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Asian Option Pricing Using Comonotonotic Bounds

Author: David Beer
Supervisor(s): Frederic Vrins

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UNIVERSITÉ CATHOLIQUE DE LOUVAIN
Louvain School of Management
Place des Doyens, 1 bte L2.01.01, 1348 Louvain-la-Neuve
Boulevard Emile Devreux 6, 6000 Charleroi, Belgique
Chaussée de Binche 151, 7000 Mons, Belgique
www.uclouvain.be/lsm

Abstract

In this thesis we review fundamental theoretical aspect ofcomonotonicity and apply a method used in insurance to find a value for an arithmetic Asian option given some parameters, such as the number of averaging day, the strike price, or the volatility. We also review and build other competing models for the valuation of Asian Option and compare the accuracy of the estimators. We find that for higher level of volatility the comonotonic approach is far more efficient, keeping up with the Monte Carlo estimator.

Keywords: Asian Option, Comonotonicity, stochastic process, Monte-Carlo

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1 Introduction

In the financial world, options are powerful resources for investors, giving them the tools to hedge risks, optimize investments, or speculate on price movements. In their most basic definition, options are contracts that give the right to the holder to buy (in the case of call option), or to sell (in the case of put option) an underlying asset, at a specific point in time. Since the establishment of the Chicago Board Options exchange in 1973, the market for options has grown tremendously, the trading volume growing from 1.12 million contracts to 3.69 billion contracts in 2017 (K. Li, 2021). Numerous contract types exists, ranging from the fairly simple vanilla options, to the more complex exotic types. Asian options fall under this exotic category. They first appear in Tokyo in 1987, when Standish and Spaghton developed an option formula to price crude oil average price options that could be widely used by the financial industry (Jiaying Han, 2022). The difference between an European and an Asian option, lie in the way the pay-off is calculated at the expiration date. Pay-off for European options only depends on the underlying asset price at the expiration date, while the pay-off for Asian options will depend on the historical average of the price, taken at specific monitoring times (On a monthly, weekly, or daily basis, or during a specific time period, etc.). This feature of the Asian option corrects one of the drawbacks of the European option: its vulnerability to sudden changes in the volatility of the underlying assets near the expiration date, which can potentially be done through market manipulation. See for example, the infamous "GameStop short squeeze" of 2021, where a large group of investors organized themselves through the social media Reddit, to push up the stock price of several securities, leading banks and brokerage services to stop the trading of the assets for a short amount of time (Davies, 2021). This makes Asian options a cheaper and more reliable financial instruments for risk managers. As a result, these options emerge as a cost-efficient hedging tool, particularly in market contexts characterized by low trading volumes, such as the corporate bond or commodity markets, or with high levels of volatility, as observed in the currency and interest rate markets for example. Consequently, in their 1998 survey of Financial Risk Management, Bodnar, et al found out that Asian options were the exotic option type preferred by non-financial firms for risk management. (Bodnar, Hayt, & Marston, 1998), (Horvath & Medvegyev, 2016)

However, the reliance of the pay-off on the average of the historical prices makes the valuation of Asian options based on an arithmetic average particularly difficult due to the non-linearity and path-dependency of the arithmetic Asian option. While it is assumed that the underlying asset follows a log-normal distribution (Jiaying Han, 2022), the arithmetic average of a series of log-normally distributed variables does not. This absence of known expression of the distribution of the average makes the arithmetic Asian option non-applicable to these models, and alternatives must be found, and given their above mentioned popularity among market practitioners, academic researches have frequently been made to try to price those options. Indeed, while Asian option could be priced using Monte-Carlo simulations, those can be time consuming and expensive in terms of computational power. Simpler and faster alternatives are thus the end goal of those researches.

In those researches on Asian option pricing, a first set of alternatives relies on approximating the distribution of the arithmetic average. One of the earliest study of this type was made by (Turnbull & Wakeman, 1991). In their paper they present a quick algorithm to price the Asian options by approximating the distribution of the arithmetic average with a log-normal one. The approach relies on the fact that, while the distribution is unknown, the moments of the arithmetic average, can easily be computed. One year later, Levy introduces a similar approach, without taking the skewness and the kurtosis into account. In (Vorst, 1992) an exact pricing formula for the arithmetic average option is derived, based on the price of the geometric average Asian option, by adjusting the strike price by the difference of expectation between the arithmetic and geometric means. Inspired by those distribution approximation methods,(Milevsky & Posner, 1998) argue that the approximating distribution should be a reciprocal gamma. Their resulting formula is similar to the Black Scholes formula. More recently Chung-lee and Chang have applied methods similar to the ones used by Turnbull and Levy, to approximate the price of Asian options in the Hull and White stochastic volatility model. Another set of alternatives consists of deriving partial differential equations, this is notably the case for (Alziary, Décamps, & Koehl, 1997),(J. Zhang, 2002) , and (Forsyth, Vetzal, & Zvan, 2002) who compare the PDE method with the augmented state space approaches of Hull and White, and Barraquand to price the option. (Benhamou, 2001) assumes that the distribution of the option is non-log-normal and adapt and improve the Fast-Fourier transform technique for the asian option distribution. Binomial tree methods can also be used, as evidenced by (Hull, [1993]). Later on, (W. Hsu & Lyuu, 2007) builds upon their results, to propose a convergent time lattice algorithm.

Finally, the price of Asian options can also be approximated by deriving lower and upper bounds on the actual price. A first wave of research papers focusing on deriving lower and upper bounds in the Back Scholes model is published from 1994 to 2006. Starting with (Curran, 1994), who derives a lower bound of the price by conditioning on the geometric average. One year later (L. Rogers & Shi, 1995), propose a lower bounds derived through the conditioning on the logarithmic geometric average. Several years later, (Kaas, Dhaene, & Goovaerts, 2000) describe lower and upper bounds for the sum of random variables using comonotonicity theory. They also define an improved comonotonic upper bound, obtained through conditioning of random variables. In their next two papers, (Dhaene, Goovaerts, Denuit, Kaas, & Vyncke, 2002),(Dhaene, Denuit, Goovaerts, Kaas, & Vyncke, 2002) apply the theory outlined in their previous work and derive analytical comonotonic upper and lower bounds for Asian options. Their lower bound is conditioned on a linear transformation of a first order approximation of the sum of the stock prices and is closely related to the one found in (L. Rogers & Shi, 1995). Right after (J. A. Nielsen & Sandmann, 2003) derive their upper bound as a weighted portfolio of European options, while they build upon the work from (L. Rogers & Shi, 1995) and (Curran, 1994) to develop their lower bound. In 2006, (Vanmaele, Deelstra, Liinev, Dhaene, & Goovaerts, 2006) establish a unifying framework for the valuation of arithmetic Asian options in the Black Scholes model by introducing bounds that unify the various approaches defined previously in the literature. They also introduce several new bounds with the aim to improve the results of the existing ones. While the framework is unified under the Black Scholes model, it is important to note that it is not an appropriate model for real markets, due to several limitations such as constant volatility, or constant rate of return. Extensions of the model that tries to better reflect the market and rectify the limitation of the model have been introduced over the years. Extensions include the inclusion of time deterministic volatility, or/and interest rates, or the redefinition of the asset price dynamics with the inclusion of stochastic elements, such as the Heston model for volatility, or the Hull-White model for interest rates. Naturally, those models are of interest, and researches on the bounds of Asian options were not limited to the Black Scholes model. Around the time of publication of (Vanmaele et al., 2006), several papers adapting the concepts of comonotonic bounds to those models were published. A defining paper on the subject was published by (Albrecher, Dhaene, Goovaerts, & Schoutens, 2005), who generalizes the findings of the previous papers made in the Black Scholes models to models following Levy processes. They derive a general comonotonic upper bound that can be produced under any Levy model. Building upon this, Albrecher later generalize the framework to any arbitrage-free model in (Albrecher, Mayer, & Schoutens, 2008) and derives model-independent comonotonic lower bounds for the price of arithmetic Asian options. Using (Albrecher et al., 2005) as a base, (J. Chen & Christian-Oliver, 2014) also derives model-free comonotonic upper bounds, and test the bounds in the context of stochastic volatility models on the Heston, CEV, and Schwartz models. Finally, we also cite (Tretiak, 2022), who adapted and studied the efficacy of the bounds found in (Vanmaele et al., 2006) when applied to a Black Scholes model extended with deterministic volatility resulting from the time-average of a stochastic volatility process.

Given the state of the literature on comonotonic bounds for Asian options reviewed above, we decide to focus on modelling the price of arithmetic Asian options. The purpose of this thesis is multiple:

- We test the bounds described in (Albrecher et al., 2008), and (J. Chen & Christian-Oliver, 2014) for a complex model not tested in either of the papers. For that we choose the hybrid Heston model initially presented in (Grzelak & Oosterlee, 2011), and refined in (Recchioni & Sun, 2016), . This model is an extension of the simple Heston model, which incorporates stochastic interest rates following a CIR process. The choice of such a model is based on the conclusions of several studies of the last 20 years (for instance; (Trolle & Schwartz, 2009),(Chiarella & Kwon, 2003),(Andersen & Piterbarg, 2007)), which show that stochastic interest rates should be incorporated into pricing models, to better capture market behavior. Studying the behavior of the bounds for this new model will allow to document the pricing of arithmetic Asian options in this more complex paradigm. To our knowledge, no valuation exercise on arithmetic Asian option has been performed yet for this model. This will therefore be the occasion to test whether the model independent bounds support the added complexity of having two stochastic processes, and to evaluate the bounds' efficiency in approximating the price.

- We test the efficiency of the bounds described in (Albrecher et al., 2008), and (J. Chen & Christian-Oliver, 2014) with regards to the bounds presented in (Vanmaele et al., 2006). For that, we will adapt the bounds of (Vanmaele et al., 2006) to a stochastic model similarly to (Tretiak, 2022). More precisely, we develop a Black Scholes extended models with time deterministic variables based on stochastic processes. The model is a time varying volatility and time varying interest rates model.

In order to better compare the extended Black Scholes model with their stochastic counterparts, the time deterministic functions are based on the CIR stochastic process which is used in both the Heston and hybrid Heston.

As mentioned above, we mainly test the efficiencies of the bounds in modelling approximation of the price. For that, we systematically compare the results of the bounds with a benchmark obtained through Monte Carlo simulation using the control variate variance reduction technique. For the bounds in the Black Scholes extended model, we use the geometric Asian option, while for the stochastic models we use the sum of European options at the n monitoring times.

Our contribution could be summarized as the following: we implement both the lower and upper model-independent comonotonic bounds to the hybrid Heston model. Similarly, we further the analysis of (Tretiak, 2022), by applying the time deterministic method to a Black Scholes model extended to both time varying volatility and interest rates. And we compare the quality of the bounds in the two models, given different market scenarios.

The results of our work is discussion inducing. The quality of the bounds in (Vanmaele et al., 2006) decrease quite rapidly with the apparation of both time varying volatility and interest rate. Different tests show that the interest rate are the main driver of this loss of quality. The model-independent bounds follow a better pattern, but are not precise enough to really be of use for practionners. Again, we seem to realize that the bounds fare better in low stochastic interest rates settings.

The rest of this thesis is organized as follows. In section 2, we describe the characteristics of the arithmetic Asian options studied in this paper; in section 3, we present the rationale behind the Monte Carlo simulations used as benchmark in the rest of the paper; in section 4, we focus on the Black Scholes extended model for volatility and interest rates. We first describe the bounds presented in (Vanmaele et al., 2006), before discussing the specificities of the extended model; in section 5; we present the model-independent lower and upper bounds, as well as the hybrid Heston models; finally in section 6, we present the results of the bounds for the two models.

2 Characteristics of Asian Options

Value of an arithmetic Asian option under Black Scholes In this section, we briefly describe the characteristics of an Asian options. While our thesis is focused on discrete time arithmetic average Asian options, we also give a description on the characteristics of the discrete time geometric average Asian option, as it will be used in the next two sections as the control variate for the Monte Carlo simulations.

Asian options are one of the most recognizable example of exotic options. They are predominantly traded over the counter or in off-exchange markets(Lu, Zhu, & Li, 2019). Leading to a lack of publicly available information on the individual contracts. Since their payoff depends on the average of the underlying asset over a certain time interval, they are path dependent instruments. As mentioned in section 1, this path dependency makes the Asian options less sensitive to changes in the asset price near the expiration date. As they are less prone to volatility, they are cheaper than vanilla European options. Depending on the type of averaging of the Asian option, the discrete time average can have one of the two expression: For arithmetic Asian option:

$$A = \frac{1}{n} \sum_{i=1}^{n} S\left(t_i\right) \tag{1}$$

For geometric Asian option:

$$G = \left[\prod_{i=1}^{n} S\left(t_i\right)\right]^{\frac{1}{n}} \tag{2}$$

$t_i$represents the price of the underling asset at time t_i, with t_i being the monitoring times over which the averaging takes place:

$$0 \leq t_1 \leq t_2 \cdots \leq t_n = T \tag{3}$$

In this thesis we consider that the monitoring times t_i are equally spaced over the wholes time period: $[t_i - t_{i+1}]$= $[t_j - t_{j+1}]$=h with h=T/n

The payoff of an arithmetic call Asian option with strike K is given by the following:

$$\left(\frac{1}{n} \sum_{i=1}^{n} S(t_i) - K\right)^{+} \tag{4}$$

where

$$x^{+} = \max\{x, 0\} \tag{5}$$

Recall that in the Black Scholes model, the price of a risky asset under the risk-neutral measure Q follows the geometric Brownian motion process:

$$\frac{\mathrm{d}S(t)}{S(t)} = r \ \mathrm{d}t + \sigma \mathrm{d}W(t), \quad t \geqslant 0, \tag{6}$$

Where σ is the volatility, r the drift equal to the risk-free interest rate, and $\{W(t), t \geqslant 0\}$ is a standard Brownian motion process under the risk-neutral measure Q. Given (6) , the random variables $S(t)/S(0)$ follow a log-normal distribution, with parameters $\left(r - \frac{\sigma^2}{2}\right) t$ and $t\sigma^2$. However, the average of a series of log-normally distributed random variable is not log-normally distributed itself. As a consequence, it is not possible to use the Black Scholes formula to derive the price of the arithmetic Asian option. Which motivates the use of the other method of valuation for the arithmetic Asian option. Under the risk-neutral expectation E^Q and risk-neutral interest rate r the risk neutral price of the arithmetic Asian option at current time t = 0 is given by:

$$\mathrm{AC}_0 = \mathrm{e}^{-rT} E^Q \left[\left(\frac{1}{n} \sum_{i=1}^{n} S(t_i) - K \right)^+ \right] \tag{7}$$

We will use the dynamics described by equation (7) to develop our benchmarks using the Monte Carlo simulations.

