_\phi(\kappa) = 2\pi k^2 \Delta h \Phi_n(\kappa) \tag{5.8}$$

The 2 dimensional PSD of the refractive index ($\Phi_n(\kappa) = \Phi_n(\kappa_x, \kappa_y)$) are already known for the different spectra defined in section 2.2.4. Applying equation (5.8) allows to derive the PSD of the phase (Φ_ϕ) according to $\kappa = \sqrt{\kappa_x^2 + \kappa_y^2}$. In equation (5.9), K , vK and mvK respectively symbolize the Kolmogorov, von Karman and modified von Karman spectrum.

$$\begin{cases} \Phi_\phi^K(\kappa_x, \kappa_y) = 0.49r_0^{-5/3} \kappa^{-11/3} \\ \Phi_\phi^{vK}(\kappa_x, \kappa_y) = \frac{0.49r_0^{-5/3}}{(\kappa^2 + \kappa_0^2)^{11/6}} \\ \Phi_\phi^{mvK}(\kappa_x, \kappa_y) = 0.49r_0^{-5/3} \frac{e^{-\left(\frac{\kappa}{\kappa_m}\right)^2}}{(\kappa^2 + \kappa_0^2)^{11/6}} \end{cases} \tag{5.9}$$

The goal is to obtain an expression depending on the frequencies ($\Phi_{\phi,f}(f_x, f_y)$) and not the wavenumbers ($\Phi_{\phi,\kappa}(\kappa_x, \kappa_y)$) as the simulation uses frequency grids. There is a factor 2π between κ and f according to $\kappa = 2\pi f$ but directly substituting f in κ and obtaining a factor $(2\pi)^{-11/3}$

is not correct. Indeed, it is the integrated values that must be equal¹. Starting back from the structure function $D_\phi(x, y)$, the FT according to (κ_x, κ_y) and (f_x, f_y) can be developed as in equation (5.10) [18]. Note that this equation is the 2 dimensional formulation of equation (2.15).

$$\begin{aligned} D_\phi(x, y) &= 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_{\phi, \kappa}(\kappa_x, \kappa_y) [1 - e^{i(\kappa_x x + \kappa_y y)}] d\kappa_x d\kappa_y \\ &= 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_{\phi, f}(f_x, f_y) [1 - e^{i2\pi(f_x x + f_y y)}] df_x df_y \end{aligned} \quad (5.10)$$

From equation (5.10), the integrated values must be equal and by applying the change of variable $\kappa = 2\pi f$, a link between $\Phi_{\phi, f}(f_x, f_y)$ and $\Phi_{\phi, \kappa}(\kappa_x, \kappa_y)$ is found in equation (5.12).

$$\begin{aligned} \Phi_{\phi, f}(f_x, f_y) [1 - e^{i2\pi(f_x x + f_y y)}] df_x df_y &= \Phi_{\phi, \kappa}(\kappa_x, \kappa_y) [1 - e^{i(\kappa_x x + \kappa_y y)}] d\kappa_x d\kappa_y \\ &= \Phi_{\phi, \kappa}(2\pi f_x, 2\pi f_y) [1 - e^{i2\pi(f_x x + f_y y)}] (2\pi)^2 df_x df_y \end{aligned} \quad (5.11)$$

$$\Phi_{\phi, f}(f_x, f_y) = (2\pi)^2 \Phi_{\phi, \kappa}(2\pi f_x, 2\pi f_y) \quad (5.12)$$

With the change of variable derived in equation (5.12), the PSD of the phase can be expressed in terms of the frequency. κ is replaced by $2\pi f$ in equations (5.9) and is then multiplied by $(2\pi)^2$. The resulting normalisation factor is $(2\pi)^{-5/3}$ and transform 0.49 into 0.023 [9].

$$\begin{cases} \Phi_\phi^K(f_x, f_y) = 0.023 r_0^{-5/3} f^{-11/3} \\ \Phi_\phi^{vK}(f_x, f_y) = \frac{0.023 r_0^{-5/3}}{(f^2 + f_0^2)^{11/6}} \\ \Phi_\phi^{mvK}(f_x, f_y) = 0.023 r_0^{-5/3} \frac{e^{-\left(\frac{f}{f_m}\right)^2}}{(f^2 + f_0^2)^{11/6}} \end{cases} \quad (5.13)$$

The variables κ_m and κ_0 defined in sections 2.2.4.2 and 2.2.4.3 are also transformed to be expressed as f_m and f_0 . Their values are computed following equation (5.14) using the previous definition of κ_m and κ_0 .

$$\begin{cases} f_m = \frac{\kappa_m}{2\pi} = \frac{5.92}{2\pi l_0} \\ f_0 = \frac{\kappa_0}{2\pi} = \frac{1}{L_0} \end{cases} \quad (5.14)$$

The spectrum used for the simulation is the modified von Karman as it takes the outer and inner scales in consideration. The mvK PSD found in equation (5.13) is plotted on Figure 5.2 using a logarithmic scale.

To resume this mathematical development, starting from the PSD of the refractive index according to the wavenumber $\Phi_n(\kappa_x, \kappa_y)$, the PSD of the phase was found $\Phi_\phi(\kappa_x, \kappa_y)$ and then transformed into the frequency domain $\Phi_\phi(f_x, f_y)$. With this PSD, the PS can be generated using the coloring noise procedure described in the next section.

¹Discussed with Florian Quatresooz and Gilles Orban de Xivry

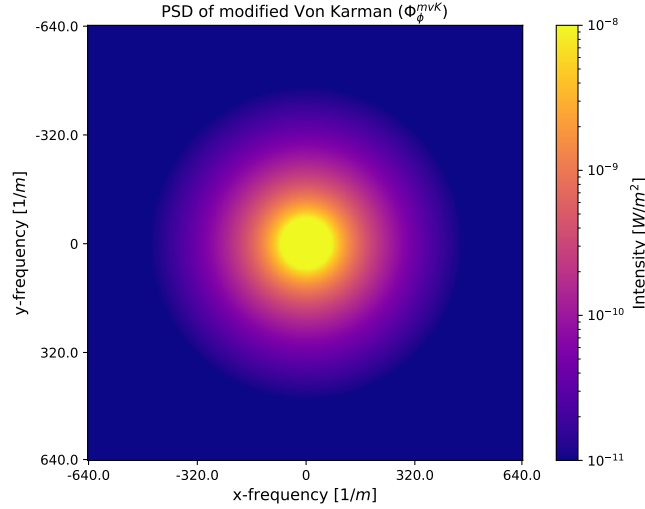


Figure 5.2: Logarithmic view of the modified von Karman power spectrum density for the phase variation.

5.2.2 Coloring noise

Knowing the power spectrum density of the modified von Karman spectrum allows to create PS matching the atmospheric turbulence [44]. The first observation is the possibility to express the phase variation ($\phi(x, y)$) by performing the inverse Fourier transform on the power spectrum ($\Phi_\phi(f_x, f_y)$) as shown in equation (5.15). The phase variation is defined for each pixel (x, y) of the image and represents the value of the phase shift at this position [15].

$$\phi(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_\phi(f_x, f_y) e^{i2\pi(f_x x + f_y y)} df_x df_y \quad (5.15)$$

The integral form of the Fourier transform can be discretized into an infinite sum of Fourier coefficients ($c_{m,n}$) multiplied by a complex exponential according to equation (5.16). f_{x_m} and f_{y_n} are the discrete frequencies along the x and y axis. These variables are the ones defined in equation (3.4) when the frequency grid was introduced. They can be written as $f_{x_m} = m\delta_{f_x}$ and $f_{y_n} = n\delta_{f_y}$

$$\phi(x, y) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_{m,n} e^{i2\pi(f_{x_m} x + f_{y_n} y)} \quad (5.16)$$

The phase variation is the results of many random inhomogeneities and its Fourier coefficients can thus be assumed Gaussian using the central limit theorem (CLT). The $c_{m,n}$ also possess a 0 mean and are often complex with both variances (real and complex) being the same and shown in equation (5.17). Finally, the $c_{m,n}$ are circular complex Gaussian because their cross-covariances are 0.

$$\langle |c_{m,n}|^2 \rangle = \Phi_\phi(f_{x_m}, f_{y_n}) \delta_{f_x} \delta_{f_y} \quad (5.17)$$

The standard deviation is simply the square root of the variance and is given by equation (5.18). In the *Python* script and more generally in this report, only square images are considered and therefore the frequency grid spacing in x and y directions is the same. This imply $\delta_{f_x} = \delta_{f_y} = \delta_f$

$$\langle |c_{m,n}| \rangle = \sqrt{\Phi_\phi(f_{x_m}, f_{y_n}) \delta_{f_x} \delta_{f_y}} = \sqrt{\Phi_\phi(f_{x_m}, f_{y_n})} \delta_f \quad (5.18)$$

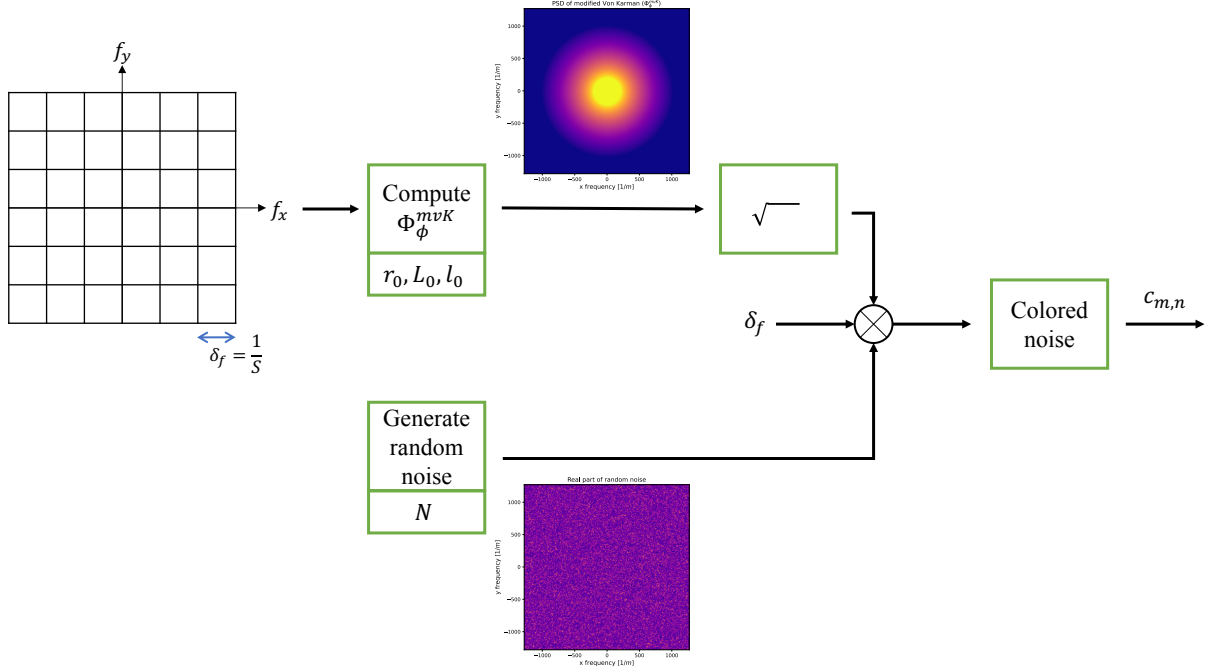


Figure 5.3: Diagram of the coloring noise process. It uses the frequency grid and a random noise grid to compute the Fourier coefficients $c_{m,n}$.

For the moment, everything is expressed for a large set of PS and statistical variables are studied. When generating only one phase screen, the impact of noise must be taken into account. It explains the apparition of the term $(\alpha + i\beta)_{m,n}$ which is the complex Gaussian random noise centered in 0 and with unitary variance for the coefficient m, n . Equation (5.19) is the colored noise.

$$c_{m,n} = (\alpha + i\beta)_{m,n} \sqrt{\Phi_\phi(f_{x_m}, f_{y_n})} \delta_f \quad (5.19)$$

The final step towards PS generation is to perform an inverse Fourier transform of the Fourier coefficients. In fact this gives 2 PS : one in the real part and the other in the complex part [20]. By convention, the real part is taken for this work. The normalisation is set to $\delta_f = 1$ as there are no correcting factor in equation (5.16) compared to the defined FT (see equation (4.14)).

$$\text{PS} = \phi(x, y) = \Re \left\{ \mathcal{F}^{-1} \left[(\alpha + i\beta)_{m,n} \sqrt{\Phi_\phi(f_{x_m}, f_{y_n})} \delta_f, 1 \right] \right\} \quad (5.20)$$

Figure 5.3 sums up the different steps towards the creation of a PS. The variables needed for each step are placed under the corresponding block. Starting from the frequency grid defined with N and S , the modified von Karman PSD is computed using the third equation in (5.13). Note that the DC component of the PSD (pixel $(\frac{N}{2}, \frac{N}{2})$) is set to 0 because otherwise the spectrum would be infinite at $f_x = f_y = 0$. At the same time, a random grid of size N is created and multiplied by the square root of the PSD and by δ_f . The output of this block is the colored noise denoted $c_{m,n}$ in the development here above (5.19). Finally, according to

equation (5.20), the IFT is performed.

At this point, phase screens representing the turbulence can be generated based on $\Phi_\phi(f_x, f_y)$. The PSD corresponds to the atmosphere depending on the considered altitude. Therefore, as the wave propagates, the PS will have changing characteristics.

5.2.3 Subharmonics

Figure 5.2 shows the PSD of the spectrum used to generate the PS. The turbulence is mainly located at the centre of the image, in the low frequencies region. Displaying the phase structure function (see Figure 5.4) for PS generated (blue curve) reveals a difference compared to the theoretical behaviour (green curve) described by equation (5.3). This difference is due to the lack of information at low frequencies. Indeed, as the spectrum is sharp near $f = 0$, the number of pixel to represent this area of the grid is too low and thus there is a need to improve the low frequencies content. The simulation for Figure 5.4 is done with a constant turbulence of $r_0 = 0.5$ m and is similar to results found in the literature [9, 18, 43].

