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Optimal Portfolio Allocation under Cumulative Prospect Theory

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Abstract

In this paper, a behavioral portfolio allocation model *à la* Cumulative Prospect Theory with loss control is numerically implemented. This optimisation problem is difficult to solve due to nonlinear objective function and nonlinear constrains. Moreover, the problem could be ill-posed. The structure of the optimal terminal wealth of this problem is a combination of three binary options whose settlements depend on the state of the market. The goal is to show the impact of each parameters (e.g. market parameters, investor's preferences parameters, initial gain or loss position) on the solution of this complex problem. Finally an application on real market data will compare the performances of the behavioral portfolio with the performances of an expected utility maximising portfolio.

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Chapter 1

Introduction

Expected Utility Theory (EUT), whose axioms have been developed by Von Neumann and Morgenstern [1], is one of the predominant models of decision making under risk. It assumes that decision makers are rational and risk averse when facing risky outcomes. It is used as decision rule in the context of portfolio selection problems. Karatzas and Shreve [2] have combined Expected Utility maximisation and martingale pricing to solve this allocation problem. First, a classical optimisation problem is solved to get the contingent claim that maximises the investor's utility. Finally, the claim is replicated by a basket of assets using martingale theory.

Nevertheless, EUT fails to describe many human behaviors : e.g. Allais [3], Ellberg [4] and economic puzzles e.g. Benartzi and Thaler [5], Sydnor [6]. In 1979, Kahneman and Tversky [7] introduced a new behavioral model: the Prospect Theory mainly based on psychological researches. In 1992 [8], they proposed an improved version: the Cumulative Prospect Theory (CPT). The main features are:

- Nonlinear decision weights: overweighting of low probabilities and underweighting of large probabilities
- Evaluating gains and losses from a reference point and not on final asset positions
- Decision makers are risk averse in the gain domain and risk seeking in the loss domain
- Loss aversion: the disappointment of losing X is greater that the satisfaction of winning X

Naturally, Cumulative Prospect Theory has been incorporated in the portfolio selection problem. Unfortunately, the new features of CPT made the optimisation problem very difficult to solve. The probability distortion can make the problem *ill-posed* in certain conditions. Then, the S-shaped value function leads to *non-convex optimisation.* In 2008, Jin and Zhou [9] proposed a behavioral portofolio selection model *à la* Cumulative Prospect Theory. They used a *Divide and Conquer* approach to reformulate the initial problem to solve it by conventional numerical methods. They showed that the optimal terminal wealth of the portfolio can be seen as a combinaison of payoffs of binary options whose settlements depend on states of the market. In certain cases, those binary options can be replicated to obtain the corresponding optimal portfolio explicitly. In 2011, Zhang [10] proposed an extension of the model to include a loss control condition which plays a crucial role when the investor suffers an initial loss position.

The main contributions of this work are:

- Implementation of a numerical solver of the CPT portfolio allocation problem (with and without loss control)
- Sensitivity analysis of the (many) parameters playing a role in this portfolio problem
- Application on the Belgium Stock Market. Comparaison between the performances of a CPT portfolio, an EUT portfolio and a benchmark (BEL20). With a particular attention on the performances during the 2020 stock market krach (COVID-19 crisis).

The rest of this work is organised as follow. First, an involved description of the weaknesses of Expected Utility Theory with various examples and the improvements of the Cumulative Prospect Theory. Second, reminders of stochastic finance and an application of martingale pricing in a EUT portfolio allocation problem. Third, the behavioral portfolio allocation problem *à la* Cumulative Prospect Theory and its numerical implementation. Finally, the application on the Belgium Stock Market.

Chapter 2

State of Art

1. Introduction

Let S denote a set of possible outcomes. For example, the set of outcomes of a football match will be:

$$S = \{\text{Team A wins, Team B wins, Draw}\} \tag{2.1}$$

For a dice roll, the set S will be:

$$S = \{1, 2, 3, 4, 5, 6\} \tag{2.2}$$

In many cases, S will be a set of monetary outcomes. As an example, consider the following game: after rolling the dice, the gambler should pay the face value if the number is odd and receive it if the number is even. The set of outcomes is thus:

$$S = \{-1, 2, -3, 4, -5, 6\} \tag{2.3}$$

A prospect (or lottery) is a contract yielding outcome x_i with probability pi. The following notation is used for the prospect:

$$(x_1, p_1; x_2, p_2; ...; x_n, p_n) \qquad \text{with} \sum_{i=1}^{n} p_i = 1, \quad p_i > 0 \quad \text{and} \, x_i \in S \tag{2.4}$$

The previous gamble is represented by the prospect:

$$(-5, \frac{1}{6}; -3, \frac{1}{6}; -1, \frac{1}{6}; 2, \frac{1}{6}; 4, \frac{1}{6}; 6, \frac{1}{6}) \tag{2.5}$$

To simplify notation:

- Null outcomes are omitted. For example the prospect $(x, p; 0, 1 - p)$ is simplified as (x, p)
- (x) denotes the riskless prospect that gives outcome x

Any prospect can be viewed as a random variable X with density function f_X and cumulative distribution function F_X. In the dice roll gamble, f_X and F_X are given by:

$$f_X = \begin{cases} -5 & \text{with} \quad p = \dfrac{1}{6} \\ -3 & \text{with} \quad p = \dfrac{1}{6} \\ -1 & \text{with} \quad p = \dfrac{1}{6} \\ 2 & \text{with} \quad p = \dfrac{1}{6} \\ 4 & \text{with} \quad p = \dfrac{1}{6} \\ 6 & \text{with} \quad p = \dfrac{1}{6} \end{cases} \qquad F_X = \begin{cases} 0 & \text{for} \quad x < -5 \\ \dfrac{1}{6} & \text{for} \quad -5 \leq x < -3 \\ \dfrac{1}{3} & \text{for} \quad -3 \leq x < -1 \\ \dfrac{1}{2} & \text{for} \quad -1 \leq x < 2 \\ \dfrac{2}{3} & \text{for} \quad 2 \leq x < 4 \\ \dfrac{5}{6} & \text{for} \quad 4 \leq x < 6 \\ 1 & \text{for} \quad x \geq 6 \end{cases} \tag{2.6}$$

2. Expected Utility (EU) Theory

2.1 Von Neumann-Morgenstern Axioms

The Expected Utility framework is based on 4 axioms called the von Neumann-Morgenstern axioms: *Completeness*, *Transitivity*, *Continuity* and *Independence*.

Axiom 1 (Completeness). *For any prospect A,B, exactly one of these holds:*

$$A \succeq B, A \sim B \, or \, A \preceq B \tag{2.7}$$

Where $A \succeq B$,$A \sim B$ and $A \preceq B$ respectfully mean that A is preferred over B, A is equivalent to B and B is preferred over A. Thus an agent can always rank a set of prospects as better, equal or worse than another.

Axiom 2 (Transitivity). *If $A \succeq B$ and $B \succeq C$ then $A \succeq C$ and similary for $\sim$ and $\preceq$.*

Transitivity assumes that an agent is able to order his preferences in a consistent way.

Axiom 3 (Continuity). *If $A \succeq B \succeq C$ then there exists a probability $p \in [0,1]$ such that:*

$$B \sim pA + (1-p)C \tag{2.8}$$

Independence of irrelevant alternatives assumes that a preference holds independently of the possibility of another outcome:

Axiom 4 (Independence). *If $A \succeq B$ then for any C and $p \in [0,1]$:*

$$pA + (1-p)C \succeq pB + (1-p)C \tag{2.9}$$

Two gambles mixed with a third one will maintain the same order of preference as when the two are presented independently of the third one.

The independence axiom leads to the axiom of reduction of compound lotteries. It is obtained by setting $A = M$, $C = N$ and $B = qX + (1-q)Y$ in the expression of the independence axiom.

Axiom 5 (Reduction of compound lotteries). *For any lotteries M, N, X, Y and for any $p, q \in [0,1]$*

$$\begin{aligned} &\textit{If} \quad M \succeq qX + (1-q)Y \\ &\textit{then} \quad pM + (1-p)N \succeq pqX + p(1-q)Y + (1-p)N \end{aligned} \tag{2.10}$$

2.2 Von Neumann-Morgenstern Theorem and utility function

The Von Neumann-Morgenstern theorem states that for any economic agent which satisfies the four axioms, the preferences can be represented in an Expected Utility form:

Theorem 1 (Von Neumann-Morgenstern). *If axioms 1-4 are satisfied, there exists a function* $u : S \to \mathbb{R}$ *such that for any prospect* A *and* B *of the form* $P : (x_1, p_1; ...; x_n, p_n)$*:*

$$A \succeq B \iff E[u(A)] \geq E[u(B)] \quad where \quad E[u(P)] = \sum_{i=1}^{n} p_i u(x_i) \tag{2.11}$$

Utility functions that satisfy theorem 1 have the following properties:

- *Non-decreasing utility.* The utility function is non-decreasing: $u' \geq 0$. This comes from the natural assumption that an agent always prefers more wealth to less wealth.
- *Linear transformation.* The utility function needs only to be determined up to linear transformations: the utility function defined by: $u^*(x) = au(x) + b$ $(a, b > 0)$ leads to the same preferences as $u(x)$:$\mathbb{E}[u(A)] \leq \mathbb{E}[u(b)] \iff \mathbb{E}[u^*(A)] \leq \mathbb{E}[u^*(b)]$.
- *Standardized utility.* From the previous property, it is thus possible to defined a standardized utility function such that: $u(x_0) = 0$ and $u'(x_0) = 1$ for some $x_0 \in \mathbb{R}$.

In the EU framework, the domain of the utility function is the final wealth position rather than the gain (or the loss) obtained from the initial wealth. Thus the prospect $(x_1, p_1; ...; x_n, p_n)$ is acceptable with initial wealth w if

$$E[u((w + x_1, p_1; ...; w + x_n, p_n))] > E[u(w)] \tag{2.12}$$

Elements of decision making under risk and an application to Constant Portfolio Allocation (continuous rebalancing of the portfolio) can be found in Appendix F.

3. Weaknesses of the Expected Utility hypothesis

The main characteristics of the Expected Utility framework are the following:

- The domain of the utility function is the final state instead of gains or losses from a reference point. Thus, when an agent has initial wealth w, the gamble X is evaluated as $u(w + X)$ instead of $u(X)$.
- The Expected Utility is linear in probabilities. The EU of the gamble X that gives outcome x_i with probability p_i is: $\mathbb{E}[u(X)] = \sum_i^n p_i \, u(w + x_i)$.
- The economic agents are risk averse. Thus the utility function is concave over the whole domain: $\dfrac{d^2u}{dx^2} < 0$.

3.1 The certainty effect and non-linear decision weights

In the EU framework, the outcomes are weighted by their probabilities. Allais [3] has shown with his famous counter-example that people tend to overweight outcomes that are certain relative to probable outcomes. Thus, *the preferences are nonlinear*: a change in probabilities from 1 to 0.99 has more impact on preferences that a change from 0.16 to 0.15.

Example of certainty effect: Allais paradox

Kahneman and Tversky [7] proposed the following alternative to the Allais version of the paradox:

$$\textbf{Problem 1} \qquad A: \begin{cases} 2500 & \text{with probability} \quad 0.33 \\ 2400 & \text{with probability} \quad 0.66 \\ 0 & \text{with probability} \quad 0.01 \end{cases} \qquad B: 2400 \quad \text{for sure} \tag{2.13}$$

$$\textbf{Problem 2} \qquad C: \begin{cases} 2500 & \text{with probability} \quad 0.33 \\ 0 & \text{with probability} \quad 0.67 \end{cases} \qquad D: \begin{cases} 2400 & \text{with probability} \quad 0.34 \\ 0 & \text{with probability} \quad 0.66 \end{cases} \tag{2.14}$$

82% of the subjects has chosen B in Problem 1 while 83% has chosen C in problem 2. Moreover 61% made the choice B and C. In fact Problem 2 is obtained from Problem 1 by eliminating a 66% chance of winning 2400. This violates the independence axioms because Problem 1 and Problem 2 should give the same preferences. Let's rewrite the two problems into a more general form:

$$\textbf{Problem 1} \qquad A: \begin{cases} X & \text{with probability} \quad p \\ P & \text{with probability} \quad 1-p \end{cases} \qquad B: \begin{cases} Y & \text{with probability} \quad p \\ P & \text{with probability} \quad 1-p \end{cases} \tag{2.15}$$

$$\textbf{Problem 2} \qquad C: \begin{cases} X & \text{with probability} \quad p \\ P' & \text{with probability} \quad 1-p \end{cases} \qquad D: \begin{cases} Y & \text{with probability} \quad p \\ P' & \text{with probability} \quad 1-p \end{cases} \tag{2.16}$$

In this form, the Independence Axiom implies either A and C or B and D. Problems 1 and 2 are given by the following parameters:

$$\begin{aligned} &X = (2500, \frac{0.33}{0.34}; 0, \frac{0.01}{0.34}) \\ &Y = (2400) \\ &P = (2400) \\ &P' = (0) \\ &p = 0.34 \end{aligned} \tag{2.17}$$

This shows that the Independence Axiom is violated. This example also violates expected utility hypothesis. According to EU, the first preference gives (with $u(0) = 0$):

$$u(2400) > 0.33u(2500) + 0.66u(2400) \rightarrow \frac{u(2400)}{u(2500)} > \frac{0.33}{0.34} \tag{2.18}$$

While the second preference gives:

$$0.33u(2500) > 0.34u(2400) \rightarrow \frac{u(2400)}{u(2500)} < \frac{0.33}{0.34} \tag{2.19}$$

which is the opposite of the first inequality.

Another situation in which the independence axiom is violated is the following:

$$\textbf{Problem 3} \qquad A : (6000, 0.45) \qquad B : (3000, 0.9) \tag{2.20}$$

$$\textbf{Problem 4} \qquad C : (6000, 0.001) \qquad D : (3000, 0.002) \tag{2.21}$$

Most people have chosen B over A and C over D. The first problem gives $0.45u(6000) < 0.9u(3000)$ which means $\dfrac{u(3000)}{u(6000)} > \dfrac{1}{2}$ while the second problem gives the inverse inequality.

The following properties can be deduced from the Allais Paradox:

- **Underweighting of high probabilities**. This is shown in problems 1 and 2. In problem 1, a certain gain of 2400 is preferred over the 99% chance to gain at least 2400. People are afraid of the 1% chance of gaining nothing. In EU theory, the 0.99 probability has a weight of 0.99. But in real life, due to this disappointment of gaining nothing, it can be stated that $w(0.99) < 0.9$ where $w(p)$ is a $[0, 1] \to [0, 1]$ weighting function. This underweighting of high probabilities leads to the certainty effect: people prefer certain gains to very probable gains with greater outcomes.

  In problem 2, people prefer a 33% chance of gaining 2500 to a 34% chance of gaining 2400. The weighting of probabilities has less impact in the range of moderate probabilities. It concludes that reducing the chance of gain of 1% from an outcome with $p = 1$ has greater impact when reducing a initial probability of $p = 0.34$.

- **Overweighting of low probabilities**. In problem 4, the effect of probability distorsion is different in the range of low probabilities: a 0.1% chance of gaining 6000 is preferred over a 0.2% chance of gaining 3000. When the probability of gains are very low, people tend to choose the option with the greater outcome. In this case, $w(0.001) > 0.001$.

Real-life examples

One interesting example is the study of Sydnor [6] who studied insurance decisions of thousand of customers of a large insurance company. The main decision was the choice of a deductible for the insurance policy. Sydnor found out that the average premium for a 500 dollars deductible was 715 dollars. However, the average premium for a 1000 dollars deductible is only 615 dollars. The annual claim rate was about 0.05. So in case of total loss, these customers agreed to pay 100 dollars a year to insure a 5% chance of paying 500 more dollars in the event of a claim.

3.2 Framing effect

In EU theory different formulations of the same problem must give the same preferences. Consider a population of 600 individuals affected by a deadly epidemic disease. People were asked to choose between two treatments: Treatment A has a $\frac{1}{3}$ probability that no one will die and a $\frac{2}{3}$ probability that everybody dies while Treatment B will result in 200 survivors (400 deaths). Two formulations were proposed: Scenario 1 frames the problem in a "gain" scenario (survivors) while Scenario 2 formulates the problem in a "loss" frame (deaths).

According to EU, the two problems should give the same preferences. Nevertheless, Treatment A was chosen by 72% of the participants in Scenario 1 but it was chosen by only 22% of the participants in Scenario 2. People make decisions depending on the formulation of the options, i.e. in a positive or a negative frame.

$$\textbf{Scenario 1} \qquad A : \begin{cases} 600\,\text{survivors with probability} & \dfrac{1}{3} \\ 0\,\text{survivors with probability} & \dfrac{2}{3} \end{cases} \qquad B : 200\,\text{survivors with certainty} \tag{2.22}$$

$$\textbf{Scenario 2} \qquad A : \begin{cases} 600\,\text{will die with probability} & \dfrac{2}{3} \\ 0\,\text{will die with probability} & \dfrac{1}{3} \end{cases} \qquad B : 400\,\text{will die with certainty} \tag{2.23}$$

3.3 Reflection effect

In the previous section, it has been shown that different formulations of the same problem can lead to different decisions. Looking at the results of the previous problem and the following table, it is possible to infer that:

- People are risk averse in gain. They prefer certain gains and they are ready pay a risk premium to ensure this certainty. It has been correctly modelled in the EU framework by using a concave utility function.
- In the domain of losses, the economic agent is risk-seeking. Risky gamble are preferred. The utility function must be convex in the loss domain, which is not the case in the EU framework.

The reflection effect can be summarized as: people are risk averse in gains and risk seeking in losses. The utility function must be concave in gains and concave in losses.

	Positive prospects	**Negative prospects**
Problem 1	(4000,0.8) $\prec$ (3000)	(-4000,0.8) $\succ$ (-3000)
Problem 2	(4000,0.2) $\succ$ (3000,0.25)	(-4000,0.2) $\prec$ (-3000,0.25)
Problem 3	(3000,0.9) $\succ$ (6000,0.45)	(-3000,0.9) $\prec$ (-6000,0.45)
Problem 4	(3000,0.002) $\prec$ (6000,0.001)	(-3000,0.002) $\succ$ (-6000,0.001)

3.4 Reference dependence

In the Expected Utility framework, utility is measured at final asset position. However, people take decisions in terms of gains and losses relative from a reference point. Consider the following problems:

1. You receive 1000 and you choose between:

$$A : \begin{cases} 1000 & \text{with probability} & 0.5 \\ 0 & \text{with probability} & 0.5 \end{cases} \qquad B : 500 \quad \text{with certainty} \tag{2.24}$$

2. You receive 2000 and choose between:

$$C : \begin{cases} -1000 & \text{with probability} \quad 0.5 \\ 0 & \text{with probability} \quad 0.5 \end{cases} \qquad D : -500 \quad \text{with certainty} \tag{2.25}$$

Considering final outcomes, the two problems are equivalent to the choice between:

$$A = C : \begin{cases} 1000 & \text{with probability} \quad 0.5 \\ 2000 & \text{with probability} \quad 0.5 \end{cases} \qquad B = D : 1500 \quad \text{with certainty} \tag{2.26}$$

The experiment showed that 84% of participants have chosen B while 69% have chosen C. This is inconsistent with Expected Utility assumptions. It is important to consider deviations from a reference point instead of final asset positions. Combined with the framing and the reflections effects, it has been shown that the choice of the “statu quo” point and the formulation of the problem are essential in decision theory.

3.5 Pseudocertainty effect

Consider the following two-stages game:

1. First stage: There is a probability of 25% to go to the second stage and 75% chance of getting nothing.
2. Second stage: If you reach the second stage, you can choose between the two following prospects:

$$A : \begin{cases} 4000 & \text{with probability} \quad 0.8 \\ 0 & \text{with probability} \quad 0.2 \end{cases} \qquad B : 3000 \quad \text{with certainty} \tag{2.27}$$

Then consider a choice between these two prospects:

$$C : \begin{cases} 4000 & \text{with probability} \quad 0.2 \\ 0 & \text{with probability} \quad 0.8 \end{cases} \qquad D : \begin{cases} 3000 & \text{with probability} \quad 0.25 \\ 0 & \text{with probability} \quad 0.75 \end{cases} \tag{2.28}$$

The results of the experiment has showed that $A \preceq B$ and $C \succeq D$. However the first problem is equivalent to the second problem:

$$A : \begin{cases} 4000 & \text{with p.} \quad (0.8) * (0.25) = 0.2 \\ 0 & \text{with p.} \quad (0.2) * (0.75) + 0.75 = 0.8 \end{cases} \qquad B : \begin{cases} 3000 & \text{with p.} \quad 1 * (0.25) = 0.25 \\ 0 & \text{with p.} \quad 0 * (0.75) + 0.75 = 0.75 \end{cases} \tag{2.29}$$

Kahneman and Tversky called this incidence the *pseudocertainty effect*. This is the tendency for people to perceive the outcome as certain while it is actually uncertain in multi-stage decisions. They made their choice in the second stage of the game without considering the uncertainty of the first stage. This also violates the EU assumption that evaluation must be done at final states instead of deviations from a reference point.

3.6 Loss aversion and the Endowment effect

Loss aversion is the fact that people are more sensitive to losses than to gains of the same magnitude. Considering a gain $X > 0$ and a value function $v(x)$, loss aversion means that:

$$v(X) < |v(-X)| \quad \text{for } X > 0 \tag{2.30}$$

For example, to compensate a loss of 100, a economic agent must win of more than 100. Kahneman asserted that most people will turn down the gamble $(-100, 0.5; 110, 0.5)$: a loss of 100 is much more painful than a gain of 110.

Loss aversion combined with reference dependance can also explain an emotional bias called the *Endowment effect*. People tend to value objects they own more highly than their market value. That means that the willingness to pay (WTP) to acquire an object is generally lower that the amount they are willing to accept (WTA) to give it up if they own it. Suppose the value function for a loss averse agent has the form (which is the case in the Cumulative Prospect Theory of Kahneman and Tversky):

$$v(x) = \begin{cases} f(x) & x \geq 0 \\ -\lambda f(-x) & x < 0 \end{cases} \tag{2.31}$$

with $\lambda > 1$ and $f(x)$ concave ($v(x)$ is concave in gains and convex in loss, it respects the reflection effect). If the object has value X, the willingness to pay to acquire this object is equal to $v(X)$ while the willingness to accept to sell this object is equal to $\lambda v(X) > v(X)$.

3.7 Summary

In this section, weaknesses of the Expected Utility have been pointed out by simple problems. The next section covers the Cumulative Prospect Theory (1992) of Kahneman and Tversky. They proposed an alternative theory of decision making under risk which handles the following topics:

1. **Non linear decision weights**. It has been shown that low probabilities tend to be overweighted while large probabilities tend to be underweighted. This leads to different decision when the probabilities are very high or very low.
2. **Reflection effect**. People tend to be risk averse in the gains domain and risk seeking in the loss domain. Combined with the non linear decision weights, it explains why people love lottery and insurance. The following risk attitudes hold:

	Gains	**Losses**
Low Probability	Risk seeking (lottery)	Risk averse (insurance)
High Probability	Risk averse	Risk seeking

3. **Reference dependence**. A prospect must be evaluated as deviation from a reference point instead of final asset position.
4. **Loss aversion**. The disappointment of losing x is greater than the satisfaction of winning x.

4. Cumulative Prospect Theory

This section will show the features of the Cumulative Prospect Theory of Kahneman and Tversky [8]. This behavioral model takes into account the weaknesses of the Expected Utility Theory with a new value function and probability distortion. First, new prospect definitions will be introduced with (discrete) numerical examples. Then, the value and probability distortion functions of the CPT will be defined with a look at their properties. Finally, the CPT will be extended to continuous prospects.

4.1 Definitions

Positive and negative parts. Let f be a prospect of the form $(x_1, p_1; ...; x_n, p_n)$. Define f^+ (resp f^-) the part of the prospect for which the outcomes are positive (resp. negative):

$$f : \begin{cases} f^+ = (x_1, p_1; ...; x_m, p_m) \\ f^- = (x_{m+1}, p_{m+1}; ...; x_n, p_n) \end{cases} \qquad x_1 \leq ... \leq x_m \leq 0 \leq x_{m+1} \leq ... \leq x_n \tag{2.32}$$

Probability weighting functions. The strictly increasing function $w^+(p) : [0,1] \to [0,1]$ maps the probability p_i to the weight w_i^+ when the outcome x_i is positive. The function $w^-(p) : [0,1] \to [0,1]$ is defined the same way when the outcome is negative. $w^+(0) = w^-(0) = 0$ and $w^+(1) = w^-(1) = 1$ are also defined.

Decision weights. Decision weights are defined by:

$$\begin{aligned} \pi_i^+ &= w^+(p_i + ... + p_n) - w^+(p_{i+1} + ... + p_n) \\ \pi_i^- &= w^-(p_1 + ... + p_i) - w^-(p_1 + ... + p_{i-1}) \\ \pi_n^+ &= w^+(p_n) \\ \pi_1^- &= w^-(p_1) \end{aligned} \tag{2.33}$$

The cumulative distribution function is thus weighted instead of the probabilities themselves. Note that unlike expected utility hypothesis, the equality $\sum_i \pi_i = 1$ no longer holds (in EU, $\pi_i = p_i$ because $w(x) = x$).

Value function. The value function v assigns the subjective value $v(x_i)$ to the outcome x_i. The value function is the equivalent of the utility function in the CPT.

CPT prospect value. The CPT value of the prospect f is given by:

$$V(f) = V(f^+) + V(f^-) \quad \text{with} \quad V(f^+) = \sum_{i=m+1}^{n} \pi_i^+ v(x_i) \quad \text{and} \quad V(f^-) = \sum_{i=1}^{m} \pi_i^- v(x_i) \tag{2.34}$$

To illustrate the model, consider the dice roll game defined earlier. The prospect f is equiprobable with outcomes $(-5, -3, -1, 2, 4, 6)$. Thus, $f^+ = (2, \frac{1}{6}; 4, \frac{1}{6}; 6, \frac{1}{6})$ and $f^- = (-5, \frac{1}{6}; -3, \frac{1}{6}; -1, \frac{1}{6})$. Equation (2.34) gives:

$$\begin{aligned} V(f^+) &= v(2)\left[w^+\left(\frac{1}{2}\right) - w^+\left(\frac{1}{3}\right)\right] + v(4)\left[w^+\left(\frac{1}{3}\right) - w^+\left(\frac{1}{6}\right)\right] + v(6)w^+\left(\frac{1}{6}\right) \\ V(f^-) &= v(-5)w^-\left(\frac{1}{6}\right) + v(-3)\left[w^-\left(\frac{1}{3}\right) - w^-\left(\frac{1}{6}\right)\right] + v(-1)\left[w^-\left(\frac{1}{2}\right) - w^-\left(\frac{1}{3}\right)\right] \end{aligned} \tag{2.35}$$

4.2 Value Function

The value function has the following form:

$$v(x) = \begin{cases} v^+(x) = x^\alpha & x \geq 0 \\ -v^-(x) = -\lambda(-x)^\beta & x < 0 \end{cases} \tag{2.36}$$

Where $\lambda \geq 1$ is the loss aversion parameter. The greater λ the greater the loss aversion. α and β ($0 < \alpha \leq 1$, $0 < \beta \leq 1$) are the risk aversion parameters. The greater α, the greater the risk aversion (in gains) while the smaller β,the greater the risk seeking (in losses). The shape of the value function with different parameters is shown in Figure 2.1. In their 1992 paper, Kahneman and Tversky proposed the following values:

$$\begin{aligned} \alpha &= 0.88 \\ \beta &= 0.88 \\ \lambda &= 2.25 \end{aligned} \tag{2.37}$$

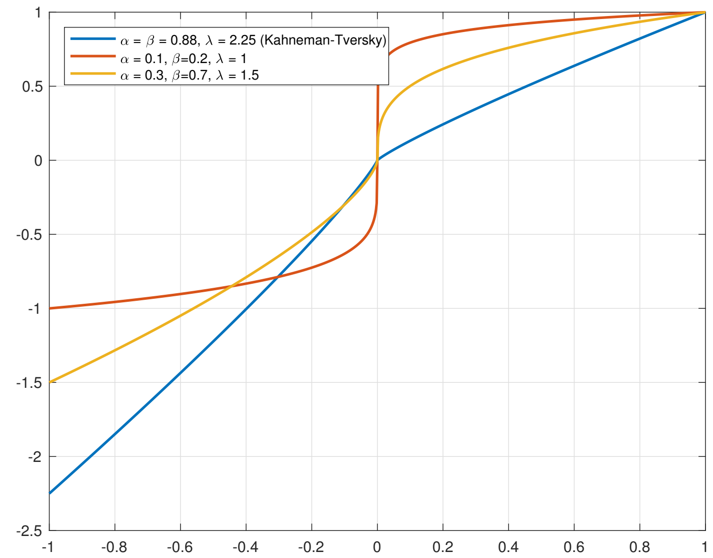

Figure 2.1: Value function with different parameters.