Value of a geometric Asian option under Black Scholes: The same way we defined the risk neutral price of the arithmetic Asian option we can also find an expression of the risk neutral price of the geometric average Asian option at current time t = 0:

$$\mathrm{GC}_0 = \mathrm{e}^{-rT} E^Q \left[\left(\left[\prod_{i=1}^{n} S\left(t_i\right) \right]^{\frac{1}{n}} - K \right)^+ \right] \tag{8}$$

Contrary to the arithmetic average, the geometric average of a series of log-normally distributed random variables follows a log-normal distribution too. In (Vorst, 1992), an exact pricing formula for the geometric average options is given. Vorst does so by taking the natural logarithm of the geometric average:

$$\ln G = \frac{1}{n} \sum_{i=1}^{n} \ln S(t_i) = \ln S_0 + \frac{1}{n} \sum_{i=1}^{n} \ln \frac{S(t_i)}{S_0} \tag{9}$$

The logarithmic values $\ln \frac{S(t_i)}{S_0}$ follow a normal distribution with the following parameters:

$$N\left(\left(r-q-\frac{1}{2}\sigma^2\right)t_i, \sigma^2 t_i\right), \quad i=1,2,\ldots,n \tag{10}$$

Since the Brownian motions have independent increments, the sum of the normally distributed random variables is normally distributed, it is then possible to find the mean and variance of lnG. Following (Vorst, 1992), (J. A. Nielsen & Sandmann, 2003) finds the following expressions for the parameters of the distribution:

$$\mu_G = \ln S_0 + \left(r - \frac{1}{2}\sigma^2\right)\frac{T+h}{2} \tag{11}$$

$$\sigma_G^2 = \sigma^2\left(h + (T-h)\frac{(2n-1)}{6n}\right) \tag{12}$$

Finally, given those parameters, the value of the geometric Asian option can be computed in the Black Scholes framework similarly to an European option:

$$GCBS_0 = \exp(-rT)\left\{\exp\left(\mu_G + \frac{1}{2}\sigma_G^2\right)\Phi\left(d_1\right) - K\Phi\left(d_2\right)\right\} \tag{13}$$

with:

$$d_1 = \frac{\mu_G - \ln K + \sigma_G^2}{\sigma_G} \tag{14}$$

$$d_2 = d_1 - \sigma_G \tag{15}$$

The expression of the risk neutral price in (13) and the Black Scholes price will be used for the Monte Carlo simulation in the Black scholes Monte Carlo simulations.

3 Monte Carlo Simulations

In this section, we describe the basic principles behind the Monte Carlo simulation. This will serve as the benchmark for the bounds we will develop later on in this paper, as we will adapt the simulations to our different models going forward.

We start by applying the naive Monte Carlo method to the Asian options. We first demonstrate the Monte Carlo simulation in the Black Scholes model but will give insights on the modifications to be done for the other models. The essence of the Monte Carlo simulation is to estimate the expected value of a random variable by drawing samples from its distribution and then computing the average over these samples.

$$\hat{S}t = \frac{1}{n}\sum_{i=1}^{n} St^{(i)} \longrightarrow E\left[S_t\right] \tag{16}$$

In the Black Scholes model, samples from the distribution of the underlying assets would be computed at time as follows:

$$\begin{gathered} s_{i_j}^{(i)} = s_{i_{j-1}} \exp\left(\left(r - q - \frac{1}{2}\sigma^2\right)(i_j - i_{j-1}) + \sigma\sqrt{i_j - i_{i-1}}Z^{i,j}\right), \\ j = 1, 2, \ldots, m, \quad i = 1, 2, \ldots, n \end{gathered} \tag{17}$$

with $s_{i,j}$ being the value of the underlying stock at time t_j during the i^{th} iteration of the simulation. And $Z^{ij} \sim N(0,1)$ From the pay-off of the arithmetic Asian option, we can then derive the estimator of the Monte Carlo simulation as follows:

$$\hat{AC}_{0,n} = \frac{e^{-rT}}{n}\sum_{i=1}^{n}\left(\frac{1}{m}\sum_{j=1}^{m} s_{i,j} - K\right)^{+} \tag{18}$$

The variance of the sample is then given by

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}\left(AC^{(i)} - \hat{AC}\right)^2 \tag{19}$$

And from the variance we can establish a confidence interval on the true value by computing the error on the estimator:

$$\left[\hat{AC} - 1.96\frac{\sigma}{\sqrt{n}}, \hat{AC} + 1.96\frac{\sigma}{\sqrt{n}}\right] \tag{20}$$

We then compute a Monte Carlo simulation on an Asian Option in the Black Scholes model with the following parameters: T= 1, $S0$=100, r = 0.09, σ = 0.3, n= 10000, h = 1/12

From this first simulation, we can see that the error decreases with K. This can be explained by the fact that as K increases, the value of the Asian option tend to 0. But while the absolute value of the error decreases, the coefficient of variation increases with K. To reduce the error, one could either increase the number of simulations, at the cost of time and computing power, or could use of the several variance techniques that exists, the main ones being the antithetic variate, and the control variate techniques. Given the conclusion of several pieces of literature on the application of either variance reduction

K	AC	error	Variance	Coefficient de variation
50	50,41	0,36	333,64	0,71
60	41,23	0,35	326,89	0,86
70	32,18	0,36	328,94	1,10
80	23,23	0,34	309,07	1,48
90	15,30	0,31	242,35	1,99
100	9,55	0,26	181,47	2,76
110	5,33	0,21	110,46	3,85
120	2,73	0,15	58,59	5,49
130	1,29	0,10	28,16	8,07
140	0,62	0,07	13,59	11,62
150	0,30	0,05	6,29	16,55

Figure 1: Table 1: Naive Monte Carlo

techniques (see (J. Zhang, 2002),(L. B. Nielsen, 2001)) , we decided to apply the control variate method in this thesis.

3.0.1 Variance Reduction

The control variates method involves the idea that when trying to estimate a specific value from data, insights from another related variable can be utilized to refine and improve the initial estimation. The main intuition behind the technique is to find a variable of control C that would be positively correlated with the variable we search to estimate. We could then define a new variable following the equation below:

$$AC' = AC - b(C - \mathbb{E}(C)) \tag{21}$$

With b a real number. Clearly, the idea is to adjust the estimate of AC given the deviation of C from its own expected value.

From there it is clear that

$$E\left(AC'\right) = E(AC) + b(E(C) - E(C)) = E(AC) = AC, \tag{22}$$

The control variable C must thus be positively correlated with AC such that $\operatorname{Cov}(AC, C) >$

0 and should also be chosen as to minimize the variance of AC':

$$\operatorname{Var}\left[AC'\right] = \operatorname{Var}[AC] - 2b\operatorname{Cov}[AC, C] + b^2\operatorname{Var}[C] \tag{23}$$

with b to be chosen to also minimize the variance of AC'. Using the OLS coefficient the optimal choice $b^\star$ can be found (Kogan):

$$b^\star = \frac{\operatorname{Cov}[AC, C]}{\operatorname{Var}[AC]} \tag{24}$$

In order to implement the control variate technique, we follow Nielsen's implementation. We thus, chose a geometric average Asian option as the control variate for the Monte Carlo simulation:

$$\mathrm{C} = \mathrm{e}^{-rT}E^Q\left[\left(\left[\prod_{i=1}^{n} S\left(t_i\right)\right]^{\frac{1}{n}} - K\right)^+\right] \tag{25}$$

Jensen's inequality shows that the geometric average is always lower the the arithmetic average. Several studies on Asian options has also used the geometric Asian options as either a direct lower bounds to price and approximate the arithmetic average (Vorst, 1992), or by conditioning on it (Curran, 1994). The positive correlation condition between AC and C is thus met. Regarding the variable b, we follow (L. B. Nielsen, 2001)'s paper which sets it to 1. Given the Black Scholes derivation of the value of a geometric Asian option, we end up with a new unbiased estimator of the form:

$$\frac{e^{-rT}}{n}\sum_{i=1}^{n}\left(\frac{1}{m}\sum_{j=1}^{m}s_{i,j} - K\right)^+ - \frac{e^{-rT}}{n}\sum_{i=1}^{n}\left(\prod_{i=1}^{m}s_{i,j}^{\frac{1}{m}} - K\right)^+ + GCBS \tag{26}$$

Finally, we compute the newly found estimator using the same values as the one for the plain Monte Carlo: T= 1, $S0$=100, r = 0.09, σ = 0.3, n= 10000, h = 1/12 Once we have applied the Monte Carlo simulation we can see that both the error and variance reduces significantly, for the whole range of K. The coefficient of variation is also greatly reduced. The results are coherent with our expectation on the use of the geometric Asian option as control variate, the errors are reduced, the Control Variate Monte Carlo simulation converges faster than the Plain one.

K	AC	error	Variance	Coefficient of variation
50	50,299	0,098	24,120	0,1948
60	41,158	0,080	16,729	0,1944
70	32,065	0,064	10,882	0,1996
80	23,334	0,048	6,177	0,2057
90	15,538	0,034	3,030	0,2188
100	94,410	0,022	1,370	0,0233
110	52,520	0,014	0,567	0,0267
120	26,970	0,010	0,308	0,0371
130	1,294	0,009	0,249	0,6957
140	0,593	0,010	0,260	1,6863
150	0,269	0,009	0,211	3,3457

Figure 2: Table 2: Monte Carlo with Control Variate

4 Comonotonic Bounds in Black-Scholes and time-deterministic models

In 2006, (Vanmaele et al., 2006) published their framework on the comonotonic bounds for Asian options, unifying and improving on previous notions and bounds derived by other researchers. (Tretiak, 2022) in their thesis adapted the bounds in (Vanmaele et al., 2006) to deterministic time-dependent volatility. We follow in their footstep and and propose to add a degree of complexity to the adaptation by add deterministic time-dependent interest rates. To better approximate the Hybrid Heston model first described in (Recchioni & Sun, 2016), we decide to model both time-dependent processes using a Cox-Ingersoll process. The goal being to analyse the robustness of the bounds derived from the Black Scholes model and the model-independent bounds on some common ground. We will first go through some theoretical concepts highlighted in (Vanmaele et al., 2006) before describing the bounds and their derivation. We will then describe the time-deterministic model we have chosen to apply the bounds to.

4.0.1 Notations

In their paper, (Vanmaele et al., 2006) use the following notations to describe the pay-off of an European-style discrete Asian option, with strike K, expiration date T, and n monitoring dates as follows:

$$\left(\frac{1}{n}\sum_{i=0}^{n-1} S\left(T-t_i\right)-K\right)_+ \tag{27}$$

$S(T-t_i)$ being the the price of a risky asset at time $T-t_i, i=0,\ldots,n-1$. and $x_+ = \max\{x,0\}$. With the price of the discrete arithmetic call option with strike K, maturity date T and n monitoring times under the risk neutral probability Q :

$$AC(n,K,T)=\frac{e^{-rT}}{n}\mathbb{E}^{\mathbb{Q}}\left[\left(\sum_{i=0}^{n-1} S\left(T-t_i\right)-nK\right)_+\right] \tag{28}$$

Furthermore, please note that the analytical formula of the bounds will already be adapted to reflect the time changing processes of the volatility and interest rates.

4.1 Bounds based on comonotonicity reasoning

In the actuarial and financial worlds, one can often find random variables under the form: $\mathbb{S}=\sum_{i=1}^{n} X_i$ where the terms X_i are not mutually independent. As only the marginal distribution functions of the random variables X_i is known, the multivariate distribution function of the random vector is not completely specified (Vanmaele et al., 2006). Finding upper and lower bounds of the sum $\mathbb{S}$ of the X_i random variables would prove useful in this case. Lower bounds would be of the form $\underline{\mathbb{S}}=\sum_{i=1}^{n}\underline{X_i}$; while upper bounds should be of the form $\overline{\mathbb{S}}=\sum_{i=1}^{n}\overline{X_i}$ for the sum $\mathbb{S}=\sum_{i=1}^{n} X_i$. The bounds should then answer the following specifications:

- the marginal distribution functions of the random variables $X_i, \underline{X_i}$ and $\overline{X_i}(i=1,\ldots,n)$ are equal,
- And the sum of random variables follow a convex order: $\underline{S} \preceq_{\mathrm{cx}} \mathbb{S} \preceq_{\mathrm{cx}} \overline{\mathbb{S}}$. This then implies that the variables have the same expectation($E[\underline{\mathbb{S}}]=E[\mathbb{S}]=E[\overline{\mathbb{S}}]$) and that the stop-loss premiums wrt to d of the random variables also follow a convex order: $E\left[(\underline{\mathbb{S}}-d)_+\right] \leqslant E[(\mathbb{S}-\ d)_+] \leqslant E\left[(\overline{\mathbb{S}}-d)_+\right]$for all $d\in\mathbb{R}$.

Out of three potential bounds highlighted in previous papers, (Vanmaele et al., 2006) chose to develop upper and lower bounds as follows:

- Based on the work of (Kaas et al., 2000), the upper bounds chosen is of the form $\overline{\mathbb{S}} := \mathbb{S}^{\mathrm{u}}$, with

$$\mathbb{S}^{\mathrm{u}}=F^{-1}_{X_1|\Lambda}(U)+F^{-1}_{X_2|\Lambda}(U)+\cdots+F^{-1}_{X_n|\Lambda}(U) \tag{29}$$

Based on the work of (Kaas et al., 2000), takes the components of the random vectors $\left(\overline{X_1}, \overline{X_2}, \ldots, \overline{X_n}\right)$ such that $\overline{X_i} := F^{-1}_{X_i|\Lambda}(U)$, where $F^{-1}_{X_i}(U)$ is the usual inverse distribution, $F^{-1}_{X_i|\Lambda}(U)$ is the random variable of the form $f_i(u, \lambda) = F^{-1}_{X_i|\Lambda=\lambda}(u)$ and U a $(0,1)$-uniform random variable independent of Λ . This bound is an improved over the upper bound developed by (Dhaene, Denuit, et al., 2002).