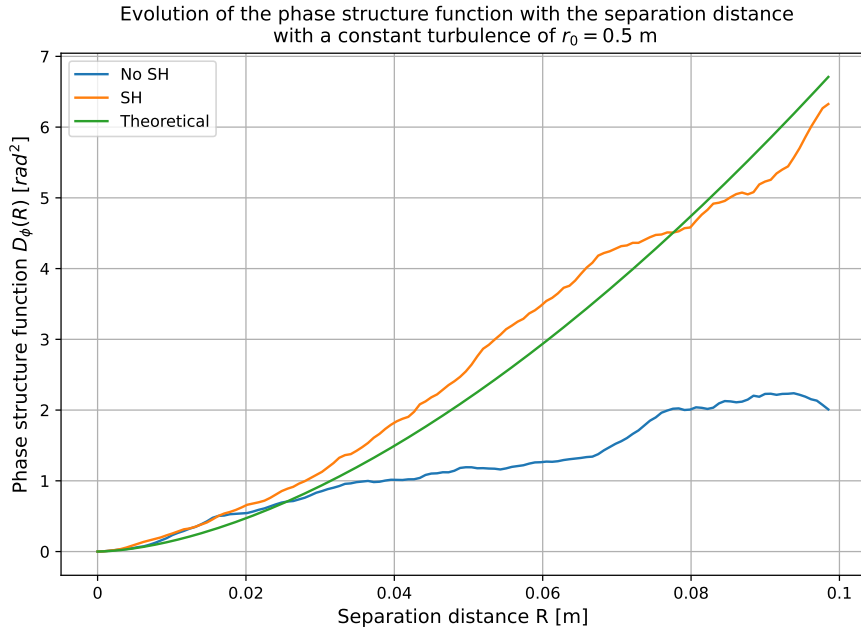


Figure 5.4: Evolution of the theoretical (green), classical (blue) and subharmonic (orange) phase structure functions depending on the distance between two points (R).

The objective is to improve the low frequency content of the generated PS to match the theoretical phase structure function. To attain this goal, a subharmonic (SH) method can be implemented. The general idea is to perform the coloring procedure explained in the previous section on a grid of size $N = 3$ pixels with frequency spacing becoming smaller and smaller as p , the degree of the subharmonic, increases. Then, with the Fourier coefficients coming from the coloring, one can perform a manual IDFT for each pixel (x, y) of the 256×256 grid. Finally, the procedure is repeated for p increased by 1 and the results are added. The block scheme is shown on Figure 5.5 [18, 55].

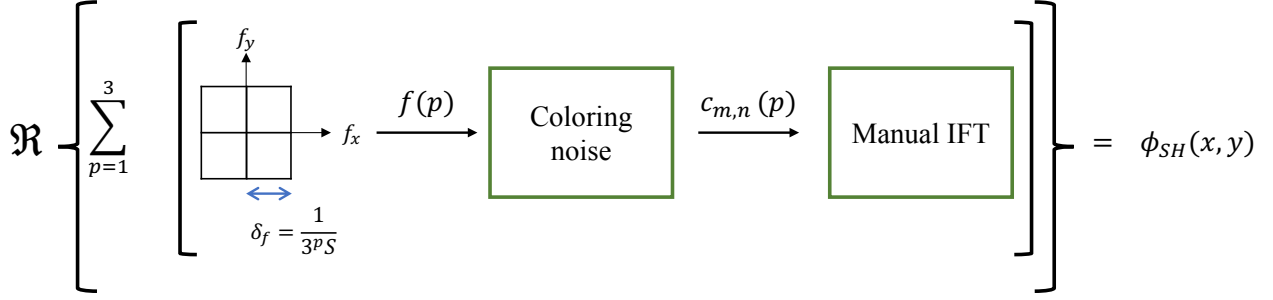


Figure 5.5: Diagram illustrating the subharmonic method. p is the degree of the SH.

With the apparition of p comes the definition of a new frequency spacing in equation (5.21). The f_x and f_y deltas are the same in both directions keeping in mind the image is a square. In the *Python* code, p ranges from 1 to 3 creating subharmonics with frequency spacing $\delta_f = \frac{1}{3S}$, $\frac{1}{9S}$ and $\frac{1}{27S}$

$$\delta_{f_x}(p) = \delta_{f_y}(p) = \delta_f(p) = \frac{1}{3^p S} \quad (5.21)$$

The new frequency grid of size 3 is defined in the f_x and f_y directions according to equation (5.22). It is a simple multiplication between the variable m (or n) indicating the pixel location and the frequency delta.

$$\begin{cases} f_{x_m}(p) = m\delta_f(p) = \frac{m}{3^p S} & m = (-1, 0, 1) \\ f_{y_n}(p) = n\delta_f(p) = \frac{n}{3^p S} & n = (-1, 0, 1) \end{cases} \quad (5.22)$$

Figure 5.5 shows the procedure to perform SH PS generation. The major difference with the classical PS generation is the manual IDFT performing the SH improvement directly in the spatial domain.

Starting with equation (5.16) described in the previous section, the same development can be done. First of all, the total phase variation (ϕ_{tot}) is the previous phase variation (ϕ) on top of which the SH phase variation (ϕ_{SH}) is added. The SH phase is then equal to equation (5.23) which resumes Figure 5.5. The DC component is also removed from the SH generation but by subtracting the ϕ_{SH} by its mean as it is already in the spatial domain.

$$\begin{aligned} \phi_{tot} &= \phi + \phi_{SH} \quad \text{with} \\ \phi_{SH}(x, y) &= \Re \left\{ \sum_{p=1}^3 \sum_{n=-1}^1 \sum_{m=-1}^1 c_{m,n}(p) e^{i2\pi(f_{x_m}(p)x + f_{y_n}(p)y)} \right\} \end{aligned} \quad (5.23)$$

The Fourier coefficients ($c_{m,n}$) are dependents on the variable p . The definition of their variance stays the same but the frequencies can be replaced by their definitions in equation (5.22). The result is shown in equation (5.24). The standard deviation is the square root of equation (5.24).

$$\begin{aligned} \langle |c_{m,n}(p)|^2 \rangle &= \Phi_\phi(f_{x_m}(p), f_{y_n}(p)) \delta_f^2(p) \\ &= \Phi_\phi\left(\frac{m}{3^p L}, \frac{n}{3^p L}\right) \delta_f^2(p) \end{aligned} \quad (5.24)$$

The impact of the noise is added to symbolise one occurrence. The Fourier coefficients can be seen as a Gaussian complex noise multiplied by their standard deviation : $c_{m,n}(p) = (\alpha + i\beta)_{m,n,p} \langle c_{m,n}(p) \rangle$.

$$c_{m,n}(p) = (\alpha + i\beta)_{m,n,p} \sqrt{\Phi_\phi\left(\frac{m}{3^p L}, \frac{n}{3^p L}\right)} \delta_f(p) \quad (5.25)$$

The last step to obtain equation (5.26) is to replace the coefficients of equation (5.25) in the SH phase variation of equation (5.23).

$$\phi_{SH}(x, y) = \sum_{p=1}^3 \sum_{n=-1}^1 \sum_{m=-1}^1 (\alpha + i\beta)_{m,n,p} \sqrt{\Phi_\phi\left(\frac{m}{3^p L}, \frac{n}{3^p L}\right)} \delta_f(p) e^{i2\pi\left(\frac{m}{3^p L}x + \frac{n}{3^p L}y\right)} \quad (5.26)$$

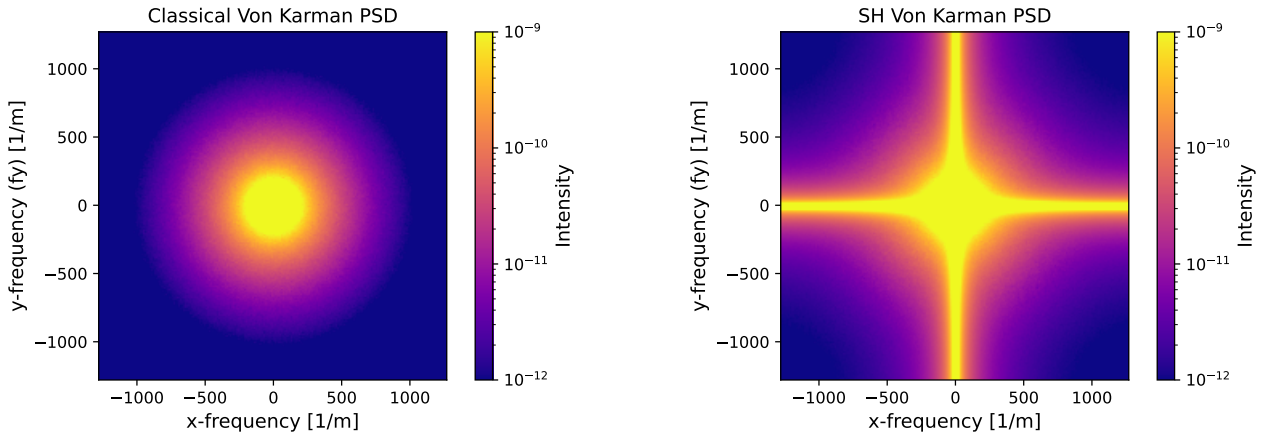


Figure 5.6: Result of the validation procedure for the classical PS (left) and the SH PS (right). The plots are in logarithmic scale.

The obtained PSD with the addition of subharmonics is shown on the right in Figure 5.6. The main difference is a growth of the intensity for the low frequencies. Computing the phase structure function for the SH PS results in the orange curve on Figure 5.4. The theoretical behaviour is matched and the generated PS correctly represents the turbulent atmosphere. For the *Python* code, the SH version of the PS is implemented to perform the turbulent propagation.

5.2.4 Validation

Validation of the phase screen is important to confirm the presence of the turbulent spectrum hidden by the noise. Indeed, if one occurrence of a PS is taken, it is impossible to say whether it represents the turbulence or not. To illustrate, Figure 5.7 shows on the left a phase screen generated using formula (5.20) and on the right using formula (5.26).

The method to validate the generation of PS is to isolated Φ_ϕ in equation (5.20). To do so, the first step is to take the Fourier transform with a coefficient of δ . Note that the coefficient could also be 1 and would result in a different normalisation at the end of the procedure. Equation (5.27) displays the result of taking the FT with use of equation (4.13).

$$\mathcal{F}\left[\mathcal{F}^{-1}\left[(\alpha + i\beta)_{m,n} \sqrt{\Phi_\phi(f_{x,n}, f_{y,m})} \delta_f, 1\right], \delta\right] = (\alpha + i\beta)_{m,n} \sqrt{\Phi_\phi(f_{x,n}, f_{y,m})} \delta_f N^2 \delta^2 \quad (5.27)$$

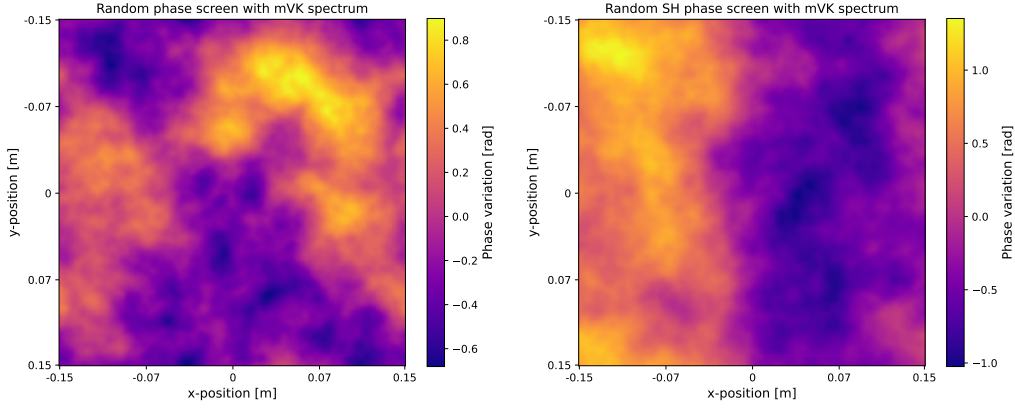


Figure 5.7: One occurrence of a phase screen (ϕ) with the hidden PSD of modified von Karman for the classical case (left) and the SH case (right).

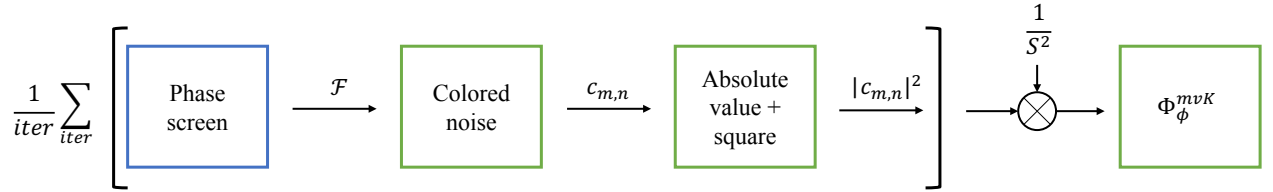


Figure 5.8: Diagram of the steps to validate the PSD of generated phase screens ($iter$ is the number of PS used for validation).

The second step is to take the absolute value and then delete the square root by taking the square of equation (5.27) to obtain equation (5.28). The absolute value is useful to let the variance appear later in the development. Afterwards, using the definitions of the variable $\delta = \frac{S}{N}$ and $\delta_f = \frac{1}{S}$ referenced in chapter 3 allows to simplify the expression to only one factor equal to S^2 .

$$|(\alpha + i\beta)_{m,n}|^2 \Phi_{\phi(f_{x_n}, f_{y_m})} \delta_f^2 N^4 \delta^4 = |(\alpha + i\beta)_{m,n}|^2 \Phi_{\phi(f_{x_n}, f_{y_m})} S^2 \quad (5.28)$$

The last step is to delete the randomness of the phase screen added by the noise. To do so, one must consider multiple phase screens and perform a mean. As stated earlier, the noise is centered around 0 with variance of 1, when taking the mean of equation (5.28), the factor $\langle |(\alpha + i\beta)_{m,n}|^2 \rangle$ appears and is by definition the variance of the noise. Substituting the variance with 1 gives the last form in equation (5.29). A normalisation of S^2 needs to be applied in order to finally obtain the PSD.

$$\langle |(\alpha + i\beta)_{m,n}|^2 \rangle \Phi_{\phi(f_{x_n}, f_{y_m})} S^2 = \Phi_{\phi(f_{x_n}, f_{y_m})} S^2 \quad (5.29)$$

Figure 5.9 illustrate on a logarithmic scale the theory and the practice. The theoretical PSD is built using the formula in equation (5.13) for the modified von Karman spectrum (mvK) whereas the right plot is the output of the procedure explained here above using $iter = 100$ PS to verify the presence of the hidden spectrum. To perform those plots, the parameters were set as follows : $r_0 = 0.2$ m, $S = 0.1$ m, $N = 256$ pixels, $l_0 = 0.001$ m and $L_0 = 100$ m.

