APPENDIX B

ROOT LOCUS: UNSTABILITY SPIN SPEED

As explained in section 3.7.4, the root locus is determined by calculating the eigenvalues of the matrix $\mathbf{A}'$ in (3.62) for a large range of spin speeds. To simplify the problem, let us consider that the detent stiffness K_d is equal to zero. It means that the matrix of the state-space model reduces to $\mathbf{A}$, given in (3.57). From this matrix, the characteristic polynomial $P(s)$ of the state-space representation is calculated as follows:

$$\begin{aligned} P(s) &= \det(s\mathbf{I} - \mathbf{A}) \\ &= \det \begin{bmatrix} s + \frac{R}{L_c} & -\omega & 0 & \frac{K_\Phi^2 N}{2L_c} \\ \omega p^2 & s + \frac{R}{L_c} & \omega p^2 \frac{K_\Phi^2 N}{2L_c} & 0 \\ 0 & 0 & s & -1 \\ -\frac{1}{M} & 0 & 0 & s + \frac{C}{M} \end{bmatrix} \\ &= s^4 + s^3 \left(\frac{C}{M} + \frac{2R}{L_c} \right) + s^2 \left(\frac{2R}{L_c} \frac{C}{M} + \frac{R^2}{L_c^2} + \omega^2 p^2 + \frac{K_\Phi^2 N}{2L_c M} \right) \\ &\quad + s \left(\frac{R^2}{L_c^2} \frac{C}{M} + \omega^2 p^2 \frac{C}{M} + \frac{R}{L_c} \frac{K_\Phi^2 N}{2L_c M} \right) + \omega^2 p^2 \frac{K_\Phi^2 N}{2L_c M}. \end{aligned} \tag{B.1}$$

Considering in addition that the damping factor C is equal to zero, the characteristic polynomial reduces to:

$$P(s) = s^4 + s^3 \left(\frac{2R}{L_c} \right) + s^2 \left(\frac{R^2}{L_c^2} + \omega^2 p^2 + \frac{K_\Phi^2 N}{2L_c M} \right) + s \left(\frac{R}{L_c} \frac{K_\Phi^2 N}{2L_c M} \right) + \omega^2 p^2 \frac{K_\Phi^2 N}{2L_c M}. \tag{B.2}$$

The eigenvalues are calculated as the roots of this polynomial. Besides, the unstability speed ω_u is defined as the spin speed where the roots cross the imaginary axis, meaning that they can be written as $s = jy$. Substituting this expression in (B.2) and equalising to zero yields:

$$y^4 - jy^3 \left(\frac{2R}{L_c} \right) - y^2 \left(\frac{R^2}{L_c^2} + p^2 \omega_u^2 + \frac{K_\Phi^2 N}{2L_c M} \right) + jy \left(\frac{R}{L_c} \frac{K_\Phi^2 N}{2L_c M} \right) + p^2 \omega_u^2 \frac{K_\Phi^2 N}{2L_c M} = 0. \tag{B.3}$$

This equation can be separated into real and imaginary parts as follows:

$$\begin{cases} 0 = y^4 - y^2 \left(\frac{R^2}{L_c^2} + p^2 \omega_u^2 + \frac{K_\Phi^2 N}{2L_c M} \right) + p^2 \omega_u^2 \frac{K_\Phi^2 N}{2L_c M} \\ 0 = -y^3 \left(\frac{2R}{L_c} \right) + y \left(\frac{R}{L_c} \frac{K_\Phi^2 N}{2L_c M} \right) \end{cases} \tag{B.4}$$

The second relation has three solutions. On the one hand, the trivial one, i.e. $y = 0$, yielding $\omega_u = 0$, which corresponds to the start of the brown curve in Fig. 3.7 for instance.

On the other hand, the two last solutions are:

$$y = \pm\sqrt{\frac{K_\Phi^2 N}{4L_c M}}. \tag{B.5}$$

As explained in section 4.4.1, it represents the maximal pulsation of the state-space variables when the bearing is stable. It can be remarked that these two solutions are complex conjugates and thus correspond respectively to the red stars ω_3 on the blue and brown curves in Fig. 3.7. It proves that the system remains unstable after the unstability speed as there are no other crosses with the imaginary axis. It also shows that the two other fast poles never become unstable. Substituting (B.5) into the first equation of the system (B.4) yields:

$$\omega_u = \frac{1}{p}\sqrt{\frac{K_\Phi^2 N}{2L_c M} + \frac{R^2}{L_c^2}}. \tag{B.6}$$

Considering that the second term of the square root is dominant, this expression can be further simplified, leading to:

$$\omega_u = \frac{1}{p}\frac{R}{L_c}. \tag{B.7}$$

Hence, in this particular case, the unstability speed corresponds to the electrical pole R/L_c.