- For the lower bound (Vanmaele et al., 2006) also chose a bound based on (Kaas et al., 2000). This bound is of the form: $\mathbb{S} := \mathbb{S}^{\ell}$, with $\mathbb{S}^{\ell}$ being the conditional expectation of $\mathbb{S}$ given the random variable Λ :

$$\mathbb{S}^{\ell} = E[\mathbb{S} \mid \Lambda] \tag{30}$$

As a lower bound following Kaas et al. [15], where $\mathbb{S}^{\ell}$ is a conditional expectation Through those two bounds, the sum $\mathbb{S}$ is bounded above and below:

$$\mathbb{S}^{\ell} \preceq_{\mathrm{cx}} \mathbb{S} \preceq_{\mathrm{cx}} \mathbb{S}^{\mathrm{u}} \tag{31}$$

The definition of the convex order, then implies that the stop-loss premiums of the random variables are also ordered for all d in $\mathbb{R}$:

$$E\left[\left(\mathbb{S}^{\ell} - d\right)_+\right] \leqslant E\left[(\mathbb{S} - d)_+\right] \leqslant E\left[(\mathbb{S}^{\mathrm{u}} - d)_+\right] \leqslant E\left[(\mathbb{S}^{\mathrm{c}} - d)_+\right] \tag{32}$$

It is finally important to note that the average of the underlying asset in the price of the Asian options can be rewritten as a sum of lognormal variables:

$$\mathbb{S} = \sum_{i=0}^{n-1} S(T - t_i) = \sum_{i=0}^{n-1} S(0) \mathrm{e}^{\left(r - \frac{\sigma^2}{2}\right)(T - t_i) + \sigma B(T - t_i)}. \tag{33}$$

$$\mathbb{S} = \sum_{i=0}^{n-1} X_i = \sum_{i=0}^{n-1} \alpha_i \mathrm{e}^{Y_i} \tag{34}$$

The theoretical principles of the two comonotonic upper bounds can thus applied to the sum of the underlying assets in order to derive bounds on the Asian options.

4.2 Lower Bounds

(Vanmaele et al., 2006) derive the lower bound

$$\mathbb{S}^{\ell} = \sum_{i=0}^{n-1} E^{Q}\left[X_i \mid \Lambda\right] = \sum_{i=0}^{n-1} \alpha_i E^{Q}\left[\mathrm{e}^{Y_i} \mid \Lambda\right] \tag{35}$$

for three possible values of the conditioning variable, under the assumption that Λ is a normally distributed conditioning variable such that $(\sigma(t_i) B(T-t_i), \Lambda)$ are bivariate normally distributed for all i and $\sigma(t)$.

The comonotonic lower bound for the option price AC(n, K, T) is then given by:

$$\begin{aligned} \mathrm{LB}\Lambda &= \frac{e^{-rT}}{n}\widetilde{\mathbb{E}}\left[\left(\sum_{i=0}^{n-1} \mathcal{S}^{l} - nK\right)_{+}\right] \\ &= \frac{S(0)}{n}\sum_{i=0}^{n-1} e^{-r_{(\Delta t_i)} t_i}\Phi\left[\sigma(\Delta t_i)\rho_{\Delta t_i}\sqrt{\Delta t_i} - \Phi^{-1}\left(F_{\mathcal{S}^{l}}(nK)\right)\right] \\ &\quad - e^{-r_{(T)}T} K\left(1 - \left(F_{\mathcal{S}^{l}}(nK)\right)\right) \end{aligned} \tag{36}$$

where $\rho_{(t\Delta_i)} = \operatorname{corr}(\sigma(t\Delta_i) B(t\Delta_i), \Lambda) \geq 0$ and $F_{\mathcal{S}^{l}}(nK)$ a solution to:

$$\begin{aligned} S(0)\sum_{i=0}^{n-1} \exp\Bigg[&\left(r_{\Delta t_i} - \frac{\sigma^2(\Delta t_i)}{2}\rho_{\Delta t_i}^2\right)\Delta t_i \\ &+ \sigma(\Delta t_i)\rho_{\Delta t_i}\sqrt{\Delta t_i}\Phi^{-1}\left(F_{\mathcal{S}^{l}}(nK)\right)\Bigg] = nK \end{aligned} \tag{37}$$

With (Δt_i) the averaging time($T - t_i$), $\Phi(\cdot)$ the cumulative distribution function (*cdf*) of a standard normal variable and $F_{\mathbb{S}^{\ell}}(\cdot)$ represents the cdf of $\mathbb{S}^{\ell}$.

The conditional variable Λ only intervenes in the function through the correlation term: $\rho_{\Delta t_i}$ Given the purpose of deriving a closed form expression for the lower bound, (Vanmaele et al., 2006), gives the following general expression for the conditional variable:

$$\Lambda = \sum_{i=0}^{n-1} \beta_i B(\Delta t_i), \quad \beta_i \in \mathcal{R}^{+} \tag{38}$$

For general positive β_i, the variance of Λ is then given by

$$\sigma_{\Lambda}^2 = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \beta_i\beta_j \min(\Delta t_i, \Delta t_j) \tag{39}$$

and the general form of the correlation is given by:

$$\begin{aligned}\rho_{\Delta t_i} &= \operatorname{corr}(\sigma B(\Delta t_i), \Lambda) \\ &= \frac{\operatorname{cov}(B(\Delta t_i), \Lambda)}{\sqrt{\Delta t_i}\sigma_\Lambda} \\ &= \frac{\sum_{j=0}^{n-1} \beta_j \min(\Delta t_i, \Delta t_j)}{\sqrt{\Delta t_i}\sigma_\Lambda} \geqslant 0.\end{aligned} \tag{40}$$

To ensure that ensure that $\mathbb{S}^\ell$ is a sum of n comonotonic random variables, (Vanmaele et al., 2006) only takes the positive β_i, implying that the $\rho_{\Delta t_i}$ are positive. The quality of the lower bound is dependent on the choice of β_i, as it directly influence the variance of the lower bound $E^Q[\mathbb{S} \mid \Lambda]$, the quality of the lower bound being judged by its variance. Indeed, maximization of the quality implies that the variance of the lower bound be made as close as possible to var^Q [S]

With this in mind, (Vanmaele et al., 2006) proposes three possible conditioning variables to define the lower bound. The resulting bounds only differ from each other due to their correlation coefficients and the variance of the conditioning variable. their correlation and variance .

The first variable is a linear transformation of a first-order approximation of the stock price, proposed by (Dhaene, Goovaerts, et al., 2002)

$$\Lambda = \sum_{i=0}^{n-1} e^{\left(r_{(\Delta t_i)} - \frac{\sigma^2(\Delta t_i)}{2}\right)\Delta t_i} B(\Delta t_i) \tag{41}$$

This lower bound has been used by already used by (Vanmaele et al., 2006) to derive their comonotonic lower bound.

Correlations and variances are defined as follows:

$$\begin{aligned}\rho_{\Delta t_i} &= \frac{\sum_{j=0}^{n-1} e^{\left(r_{(\Delta t_j)} - \frac{\sigma^2(\Delta t_j)}{2}\right)\Delta t_j} \min(\Delta t_i, \Delta t_j)}{\sqrt{\Delta t_i}\sigma_\Lambda} \\ \sigma_\Lambda^2 &= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} e^{\left(r_{(\Delta t_i)} - \frac{\sigma^2(\Delta t_i)}{2}\right)\Delta t_i + \left(r_{(\Delta t_j)} - \frac{\sigma^2(\Delta t_j)}{2}\right)\Delta t_j} \min(\Delta t_i, \Delta t_j)\end{aligned} \tag{42}$$

The second one, is the standardized logarithm of the geometric average

$\mathbb{G} = \sqrt[n]{\prod_{i=0}^{n-1} S(\Delta t_i)}$ proposed by (J. A. Nielsen & Sandmann, 2003), and (L. Rogers & Shi, 1995):

$$\Lambda = \frac{\ln \mathcal{G} - \widetilde{\mathbb{E}}[\ln \mathcal{G}]}{\sqrt{\widetilde{\operatorname{var}}[\ln \mathcal{G}]}} = \frac{\sum_{i=0}^{n-1} B(\Delta t_i)}{\sqrt{\widetilde{\operatorname{var}}\left[\sum_{i=0}^{n-1} B(\Delta t_i)\right]}} \tag{43}$$

With

$$\widetilde{\text{var}}\left[\sum_{i=0}^{n-1} B(\Delta t_i)\right] = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \min(\Delta t_i, \Delta t_j) \tag{44}$$

and correlation:

$$\rho_{\Delta t_i} = \frac{\sum_{j=0}^{n-1} \min(\Delta t_i, \Delta t_j)}{\sqrt{\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \min(\Delta t_i, \Delta t_j)}} \tag{45}$$

Finally, (Vanmaele et al., 2006) proposes the conditioning variables:

$$\Lambda = \sum_{k=1}^{T} \beta_k W_k, \tag{46}$$

$$\text{with } W_k \text{ i.i.d. } N(0,1) \text{ such that } B(\Delta t_i) \stackrel{d}{=} \sum_{k=1}^{\Delta t_i} W_k, i = 0, \ldots n-1,$$

(Vanmaele et al., 2006) identifies this conditioning variable as the one producing the most accurate bounds, be it on lower or upper bounds.

$$r_i = \rho_{\Delta t_i} \quad = \frac{\text{cov}\left(B(\Delta t_i), \Lambda\right)}{\sqrt{\Delta t_i}\sigma_\Lambda} \quad = \frac{\Delta t_i}{\sqrt{\Delta t_i}\sqrt{T}} \quad = \frac{\sqrt{\Delta t_i}}{\sqrt{T}}, \quad i = 0, 1, \ldots, n-1. \tag{47}$$

4.3 Improved comonotonic upper bound

Similarly to the lower bound, (Vanmaele et al., 2006) considers conditioning on a normal random variable Λ Through the following inequality, they derive the improved comonotonic upper bound:

$$AC(n, K, T) = \frac{e^{-rT}}{n}\mathbb{E}^Q\left[((\mathbb{S}^{\text{u}} - nK)_+)\right] \leq \frac{e^{-rT}}{n}\mathbb{E}^Q\left[(\mathbb{S}^{\text{u}} - nK)_+\right] \tag{48}$$

where previously, we defined $\mathbb{S}^{\text{u}} = \sum_{i=0}^{n-1} F^{-1}_{X_i|\Lambda}(U) = \sum_{i=0}^{n-1} F^{-1}_{\alpha_i e^{Y_i|\Lambda}}(U)$ for a $(0,1)$-uniform random variable.

The improved comonotonic upper bound is as follows:

$$\text{ICUB}\,\Lambda = \frac{e^{-rT}}{n}\widetilde{\mathbb{E}}\left[(\mathcal{S}^u - nK)_+\right]$$

$$= \frac{e^{-r_{(T)}T}}{n}\sum_{i=0}^{n-1} S(0)e^{-r(\Delta t_i)}e^{-\frac{\sigma^2(\Delta t_i)}{2}\rho^2_{\Delta t_i}\Delta t_i}$$

$$\times \int_0^1 e^{\rho_{\Delta t_i}\sigma(\Delta t_i)\sqrt{\Delta t_i}\Phi^{-1}(v)}\Phi\left(\sqrt{1-\rho^2_{\Delta t_i}}\sigma(\Delta t_i)\sqrt{\Delta t_i} - \Phi^{-1}\left(F_{\mathcal{S}^u|V=v}(nK)\right)\right) dv$$

$$-e^{-r_{(T)}T}K\left(1 - F_{\mathcal{S}^u}(nK)\right)$$

With

$$V=\Phi\left(\frac{\Lambda-E[\Lambda]}{\sigma_\Lambda}\right) \tag{49}$$

is a uniform $(0,1)$ random variable, $\rho_{T-i}=\operatorname{corr}(\sigma B(\Delta t_i),\Lambda)$, and

$$F_{\mathbb{S}^u}(nK)=\int_0^1 F_{\mathbb{S}^u|V=v}(nK)\mathrm{d}v \tag{50}$$

Furthermore, the conditional distribution $F_{\mathbb{S}^u}\mid V=v(nK)$ follows from

$$\begin{aligned}&S(0)\sum_{i=0}^{n-1}\exp\left[\left(r_{(\Delta t_i)}-\frac{\sigma^2(\Delta t_i)}{2}\right)\Delta t_i\right]\\&\times\exp\left[\rho_{\Delta t_i}\sigma(\Delta t_i)\sqrt{\Delta t_i}\Phi^{-1}(v)+\sqrt{1-\rho^2_{\Delta t_i}}\sigma(\Delta t_i)\sqrt{\Delta t_i}\Phi^{-1}\left(F_{\mathcal{S}^l}(nK)\right)\right]=nK\end{aligned} \tag{51}$$

4.4 Bounds based on the Rogers and Shi approach

Here (Vanmaele et al., 2006), derives upper bounds based on the lower bounds. For that they apply the following general inequality for any random Y and Z as described by (L. C. G. Rogers & Shi, 1995),

$$0\leqslant E\left[E\left[Y_+\mid Z\right]-E[Y\mid Z]_+\right]\leqslant\frac{1}{2}E[\sqrt{\operatorname{var}(Y\mid Z)}]. \tag{52}$$

(Vanmaele et al., 2006) derives upper bonds by applying the inequality to the case of Y being $\sum_{i=0}^{n-1}S(\Delta t_i)-nK$. They therefore obtain an error bound for the difference of the option price and its lower bound:

$$0\leq\mathbb{E}^Q\left[\mathbb{E}^Q\left[(\mathbb{S}-nK)_+\mid\Lambda\right]-\mathbb{E}^Q\left(\mathbb{S}^l-nK\right)_+\right]\leq\frac{1}{2}\mathbb{E}^Q[\sqrt{\operatorname{var}^Q(\mathbb{S}\mid\Lambda)}] \tag{53}$$

Therefore, the upper bond is given by

$$\mathrm{UB}\Lambda=\frac{\mathrm{e}^{-rT}}{n}\left\{E^Q\left[\left(\mathbb{S}^\ell-nK\right)_+\right]+\varepsilon\right\}, \tag{54}$$

The error bound can be computed with the following:

$$\begin{aligned}\alpha_i\alpha_j&=S(0)^2\exp\left[\left(r_{(\Delta t_i)}-\frac{\sigma^2(\Delta t_i)}{2}\right)\Delta t_i+\left(r_{(\Delta t_j)}-\frac{\sigma^2(\Delta t_j)}{2}\right)\Delta t_j\right]\\\sigma_{ij}&=\sqrt{\Delta t_i+\Delta t_j+2\min(\Delta t_i,\Delta t_j)}\\r_{ij}&=\frac{\sqrt{\Delta t_i}}{\sigma_{ij}}\rho_{\Delta t_i}+\frac{\sqrt{\Delta t_j}}{\sigma_{ij}}\rho_{\Delta t_j}\end{aligned} \tag{55}$$

This error terms is independent of the strike K, (Vanmaele et al., 2006) derives the next bound by making the error term dependent on K, by choosing a specific constant d_Λ such that $\Lambda \geqslant d_\Lambda$ implies that $\mathbb{S} \geqslant nK$ This algorithm was first presented by (L. C. G. Rogers & Shi, 1995), (Vanmaele et al., 2006) generalizing the approach to any normally distributed conditioning random variable and not just on the geometric average. Given the two assumptions:

1. Λ is a normally distributed conditioning variable such that $(\sigma_{(\Delta ti)} B(\Delta ti), \Lambda)$ are bivariate normally distributed for all i.
2. there exists a $d_\Lambda \in \mathbb{R}$ such that $\Lambda \geqslant d_\Lambda$ implies that $\mathbb{S} \geqslant nK$