The two plots on Figure 5.9 look similar and thus confirm the presence of the spectrum in each PS but a closer look on the right image shows the non perfect behaviour. Indeed, with 100

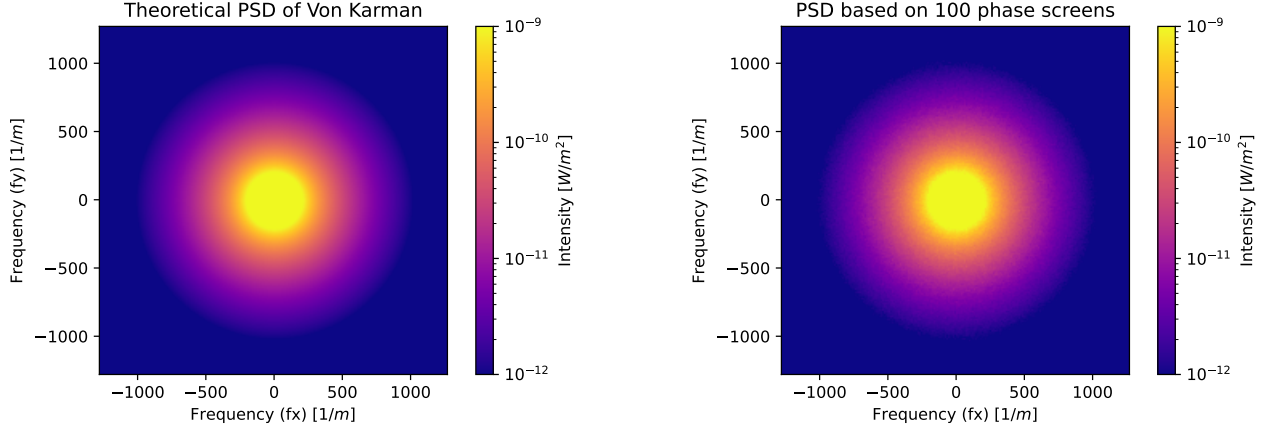


Figure 5.9: Logarithmic plots of the theoretical (left) and simulated (right) PSD of the modified von Karman spectrum.

PS, the variance tends to 1 but does not reach it inducing some irregularities. If the number of PS taken to perform the mean would tend to ∞ , then both images would be exactly the same.

With the section just presented, the PS have been defined and validated. It is now possible to perform complete simulations of propagation through atmospheric turbulence. The last section in this chapter analyses the received intensities at the output plane and aims at finding an appropriate distribution to represent it.

5.3 Distribution of the received intensity

The theoretical background of the turbulence, the creation of phase screens and their validation have been explained. This last section looks at the distribution of the received intensity after simulations and at a possible match with a theoretical distribution.

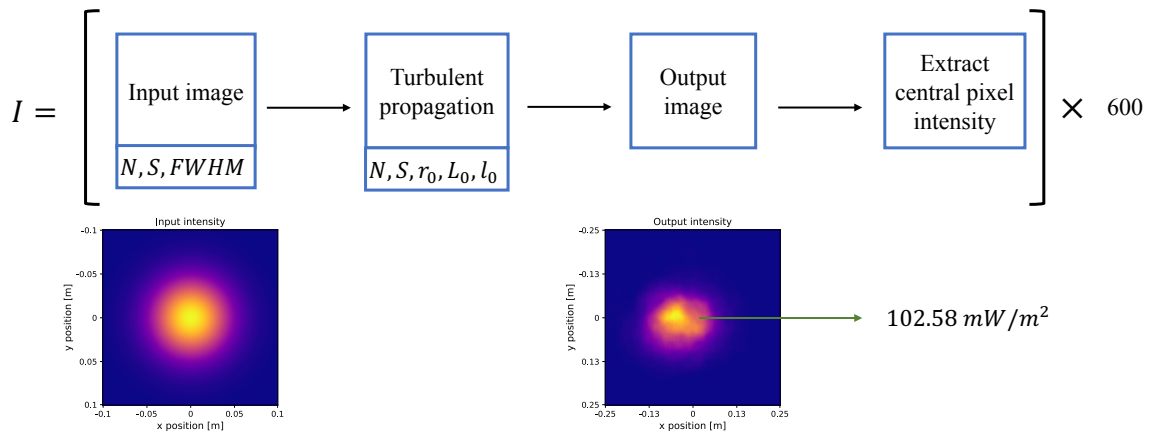


Figure 5.10: Block scheme for the creation of the central pixel received intensity array (I) using 600 simulations. The parameters are $N = 256$ pixels, $S_0 = 20$ cm and $\text{FWHM}_0 = 5$ cm.

The general idea is to perform a large number of simulations and look at the received intensity. In this report, 600 simulations and therefore intensities are collected and stored in an array

(I). Figure 5.10 shows the procedure applied to obtain the intensities. It always starts with the same input image : $S_0 = 20$ cm, $N = 256$ pixels and an initial FWHM₀ of 5 cm. A turbulent PS propagation is then performed using constant r_0 and the output image is found. The last step is to extract the value of the intensity. To know which intensity out of the 256×256 pixels constituting the output to choose, the hypothesis of perfect tracking has been set in this report (see section 3.4). This means the satellite is always align with the transmitter and that only the central pixel ($x = 0, y = 0$) is captured by the aperture of the telescope.

The general idea is to analyse the fluctuations of the received intensity at the centre of the image. Those fluctuations can then be compared to known distributions. A knowledge of the type of distribution the turbulence follows could allow scientists to improve the communication link.

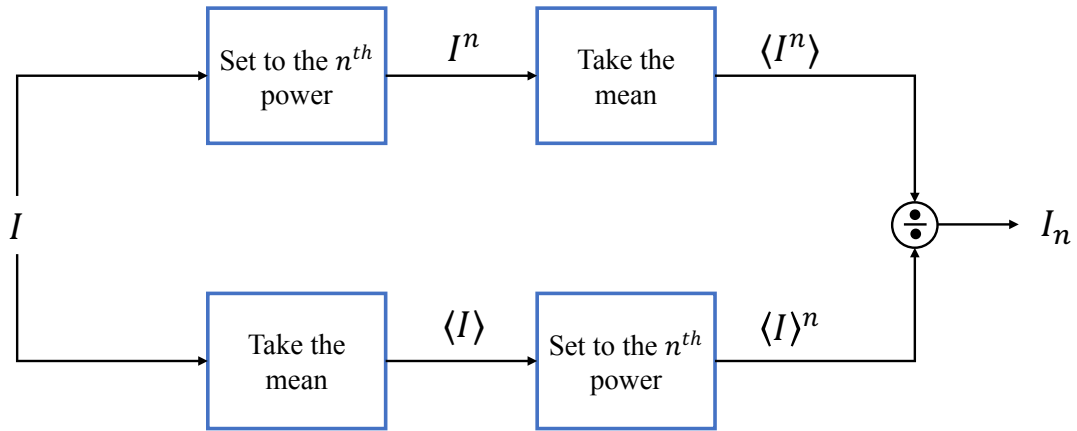


Figure 5.11: Block scheme for the computation of the high order moments I_n using the central pixel received intensity array I .

In this report, 3 distributions selected based on the literature are considered. The distributions are the log-normal, the log-normal exponentially modulated and the K distribution [15, 56]. In order to find the appropriate distribution to match the atmospheric turbulence, the high order moments of the intensity array received (I_n) are compared to the second order (I_2). By definition, the moments of order n are described according to equation (5.30) [57] where I is the array of 600 values extracted from the simulations. The method applied to obtain I_n is shown on Figure 5.11 and is applied for $n = \{2, 3, 4, 5\}$.

$$I_n = \frac{\langle I^n \rangle}{\langle I \rangle^n} \quad (5.30)$$

With theoretical curves describing the evolution of I_n with respect to I_2 for different distributions and by plotting the I_n and I_2 obtained from the simulations, one can evaluate the fitting.

For the log-normal distribution, the link between the second order and the higher order moments is given by equation (5.31). In this section, the order analysed are $n = 3, 4$ and 5 . The log-normal distribution is studied for $I_2 > 1$ otherwise the I_n would tend to 0 as n increases [57].

$$I_n = I_2^{n(n-1)/2} \quad (5.31)$$

The second distribution is the log-normally modulated exponential. It uses a relation between the Rytov variance (σ_R^2) and the scintillation index (σ_I^2) as shown in equation (5.32) [57].

$$e^{\sigma_R^2} = \frac{\sigma_I^2 + 1}{2} \quad \text{and} \quad \sigma_I^2 = I_2 - 1 \quad \text{thus} \quad e^{\sigma_R^2} = \frac{I_2}{2} \quad (5.32)$$

The expression of the n^{th} moment as a function of I_2 is displayed in equation (5.33). It can be simplified using the link explained in equation (5.32). The obtained link between I_n and I_2 is similar to the one of the log-normal distribution except for the factorial in front and the factor $\frac{1}{2}$. The distribution is analysed for value of I_2 larger than 2 otherwise the high order moments would tend to 0.

$$\begin{aligned} I_n &= n! e^{\sigma_R^2 n(n-1)/2} \\ &= n! \left(e^{\sigma_R^2} \right)^{n(n-1)/2} \\ &= n! \left(\frac{I_2}{2} \right)^{n(n-1)/2} \end{aligned} \quad (5.33)$$

The last distribution is the K distribution. It uses the gamma function with the expression in equation (5.34) [58].

$$\Gamma(n) = (n-1)! \quad \text{where } n > 0 \quad (5.34)$$

The I_n of the K distribution evolves with I_2 according to equation (5.35). With use of the definition of the gamma function, one can see observe that $I_2 > 2$ otherwise the argument of the Gamma function would be negative.

$$I_n = \frac{n! \Gamma\left(\frac{2}{I_2-2} + n\right)}{\left(\frac{2}{I_2-2}\right)^n \Gamma\left(\frac{2}{I_2-2}\right)} \quad (5.35)$$

Figure 5.12 plots the logarithm of the n^{th} order for the 3 distributions. As explained previously, only the log-normal distribution goes beneath $I_2 = 2$. The two others are only defined for values greater than 2. In order to choose the appropriate distribution to represent the turbulence, simulations at constant turbulence are performed. It has no physical meaning as the turbulence varies as shown on Figure 5.1 but it is a great way to distinguish the weak and strong turbulent regimes. The array of intensities (I) received from the simulations is used to compute the I_n at different levels of turbulence. Those points are reported on Figure 5.12 for values of r_0 equal to 0.1 m, 0.3 m, 0.5 m and 1 m.

For a large turbulence ($r_0 = 0.1$ m), the high order moments seem to follow a K distribution whereas at weak turbulence the behaviour matches the log-normal distribution. The right graph on Figure 5.12 is a zoom on the values of I_2 ranging from 1 to 1.4. On this graph, the points are for the turbulence r_0 equal to 0.3 m, 0.5 m and 1 m.

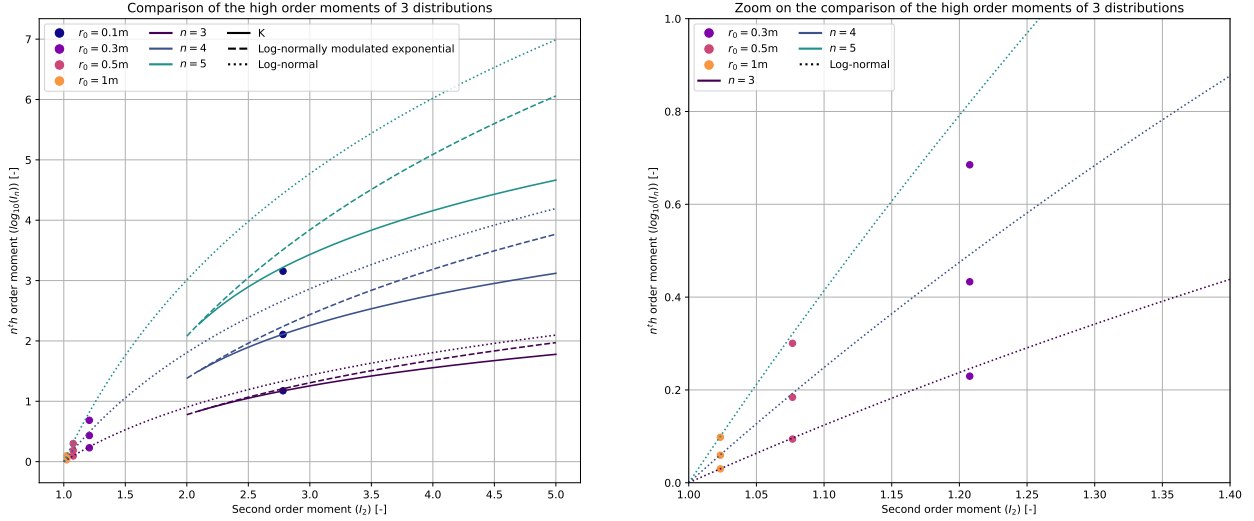


Figure 5.12: Comparison of the moments of order 3 (I_3), 4 (I_4) and 5 (I_5) with the second order (I_2) for three distribution : K distribution (line), log normally modulated exponential (long dotted line) and log normal (short dotted line). Simulated moments are shown by points for different levels of turbulence.

To sum up, at weak turbulence the received intensity tends to follow a log-normal distribution and in the strong turbulence regime, the K distribution seems to govern. The strong regime is not further investigated. The behaviour at weak turbulence as well as the non fitting behaviour of the log-normal distribution at strong turbulence were predicted by Andrews [15]. To prove the case, Table 5.2 is provided to see the error between the simulated moment and the theoretical value for a log normal distribution. It is clear that for $r_0 = 0.1$ m, the selected distribution is not appropriate.

r_0		m	0.1	0.3	0.5	1
I_2		W/m ²	2.78	1.21	1.08	1.02
I_3	Simulated	W/m ²	14.93	1.7	1.24	1.07
	Theoretical		21.53	1.76	1.25	1.07
	Difference		6.6	0.06	0.01	0.0
I_4	Simulated	W/m ²	128.18	2.71	1.53	1.15
	Theoretical		463.56	3.1	1.56	1.15
	Difference		335.38	0.39	0.03	0.0
I_5	Simulated	W/m ²	1430.72	4.84	2.0	1.25
	Theoretical		27765.35	6.6	2.09	1.26
	Difference		26334.63	1.76	0.09	0.01

Table 5.2: Values of the simulated, theoretical high order moments and their difference for different levels of constant turbulence ($r_0 = 0.1$ m, 0.3 m, 0.5 m and 1 m).

For the work done in this report, only the weak regime is of interest. Indeed, as stated before, a constant r_0 has no physical sense. To get an idea of the order of magnitude of the moments of a propagation using an existing profile, the I_2 is computed. It ranges between 1.041 and 1.442 depending on the wind speed and the ground turbulence. The upper value is for a wind speed

of 100 m/s and can be considered as a extreme situation. I_2 thus stays in the weak regime where the log-normal distribution is in play.