The value function handles the required specifications:

- **Reference dependence**. The value function pass through the origin, which represent the statu quo. Positive values represent gains while negatives values represent losses.

- **Reflection Effect**. The value function is concave in gains and convex in losses.
- **Loss aversion**. It is clear that the slope of the value function is steeper in losses than in gains. Thus it can handle the loss aversion and the endowment effect.

4.3 Weighting function

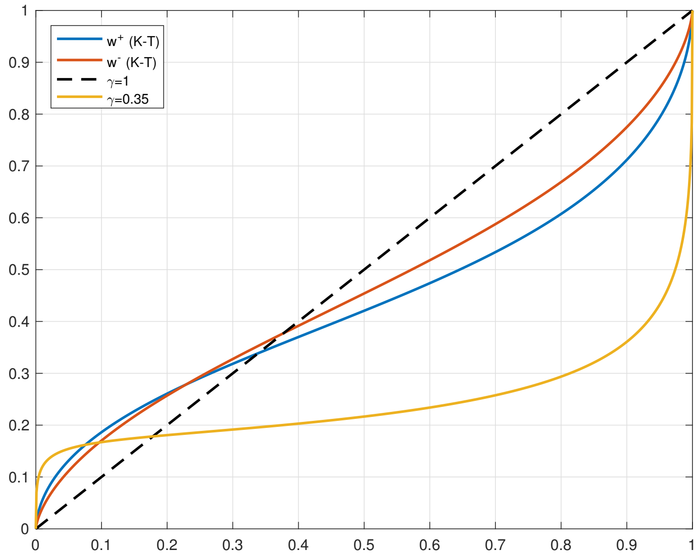

Figure 2.2: Weighting function with different parameters.

In Cumulative Prospect Theory, the non-linear weighting function directly applies to the cumulative distribution function. By doing this, the tails of any distribution will be overweighted. In CPT, the weighting function depends upon one parameter which is different for gains and losses. Figure 2.2 shows the weighting function for different values of the parameter. Kahneman and Tversky proposed the following form:

$$\begin{cases} w^+(p) = \dfrac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{1/\gamma}} \quad w^-(p) = \dfrac{p^\delta}{(p^\delta + (1-p)^\delta)^{1/\delta}} \\ \gamma = 0.61 \\ \delta = 0.69 \end{cases} \tag{2.38}$$

4.4 Extension to continuous prospects

In this section, the discrete version of CPT (equation (2.34)) will be extended to continuous prospects. Let's define the random variable X with cumulative distribution function $F_X(x)$ and complementary cdf $\overline{F}_X(x) = 1 - F_X(x)$. The probability distorsion functions (or weighting functions) $T_+ : [0,1] \to [0,1]$ and $T_- : [0,1] \to [0,1]$ are given by:

$$\begin{aligned} T_+(F_X(x)) &= \frac{F_X(x)^\gamma}{(F_X(x)^\gamma + \overline{F}_X(x)^\gamma)^{1/\gamma}} \\ T_-(F_X(x)) &= \frac{F_X(x)^\delta}{(F_X(x)^\delta + \overline{F}_X(x)^\delta)^{1/\delta}} \end{aligned} \tag{2.39}$$

The continuous version of the CPT objective function is given by:

$$V^{CPT}(X) = \int_0^\infty T_+(\overline{F}_X(x))dv^+(x) - \int_0^\infty T_-(F_X(x))dv^-(x) \tag{2.40}$$

Chapter 3

Portfolio allocation

In this chapter, the martingale method for solving portfolio allocation problems will be applied to the Expected Utility Theory framework. The first section is a reminder of elements of stochastic finance and important related theorems (e.g. Girsanov's theorem, fundamental theorem of asset pricing,...). The chapter then continues with the market model (SDE's, pricing kernel, fund equation,...). Finally, the martingale method is introduced with an direct application to portfolio selection in EUT.

1. Elements of Stochastic Finance

This section will introduce some definitions of stochastic finance (e.g. wealth process, arbitrage, replicating strategy,...) and important related theorems[1].

1.1 Definitions

- Decisions are made in a finite time horizon $\mathbb{T} = [0, T]$ with T fixed and $T < \infty$.
- Let's consider $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in\mathbb{T}})$ a filtered probability space.
- The market is composed of $N+1$ assets S_t^i. We define the vector of assets $\boldsymbol{S_t} = (S_t^0, ...S_t^N)'$. One of these assets is risk-free and called the bank account. Its process is denoted by S_t^0.
- The number of unit of asset S_t^i held is denoted by ϕ_t^i. The vector $\boldsymbol{\phi_t} = (\phi_t^0, ...\phi_t^N)'$ is called a **strategy**.

The **Wealth Process** of the strategy $\boldsymbol{\phi_t}$ is defined as

$$V_t(\boldsymbol{\phi_t}) = \boldsymbol{\phi_t} \bullet \boldsymbol{S_t} \tag{3.1}$$

(The expression $\mathbf{a} \bullet \mathbf{b}$ is the dot product of vectors $\mathbf{a}$ and $\mathbf{b}$). A strategy is **self-financing** if the quantities invested can be rebalanced at each date $t \in \mathbb{T}$ without exogenous infusion or withdrawal of funds. Mathematically, a strategy is self-financing if:

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[1] This section is inspired by the courses LACTU2070 (Prof. Donatien Hainaut) and LACTU2240 (Prof. Pierre Ars).

$$dV_t = \boldsymbol{\phi_t} \bullet \boldsymbol{dS_t} \tag{3.2}$$

In other words, the variation of wealth is only due to the variation of the asset values:

$$V_t = V_0 + \int_0^t \boldsymbol{\phi_s} \bullet \boldsymbol{dS_s} \tag{3.3}$$

The self-financing strategy $\boldsymbol{\phi_t}$ is said to be an **arbitrage** if the following conditions are met:

1. $P[V_0(\boldsymbol{\phi_t}) = 0] = 1$
2. $P[V_T(\boldsymbol{\phi_t}) \geq 0] = 1$
3. $P[V_T(\boldsymbol{\phi_t}) > 0] > 0$

A strategy is an arbitrage if with zero initial wealth (condition 1) it is possible to have a strictly positive terminal wealth (condition 3) without any possibility to have a negative final wealth (condition 2).

Let's consider the following definitions:

- A **Contingent Claim** X that settles at time T is a $\mathcal{F}_T$-measurable random variable whose payoff depends on certain conditions. Options are the most famous examples of contingent claims. The set of contingent claims is noted $\mathcal{X}$.
- A **Replicating Strategy** for the claim $X \in \mathcal{X}$ is a self-financing strategy $\boldsymbol{\phi_{X,t}}$ such that X has the same cash-flows as $V(\boldsymbol{\phi_{X,t}})$.
- A claim is said to be **attainable** if it admits at last one replicating strategy.

Contingent claims can thus be priced by a method called *replicating portfolio*. This method will be illustrated by pricing call option with the *single step binomial tree* model (Appendix D)

1.2 Important theorems

The following theorems will be considered without demonstration.

Theorem 2 (**Complete Market**). *A market $\mathcal{M}$ is complete if every contingent claim $X \in \mathcal{X}$ is attainable in $\mathcal{M}$. Thus for any contingent claim X, there exists at least one replicating strategy such that $V_T(\boldsymbol{\phi_T}) = X$.*

Theorem 3 (**Arbitrage free Market**). *If a market $\mathcal{M}$ is arbitrage free then any contingent claim $X \in \mathcal{X}$ is uniquely replicated in $\mathcal{M}$.*

Theorem 4 (**Martingale Measure**). *A probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F}_T)$ equivalent to $\mathbb{P}$ is called a martingale measure for the market $\mathcal{M}$ if for every strategy $\boldsymbol{\phi_t}$, the discounted wealth process $(S_t^0)^{-1} V_t(\boldsymbol{\phi_t})$ is a $\mathbb{Q}$-martingale w.r.t $\mathcal{F}_T$:*

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{V_t(\boldsymbol{\phi_t})}{S_t^0} | \mathcal{F}_s\right] = \frac{V_s(\boldsymbol{\phi_s})}{S_t^0} \quad \textit{for} \quad s < t. \tag{3.4}$$

The class of martingale measures is noted $\mathcal{Q}$

Theorem 5 (**Fundamental theorem of asset pricing**). *A market $\mathcal{M}$ is arbitrage free $\iff$ the class $\mathcal{Q}$ is not empty. Any contingent claim $X \in \mathcal{X}$ can thus be priced by:*

$$X_t = S_t^0 \, \mathbb{E}^{\mathbb{Q}} \left[\frac{X_T}{S_T^0} | \mathcal{F}_t \right] \tag{3.5}$$

Theorem 6 (**Completeness and risk neutral measure**). *An arbitrage free market $\mathcal{M}$ is complete $\iff$ there exists a unique martingale measure $\mathbb{Q}$ for $\mathcal{M}$. This measure is called the risk neutral measure*

Theorem 7 (**Girsanov's theorem**). *Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in\mathbb{T}})$ a filtered probability space on which $\boldsymbol{W_t}$ is a m-Brownian motion. Let $\boldsymbol{\theta_t}$ be a m vector of adapted processes which satisfies the Novikov condition: $\mathbb{E}\left[\exp\left\{\frac{1}{2}\int_0^T \|\boldsymbol{\theta_s}\|^2 ds\right\}\right] < \infty$. The following process:*

$$Z_t = \exp\left(- \int_0^t \boldsymbol{\theta_s} \bullet \boldsymbol{dW_s} - \frac{1}{2}\int_0^t \|\boldsymbol{\theta_s}\|^2 ds \right) \tag{3.6}$$

is a martingale. Then the m-process $\boldsymbol{W_t^{\mathbb{Q}}}$ defined by :

$$W_t^{i,\mathbb{Q}} = W_t^i + \int_0^t \theta_s^i ds \quad i = 1...m \tag{3.7}$$

is a m-Brownian motion under the equivalent measure $\mathbb{Q}$ whose Radon-Nikodym derivative w.r.t $\mathbb{P}$ is given by:

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_T} = Z_T \tag{3.8}$$

Theorem 8 (**Martingale Representation Theorem**). *Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in\mathbb{T}})$ a filtered probability space on which $\boldsymbol{W_t}$ is a m-Brownian motion. Let M_t be a $\mathcal{F}_t$-adapted martingale. There exists a unique m vector of $\mathcal{F}_t$-adapted processes $\mathbf{h_t}$ such that:*

$$M_t = M_0 + \int_0^t \mathbf{h_s} \bullet \mathbf{dW_s} \tag{3.9}$$

2. Market Model

For the next sections the following assumptions are made:

- Decisions are made in a finite time horizon $\mathbb{T} = [0, T]$ with T fixed and $T < \infty$.
- Let's consider $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in\mathbb{T}})$ a filtered probability space on which is defined a N-Brownian motion $(\boldsymbol{W_t})_{t\in\mathbb{T}} = (W_t^i, ..., W_t^N)'$.
- The market $\mathcal{M}$ is complete and arbitrage-free. Thus the risk neutral measure $\mathbb{Q}$ is unique. Moreover, the discounted payoff of any contingent claim $X \in \mathcal{X}$ is a martingale and any contingent claim is uniquely replicated in $\mathcal{M}$ by a self-financing strategy.
- The bank account S_t^0 follows the following dynamic:

  $$\frac{dS_t^0}{S_t^0} = r_t dt \quad S_0^0 = s^0 > 0 \tag{3.10}$$

  where the risk-free rate r_t is a $\mathcal{F}_t$-adapted process. This means that:

  $$\frac{S_t^0}{S_0^0} = e^{\int_0^t r_s ds} \tag{3.11}$$

- The market is composed of N stocks, S_t^i for $i = 1...N$ which are driven by the following SDE's:
$$\frac{dS_t^i}{S_t^i} = \mu_t^i dt + \sum_{j=1}^{N} \sigma_t^{ij} dW_t^j \quad S_0^i = s^i > 0 \tag{3.12}$$
where the processes μ_t^i (the return) and σ_t^{ij} (the volatility) are $\mathcal{F}_t$-adapted and sufficiently regular such that the solutions of the SDE's exist.

- The following notations are used: $\boldsymbol{\mu_t} = (\mu_t^1, ..., \mu_t^N)'$ and $\boldsymbol{\sigma_t} = \begin{pmatrix} \sigma_t^{11} & ... & \sigma_t^{1N} \\ \vdots & \ddots & \vdots \\ \sigma_t^{N1} & ... & \sigma_t^{NN} \end{pmatrix}$

  It is assumed that $\boldsymbol{\sigma_t}$ is invertible.

- Under regularity conditions (continuity, differentiability, Novikov), it can be shown by Girsanov's theorem that there exists the N-dimensional $\mathcal{F}_t$-adapted process given by
$$\boldsymbol{\theta_t} = \boldsymbol{\sigma_t}^{-1} (\boldsymbol{\mu_t} - \mathbf{1}\, r_t) \tag{3.13}$$
such that $\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_T} = Z_T$ where
$$Z_t = \exp\left(- \int_0^t \boldsymbol{\theta_s} \bullet \boldsymbol{dW_s} - \frac{1}{2} \int_0^t \|\boldsymbol{\theta_s}\|^2 ds \right) \tag{3.14}$$
According to the assumptions made (the market is arbitrage free), the fundamental theorem of asset pricing can be applied. This means that any contingent claim can be priced by martingale:
$$X_0 = S_0^0 \mathbb{E}^{\mathbb{Q}} \left[(S_T^0)^{-1} X_T | \mathcal{F}_0 \right] = \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_0^T r_s ds} X_T \right] = \mathbb{E}^{\mathbb{P}} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} e^{-\int_0^T r_s ds} X_T \right] \tag{3.15}$$
Defining the **Pricing Kernel** (i.e. **Deflator**):
$$\rho_t = \exp\left(- \int_0^t \boldsymbol{\theta_s} \bullet \boldsymbol{dW_s} - \int_0^t \left[r_s + \frac{1}{2} \|\boldsymbol{\theta_s}\|^2 \right] ds \right) \tag{3.16}$$
The pricing kernel follows the dynamic:
$$\frac{d\rho_t}{\rho_t} = -r_t dt - \boldsymbol{\theta_t} \bullet \boldsymbol{dW_t} \tag{3.17}$$
The **Budget Constrain** is finally defined as:
$$\mathbb{E}^{\mathbb{P}} \left[\rho_T X_T \right] = X_0 \tag{3.18}$$

- The portfolio vector is denoted $\boldsymbol{\pi_t} = (\pi_t^1, ..., \pi_t^N)'$ where π_t^i represent the fraction of the fund X_t invested in stock i. The fraction of X_t invested in the bank account is thus given by $1 - \sum_{i=1}^N \pi_t^i$. The strategy $\boldsymbol{\pi_t}$ is supposed to be self-financing.

- The dynamic of the fund X_t composed of the $N+1$ assets is thus given by (generalisation of the constant portfolio allocation of Appendix F):
$$\frac{dX_t}{X_t} = \left[r_t + (\boldsymbol{\mu_t} - \mathbf{1}\, r_t) \bullet \boldsymbol{\pi_t} \right] + \boldsymbol{\pi_t} \bullet (\boldsymbol{\sigma_t}\, \boldsymbol{dW_s}) \quad X_0 > 0 \tag{3.19}$$

3. Martingale method: Expected Utility framework

The goal is to find the portfolio $\boldsymbol{\pi_t}$ that maximise the agent's utility of the terminal wealth: $u(F_T)$. In this problem, CRRA utility is assumed:

$$u(x) = \frac{x^\gamma}{\gamma} \tag{3.20}$$

In this section, the martingale approach will be introduced to solve the following optimisation problem:

$$\boxed{\begin{aligned} &\underset{\boldsymbol{\pi_t}}{\text{argmax}} \quad \mathbb{E}^{\mathbb{P}}[u(X_T)] \\ &\text{subject to} \quad \text{equation (3.19)} \end{aligned}} \tag{3.21}$$

The martingale method consists of two steps:

1. Find the contingent claim X_T^* that maximises the agent's utility. Because the initial capital X_0 is given, the optimal claim must respect the budget constrain. This is the **Optimisation Problem**.
2. Then X_T^* can be replicated because the market is complete. The corresponding replicating strategy will give the optimal portfolio. This is the **Replication Problem**.

Problem (3.21) can be reformulated:

$$\boxed{\begin{aligned} &\underset{X_T}{\text{argmax}} \quad \mathbb{E}^{\mathbb{P}}\left[u(X_T)\right] = \mathbb{E}^{\mathbb{P}}\left[\frac{(X_T)^\gamma}{\gamma}\right] \\ &\text{s. t.} \quad \mathbb{E}^{\mathbb{P}}\left[\rho_T X_T\right] = X_0 \end{aligned}} \tag{3.22}$$

3.1 The Optimisation Problem

This optimisation problem can be solved by Lagrange multipliers. The Lagrangian is given by:

$$\mathcal{L} = \mathbb{E}^{\mathbb{P}}\left[u(X_T)\right] - \lambda\left[\mathbb{E}^{\mathbb{P}}\left[\rho_T X_T\right] - X_0\right] = \mathbb{E}^{\mathbb{P}}\left[u(X_T) - \lambda\left[\rho_T X_T - X_0\right]\right] \tag{3.23}$$

Derivating with respect to X_T gives:

$$\frac{d\mathcal{L}}{dX_T} = 0 \iff u'(X_T) = \lambda\rho_T \tag{3.24}$$

With CRRA utility, $u'(x) = x^{\gamma-1}$, the optimal terminal wealth X_T^* is thus

$$X_T^* = (\lambda\rho_T)^{\frac{1}{\gamma-1}} \tag{3.25}$$

The value of the Lagrange multiplier λ is found by injecting the previous result into the budget constrain:

$$\mathbb{E}^{\mathbb{P}}\left[\rho_T X_T^*\right] = \mathbb{E}^{\mathbb{P}}\left[\rho_T\left(\lambda\rho_T\right)^{\frac{1}{\gamma-1}}\right] = \lambda^{\frac{1}{\gamma-1}}\mathbb{E}^{\mathbb{P}}\left[\left(\rho_T\right)^{\frac{\gamma}{\gamma-1}}\right] = X_0 \tag{3.26}$$

Let's define the following process:

$$H_t = \mathbb{E}^{\mathbb{P}}\left[\left(\frac{S_t^0}{S_T^0}\frac{Z_T}{Z_t}\right)^{-\frac{\gamma}{1-\gamma}} |\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{P}}\left[\left(\frac{\rho_T}{\rho_t}\right)^{-\frac{\gamma}{1-\gamma}} |\mathcal{F}_t\right] \tag{3.27}$$

With initial value $H_0 = \mathbb{E}^{\mathbb{P}}\left[\left(\rho_T\right)^{-\frac{\gamma}{1-\gamma}}\right]$.

The expression of the Lagrange multiplier can be rewritten:

$$\lambda^{-\frac{1}{1-\gamma}} = \frac{X_0}{H_0} \tag{3.28}$$

The optimal terminal wealth is thus given by:

$$X_T^* = \frac{X_0}{H_0}\left(\rho_T\right)^{-\frac{\gamma}{1-\gamma}} \tag{3.29}$$

3.2 The Replication Problem

Referring to Appendix E, the price process of X_T^* is given by:

$$X_t^* = X_0\,\frac{H_t}{H_0}\left(\rho_t\right)^{-\frac{1}{1-\gamma}} \tag{3.30}$$

It has been shown in Appendix E that the discounted price process $(S_t^0)^{-1}X_t^*$ is a martingale under $\mathbb{Q}$:

$$\frac{d\left((S_t^0)^{-1}X_t^*\right)}{(S_t^0)^{-1}X_t^*} = \frac{1}{1-\gamma}\left(\boldsymbol{\theta_t} + \boldsymbol{\sigma_t^H}\right) \bullet \boldsymbol{dW_t^{\mathbb{Q}}} \tag{3.31}$$

The discounted price process of the fund $(S_t^0)^{-1}X_t$ (3.19) is also a martingale under $\mathbb{Q}$:

$$\frac{d\left((S_t^0)^{-1}X_t\right)}{\left(S_t^0\right)^{-1}X_t} = \boldsymbol{\pi_t} \bullet \left(\boldsymbol{\sigma_t}\,\boldsymbol{dW_t^{\mathbb{Q}}}\right) \tag{3.32}$$

Because the martingales $(S_t^0)^{-1}X_t^*$ and $(S_t^0)^{-1}X_t$ are the same, their diffusion term should be equal (Martingale Representation Theorem):

$$\boxed{\boldsymbol{\pi_t^*} = \frac{1}{1-\gamma}\left(\boldsymbol{\mu_t} - \mathbf{1}\,r_t\right)\left(\boldsymbol{\sigma_t}\boldsymbol{\sigma_t'}\right)^{-1} + \boldsymbol{\sigma_t^H}\,\boldsymbol{\sigma_t^{-1}}} \tag{3.33}$$

The solution is composed of two terms:

- The first term is the "classical" solution of the optimal portfolio problem in the EU framework.
- In [11], the last term is called the "Hedging demand for parameter risk". This comes from the fact that μ_t, r_t and σ_t are stochastic processes

If the following assumptions are made:

- The risk free rate is time independent: $r_t = r$
- The returns μ_t^i and the volatilities σ_t^{ij} are also time independent.
- The process $\boldsymbol{\theta_t}$ is thus a time independent vector: $\boldsymbol{\theta} = \boldsymbol{\sigma}^{-1}\left(\boldsymbol{\mu} - \mathbf{1}r\right)$

the second term vanishes and the solution is the classic one. Indeed, in this case the process H_t becomes deterministic, the term $\boldsymbol{\sigma^H}$ is thus zero (see Appendix E) . Note that in the unidimensional case, the solution above is the same as the one in section 2. where the constant portfolio allocation has been introduced.

Chapter 4

Portfolio selection in Cumulative Prospect Theory

This chapter aims to find an expression for the optimal wealth of the portfolio selection problem in the Cumulative Prospect Theory framework. Unlike the Expected Utility framework, the CPT allocation problem can be *ill-posed.* Moreover, due to the S-shaped utility function and the probability distorsion, the objective functional isn't globally concave, which implies that Lagrange multipliers cannot be used.
The structure of this chapter will be:

1. The difficulties of the CPT portfolio selection will be presented.
2. The second section explains a two-steps algorithm consisting of three sub-problems to tackle the difficulties.
3. The next sections will develop the three sub-problems.
4. Finally, the (elegant) structure of the terminal wealth will be expressed with an interpretation of its appealing features.

1. Difficulties in CPT selection problem: ill-Posedness

Remember the CPT objective function (2.40):

$$V^{CPT}(X) = V^{+}(X) + V^{-}(X) = \int_0^\infty T^{+}(\overline{F}_X(x))du^{+}(x) - \int_0^\infty T^{-}(F_X(x))du^{-}(x) \tag{4.1}$$

The optimal terminal is given by the following optimisation problem (see (3.22) for derivation):

$$\boxed{\begin{aligned} &\underset{X_T}{\text{argmax}} \quad V^{CPT}(X_T) \\ &\text{s. t.} \quad \mathbb{E}^{\mathbb{P}}\left[\rho_T X_T\right] = x_0 \end{aligned}} \tag{4.2}$$

A problem is well-posed if the supremum of its objective is finite. A optimal solution will be found only if the problem is known a priori to be well-posed. In Expected Utility theory, the utility function is assumed to be globally concave which guarantees the problem to be well-posed.

In Cumulative Prospect Theory, probability distorsion and the S-shape of the utility function can lead to ill-posed problem.

Theorem 9 (Ill-Posedness)**.** *The problem* (4.2) *is ill-posed if there exists a $\mathcal{F}_T$-measurable random variable X_T such that $\mathbb{E}^{\mathbb{P}}\left[\rho_T X_T\right] < \infty$ and $V^+(X_T) = \infty$.*

Proof. See Theorem 3.1 in Jin and Zhou [9]. □

This theorem means that the problem is ill-posed if it exists a non-negative contingent claim with finite price at time $t = 0$ ($\mathbb{E}^{\mathbb{P}}\left[\rho_T X_T\right] < \infty$) leading to infinite prospect value at time $t = T$ ($V^+(X_T) = \infty$). Economically, the investor purchases such a claim initially and then reaches the infinite prospect value at the end.

2. Algorithm to solve the CPT allocation problem

In [10], Zhang, Jin and Zhou called this method the **Divide and Conquer approach**:

1. **Divide**: The problem is split into two Choquet optimisation sub-problems: the *Positive part problem* and the *Negative part problem.* The split and the solutions of those problems depends on a set of parameters (A, x_+) (defined below).
2. **Conquer**: When the sub-problems are solved, the best set of parameters (A, x_+) is found to solve another optimisation problem equivalent to the initial one.

2.1 Step 1: Divide

In step 1, the first two sub-problems are solved. If X_T is a feasible solution (i.e. if X_T satisfies the contrains), then it can be split in X_T^+ - which defines the event $A = \{\omega : X_T \geq 0\}$ and initial price $x_+ = \mathbb{E}^{\mathbb{P}}\left[\rho_T X_T^+\right]$ - and X_T^- which correspond to A^c and initial price $x_- = x_+ - x_0$.
The two sub-problems are:

Positive part problem

Given parameters (A, x_+) the positive part problem is:

$$\boxed{\begin{aligned} &\underset{X_T^+}{\text{argmax}} \quad V_+(X_T^+) = \int_0^{\infty} T_+\left(\mathbb{P}[u_+(X_T^+) > y]\right) dy \\ &\text{s. t.} \quad \mathbb{E}^{\mathbb{P}}\left[\rho_T X_T^+\right] = x_+, \quad X_T^+ \geq 0 \text{ a.s.}, \quad X_T^+ = 0 \text{ a.s. on } A^c, \end{aligned}} \tag{4.3}$$

Let $v_+(A, x_+)$ denotes the optimal value of problem (4.3). Three cases are possible:

1. $\mathbb{P}[A] > 0$ then the feasible region of (4.3) is non empty. For exemple:

$$X_T^+ = \frac{x_+ \mathbf{1}_A}{\rho_T \mathbb{P}[A]} \tag{4.4}$$

   is feasible.

   Proof. Injecting the expression in the budget equation of (4.3) proves the result. □

The optimal value $v_+(A, x_+)$ is thus the supremum of (4.3)

2. $\mathbb{P}[A] = 0$ and $x_+ = 0$ the only feasible solution is $X_T = 0$ and $v_+(A, x_+) = 0$
3. $\mathbb{P}[A] = 0$ and $x_+ > 0$, there is no feasible solution of (4.3) and $v_+(A, x_+) = -\infty$ is imposed.