Then an upper bound of the option price AC is given by the

$$\mathrm{UB}\Lambda_d = \mathrm{LB}\Lambda + \frac{e^{-rT}}{n} \epsilon\left(d_\Lambda\right) \tag{56}$$

with the error bound:

$$\begin{aligned} \epsilon\left(d_\Lambda\right) = \frac{S(0)}{2} \left\{\Phi\left(d_\Lambda^*\right)\right\}^{\frac{1}{2}} \times & \left[\sum_{j=0}^{n-1} \sum_{i=0}^{n-1} e^{r(2T - t_i - t_j) + \sigma(\Delta t_i)\rho_{\Delta t_i}\sigma(\Delta t_j)\rho_{\Delta t_j}\sqrt{\Delta t_i}\sqrt{\Delta t_j}} \right. \\ & \times \Phi\left(d_\Lambda^* - \left(\sigma\left(\Delta t_i\right)\rho_{\Delta t_i}\sqrt{\Delta t_i} + \sigma\left(\Delta t_j\right)\rho_{\Delta t_j}\sqrt{\Delta t_j}\right)\right) \\ & \left. \times \left(e^{\sigma(\Delta t_i)\sigma(\Delta t_j)\left(\min(\Delta t_i, \Delta t_j) - \rho_{\Delta t_i}\rho_{\Delta t_j}\sqrt{\Delta t_i}\sqrt{\Delta t_j}\right)} - 1\right)\right]^{\frac{1}{2}} \end{aligned} \tag{57}$$

with $d_\Lambda^* = \left(d_\Lambda - E^Q[\Lambda]\right)/\sigma_\Lambda, \Phi(\cdot)$ the standard normal cdf

and $\rho_{\Delta t_i} = \mathrm{corr}(\sigma_{(\Delta t_i)} B(\Delta t_i), \Lambda) \geqslant 0$.

d_{GA}^* being equal to

$$d_{GA} = \frac{n \ln \frac{K}{S(0)} - \sum_{i=0}^{n-1} \left(r_{(\Delta t_i)} - \frac{\sigma^2(t_i)}{2}\right)(\Delta t_i)}{\sigma(T) \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \min\left(\Delta t_i, \Delta t_j\right)} \tag{58}$$

d_{FA}^* being equal to

$$d_{\mathrm{FA}} = \frac{nK - \sum_{i=0}^{n-1} S(0) \mathrm{e}^{\left(r_{(\Delta t_i)} - \frac{\sigma(\Delta t_i)^2}{2}\right)(\Delta t_i)}}{S(0)\sigma(T)\sqrt{\sigma_\Lambda^2}} \tag{59}$$

In both case the error bound now takes into account the strike price. The newly defined error bound helps strengthen the error bound and creates sharper upper bound.

4.5 Partially exact/comonotonic upper bound

Starting from the work in (Curran, 1994), (Vanmaele et al., 2006) derives the partially/exact comonotonic upper bound. The method consists of deriving an improved comonotonic bounds by conditioning on some normally distributed random variable and to also decompose the calculations into exact and approximate parts. For a given conditioning random variable Λ the partially/exact comonotonic upper bound will be a more sharper upper bound than it's related improved comonotonic upper bound.

$$\begin{aligned}
\text{PECUB}\,\Lambda &= \frac{S(0)}{n}\sum_{i=0}^{n-1} e^{-r_{(\Delta ti)}t_i}\Phi\left(\rho_{\Delta t_i}\sigma(\Delta t_i)\sqrt{\Delta t_i} - d_\Lambda^*\right) - e^{-r_{(T)}T}K\Phi(-d_\Lambda^*) \\
&+ \frac{S(0)}{n}\sum_{i=0}^{n-1} e^{-r_{(\Delta t_i)}t_i}e^{-\frac{-\sigma^2(\Delta t_i)}{2}\rho_{\Delta t_i}\Delta t_i} \\
&\times \int_0^{\Phi(d_\Lambda^*)} e^{\rho_{\Delta t_i}\sigma(\Delta t_i)\sqrt{\Delta t_i}\Phi^{-1}(v)}\Phi\left(\sqrt{1-\rho_{\Delta t_i}^2}\sigma(\Delta t_i)\sqrt{\Delta t_i} - \Phi^{-1}\left(F_{\mathcal{S}^u|V=v}(nK)\right)\right)dv \\
&- e^{-r_{(T)}T}K\left(\Phi(d_\Lambda^*) - \int_0^{\Phi(d_\Lambda^*)} F_{\mathcal{S}^u|V=v}(nK)dv\right).
\end{aligned} \tag{60}$$

4.6 Adaptation of the bounds to a time deterministic model

The bounds described in this thesis were derived from the Black Scholes models, which assumes constant volatility and interest rates. However, since the inception of the Black Scholes model, several studies have shown that these assumptions do not hold in the real economy. Halfway through a real stochastic process, we extend the Black Scholes model to time deterministic volatility and interest rates based on stochastic processes. In order to create a coherent comparison with the hybrid Heston model detailled in the next section, we decide to model both the volatility and the interest rates using a Cox-Ingersoll-Ross process (CIR).

Characteristics of the CIR model The CIR model was developed initially in 1985, by (Cox, Ingersoll, & Ross, 1985) as a spin-off of the Vasicek Interest rate model. One of the main drawbacks of the Vasicek stochastic model for interest rates being that rates can become negative. The CIR correct this by introducing a square root element into the stochastic equation:

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dW(t) \tag{61}$$

With κ, θ, σ being positive constants. κ is the speed of mean reversion of the process to its long term mean; θ its long term mean, and σ the volatility of the process, *W(t)* a Brownian motion. The expectation and variance of the process are widely known():

$$\begin{cases} \mathbb{E}[r_t] = e^{-\kappa t} r_0 + \theta \left(1 - e^{-\kappa t}\right), \\ \mathrm{Var}[r_t] = \frac{\sigma^2}{\kappa} r_0 \left(e^{-\kappa t} - e^{-2\kappa t}\right) + \frac{\theta\sigma^2}{2\kappa} \left(1 - e^{-\kappa t}\right)^2. \end{cases} \tag{62}$$

the same applies for the model of the volatility:

$$\begin{cases} \mathbb{E}[\sigma_t] = e^{-\kappa t} v_0 + \theta \left(1 - e^{-\kappa t}\right), \\ \mathrm{Var}[\sigma_t] = \frac{\sigma^2}{\kappa} \sigma_0 \left(e^{-\kappa t} - e^{-2\kappa t}\right) + \frac{\theta\sigma^2}{2\kappa} \left(1 - e^{-\kappa t}\right)^2. \end{cases} \tag{63}$$

We extend the Black Scholes model to a time-varying model on the volatility and the interest rates by using the time-average value of the expected value of the stochastic process. Therefore for the volatility:

$$\sigma(t) = \frac{1}{t} \int_0^t \mathbb{E}\left[\sigma_s\right] ds \tag{64}$$

for the interest rates:

$$r(t) = \frac{1}{t} \int_0^t \mathbb{E}\left[r_s\right] ds \tag{65}$$

The value of $E[r_s]$, $E[\sigma_s]$, shows us that the stochastic volatility doesn't play any role in this model. Instead, the volatility and interest rates are functions of the speed of reversion, the initial level of volatility or interest rates, and the level of the long term mean. In our model, if the value at 0 (v_0, r_0) is equal to its long term mean, the system reverts back to a constant volatility/interest rate model: κ is the variable that will dictate dictating the rate of reversion, and thus the changes in the time-deterministic parameter over time. For value at $0 = 0.5$, $\theta = 0.02$ and $\kappa = [0.1, \text{ or } 0.9]$, T = 1. We remark that going from 0.1 to 0.9, the difference in rate at which the process reverts at the long-term mean is significant. Analysing the robustness of the bounds by varying the mean reversion rate seems to be the more impactful.

Adaptation of the Monte-Carlo simulation :

Given our hybrid deterministic model, the price process in the Monte Carlo simulation

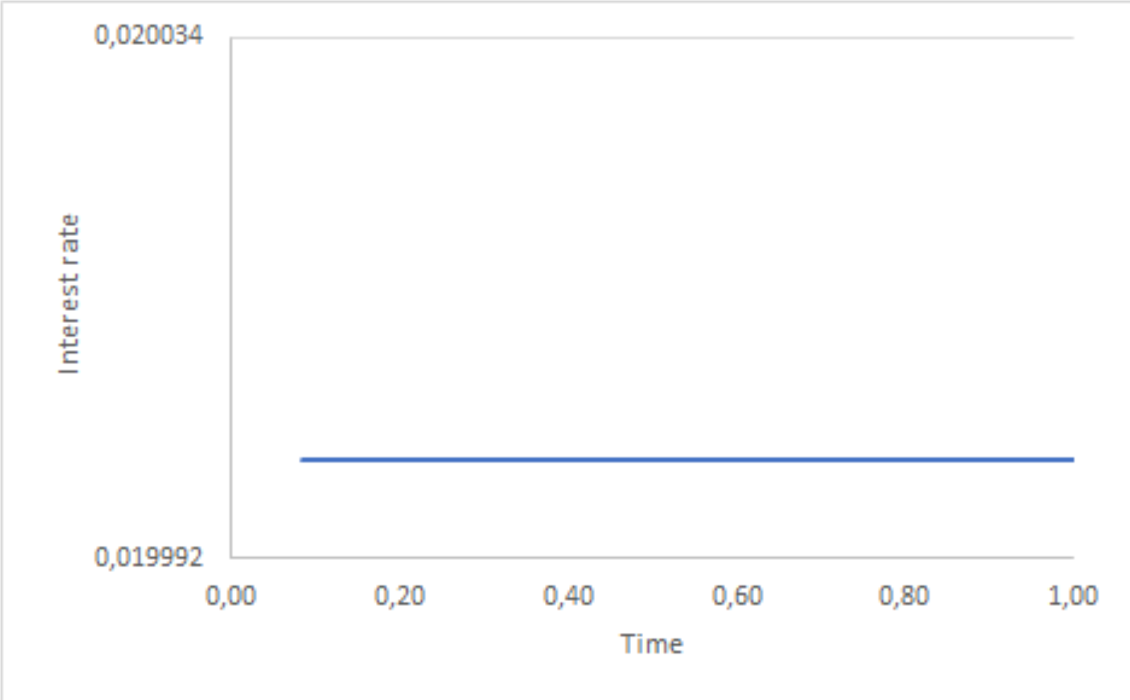

Figure 3: Time deterministic process with θ = value at 0

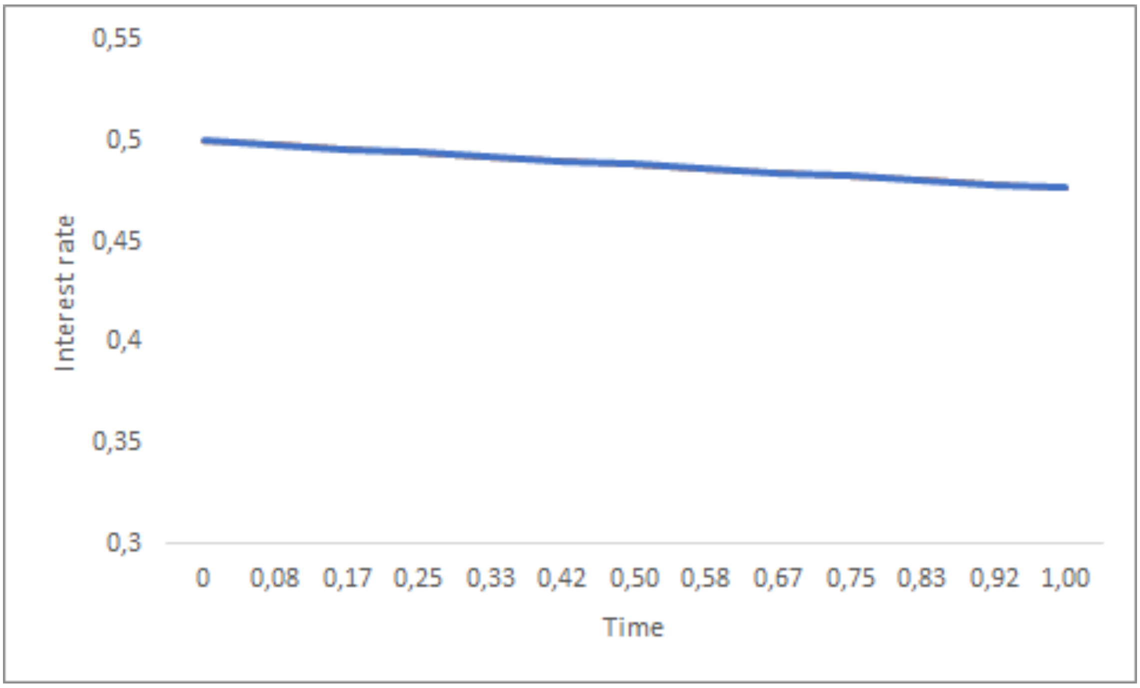

Figure 4: Time deterministic process $\kappa = 0.1$

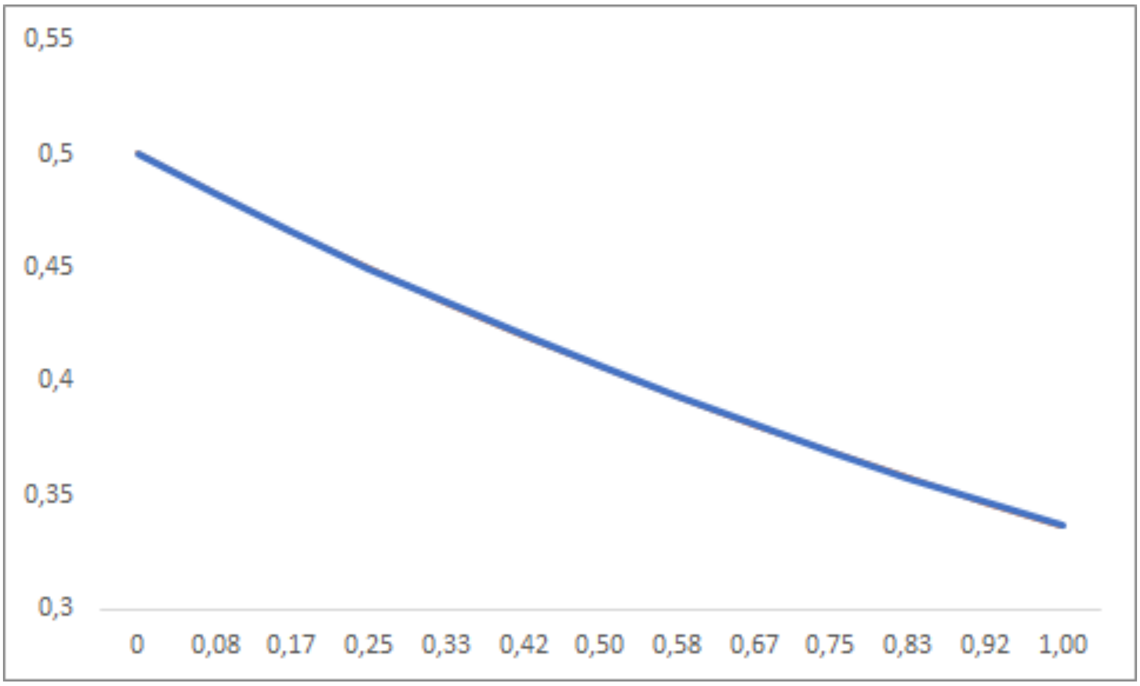

Figure 5: Time deterministic process $\kappa = 0.9$

must be slightly adapted. The discretization of the price becomes:

$$S_{t+\delta} = S_t e^{\left(\hat{r(t)} - \frac{\hat{\sigma}^2(t)}{2}\right) + \sqrt{\hat{\sigma}(t)}Z}$$

with

$$\hat{r(t)} = \frac{1}{\delta} \int_t^{t+\delta} \mathbb{E}\left[r_s\right] ds \tag{66}$$

and

$$\hat{\sigma(t)} = \frac{1}{\delta} \int_t^{t+\delta} \mathbb{E}\left[\sigma_s\right] ds \tag{67}$$

$Z \backsim \mathcal{N}(0,1), \delta = t_{i+1} - t_i$ and $\mathbb{E}[r_s]$, $\mathbb{E}[\sigma_s]$ given by (59), and (60).