5.3.1 Log-normal distribution

The previous section asserted the output intensity to behave as a log-normal distribution in the weak regime. The goal now is to fit the data received with the correct log-normal parameters. To do so, the maximum likelihood estimator (MLE) is used to compute the log-normal estimators. To begin with, the probability density function (PDF) of the studied distribution is given by equation (5.36) where in the current case, the x_i are the values of intensities at the output [59]. The parameters estimated by the MLE are μ and σ , the mean and standard deviation.

$$f_X(x_i|\mu, \sigma^2) = \frac{1}{x_i \sigma \sqrt{2\pi}} e^{-\frac{(\ln(x_i) - \mu)^2}{2\sigma^2}} \quad (5.36)$$

A MLE is done by multiplying the PDF's for all x_i . With the knowledge of the PDF form in equation (5.36), equation (5.37) is found [59].

$$L(x_1, x_2, \dots, x_N) = \prod_{i=1}^N f_X(x_i) = \left(\frac{1}{\sigma \sqrt{2\pi}} \right)^N \left(\prod_{i=1}^N \frac{1}{x_i} \right) e^{-\frac{1}{2\sigma^2} \sum_{i=1}^N (\ln(x_i) - \mu)^2} \quad (5.37)$$

The objective is to find the optimum of this function for μ and σ . The derivative of the MLE according to both those variables is done to find $\hat{\mu}$ and $\hat{\sigma}^2$. Equations (5.38) and (5.39) develop the mathematical expressions of the two derivatives.

$$\frac{\partial L}{\partial \mu} = 0 \iff \frac{2 \sum_{i=1}^N (\ln(x_i) - \mu)}{2\sigma^2} = 0 \quad (5.38)$$

$$\begin{aligned} \frac{\partial L}{\partial \sigma} = 0 \iff & N \left(\frac{1}{\sigma \sqrt{2\pi}} \right)^{N-1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^N (\ln(x_i) - \mu)^2} + \\ & \left(\frac{1}{\sigma \sqrt{2\pi}} \right)^N e^{-\frac{1}{2\sigma^2} \sum_{i=1}^N (\ln(x_i) - \mu)^2} - \frac{1}{2} \sum_{i=1}^N (\ln(x_i) - \mu)^2 \frac{-2}{\sigma^3} = 0 \end{aligned} \quad (5.39)$$

By simplifying the equations and isolating the parameters of interest, the two estimators are found in equation (5.40). Note that they are equivalent to the definitions of the mean and variance for a normal distribution with the only difference that the logarithm of the values is taken [59].

$$\begin{cases} \hat{\mu} = \frac{\sum_{i=1}^N \ln(X_i)}{N} \\ \hat{\sigma}^2 = \frac{\sum_{i=1}^N (\ln(X_i) - \hat{\mu})^2}{N} \end{cases} \quad (5.40)$$

At this point, a log-normal distribution can be plotted using the estimators $\hat{\mu}$ and $\hat{\sigma}^2$ computed based on the 600 simulations. Afterwards, the created distribution can be superimposed onto the probability density function (PDF) of the data. The goal is to compare both plots to confirm if the log-normal distribution is appropriate to represent the distribution of the received intensity at the central pixel of the output.

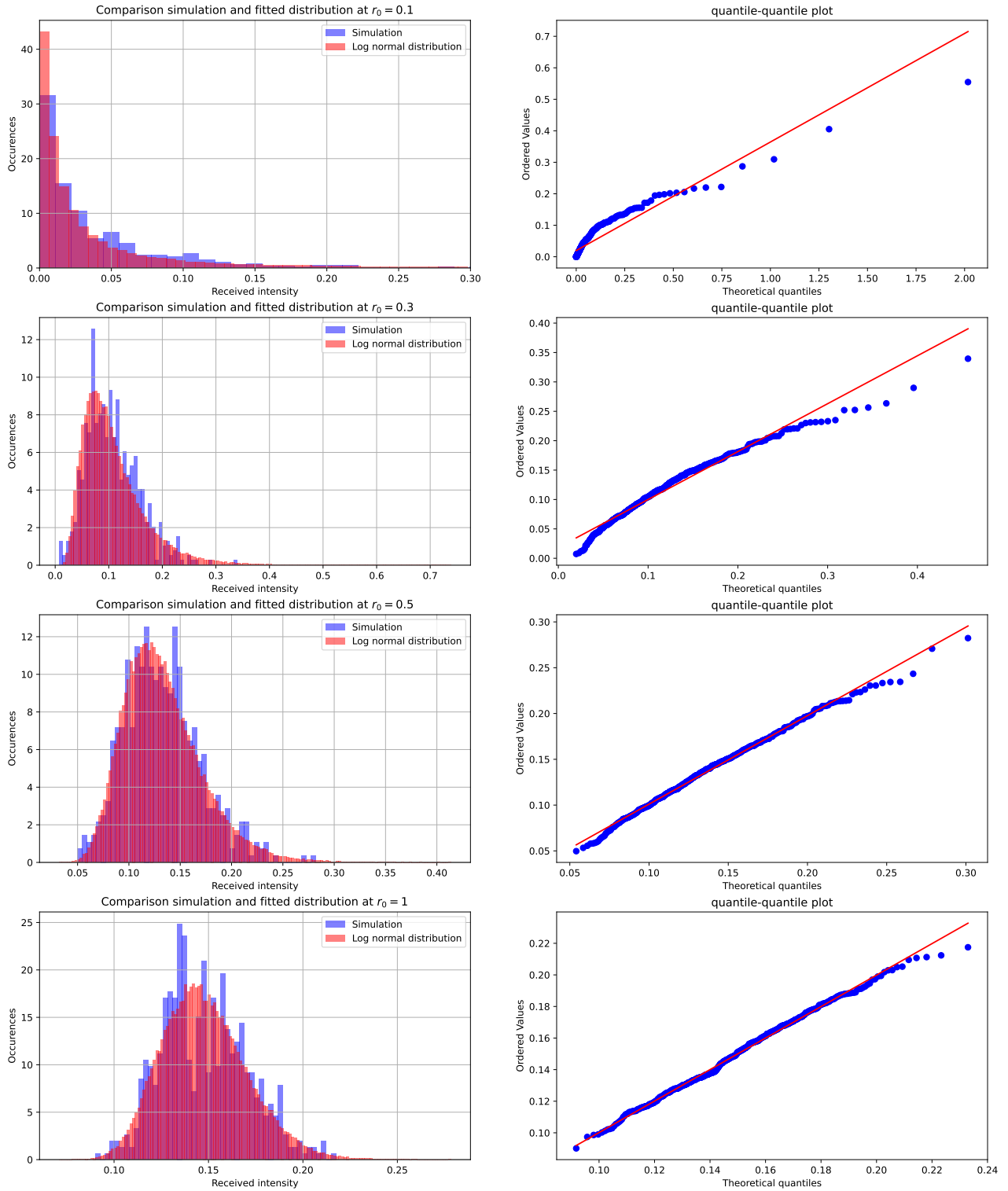


Figure 5.13: On the left : comparison between the simulated (blue) and fitted using MLE (red) distributions of the central pixel received intensity. On the right : quantile-quantile plots (qqplots) of the simulated data.

To allow a comparison, quantile-quantile plots (QQplots) are generated. The principle is to draw a red line representing the distribution of the quantile and then to place the data. If the log-normal distribution fits well the PDF of the data, then the points would be on the red line. Otherwise, this distribution is not appropriate. Figure 5.13 shows on the left the superposition of the PDF of the dataset (blue) and the PDF generated with the estimators (red) and on the right the QQplot. Different degrees of turbulence are analysed in order to confirm the intuition developed previously.

The turbulences analysed are for r_0 of 0.1 m, 0.3 m, 0.5 m and 1 m which are the values analysed in Table 5.2 and on Figure 5.12. As predicted previously, for a strong turbulence, the log-normal distribution is not adequate. Indeed, when looking at the QQplot of $r_0 = 0.1$ m, the blue dots do not fit well on the red line. The more the turbulence decreases, the more adequate the fitting becomes. This statement corresponds to the results observed by the moment matching procedure visible on Figure 5.12 in section 5.3.

The main elements to keep from this section are the presence of two turbulence regimes : weak and strong. The strong regime seems to follow a K distribution but is not studied in details because real profiles stay in the weak region. This region follows a log-normal distribution.

5.4 Example

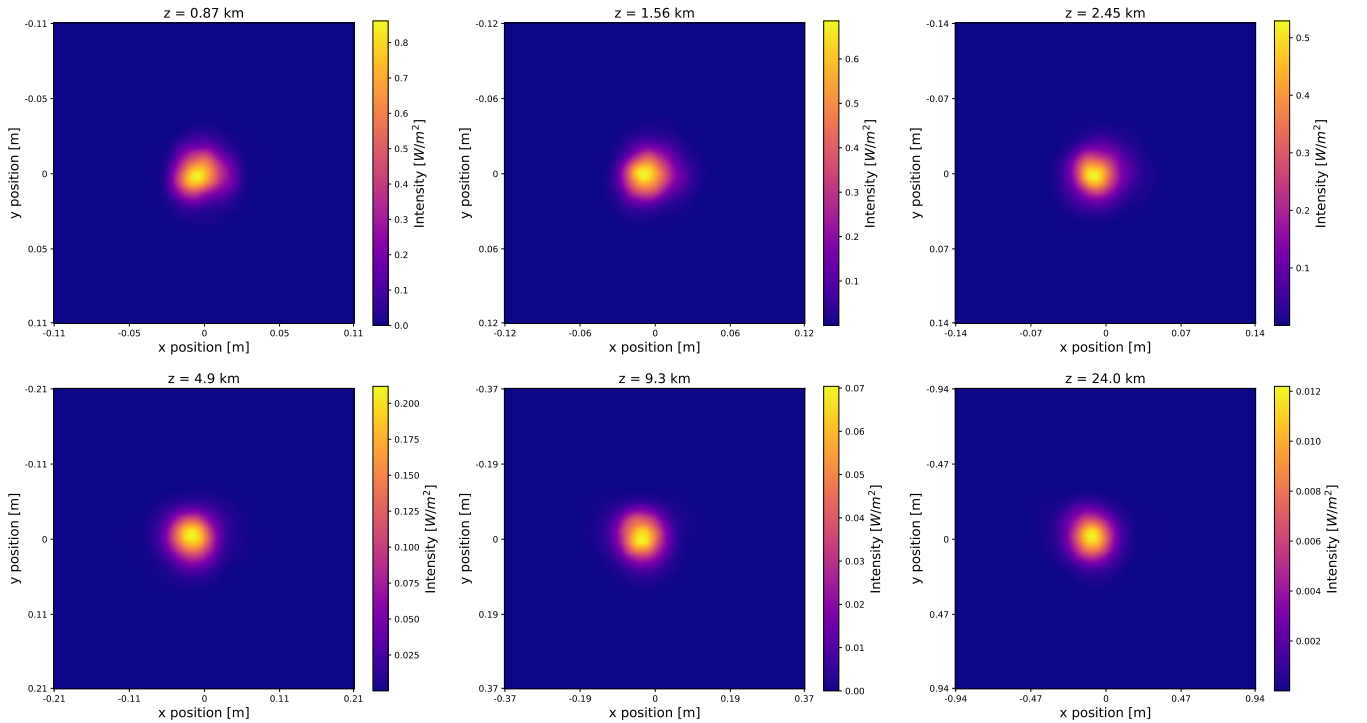


Figure 5.14: Evolution of the intensity of an optical field propagated over 24 km using $n = 14$ PS. The input plane parameters are : $N = 256$ pixels, $S_0 = 20$ cm, $\text{FWHM}_0 = 2.5$ cm. The turbulence follows a HV profile with $w = 21$ m/s and $A = 1.7 \times 10^{-14} \text{ m}^{-2/3}$.

The entire chain of propagation is now explained and turbulent simulations can be performed by applying the method described in chapter 3. To illustrate, the intensity of a turbulent

propagation is shown on Figure 5.14 for a HV profile with $w = 21$ m/s and $A = 1.7 \times 10^{-14} \text{ m}^{-2/3}$. At low altitude (870 m), the turbulence is strong as shown by the profile in Figure 5.1 resulting in a deformation visible on the top left plot in Figure 5.14. A typical output is displayed at an altitude of 24 km on the bottom right. The Gaussian is slightly shifted from the center thus one can understand the central pixel received intensity fluctuates.

In the next chapter, turbulent propagations at constant r_0 are simulated and several measurements are taken. This chapter is intended to show the impact the degree of turbulence has on the propagated image with the use of metrics.

Chapter 6

Measurements

The previous chapters allowed for building a framework to propagate in a turbulent atmosphere. They explained the theoretical background and stated the problem (chapter 2 and chapter 3). The free propagation to go from a phase screen to the next one is explained in chapter 4 and the generation of accurate PS is developed in chapter 5.

With working simulations, a set of measurements can be studied. In this chapter, simulations with a constant r_0 over a 24 km path are performed. Chapter 5 displayed turbulence profiles thus constant r_0 has no physical meaning but it allows to study multiple regimes of turbulence and compare them. The impact of real profiles is done in the next chapter to analyse both the impact of the wind and the ground turbulence on the measurements.

The first measurement analysed is the probability density function (PDF) of the central pixel output intensity. It was already used in the previous chapter to assess the log-normal distribution fitting. In this chapter, the PDF is used to compute several other measurements. Afterwards, cumulative density functions (CDF) are computed.

Since the PDF is known, mean bit error rate (BER) evolution as a function of the mean signal to noise ratio (SNR) is examined. Optical communication is intended for data transmission so it is interesting to compute telecommunication measurements.

Beam wandering is an important method for analysing random phenomena, it permits the investigation of measurements with similar results independent of the random turbulence applied. With the new method, radius growth is computed and compared to theoretical formulas.

Finally, the scintillation index and Strehl ratio are researched. Those metrics quantify the quality of the optical path. They both use the PDF already discussed.

6.1 PDF and CDF

The method to collect the data for the constitution of the PDF is already explained on Figure 5.10 in section 5.3. With the array of received intensity of the central pixel (I), the PDF is created by counting the number of occurrences received for each interval of intensity. The resulting PDFs after turbulent propagation for different r_0 values are displayed on Figure 6.1.