Note that the parameter $A = \{\omega : X_T \geq 0\}$ does not appear explicitly in Problem (4.3). Let's consider the following sketch of proof (inspired by Jin and Zhou [9]) to rewrite problem (4.3) in terms of (A, x_+). First, define the distortion function $T_A(x) := T_+(x\mathbb{P}[A])/T_+(\mathbb{P}[A])$. Using the law of total probabilities, the objective function (4.3) can be rewritten conditionally on A:

$$T_+\left(\mathbb{P}[u_+(X_T^+) > y]\right) = T_+\left(\mathbb{P}[u_+(X_T^+) > y|A]\,\mathbb{P}[A]\right) = T_+(\mathbb{P}[A])\,T_A\left(\mathbb{P}[u_+(X_T^+) > y|A]\right), \quad (4.5)$$

and $\{\omega : u_+(X_T^+) > y\} \subset A$. Using the law of total expectation, the budget constrain becomes:

$$\mathbb{E}^{\mathbb{P}}\left[\rho_T X_T^+\right] = \mathbb{E}^{\mathbb{P}}\left[\rho_T X_T^+|A\right]\ \mathbb{P}[A] \quad (4.6)$$

Considering Problem (4.3) in the conditional probability space $(\Omega \cap A, \mathcal{F} \cap A, \mathbb{P}_A := \mathbb{P}[\cdot|A])$ gives:

$$\boxed{\begin{aligned} &\underset{Y}{\operatorname{argmax}} \quad V_+(Y) = T_+(\mathbb{P}[A]) \int_0^\infty T_A\left(\mathbb{P}_A[u_+(Y) > y]\right) dy \\ &\text{s. t.} \quad \mathbb{E}_A^{\mathbb{P}}\left[\rho_T Y\right] = \frac{x_+}{\mathbb{P}[A]}, \quad Y \geq 0 \text{ a.s.,} \end{aligned}} \quad (4.7)$$

If Y^* is optimal for Problem (4.7) then $(X_T^+)^* = Y^*\mathbf{1}_A$ is optimal for Problem (4.3). The Choquet maximisation problem (4.7) is solved in Appendix A.

Negative part problem

Given parameters (A, x_+) the negative part problem is:

$$\boxed{\begin{aligned} &\underset{X_T^-}{\operatorname{argmin}} \quad V_-(X_T^-) = \int_0^\infty T_-\left(\mathbb{P}[u_-(X_T^-) > y]\right) dy \\ &\text{s. t.} \quad \mathbb{E}^{\mathbb{P}}\left[\rho_T X_T^-\right] = x_- = x_+ - x_0, \quad X_T^- \geq 0 \text{ a.s.,} \quad X_T^- = 0 \text{ a.s. on } A, \end{aligned}} \quad (4.8)$$

Similarly to the positive part problem, $v_-(A, x_+)$ is defined as the optimal value of problem (4.8). The three possibilities are:

1. $\mathbb{P}[A] > 0$ then the feasible region of (4.8) is non empty and the optimal value $v_-(A, x_+)$ is thus the infimum of (4.8)
2. $\mathbb{P}[A]$ and $x_+ = x_0$ the only feasible solution is $X_T = 0$ and $v_-(A, x_+) = 0$
3. $\mathbb{P}[A]$ and $x_+ \neq 0$, there is no feasible solution of (4.8) and we define $v_-(A, x_+) = \infty$

Along with the positive part problem, the following distorsion function is defined: $T_{A^c}(x) := T_-(x\mathbb{P}[A^c])/T_-(\mathbb{P}[A^c])$. Under the conditional probability space $(\Omega \cap A^c, \mathcal{F} \cap A^c, \mathbb{P}_{A^c} := \mathbb{P}[\cdot|A^c])$, Problem (4.8) becomes:

$$\boxed{\begin{aligned} &\underset{Y}{\operatorname{argmax}} \quad V_-(Y) = T_-(\mathbb{P}[A^c]) \int_0^\infty T_{A^c}\left(\mathbb{P}_{A^c}[u_+(Y) > y]\right) dy \\ &\text{s. t.} \quad \mathbb{E}^{\mathbb{P}}_{A^c}\left[\rho_T Y\right] = \frac{x_+ - x_0}{\mathbb{P}[A^c]}, \quad Y \geq 0 \text{ a.s.,} \end{aligned}} \tag{4.9}$$

If Y^* is optimal for Problem (4.9) then $(X_T^-)^* = Y^* \mathbf{1}_{A^c}$ is optimal for Problem (4.8). The Choquet minimisation problem (4.9) is solved in Appendix B.

2.2 Step 2: Conquer

In step 2, the solutions of Step 1 are combined to solve the original problem:

$$\boxed{\begin{aligned} &\underset{(A,x_+)}{\operatorname{argmax}} \quad v_+(A, x_+) - v_-(A, x_+) \\ &\text{s. t.} \quad A \in \mathcal{F}_T,\ x_+ \geq x_0^+ = \max(x_0, 0) \end{aligned}} \tag{4.10}$$

The problem (4.10) consists in finding the optimal event A that splits the good states and the bad states. The price of the gains x_+ is also determined.

2.3 Splitting justification

The following propositions justifiy the splitting of problem (4.2):

Proposition 1. *Problem* (4.2) *is ill-posed* $\iff$ *problem* (4.10) *is ill-posed.*

Proposition 2. *Given* X_T^**, define the event* $A^* := \{\omega : X^* \geq 0\}$ *and* $x_+^* := \mathbb{E}\left[\rho_T (X_T^*)^+\right]$*. Then* X^* *is optimal for problem* (4.2) $\iff$ (A^*, x_+^*) *are optimal for problem* (4.10) *and* $(X_T^*)^+$ *and* $(X_T^*)^-$ *are optimal for problems* (4.3) *and* (4.8) *respectively.*

Thus, problem (4.2) is completely equivalent to the set of problem (4.3), (4.8) and (4.10). Hence, solution of problem (4.2) can be obtained via the solution of problems (4.3), (4.8) and (4.10).

Remember that the decision variables of problem (4.10) are a real number x_+ and a random event A. The following theorem shows that it is sufficient to consider events of the form $A = \{\omega : \rho_T \leq c\}$ for some real number c:

Theorem 10. *Define* $\bar{\rho}_T := \sup\{x \in \mathbb{R} : \mathbb{P}[\rho_T > x] > 0\}$ *and* $\underline{\rho}_T := \inf\{x \in \mathbb{R} : \mathbb{P}[\rho_T < x] > 0\}$*.For any feasible pair* (A, x_+) *of problem* (4.10)*, there exists a real number* $c \in \left[\underline{\rho}_T, \bar{\rho}_T\right]$ *such that* $\overline{A} := \{\omega : \rho_T \leq c\}$ *satisfies:*

$$v_+(\overline{A}, x_+) - v_-(\overline{A}, x_+) \geq v_+(A, x_+) - v_-(A, x_+) \tag{4.11}$$

To simplify notations, $v_+(c, x_+)$ and $v_-(c, x_+)$ will be used to denote $v_+(\{\omega : \rho_T \le c\}, x_+)$ and $v_-(\{\omega : \rho_T \le c\}, x_+)$ respectively. Using Theorem 10, one may replace problem (4.10) by:

$$\boxed{\begin{array}{ll} \underset{(c,x_+)}{\operatorname{argmax}} & v_+(c, x_+) - v_-(c, x_+) \\ \text{s. t.} & \underline{\rho} \le c \le \overline{\rho},\ x_+ \ge x_0^+ \end{array}} \tag{4.12}$$

which is clearly a much simpler problem. The following theorem discloses the form of the general solution of the CPT model:

Theorem 11. *Given X_T^*, define $c^* := F_\rho^{-1}(P[X_T^* \ge 0])$ and $x_+^* := \mathbb{E}\left[\rho_T (X_T^*)^+\right]$ where $F_\rho(\cdot)$ is the cdf of ρ_T. Then X_T^* is optimal for problem (4.2) $\iff$ (c^*, x_+^*) is optimal for problem (4.12) and $(X_T^*)^+ \mathbf{1}_{\rho_T \le c^*}$ and $(X_T^*)^- \mathbf{1}_{\rho_T > c^*}$ are respectively optimal for problems (4.3) and (4.8) with parameters $(\{\omega : \rho_T \le c^*\}, x_+^*)$. Moreover, in this case $\{\omega : \rho_T \le c^*\}$ and $\{\omega : X_T^* \ge 0\}$ are identical.*

This theorem shows that the optimal wealth is the payoff of a combination of two binary options characterised by the single number c^*.

3. Final solution of the CPT allocation problem

In the *Divide and Conquer approach* introduced previously, two Choquet optimisation problems must be solved. Appendices A and B are dedicated to those resolutions. They are mainly based on the detailed paper of Jin and Zhou [9]. The appendices do not pretend to be as exhaustive as Jin and Zhou's paper, their goal being to give the reader a general idea of the resolution strategies.

Combining the derivations of the Positive Part (Theorem 16 in Appendix A) and Negative Part (Theorem 17 in Appendix B) problems, the *Conquer* subproblem (Problem (4.12)) can be rewritten:

Theorem 12 (CPT allocation). *The CPT allocation problem consists in finding the optimal couple (c^*, x_+^*) that solves:*

$$\begin{aligned} \underset{(c,x_+)}{\operatorname{argmax}} \quad V_{CPT} = & \mathbb{E}\left[u_+\left((u_+')^{-1}\left(\frac{\lambda \rho_T}{T_+'(F_\rho(\rho_T))}\right)\right) T_+'(F_\rho(\rho_T))\, \mathbf{1}_{\rho_T \le c}\right] \\ & - u_-\left(\frac{x_+ - x_0}{\mathbb{E}\left[\rho_T \mathbf{1}_{\rho_T > c}\right]}\right) T_-(P[\rho_T > c]) \end{aligned} \tag{4.13}$$

where $\lambda > 0$ satisfies: $\mathbb{E}\left[\rho_T (u_+')^{-1}\left(\frac{\lambda \rho_T}{T_+'(F_\rho(\rho_T))}\right) \mathbf{1}_{\rho_T \le c}\right] = x_+$. If (c^*, x_+^*) solves the above problem, the optimal contingent claim is given by:

$$X_T^* = (u_+')^{-1}\left(\frac{\lambda \rho_T}{T_+'(F_\rho(\rho_T))}\right) \mathbf{1}_{\rho_T \le c^*} - \frac{x_+^* - x_0}{\mathbb{E}\left[\rho_T \mathbf{1}_{\rho_T > c^*}\right]} \mathbf{1}_{\rho_T > c^*} \tag{4.14}$$

Solution (4.14) has some notable features:

- The terminal wealth X_T^* is the payoff of a combination of two binary options that can be priced explicitly in a certain framework.

- The terminal wealth X_T^* being a gain or a loss is only determined by the pricing kernel ρ_T being higher or lower than a certain threshold c^*.
- In good states of the market ($\rho_T \leq c^*$), the optimal strategy leads to a gain with respect to the reference wealth level while it leads to a loss in bad states ($\rho_T > c^*$)

To archive this, the economic agent should:

1. Buy a contingent claim with payoff $(u'_+)^{-1}\left(\frac{\lambda\rho_T}{T'_+(F_\rho(\rho_T))}\right)\mathbf{1}_{\rho_T\leq c^*}$ that costs x_+^* at $t=0$.
2. Remember that $x_+^* > x_0$. To fund the amount $x_+^* - x_0$, the economic agent will take a short position in contingent claim with payoff $\frac{x_+^*-x_0}{\mathbb{E}\left[\rho_T\mathbf{1}_{\rho_T>c^*}\right]}\mathbf{1}_{\rho_T>c^*}$

Hence, the economic agent gambles on good states of the world to archive gains w.r.t reference wealth level while accepting a fixed loss in case of bad states.

4. Application to CRRA utility function

In this section, an exemple with CRRA utility functions will be solved. Kahneman and Tversky [8] used the following:

$$\begin{aligned} u_+(x) &= x^\alpha, \\ u_-(x) &= kx^\alpha, \end{aligned} \tag{4.15}$$

where $0 < \alpha < 1$ is the risk aversion coefficient and $k > 1$ is the loss aversion coefficient. It follows that $u'_+(x) = \alpha x^{\alpha-1}$, $\left(u'_+(x)\right)^{-1} = (x/\alpha)^{1/(\alpha-1)}$ and $u_+\left(\left(u'_+(x)\right)^{-1}\right) = (x/\alpha)^{\alpha/(\alpha-1)}$

The optimal solution of the positive part problem (Theorem 16 in Appendix A) with CRRA utility functions is:

$$(X_T^*)^+(\lambda) = (u'_+)^{-1}\left(\frac{\lambda\rho_T}{T'_+\left(F_\rho(\rho_T)\right)}\right)\mathbf{1}_{\rho_T\leq c} = \left(\frac{\lambda\rho_T}{\alpha\, T'_+\left(F_\rho(\rho_T)\right)}\right)^{1/(\alpha-1)}\mathbf{1}_{\rho_T\leq c} \tag{4.16}$$

Note that the Lagrange multiplier λ is still unknown. To find its value, the budget equation (Theorem 16) must be solved:

$$x_+ = \mathbb{E}\left[\rho_T X_T^*(\lambda)\right] = \mathbb{E}\left[\rho_T\left(\frac{\lambda\rho_T}{\alpha\, T'_+\left(F_\rho(\rho_T)\right)}\right)^{1/(\alpha-1)}\mathbf{1}_{\rho_T\leq c}\right] \tag{4.17}$$

To have lighter notations, define:

$$\psi(c) \coloneqq \mathbb{E}\left[\rho_T\left(\frac{T'_+(F_\rho(\rho_T))}{\rho_T}\right)^{1/(1-\alpha)}\mathbf{1}_{\rho_T\leq c}\right] \tag{4.18}$$

The budget equation becomes: $x_+ = \psi(c)\left(\dfrac{\lambda}{\alpha}\right)^{1/(\alpha-1)}$. The solution of this equation is:

$$\lambda(c, x_+) = \alpha\left(\frac{x_+}{\psi(c)}\right)^{\alpha-1} \tag{4.19}$$

With this value of λ, it is now possible to rewrite the optimal solution of the positive part problem:

$$(X_T^*)^+(c, x_+) = \frac{x_+}{\psi(c)} \left(\frac{T_+'(F_\rho(\rho_T))}{\rho_T} \right)^{1/(1-\alpha)} \tag{4.20}$$

Using Theorem 16 again, the optimal value of the objective function is given by:

$$v_+(c, x_+) = x_+^\alpha \, \psi(c)^{1-\alpha} \tag{4.21}$$

Now, CRRA utility must be applied to the Negative Part problem. The optimal value of the functional (Theorem 17) gives:

$$v_-(c, x_+) = \frac{k \, T_- \left(1 - F_\rho(c)\right)}{\left(\mathbb{E}\left[\rho_T \mathbf{1}_{\rho_T > c^*}\right]\right)^\alpha} \, (x_+ - x_0)^\alpha \tag{4.22}$$

Combining the previous results, CRRA utility turns the *Conquer problem* (Theorem 12) into:

$$\boxed{\begin{aligned} &\underset{(c, x_+)}{\operatorname{argmax}} \quad v(c, x_+) = x_+^\alpha \, \psi(c)^{1-\alpha} - \frac{k \, T_- \left(1 - F_\rho(c)\right)}{\left(\mathbb{E}\left[\rho_T \mathbf{1}_{\rho_T > c^*}\right]\right)^\alpha} \, (x_+ - x_0)^\alpha \\ &\text{s. t.} \quad 0 \leq c \leq \infty, \; x_+ \geq x_0^+ \end{aligned}} \tag{4.23}$$

5. Case with initial gain position ($x_0 > 0$)

To solve this problem, the objective function of problem (4.23) has to be reformulated:

$$v(c, x_+) = \psi(c)^{1-\alpha} \left[x_+^\alpha - \kappa(c)(x_+ - x_0)^\alpha \right] \quad \kappa(c) := \frac{k \, T_- \left(1 - F_\rho(c)\right)}{\psi(c)^{1-\alpha} \left(\mathbb{E}\left[\rho_T \mathbf{1}_{\rho_T > c}\right]\right)^\alpha} \tag{4.24}$$

Before solving problem (4.23), the following (simple) optimisation problem must be solved:

$$\underset{x \geq x_0}{\operatorname{argmax}} \quad f(x) = x^\alpha - \kappa (x - x_0)^\alpha \tag{4.25}$$

with $\kappa \geq 0$ fixed. The derivative of the objective function reads: $f'(x) = \alpha(x^{\alpha-1} - \kappa(x - x_0)^{\alpha-1})$. Since $\alpha - 1 < 0$ and $x \geq x_0 \geq 0$, it follows that $x^{1-\alpha} \leq (x - x_0)^{1-\alpha}$. The solution depends on the value of κ:

(i) If $\kappa \geq 1$ then $f'(x) \leq 0 \; \forall x \geq x_0$. Therefore, x_0 is optimal and $\sup_{x \geq x_0} = x_0^\alpha$.

(ii) If $\kappa < 1$, the objective function can be expressed as $f(x) = x^\alpha \left[1 - \kappa(1 - x_0/x)^\alpha\right]$. Then, $\sup_{x \geq x_0} = +\infty$ because $\lim_{x \to \infty} f(x) = +\infty$.

With this results, the supremum of (4.24) can be obtained:

(i) If $\inf_{c\geq 0}\kappa(c)\geq 1$ then

$$\begin{aligned}\sup_{c>0,x_+\geq x_0} v(c,x_+) &= \sup_{c>0}\left[\psi(c)^{1-\alpha}\sup_{x_+\geq x_0}\left(x_+^\alpha-\kappa(c)(x_+-x_0)^\alpha\right)\right]\\ &= \sup_{c>0}\left[\psi(c)^{1-\alpha}\,x_0^\alpha\right]=\psi(\infty)^{1-\alpha}\,x_0^\alpha\\ &= v_+(\infty,x_0)\end{aligned}\tag{4.26}$$

The couple (∞,x_0) gives the optimal contingent claim $X_T^*=(X_T^*)^+(\infty,x_0)$.

(ii) If $\inf_{c\geq 0}\kappa(c)<1$ then there exists $c_0>0$ such that $\kappa(c_0)<1$ and:

$$\begin{aligned}\sup_{c>0,x_+\geq x_0} v(c,x_+)\geq \sup_{x_+\geq x_0} v(c_0,x_+) &= \psi(c_0)^{1-\alpha}\sup_{x_+\geq x_0}\left(x_+^\alpha-\kappa(c_0)(x_+-x_0)^\alpha\right)\\ &=+\infty\end{aligned}\tag{4.27}$$

Hence, the problem is ill-posed if $\inf_{c\geq 0}\kappa(c)<1$.

The following theorem (Theorem 9.1 in [9]) concludes this section:

Theorem 13 (Optimal wealth with CRRA utility and positive initial wealth)**.** *Assume that* $x_0\geq 0$ *and let* $\kappa(c):=\frac{k\,T_-(1-F_\rho(c))}{\psi(c)^{1-\alpha}\left(\mathbb{E}[\rho_T\mathbf{1}_{\rho_T>c}]\right)^\alpha}$. *The CPT value with CRRA utility function is given by:*

$$v(c,x_+)=\psi(c)^{1-\alpha}\left[x_+^\alpha-\kappa(c)(x_+-x_0)^\alpha\right]\tag{4.28}$$

where $\psi(c):=\mathbb{E}\left[\rho_T\left(\frac{T_+'(F_\rho(\rho_T))}{\rho_T}\right)^{1/(1-\alpha)}\mathbf{1}_{\rho_T\leq c}\right]$.

(i) If $\inf_{c\geq 0}\kappa(c)\geq 1$*, the couple* (∞,x_0) *maximises* (4.28) *with terminal wealth:*

$$X_T^*=\frac{x_0}{\psi(\infty)}\left(\frac{T_+'(F_\rho(\rho_T))}{\rho_T}\right)^{1/(1-\alpha)}\tag{4.29}$$

(ii) If $\inf_{c\geq 0}\kappa(c)<1$*, the problem is ill-posed.*

6. Positive distortion function: Lognormal Pricing Kernel

Assume that the cdf of ρ, $F_\rho(\cdot)$, is twice differentiable and $F_\rho'(x)>0\,\forall x>0$. It is furthermore assumed that $T_+(\cdot)$ is twice differentiable on $(0,1)$. Denote $x=F_\rho^{-1}(z)$ or $z=F_\rho(x)$. Set $H(x):=T_+(F(x))$ which is the *distorted distribution of* ρ. In [9], Jin and Zhou defined the following:

$$j(x):=x\frac{H''(x)}{H'(x)}-x\frac{F_\rho''(x)}{F_\rho'(x)}=x\left[(lnH'(x))'-(lnF_\rho'(x))'\right]\quad x>0\tag{4.30}$$

They showed that a S-shaped distortion function $T_+(\cdot)$ satisfies the monotonicity of $F_\rho^{-1}(z)/T_+(z)$ (this is an important hypothesis for the resolution of the Choquet maximisation problem, see Appendix A) if there exists $c_0>0$ such that

$$j(x) \leq 0 \;\forall x \in (0, c_0] \quad \text{and} \quad 0 \leq j(x) \leq 1 \;\forall x \in (c_0, +\infty] \tag{4.31}$$

The pricing kernel follows a lognormal distribution with parameters (μ_t, σ_t^2):

$$\rho(t,T) = \frac{\rho_T}{\rho_t} \sim Log\mathcal{N}(\mu_t, \sigma_t^2) \quad \text{with} \quad \mu_t := -(r + \frac{\theta^2}{2})(T-t), \sigma_t^2 := \theta^2(T-t) \tag{4.32}$$

The cdf is $F_\rho(x) = \Phi\left(\frac{ln(x)-\mu_t}{\sigma_t}\right)$. Let's consider the simplest form of $j(x)$:

$$j(x) = a\mathbf{1}_{0<x\leq c_0} + b\mathbf{1}_{x>c_0} \quad \text{with} \quad c_0 > 0, a < 0, 0 < b < 1 \tag{4.33}$$

The function $g(q) : [0,1] \to [0,1]$ will be introduced to have lighter notation:

$$g(q,k) = \Phi\left(\Phi^{-1}(q) - k\right) \tag{4.34}$$

Note that $g(\cdot)$ satisfies the properties of a distortion function.

The distortion function $T_+(\cdot)$ is thus given by:

$$T_+(z) = \begin{cases} ke^{a\mu_0 + a^2\sigma_0^2/2}\, g\left(z, a\sigma_0\right) & 0 < z < F_\rho(c_0) := z_0 \\ ke^{a\mu_0 + a^2\sigma_0^2/2}\, g\left(z_0, a\sigma_0\right) & \\ \qquad + (c_0^{a-b}k)e^{b\mu_0 + b^2\sigma_0^2/2}\left[g\left(z, b\sigma_0\right) - g\left(z_0, b\sigma_0\right)\right] & z_0 < z \leq 1 \end{cases} \tag{4.35}$$

and $k^{-1} = e^{a\mu_0 + a^2\sigma_0^2/2}\, g\left(z_0, a\sigma_0\right) + c_0^{a-b}e^{b\mu_0 + b^2\sigma_0^2/2}\left[1 - g\left(z_0, b\sigma_0\right)\right]$

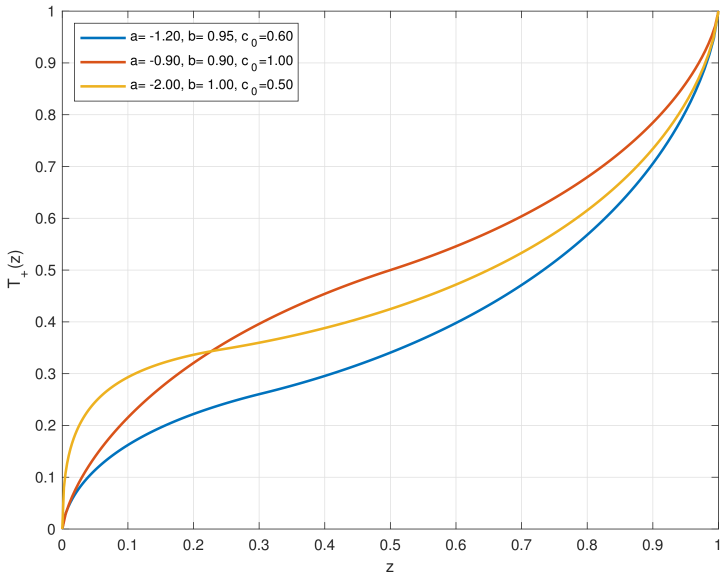

Figure 4.1: Examples of distortion function $T_+(z)$ with $\mu_0 = 0$ and $\sigma_0 = 1$.

It follows from (4.35) that

$$\frac{T_+'\left(F_\rho(\rho_T)\right)}{\rho_T} = k\rho_T^{a-1}\mathbf{1}_{\rho_T \leq c_0} + kc_0^{a-b}\rho_T^{b-1}\mathbf{1}_{\rho_T > c_0} \tag{4.36}$$

Then,

$$\psi(\infty) = \mathbb{E}\left[\left(\frac{T'_+\left(F_\rho(\rho_T)\right)}{\rho_T}\right)^{\frac{1}{1-\alpha}} \rho_T\right] = k^{1-\alpha}\,\mathbb{E}\left[\rho_T^{\frac{a-\alpha}{1-\alpha}}\mathbf{1}_{\rho_T\le c_0} + c_0^{\frac{a-b}{1-\alpha}}\rho_T^{\frac{b-\alpha}{1-\alpha}}\mathbf{1}_{\rho_T>c_0}\right] := k^{1-\alpha}\,\gamma \tag{4.37}$$

where $\gamma := \mathbb{E}\left[\rho_T^{\frac{a-\alpha}{1-\alpha}}\mathbf{1}_{\rho_T\le c_0} + c_0^{\frac{a-b}{1-\alpha}}\rho_T^{\frac{b-\alpha}{1-\alpha}}\mathbf{1}_{\rho_T>c_0}\right]$ has been defined.

Going back to Theorem 14 and applying the previous results, the optimal wealth is given by the following combination of binary options:

$$\boxed{X_T^* = \frac{x_0}{\gamma}\left[k\rho_T^{a-1}\mathbf{1}_{\rho_T\le c_0} + kc_0^{a-b}\rho_T^{b-1}\mathbf{1}_{\rho_T>c_0}\right]} \tag{4.38}$$

Such options can be easily replicated (see Appendix C).

7. Case with initial loss position ($x_0 < 0$) and loss control

When $x_0 > 0$, the couple (∞, x_0) solves problem (4.23). Such a simple solution does not exist when the investor suffers an initial loss position ($x_0 < 0$). Nevertheless, using the same approach as the case $x_0 > 0$, problem 4.23 can be simplified (see Theorem 9.2 in [9]):

Theorem 14 (Optimal wealth with CRRA utility and negative initial wealth)**.** *Assume that* $x_0 \le 0$ *and let* $\kappa(c) := \frac{k\,T_-(1-F_\rho(c))}{\psi(c)^{1-\alpha}\left(\mathbb{E}[\rho_T\mathbf{1}_{\rho_T>c}]\right)^{\alpha}}$.

(i) If $\inf_{c\ge0}\kappa(c) > 1$*, the problem admits an optimal portfolio if and only if:*

$$\operatorname*{argmin}_{c\ge0}\left[\left(\frac{kT_-(1-F_\rho(c))}{\left(\mathbb{E}\left[\rho_T\mathbf{1}_{\rho_T>c}\right]\right)^{\alpha}}\right)^{1/(1-\alpha)} - \psi(c)\right] \neq \emptyset \tag{4.39}$$

If c^* *solves* (4.39) *then* $x_+^* = \frac{-x_0}{k(c^*)^{1/(1-\alpha)}-1}$ *and the optimal terminal wealth is given by:*

$$X_T^* = \frac{x_+^*}{\psi(c^*)}\left(\frac{T'_+(F_\rho(\rho_T))}{\rho_T}\right)^{1/(1-\alpha)}\mathbf{1}_{\rho_T\le c^*} - \frac{x_+^* - x_0}{\mathbb{E}\left[\rho_T\mathbf{1}_{\rho_T>c^*}\right]}\mathbf{1}_{\rho_T>c^*} \tag{4.40}$$

(ii) If $\inf_{c\ge0}\kappa(c) = 1$*, the supremum value of the problem is* 0 *which doesn't admit any portfolio.*

(iii) If $\inf_{c\ge0}\kappa(c) < 1$*, the problem is ill-posed.*

In this case, the two-dimensional optimisation problem (4.23) is solved in two steps:

1. Solve the one-dimensional optimisation problem (4.39) to find the optimal state c^*
2. The $t = 0$ price of the gain contingent claim x^* is then given by $x_+^* = \frac{-x_0}{k(c^*)^{1/(1-\alpha)}-1}$.