5 Comonotonic model independent bounds

In this section, we review the model independent bounds developped by (Albrecher et al., 2008) (for the lower bounds) and (J. Chen & Christian-Oliver, 2014) (for the upper bound). The bounds derived are model independent in the fact that they are supposed to work for any models satisfying the two following conditions:

- Be an arbitrage-free model
- Having access to closed-form formula for European options. As the bounds are achieved through sub and super-replicating strategies involving buying an optimal

number of European options.

We then apply their methodology to the hybrid Heston model introduced by (Grzelak & Oosterlee, 2011) and reviewed by (Recchioni & Sun, 2016) . Finally, we specifiy the Monte Carlo simulation of the hybrid Heston model, with different control variates as the one previously shown: the complexity of the model not allowing for a closed form solution of the geometric average Asian option, we choose the average of European options summed over the averaging times of the Asian option.

5.1 Model Independent lower bounds

We first look at the derivation of lower bounds made by (Albrecher et al., 2008). Please note that although they derive three lower bounds, for our analysis, we will only consider their first two. The third bounds would potentially be the subject of another paper.

(Albrecher et al., 2008) starts from the methods developed by (Curran, 1994), and (L. C. G. Rogers & Shi, 1995)to derive lower bounds in the Black Scholes model which used Jensen's inequality to define a lower bond on the sum of random variables S:

$$\sum_{i=1}^{n} S_{t_i} \geq_{cx} \sum_{i=1}^{n} \mathbb{E}\left[S_{t_i} \mid Z\right] \tag{68}$$

With Z an arbitrary random variable and $\geq_{cx}$ the convex ordering relation. Using the concept of comonotonicity of a random vector, (Albrecher et al., 2008) chooses the random variable $Z = S_1$ for their first bound. We thus have a first lower approximation given the conditional variable:

$$\sum_{i=1}^{n} \mathbb{E}\left[S_i \mid S_1\right] = \sum_{i=1}^{n} S_1 e^{r(t_i - t_1)} := S^l.$$

Since $e^{r(t_i-t_1)}S_1$ is a non decreasing function of S_1 for every i,
the random vector $\left(S_1, e^{r(t_2-t_1)}S_1, \ldots, e^{r(t_n-t_1)}S_1\right)$ is comonotone, and standard comonotonicity theory can be applied:

$$\mathbb{E}\left[\left(\frac{1}{n}S^l - K\right)^+\right] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[\left(e^{r(t_i-t_1)}S_1 - F^{-1}_{e^{r(t_i-t_1)}S_1}\left(F_{S^l}(nK)\right)\right)^+\right]. \tag{69}$$

Deriving the expression further, the lower bound with conditionning variable S_1 can be modelled as the following:

$$\begin{aligned} \mathrm{AC}(K,n) &\geq \mathbb{E}\left[e^{-rT}\frac{1}{n}\left(S^{l}-nK\right)^{+}\right] \\ &= \frac{1}{n}\mathrm{C}\left(\frac{nK}{\sum_{j=1}^{n}e^{r(t_j-t_1)}},t_1\right)\sum_{i=1}^{n}e^{-r(T-t_i)} =: \mathrm{LB}_1, \end{aligned} \tag{70}$$

With $\mathrm{C}\left(K,t_1\right)$ the price of a European call option at time 0 with strike $K$and maturity t_1. This first naive lower bound implies a simple sub-replicating trading strategy.

S_1 as the conditionning variable does not necessarily leads to the optimal lower bound. Results from (70) can be generalized to the other prices at time t_i, for each t_i, a corresponding lower bound can be computed:

$$\mathrm{AC}(K,n) \geq \frac{e^{-rT}}{n}\mathrm{C}\left(\tilde{c}_t^{(1)},t\right)\sum_{i=j(t)}^{n}e^{rt_i} =: \mathrm{LB}_t^{(1)}, \tag{71}$$

With $\tilde{c}_t^{(1)}$ the optimal choice for c given t_i:

$$\tilde{c}_t^{(1)} = \frac{nK-\sum_{i=1}^{j(t)-1}e^{rt_i}S_0}{\sum_{i=j(t)}^{n}e^{r(t_i-t)}}. \tag{72}$$

From (Albrecher et al., 2008) , the optimal lower bound, will be the maximum of the $\mathrm{LB}_t^{(1)}$ bounds.

A second type of lower bounds were derived by (Albrecher et al., 2008), it is given below for information but is not part of the analysis of this thesis. Further work on the subject could include them:

$$\begin{aligned} \mathrm{AC}(K,n) &\geq \max_{0\leq t\leq T}\mathrm{LB}_t^{(2)} \\ &= \frac{e^{-rT}}{n}\max_{0\leq t\leq T}\left(\sum_{i=1}^{j(t)-1}\mathbb{E}\left[S_0\left(\left(\frac{S_t}{S_0}\right)^{t_i/t}-\left(\frac{\tilde{c}_t^{(2)}}{S_0}\right)^{t_i/t}\right)^{+}\right]+\sum_{i=j(t)}^{n}e^{rt_i}\mathrm{C}\left(\tilde{c}_t^{(2)},t\right)\right), \end{aligned} \tag{73}$$

with $\tilde{c}_t^{(2)}$, which solves the following equation:

$$nK-\sum_{i=1}^{j(t)-1}S_0\left(\frac{\tilde{c}_t^{(2)}}{S_0}\right)^{t_i/t}-\tilde{c}_t^{(2)}\sum_{i=j(t)}^{n}e^{r(t_i-t)}=0 \tag{74}$$

5.2 Model independent upper bound

To derive an model independent bound on Asian option, (J. Chen & Christian-Oliver, 2014) follow and generalize the approach previously outlined by (Albrecher et al., 2005) in for Levy type jump models.

Expressing the value of an Asian option at time t:

$$AC_t = \frac{e^{(-r(T-t))}}{n}\mathbb{E}^{\mathbb{Q}}\left[\left(\sum_{i=1}^{n} S_{t_i} - nK\right)^{+} \mid \mathcal{F}_t\right] \tag{75}$$

And using the convexity of the payoff function of the Asian option, the following inequality holds for any $K_1, K_2, \cdots, K_n$ with $K = \sum_{i=1}^{n} K_k$

$$\begin{aligned}\left(\sum_{i=1}^{n} S_{t_i} - nK\right)^{+} &= ((S_{t_1} - nK_1) + (S_{t_2} - nK_2) + \cdots + (S_{t_n} - nK_n))^{+} \\ &\leq \sum_{i=1}^{n} (S_{t_i} - nK_i)^{+}\end{aligned} \tag{76}$$

From this inequality, the expression of the upper bound at time 0 is:

$$\begin{aligned}AC_0(K,T) &= \frac{e^{(-rT)}}{n}\mathbb{E}^{\mathbb{Q}}\left[\left(\sum_{i=1}^{n} S_{t_i} - nK\right)^{+} \mid \mathcal{F}_0\right] \\ &\leq \frac{e^{(-rT)}}{n}\sum_{i=1}^{n}\mathbb{E}^{\mathbb{Q}}\left[(S_{t_i} - nK_i)^{+} \mid \mathcal{F}_0\right] \\ &= \frac{e^{(-rT)}}{n}\sum_{i=1}^{n} e^{(rt_i)} C_0\left(\kappa_i, t_i\right)\end{aligned} \tag{77}$$

With $C_0(k_i, t_i)$ the price of an European call option at time 0, with maturity t_i and strike κ_i. This expression holds for any combination of κ_i ¿ 0 such that $\sum_{i=1}^{n} \kappa_i = nK$. Expression (77) is an upper bound of the Asian option, using a super-replicating strategy. Similarly to the work done on the model independent lower bounds, this general upper bound must be optimized. This is done by finding the right combination of κ_i. Using stop-loss transform and comonotonic theory, (J. Chen & Christian-Oliver, 2014) finds a solution for the optimisation of the κ_i combination:

$$\kappa_i = F^{-1}\left(F_{S^c}(nK); t_i\right), \quad k = 1, \cdots, n \tag{78}$$

With where $F\left(x_k; t_k\right)$, the conditional distribution of S_{t_i} with respect to the σ-algebra of initial information $\mathcal{F}_0$ under the risk-neutral measure $\mathbb{Q}$ and F_{S^c} follows:

$$F_{S^c}^{-1}(x) = \sum_{i=1}^{n} F_{x_i}^{-1}(x), \quad x \geq 0 \tag{79}$$

with $F_{x_k}^{-1}(x)$ the inverse distribution function of $F(x_k; t_k)$ with respect to the argument x_i

(J. Chen & Christian-Oliver, 2014) then apply the method to three stochastic models: Heston, CEV, and Schwartz two-factor. To derive this upper bound, an explicit expression of the distribution function of the price S_t is needed. This is a piece of information that is lacking in the hybrid Heston model. Thus we followed (J. Chen & Christian-Oliver, 2014)'s proposed algorithm to obtain the distribution function via Monte Carlo simulation (see Annex).

5.3 The hybrid Heston model

(Recchioni & Sun, 2016) present in their paper a variation of the hybrid Heston model illustrated first by (Grzelak & Oosterlee, 2011). One advantage of this adaptation is to be analytically tractable, and the price of European call option, one of our pre-requisite to apply the independent bounds model, is derived analytically.

The relaxation of the assumption of constant volatility starts and incorporation of stochastic volatility into models starts around the end of the 80s and several models are proposed during that period to incorporate this variable into models: such as the Hull and White, Stein-Stein, Heston, Ball and Roma models. Among other, the Heston model allows for closed-form formulas for option pricing and can describe asset behavior accurately in conditions where the assumption of constant interest rates is realistic.

However, in the last 20 years, studies have been made on the relaxation of the assumption of constant interest rates. (Chiarella & Kwon, 2003), (Trolle & Schwartz, 2009), (Andersen & Piterbarg, 2007), all arrive at the conclusion that stochastic interest rates should be incorporated to models to better capture the bond yield behavior.

In view of the above, we decided to add that complexity to the model that would be introduced to the model-independent techniques described in the previous sub-sections. Those hybrid models are still quite recent, and extensive literature can be quite difficult to find. The choice therefore has two objectives: further the literature on the subject, and to try to apply the methods to a good realistic model.

Given the need for a computable European option, (Recchioni & Sun, 2016)'s model seemed a good fit to reach those objectives.

The hybrid Heston SDE model The model incorporates both stochastic volatility and interest rates using CIR processes. It generalizes the Heston model within a framework of stochastic interest rates. (Recchioni & Sun, 2016)'s variation of (Grzelak & Oosterlee, 2011) has the following stochastic differential equations:

$$
\begin{aligned}
dS_t &= S_t r_t dt + S_t \sqrt{v_t} dW_t^{p,v} + S_t \Delta \sqrt{v_t} dW_t^v + S_t \Omega_t \sqrt{r_t} dW_t^{p,r}, t > 0, \\
dv_t &= \chi \left(v^* - v_t\right) dt + \gamma \sqrt{v_t} dW_t^v, \quad t > 0, \\
dr_t &= \lambda \left(\theta - r_t\right) dt + \eta \sqrt{r_t} dW_t^r, \quad t > 0
\end{aligned} \tag{80}
$$

Δ is a positive constant, Ω_t is a positive function and the constants θ, v^*, are the long term means of the interest rate and volatility process respectively; λ, χ, their mean reversion speed, and γ, η their volatility.

$W_t^{p,v}, W_t^{p,r}, W_t^v, W_t^r$ are standard Wiener processes with correlation structure as follows:

$$
\begin{aligned}
E\left(dW_t^{p,v} dW_t^v\right) &= \rho_{p,v} dt, t > 0, \\
E\left(dW_t^{p,v} dW_t^r\right) &= 0, t > 0, \\
E\left(dW_t^{p,r} dW_t^r\right) &= \rho_{p,r} dt, t > 0, \\
E\left(dW_t^{p,r} dW_t^v\right) &= 0, t > 0, \\
E\left(dW_t^r dW_t^v\right) &= 0, t > 0,
\end{aligned} \tag{81}
$$

$W_t^{p,r}$ is introduced by (Recchioni & Sun, 2016), to introduce direct correlation between the price process and the interest rate.