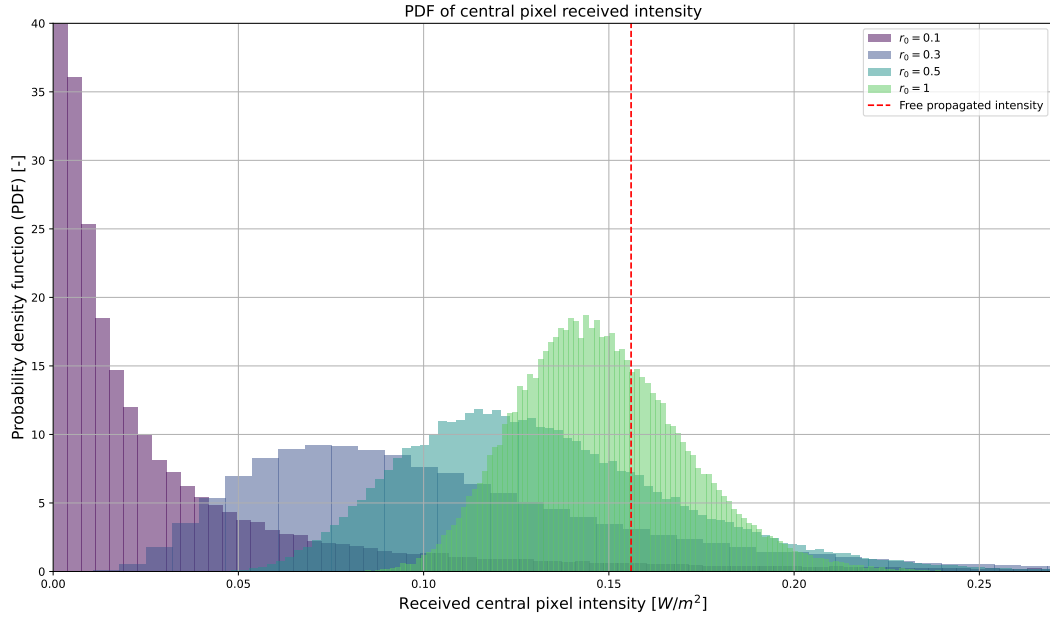


Figure 6.1: Probability density functions (PDF) of the received intensity of the central pixel for different constant turbulence. $r_0 = 0.1$ m is purple, $r_0 = 0.3$ m is dark blue, $r_0 = 0.5$ m is blue and $r_0 = 1$ m is green. The red dotted line indicates the free propagated intensity (I_{free}).

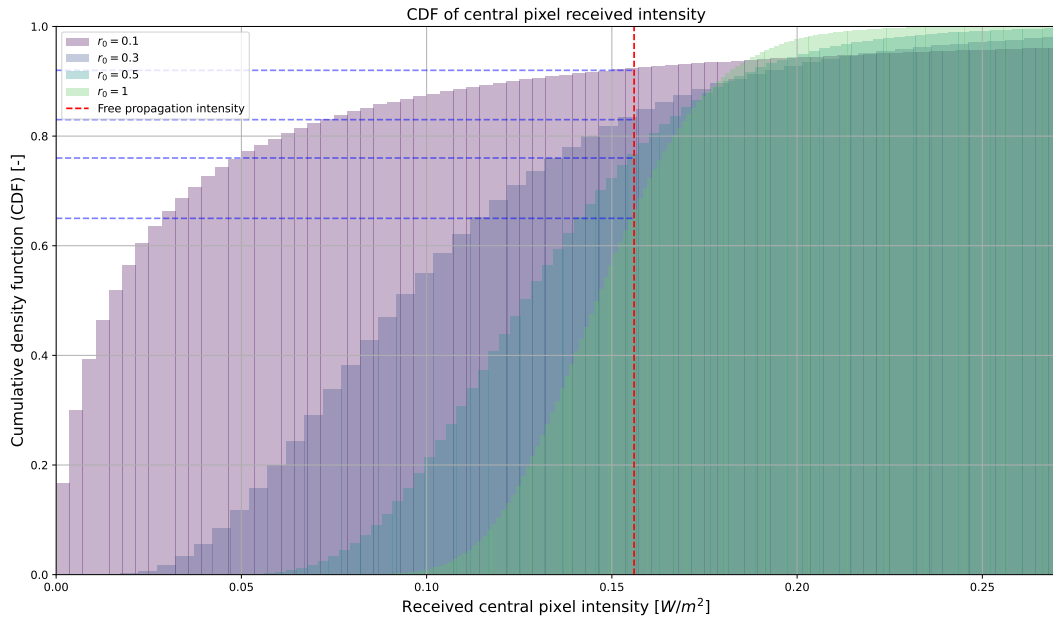


Figure 6.2: Cumulative density functions (CDF) of the received intensity of the central pixel for different constant turbulence. $r_0 = 0.1$ m is purple, $r_0 = 0.3$ m is dark blue, $r_0 = 0.5$ m is blue and $r_0 = 1$ m is green. The red dotted line indicates the free propagated intensity (I_{free}) and the blue dotted lines the value of the CDF for this intensity. The blue dotted lines are the probability of fade.

The received intensities have more chance of getting close to 0 for a large turbulence ($r_0 = 0.1$ m) as represented by the purple histogram. Indeed, large turbulence is more likely to have destructive interference causing smaller received intensities. The lower the turbulence the closer the received intensity is to the free propagated intensity. When considering no atmosphere and using the same propagation parameters, the received intensity value at the output would be 156 mW/m^2 . This value is only due to diffraction phenomenon. The PDF for $r_0 = 1$ m presents many received intensities close to the free propagate value.

The cumulative density function (CDF) can be computed using the PDF. This measurement is also called the probability of fade as it gives the percentage of intensities lower than a threshold intensity (I_T). Equation (6.1) is a formalisation of the definition [40].

$$P(I \leq I_T) = \int_0^{I_T} p_I(I) dI \quad (6.1)$$

For example, considering the free propagation received intensity (156 mW/m^2) as threshold for $I_T =$ (red dotted line on Figures 6.1 and 6.2) gives the following probability of fade : $P_{r_0=0.1}(I \leq 0.156) = 92\%$, $P_{r_0=0.3}(I \leq 0.156) = 83\%$, $P_{r_0=0.5}(I \leq 0.156) = 76\%$ and $P_{r_0=1}(I \leq 0.156) = 65\%$. These results are visible on the y-axis of Figure 6.2 for each r_0 values (blue dotted lines). It represents the probability to receive an intensity lower than in the free propagate case. It is a good way to show that the large turbulence has a worse impact on the transmission of information.

6.2 BER - SNR

The bit error rate (BER) is a measurement of the quality of the optical communication. A small BER indicates few errors while a high BER is a sign of bad transmission. For example a BER of 10^{-3} means that in average, an error occurs every 1000 sent bits.

The signal to noise ratio (SNR) is a measure of the signal power compared to the noise in the background. It is usually evaluated in the decibel scale (logarithmic) with the following transformation. A signal 100 times more powerful than the noise would result in a SNR of 20 dB according to equation (6.2).

$$\text{SNR}_{dB} = 10 \log_{10}(\text{SNR}) \quad (6.2)$$

The BER and the SNR can be related using equation (6.3) [40]. The formula expresses the mean BER as a function of the mean SNR (in real values), by integrating over all the received intensities. Therefore, the computation uses the received intensity array I .

$$\langle \text{BER} \rangle = \frac{1}{2} \int_0^\infty p_I(I) \text{erfc} \left(\frac{\langle \text{SNR} \rangle I}{2\sqrt{2}\langle I \rangle} \right) dI \quad (6.3)$$

To avoid dealing with an integral where the increments dI are not constant, the BER can be evaluated through an expectation of a certain function f depending on the mean SNR and the array of received intensities (I) as shown by equation (6.4). The function returns the mean of the function f evaluated for all values of I at a certain mean SNR. The procedure is then repeated for all SNR possible to obtain one curve on the graph in Figure 6.3.

$$\langle \text{BER} \rangle = \mathbb{E} \left[f(I, \langle \text{SNR} \rangle) \right] \quad \text{with } f(I, \langle \text{SNR} \rangle) = \frac{1}{2} \text{erfc} \left(\frac{\langle \text{SNR} \rangle I}{2\sqrt{2}\langle I \rangle} \right) \quad (6.4)$$

Plotting the formula in equation (6.4) for different constant values of r_0 gives Figure 6.3. The mean signal to noise ratio ($\langle \text{SNR} \rangle$) considered ranges from 1 to 100 thus $\langle \text{SNR}_{dB} \rangle$ from 0 to 20.

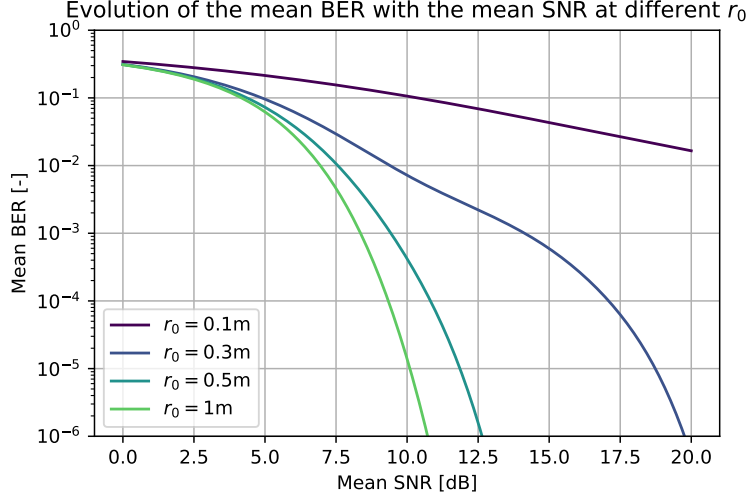


Figure 6.3: Mean bit error rate ($\langle \text{BER} \rangle$) as a function of the mean signal to noise ratio ($\langle \text{SNR} \rangle$) for different constant turbulence $r_0 = 0.1 \text{ m}$, 0.3 m , 0.5 m and 1 m .

The high turbulence regime causes lots of error even when the $\langle \text{SNR} \rangle$ reaches 20 dB as represented by the purple curve. For this $\langle \text{SNR} \rangle$, in average, more than 1 error is committed every 100 sent bits at a constant turbulence of $r_0 = 0.1 \text{ m}$. The reduction of turbulence implies a lower $\langle \text{SNR} \rangle$ to achieve identical $\langle \text{BER} \rangle$. For example, to achieve a $\langle \text{BER} \rangle$ of 10^{-3} , the required $\langle \text{SNR} \rangle$ for turbulence of $r_0 = 0.3 \text{ m}$, $r_0 = 0.5 \text{ m}$ and $r_0 = 1 \text{ m}$ are respectively 14.15 dB, 9.46 dB and 8.42 dB.

Another point of view is to fix the $\langle \text{SNR} \rangle$ at a given value and to look at the corresponding $\langle \text{BER} \rangle$. Table 6.1 displays the bit error rate for the four considered turbulences from strong to weak at four mean SNR values (5 dB, 10 dB, 15 dB and 20 dB). Going from left to right and from top to bottom, the mean BER decreases. For example, at a $\langle \text{SNR} \rangle$ of 20 dB and a weak turbulence of $r_0 = 1 \text{ m}$, there is almost no error ($\langle \text{BER} \rangle \approx 10^{-207}$).

	$\langle \text{BER} \rangle$			
	$\langle \text{SNR} \rangle = 5 \text{ dB}$	$\langle \text{SNR} \rangle = 10 \text{ dB}$	$\langle \text{SNR} \rangle = 15 \text{ dB}$	$\langle \text{SNR} \rangle = 20 \text{ dB}$
$r_0 = 0.1 \text{ m}$	2.14×10^{-1}	1.06×10^{-1}	4.3×10^{-2}	1.65×10^{-2}
$r_0 = 0.3 \text{ m}$	9.49×10^{-2}	7.17×10^{-3}	5.88×10^{-4}	5.122×10^{-7}
$r_0 = 0.5 \text{ m}$	7.18×10^{-2}	4.15×10^{-4}	2.51×10^{-12}	2.045×10^{-81}
$r_0 = 1 \text{ m}$	6.1×10^{-2}	1.358×10^{-5}	3.561×10^{-25}	5.913×10^{-207}

Table 6.1: Mean bit error rate ($\langle \text{BER} \rangle$) for 4 mean signal to noise ratio ($\langle \text{SNR} \rangle$) values (5 dB, 10 dB, 15 dB and 20 dB) and at 4 turbulence levels (0.1 m, 0.3 m, 0.5 m and 1 m).

6.3 Beam wandering

This section investigates a method to allow computation of the radius growth caused by the turbulence. This growth is not always the same when simulating several times thus the results are not conclusive. Indeed, the turbulence is random due to the noise coloring procedure (see section 5.2.2). To delete the randomness induced by the turbulence while keeping its impact on the growth of the radius, beam wandering is performed.

The fluctuations in the output image are caused by the turbulence applied during the propagation. The randomness of the PS causes the center of the image to move. Moreover, the radius computed at the output image is not constant between each simulation. Figure 6.4 shows the circle computed for one output on the left and then the beam wandering on the right. One can understand the necessity of performing long exposure (LE) in order to collect enough outputs and delete the randomness caused by the turbulence. Figure 6.4 really show why the radius increase with the turbulence. The output can be shifted from the center and spread more than in the free propagation case. With beam wandering average are performed and the more outputs are available, the closest the center will be to (0,0) and the radius value will tend to a value called w_{LE} , the long exposure intensity radius [60].

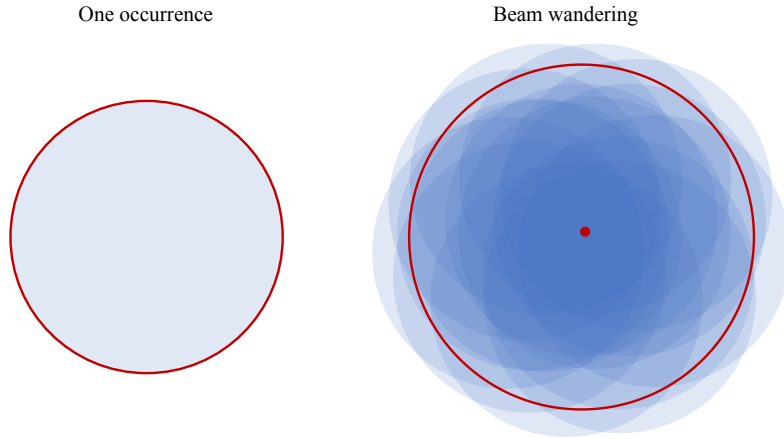


Figure 6.4: Difference between the radius computed with (left) and without (right) beam wandering.

The idea of beam wandering is to study parameters such as the growth of the radius without the random factor causing variation in the measurements. To do so, beam wandering takes multiple output images and averages them together. This method is also called long exposure (LE). By taking multiples outputs, the final resulting image gets closer and closer to the sent image.

As an example, Figure 6.5 shows a Gaussian beam being transmitted with a really strong constant turbulence of $r_0 = 0.01$ m. Such a turbulence never occurs over the entire propagation but is really helpful to illustrate the importance of beam wandering. The propagation distance is $z = 1.2$ km with the following input parameters : $S_0 = 20$ cm, $N = 256$ pixels and $\text{FWHM}_0 = 2.5$ cm. From left to right, the number of outputs taken for the long exposure increases. A strong turbulence is applied thus with only $n_{LE} = 50$, the average image is still close to a speckle. As the number n_{LE} increases, the average output look similar to the initial output, a Gaussian beam.