In their 2011 paper, Zhang, Jin and Zhou [10] extended the Jin and Zhou [9] model to include a pre-specified upper bound on the losses. The initial problem(4.2) becomes :

$$\boxed{\begin{aligned} &\underset{X_T}{\operatorname{argmax}} \quad V^{CPT}(X_T) \\ &\text{s. t.} \quad \mathbb{E}^{\mathbb{P}}\left[\rho_T X_T\right] = x_0, \quad X_T \geq -L \end{aligned}} \tag{4.41}$$

where the maximum potential loss, L, is specified exogenously. The *Divide and Conquer approach* was also used to solve (4.41). The positive part problem is not affected by the new constrain, Theorem 16 can thus be reused. The only difference lies in the negative part problem where there is the additional upper bound constrain on X_-:

$$\boxed{\begin{aligned} &\underset{X_-}{\operatorname{argmin}} \quad V_-(X_-) = \int_0^\infty T_-\left(P[u_-(X_-) > y]\right) dy \\ &\text{s. t.} \quad \mathbb{E}^{\mathbb{P}}\left[\rho_T X_-\right] = x_- = x_+ - x_0, \quad X_- \leq L \end{aligned}} \tag{4.42}$$

The approach used to solve (4.42) is similar to the one used earlier. See [10] for the resolution of this problem. Let's compare the solutions of the negative part problem with and without loss control:

$$\begin{aligned} X^*_{T,-}(c^*) &= \frac{x_+ - x_0}{\mathbb{E}\left[\rho_T \mathbf{1}_{\rho_T > c^*}\right]} \mathbf{1}_{\rho_T > c^*} \quad \text{without loss control} \\ X^*_{T,-}(c^*, c_2^*) &= \frac{x_+ - x_0 - L\mathbb{E}\left[\rho_T \mathbf{1}_{\rho_T > c_2^*}\right]}{\mathbb{E}\left[\rho_T \mathbf{1}_{\rho_T \in (c^*, c_2^*]}\right]} \mathbf{1}_{\rho_T \in (c^*, c_2^*]} + L\mathbf{1}_{\rho_T > c_2^*} \quad \text{with loss control} \end{aligned} \tag{4.43}$$

The novelties of the portfolio allocation problem with loss control are:

- The presence of a third state of the world: $\rho_T > c_2^*$. The optimal values c^* and c_2^* define the three states: the gain state ($\rho_T < c^*$), the moderate loss state ($\rho_T \in (c^*, c_2^*]$) and the maximum loss state ($\rho_T > c_2^*$).
- This implies the presence of a third binary option: $L\mathbf{1}_{\rho_T > c_2^*}$.
- The initial price of the moderate loss contingent claim, $x_+ - x_0$ is reduced by the initial price of the maximum loss contingent claim $L\mathbb{E}\left[\rho_T \mathbf{1}_{\rho_T > c_2^*}\right]$.

The optimal values $(c, c2, x_+)$ are obtained by solving the problem:

Theorem 15 (Optimal wealth with CRRA utility and negative initial wealth and loss control)**.**

$$\begin{aligned}\underset{(c,c_2,x_+)}{\operatorname{argmax}} \quad & x_+^{\alpha}\,\psi(c)^{1-\alpha} - \frac{k\;(x_+ - x_0 - L\mathbb{E}\left[\rho_T \mathbf{1}_{\rho_T>c_2}\right])^{\alpha}}{\left(\mathbb{E}\left[\rho_T \mathbf{1}_{\rho_T\in(c,c_2]}\right]\right)^{\alpha}} \\ & \left[T_-\left(1-F_\rho(c)\right) - T_-\left(1-F_\rho(c_2)\right)\right] - k\,L^{\alpha}\,T_-\left(1-F_\rho(c_2)\right) \qquad (4.44)\\ s.\ t. \quad & 0 \le c \le c_2 \le \infty \\ & max\left(x_0^+, x_0 + L\,\mathbb{E}\left[\rho_T \mathbf{1}_{\rho_T>c_2}\right]\right) \le x_+ \le x_0 + L\,\mathbb{E}\left[\rho_T \mathbf{1}_{\rho_T>c_2}\right]\end{aligned}$$

The optimal terminal wealth is given by:

$$X_T^* = X_+^* \mathbf{1}_{\rho_T \le c^*} - \frac{k\;\left(x_+ - x_0 - L\mathbb{E}\left[\rho_T \mathbf{1}_{\rho_T>c_2^*}\right]\right)}{\left(\mathbb{E}\left[\rho_T \mathbf{1}_{\rho_T\in(c^*,c_2^*]}\right]\right)} \mathbf{1}_{\rho_T\in(c^*,c_2^*]} - L\mathbf{1}_{\rho_T>c_2^*} \qquad (4.45)$$

Note that Problem (4.44) is a natural extension of Problem (4.23): each term is composed of the utility ($u(x) = x^{\alpha}\,\mathbf{1}_{x\ge 0} - kx^{\alpha}\,\mathbf{1}_{x<0}$) of its respective terminal payoff multiplied by its *distorted probability weight* defined as: $w\left(z_1, z_2\right) := T_-\left(1-F_\rho(z_1)\right) - T_-\left(1-F_\rho(z_2)\right)$.

The following table summarises the payoffs, initial price and distorted probability weights of each binary option:

	Gain $\rho_T \le c$	**Moderate loss** $\rho_T \in (c, c_2]$	**Maximum loss** $\rho_T > c_2$
Initial price	x_+	$x_+ - x_0 - L\mathbb{E}\left[\rho_T \mathbf{1}_{\rho_T>c_2}\right]$	$L\mathbb{E}\left[\rho_T \mathbf{1}_{\rho_T>c_2}\right]$
Terminal payoff	$X_+ \mathbf{1}_{\rho_T\le c^*}$	$\frac{x_+ - x_0 - L\mathbb{E}\left[\rho_T \mathbf{1}_{\rho_T>c_2}\right]}{\mathbb{E}\left[\rho_T \mathbf{1}_{\rho_T\in(c,c_2]}\right]}\,\mathbf{1}_{\rho_T\in(c^*,c_2^*]}$	$L\mathbf{1}_{\rho_T>c_2^*}$
Dist. prob. weight	$T_-\left(1-F_\rho(c)\right)$	$T_-\left(1-F_\rho(c)\right) - T_-\left(1-F_\rho(c_2)\right)$	$T_-\left(1-F_\rho(c_2)\right)$

Table 4.1: Summary of the different binary options

7.1 Numerical resolution of the CPT problem with loss control

The numerical resolution of problem with loss control (Problem (4.44)) is far more difficult than the problem without it (Problem (4.2)). Indeed, Problem (4.2) is a two-dimensional optimisation problem with linear constrains: the feasible region is a rectangle. Moreover, Problem (4.2) can be turned into Problem (4.39) which is even simpler to solve.

The main difficulty of the numerical resolution of Problem (4.44) is the treatment of the non-linear constrains:

$$\max\left(x_0^+, x_0 + L\,\mathbb{E}\left[\rho_T \mathbf{1}_{\rho_T>c_2}\right]\right) \le x_+ \le x_0 + L\,\mathbb{E}\left[\rho_T \mathbf{1}_{\rho_T>c_2}\right] \qquad (4.46)$$

The *numerical trick* is to add an artificial upper bound constrain on c and c_2. The constrain $0 \le c \le c_2 \le \infty$ is modified into:

$$0 \le c \le c_2 \le c_{max} \qquad (4.47)$$

where $c_{max} = F_\rho^{-1}(1 - tol)$ and tol has been fixed to 10^{-6}.

Non-linear convex optimisation problems like Problem (4.44) can be solved by a class of numerical methods called *Interior-point methods (IPM)*. This topic goes far beyond the scope of this work. Note that the general idea of IPM's is to iteratively approach the optimal solution from the interior of the feasible region. To solve this NLP, the built-in Matlab constrained optimisation solver was used.

7.2 Numerical illustration

For this illustration, the pricing kernel follows a lognormal distribution with parameters:

$$log\, \rho_T \sim \mathcal{N}(-0.03, 0.04) \tag{4.48}$$

The economic agent begins with a loss position: $x_0 = -1$. He has a CRRA utility function with risk aversion parameter $\alpha = 0.88$ and the loss aversion parameter is equal to $k = 2.25$ (Kahneman and Tversky 1992 [8]). The positive distorsion function $T_+(\cdot)$ follows (4.35) (Jin and Zhou [9]) with parameters: $a = -1.5$, $b = 0.9$ and $c_0 = 0.89$ (note that $z_0 = F_\rho(c_0) = 1/3$). With those parameters, the value of the artificial upper bound is $c_{max} = 2.511$.

To ensure well-posedness, Escobar [12] proposed the the following negative distortion function:

$$T_-(z) = \left(\Phi\left[\Phi^{-1}(z) - \delta\sigma\right]\right)^{\alpha_-} \tag{4.49}$$

This function is reverse S-shaped and satisfies the monotonicity condition if $\alpha_- \in (0,1)$ and $\delta \in (0,1]$. For that purpose, $\alpha_- = 0.8$ and $\delta = 0.1$ have been chosen.
The first step is to ensure the well-posedness of the problem. This is satisfied if $\inf_{c>0} \kappa(c) > 1$. Figure 4.2 represents the function $\kappa(c)$. Since $\inf_{c>0} \kappa(c) = 1.0303$, the problem is well-posed.

Without loss control, the optimal values are $(c, x_+) = (1.344, 3.5453)$. The corresponding terminal wealth, X_T is showed in Figure 4.3. In this settings, at $t = 0$, the investor short-sells the loss binary option at $x_+ - x_0 = 4.5453$ to get enough funds (he suffers a loss position $x_0 = -1$) to buy the gain binary option at $x_+ = 3.5453$.
The probability to be in the gain state at maturity is $\mathbb{P}[\rho_T \leq 1.344] = \Phi\left(\frac{log\, 1.344 + 0.03}{0.2}\right) = 0.9483$. The price of the binary option $\mathbf{1}_{\rho_T > 1.344}$ at $t = 0$ is $\mathbb{E}\left[\rho_T \mathbf{1}_{\rho_T > 1.344}\right] = 0.0758$. The maximum (endogenous) potential loss is thus:

$$X_T^{loss} = \frac{x_+ - x_0}{\mathbb{E}\left[\rho_T \mathbf{1}_{\rho_T > 1.344}\right]} = 59.94 \tag{4.50}$$

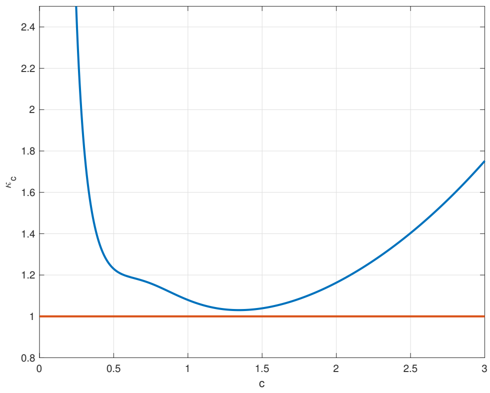

Figure 4.2: Plot of $\kappa(c)$. Problem is well-posed when $\inf_{c>0}\kappa(c) \geq 1$

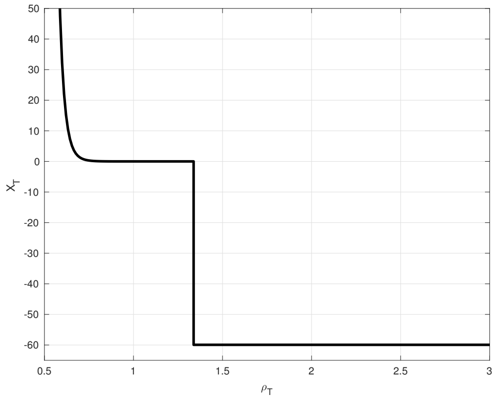

Figure 4.3: Optimal terminal wealth, X_T, without loss control.

This shows the weakness of the endogenous imposition of the maximum loss: the potential loss ($X_T = 59.94$) is much greater than the initial loss position ($x_0 = -1$). Note that the probability to get into this catastrophic state is not negligible: 5.17%. Even if the constant loss in bad states is an appealing feature, the possibility to impose the maximum loss exogenously seems to be a better one.

Let's consider the same problem with loss control. An exogenous maximum loss $L = 5$ has been imposed. The optimal solution of this problem is $(c, c_2, x_+) = (1.0893, \infty, 0.7470)$. Surprisingly, the solution has exactly the same structure as the problem without loss control ($c_2 = \infty$). The difference is that the couple (c, x_+) is set such that the maximum potential loss $(x_+ - x_0)/\mathbb{E}\left[\rho_T \mathbf{1}_{\rho_T > c}\right] = L$. Note that a solution of the form (c, ∞, x_+) is equivalent to (c, c, x_+) because it has been defined that (see [10]):

$$\frac{x_+ - x_0 - L\mathbb{E}\left[\rho_T \mathbf{1}_{\rho_T > c_2}\right]}{\mathbb{E}\left[\rho_T \mathbf{1}_{\rho_T \in (c, c_2]}\right]} := 0 \quad \text{when } c = c_2 \tag{4.51}$$

Note that the probability to be in the gain state is equal to $\mathbb{P}[\rho_T \leq 1.0893] = \Phi\left(\frac{log\, 1.0893 + 0.03}{0.2}\right) = 0.7182$ which is much smaller than the case without loss control. Figure 4.4 shows the terminal wealth with and without loss control. As pointed out earlier, the probability to get in the loss state is bigger when loss control is applied. Note that the gains are smaller in the case with loss control. This is the price to pay to limit his losses in case of bad market.

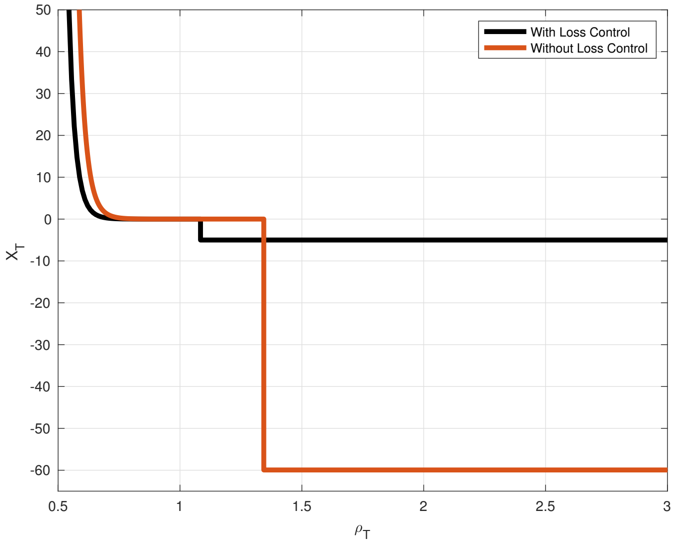

Figure 4.4: Optimal terminal wealth, X_T, with and without loss control.

Note that the above solution is still a two states payoff (in fact, the gain claim is composed of two pieces but it is considered as a unique part). The previous parameters were used to show the need of the exogenous imposition of the maximum loss as it has been shown that this maximum amount can be huge when imposed endogenously. Using other parameters, a three states payoff can be easily obtained.

The pricing kernel still follows a lognormal distribution but other parameters are used:

$$log\, \rho_T \sim \mathcal{N}(-0.0982, 0.1725) \tag{4.52}$$

The parameters of the lognormal distribution were computed using real market data (see next section). The initial loss position remains the same: $x_0 = -1$. The risk aversion coefficient has been changed to $\alpha = 0.7$ but the loss aversion parameters is unchanged: $k = 2.25$. The positive distorsion function, $T_+(\cdot)$, has new parameters: $a = -0.2$, $b = 0.5$ and $c_0 = 0.685$ ($z_0 = F_\rho(c_0) = 1/4$). Finally, the parameters of the negative distorsion function, $T_+(\cdot)$, are $\alpha_- = 0.35$ and $\delta = 0.1$.

When the maximum loss is set to $L = 5$, the optimal solution is given by the vector $(c, c_2, x_+) = (0.9964, 1.8338, 0.3006)$. The related probabilities are $\mathbb{P}[\rho_T \leq 0.9964] = 0.5901$ and $\mathbb{P}[\rho_T \leq 1.8338] = 0.9551$. Using Theorem 15 and the optimal (c, c_2, x_+), the terminal wealth is given by (see Figure 4.5):

$$X_T = \begin{cases} 0.0995\, \rho_T^{-4} & \rho_T \leq 0.685 \\ 0.2405\, \rho_T^{-5/3} & 0.685 < \rho_T \leq 0.9964 \\ -1.721 & 0.9964 < \rho_T \leq 1.8338 \\ -5 & \rho_T > 1.8338 \end{cases} \tag{4.53}$$

The probability to be in the gain state ($\rho_T \leq c$) is lower that in the other cases and the probability to hit the maximum loss is note negligible ($\mathbb{P}[\rho_T > c_2] = 4.49\%$)

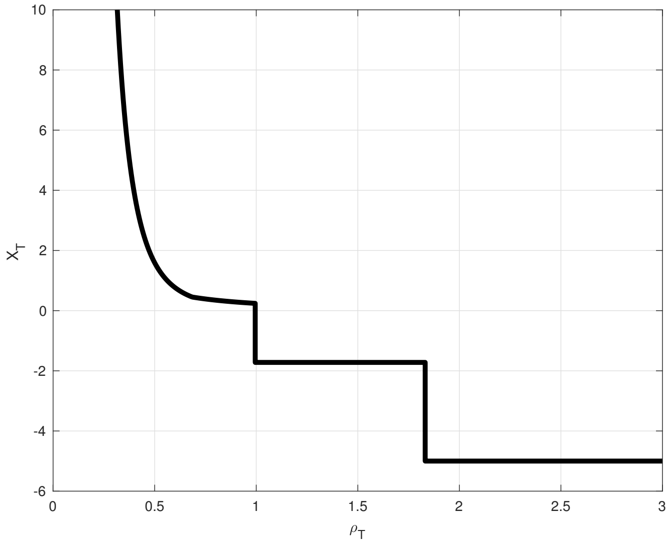

Figure 4.5: 3 pieces terminal wealth X_T.

Let's run the same case without loss control. Optimal values are $(c, x_+) = (0.6193, 0.052)$. The comparaison is depicted in Figure 4.6. Loss control enables other possibilities:

- The probability to end in the gain state is higher in the case with loss control ($\mathbb{P}[\rho_T \leq c] = 51.09\%$) than without ($\mathbb{P}[\rho_T \leq c] = 17.95\%$).
- The terminal wealth in the gain state in higher in the case with loss control.
- The price to pay is that the moderate loss and the maximum loss in the case with loss control ($X_T = -1.721$ and $X_T = -5$ respectively) are higher that the maximum loss in the case ($X_T = -1.172$) without it.

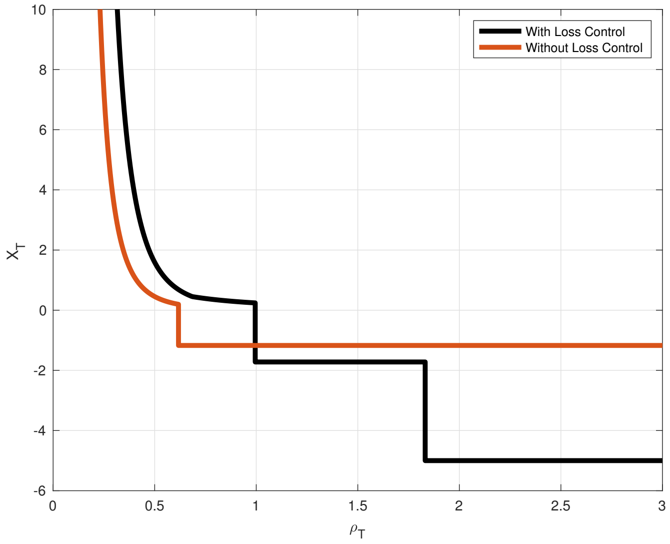

Figure 4.6: Terminal wealth with and without loss control (3 pieces solution).

Chapter 5

Results

1. Introduction

In this section, the previous results will be applied to real market data. First of all, the stocks models (constant market parameters) will be calibrated based on daily quotes ($\Delta t = 1/252$). Then, the investment strategy will be studied for the period going from 01/01/2019 to 01/04/2020.

The portfolio will be composed of the three biggest stocks that make up the BEL20 (BFX) index: *Anheuser-Busch InBev* (ABI), *ING Groep NV* (INGA) and *KBC GR* (KBC). The risk free rate, r, will be fixed to 0.6%. The quotes are shown in Figure 5.1.

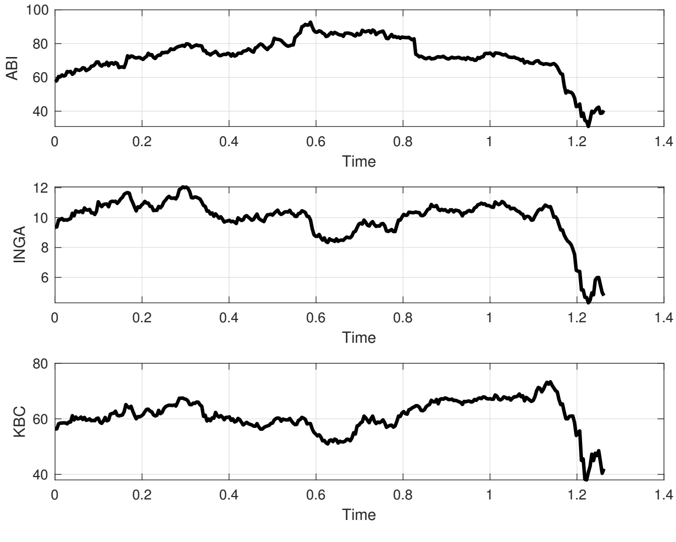

Figure 5.1: ABI, INGA and KBC quotes.

As an appetizer, consider the portfolio performances (with a 10 years investment horizon) of Figures 5.2 and 5.4 (the corresponding portfolios are shown in Figures 5.3 and 5.5 respectively). The first performance is linked to a moderate risk averse investor while the second one is linked to a greedier investor. These plots compare the performances between the CPT strategy and the EUT strategy. The BEL20 index (BFX) is also plotted as a benchmark.

The *defensive* CPT portfolio did not perform as well as the EUT investor during a good market period (first half of 2019) but performed better than the BFX. However, during the COVID-19 crisis, the value of the CPT portfolio went down to 90% of the initial investment (BFX performed the same) while the EUT investor went down to 19% of the initial investment.

The performances of the *more aggressive* CPT portfolio are slightly lower than the ones of the EUT portfolio in 2019 (note that they still outperformed the BFX). The critical point is the capital at the end of the study period: the value of the CPT portfolio reached 60% of the initial capital while the value CPT portfolio dropped to 14% of the initial investment.

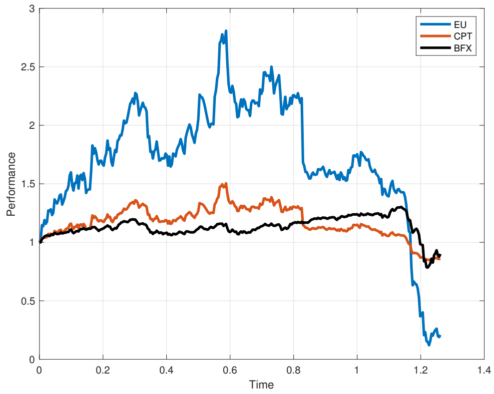

Figure 5.2: Performance of the portfolio with $\alpha = 0.5$, $a = -0.2$ and $b = 1$ when $T = 10$

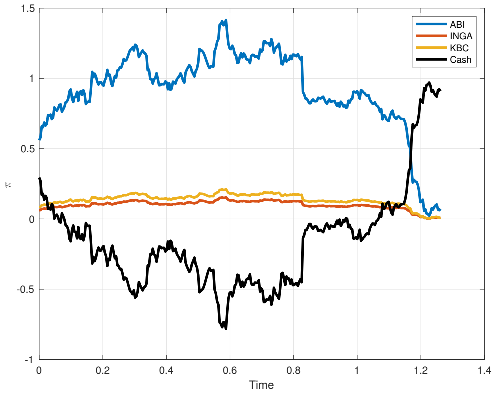

Figure 5.3: Composition of the portfolio with $\alpha = 0.5$, $a = -0.2$ and $b = 1$ when $T = 10$

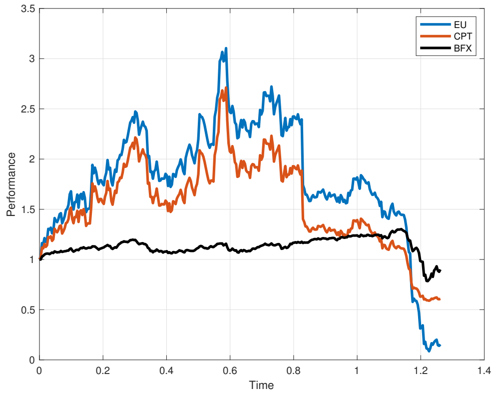

Figure 5.4: Performance of the portfolio with $\alpha = 0.55$, $a = -0.5$ and $b = 1$ when $T = 10$

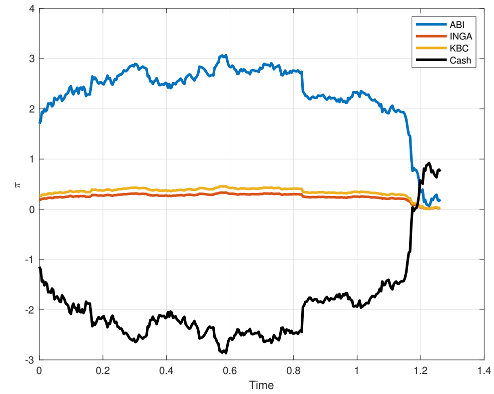

Figure 5.5: Composition of the portfolio with $\alpha = 0.55$, $a = -0.5$ and $b = 1$ when $T = 10$

2. Model calibration

As explained in the introduction the log-returns are supposed to be multivariate normal:

$$log\left(\frac{S_t}{S_0}\right) \sim \mathcal{N}\left(\left(\boldsymbol{\mu} - \frac{1}{2}diag(\Sigma)\right)t, \Sigma t\right) \tag{5.1}$$

Defining L, the Cholesky decomposition of the cov. matrix: $\mathbf{L}\,\mathbf{L}' = \Sigma$, the SDE's of the stocks can be written in the matrix form:

$$\frac{\mathbf{dS_t}}{\mathbf{S_t}} = \boldsymbol{\mu}\, dt + \mathbf{L} \cdot \mathbf{dW_t} \tag{5.2}$$

where $\mathbf{W_t}$ is a vector of independent brownian motions. The market price of risk vector $\boldsymbol{\theta}$ is obtained by solving:

$$\mathbf{L}\boldsymbol{\theta} = \boldsymbol{\mu} - r\,\mathbf{1} \tag{5.3}$$

Finally, the pricing kernel is found by solving the SDE:

$$\frac{d\rho_t}{\rho_t} = -r\, dt - \boldsymbol{\theta} \cdot \mathbf{dW_t} \tag{5.4}$$

The process is thus:

1. Estimate the parameters of a multivariate normal distribution ($\hat{\mu}$ and Σ) using daily stock log-returns (before the study period).
2. Calculate the vector of means $\boldsymbol{\mu}$ from the covariance matrix and observed mean returns.
3. Perform the Cholesky decomposition $\mathbf{L}$ of the covariance matrix.
4. Solve system (5.3) to obtain the market price of risk vector
5. Find the corresponding brownian motion vector $\mathbf{W_t}$ during the period of study
6. Compute the pricing kernel

The resulting pricing kernel is shown in Figure 5.6. This shows two big features: firstly, the well performing market in 2019 and secondly, the 2020 stock market krach during the COVID-19 crisis.

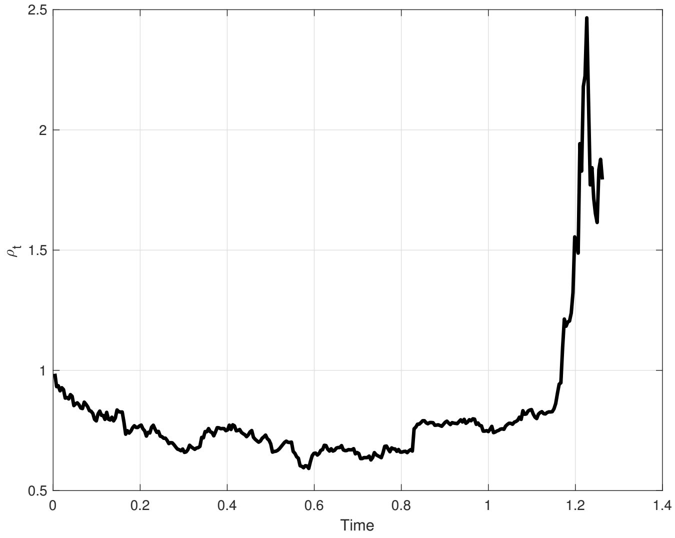

Figure 5.6: Pricing Kernel ρ.