Monte Carlo Simulation Taking the log-price of the price process $x_t = \ln\left(S_t/S_0\right), t > 0$, and assuming that Ω_t is a positive constant we can specifiy the discretization of the processes for our Monte Carlo simulation:

$$
\begin{aligned}
x_{t+\delta} &= x_t * e^{\left[r_t - \frac{1}{2}\left(\tilde{\psi} v_t + \Omega^2 r_t\right)\right]\delta + \sqrt{v_t}\sqrt{\delta} Z_t^{p,v} + \Delta\sqrt{v_t}\sqrt{\delta} Z_t^v + \Omega\sqrt{r_t}\sqrt{\delta} Z_t^{p,r}}, t > 0, \\
v_{t+\delta} &= v_t + \chi\left(v^* - v_t\right)\delta + \gamma\sqrt{v_t}\sqrt{\delta} Z_t^v, \qquad t > 0, \\
r_{t+\delta} &= r_t + \lambda\left(\theta - r_t\right)\delta + \eta\sqrt{r_t}\sqrt{\delta} Z_t^r, \qquad t > 0,
\end{aligned} \tag{82}
$$

We keep using the control variate technique for variance reduction. However, no closed-formula are available for goemetric Asian option. Taking full advantage of the derivation of (Recchioni & Sun, 2016) on the European Option, we follow (Vrins, 2020), and choose as control variable:

$$C = \frac{1}{n} \sum_{i=1}^{n} \left(S_{t_i} - K \right)^+ e^{-r t_i} \tag{83}$$

The formula of the European call option derived by (Recchioni & Sun, 2016) :

$$\begin{aligned}
C_A \left(S_0, T, E, r_0, v_0 \right) = e^{-r_0 \frac{T}{(1+e^{\lambda T})}} \frac{S_0}{2\pi} \int_{-\infty}^{+\infty} dk \frac{\left(\frac{S_0}{E}\right)^{(1-\imath k)}}{-k^2 - 3\imath k + 2} e^{-2\lambda\theta(\zeta_{q,r}+\mu_{q,r})\tau/\eta^2} \\
\left\{ e^{-\left(2\chi v^*/\gamma^2\right) \ln\left(s_{q,v,b}/(2\zeta_{q,v})\right)} e^{-\left(2v_0/\gamma^2\right)\left(\zeta_{q,v}^2 - \mu_{q,v}^2\right) s_{q,v,g}/s_{q,v,b}} e^{-2\chi v^*(\zeta_{q,v}+\mu_{q,v})T/\gamma^2} \right\} \cdot \\
e^{-\left(2\lambda\theta/\eta^2\right) \ln\left(s_{q,r,b}/(2\zeta_{q,r})\right)} e^{-\left(2r_0/\eta^2\right)\left(\zeta_{q,r}^2 - \mu_{q,r}^2\right) s_{q,r,g}/s_{q,r,b}} \left(\frac{M_{q,r}}{M_{q,r} + \frac{T e^{\lambda T}}{(1+e^{\lambda T})}} \right)^{\nu_r + 1} e^{-\left(\frac{T e^{\lambda T}}{(1+e^{\lambda T})}\right)\left(\frac{M_{q,r} \tilde{r}_q}{M_{q,r} + \frac{T e^{\lambda T}}{(1+e^{\lambda T})}}\right)} \\
S_0, T, E, r_0, v_0, \quad q = 2,
\end{aligned} \tag{84}$$

The list of explanation of the term is presented in the annex.

6 Numerical Results

In this section, we look at the robustness of the bounds in the hybrid models. For the bounds in the Black Scholes extended model, we test for the impact of the process intensity by making the process λ vary from three possible values [0, 0.45, 0.9] for the interest rate and the volatility. The values correspond to a non-time dependent state, a mid-level intensity of mean reversion and finally high level of mean reversion. For the bounds in the hybrid Heston model, from a base case scenario we do two variation for each variable too. The models being more complex, it is less easy to single-out one variable of the processes to drive significant changes to the bounds. Thus for each variable, we evaluate the bounds over two steps along either r or σ, each step, the target related variable (mean reversion, long term mean) are divided by ten. A lower enough value would be equivalent to non-stochastic variation of that variable.

Starting from the bounds in the extended Black-Scholes model, we can see that overall the quality of the bounds worsen with the introduction of the time-deterministic interest rates and volatility. The main driver of the decrease in quality of the bounds seems to be the interest rate. Indeed, under the no time deterministic interest rate regime, the bounds behave as intended. Furthermore, the computation of some bounds has some difficulties in outputting an approximation of the price under the high interest rate regime. This is notably the case for the PECUBs bounds, which stopped to return values after some value for K in that configuration. The UBd bounds also failed to deliver output after some values of K. Overall, the lower bounds keeps their characteristic of lower bounds of the price, despite the overall lack of accuracy as K increases. As the upper are derived from, through a correction terms, it is not surprising to see that the upper bounds are so far from the MT price. Looking at the effect of volatility, we do not see the amelioration of the approximation with the no time-deterministic volatility regime, as we could see with the interest rates. The volatility seems to not impact the quality of the bounds as much as the interest rate. The changes in quality between the low and high volatility regimes is not as drastic as the ones from the interest rates. We fail to explain the unforeseen impact of the interest rate on the quality of the bounds. Two hypothesis could give an answer to this case:

- The most simple one would be a programming error of the dynamics of the interest rate which could either impact the MC price or the price delivered by the bounds;
- The bounds are not tailored for time-varying interest rates, as possibly indicated by the fact that in Python, some bounds will not be able to compute any results for values if values of r are high enough.

We now look at the bounds for the hybrid Heston model. We first remark that the price is bounded by the bounds albeit with low accuracy. We can again see that overall the bounds increases in quality as the interest rate process losses in predominance in the model. The lower bound quality improve at each step towards low interest rates, and the three bounds perform at their best in the low regime of interest rates. Other than that, the bounds behave as they are described in their respective papers: LB1 is always lower than LBT as LBT is the maximal lower bound. However, having the price of the asian option being bounded between a [-10; +10] interval is not very efficient for market practitioner. We again have two hypothesis to explain the lack accuracy of the bounds:

- Despite our best efforts, we have failed to replicate the lowering in variance of MC simulations as we did in the Black Scholes model and its extension, the error hovering around 1. This could impact the approximation of the price significantly. However please note that the price is still bounded over large intervals. While one of the bound could be revealed as more precise after potential correction of the error, this also implies that the other one is of lesser quality.
- The second hypothesis is that by design the bounds are less precise. Indeed, recall

that the bounds are model-independent and are therefore not tailored to the testing model. As we have shown, they still bound the price but with far less accuracy than model-specific ones.

Seeing as the bounds in the hybrid Heston are also more prone to the change of interest rate regimes, one could also ponder if the bounds are not just too sensitive to this kind of model.

7 Conclusion

We have tried to look at the robustness of the bounds described in (Vanmaele et al., 2006), (Albrecher et al., 2008), and (J. Chen & Christian-Oliver, 2014) when adapted to models with stochastic interest rates and volatility. For that, we first have developed Monte Carlo simulations with different control variates per model: a geometric Asian option for the extended BLack Scholes model, and a sum of European options for the hybrid Heston. Both models uses the Cox-Ingersoll-Ross model to simulate the stochastic processes around the volatility and the interest rates. The Monte Carlo price is then used as benchmark for the evaluation of the quality of the bounds.

On a second part, we adapt the Black Scholes model to have time-deterministic volatility and interest rates in order to test for the resilience of the bounds in (Vanmaele et al., 2006). We also apply the methodologies of (Albrecher et al., 2008), and (J. Chen & Christian-Oliver, 2014) on a new model, the hybrid Heston model, to see to which extend model-independent bounds can be accurate. So far the results are mitigated, the bounds presented in (Vanmaele et al., 2006) lose much of their accuracy and properties when under a fast mean reverting interest process. We have yet to find a reason for that, our main hypothesis for the moment being i) a faulty implementation of the interest rate process ii) the non-compatibility of the bounds with time-varying interest rates. ii) is supported by the fact that several bounds' computing programs cease to even deliver results for high enough level of interest rates. Similarly, we confirm the feasibility of the model-independent lower and upper bounds, however with the caveat that they lack accuracy. Given the difficulty to lower our error to less than 1 in the Monte Carlo simulation of the hybrid Heston model, the lack of precision in our benchmark price might cause some of the accuracy's issue. However, the price is more or less located at the center of the interval between the upper and lower bounds. If the decrease in error were to lead to a new price, one of the two bounds would be revealed as far less performing than the other. As was shown with

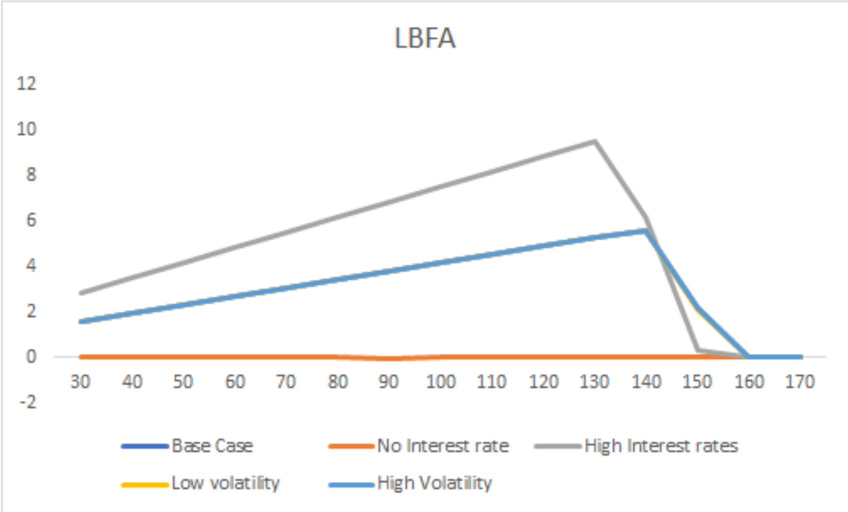

Figure 6: LB FA

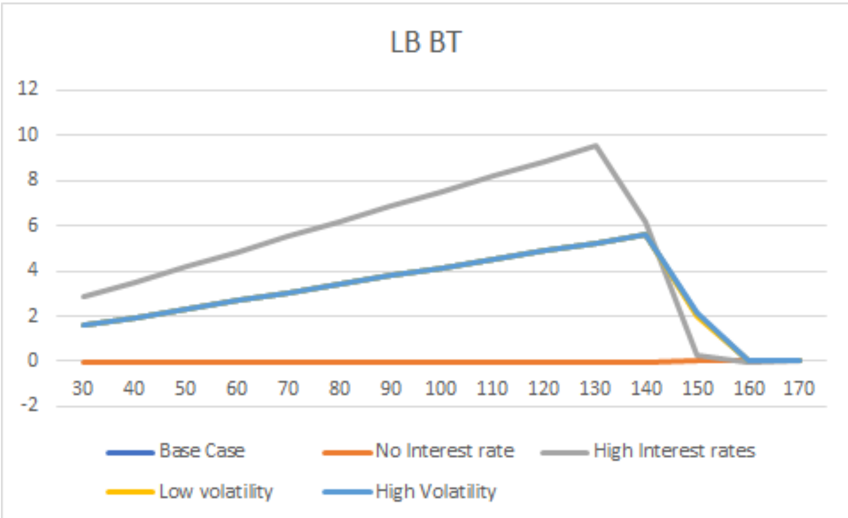

Figure 7: LB BT

the bounds in the Black Scholes extended model, the bounds far better in low stochastic interest rates settings, which again opens the discussion on the viability of those methods for varying interest rates settings. The results of this thesis opens the discussion for more researches, on the bounds described in this thesis and their applicability to interest rates models. Further work on our end would involve improving the Monte Carlo simulation in the hybrid Heston model in order to rule out the Monte Carlo price as one of the driver of the inaccuracy of both bounds. Furthermore, applying the bounds in (Vanmaele et al., 2006) to other interest rate model and compare the results with the one of this thesis in order to start forming a clear opinion on the lack of performance highlighted in this thesis.