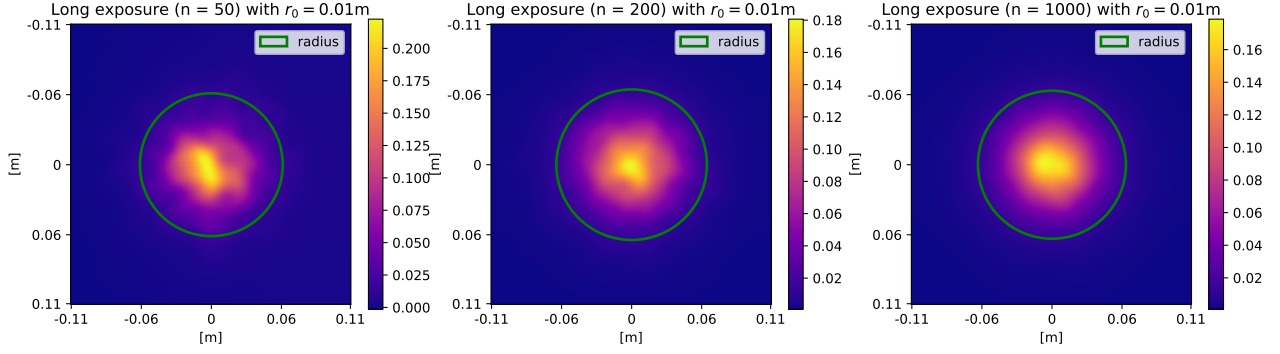


Figure 6.5: Impact of the number of outputs (n_{LE}) taken for the long exposure (LE) of an image with parameters $S_0 = 20$ cm, $N = 256$ pixels and $\text{FWHM}_0 = 2.5$ cm propagated at a constant turbulence $r_0 = 0.01$ m over $z = 1.2$ km.

With the theory of long exposure output, growth of radius can be computed. Indeed, after LE, all the irregularities are attenuated and a circular shape appears as shown on the right in Figure 6.4.

6.4 Evolution of the radius

6.4.1 Radius computation

Radius computation is possible thanks to beam wandering. Using the long exposure image at the receiver, the intensity radius w_i for each layer i is computed. To do so, the first step is to find the center of the circle. Indeed, turbulence can cause movement of the propagated image as shown on Figure 6.5. The center $\boldsymbol{\mu} = (\mu_x, \mu_y)$ is given by equation (6.5) which sums all the distance vectors $\boldsymbol{r} = (x, y)$ and weights them using a function denoted $\hat{f}(\boldsymbol{r})$ [61].

$$\boldsymbol{\mu}(\boldsymbol{r}) = \sum_{\boldsymbol{r}} \boldsymbol{r} \hat{f}(\boldsymbol{r}) \quad (6.5)$$

The function $\hat{f}(\boldsymbol{r})$ returns the pixel value at the location $\boldsymbol{r}$ normalised by the sum of all the image pixels. With the definition of $\hat{f}(\boldsymbol{r})$ given in equation (6.6), the result obtained for $\boldsymbol{\mu}$ will not need normalisation.

$$\hat{f}(\boldsymbol{r}) = \frac{f(\boldsymbol{r})}{\sum_{\boldsymbol{r}} f(\boldsymbol{r})} \quad (6.6)$$

With the knowledge of the center, the radius is the last unknown needed to draw the circle where the intensity has decreased by a factor $1/e^2$. In section 3.2, equation (3.9) demonstrated the link between the intensity radius and the variance. The same method is applied here to find w_i using σ_i^2 .

The variance is obtained by performing the sum in equation (6.7). Each position vector is shifted to match the new center and the function $\hat{f}$ is once again used to weight the contribution of each pixel. In the classical definition of the variance, there is no factor $\frac{1}{2}$ but because a 2 dimensional image is considered, this factor must be taken into account [61].

$$\begin{aligned}
 \sigma^2 &= \frac{1}{2} \sum_{\mathbf{r}} \|\mathbf{r} - \boldsymbol{\mu}(\mathbf{r})\|^2 \hat{f}(\mathbf{r}) \\
 &= \frac{1}{2} \sum_x \sum_y ((x - \mu_x)^2 + (y - \mu_y)^2) \hat{f}(x, y)
 \end{aligned} \tag{6.7}$$

The procedure is applied on the input amplitude and intensity of the optical field (see Figure 6.6). It allows for a sanity check as well as a representation of the radius. Now the radius growth can be studied for different turbulences.

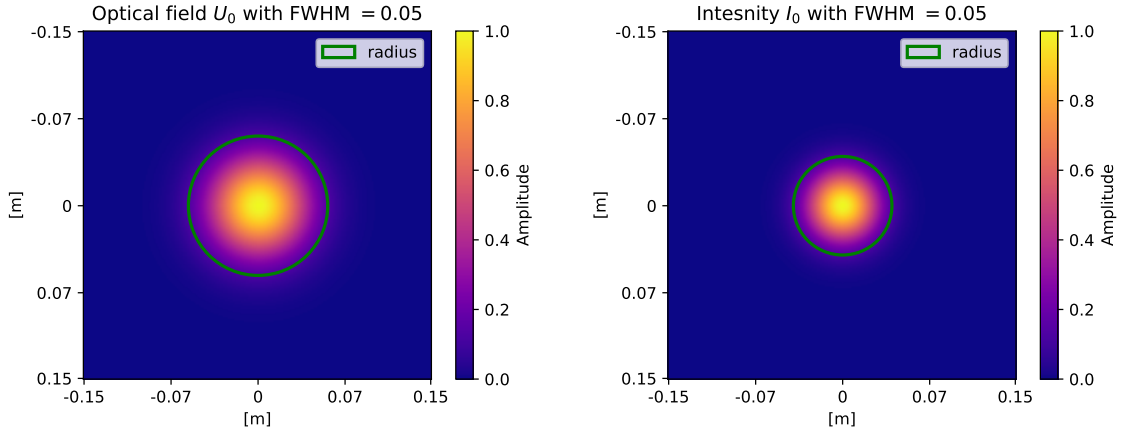


Figure 6.6: Representation of the input amplitude (left) and intensity (right) with a FWHM of 5 cm. The green circles show the intensity radius.

6.4.2 Radius after beam wandering

The previous section introduced the computation procedure to obtain the center and the radius of a 2 dimensional image. With this theory in mind, the growth of the radius for propagation at a constant turbulence is studied and compared for different turbulences.

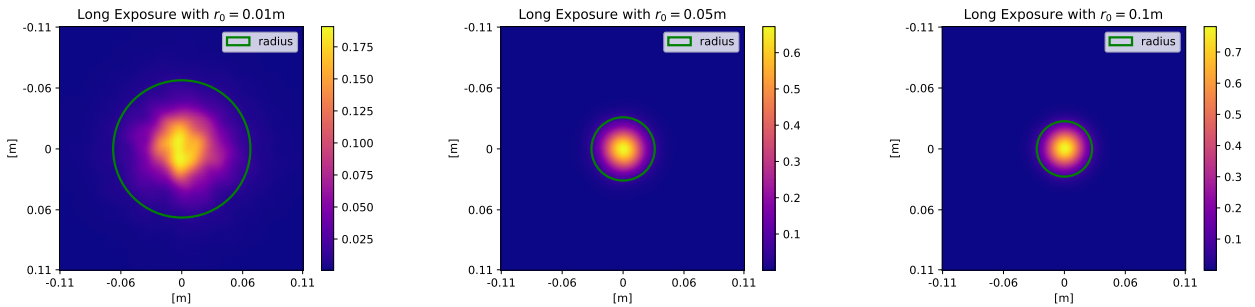


Figure 6.7: Long exposure radius for three level of turbulence ($r_0 = 0.01$ m, 0.05 m and 0.1 m) using $n_{LE} = 50$. The green circle indicates the radius.

Figure 6.7 gives a first intuition about the radius growth by displaying three turbulence levels ($r_0 = 0.01$ m, 0.05 m and 0.1 m) with their associated radius. The radius for turbulence $r_0 = 0.05$ m and 0.1 m are quite similar whereas the radius for $r_0 = 0.01$ m is much larger. The

insight is a non linear evolution of long exposure radius (w_{LE}) with the turbulence. To confirm this, a study is conducted to obtain a graph of the w_{LE} as a function of the turbulence r_0 . One objective is also to compare the obtained curve with theoretical evolution and therefore, an analytical formula approaching w_{LE} is developed.

The theoretical formula for the long exposure intensity radius ($w_{LE,i}$) at a distance z_i is given in equation (6.8) [15]. It multiplies the free propagated radius at distance z_i already described in section 3.3.1 by a square root factor. This factor contains the variables σ_R , Λ , Q_m and Q_0 which are defined in equation (6.8). They depend on the slab parameters (Δh_i , w_i and $C_{n,i}^2$).

$$w_{LE,i} \cong w_i \sqrt{1 + 3.54\sigma_R^2 \Lambda^{5/6} \left[\frac{(1 + 0.31\Lambda Q_m)^{5/6}}{(\Lambda Q_m)^{5/6}} - 0.36(\Lambda Q_0)^{1/6} \right]}$$

$$\begin{cases} \sigma_R = 1.23C_{n,i}^2 k^{7/6} \Delta h_i^{11/6} \\ \Lambda = \frac{2\Delta h_i}{kw_i^2} \\ Q_m = \frac{\Delta h_i \kappa_m^2}{k} \\ Q_0 = \frac{\Delta h_i \kappa_0^2}{k} \end{cases} \quad (6.8)$$

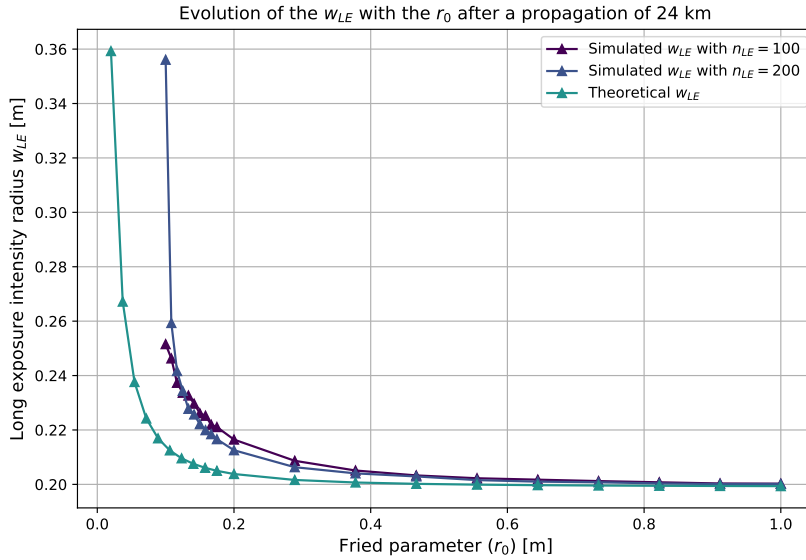


Figure 6.8: Theoretical and simulated evolutions of the LE intensity radius for different r_0 and for $n_{LE} = 100$ and 200 . The propagation is done on 24km with the following input parameters : $S_0 = 20$ cm, $N = 256$ pixels and $FWHM_0 = 2.5$ cm. The turbulence parameters are $L_0 = 100$ m and $l_0 = 1$ mm.

Figure 6.8 shows the theoretical formula and two simulations using n_{LE} of 100 iterations and 200 iterations after a propagation of 24km. At low turbulence, the two simulations align with the theoretical curve. As the turbulence increases, the curves separate. The difference observed

between the two simulations around $r_0 = 0.1$ m is due to the inaccuracy of not having enough iterations for the beam wandering. On the other hand, the theoretical curve does not match the simulations when the turbulence is too strong but still follows the same dynamic. A reason for this disparity is the regime where equation (6.8) is valid. Indeed, the sign $\cong$ is due to approximation of hypergeometric functions [15].

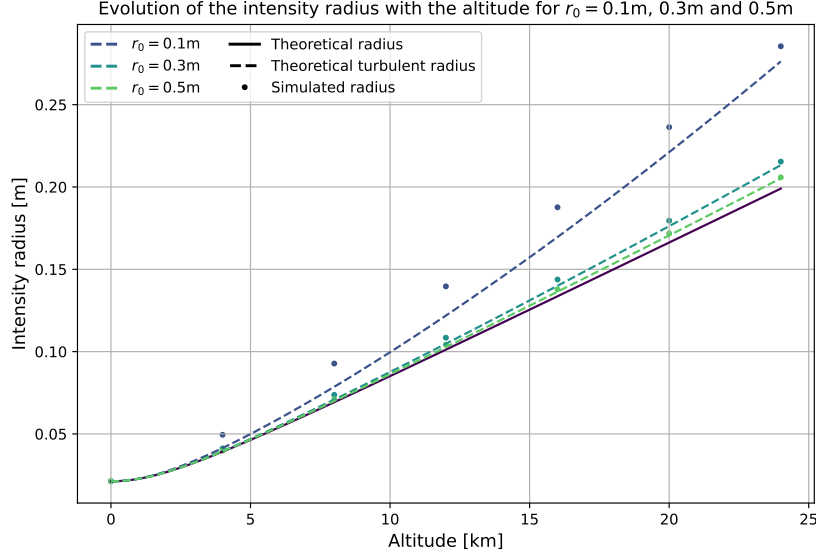


Figure 6.9: Growth of the radius for different levels of constant turbulence ($r_0 = 0.1$ m, 0.3 m and 0.5 m). The full line and dotted lines respectively represent the theoretical free propagated and turbulent radius. The dots are the simulated values with *Python* for the different turbulences.

The radius at the end of the propagation has been analysed but another point of view is to visualise the growth while it propagates at different altitudes. Once again, different r_0 can be compared to the theoretical formula in equation (6.8). Figure 6.9 shows the intensity radius evolution for the free propagation case (purple line). It also represents with the dotted lines the theoretical radius evolution at three constant turbulence levels ($r_0 = 0.5$ m, 0.3 m and 0.1 m). Finally the simulations results for the same three turbulent levels are represented by the points. The non-linear behaviour is once again observable.

6.5 Scintillation index

The scintillation index is a metric to measure the quality of the communication link. It was already explained in section 5.3 when talking about second order moment. Indeed, as mentioned by definition (6.9), the scintillation index (σ_I^2) is the second order moment minus 1 [15].

To measure the σ_I^2 via the simulations, one must use the received intensity array (I). It is the same as for the $\langle \text{BER} \rangle - \langle \text{SNR} \rangle$ curves and obtained according to the procedure explained in section 5.3.