To check whether the log-returns are normally distributed or not, a Jarque-Bera test can be performed (H_0: data are normally distributed). With a 5% significance level, the critical value is $JB = 5.7842$. The Ljung-Box test (Portmanteau test) can be used to check if the data (of each stock, note between stocks) are independently distributed (H_0: data are independently distributed). The critical value at 5% level of significance is $LB = 31.4104$. It is clear that the convenient hypothesis of normally and independently distributed log-returns is rejected. However, this hypothesis was made to show that an analytical expression for the optimal portfolio is available when the pricing kernel is lognormally distributed. Further work on this subject can involve more precise models.

	ABI	**INGA**	**KBC**
Jarque-Bera test	3189	6354	8668
Portmanteau test	38.34	66.4486	53.4096

Table 5.1: Summary of test statistics.

3. Some reminders

The optimal terminal wealth is given by:

$$X_T^* = \frac{x_+^*}{\psi(c^*)} \left(\frac{T'_+(F_\rho(\rho_T))}{\rho_T} \right)^{1/(1-\alpha)} \mathbf{1}_{\rho_T \leq c^*} - \frac{x_+^* - x_0}{\mathbb{E}\left[\rho_T \mathbf{1}_{\rho_T > c^*}\right]} \mathbf{1}_{\rho_T > c^*} \tag{5.5}$$

This expression can be further simplified using Jin and Zhou [9] distortion function $T_+(\cdot)$ and lognormal pricing kernel $\rho(t,T)$:

$$X_T^* = \frac{x_+^* \, k^{1/(1-\alpha)}}{\psi(c^*)} \left[\rho_T^{\frac{a-1}{1-\alpha}} \mathbf{1}_{\rho_T \leq c_0} + \rho_T^{\frac{b-1}{1-\alpha}} \mathbf{1}_{\rho_T \in (c_0, c^*)} \right] \mathbf{1}_{\rho_T \leq c^*} - \frac{x_+^* - x_0}{\mathbb{E}\left[\rho_T \mathbf{1}_{\rho_T > c^*}\right]} \mathbf{1}_{\rho_T > c^*} \tag{5.6}$$

The terminal wealth is thus the combination of payoffs of binary options on truncated lognormal variables $\rho^\eta_{\rho \in (z1, z2)}$ which can be easily replicated (see Appendix C). When $x_0 > 0$, it has been shown that $(x_+^*, c^*) = (x_0, \infty)$ and

$$X_T^* = \frac{x_0 \, k^{1/(1-\alpha)}}{\psi(\infty)} \left[\rho_T^{\frac{a-1}{1-\alpha}} \mathbf{1}_{\rho_T \leq c_0} + c_0^{\frac{a-b}{1-\alpha}} \rho_T^{\frac{b-1}{1-\alpha}} \mathbf{1}_{\rho_T > c_0)} \right] \tag{5.7}$$

Defining the process x_t^1 that replicates the claim $\rho_T^{\frac{a-1}{1-\alpha}} \mathbf{1}_{\rho \in (0, c_0)}$ and the process x_t^2 replicates the claim $\rho_T^{\frac{b-1}{1-\alpha}} \mathbf{1}_{\rho \in (c_0, c^*)}$, the price process of the portfolio x_t and the stocks portfolio π_t are given by:

$$\begin{aligned} x_t &= \frac{x_0 \, k^{1/(1-\alpha)}}{\psi(\infty)} \left[x_t^1 + c_0^{\frac{a-b}{1-\alpha}} x_t^2 \right] \\ \boldsymbol{\pi_t} &= \left[\frac{(1-a) x_t^1 + c_0^{\frac{a-b}{1-\alpha}} (1-b) x_t^2}{x_t^1 + c_0^{\frac{a-b}{1-\alpha}} x_t^2} \right] \frac{1}{1-\alpha} (\boldsymbol{\mu} - \mathbf{1} r)(\boldsymbol{\sigma\sigma'})^{-1} \end{aligned} \tag{5.8}$$

The optimal portfolio of a maximising expected utility investor is given by: $\boldsymbol{\pi}^{\mathbf{EUT}} = \frac{1}{1-\alpha}(\boldsymbol{\mu} - \mathbf{1} r)(\boldsymbol{\sigma\sigma'})^{-1}$. From that, it can be deduced that:

$$\frac{\boldsymbol{\pi}_{\mathbf{CPT}}}{\boldsymbol{\pi}_{\mathbf{EUT}}} = \left[\frac{(1-a)x_t^1 + c_0^{\frac{a-b}{1-\alpha}}(1-b)x_t^2}{x_t^1 + c_0^{\frac{a-b}{1-\alpha}} x_t^2} \right] \tag{5.9}$$

Several parameters will influence the portfolio π_t:

1. The time horizon T. The pricing kernel is given by $\rho(t,T) = \dfrac{\rho_T}{\rho_t} \sim Log\mathcal{N}(\mu_t, \sigma_t^2)$ with $\mu_t = -(r + \frac{1}{2}\|\theta^2\|)(T-t)$ and $\sigma_t = \|\theta\|\sqrt{T-t}$. The *time to maturity* $T-t$ will directly influence the parameters of the distribution. First of all, the threshold $c_0 = F^{-1}_{\rho(0,T)}(z_0)$ will be impacted. The inflexion point of the distortion function $T_+(\cdot)$, z_0, is considered to be constant and equal to $z_0 = 0.25$. c_0 will thus depend on the time horizon T.
   Changing the distribution of the pricing kernel will also change the price process, x_t^1 and x_t^2, of the binary options.

2. The *hope parameter* $a < 0$ (name given by He [13]). In the CPT allocation problem, the investor will overweight his portfolio in stocks comparing to the EUT investor when the market is good(ρ_t is very low). Indeed,

   $$lim_{\rho \to 0} \frac{\pi_{CPT}}{\pi_{EUT}} = 1 - a \geq 1 \tag{5.10}$$

   In this case, the CPT investing strategy will be the same as the EUT strategy with a equivalent risk aversion coefficient $\alpha^* = (\alpha - a)/(1-a) \geq \alpha$. In case of bull market, the CPT investor will act as a EUT investor with higher CRRA coefficient, which means a lesser risk aversion. The hope parameter will thus represent the willingness to invest in risky stocks when the market is good.

3. The *fear parameter* $b \in [0,1]$ (name given by He [13]). Conversely, when the market is bad (ρ_t is very high), the CPT investor will underweight his stocks portfolio comparing to the EUT investor:

   $$lim_{\rho \to \infty} \frac{\pi_{CPT}}{\pi_{EUT}} = 1 - b \in [0,1] \tag{5.11}$$

   When $b = 1$, the investor will close all his positions in stock when the market is bad. Along with the hope parameter, in case of very bad market, the CPT strategy is the same as the EUT strategy with a equivalent risk aversion coefficient equal to $\alpha^* = (\alpha - b)/(1-b) \leq \alpha$.

4. The risk aversion parameter α. This parameter will also influence the ratio $\dfrac{\pi_{CPT}}{\pi_{EUT}}$ because the binary options depend on this parameter. It will thus influence x_t^1 and x_t^2.

4. Investment horizon T and time to maturity $T - t$

In this section, the sensitivity analysis of the parameters T and $T-t$ will be performed. Figures 5.7 and 5.8 show the ratio $\pi_{CPT}/\pi_{EUT}(\rho)$ for different values of $T-t$ when the time horizon is equal to $T = 2$ and $T = 10$ respectively. The parameters are $a = -0.2$, $b = 0.5$, $\alpha = 0.5$ and $c_0 = 0.685|_{T=2}$ and $c_0 = 0.327|_{T=10}$.

The relation $\pi_{CPT}/\pi_{EUT}(\rho)$ is reverse S-shaped. As said earlier, the asymptotic values are $1-a$ when $\rho_t \to 0$ and $1-b$ when $\rho_t \to \infty$. When the investment horizon, T increases, the ratio of the portfolios presents thicker tails. The thickness of the tails also depends on the time to maturity: the tails are thiner when t approach the horizon investment T. Obviously, when $t \to T$ the relation $\pi_{CPT}/\pi_{EUT}(\rho)$ becomes a reverse step function with discontinuity at $\rho = c_0$ and steps equal to $1-a$ and $1-b$. The right tail appears to be thicker that the left tail and that effect is greater when the investment horizon increases.

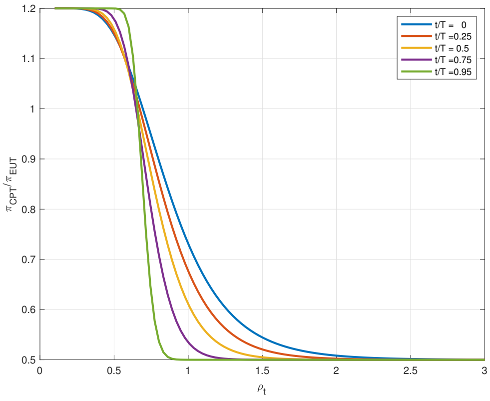

Figure 5.7: Time horizon $T = 2$

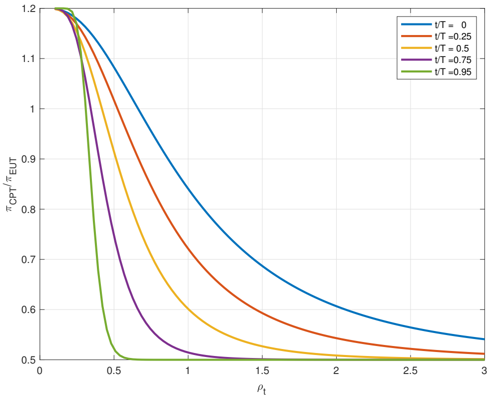

Figure 5.8: Time horizon $T = 10$

Figures 5.9 and 5.10 show the same relation with $\alpha = 0.7$, the investor is thus *greedier*. In this case, the tails are quite thicker than the case $\alpha = 0.5$. This is ever more pronounced when $T = 10$: at $t = 0.25T$, the investor will overweight his portfolio in risky assets when $\rho_t \leq 1.25$ if $\alpha = 0.7$ but only when $\rho_t \leq 0.55$ if $\alpha = 0.5$.

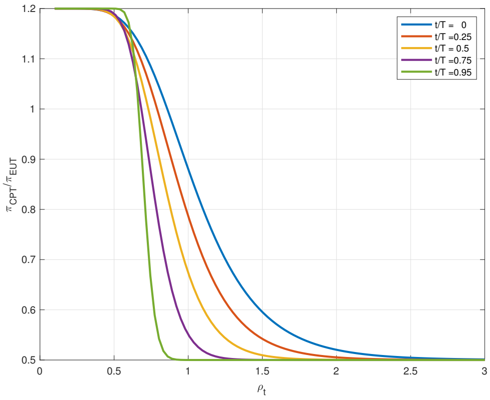

Figure 5.9: Time horizon $T = 2$

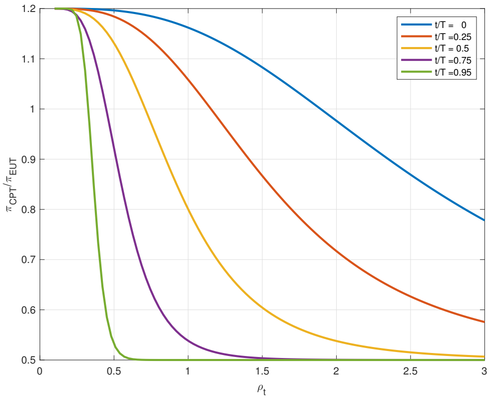

Figure 5.10: Time horizon $T = 10$

5. The hope parameter a

In this section, the performance of the strategy will be studied with different values of the hope parameter a. The EU strategy is independent of the investment horizon T, the time to maturity $T-t$ and the state of the market ρ_t. The EU strategy is performing very well during the 2019 year (when the market is quite good) but it is catastrophic during the 2020 stock market krach: about 75% of the initial investment is lost. Note that the risk aversion coefficient is moderate ($\alpha = 0.5$), a larger value leads to total loss of the investment (note that for higher values of α, the problem is ill-posed) . The proportion in cash is negative, which means that the investor is borrowing money to invest in stocks. The borrowed amount is equal to 180% of his starting capital which is huge for such a moderate aversion coefficient.

The CPT portfolio is overperformed by the EUT investor when the market is good but has offered a good capital protection during the COVID-19 crisis (this is also due the fear parameter $b = 1$ which means closing all positions in risky assets when the market is too bad). Note that when $T = 10$ and $a = -0.8$, the CPT investor will take more leverage than the EUT investor before the 2020 krach (see Figure 5.13) which will lead to better performance (see Figure 5.14). During the COVID-19 crisis, the CPT investor will close a large part of his positions in stocks leading to a loss of 50% of his initial capital (to compare with the 75% loss of the EUT investor).

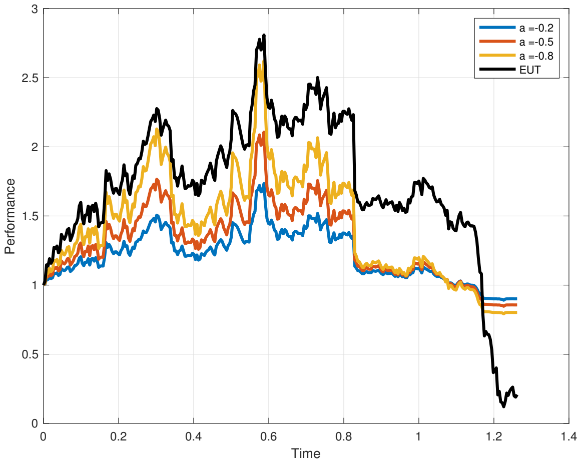

Figure 5.11: Wealth process for different values of a when $T = 2$ ($\alpha = 0.5$, $b = 1$).

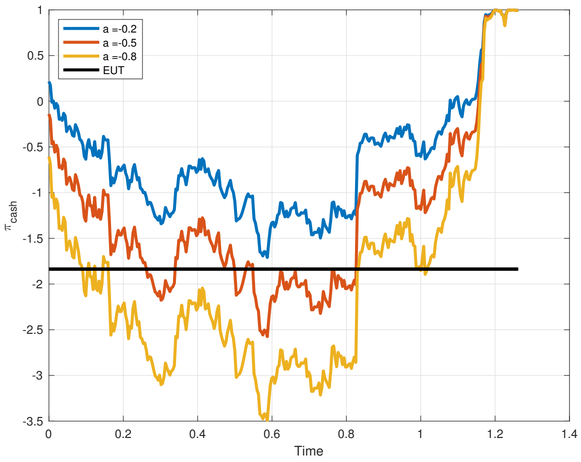

Figure 5.12: Cash portfolio for different values of a when $T = 2$ ($\alpha = 0.5$, $b = 1$).

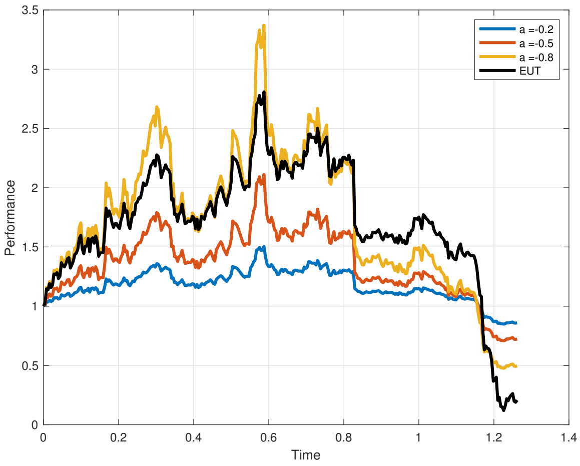

Figure 5.13: Wealth process for different values of a when $T = 10$ ($\alpha = 0.5$, $b = 1$).

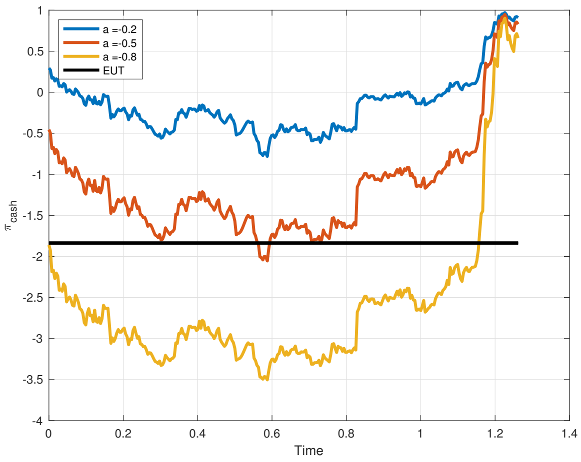

Figure 5.14: Cash portfolio for different values of a when $T = 10$ ($\alpha = 0.5$, $b = 1$).

6. The fear parameter b

As said earlier, when the market becomes too bad (ρ_t is very high), the proportion of the portfolio invested in stocks becomes independent of ρ_t and t and equal to

$$\boldsymbol{\pi} = (\boldsymbol{\mu} - r\mathbf{1})(\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1}\frac{1-b}{1-\alpha} \tag{5.12}$$

The proportion invested in cash is then given by $\pi_0 = 1 - \boldsymbol{\pi} \cdot \mathbf{1}$. With $\alpha = 0.5$ and $T = 2$, $\pi_0 = 1 - 2.836(1 - b)$. In Figure 5.16, the market has reached this state of badness during the COVID-19 crisis and the investor made up his portfolio in accordance with the preceding formula.

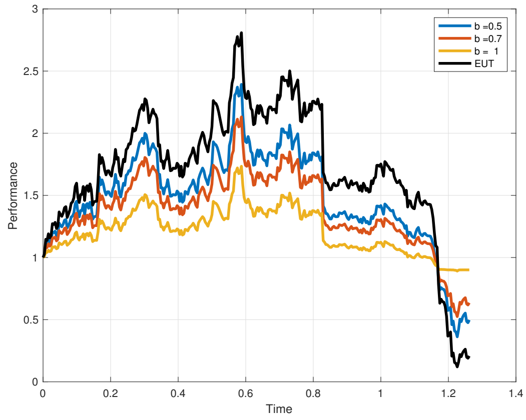

Figure 5.15: Wealth process for different values of b when $T = 2$ ($\alpha = 0.5$, $a = -0.2$).

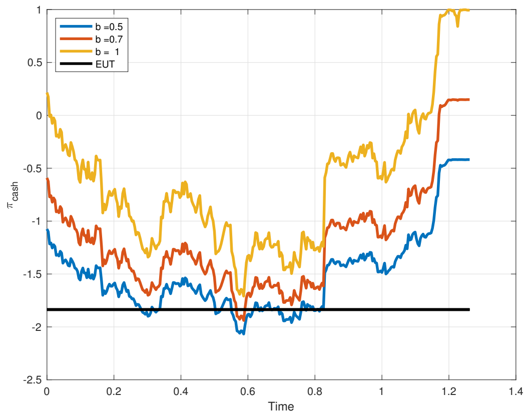

Figure 5.16: Cash portfolio for different values of b when $T = 2$ ($\alpha = 0.5$, $a = -0.2$).

The main feature of the fear parameter b is to reduce the exposition in risky assets. A larger value of this parameter, $b \to 1$, will increase the proportion of the portfolio invested in cash (compared to EUT) which can lead to a less performing investment during a period of bull market (see Figure 5.17 and 5.18). However losses can be drastically reduced during a period of bear market thanks to this effect of closing a large part of the positions in stocks.

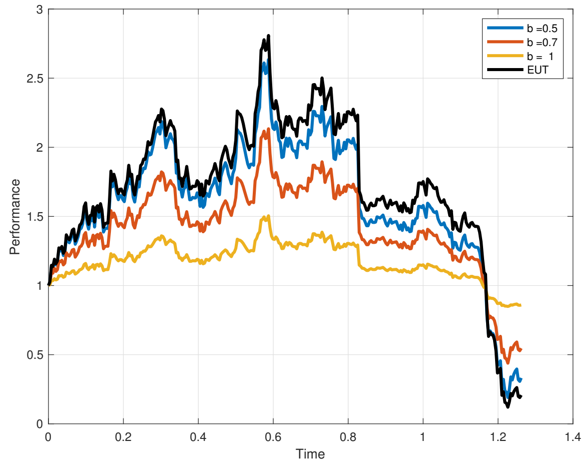

Figure 5.17: Wealth process for different values of b when $T = 10$ ($\alpha = 0.5$, $a = -0.2$).

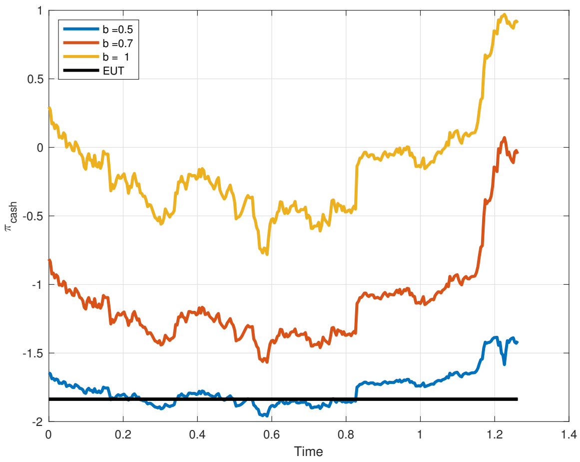

Figure 5.18: Cash portfolio for different values of b when $T = 10$ ($\alpha = 0.5$, $a = -0.2$).

7. Risk aversion coefficient α

The risk aversion coefficient plays a complex role in the CPT allocation problem especially in the well-posedness of the initial problem. Note that for $T = 10$ and $\alpha = 0.7$ (see Figure 5.21 and 5.22), the problem is at the limit of the well-posedness. As expected, a larger value of this coefficient (less risk aversion) will lead to a higher proportion of the portfolio invested in stocks.

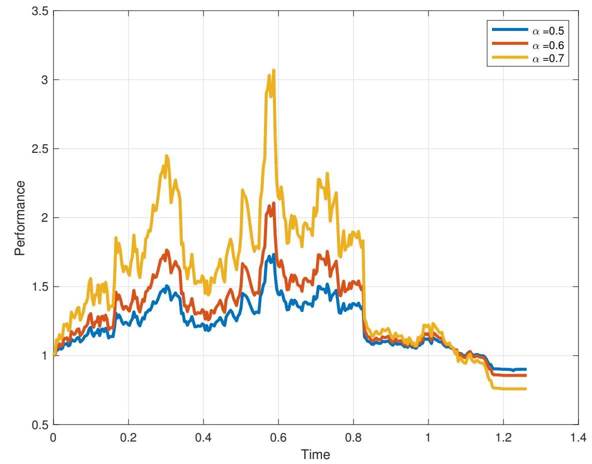

Figure 5.19: Wealth process for different values of α when $T = 2$ ($a = -0.2$, $b = 1$).

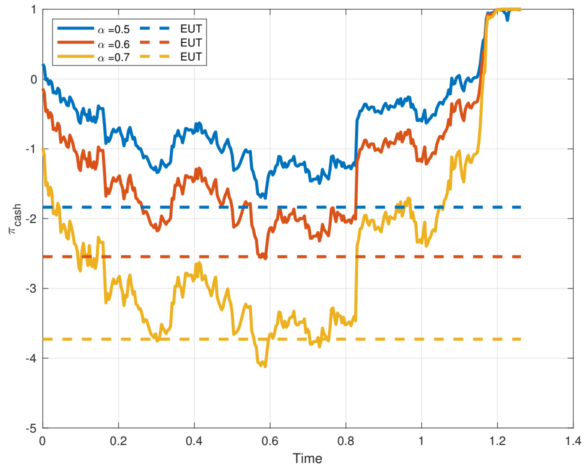

Figure 5.20: Cash portfolio for different values of α when $T = 2$ ($a = -0.2$, $b = 1$).

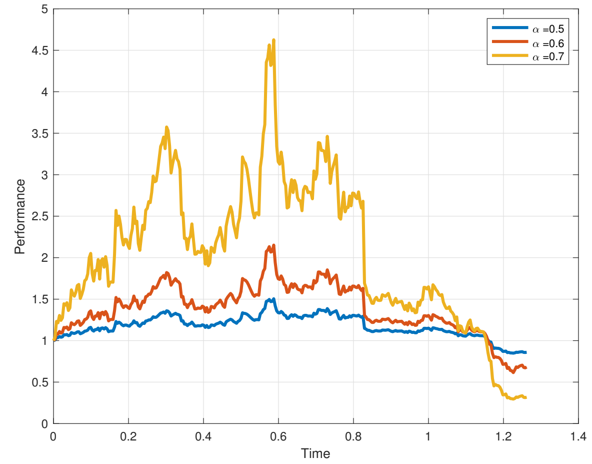

Figure 5.21: Wealth process for different values of α when $T = 10$ ($a = -0.2$, $b = 1$).

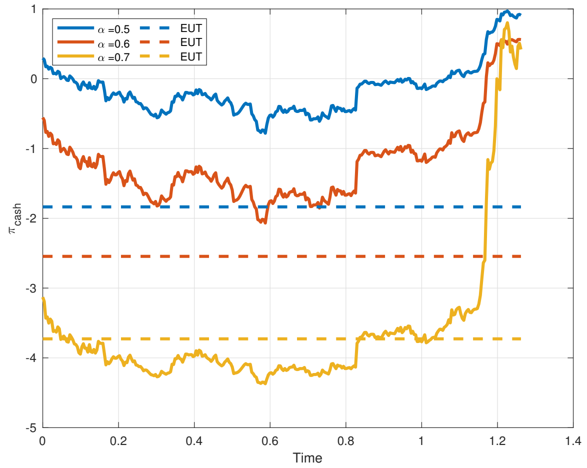

Figure 5.22: Cash portfolio for different values of α when $T = 10$ ($a = -0.2$, $b = 1$).

8. Strategy with initial loss: $x_0 = -1$

8.1 Case without Loss Control

When $x_0 < 0$, the economic agent needs to take leverage to reach his objective. It has been already shown that the maximal possible loss is equal to $X_- = \frac{x_+^* - x_0}{\mathbb{E}\left[\rho_T \mathbf{1}_{\rho_T > c^*}\right]}$. The maximum possible loss, X_- for different values of the risk aversion coefficient (α) and investment horizon (T) is depicted in the table below.

	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$
$T = 1.3$	2.4031	1.2025	1.0629
$T = 10$	71.5936	5.9242	2.9601

The maximum possible loss, X_-, has higher value for lower of α. When $\alpha = 0.5$ and $T = 10$, X_- is very huge compared to the value of the initial position x_0. When $T = 1.3$ (note that the study period ends at $T = 1.25$), the market is going to end with high probability in the bad states of the world ($\rho > c^*$) which means that the investor will suffer a loss position equal to X_-. In 2019 (when the market is good), the portfolio with lower α is the best performing but it is catastrophic during the COVID-19 crisis. When $T = 10$, the value of the portfolio felt to -9.8 while the initial capital is equal to $x_0 = -1$. However, the investment horizon is quite long and the market has good chances to retrieve its levels before the crisis.

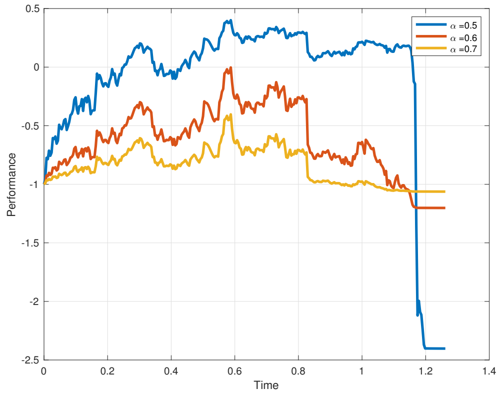

Figure 5.23: Wealth process for different values of α when $T = 1.3$ and initial loss position: $x_0 = -1$.