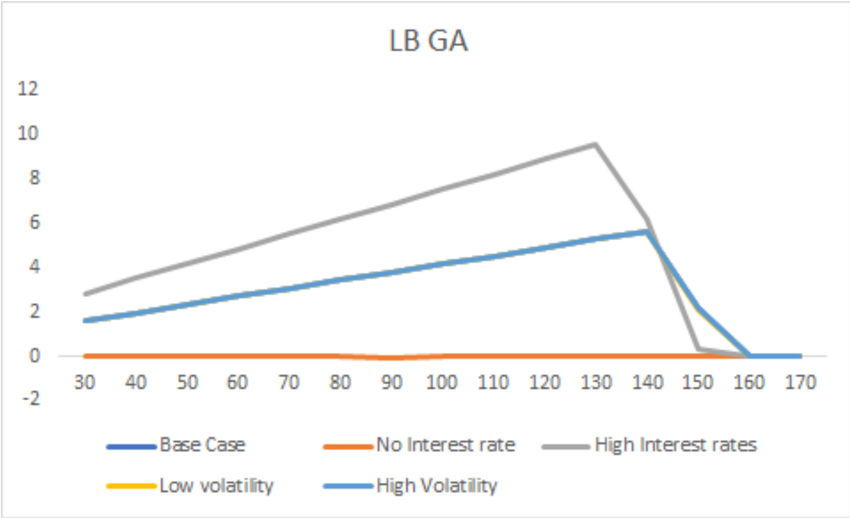

Figure 8: LB GA

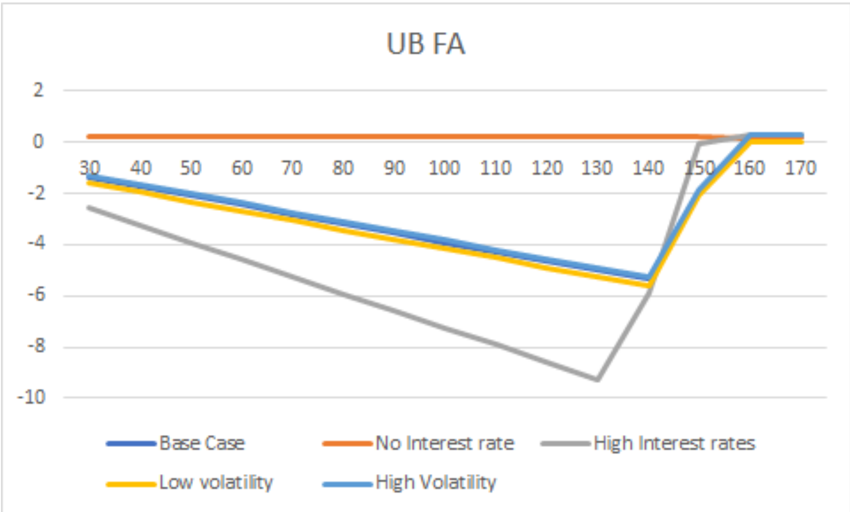

Figure 9: UB FA

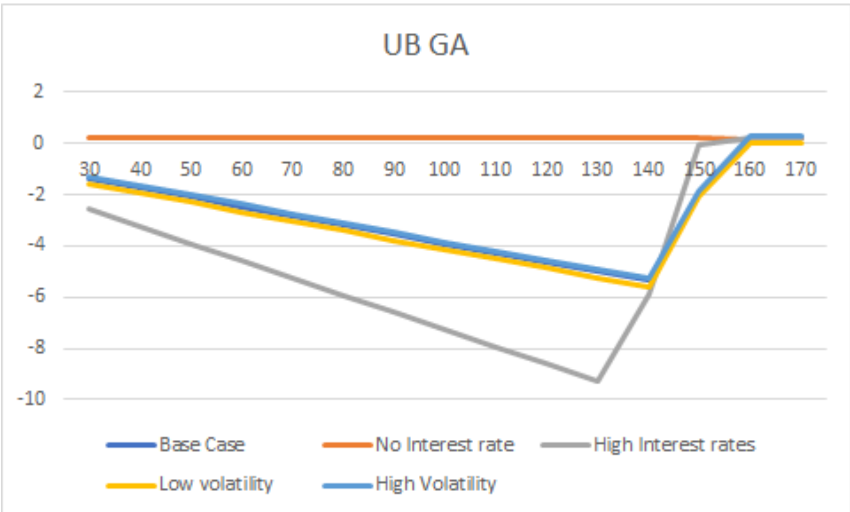

Figure 10: UB GA

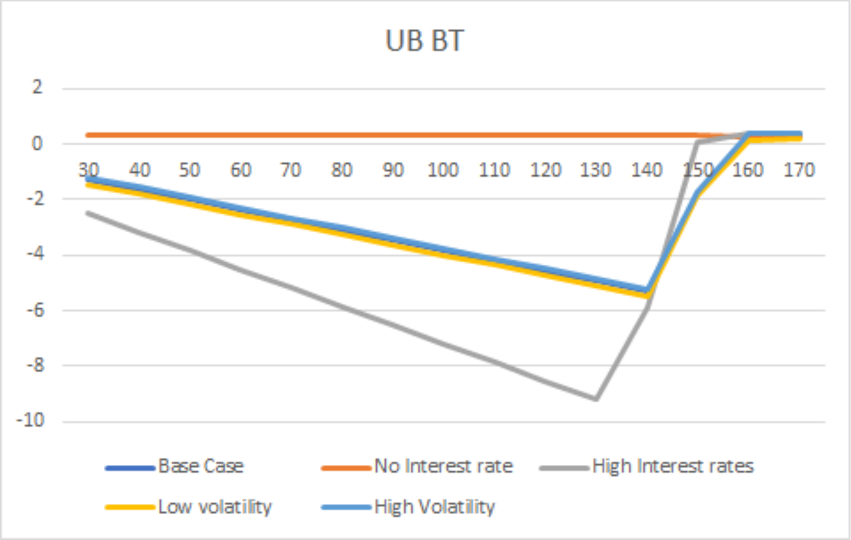

Figure 11: Enter Caption

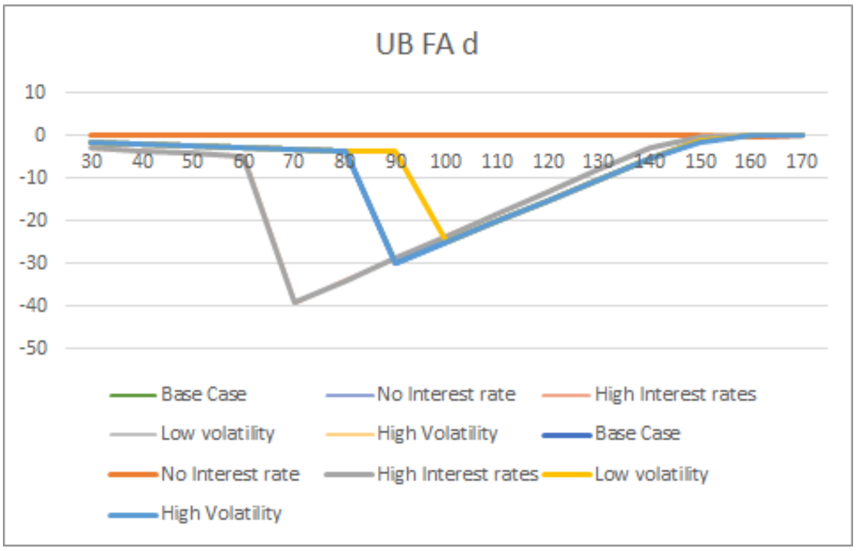

Figure 12: UB FA_d

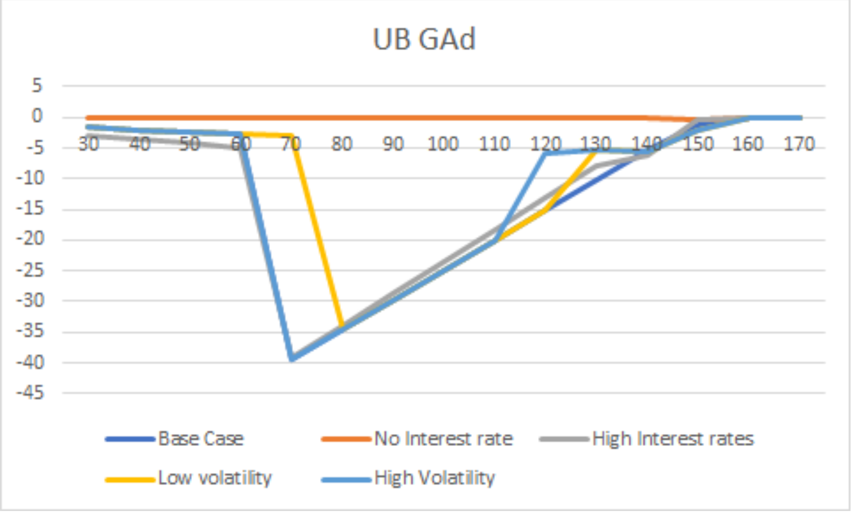

Figure 13: UB GA_d

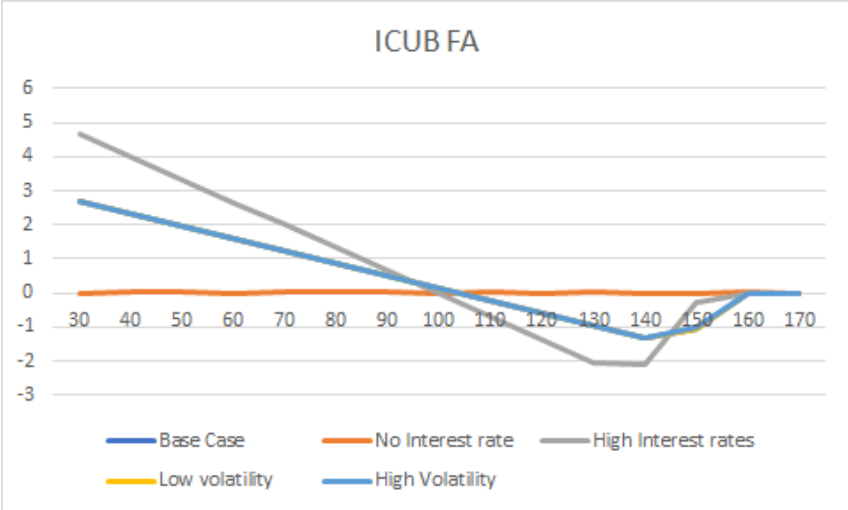

Figure 14: ICUB FA

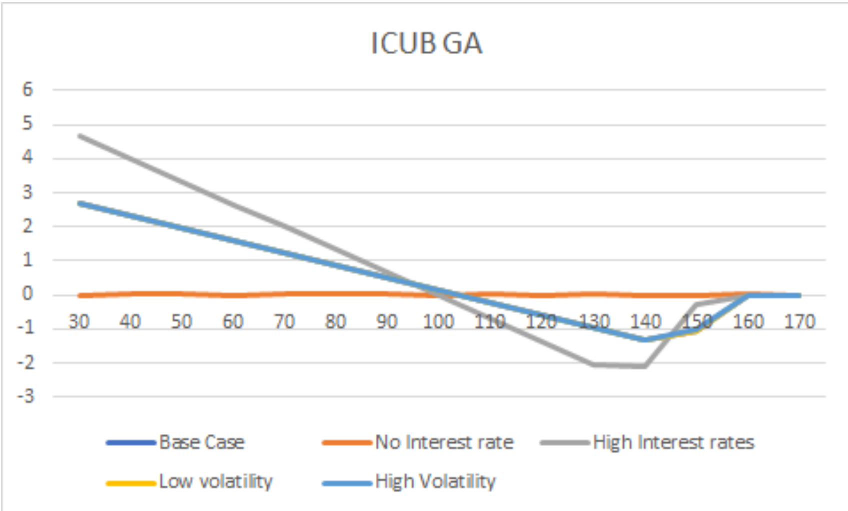

Figure 15: ICUB GA

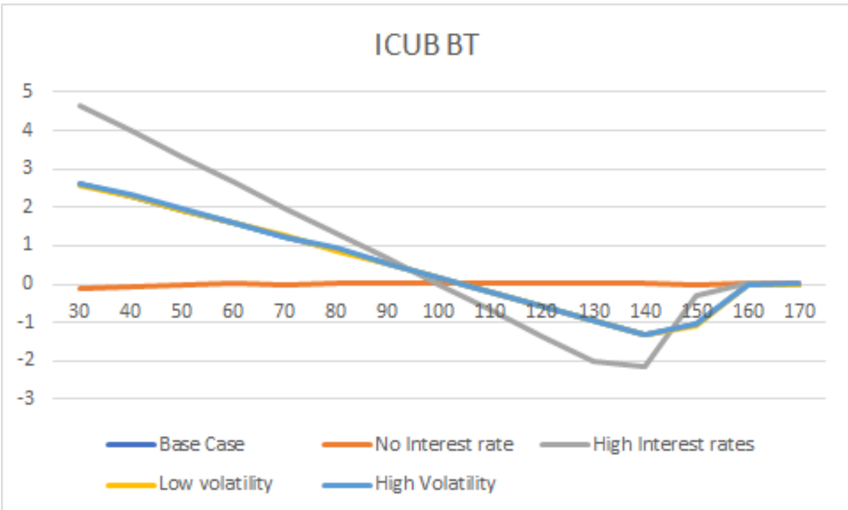

Figure 16: ICUB BT

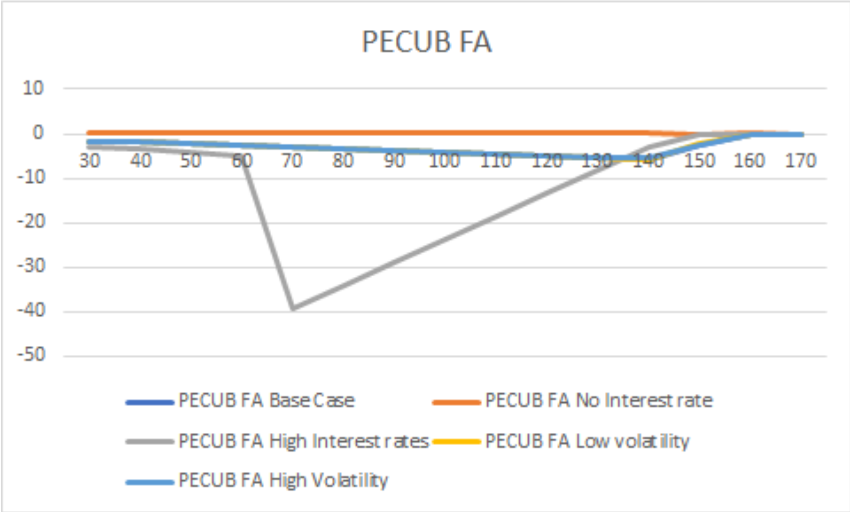

Figure 17: PECUB FA

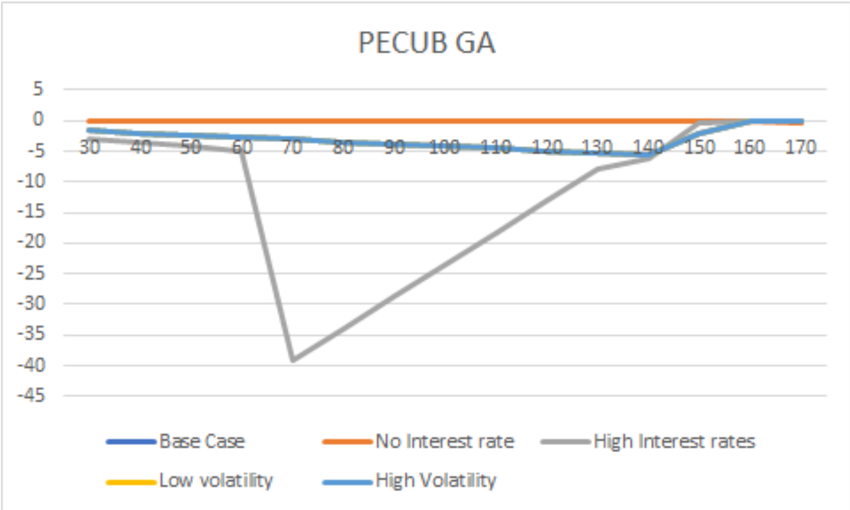

Figure 18: PECUB GA

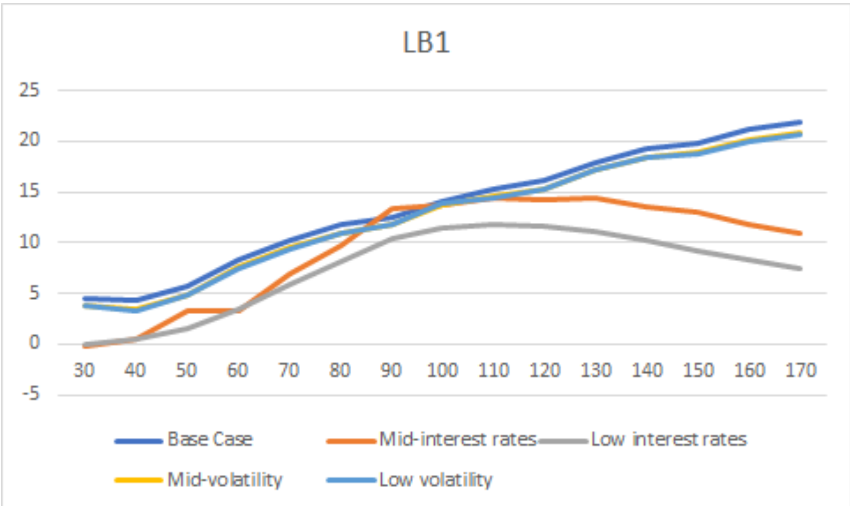

Figure 19: Model independent naive lower bound

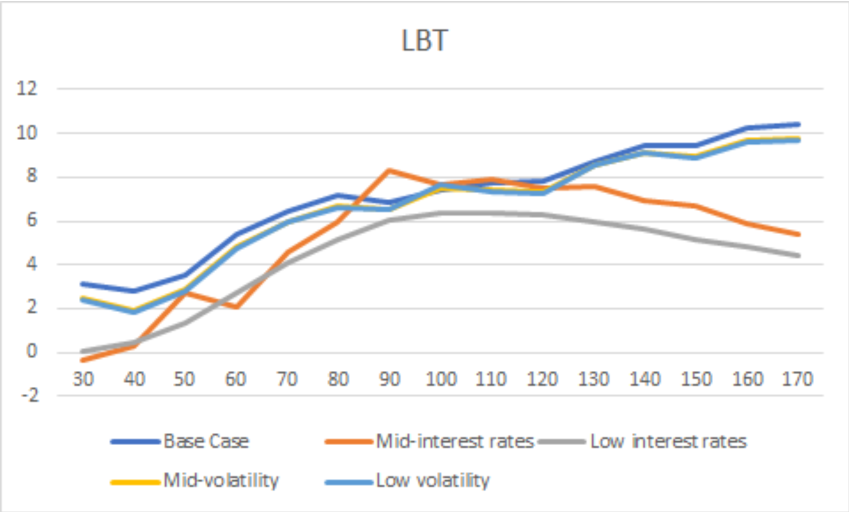

Figure 20: Model independent lower bound

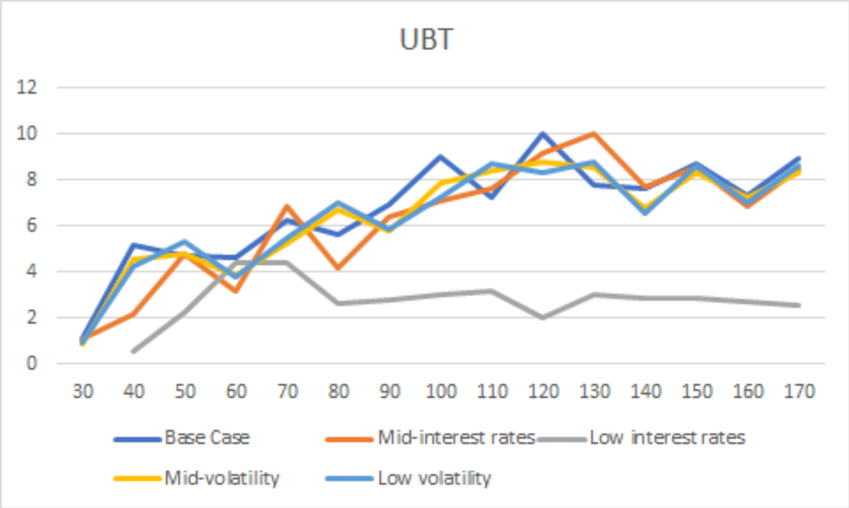

Figure 21: Model independent lower bound

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K	AC	Error	Variance	Coefficient de variation
50	49,93	4,14	44816,35	8,29
60	48,63	4,67	56807,88	9,60
70	49,41	7,22	135711,39	14,61
80	52,07	7,13	132170,72	13,68
90	44,67	4,48	52255,85	10,03
100	40,55	3,85	38702,73	9,49
110	41,71	4,45	51535,26	10,67
120	39,93	5,14	68812,95	12,88
130	41,69	5,78	86972,80	13,86
140	38,60	4,11	44168,64	10,65
150	35,84	3,49	31876,57	9,74

Figure 22: Naive Monte Carlo Heston

A Example of result table for the black scholes extended model

B Example of result table for the hybrid Heston model

C European option under the hybrid Heston model

$\zeta_{q,r}$, $\mu_{q,v}$, $\mu_{q,r}$, $s_{q,v,b}$, $s_{q,r,b}$, $s_{q,v,g}$, $s_{q,r,g}$, $\tilde{v}_q$, $\tilde{r}_q$, $M_{q,v}$ and $M_{q,r}$ are as follows :

K	AC	Error	Variance	Coefficient de variation
50	54,81	1,55	6259,99	2,83
60	52,29	1,34	4677,96	2,56
70	52,29	1,44	5391,20	2,75
80	49,19	0,99	2530,59	2,00
90	48,60	1,05	2871,04	2,16
100	47,42	1,08	2871,04	2,28
110	45,08	1,07	2956,07	2,36
120	44,64	1,32	4516,29	2,95
130	42,36	1,25	4089,90	2,96
140	41,39	1,59	6601,55	3,85
150	40,98	1,45	5495,38	3,54

Figure 23: Monte Carlo with Sum of European Options as Control Variate

Lambda R 0,9

Monte Carlo	LB FA	LB GA	LB BT	UB FA	UB GA	UB BT
59,91551204	57,07328664	57,07318928	57,07328638	57,33722238	57,33251927	57,42228976
54,72032363	51,21252952	51,2125295	51,21252921	51,47646524	51,4717624	51,56151357
49,52798236	45,35177239	45,35177239	45,35176429	45,61570814	45,61100535	45,70075516
44,33902662	39,491015	39,49101526	39,4910002	39,75495101	39,75024825	39,84001817
39,14474594	33,63025813	33,63025813	33,63025813	33,89419388	33,88949081	33,97926124
33,9538323	27,76950095	27,769501	27,769501	28,03343675	28,02873399	28,11850412
28,75947868	21,90874387	21,90874387	21,90874387	22,17267962	22,16797637	22,25774699
23,57270958	16,04798674	16,04798674	16,04798674	16,31192249	16,30721973	16,39698986
18,37783003	10,18722961	10,18722961	10,18722961	10,45116536	10,4464626	10,53623273
13,18759115	4,326472486	4,326472486	4,326472486	4,590408234	4,585705475	4,6754756
7,994102083	-1,532925507890539	-1,532949166504693	-1,5340108058962443	-1,26898975932515	-1,273716176825811	-1,185007691609504
2,929694926	-3,207048724388656	-3,2094246523233423	-3,2864786375504735	-2,943112975823274	-2,9501916626444746	-2,9374755232637404
0,282452944	-0,005454752581366039	-0,005348568944804143	-0,0016840298296283314	0,258480996	0,253884421	0,347319084
6,51E-05	-2,6367360606435266e-09	-2,420498745560651e-09	-4,460466402710721e-11	0,263935746	0,259232987	0,349003114
3,37E-20	-1,4904632268395583e-13	-1,1081049237387699e-15	-1,2521312257146163e-15	0,263935749	0,25923299	0,349003114

Monte Carlo	UB FA_d	UB GA_d	ICUB FA	ICUB GA	ICUB BT	PECUB FA_d	PECUB GA_d
59,91551204	57,07328663	57,07326982	64,58044774	64,58042539	64,55108171	57,07328663	57,07326982
54,72032363	51,21252448	51,21252952	58,71963236	58,71968228	58,70380164	51,21252448	51,21252952
49,52798236	45,35177239	45,35177238	52,85889794	52,85891041	52,8371925	45,35177239	45,35177238
44,33902662	39,49101525	39,49101524	46,99813882	46,99817118	46,97906569	39,49101525	39,49101524
39,14474594	0	0	41,13736009	41,13742573	41,13136843	0	0
33,9538323	0	0	35,27662477	35,27667346	35,27673982	0	
28,75947868	0	0	29,41588536	29,41595443	29,41584432	0	
23,57270958	0	0	23,55515808	23,55518699	23,55575173	0	
18,37783003	0	0	17,69439734	17,6944267	17,69459632	0	
13,18759115	0	0	11,8336395	11,83366688	11,83392212	0	
7,994102083	0	0	5,972884225	5,972914676	5,973155997	0	
2,929694926	0	-3,194386047843878	0,808231756	0,810596068	0,77895513	0	-3,194386047843878
0,282452944	0	0,019688269	4,95E-05	2,58E-05	-0,0038873028963962303	0	0,019688269
6,51E-05	0	0,025040749	0,000795298	8,18E-09	0,005881972	0	0,025040749
3,37E-20	0	0,025040751	-0,00067053363509346 75	2,27E-05	2,29E-08	0	0,025040751