A scintillation index close to 0 implies a weak turbulence on the optical path as $\langle I^2 \rangle \approx \langle I \rangle^2$. When those two quantities are close, it symbolises the uniformity of the received intensity as

the impact of the mean disappears. On the other hand, when the turbulence is strong, the σ_I^2 gets larger.

$$\sigma_I^2 = \frac{\langle I^2 \rangle - \langle I \rangle^2}{\langle I \rangle^2} = \frac{\langle I^2 \rangle}{\langle I \rangle^2} - 1 \quad (6.9)$$

An analytical formula for the scintillation index exists and is shown in equation (6.10) [62]. It is a variable of the turbulence considered and the propagated distance. To validate the simulations and to see the evolution of σ_I^2 with r_0 , Figure 6.10 is plotted.

$$\sigma_I^2 = 1.229 C_n^2 k^2 z_i \left(\frac{z_i}{k} \right)^{5/6} \quad (6.10)$$

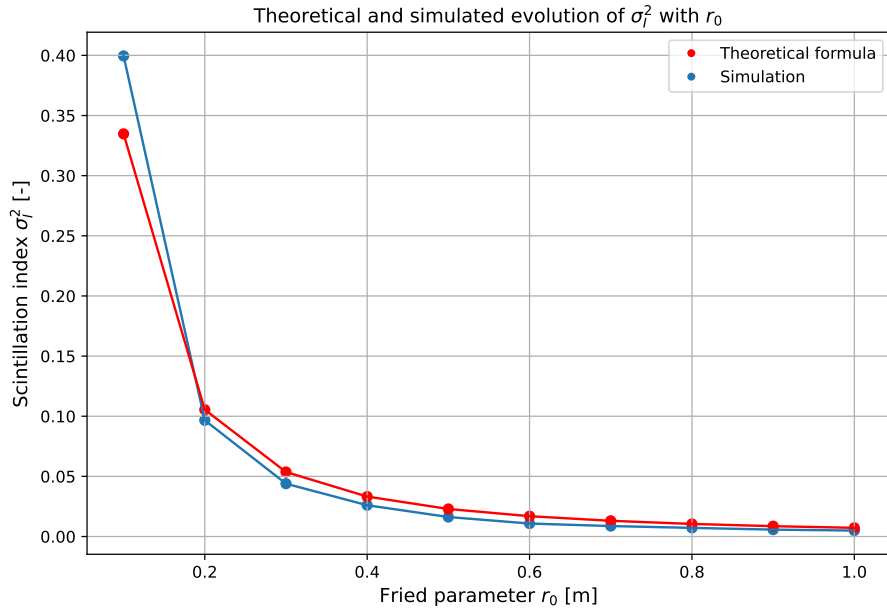


Figure 6.10: Evolution of the scintillation index (σ_I^2) according to the Fried parameter r_0 . The theoretical behaviour is in red and the simulated in blue.

Figure 6.10 attests the theoretical and simulated curves match for the range of r_0 considered. The main observation is the small separation and then the crossing of the two curves when the turbulence increases.

6.6 Strehl ratio

The Strehl ratio (SR) is a normalisation of the mean received intensity of the central pixel by the intensity obtained in the case of free propagation like equation (6.11) shows [15]. Once again, the procedure to obtain the vector I is the same as illustrated in section 5.3. For the free propagation intensity (I_{free}), one propagation is performed with the same parameters as for the simulations but without turbulence. The intensity attenuation is thus only linked to the diffraction.

$$SR = \frac{\langle I \rangle}{I_{\text{free}}} \quad (6.11)$$

The value of the Strehl ratio is between 0 and 1 where 0 is synonym of a very bad transmission and 1 is synonym of a perfect transmission. Looking at the formula, the perfect case occurs when in average the received intensity is equal to the free propagation received intensity.

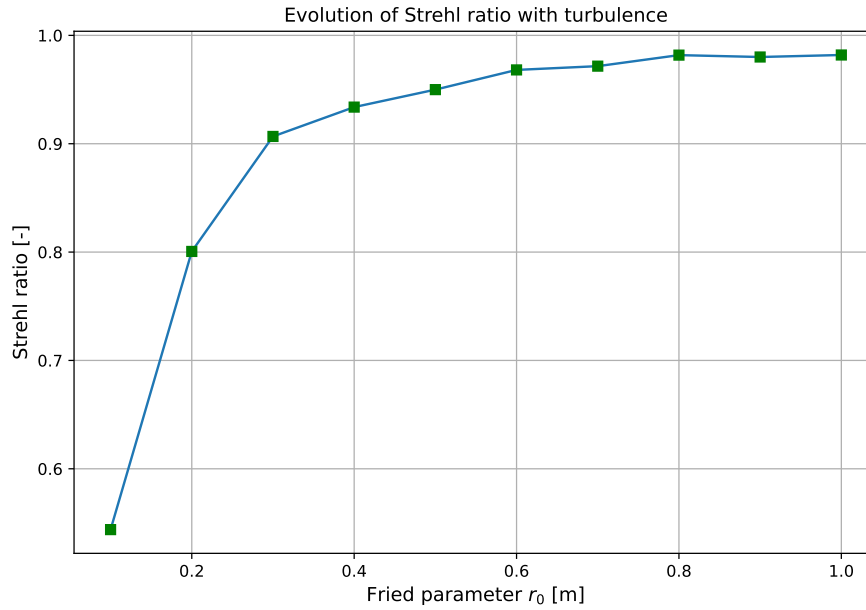


Figure 6.11: Simulated evolution of the Strehl ratio with the Fried parameter. The SR is computed for r_0 between 0.1 m and 1 m.

Figure 6.11 shows the Strehl ratio as a function of the turbulence. For constant Fried parameter above 0.3 m, the Strehl ratio stays above 90% and the transmission does not present too many errors. When the turbulence increases higher than $r_0 = 0.3$ m, the Strehl ratio drops fast to attain around 55% at a $r_0 = 0.1$ m. For r_0 higher than 1, the SR would tend to 1. This evolution points out the change in behaviour and shows the presence of two regimes.

Chapter 7

Impact of profile

This chapter focus on the impact of the wind velocity and ground turbulence on the turbulent profile of Hufnagel-Valley 5/7. It is a real profile and not a constant r_0 profile as previously developed. The purpose is to evaluate the influence of these 2 factors on the studied metrics.

The chapter starts with the C_n^2 evolution with the altitude in the different analysed cases. The impact on the measurements is then studied. The PDF and CDF are shown. The BER and SNR are also plotted followed by the radius evolution. Finally, the scintillation index and Strehl ratio are computed. All the procedure to compute the metrics are the same as in chapter 6.

7.1 The new profiles

The turbulent profiles are Hufnagel-Valley (HV) profiles already presented in section 5.1 when describing the generation of PS. On Figure 7.1, two plots are presented. On the left, three wind speeds (w) of 20 m/s, 50 m/s and 100 m/s and on the right three ground turbulences (A) of $1.7 \times 10^{-13} \text{ m}^{-2/3}$, $1.7 \times 10^{-14} \text{ m}^{-2/3}$ and $1.7 \times 10^{-17} \text{ m}^{-2/3}$.

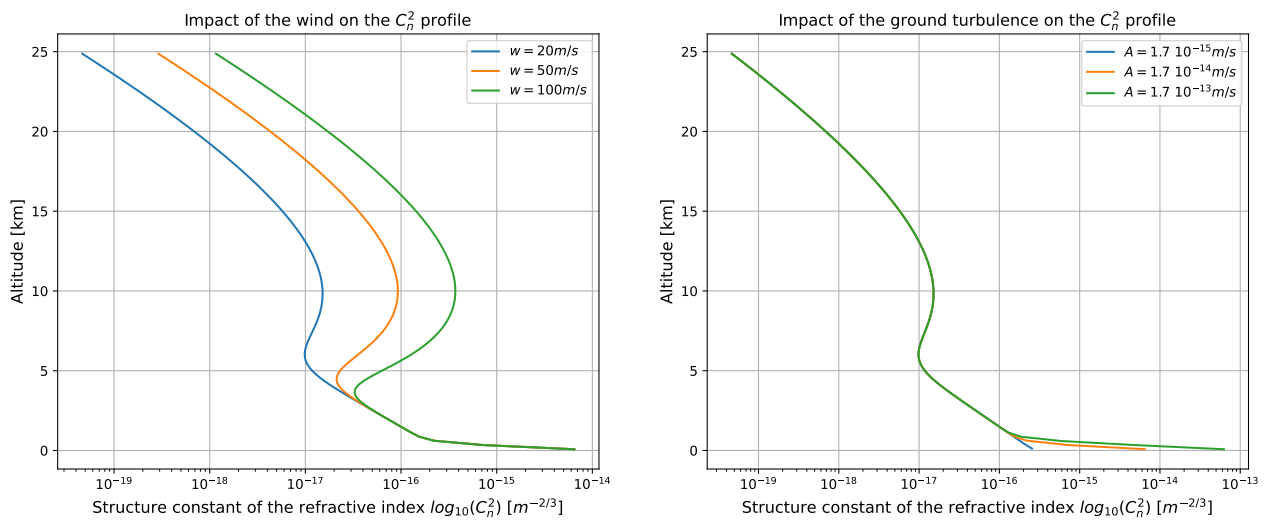


Figure 7.1: 6 turbulence profiles analysed. The left graph is for varying wind velocities of $w = 20 \text{ m/s}$, $w = 50 \text{ m/s}$ and $w = 100 \text{ m/s}$. The right graph is for different ground turbulences of $A = 1.7 \times 10^{-13} \text{ m}^{-2/3}$, $A = 1.7 \times 10^{-14} \text{ m}^{-2/3}$ and $A = 1.7 \times 10^{-17} \text{ m}^{-2/3}$.

The effect of the wind on the HV profile starts at an altitude of around 5 km and grows until 10 km. Then the spacing between the 3 profiles stays roughly constant. The ground turbulence impact is visible only near the surface but the variation of C_n^2 is important. The first PS generated will therefore induce much more turbulence in the presence of strong A . The HV profile at higher altitude is independent of A .

7.2 Impact on the measurements

In this section, the impacts of the wind and the ground turbulence on the measurement are analysed. To guarantee a change in behaviour, extreme values for the wind and ground turbulence are considered. For example, the strong wind is set to $100 \text{ m/s} = 360 \text{ km/h}$.

7.2.1 PDF and CDF

The impact of the wind (w) and the ground turbulence (A) on the probability density function (PDF) and the cumulative density function (CDF) is a modification of the histograms (for the PDF) and the curves (for the CDF).

For the PDF, Figure 7.2 shows three cases (blue = low turbulence, orange = medium turbulence and green = strong turbulence) for both parameters (left = wind and right = ground turbulence). The change in the PDF is a logical phenomenon. An increase of the turbulence perturbs more the optical propagated field and result in a spreading or shifting in the received intensity.

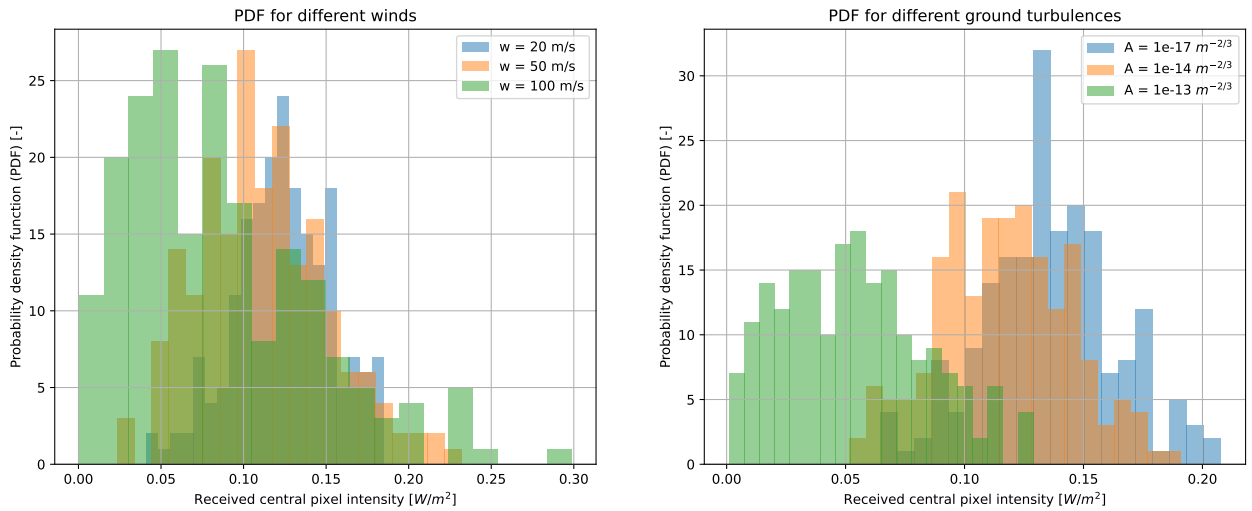


Figure 7.2: Probability density function of the 6 turbulence profiles. The left graph is for varying wind velocities of $w = 20 \text{ m/s}$, $w = 50 \text{ m/s}$ and $w = 100 \text{ m/s}$. The right graph is for different ground turbulences of $A = 1.7 \times 10^{-13} \text{ m}^{-2/3}$, $A = 1.7 \times 10^{-14} \text{ m}^{-2/3}$ and $A = 1.7 \times 10^{-17} \text{ m}^{-2/3}$.

The PDF does not vary in the same manner for the two parameters. For the wind (left), the histograms are spread over a larger set of received intensities as the wind increases whereas for the ground turbulence, the histograms are shifted towards the lower received intensities. Indeed, based on simulations, the received intensities oscillate between 0 W/m^2 and 0.3 W/m^2

for the wind and between 0 W/m^2 and 0.13 W/m^2 for the ground turbulence. For the ground turbulence (right), the support for the received intensities stays constant ($\approx 0.15 \text{ W/m}^2$) but is shifted towards the lower intensities as the turbulence increases.

The observation can also be made through Figure 7.3 plotting the CDF of the above PDF.

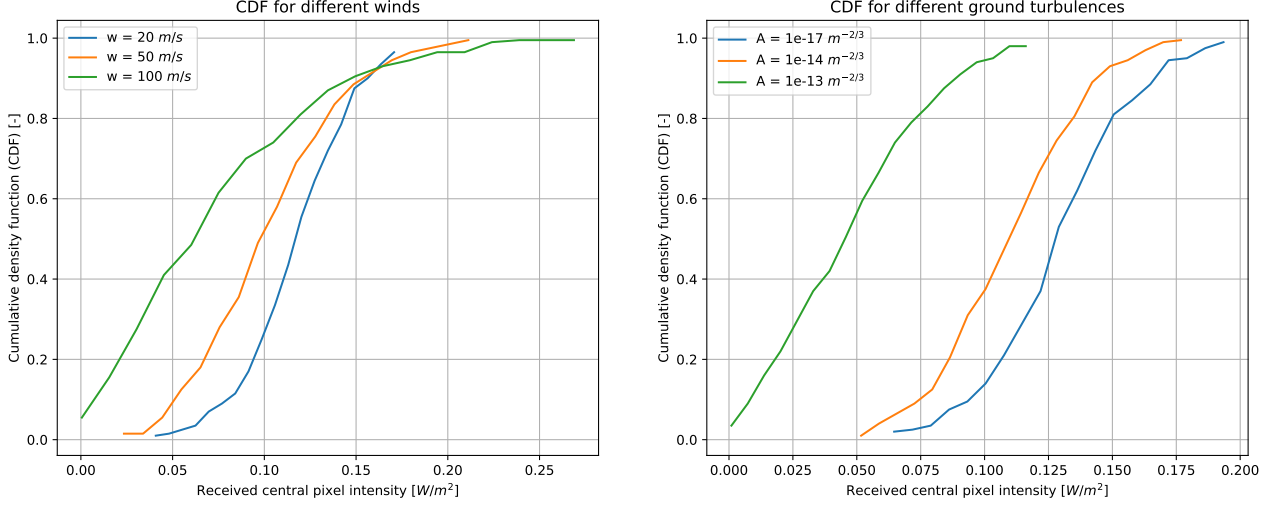


Figure 7.3: Cumulative density function of the 6 turbulence profiles. The left graph is for varying wind velocities of $w = 20 \text{ m/s}$, $w = 50 \text{ m/s}$ and $w = 100 \text{ m/s}$. The right graph is for different ground turbulences of $A = 1.7 \times 10^{-13} \text{ m}^{-2/3}$, $A = 1.7 \times 10^{-14} \text{ m}^{-2/3}$ and $A = 1.7 \times 10^{-17} \text{ m}^{-2/3}$.