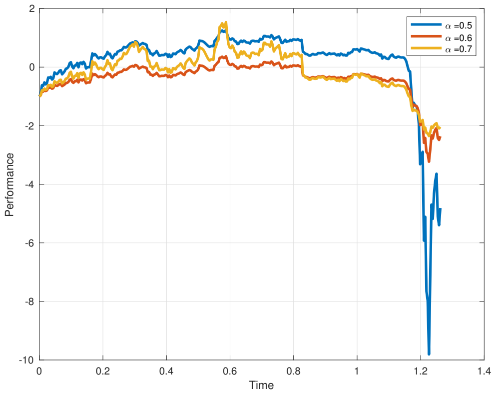

Figure 5.24: Wealth process for different values of α when $T = 10$ and initial loss position: $x_0 = -1$.

8.2 Case with Loss Control

In this case, an exogenous maximum loss $L = 1.3$ have been imposed. The strategy has been computed using the same parameters as before but the risk aversion parameter has been fixed to $\alpha = 0.5$. Figures 5.26 and 5.26 show the corresponding performances when $T = 1.3$ and $T = 10$ respectively. The corresponding optimal values are $(c, c_2, x_+) = (1.0225, \infty, 0.0536)$ for $T = 1.3$ and $(c, c_2, x_+) = (0.5858, \infty, 0.0233)$ for $T = 10$. When $T = 1.3$ (close to maturity), the maximum loss is hit. As expected, the strategy with loss control is performing not as good as the strategy without it when the market is good. When $T = 10$, the loss control protection is running well but the strategy is not performing at all: the performance curve is "flat".

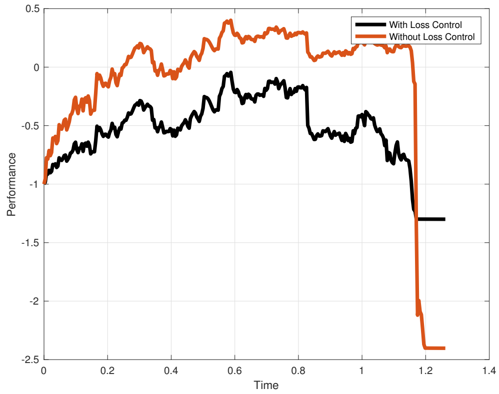

Figure 5.25: Wealth process with $T = 1.3$ and maximum loss $L = 1.3$.

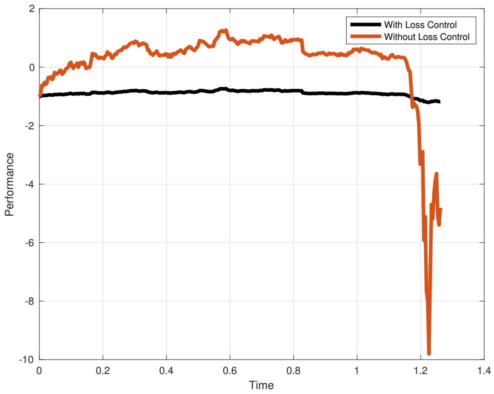

Figure 5.26: Wealth process with $T = 10$ and maximum loss $L = 1.3$.

Chapter 6

Concluding remarks

In this paper, the portfolio selection model *à la* Cumulative Prospect Theory of Jin and Zhou [9] and the loss control extension of Zhang [10] have been implemented. Although this problem is difficult to solve (ill-posedness, non-convex optimisation, nonlinear programming), the solution pattern is quite simple: it is a combinaison of binary options, the underlying being the pricing kernel (i.e. the state of the market). The economic interpretation is a gambling strategy based on good states of the market: if the investor begins in a gain position, he will invest his extra money like a EUT agent in a *gain contingent claim* (binary option). If he suffers a loss position, he needs to short-sell the *loss contingent claim* for the purpose of getting enough funds to buy the *gain contingent claim*. In the case with loss control, the investors short-sells a third contingent claim and the solution has a three pieces shape: a gain domain (good market) and two constant loss domains (bad market and very bad market). It has been showed that the investor who is selling the loss contingent claim is exposed to dramatic losses if the market goes very bad. This is evident that the exogenous imposition of the maximum loss (loss control) is a key feature of this portfolio allocation problem.

Many parameters are involved in this CPT allocation problem: some come from the profile of the investor (distortion function, risk aversion, investment horizon,...) others from the market itself. A sensitivity analysis has been performed on all the parameters. An interesting feature is that in case of very good market, the portfolio allocation becomes independent of the market evolution and time. The investor behaves like a EUT investor with a lower risk aversion, the equivalent risk aversion coefficient being determined by the input parameters. The same feature appears when the market is very bad with the particularity that the investor closes all his positions in risky assets when particular parameters are used.

The sensitivity analysis has been performed on real data coming from the Belgium Stock Market. During this analysis, three performances were compared to each others: the performances of the CPT portfolio, the performances of the EUT portfolio and the performances of the BEL20 index. The EUT portfolio performed badly during the COVID-19 crisis due to its inflexibility: it is completely independent of the state of the market. This investment strategy cannot respond to important event such as the stock market krach. By contrast, the flexible CPT portfolio and its great reactivity successfully managed this difficult period. For the most defensive investors, the CPT outperformed the market after the stock market krach (but their performances during good market period were limited).

The case study on real data supposed constant market and geometric brownian motion dynamic for the stocks. This is well-known to be a strong hypothesis because logreturns are non-normal and present volatility clustering. Such analytic solutions (and available analytic form for the replicating strategy) can only be obtained when this hypothesis is made. Further work can involve more precise models (e.g stochastic volatility models like Heston's) although the numerical solution might be very difficult to obtain.

Appendix A

Positive part problem

This chapter is inspired by Appendix C of Jin and Zhou [9].
First, let's consider the general maximisation problem involving the Choquet integral:

$$\boxed{\begin{aligned} &\underset{X}{\operatorname{argmax}} \quad V_1(X) = \int_0^\infty T\left(P[u(X) > y]\right) dy \\ &\text{s. t.} \quad \mathbb{E}\left[\xi X\right] = a,\ X \geq 0 \text{ a.s.,} \end{aligned}} \tag{A.1}$$

where ξ is a given strictly positive continuous random variable with cdf $F_\xi(\cdot)$, $a \geq 0$, $T(\cdot)$ is a distorsion function and $u(\cdot)$ is a utility function.

The main difficulty is that (A.1) is a non-convex problem with a constrain. Thus Lagrange multiplier does not apply directly. The approach is to change the decision variable and turn (A.1) into a convex optimization problem through a series of transformations.

Quantile formulation

The idea behind the quantile formulation is to change the decision variable X by its cumulative distribution function. This is illustrated through the following lemma:

Lemma 1 (Quantile formulation). *If problem* (A.1) *admits an optimal solution* X^* *whose distribution function is* $G_{X^*}(\cdot)$ *then* $X^* = G_{X^*}^{-1}(1 - F_\xi(\xi))$

Lemma 1 implies that the solution of (A.1) must be anti-comonotonic with ξ. Define $Z := 1 - F_\xi(\xi)$, then $Z \sim Unif(0,1)$ and $\xi = F_\xi^{-1}(1 - Z)$. Thus from Lemma (1), one only needs to find variables of the form $G_{X^*}^{-1}(Z)$ in order to solve (A.1). Introducing the following problem:

$$\boxed{\begin{aligned} &\underset{G_X}{\operatorname{argmax}} \quad V_1(G_X) = \int_0^\infty T\left(P[u(G_X^{-1}(Z)) > y]\right) dy \\ &\text{s. t.} \quad \mathbb{E}\left[F_\xi^{-1}(1 - Z) G_X^{-1}(Z)\right] = a,\ G_X(\cdot) \text{ is a cdf} \end{aligned}} \tag{A.2}$$

the following result holds:

Proposition 3. *If* $G_X^*(\cdot)$ *is optimal for* (A.2) *then* $X^* := (G_X^*)^{-1}(Z)$ *is optimal for* (A.1). *Conversely, if* X^* *is optimal for* (A.1) *then its cdf* $G_X^*(\cdot)$ *is optimal for* (A.2) *and* $X^* = (G_X^*)^{-1}(Z)$

The objective function of (A.2) can be rewritten. Defining $\overline{T}(x) \coloneqq T(1-x)$:

$$\begin{aligned} V_1(G_X) &= \int_0^\infty T\left(P[u(G_X^{-1}(Z)) > y]\right) dy = \int_0^\infty \overline{T}\left(P[u(G_X^{-1}(Z)) \leq y]\right) dy \\ &= \int_0^\infty T\left(P[Z \leq G_X(u^{-1}(y))]\right) dy = \int_0^\infty T\left(G_X(u^{-1}(y))\right) dy \quad \Big(Z \sim Unif(0,1)\Big) \\ &= \int_0^1 u\left(G_X^{-1}(\overline{T}^{-1}(t))\right) dt \quad \Big(\text{by integration of inverse functions }^{1}\Big) \\ &= -\int_0^1 u\left(G_X^{-1}(s)\right) \overline{T}'(s) ds \quad \left(s \coloneqq \overline{T}^{-1}(t),\, ds = \frac{1}{\overline{T}'(t)} dt\right) \\ &= \int_0^1 u\left(G_X^{-1}(s)\right) T'(1-s) ds \\ &= \mathbb{E}\left[u\left(G_X^{-1}(Z)\right) T'(1-Z)\right] \end{aligned} \tag{A.3}$$

Denoting the set of quantile functions by $\mathcal{G} = \{g : [0,1] \to \mathbb{R}^+ \text{is not decreasing with } g(0) = 0\}$ and considering $g(\cdot) = G_X^{-1}(\cdot)$, problem (A.2) can be rewritten:

$$\boxed{\begin{aligned} &\underset{g(\cdot)}{\text{argmax}} \quad \overline{V}_1(g) = \mathbb{E}\left[u\left(g(Z)\right) T'(1-Z)\right] \\ &\text{s. t.} \quad \mathbb{E}\left[F_\xi^{-1}(1-Z) g(Z)\right] = a,\ g \in \mathcal{G} \end{aligned}} \tag{A.4}$$

The objective function $\overline{V}_1(g)$ is now concave because $T'(\cdot) > 0$ and $u(\cdot)$ is concave. Moreover, the constrain in linear in g. Problem (A.4) can thus be solved by Lagrange multipliers:

$$\begin{aligned} \underset{g \in \mathcal{G}}{\text{argmax}} \quad \mathcal{L} &= \mathbb{E}\left[u\left(g(Z)\right) T'(1-Z) - \lambda F_\xi^{-1}(1-Z) g(Z)\right] \\ \frac{\partial \mathcal{L}}{\partial g} &= u'\left(g(Z)\right) T'(1-Z) - \lambda F_\xi^{-1}(1-Z) \end{aligned} \tag{A.5}$$

The optimal function $g^*(\cdot)$ is thus given by:

$$\frac{\partial \mathcal{L}}{\partial g} = 0 \iff g^*(z) = (u')^{-1}\left(\frac{\lambda F_\xi^{-1}(1-z)}{T'(1-z)}\right) \tag{A.6}$$

If $F_\xi^{-1}(z)/T'(z)$ is non-decreasing then $g(z)$ is non-decreasing and it solves (A.4). Using Lemma 1, the optimal contingent claim satisfying (A.1) is given by:

$$X^* = (u')^{-1}\left(\frac{\lambda \xi}{T'\left(F_\xi(\xi)\right)}\right) \tag{A.7}$$

Applying this results to problem (4.7) gives:

$$Y^* = (u'_+)^{-1}\left(\frac{\bar{\lambda} \rho_T}{T'_A\left(F_A(\rho_T)\right)}\right) \tag{A.8}$$

---

[1] If $f(a) = c$ and $f(b) = d$ then $\int_a^b f(x)dx + \int_c^d f^{-1}(x)dx = bd - ac$. See [14]

Given $A := \{\omega : \rho_T \leq c\}$, $F_A(x) := \mathbb{P}_A[\rho_T \leq x] = \dfrac{\mathbb{P}[\rho_T \leq x \wedge c]}{\mathbb{P}[\rho_T \leq c]} = \dfrac{F_\rho(x \wedge c)}{\mathbb{P}[A]}$. The derivative of $T_A(\cdot)$ is given by: $T'_A(x) = \dfrac{\mathbb{P}[A]}{T_+(\mathbb{P}[A])}\, T'_+(x\,\mathbb{P}[A])$. Because $\rho_T \leq c$ on A, it follows that:

$$\frac{\rho_T}{T'_A\left(F_A(\rho_T)\right)} = \frac{\rho_T}{T'_+(F_\rho(\rho_T))}\,\frac{T_+(\mathbb{P}[A])}{\mathbb{P}[A]} \tag{A.9}$$

Defining $\lambda := \dfrac{T_+(\mathbb{P}[A])}{\mathbb{P}[A]}\,\bar{\lambda}$, the optimal solution Y^* can be rewritten:

$$Y^* = (u'_+)^{-1}\left(\frac{\lambda\rho_T}{T'\left(F_\rho(\rho_T)\right)}\right) \tag{A.10}$$

By optimality of $(X_T^+)^* = Y^*\,\mathbf{1}_A$:

$$(X_T^+)^* = (u'_+)^{-1}\left(\frac{\lambda\rho_T}{T'\left(F_\rho(\rho_T)\right)}\right)\mathbf{1}_{\rho_T \leq c} \tag{A.11}$$

The following theorem summarises the previous results:

Theorem 16 (Positive part problem)**.** *Given parameters (c, x_+), the positive part problem is the following Choquet maximisation problem:*

$$\begin{aligned} &\underset{X_T^+}{\text{argmax}} \quad V_+(X_T^+) = \int_0^\infty T_+\left(P[u_+(X_T^+) > y]\right) dy \\ &s.\ t. \quad \mathbb{E}^{\mathbb{P}}\left[\rho_T X_T^+\right] = x_+, \quad X_T^+ \geq 0 \ \ a.s., \quad X_T^+ = 0 \ \ a.s. \ \ on \ A^c, \end{aligned} \tag{A.12}$$

The optimal contingent claim X_T^* maximising the objective functional is given by:

$$(X_T^+)^*(\lambda) = (u'_+)^{-1}\left(\frac{\lambda\rho_T}{T'_+\left(F_\rho(\rho_T)\right)}\right)\mathbf{1}_{\rho_T \leq c} \tag{A.13}$$

with the optimal value of the objective functional $v_+(c, x_+)$:

$$v_+(c, x_+) = \mathbb{E}\left[u_+\left((u'_+)^{-1}\left(\frac{\lambda\rho_T}{T'_+\left(F_\rho(\rho_T)\right)}\right)\right) T'_+\left(F_\rho(\rho_T)\right)\mathbf{1}_{\rho_T \leq c}\right] \tag{A.14}$$

where $\lambda > 0$ is the unique real number satisfying $\mathbb{E}\left[\rho_T (X_T^+)^*(\lambda)\right] = x_+$.

Appendix B

Negative part problem

This chapter is inspired by Appendix D of Jin and Zhou [9].
A general minimisation problem involving a Choquet integral is given by:

$$\boxed{\begin{aligned} &\underset{X}{\text{argmin}} \quad V_2(X) = \int_0^\infty T\left(P[u(X) > y]\right) dy \\ &\text{s. t.} \quad \mathbb{E}\left[\xi X\right] = a,\ X \geq 0 \text{ a.s. } , \end{aligned}} \tag{B.1}$$

Similarly to the positive part problem, using quantile formulation, problem (B.1) can be rewritten:

$$\boxed{\begin{aligned} &\underset{g(\cdot)}{\text{argmin}} \quad \overline{V}_2(g) = \mathbb{E}\left[u\left(g(Z)\right) T'(1-Z)\right] \\ &\text{s. t.} \quad \mathbb{E}\left[F_\xi^{-1}(1-Z)g(Z)\right] = a,\ g \in \mathcal{G} \end{aligned}} \tag{B.2}$$

Since the above problem is to *minimise* a concave objective function, its solution has different structure compared to the positive part problem and cannot be solved by Lagrange multipliers. In linear programming, the solution of (B.2) should be a *corner point solution.* Solution to such a problem goes beyond the scope of this work. The following result will be considered without proof [1]:

Proposition 4. *The form of the optimal solution for problem* (B.2) *(if it exists) is:*

$$g^*(z) = \frac{a}{\mathbb{E}\left[F_\xi^{-1}(Z)\mathbf{1}_{(b,1)}(Z)\right]} \mathbf{1}_{(b,1)}(z) \tag{B.3}$$

with some $b \in [0, 1)$.

Using Proposition 4, problem (B.2) can be rewritten:

$$\boxed{\begin{aligned} &\underset{g(\cdot)}{\text{argmin}} \quad \widetilde{V}_2(g) = \mathbb{E}\left[u\left(g(Z)\right) T'(1-Z)\right] \\ &\text{s. t.} \quad g(\cdot) = \frac{a}{\mathbb{E}\left[F_\xi^{-1}(Z)\mathbf{1}_{(b,1)}(Z)\right]} \mathbf{1}_{(b,1)}(\cdot) \end{aligned}} \tag{B.4}$$

---

[1] Proof of this proposition can be found in [9]

The functional can be expressed as:

$$\begin{aligned}\widetilde{V}_2(g) &= \mathbb{E}\left[u\left(g(Z)\right)T'(1-Z)\right] = \mathbb{E}\left[u\left(\frac{a}{\mathbb{E}\left[F_\xi^{-1}(Z)\mathbf{1}_{(b,1)}(Z)\right]}\mathbf{1}_{(b,1)}(Z)\right)T'(1-Z)\right]\\ &= \int_b^1 u\left(\frac{a}{\mathbb{E}\left[F_\xi^{-1}(Z)\mathbf{1}_{(b,1)}(Z)\right]}\right)T'(1-t)dt\\ &= u\left(\frac{a}{\mathbb{E}\left[F_\xi^{-1}(Z)\mathbf{1}_{(b,1)}(Z)\right]}\right)T(1-b)\\ &= u\left(\frac{a}{\mathbb{E}\left[\xi\mathbf{1}_{\xi>c}\right]}\right)T(P[\xi>c]) \quad c := F_\xi^{-1}(b)\end{aligned} \tag{B.5}$$

The goal is to find the optimal c^* that solves the problem:

$$\underset{0\le c<\bar{\xi}}{\operatorname{argmin}} \quad u\left(\frac{a}{\mathbb{E}\left[\xi\mathbf{1}_{\xi>c}\right]}\right)T(P[\xi>c]) \tag{B.6}$$

Using the quantile formulation (Lemma 1) and the form of the quantile function (Proposition 4), the optimal contingent claim has the form:

$$X_T^* = \frac{a}{\mathbb{E}\left[\xi\mathbf{1}_{\xi>c^*}\right]}\mathbf{1}_{\xi>c^*} \tag{B.7}$$

where c^* minimises (B.6).

Applying the previous results to the negative part problem gives:

Theorem 17 (Negative part problem)**.** *Given parameters (c, x_+), the negative part problem is the following Choquet minimisation problem:*

$$\begin{aligned}&\underset{X_T^-}{\operatorname{argmin}} \quad V_-(X_T^-) = \int_0^\infty T_-\left(P[u_-(X_T^-)>y]\right)dy\\ &s.\ t. \quad \mathbb{E}^{\mathbb{P}}\left[\rho_T X_T^-\right] = x_- = x_+ - x_0, \quad X_T^- \ge 0 \ a.s., \quad X_T^- = 0 \ a.s.\ on\ A,\end{aligned} \tag{B.8}$$

The optimal contingent claim X_T^* minimising the objective functional is given by:

$$(X_T^-)^*(c^*) = \frac{x_+ - x_0}{\mathbb{E}\left[\rho_T\mathbf{1}_{\rho_T>c^*}\right]}\mathbf{1}_{\rho_T>c^*} \tag{B.9}$$

with the optimal value of the objective functional $v_-(c, x_+)$:

$$v_-(c,x_+) = u_-\left(\frac{x_+ - x_0}{\mathbb{E}\left[\rho_T\mathbf{1}_{\rho_T>c^*}\right]}\right)T_-(P[\rho_T>c^*]) \tag{B.10}$$

where c^* minimises: $u_-\left(\frac{x_+-x_0}{\mathbb{E}\left[\rho_T\mathbf{1}_{\rho_T>c}\right]}\right)T_-(P[\rho_T>c])$

Appendix C

Replication of truncated lognormal binary options

The pricing kernel $\rho(t,T) = \frac{\rho_T}{\rho_t} \sim Log\mathcal{N}\left(\mu_\rho(T-t), \sigma_\rho^2(T-t)\right)$ is lognormally distributed with $\mu_\rho = -(r + \frac{1}{2}\|\boldsymbol{\theta}\|^2)$ and $\sigma_\rho = \|\boldsymbol{\theta}\|$. Its SDE is given by:

$$\frac{d\rho_t}{\rho_t} = -r\,dt - \boldsymbol{\theta} \bullet \mathbf{dW_t} \tag{C.1}$$

with $\boldsymbol{\theta} = \boldsymbol{\sigma}(\boldsymbol{\mu} - \mathbf{1}\,r)$. The random variable $\rho_T^\eta\,\mathbf{1}_{\rho_T \in (z_1,z_2)}$ is a *truncated lognormal random variable.* Its expected value is easily computed:

$$\mathbb{E}\left[\rho_T^\eta\,\mathbf{1}_{\rho_T \in (z_1,z_2)}\right] = e^{\eta\mu_\rho + \frac{1}{2}\eta^2\sigma_\rho^2}\left[\Phi\left(\frac{log\, z_2 - \mu_\rho}{\sigma_\rho} - \eta\sigma_\rho\right) - \Phi\left(\frac{log\, z_1 - \mu_\rho}{\sigma_\rho} - \eta\sigma_\rho\right)\right], \tag{C.2}$$

where $\Phi(\cdot)$ is the cdf of a standard normal variable. The first term is the expectation of the non-truncated lognormal variable ρ_T^η while the term in brackets represents the truncation.
Using the fundamental theorem of asset pricing, the price process, X_t, of the contingent claim $\rho_T^\eta\,\mathbf{1}_{\rho_T \in (z_1,z_2)}$ is given by:

$$\begin{aligned} X_t &= \mathbb{E}^{\mathbb{P}}\left[\rho_T^\eta\,\mathbf{1}_{\rho_T \in (z_1,z_2)}\,\rho(t,T)|\mathcal{F}_t\right] \\ &= \rho_t^\eta \int_{z_1/\rho_t}^{z_2/\rho_t} u^\eta \frac{1}{\sqrt{2\pi}\,\sigma_\rho\sqrt{T-t}}\,\frac{1}{u} e^{-\frac{1}{2}\left(\frac{log\, u - \mu_\rho(T-t)}{\sigma_\rho\sqrt{T-t}}\right)^2} du, \\ &= \frac{\rho_t^\eta}{\sigma_\rho\sqrt{T-t}} \int_{z_1/\rho_t}^{z_2/\rho_t} u^\eta\,\phi\left(\frac{log\, u - \mu_\rho(T-t)}{\sigma_\rho\sqrt{T-t}}\right) du := f(t,\rho_t), \end{aligned} \tag{C.3}$$

where $\phi(\cdot)$ is the pdf of a standard normal variable. According to the martingale representation theorem the diffusion part of $f(t,\rho_t)$ (after applying Itô's lemma) should be equal to the diffusion part of the mutual fund process (equation (3.19)):

$$-\frac{\partial f(t,\rho_t)}{\partial \rho}\,\rho_t\,\boldsymbol{\theta} = X_t\boldsymbol{\pi_t}'\,\boldsymbol{\sigma} \tag{C.4}$$

The general formula of the replicating strategy of a contingent claim whose price process is only function of time and pricing kernel, $X_t = f(t,\rho_t)$, is given by:

$$\boxed{\boldsymbol{\pi_t}\, X_t = -\rho_t \, \frac{\partial f(t,\rho_t)}{\partial \rho} \, (\boldsymbol{\sigma\sigma'})^{-1}(\boldsymbol{\mu} - \mathbf{1}\, r)} \tag{C.5}$$

In this case,

$$\frac{\partial f(t,\rho_t)}{\partial \rho} = \frac{\eta}{\rho_t}\, f(t,\rho_t) + \frac{\rho_t^{\eta}}{\sigma_\rho\sqrt{T-t}} \frac{\partial}{\partial \rho} \int_{z_1/\rho_t}^{z_2/\rho_t} u^{\eta}\, \phi\left(\frac{\log u - \mu_\rho(T-t)}{\sigma_\rho\sqrt{T-t}}\right) du \tag{C.6}$$

Using Leibniz's integral rule: $\frac{d}{dx}\int_{a(x)}^{b(x)} g(s)ds$ with $a(x) = z_1/x$, $b(x) = z_2/x$ and $g(s) = s^{\eta}\, \phi\left(\frac{\log s - \mu_\rho(T-t)}{\sigma_\rho\sqrt{T-t}}\right)$ gives:

$$\begin{aligned}
\frac{\partial f(t,\rho_t)}{\partial \rho} =& \frac{\eta}{\rho_t}\, f(t,\rho_t) + \frac{\rho_t^{\eta}}{\sigma_\rho\sqrt{T-t}}\, \frac{1}{\rho_t^{\eta}} \\
&\left[-\frac{z_2}{\rho_t^2}\, z_2^{\eta}\phi\left(\frac{\log z_2 - \log \rho_t - \mu_\rho(T-t)}{\sigma_\rho\sqrt{T-t}}\right) - (-\frac{z_1}{\rho_t^2})z_1^{\eta}\phi\left(\frac{\log z_1 - \log \rho_t - \mu_\rho(T-t)}{\sigma_\rho\sqrt{T-t}}\right)\right] \\
=& \frac{\eta}{\rho_t}\, X_t \\
&- \frac{1}{\rho_t^2\, \sigma_\rho\sqrt{T-t}} \left[z_2^{\eta+1}\phi\left(\frac{\log\left(z_2/\rho_t\right) - \mu_\rho(T-t)}{\sigma_\rho\sqrt{T-t}}\right) - z_1^{\eta+1}\phi\left(\frac{(z_1/\rho_t) - \mu_\rho(T-t)}{\sigma_\rho\sqrt{T-t}}\right)\right]
\end{aligned} \tag{C.7}$$

Putting this result into (C.5):

$$\boxed{\begin{aligned}
\boldsymbol{\pi_t}\, X_t = \Bigg\{ & \frac{1}{\rho_t\, \sigma_\rho\sqrt{T-t}} \left[z_2^{\eta+1}\phi\left(\frac{\log\left(z_2/\rho_t\right) - \mu_\rho(T-t)}{\sigma_\rho\sqrt{T-t}}\right) - z_1^{\eta+1}\phi\left(\frac{(z_1/\rho_t) - \mu_\rho(T-t)}{\sigma_\rho\sqrt{T-t}}\right)\right] \\
& -\eta X_t \Bigg\}\, (\boldsymbol{\sigma\sigma'})^{-1}(\boldsymbol{\mu} - \mathbf{1}\, r)
\end{aligned}} \tag{C.8}$$

When $z_2 = \infty$ then $db(x)/dx = 0$ and:

$$\boldsymbol{\pi_t}\, X_t = \left[-\frac{1}{\rho_t\, \sigma_\rho\sqrt{T-t}}\, z_1^{\eta+1}\phi\left(\frac{(z_1/\rho_t) - \mu_\rho(T-t)}{\sigma_\rho\sqrt{T-t}}\right) - \eta X_t\right] (\boldsymbol{\sigma\sigma'})^{-1}(\boldsymbol{\mu} - \mathbf{1}\, r) \tag{C.9}$$

Note that the integral of X_t can be simplified:

$$\begin{aligned}
X_t &= \frac{\rho_t^{\eta}}{\sigma_\rho\sqrt{T-t}} \int_{z_1/\rho_t}^{z_2/\rho_t} u^{\eta}\, \phi\left(\frac{\log u - \mu_\rho(T-t)}{\sigma_\rho\sqrt{T-t}}\right) du \\
&= \rho_t^{\eta}\, \mathbb{E}^{\mathbb{P}}\left[\rho(t,T)^{\eta+1}\, \mathbf{1}_{\rho(t,T)\in(z_1/\rho_t, z_2/\rho_t)}\right]
\end{aligned} \tag{C.10}$$