Figure 24: Results from scenario with high mean reversion: n=12;r0= 0.8,v0=0.03, λV=0.45, θV=0.05, σV= 0.6 ,λR=0.9, θR=0.02, σR= 0.01, T = 1; S0 = 100

Scenario de base — Changement pour r — IR thêta 0,0457 — IR sigma 0,1 — r0 0,8596

K	MC	error	LB 1		LB_T	time of LBt		UB	
30	61,97637021	0,990151659	57,47039882	4,505971394	58,85358807	2	3,122782139	63,04791188	1,07154167
40	57,66382439	1,259940952	53,24573433	4,418090056	54,87003754	2	2,793786852	62,80842759	5,144603195
50	54,92896496	1,188713684	49,19761417	5,731350793	51,42145323	3	3,507511736	59,67540712	4,746442153
60	53,74604314	1,164708302	45,36755636	8,378486778	48,34807275	3	5,397970387	58,36419411	4,618150979
70	52,02476103	1,009041083	41,77956254	10,2451985	45,59326593	3	6,431495103	58,2982656	6,273504565
80	50,26364172	0,930652077	38,44342868	11,82021305	43,10965332	3	7,153988407	55,94392006	5,680278334
90	47,8437504	1,094244567	35,35866007	12,48509032	40,99685558	4	6,846894814	54,76029491	6,916544517
100	46,58218306	1,195355725	32,51785391	14,06432916	39,13535137	4	7,446831694	55,62836068	9,046177622
110	45,16615606	1,295475838	29,90929716	15,25685889	37,44327678	4	7,722879281	52,4527384	7,286582342
120	43,75672116	1,237906916	27,51884289	16,23787827	35,89582241	4	7,86089875	53,74387273	9,987151565
130	43,21532522	1,086235322	25,33121152	17,88411369	34,47322234	4	8,742102876	51,04552591	7,830200689
140	42,58490192	0,928945847	23,33086361	19,25403831	33,15945867	4	9,425443253	50,22555084	7,640648919
150	41,39566417	1,067311611	21,50256462	19,89309955	31,94136159	4	9,454302573	50,13460112	8,738936957
160	41,04779897	0,948929661	19,83173373	21,21606524	30,80796724	4	10,23983173	48,36458523	7,316786262
170	40,20339844	0,9117292	18,30464405	21,89875439	29,81858649	5	10,38481195	49,12024908	8,916850644

Changement pour r — IR thêta 0,000457 — IR sigma 0,001 — R0 0,008569

K	MC	error	LB 1		LBt	time		UB	
30	69,94511165	0,038734634	69,8715045	0,07360715	69,88686767	2	0,058243989	0	-69,94511165
40	60,47051644	0,051582056	59,95440466	0,516111782	59,98177451	3	0,48874193	61,00586675	0,535350312
50	51,60785504	0,064289844	50,06260691	1,545248125	50,24708939	3	1,360765646	53,85185974	2,2440047
60	43,78782137	0,071247238	40,33810579	3,449715586	41,07189349	3	2,715927881	52,23910251	8,45128114
70	36,95739341	0,077150741	31,14235543	5,815037974	32,88909224	4	4,068301165	51,37941792	14,42202451
80	31,21632509	0,080795905	22,96886901	8,247456082	26,06729018	4	5,14903491	33,88656179	2,670236697
90	26,55340863	0,08105546	16,20837698	10,34503164	20,52460213	4	6,028806493	53,36937859	26,81596996
100	22,44175896	0,085859764	10,99319706	11,4485619	16,11561992	4	6,326139042	25,46164229	3,019883331
110	19,02569738	0,093859949	7,20857467	11,81712271	12,65176129	4	6,373936091	22,17227162	3,146574242
120	16,26130235	0,097261217	4,597749716	11,66355264	9,984653716	5	6,276648636	51,26287182	35,00156947
130	13,94885359	0,103792495	2,868424683	11,08042891	7,961775353	5	5,987078237	16,98897755	3,040123964
140	12,04519181	0,110273126	1,759010808	10,286181	6,38287718	5	5,662314629	14,94157309	2,896381277
150	10,31232367	0,119966472	1,064678892	9,247644774	5,145258035	5	5,167065631	13,20299857	2,890674906
160	8,994659886	0,124314837	0,63824543	8,356414456	4,170365324	5	4,824294562	11,69940438	2,704744491
170	7,834667865	0,130197232	0,38001733	7,454650535	3,398379568	5	4,436288296	10,37268094	2,538013072

Figure 25: Base case scenario + Low intensity IR process

$$\mu_{q,v} = -\frac{1}{2}\left(\chi + (\imath k - q)\gamma\tilde{\rho}_{p,v}\right),$$

$$\zeta_{q,v} = \frac{1}{2}\left[4\mu_{q,v}^2 + 2\gamma^2\varphi_q(k)\tilde{\psi}\right]^{1/2},$$

$$s_{q,v,g} = 1 - e^{-2\zeta_{q,v}\tau},$$

$$s_{q,v,b} = \left(\zeta_{q,v} + \mu_{q,v}\right)e^{-2\zeta_{q,v}\tau} + \left(\zeta_{q,v} - \mu_{q,v}\right),$$

$$M_{q,v} = \frac{2}{\gamma^2}\frac{s_{q,v,b}}{s_{q,v,g}}\tilde{v}_q = \frac{4\zeta_{q,v}^2 v e^{-2\zeta_{q,v}\tau}}{s_{q,v,b}^2}M_{q,v}\tilde{v}_q = \frac{8}{\gamma^2}\frac{\zeta_{q,v}^2 v e^{-2\zeta_{q,v}\tau}}{s_{q,v,g}s_{q,v,b}},$$

$$\mu_{q,r} = -\frac{1}{2}\left(\lambda + (\imath k - q)\eta\Omega\rho_{p,r}\right),$$

$$\zeta_{q,r} = \frac{1}{2}\left[4\mu_{q,r}^2 + 2\eta^2\left(\varphi_q(k)\Omega^2 - q + \imath k\right)\right]^{1/2},$$

$$s_{q,r,g} = 1 - e^{-2\zeta_{q,r}\tau},$$

$$s_{q,r,b} = \left(\zeta_{q,r} + \mu_{q,r}\right)e^{-2\zeta_{q,r}\tau} + \left(\zeta_{q,r} - \mu_{q,r}\right),$$

$$M_{q,r} = \frac{2}{\eta^2}\frac{s_{q,r,b}}{s_{q,r,g}} \quad \tilde{r}_q = \frac{4\zeta_{q,r}^2 r e^{-2\zeta_{q,r}\tau}}{s_{q,r,b}^2}M_{q,r}\tilde{r}_q = \frac{8}{\eta^2}\frac{\zeta_{q,r}^2 r e^{-2\zeta_{q,r}\tau}}{s_{q,r,g}s_{q,r,b}}$$

$$\varphi_q(k) = \frac{k^2}{2} + \imath\frac{k}{2}(2q-1) - \frac{1}{2}\left(q^2 - q\right), k \in \mathbf{R}$$

$$\tilde{\rho}_{p,v} = \rho_{p,v} + \Delta$$

$$\tilde{\psi} := 1 + \Delta^2 + 2\Delta\rho_{p,v}$$