Figure 7.3 displays the CDF for the three turbulence levels and for the two parameters. The main observation based on the PDF plots is clearly visible via the CDF plots. The spreading for the various wind velocities as every curves is intersecting and the shift for the ground turbulence as the curves are almost identical but not crossing.

7.2.2 BER - SNR

In chapter 6, the mean SNR influence on the mean BER was studied at constant r_0 . On Figure 7.4, the evolution of the $\langle \text{BER} \rangle$ with $\langle \text{SNR} \rangle$ is drawn. For both parameters (wind and ground turbulence), the strong turbulence (green) exhibits high bit error rate for the signal to noise ratio range considered. This regime will be analysed apart from the two others.

The shape of the blue and orange curves for the two graphs on Figure 7.4 can be explained using the following reasoning based on the CDF :

- The observations made during the study of the CDF for the different winds was the increase of the received intensity support. When the set of reachable intensities gets wider, the number of errors committed increases. Therefore, the $\langle \text{BER} \rangle$ is higher for a same $\langle \text{SNR} \rangle$ and the performances are worse. Indeed, the left graph on Figure 6.3 displays the blue curve lower than the orange.
- For the mean BER evolution with the mean SNR for ground turbulence, the blue and orange curves behave in the same way and are very close to each other. The small visible shift is due to the shift observed in the CDF curves (see right graph on Figure 7.3).

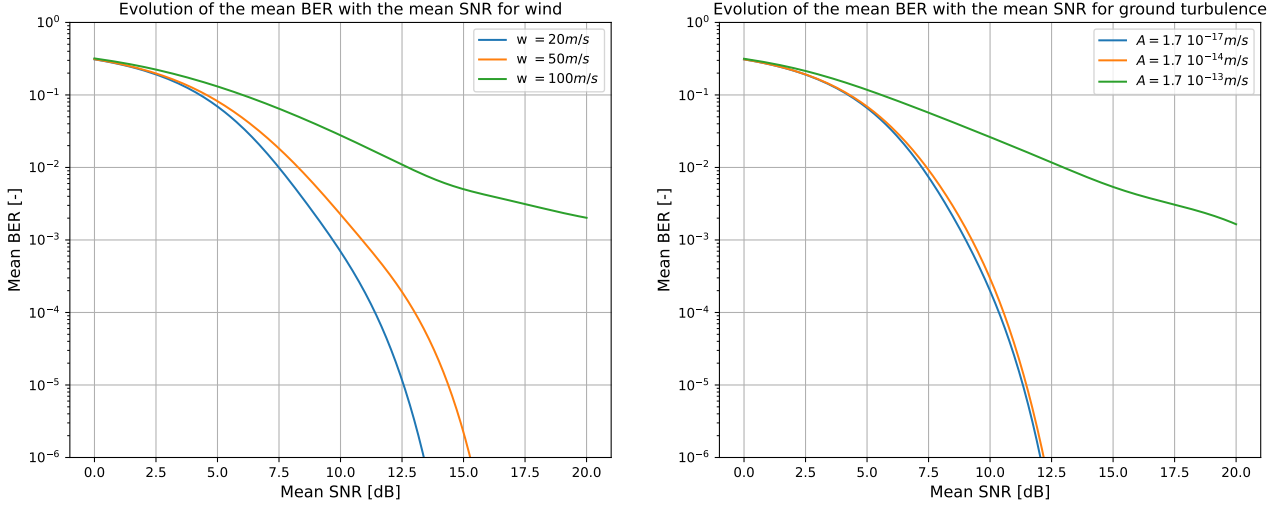


Figure 7.4: Mean bit error rate with respect to the mean signal to noise ratio for the 6 turbulence profiles. The left graph is for varying wind velocities of $w = 20 \text{ m/s}$, $w = 50 \text{ m/s}$ and $w = 100 \text{ m/s}$. The right graph is for different ground turbulences of $A = 1.7 \times 10^{-13} \text{ m}^{-2/3}$, $A = 1.7 \times 10^{-14} \text{ m}^{-2/3}$ and $A = 1.7 \times 10^{-17} \text{ m}^{-2/3}$.

The green curves are in the strong regime and lots of errors appears. Increasing the $\langle \text{SNR} \rangle$ allows for some improvement of the $\langle \text{BER} \rangle$ but for the mean SNR range considered, it is not possible to reach a lower mean BER than 1×10^{-3} .

7.2.3 Radius evolution

The third measurement is the growth of the intensity radius. As previously mentioned in multiple sections, the radius undergoes the phenomenon of diffraction on top of which turbulence is added. Section 6.4 showed that the stronger the turbulence, the larger the radius. In the same section the importance of beam wandering for radius computation was explained.

For the simulation performed here, the number of outputs taken to perform LE is 50 and the input parameters are $S_0 = 20 \text{ cm}$, $N = 256$ pixels and $\text{FWHM}_0 = 2.5 \text{ cm}$. The propagation distances between each PS are the ones in Table 4.1 to avoid aliasing.

Figure 7.5 displays the radius intensity as a function of the altitude for the 6 analysed cases. The markers are located at the altitudes where the radius are computed. As predicted, the turbulence increases the long exposure intensity radius.

For wind speed of 20 m/s , 50 m/s and 100 m/s , the three curves are very similar. The radius increases slightly with the wind velocity as the path is more turbulent. Looking closely, the separation between the three curves occurs between 10 km and 15 km thus after the wind shear altitude (cfr left graph of Figure 7.1).

For the ground turbulence, two regimes are distinguished. The blue and orange curves indicating $A = 1.7 \times 10^{-17} \text{ m}^{-2/3}$ and $1.7 \times 10^{-14} \text{ m}^{-2/3}$ are close while for $A = 1.7 \times 10^{-13} \text{ m}^{-2/3}$ the radius explodes. In that case, the error done at ground altitude is too important and is propagated until $z = 24 \text{ km}$.

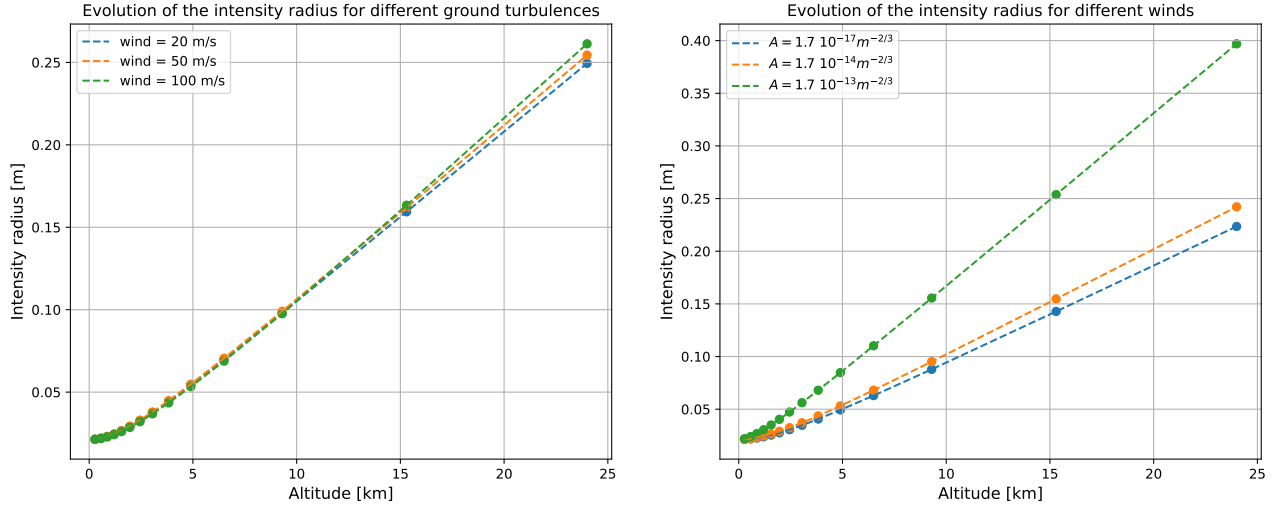


Figure 7.5: Evolution of the intensity radius for the 6 turbulence profiles. The marker are the altitude where the radius was computed. The left graph is for varying wind velocities of $w = 20 \text{ m/s}$, $w = 50 \text{ m/s}$ and $w = 100 \text{ m/s}$. The right graph is for different ground turbulences of $A = 1.7 \times 10^{-13} \text{ m}^{-2/3}$, $A = 1.7 \times 10^{-14} \text{ m}^{-2/3}$ and $A = 1.7 \times 10^{-17} \text{ m}^{-2/3}$.

7.2.4 Scintillation index and Strehl ratio

The two last measurements analysed are the scintillation index and the Strehl ratio. The results for the 6 cases are displayed in Table 7.1. As a reminder, a Strehl ratio close to one and a scintillation index close to 0 are synonyms of good performances.

Analysed case	Strehl ratio [-]	Scintillation index [-]
$w = 20 \text{ m/s}$	0.799	0.055
$w = 50 \text{ m/s}$	0.716	0.123
$w = 100 \text{ m/s}$	0.548	0.442
$A = 1 \times 10^{-17} \text{ m}^{-2/3}$	0.873	0.041
$A = 1 \times 10^{-14} \text{ m}^{-2/3}$	0.747	0.056
$A = 1 \times 10^{-13} \text{ m}^{-2/3}$	0.342	0.314

Table 7.1: Strehl ratio and scintillation index for the 6 analysed profiles. In blue the weak turbulence, in orange the medium and in green the strong turbulence.

The values of the scintillation index oscillate between 0.041 for the weak ground turbulence and 0.442 for the strong wind. The order of magnitude can change by a factor 10 depending on the turbulence applied. For the Strehl ratio, the minimal and maximal values are 0.342 and 0.873 respectively for the weak and strong ground turbulence.

To study the evolution of the scintillation and Strehl ratios with the wind velocities, the ratios are normalised by their associated wind speed. The results of the normalised scintillation index are $\sigma_{I,norm}^2 = (0.0027; 0.00246; 0.00442)$ and of the normalised Strehl ratio are $SR_{norm} = (0.04, 0.014, 0.0055)$. For $\sigma_{I,norm}^2$, the two first values are close while the last is twice higher. In the weak turbulence regime, the $\sigma_{I,norm}^2$ seems constant but once in the strong

regime, the scintillation index increases faster than the wind velocity. The observation is a faster decrease of the performance in this regime. For SR_{norm} , the normalised ratios always decrease but the first drop is of a factor 2.85 while the second with a factor 25.45.

The evolution of the Strehl ratio and scintillation index with the ground turbulence once again reveals the presence of two regimes. Between $A = 1 \times 10^{-17} \text{ m}^{-2/3}$ and $1 \times 10^{-14} \text{ m}^{-2/3}$ both metrics stay more or less constant but when A reaches $1 \times 10^{-13} \text{ m}^{-2/3}$, the Strehl ratio is divided by 2 and the scintillation index multiplied by 7.

Finally, comparing the Strehl ratios obtained with the curve in Figure 6.11 shows that the green cases can be seen as equivalent to applying a constant turbulence of $r_0 = 0.1 \text{ m}$.

To sum up, the analysis of both the wind velocity and ground turbulence indicates the presence of the weak and strong regime. It was already mentioned in the previous metrics and studied in section 5.3.

Chapter 8

Conclusion

8.1 Summary of contributions

The main contribution of this master thesis is a working *Python* simulation tool allowing for turbulent propagation using the phase screen method in order to improve the Earth to satellite communication link.

The first major impact is the free propagation with varying grid spacing δ evolving at the same rate as the intensity radius. This allows to propagate freely without aliasing in the scope of this work. The study on aliasing demonstrated the importance of the evolution of δ and the necessity to have close to zero values at the edges of the grid. Once this is respected, one can propagate freely in a random number of propagations since in the scope of this project, there is no degradation.

The second major impact is the study of the distribution of the received intensity as well as the study of the metrics. Chapter 5 shows the presence of two turbulence regimes through the matching of the high order moments. The distribution at weak intensity was found to follow a log-normal behaviour as developed in the literature [57]. Afterwards, multiple metrics were investigated at fixed turbulence to demonstrate the impact of weak and strong regimes on the measurements. The study of physical profiles under several conditions was also conducted. Each time, two turbulent regimes are revealed in the metrics.

8.2 Further research directions

Phase screen propagation is a vast subject and multiple research directions are possible. The main directions are described hereunder.

- The free propagation described in this work presents no aliasing under certain conditions. However, at the end of chapter 4, these conditions were discussed and found to be guidelines instead of strict conditions. To pursue this work, an analysis of the limitations of the conditions of free propagation could be done by simulations of various situations.
- In the framework of this report the images analysed are of size $N \times N = 256 \times 256$ pixels. If a higher resolution is needed, the computational time will strongly increase causing long

simulations. To avoid this undesired effect, optimised phase screen propagation method could be investigated. With a higher resolution, real information could be propagated to see the attainable performance on a real transmission.

- The study conducted in this report only considers the HV profiles with varying wind and ground turbulence. The study of the impact of other profiles as well as the uplink and downlink communications could be interesting.

8.3 Beyond the framework of this document

To go beyond the framework of this document, two ideas are given. The first one is about adaptive optics (AO) and the second one suggests to combine two master thesis.

- AO is a method to correct the received distorted wavefront to improve the quality of the image. Classical AO is done using sensor and deformable mirrors [63]. There are many of methods to perform AO such as LGAO (laser guide AO), MCAO (multi-conjugate AO), ...
- Performing a real transmission is difficult because it needs to have access to a satellite. To overcome this issue, an idea could be to combine this master thesis with the master thesis of J-B. Zimer. By applying the phase screen procedure in optical fiber, it could be possible to simulate a transmission.

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