Defining $\mu_\rho(t) = \mu_\rho\,(T-t)$ and $\sigma_\rho(t) = \sigma_\rho\sqrt{T-t}$, the expected value can be evaluated using the formula (C.2) for a truncated lognormal random variable :

$$\begin{aligned} X_t =& \rho_t^{\eta}\, e^{(\eta+1)\mu_\rho(t)+\frac{1}{2}(\eta+1)^2\sigma_\rho(t)^2} \left[\Phi\left(\frac{log\,(z_2/\rho_t)-\mu_\rho(t)}{\sigma_\rho(t)}-(\eta+1)\sigma_\rho(t)\right)\right. \\ &\left.-\Phi\left(\frac{log\,(z_1/\rho_t)-\mu_\rho(t)}{\sigma_\rho(t)}-(\eta+1)\sigma_\rho(t)\right)\right] \end{aligned} \tag{C.11}$$

For exemple, let's replicate the terminal wealth of the CPT portfolio when $x_0 > 0$:

$$X_T = \rho_T^{(a-1)/(1-\alpha)}\, \mathbf{1}_{\rho_T \le c_0} + c_0^{(a-b)/(1-\alpha)}\rho_T^{(b-1)/(1-\alpha)}\, \mathbf{1}_{\rho_T > c_0} \tag{C.12}$$

If the price processes of the two binary options are designated by x_t^1 and x_t^2 respectively:

$$\boldsymbol{\pi}_\mathbf{t}^\mathbf{1}\, x_t^1 = \left[\frac{1}{\rho_t\,\sigma_\rho\sqrt{T-t}}\, c_0^{(a-\alpha)/(1-\alpha)}\phi\left(\frac{(c_0/\rho_t)-\mu_\rho(T-t)}{\sigma_\rho\sqrt{T-t}}\right)-\frac{a-1}{1-\alpha}x_t^1\right]\,(\boldsymbol{\sigma\sigma'})^{-1}(\boldsymbol{\mu}-\mathbf{1}\,r) \tag{C.13}$$

$$\begin{aligned} \boldsymbol{\pi}_\mathbf{t}^\mathbf{2}\, x_t^2 &= \left[-\frac{c_0^{(a-b)/(1-\alpha)}}{\rho_t\,\sigma_\rho\sqrt{T-t}}\, c_0^{(b-\alpha)/(1-\alpha)}\phi\left(\frac{(c_0/\rho_t)-\mu_\rho(T-t)}{\sigma_\rho\sqrt{T-t}}\right)-c_0^{(a-b)/(1-\alpha)}\,\frac{b-1}{1-\alpha}x_t^2\right]\,(\boldsymbol{\sigma\sigma'})^{-1}(\boldsymbol{\mu}-\mathbf{1}\,r) \\ &= \left[-\frac{1}{\rho_t\,\sigma_\rho\sqrt{T-t}}\, c_0^{(a-\alpha)/(1-\alpha)}\phi\left(\frac{(c_0/\rho_t)-\mu_\rho(T-t)}{\sigma_\rho\sqrt{T-t}}\right)-c_0^{(a-b)/(1-\alpha)}\,\frac{b-1}{1-\alpha}x_t^2\right]\,(\boldsymbol{\sigma\sigma'})^{-1}(\boldsymbol{\mu}-\mathbf{1}\,r) \end{aligned} \tag{C.14}$$

Then,

$$\boldsymbol{\pi}_\mathbf{t}\, X_t = \boldsymbol{\pi}_\mathbf{t}^\mathbf{1}\, x_t^1 + \boldsymbol{\pi}_\mathbf{t}^\mathbf{2}\, x_t^2 = \left[(1-a)\,x_t^1 + c_0^{(a-b)/(1-\alpha)}\,(1-b)\,x_t^2\right]\,\frac{1}{1-\alpha}\,(\boldsymbol{\sigma\sigma'})^{-1}(\boldsymbol{\mu}-\mathbf{1}\,r) \tag{C.15}$$

The optimal portfolio is obtained by dividing the previous expression by $X_t = x_t^1 + c_0^{(a-b)/(1-\alpha)}\,x_t^2$:

$$\boxed{\boldsymbol{\pi}_\mathbf{t} = \frac{\left[(1-a)\,x_t^1 + c_0^{(a-b)/(1-\alpha)}\,(1-b)\,x_t^2\right]}{\left[x_t^1 + c_0^{(a-b)/(1-\alpha)}\,x_t^2\right]}\,\frac{1}{1-\alpha}\,(\boldsymbol{\sigma\sigma'})^{-1}(\boldsymbol{\mu}-\mathbf{1}\,r)} \tag{C.16}$$

Appendix D

Single step binomial tree

Let's consider two times: $t = 0$ and $t = T$. The market is composed of two assets:

- A bank account with risk free rate r
- A risky asset with initial value S_0 at $t = 0$. At time $t = T$ the asset price will be either $S_u = S_0 u$ or $S_d = S_0 d$ with $d < 1 < u$.

The space of probability is thus given by the two states $\Omega = \{\omega_u, \omega_d\}$. The filtration is given by $\mathcal{F} = \{\mathcal{F}_0, \mathcal{F}_T\}$ with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \{\emptyset, \Omega, \omega_u, \omega_d\}$.

Let's consider a european call option X with strike K and maturity T on this risky asset. The two possible terminal payoffs X_T,of this option are $X_u = (S_u - K)_+$ and $X_d = (S_u - K)_+$. The goal is to find the price at $t = 0$, X_0 of this option.

The method consists in finding a replicating strategy $\boldsymbol{\phi_{X,t}} = (\alpha, \beta)$ that replicates the cashflows $X_T = \{X_u, X_d\}$. α is the number of units invested in the bank account while β is the number of shares of the risky asset. Equaling the payoffs gives the following system of equations:

$$\begin{aligned} \alpha e^{rT} + \beta S_u &= X_u \\ \alpha e^{rT} + \beta S_d &= X_d \end{aligned} \tag{D.1}$$

The solutions of this system are:

$$\begin{aligned} \alpha &= e^{-rT} \frac{X_d S_u - X_u S_d}{S_u - S_d} \\ \beta &= \frac{X_u - X_d}{S_u - S_d} \end{aligned} \tag{D.2}$$

The price at $t = 0$ of the call option is thus the initial wealth of the replicating portfolio: $X_0 = V_0 = \alpha + \beta S_0$

$$X_0 = e^{-rT} \frac{X_d u - X_u d}{u - d} + \frac{X_u - X_d}{u - d} \tag{D.3}$$

This expression is rewritten in the more canonical form:

$$X_0 = e^{-rT}\,(q\,X_u + (1-q)\,X_d) = \mathbb{E}^{\mathbb{Q}}[e^{-rT}X_T] \quad q = \frac{e^{rT}-d}{u-d} \tag{D.4}$$

The price of the option is the expectation of the discounted payoff under an equivalent measure of probability $\mathbb{Q}$ on Ω where $\mathbb{E}^{\mathbb{Q}}[\omega_u] = q$ and $\mathbb{E}^{\mathbb{Q}}[\omega_d] = 1 - q$

Appendix E

Martingale method in EUT: The Replication Problem

The optimal terminal wealth in the EU framework with CRRA utility is given by:

$$X_T^* = \frac{X_0}{H_0} \left(\rho_T\right)^{-\frac{\gamma}{1-\gamma}} \tag{E.1}$$

H_t was defined in (3.27). For the rest of the section, $\beta = \dfrac{\gamma}{1-\gamma}$ will to used to have lighter notation. Remember that under $\mathbb{Q}$, discounted assets are martingale. From this, the optimal wealth process is given by:

$$X_t^* = S_t^0 \, \mathbb{E}^{\mathbb{Q}} \left[\left(S_T^0\right)^{-1} X_T^* | \mathcal{F}_t \right] \tag{E.2}$$

Using previous results, the optimal wealth process becomes:

$$X_t^* = \frac{X_0}{H_0} S_t^0 \, \mathbb{E}^{\mathbb{Q}} \left[\left(S_T^0\right)^{-1} \left(\left(S_T^0\right)^{-1} Z_T \right)^{-\frac{\gamma}{1-\gamma}} | \mathcal{F}_t \right] \tag{E.3}$$

Because $\dfrac{d\mathbb{Q}}{d\mathbb{P}} = Z_T$, using Bayes' Theorem, the previous expression can be rewritten:

$$X_t^* = \frac{X_0}{H_0} S_t^0 \frac{\mathbb{E}^{\mathbb{P}} \left[\left(\rho_T\right)^{-\beta} | \mathcal{F}_t \right]}{\underbrace{\mathbb{E}^{\mathbb{P}} \left[Z_T | \mathcal{F}_t \right]}_{Z_t}} = \frac{X_0}{H_0} \left(\rho_t\right)^{-1} \mathbb{E}^{\mathbb{P}} \left[\left(\rho_T\right)^{-\beta} | \mathcal{F}_t \right] \tag{E.4}$$

Multiplying and dividing by $\left(\rho_t\right)^{-\beta}$:

$$X_t^* = X_0 \frac{H_t}{H_0} \left(\rho_t\right)^{-\frac{1}{1-\gamma}} \tag{E.5}$$

1. The dynamics of H_t

Let's define the following Radon-Nikodym derivative:

$$\frac{d\mathbb{Q}^0}{d\mathbb{P}} = Z_T^0 = \exp\left(- \int_0^t \beta \boldsymbol{\theta_s} \bullet \boldsymbol{dW_s} - \frac{1}{2} \int_0^t \beta^2 \left\| \underline{\boldsymbol{\theta_s}} \right\|^2 ds \right) \tag{E.6}$$

It is clear that Z_t^0 is a martingale with dynamics:

$$\frac{dZ_t^0}{Z_t^0} = -\beta \boldsymbol{\theta_s} \bullet \boldsymbol{dW_s} \tag{E.7}$$

Remember that:

$$Z_t^{-\beta} = \exp\left(-\int_0^t \beta \boldsymbol{\theta_s} \bullet \boldsymbol{dW_s} - \frac{1}{2}\int_0^t \beta \|\underline{\boldsymbol{\theta_s}}\|^2 ds\right) \tag{E.8}$$

which can be expressed:

$$Z_t^{-\beta} = Z_t^0 \exp\left[\frac{1}{2}\int_0^t \frac{\beta}{1-\gamma}\|\underline{\boldsymbol{\theta_s}}\|^2 ds\right] \tag{E.9}$$

It is possible to rewrite the ratio in (3.27):

$$\left(\frac{S_t^0}{S_T^0}\frac{Z_T}{Z_t}\right)^{-\beta} = \frac{Z_T^0}{Z_t^0}\exp\left[\beta\int_t^T \left(r_s + \frac{\beta}{1-\gamma}\|\underline{\boldsymbol{\theta_s}}\|^2\right) ds\right] \tag{E.10}$$

For any random variable Y, we have:

$$\mathbb{E}^{\mathbb{P}}\left[Y|\mathcal{F}_t\right] = \frac{\mathbb{E}^{\mathbb{Q}^0}\left[Y\left(\frac{d\mathbb{Q}^0}{d\mathbb{P}}\right)^{-1}|\mathcal{F}_t\right]}{\mathbb{E}^{\mathbb{Q}^0}\left[\left(\frac{d\mathbb{Q}^0}{d\mathbb{P}}\right)^{-1}|\mathcal{F}_t\right]} = \frac{\mathbb{E}^{\mathbb{Q}^0}\left[Y\left(Z_T^0\right)^{-1}|\mathcal{F}_t\right]}{\underbrace{\mathbb{E}^{\mathbb{Q}^0}\left[\left(Z_T^0\right)^{-1}|\mathcal{F}_t\right]}_{Z_t^0}} = \mathbb{E}^{\mathbb{Q}^0}\left[Y\frac{Z_t^0}{Z_T^0}|\mathcal{F}_t\right] \tag{E.11}$$

The process H_t can be expressed as:

$$H_t = \mathbb{E}^{\mathbb{Q}^0}\left[\exp\left\{\beta\int_t^T\left(r_s + \frac{1}{2(1-\gamma)}\|\underline{\boldsymbol{\theta_s}}\|^2\right)ds\right\}|\mathcal{F}_t\right] \tag{E.12}$$

Let's consider the following Lemma:

Lemma 2. *The SDE of process H_t follows:*

$$\frac{dH_t}{H_t} = \mu_t^H dt + \boldsymbol{\sigma_t^H} \bullet \boldsymbol{dW_t} \tag{E.13}$$

Proof. It is possible to rewrite H_t as:

$$H_t = \mathbb{E}^{\mathbb{Q}^0}\left[\exp\left(\int_t^T h_s ds\right)|\mathcal{F}_t\right] \tag{E.14}$$

where h_s is defined in (E.12). Using one more time the abstract Bayes' Theorem:

$$H_t = \mathbb{E}^{\mathbb{P}}\left[\exp\left(\int_0^T h_s ds\right)|\mathcal{F}_t\right]\frac{\exp\left(-\int_0^t h_s ds\right)}{Z_t^0} \tag{E.15}$$

It can be further rewritten:

$$H_t = M_t\frac{Y_t}{Z_t^0} \tag{E.16}$$

where M_t is a martingale with dynamics $dM_t = \boldsymbol{a_t} \bullet \boldsymbol{dW_t}$ for some process a_t. The dynamic of the process Y_t is simply: $dY_t = -h_t Y_t dt$. The dynamic of Z_t^0 was given as : $dZ_t^0 = -Z_t^0 \beta\, \boldsymbol{\theta_t} \bullet \boldsymbol{dW_t}$. Using Ito's Lemma to find the dynamics of H_t will conclude the proof. ☐

2. Replication problem: The optimal portfolio

Remember that the dynamic of the fund is given by 3.19. Under the risk neutral measure $\mathbb{Q}$, the discounted fund process $\left(S_t^0\right)^{-1} F_t$ is martingale:

$$\frac{d\left(\left(S_t^0\right)^{-1} F_t\right)}{\left(S_t^0\right)^{-1} F_t} = \boldsymbol{\pi_t} \bullet \left(\boldsymbol{\sigma_t} \, \boldsymbol{dW_t^{\mathbb{Q}}}\right) \tag{E.17}$$

The dynamic of the optimal wealth is given by (E.5). The discounted optimal wealth process is thus:

$$(S_t^0)^{-1} X_t^* = X_0 \frac{H_t}{H_0} \left((S_t^0)^{-1}\right)^{-\beta} (Z_t)^{-\frac{1}{1-\gamma}} \tag{E.18}$$

Using Ito's Lemma and the dynamics of H_t and Z_t, the dynamic of $(S_t^0)^{-1} X_t^*$ has the following form:

$$\frac{d\left((S_t^0)^{-1} X_t^*\right)}{(S_t^0)^{-1} X_t^*} = [...] \, dt + \frac{1}{1-\gamma}\left(\boldsymbol{\theta_t} + \boldsymbol{\sigma_t^H}\right) \bullet \boldsymbol{dW_s} \tag{E.19}$$

Under the risk neutral measure $\mathbb{Q}$, this process is a martingale:

$$\frac{d\left((S_t^0)^{-1} X_t^*\right)}{(S_t^0)^{-1} X_t^*} = \frac{1}{1-\gamma}\left(\boldsymbol{\theta_t} + \boldsymbol{\sigma_t^H}\right) \bullet \boldsymbol{dW_t^{\mathbb{Q}}} \tag{E.20}$$

The optimal portfolio is finally obtained by identifying the terms in (E.17) and (E.20):

$$\boldsymbol{\pi_t^*} = \frac{1}{1-\gamma}\left(\boldsymbol{\theta_t} + \boldsymbol{\sigma_t^H}\right) \boldsymbol{\sigma_t^{-1}} \tag{E.21}$$

By expressing the terms explicitly, the optimal portfolio process is:

$$\boxed{\boldsymbol{\pi_t^*} = \frac{1}{1-\gamma}\left(\boldsymbol{\mu_t} - \mathbf{1}\, r_t\right) \left(\boldsymbol{\sigma_t \sigma_t'}\right)^{-1} + \boldsymbol{\sigma_t^H} \, \boldsymbol{\sigma_t^{-1}}} \tag{E.22}$$

Appendix F

Elements of decision making under risk

1. Definitions of decision making under risk

Risk aversion. An individual is risk averse if he prefers a riskless prospect (x) to any risky prospect with expected value x:

$$u(w + \mathbb{E}[X]) > \mathbb{E}[u(w + X)] \quad \text{for a random variable} \, X \tag{F.1}$$

Utility function of a risk averse agent. Following Jensen's inequality, the utility function of a risk-averse individual must be concave:

$$\frac{d^2u}{dx^2} < 0 \tag{F.2}$$

Risk Premium. The risk premium π is defined as the maximum amount the economic agent agrees to pay to replace the random variable X by its expected value $\mathbb{E}[X]$. It is the solution of the following equation:

$$u(w + \mathbb{E}[X] - \pi) = \mathbb{E}[u(w + X)] \tag{F.3}$$

The risk premium is dependent of the wealth level w and risk X: $\pi = \pi(w, X)$. Suppose a tossing game where the player receives 100 if a head appears and 0 otherwise. With initial wealth 100, and logarithmic utility function, the risk premium is given by:

$$\begin{aligned} &ln(100 + 50 - \pi) = \frac{1}{2}[ln(100) + ln(200)] \\ &\pi \approx 8.5786 \end{aligned} \tag{F.4}$$

This economic agent agrees to turn down the gamble if he's certain to get an amount greater than 41.4214 ($= E[X] - \pi$). Even if the expectation of the gamble is 50, he agrees to receive less than 50 (but greater than 41.4214) with certainty instead of playing the tossing game.

Arrow-Pratt Approximation. For an initial wealth level w, a risk X is said to be small if the variance of X is sufficiently small. For small risks, the following local approximation of the risk premium holds:

$$\pi(w, X) \approx \frac{1}{2} V[X] r_{abs}(w + \mathbb{E}[X]) \tag{F.5}$$

where $r_{abs}(x)$ is defined as the absolute risk aversion.
Proof. The left-hand side of (F.3) can be seen as a function $\pi \to u(w+\mathbb{E}[X]-\pi)$. A first order Taylor expansion of this function around $w+\mathbb{E}[X]$ is given by:

$$u(w+\mathbb{E}[X]-\pi) \approx u(w+\mathbb{E}[X]) - \pi\, u'(w+\mathbb{E}[X]) \tag{F.6}$$

Because X is small, it is correct to approximate around $\mathbb{E}[X]$. A second order Taylor approximation of the right-hand side of (F.3) around $\mathbb{E}[X]$ is given by:

$$u(w+X) \approx u(w+\mathbb{E}[X]) + u'(w+\mathbb{E}[X])\,(X-\mathbb{E}[X]) + \frac{1}{2}u''(w+\mathbb{E}[X])\,(X-\mathbb{E}[X])^2 \tag{F.7}$$

The expected value of each term is given by:

$$\begin{aligned}
&\mathbb{E}[u(w+\mathbb{E}[X])] = u(w+\mathbb{E}[X])\\
&\mathbb{E}[u'(w+\mathbb{E}[X])\,(X-\mathbb{E}[X])] = u'(w+\mathbb{E}[X])\,\underbrace{\mathbb{E}[X-\mathbb{E}[X]]}_{\mathbb{E}[X]-\mathbb{E}[X]} = 0\\
&\mathbb{E}[\frac{1}{2}u''(w+\mathbb{E}[X])\,(X-\mathbb{E}[X])^2] = \frac{1}{2}u''(w+\mathbb{E}[X])\,\underbrace{\mathbb{E}[(X-\mathbb{E}[X])^2]}_{V[X]}
\end{aligned} \tag{F.8}$$

Equation (F.3) can be approximated by:

$$u(w+\mathbb{E}[X]) - \pi\, u'(w+\mathbb{E}[X]) \approx u(w+\mathbb{E}[X]) + \frac{1}{2}u''(w+\mathbb{E}[X])\,V[X] \tag{F.9}$$

Which gives the expected result:

$$\pi \approx -\frac{1}{2}\frac{u''(w+\mathbb{E}[X])}{u'(w+\mathbb{E}[X])}V[X] \tag{F.10}$$

■

Absolute Risk Aversion. The absolute risk aversion (ARA) function is given by:

$$r_{abs}(x) = \frac{-u''(x)}{u'(x)} = -\frac{d}{dx} ln\, u'(x) \tag{F.11}$$

Relative Risk Aversion. The relative risk aversion (RRA) measures risk aversion relative to current wealth. RRA function is given by:

$$r_{rel}(x) = \frac{-x\, u''(x)}{u'(x)} \tag{F.12}$$

With these functions called the Arrow-Pratt measures of risk aversion, it is possible to obtain several utility function. The utility function will be standartized to have $u(0) = 0$ and $u'(0) = 1$.

Risk neutrality. If $r_{rel} = 0$, the utility is linear: $u(x) = x$. The agent is thus risk neutral: he is indifferent between the r.v.X and $\mathbb{E}[X]$. Risk neutral agents have $\pi(w, Z) = 0$ and maximise the expected value.

Constant absolute risk aversion (CARA). If the ARA function is constant, the utility function has an exponential form:

$$\begin{aligned} r_{abs}(x) &= c \\ u(x) &= \frac{1}{c}(1 - e^{-cx}) \end{aligned} \tag{F.13}$$

Constant relative risk aversion (CRRA). If the CRRA function is constant, the utility function has a power law form:

$$\begin{aligned} r_{rel}(x) &= 1 - \gamma \quad \text{with}\, \gamma \leq 1 \\ r_{abs}(x) &= \frac{1-\gamma}{x} \\ u(x) &= \frac{1}{\gamma} x^{\gamma} \end{aligned} \tag{F.14}$$

Logarithmic utility. CRRA utility with $\gamma \to 0$ leads to logarithmic utility:

$$\begin{aligned} r_{rel}(x) &= 1 - \gamma \quad \text{with}\, \gamma \to 0 \\ u(x) &= ln(x) \end{aligned} \tag{F.15}$$

In economics, CRRA utilities are also called isoelastic utility functions and have the general form:

$$u(x) = \begin{cases} \frac{1}{\gamma} x^{\gamma} & \text{if}\, \gamma \neq 0 \\ ln(x) & \text{if}\, \gamma = 0 \end{cases} \tag{F.16}$$

Hyperbolic absolute risk aversion (HARA). The HARA utility exhibits the following form:

$$\begin{aligned} r_{abs}(x) &= \frac{1}{ax + b} \\ u(x) &= \frac{(x - x_s)^{1-R}}{1 - R} \end{aligned} \tag{F.17}$$

with $R = 1/a$ and $x_s = -b/a$. CARA utilities are obtained with $a = 0$ while CRRA utilities are obtained with $b = 0$.

2. Example: Constant portfolio allocation

The following example shows how to balance a portfolio between risky and riskless assets by maximizing the expected utility of an economic agent.
Suppose a market with one risky asset S_t and one riskless asset B_t. They are modelled by the following stochastic differential equations (SDE's):

$$\begin{aligned} dB_t &= rB_t dt \\ dS_t &= \mu S_t\, dt + \sigma S_t dW_t \end{aligned} \tag{F.18}$$

Where r is the constant risk-free rate, μ is the return of the risky asset, σ its volatility and W_t is a one dimensional brownian motion. Obviously, $\mu > r > 0$ must hold, otherwise the agent will invest all his money in the riskless asset. F_t denotes the value at time t of a portfolio composed of the two assets. This portfolio is *self-financing.* A portfolio F_t composed of n assets S_t^i is said to be self-financing if:

$$dF_t = \sum_i^n \phi_t^i \, dS_t^i \tag{F.19}$$

Where ϕ_i is the number of unit of asset S_t^i. Thus the value of the portfolio changes if the value of the assets changes. There is no infusion of external funds. *Constant portfolio allocation hypothesis* supposes that at each time t and for $x \in [0,1]$:

- The amount $x\,F_t$ is invested in asset S_t
- The amount $(1-x)\,F_t$ is invested in the riskless asset B_t.

The goal is to find the optimal proportion x for an investor with utility function $u(x)$.

Let α_t and β_t be the number of unit of assets S_t and B_t respectively. The portfolio value F_t can be split:

$$\begin{aligned} x\,F_t &= \alpha_t\, S_t \\ (1-x)\,F_t &= \beta_t\, B_t \end{aligned} \tag{F.20}$$

Thus, $F_t = \alpha_t S_t + \beta_t B_t$. Because the portfolio is self-financing, the following holds:

$$\begin{aligned} dF_t &= \alpha_t dS_t + \beta_t dB_t \\ &= \alpha_t(\mu S_t dt + \sigma S_t dW_t) + \beta_t(rB_t dt) \\ &= xF_t(\mu dt + \sigma dW_t) + (1-x)F_t r dt \\ \frac{dF_t}{F_t} &= \underbrace{[x\mu + (1-x)r]}_{\mu_x}\, dt + \underbrace{x\sigma}_{\sigma_x} dW_t \end{aligned} \tag{F.21}$$

From Itô calculus it is known that a process with dynamics (F.21) follows:

$$F_t = F_0\, e^{(\mu_x - \frac{1}{2}\sigma_x^2)t + \sigma_x W_t} \tag{F.22}$$

The goal is to solve the optimization problem:

$$\underset{x\in[0,1]}{\text{argmax}}\,\mathbb{E}[u(F_t)] \tag{F.23}$$

2.1 Logarithmic utility: $u(x) = ln(x)$

In this case the utility of F_t is given by:

$$u(F_t) = ln(F_0)\left[(\mu_x - \frac{1}{2}\sigma_x^2)t + \sigma_x^2 W_t\right] = ln(F_0)\left[(x\mu + (1-x)r - \frac{1}{2}x^2\sigma^2)t + x^2\sigma^2 W_t\right] \tag{F.24}$$

Because $\mathbb{E}[W_t] = 0$, the expected value of the random variable $u(F_t)$ follows:

$$\mathbb{E}[u(F_t)] = ln(F_0)\left[x\mu + (1-x)r - \frac{1}{2}x^2\sigma^2\right] t \tag{F.25}$$

The derivative with respect to x:

$$\frac{\partial}{\partial x}\mathbb{E}[u(F_t)] = 0 \iff \mu - r - x\sigma^2 = 0 \tag{F.26}$$

Thus for a logarithmic utility, the optimal x is given by:

$$x_{opt} = \frac{\mu - r}{\sigma^2} \tag{F.27}$$

2.2 Power Law utility: $u(x) = \dfrac{x^\gamma}{\gamma}$

The utility is given by:

$$u(F_t) = \frac{1}{\gamma}F_0^\gamma\, e^{\gamma\left[(\mu_x - \frac{1}{2}\sigma_x^2)t + \sigma_x W_t\right]} = \frac{1}{\gamma}F_0^\gamma\, e^{\gamma\left[(\mu_x - \frac{1}{2}\sigma_x^2)t\right]}e^{\gamma\sigma_x W_t} \tag{F.28}$$

$e^{\gamma\sigma_x W_t}$ follows a log-normal distribution because $W_t \sim \mathcal{N}(0,t)$. The expected value is:

$$\mathbb{E}[e^{\gamma\sigma_x W_t}] = e^{\frac{1}{2}\gamma^2\sigma_x^2 t} \tag{F.29}$$

Using this result, the expected utility of the portfolio value can be expressed as:

$$\mathbb{E}[u(F_t)] = \frac{1}{\gamma}F_0^\gamma\, e^{\gamma\left[(\mu_x - \frac{1}{2}\sigma_x^2)t\right]}e^{\frac{1}{2}\gamma^2\sigma_x^2 t} = \frac{1}{\gamma}F_0^\gamma\, e^{\gamma\mu_x t + \frac{1}{2}\gamma\sigma_x^2 t(\gamma - 1)} \tag{F.30}$$

Derivating w.r.t x gives:

$$\frac{\partial}{\partial x}\mathbb{E}[u(F_t)] = 0 \iff (\mu - r)\gamma + \gamma(\gamma - 1)\sigma^2 x = 0 \tag{F.31}$$

Which gives the optimal value of x:

$$x_{opt} = \frac{1}{1-\gamma}\frac{\mu - r}{\sigma^2} \tag{F.32}$$

2.3 Linear utility: $u(x) = ax + b$

It is clear that $u(F_t)$ follows a log-normal distribution (because $log\, F_t \sim \mathcal{N}(\mu_x - \frac{1}{2}\sigma_x^2, \sigma_x^2)$). The expected value is simply given by:

$$\mathbb{E}[u(F_t)] = e^{\mu_x t} = e^{[x\mu + (1-x)r]t} \tag{F.33}$$

And the derivative:

$$\frac{\partial}{\partial x}\mathbb{E}[u(F_t)] = (\mu - r)t\, e^{[x\mu+(1-x)r]t} > 0 \quad \forall x \tag{F.34}$$

And finally:

$$\underset{x\in[0,1]}{\operatorname{argmax}}\, \mathbb{E}[u(F_t)] = 1 \tag{F.35}$$

Thus a risk neutral agent will invest all his money in the risky asset.

Appendix G

Codes

1. Market parameters fitting

```
close all;
clear variables;
%% Estimate the parameters of the stocks SDE's (Geometric BM)
stock_info = readmatrix('stock_info2.csv','OutputType','string');
nb_samples = getNbSamples(stock_info);
N = length(nb_samples);
dt = 1/252;
r = 0.006;

n_samples = 10/dt;min(nb_samples);
stock_mat = extractData(stock_info,n_samples);

returns_mat = diff(log(stock_mat));
return_mean = mean(returns_mat);
sigma_mat = cov(returns_mat)/dt; % Sample covariance
vol_mat = chol(sigma_mat,'lower'); % Cholesky decomposition

mu = return_mean/dt+0.5*ones(1,3)*sigma_mat*ones(3,1);
theta = vol_mat\(mu-r)';

function nb_samples = getNbSamples(stock_info)

N = length(stock_info);
nb_samples = zeros(1,N);
for i=1:N
   data = readmatrix(stock_info(i));
   stock = data(:,5);
   stock = stock(~isnan(stock));
   nb_samples(i) = length(stock);
end
end

function stock_mat = extractData(stock_info,n_samples)

N = length(stock_info);
stock_mat = zeros(n_samples,N);
for i=1:N
   data = readmatrix(stock_info(i));
   stock = data(:,5);
   stock = stock(~isnan(stock));
   stock_mat(:,i) = stock(length(stock)-n_samples+1:end);
```

```
end

end
```

2. Pricing kernel

```
close all;
clear variables;
%% Compute the pricing kernel rho_t for the period 01/01/2019 - 01/04/2020
stock_info = readmatrix('2019_2020/stock_info2.csv','OutputType','string');
nb_samples = getNbSamples(stock_info);
N = length(nb_samples);
dt = 1/252;
%% Fitted parameters (on daily basis)
r = 0.006*dt;
mu_s = [0.0687 0.0458 0.0693]*dt;
sigma_s = [0.2223,0,0;0.0742,0.3505,0;0.1329,0.0645,0.3641]*sqrt(dt);
theta = sigma_s\(mu_s-r)';
theta_n = norm(theta);
mu_rho = -r - 0.5*theta_n^2;
sigma_rho = theta_n;

n_samples = min(nb_samples);
stock_mat = extractData(stock_info,n_samples);

returns_mat = diff(log(stock_mat));
return_mean = mean(returns_mat);
%% Extract BM's and pricing kernel
w = zeros(n_samples-1,3);
for i = 1:n_samples-1
   w(i,:) = sigma_s\(returns_mat(i,:) - mu_s)';
end

t = (1:1:n_samples-1)';
diff = cumsum(w)*theta;
drift = mu_rho*t;

rho = exp(drift - diff);

figure
plot(t*dt,rho,'k','LineWidth',2.5);
xlabel('Time');
ylabel('\rho_t');
grid;

figure
plot(dt*(1:1:length(rho_scenario)),rho_scenario,'k','LineWidth',2.5);
xlabel('Time');
ylabel('\rho_t');
grid;


figure
subplot(3,1,1);
plot([0 ;t]*dt,stock_mat(:,1),'k','LineWidth',2.5);
xlabel('Time')
ylabel('ABI');
grid;
subplot(3,1,2);
plot([0 ;t]*dt,stock_mat(:,2),'k','LineWidth',2.5);
xlabel('Time')
```

```
ylabel('INGA');
grid;
subplot(3,1,3);
plot([0 ;t]*dt,stock_mat(:,3),'k','LineWidth',2.5);
xlabel('Time')
ylabel('KBC');
grid;

%save('rho_2019_2020_bis','rho');

%% Jarque-Bera and portmanteau test
JB_val = zeros(1,3);
LB_val = zeros(1,3);

for i=1:3
    [~,~,JB_val(i),critical_JB] = jbtest(returns_mat(:,i)-return_mean(i));
    [~,~,LB_val(i),critical_LB] = lbqtest(returns_mat(:,i)-return_mean(i),'Alpha',0.05);
end

%% Functions

function nb_samples = getNbSamples(stock_info)

N = length(stock_info);
nb_samples = zeros(1,N);
for i=1:N
   data = readmatrix(strcat('2019_2020/',stock_info(i)));
   stock = data(:,5);
   stock = stock(~isnan(stock));
   nb_samples(i) = length(stock);
end
end

function stock_mat = extractData(stock_info,n_samples)

N = length(stock_info);
stock_mat = zeros(n_samples,N);
for i=1:N
   data = readmatrix(strcat('2019_2020/',stock_info(i)));
   stock = data(:,5);
   stock = stock(~isnan(stock));
   stock_mat(:,i) = stock(length(stock)-n_samples+1:end);
end

end
```

3. CPT portfolio

```
close all;
clear variables;

%% Agent's parameters

N=319;
a = -0.2;%[-0.2 -0.5 -0.8]';
alpha = 0.5;%[0.5 0.6 0.7]';
T = 1.3;
b=1;%[0.5 0.7 1]';
x0=-1;
M = 1;
str = strcat(repmat('\alpha = ',M,1) , num2str(alpha));
```

```
ptf_val = zeros(N,M);
ptf_cash = zeros(N,M);
ptf_eut_cash=(zeros(1,M));
%ptf_stock = zeros(N,M);
for i=1:M % Call to main function
[t,ptf_val_eut,ptf_val(:,i),BFX,ptf_stock,ptf_cash(:,i),ptf_eut_cash(i)] = calcPerf(alpha(i),T,a,b
end

figure
plot(t,ptf_val,'LineWidth',2.5);
xlabel('Time');
ylabel('Performance')
legend(str);
grid;

figure
plot(t,ptf_cash,'LineWidth',2.5);hold on;
ax = gca;
ax.ColorOrderIndex = 1;
plot(t,ones(N,1)*ptf_eut_cash,'--','LineWidth',2.5);
xlabel('Time');
ylabel('\pi_{cash}')
%legend(vertcat(str,repmat('EUT        ',M,1)),'Location','northwest','NumColumns',2);
legend(str,'Location','northwest')
%legend(vertcat(str,repmat('EUT        ',M,1)));
grid;
%saveas(gcf,'plots/b1_T2_a02_ptf','epsc');
%save('withLossControl','t','ptf_val');

%% Main function
function [t,ptf_val_eut,ptf_val,BFX,ptf_stock,ptf_cash,ptf_eut_cash] = calcPerf(alpha,T,a,b,x0)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%% Market Parameters %%%%%%%%%%%%%%%%%%%%%%%%%%%

r=0.006;
mu_s = [0.0687 0.0458 0.0693];
sigma_s = [0.2223,0,0;0.0742,0.3505,0;0.1329,0.0645,0.3641];
theta = sigma_s\(mu_s-r)';
theta_n = norm(theta);
mu_rho = (-r - 0.5*theta_n^2)*T;
sigma_rho = theta_n*sqrt(T);
dt = 1/252;

mu_rho_bis=(-r - 0.5*theta_n^2);
sigma_rho_bis=theta_n;


z0=1/4;
c_0=exp(mu_rho+sigma_rho*norminv(z0));
k_loss = 2.25;
delta_neg=0.1;
alpha_neg=0.35;

%%%%%%%%%%%%%%%%%%%%%%%%%% Check well-posedness %%%%%%%%%%%%%%%%%%%%%%%%%%%

N = 500;
c = linspace(0.5,5,N);
k_c = getK_c(c,a,b,c_0,alpha,mu_rho,sigma_rho,k_loss,delta_neg,alpha_neg);
min(k_c)
figure
plot(c,k_c,c,ones(1,N),'LineWidth',2)
%axis([0 3 0.8 2.5])
xlabel('c')
```

```
ylabel('\kappa_c');
grid;

%%%%%%%%%%%%%%%%%%% Numerical solver of the NLP %%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%
%%% This is the problem without loss control
%%% With loss control:Copy-paste the solver with loss control and add
%%% the third decision variable



% options = optimoptions(@fmincon,'Algorithm','interior-point',...
%     'OutputFcn', {@myplotx});

%Objective function
fun = @(x) -obj_fun_fmincon(x,a,b,c_0,alpha,mu_rho,sigma_rho,k_loss,delta_neg,alpha_neg,x0);
% Linear inequality constrains
A_fmincon = [-1 0; 0 -1];
b_fmincon = [0;-max(x0,0)];
%Inital values
x_init_fmincon = [2 abs(x0)];
%Solve
x_fmincon = fmincon(fun,x_init_fmincon,A_fmincon,b_fmincon);
%Optimal values
c_opt_min = x_fmincon(1);
x_opt_min = x_fmincon(2);

%%%%%%%%%%%%%%%%%%% Some variables for plots and reporting %%%%%%%%%%%%%%%%%%%%%%%%%%%

rho = linspace(0.1,3,N);
rho_down = rho(rho<=c_opt_min);
rho_up = rho(rho>c_opt_min);

X_opt_gain_part = optiClaim_gain(rho_down,c_opt_min,a,b,c_0,alpha,mu_rho,sigma_rho,x_opt_min);
X_opt_loss_part = optiClaim_loss(x0,x_opt_min,c_opt_min,mu_rho,sigma_rho);

rho_plot = [rho_down rho_down(end) rho_up];
X_opt_plot = [X_opt_gain_part -X_opt_loss_part*ones(1,length(rho_up)+1)];
figure
plot(rho_plot,X_opt_plot,'k','LineWidth',2.5)
%axis([0.5 3 -65 50]);
axis([0 3 -6 10]);
xlabel('\rho_T');
ylabel('X_T')
grid;
p_noLossControl = normcdf((log(c_opt_min)-mu_rho)/sigma_rho);
%save('withoutLossControl','rho_plot','X_opt_plot','c_opt_min','x_opt_min');

%%%%%%%%%%%%%%%%%%% Inputs for replication problem %%%%%%%%%%%%%%%%%%%%%%%%%%%

sisi = load('rho_2019_2020_bis.mat');
rho = [1 (sisi.rho)'];

time = 0:1:length(rho)-1;
t = time*dt;

x_gain = x_opt_min*getK(a,b,c_0,mu_rho,sigma_rho)^(1/(1-alpha))/evalPhi(a,b,c_0,alpha,c_opt_min,mu
x_loss = X_opt_loss_part;

if(x0>0) x_loss=0; end

% The replicating strategy, see function replicating_ptfLoss
[lvg,value_ptf] = replicating_ptfLoss(rho,mu_rho_bis,sigma_rho_bis,a,b,alpha,c_0,c_opt_min,t,T,x_
```

```

ptf_eut_stock = ((sigma_s*sigma_s')\(mu_s-r)')'/(1-alpha);
ptf_eut_cash = 1 - sum(ptf_eut_stock,2);
ptf_stock = lvg'*ptf_eut_stock*(1-alpha);
ptf_cash = 1 - sum(ptf_stock,2);

%%%%%%%%%%%%%%%%%%% Price process %%%%%%%%%%%%%%%%%%%%%%%%%%%

stock_info = readmatrix('2019_2020/stock_info2.csv','OutputType','string');
nb_samples = getNbSamples(stock_info);
N = length(nb_samples);
r = 0.006;
dt = 1/252;

n_samples = min(nb_samples);
stock_mat = extractData(stock_info,n_samples);

benchmark = readmatrix('2019_2020/benchmark.csv','OutputType','string');
BFX = extractData(benchmark,n_samples);

init_pos = x0*ptf_stock(1,:)./stock_mat(1,:);

t = (0:1:n_samples-1)'*dt;
pos = zeros(length(t),3);
ptf_val = zeros(length(t),1);
cash_val = zeros(length(t),1);
ptf_val(1) = x0;
cash_val(1) = x0*ptf_cash(1);
pos(1,:) = init_pos;

pos_eut = zeros(length(t),3);
ptf_val_eut = zeros(length(t),1);
cash_val_eut = zeros(length(t),1);
ptf_val_eut(1) = 1;
cash_val_eut(1) = ptf_eut_cash;
pos_eut(1,:) = ptf_eut_stock./stock_mat(1,:);


for i = 2:length(t)
   pos(i,:) = ptf_stock(i,:)./stock_mat(i,:)*value_ptf(i);
   cash_val(i) = ptf_cash(i)*value_ptf(i);
   ptf_val(i) = value_ptf(i);
   ptf_val_eut(i) = pos_eut(i-1,:)*stock_mat(i,:)' + cash_val_eut(i-1)*exp(r*dt);
   pos_eut(i,:) = ptf_eut_stock./stock_mat(i,:)*ptf_val_eut(i);
   cash_val_eut(i) = ptf_eut_cash*ptf_val_eut(i);
end

BFX = BFX/BFX(1);


end

%% Useful function for the gain contingent claim

function phi = evalPhi(a,b,c_0,alpha,c,mu,sigma)

phi = zeros(size(c));
k = getK(a,b,c_0,mu,sigma);
phi(c<=c_0) = k^(1/(1-alpha)) * momentLogN(mu,sigma,(a-alpha)/(1-alpha),1e-9,c(c<=c_0));
phi(c>c_0) = k^(1/(1-alpha))*(momentLogN(mu,sigma,(a-alpha)/(1-alpha),1e-9,c_0) +...
    c_0.^((a-b)/(1-alpha)) * momentLogN(mu,sigma,(b-alpha)/(1-alpha),c_0,c(c>c_0)));
end

```

```
%% Useful function for the gain contingent claim

function k = getK(a,b,c_0,mu_0,sigma_0)

tmp_a = exp(a*mu_0 + a^2*sigma_0^2/2);
tmp_b = exp(b*mu_0 + b^2*sigma_0^2/2);
tmp_cdf_a = normcdf((log(c_0)-mu_0-a*sigma_0^2)/sigma_0);
tmp_cdf_b = normcdf((log(c_0)-mu_0-b*sigma_0^2)/sigma_0);
k = (tmp_a*tmp_cdf_a + c_0^(a-b)*tmp_b*(1-tmp_cdf_b))^-1;
end

%% Expectation of a lower truncated random variable

function e = expectationRho(c,mu,sigma)
e=momentLogN(mu,sigma,1,c,1e9);

end

%% kappa(c) --> to check well posedness and useful for the objective function

function k =getK_c(c,a,b,c_0,alpha,mu,sigma,k_loss,delta_neg,alpha_neg)

phi = evalPhi(a,b,c_0,alpha,c,mu,sigma);
expectation = expectationRho(c,mu,sigma);
%T = getDistorsion(z,a,b,c_0,mu,sigma);
T = distorsionNegativeCDF(c,mu,sigma,delta_neg,alpha_neg);
k = k_loss*T./phi.^(1-alpha)...
    ./expectation.^alpha;

end

%% Expectation of a truncated lognormal variable

function y = momentLogN(mu,sigma,eta,z1,z2)

expectation = exp(mu*eta + 0.5.*sigma.^2.*eta.^2);
y = expectation .* (normcdf((log(z2)-mu)./sigma - eta*sigma) - normcdf((log(z1)-mu)./sigma - eta*s
end

%% Negative distortion function: Escobar 2019

function T = distorsionNegativeCDF(c,mu,sigma,delta_neg,alpha_neg)

T = (normcdf(-log(c)/sigma + (mu/sigma^2-delta_neg)*sigma)).^alpha_neg;

end

%% Terminal wealth of the gain contingent claim

function X = optiClaim_gain(rho,c,a,b,c_0,alpha,mu,sigma,x0)

phi = evalPhi(a,b,c_0,alpha,c,mu,sigma);
k = getK(a,b,c_0,mu,sigma);
gamma = phi/k^(1/(1-alpha));

X = zeros(size(rho));
X(rho<=c_0) = x0/gamma*rho(rho<=c_0).^((a-1)/(1-alpha));
X(rho>c_0) = x0/gamma*c_0.^((a-b)/(1-alpha))*rho(rho>c_0).^((b-1)/(1-alpha));
end

%% Terminal wealth of the loss contingent claim

function X = optiClaim_loss(x0,x_opt,c_opt,mu,sigma)
```

```

X = (x_opt - x0)/expectationRho(c_opt,mu,sigma);

end

%% Objective function

function y  = obj_fun_fmincon(x,a,b,c_0,alpha,mu,sigma,k_loss,delta_neg,alpha_neg,x0)
c= x(1);
x_p = x(2);
phi = evalPhi(a,b,c_0,alpha,c,mu,sigma);
k_c = getK_c(c,a,b,c_0,alpha,mu,sigma,k_loss,delta_neg,alpha_neg);
y = phi.^(1-alpha) .*(x_p.^alpha - k_c.*(x_p - x0).^alpha);

end

%% Price process of a binary option (lognormal pricing kernel)

function x = priceProcess(rho,mu,sigma,eta,z1,z2,t,T)

mu_t = mu*(T-t);
sigma_t = sigma*sqrt(T-t);
x = rho.^eta .* momentLogN(mu_t,sigma_t,eta+1,z1./rho,z2./rho);

end

%% Replicate the risky part of the price process

function [ptf,value] = replicating_ptfLoss(rho,mu,sigma,a,b,alpha,c_0,c,t,T,x_gain,x_loss)

eta_1 = (a-1)/(1-alpha);
eta_2 = (b-1)/(1-alpha);
mu_t = mu*(T-t);
sigma_t = sigma*sqrt(T-t);

x1 = priceProcess(rho,mu,sigma,eta_1,1e-5,c_0,t,T);
x2 = priceProcess(rho,mu,sigma,eta_2,c_0,c,t,T);


tmp = 1./(rho.*sigma_t)*c_0^((a-b)/(1-alpha))*c^((b-alpha)/(1-alpha)).*normpdf((log(c./rho)-mu_t).
% Replicate the loss part
value_loss = -c./(rho.*sigma_t).* normpdf((log(c./rho)-mu_t)./sigma_t);
% Replicate the gain part
value_gain = (1-a)/(1-alpha)*x1 + c_0^((a-b)/(1-alpha))*(1-b)/(1-alpha)*x2+ tmp;
% Price process of the gain part
price_gain = (x1 + c_0^((a-b)/(1-alpha)) *x2 );
% Price process of the loss part
price_loss = priceProcess(rho,mu,sigma,0,c,1e20,t,T); %Cash-or-nothing binary option
ptf = (x_gain*value_gain - x_loss*value_loss)./(x_gain*price_gain - x_loss*price_loss);
value = x_gain*price_gain - x_loss*price_loss;

end
%% Extract data
function nb_samples = getNbSamples(stock_info)

N = length(stock_info);
nb_samples = zeros(1,N);
for i=1:N
   data = readmatrix(strcat('2019_2020/',stock_info(i)));
   stock = data(:,5);
   stock = stock(~isnan(stock));
   nb_samples(i) = length(stock);
end
```

```
end

function stock_mat = extractData(stock_info,n_samples)

N = length(stock_info);
stock_mat = zeros(n_samples,N);
for i=1:N
   data = readmatrix(strcat('2019_2020/',stock_info(i)));
   if(stock_info(i) == "BFX.csv")
       stock = data(:,2);
   else
   stock = data(:,5);
   end
   stock = stock(~isnan(stock));
   stock_mat(:,i) = stock(length(stock)-n_samples+1:end);
end

end

function stop = myplotx(x, optimValues, state)
     persistent d1 d2
     if ~nargin % example reset mechanism
       d1=[];
       d2=[];
     end
     if strcmp(state,'iter')
      d1=[d1,x(1)];
      d2=[d2,x(2)];
      plot(d1.',d2.','diamond');
      grid on
      hold on
     end
     stop = 0;
end
```

4. Optimisation problem with loss control

```
close all;
clear variables;

%% Market parameters
r=0.006;
mu_s = [0.0687 0.0458 0.0693];
sigma_s = [0.2223,0,0;0.0742,0.3505,0;0.1329,0.0645,0.3641];
theta = sigma_s\(mu_s-r)';
theta_n = norm(theta);
T =10;
mu_rho = (-r - 0.5*theta_n^2)*T;
sigma_rho = theta_n*sqrt(T);
dt = 1/252;

mu_rho_bis=(-r - 0.5*theta_n^2);
sigma_rho_bis=theta_n;

%% Investor parameters
z0=1/4;
a=-0.2;
b=1;
c_0=exp(mu_rho+sigma_rho*norminv(z0));
alpha = 0.5;
k_loss = 2.25;
```

```matlab

delta_neg=0.1;
alpha_neg=0.35;

%% Well-posedness
N = 500;
x0 = -1;
L = 1.3;

c = linspace(0.2,3,N);
k_c = getK_c(c,a,b,c_0,alpha,mu_rho,sigma_rho,k_loss,delta_neg,alpha_neg);
min(k_c)
figure
plot(c,k_c,c,ones(1,N),'LineWidth',2)
xlabel('c')
ylabel('\kappa_c');
grid;

%% Numerical solver
% Objective function
fun = @(x) -obj_fun_lossC_fmincon(x,a,b,c_0,alpha,mu_rho,sigma_rho,k_loss,delta_neg,alpha_neg,x0,
% Nonlinear constrains
nonlcon = @(x) nonlinConstraints(x,mu_rho,sigma_rho,x0,L);
% Artificial constrain
rho_lim = exp(mu_rho+sigma_rho*norminv(1-1e-6));
A_fmincon = [];
b_fmincon = [];
Aeq = [];
beq = [];
lb = [0 0 0];
ub = [rho_lim x0+L rho_lim];
%options = optimoptions(@fmincon,'Algorithm','interior-point',...
    %'OutputFcn', {@myplotx2});
c1_init=1;
c2_init=1.5;
x_p_init = x0 + L*expectationRho(c1_init,mu_rho,sigma_rho);


x_init_fmincon = [c1_init x_p_init c2_init];
x_fmincon = fmincon(fun,x_init_fmincon,A_fmincon,b_fmincon,Aeq,beq,lb,ub,nonlcon);
% Optimal values
c_opt_min = x_fmincon(1);
x_opt_min = x_fmincon(2);
c2_opt_min = x_fmincon(3);

%% Some variables and plots

rho = linspace(0.1,3,N);
rho_down = rho(rho<=c_opt_min);
rho_up = rho(rho>c_opt_min & rho<=c2_opt_min);
rho_up2 = rho(rho>c2_opt_min);

X_opt_gain_part = optiClaim_gain(rho_down,c_opt_min,a,b,c_0,alpha,mu_rho,sigma_rho,x_opt_min);
X_opt_loss_part = optiClaim_loss_bis(x0,x_opt_min,c_opt_min,c2_opt_min,mu_rho,sigma_rho,L);

rho_plot = [rho_down rho_down(end) rho_up rho_up(end) rho_up2 ];
X_opt_plot = [X_opt_gain_part -X_opt_loss_part*ones(1,length(rho_up)+1) -L*ones(1,length(rho_up2)

x_gain = x_opt_min*getK(a,b,c_0,mu_rho,sigma_rho)^(1/(1-alpha))/evalPhi(a,b,c_0,alpha,c_opt_min,mu
x_loss = X_opt_loss_part;
eta1 = (a-1)/(1-alpha);
eta2 = (b-1)/(1-alpha);

```

```
X1=load('X_plot');
rho_no_loss = X1.rho_plot;
X_opt_no_loss = X1.X_opt_plot;

figure
plot(rho_plot,X_opt_plot,'k',rho_no_loss,X_opt_no_loss,'LineWidth',3.5);
xlabel('\rho_T');
ylabel('X_T');
axis([0 3 -6 10]);
grid;
legend('With Loss Control','Without Loss Control');

result.c_opt = c_opt_min;
result.c2_opt = c2_opt_min;
result.x_opt = x_opt_min;
result.X_loss = X_opt_loss_part;

plotVar.rho = rho_plot;
plotVar.X = X_opt_plot;

p_LossControl = normcdf((log(c_opt_min)-mu_rho)/sigma_rho);
p_LossControl2 = normcdf((log(c2_opt_min)-mu_rho)/sigma_rho);
%save('withLossControl','rho_plot','X_opt_plot','c_opt_min','x_opt_min','c2_opt_min');
%save('withLossControl','param','result','plotVar');

%% Functions

function phi = evalPhi(a,b,c_0,alpha,c,mu,sigma)

phi = zeros(size(c));
k = getK(a,b,c_0,mu,sigma);
phi(c<=c_0) = k^(1/(1-alpha)) * momentLogN(mu,sigma,(a-alpha)/(1-alpha),1e-9,c(c<=c_0));
phi(c>c_0) = k^(1/(1-alpha))*(momentLogN(mu,sigma,(a-alpha)/(1-alpha),1e-9,c_0) +...
    c_0.^((a-b)/(1-alpha)) * momentLogN(mu,sigma,(b-alpha)/(1-alpha),c_0,c(c>c_0)));
end


function k = getK(a,b,c_0,mu_0,sigma_0)

tmp_a = exp(a*mu_0 + a^2*sigma_0^2/2);
tmp_b = exp(b*mu_0 + b^2*sigma_0^2/2);
tmp_cdf_a = normcdf((log(c_0)-mu_0-a*sigma_0^2)/sigma_0);
tmp_cdf_b = normcdf((log(c_0)-mu_0-b*sigma_0^2)/sigma_0);
k = (tmp_a*tmp_cdf_a + c_0^(a-b)*tmp_b*(1-tmp_cdf_b))^-1;
end

function e = expectationRho(c,mu,sigma)
e=momentLogN(mu,sigma,1,c,1e9);

end

function e = expectationRho2(c,c2,mu,sigma)
e=momentLogN(mu,sigma,1,c,c2);

end

function k =getK_c(c,a,b,c_0,alpha,mu,sigma,k_loss,delta_neg,alpha_neg)

phi = evalPhi(a,b,c_0,alpha,c,mu,sigma);
expectation = expectationRho(c,mu,sigma);
%T = getDistorsion(z,a,b,c_0,mu,sigma);
T = distorsionNegativeCDF(c,mu,sigma,delta_neg,alpha_neg);
k = k_loss*T./phi.^(1-alpha)...
```

```
    ./expectation.^alpha;

end

function y = momentLogN(mu,sigma,eta,z1,z2)

expectation = exp(mu*eta + 0.5*sigma^2*eta^2);
y = expectation .* (normcdf((log(z2)-mu)/sigma - eta*sigma) - normcdf((log(z1)-mu)/sigma - eta*sig
end

function T = distorsionNegativeCDF(c,mu,sigma,delta_neg,alpha_neg)

T = (normcdf(-log(c)/sigma + (mu/sigma^2-delta_neg)*sigma)).^alpha_neg;

end

function X = optiClaim_gain(rho,c,a,b,c_0,alpha,mu,sigma,x0)

phi = evalPhi(a,b,c_0,alpha,c,mu,sigma);
k = getK(a,b,c_0,mu,sigma);
gamma = phi/k^(1/(1-alpha));

X = zeros(size(rho));
X(rho<=c_0) = x0/gamma*rho(rho<=c_0).^((a-1)/(1-alpha));
X(rho>c_0) = x0/gamma*c_0.^((a-b)/(1-alpha))*rho(rho>c_0).^((b-1)/(1-alpha));
end

function X = optiClaim_loss(x0,x_opt,c_opt,mu,sigma)

X = (x_opt - x0)/expectationRho(c_opt,mu,sigma);

end

function X = optiClaim_loss_bis(x0,x_opt,c_opt,c2_opt,mu,sigma,L)

e = expectationRho(c2_opt,mu,sigma);
e2 = expectationRho2(c_opt,c2_opt,mu,sigma);
X = (x_opt - x0 - L*e)/e2;

end

function y  = obj_fun_fmincon(x,a,b,c_0,alpha,mu,sigma,k_loss,delta_neg,alpha_neg,x0)
c= x(1);
x_p = x(2);
phi = evalPhi(a,b,c_0,alpha,c,mu,sigma);
k_c = getK_c(c,a,b,c_0,alpha,mu,sigma,k_loss,delta_neg,alpha_neg);
y = phi.^(1-alpha) .*(x_p.^alpha - k_c.*(x_p - x0).^alpha);

end

function y  = obj_fun_lossC_fmincon(x,a,b,c_0,alpha,mu,sigma,k_loss,delta_neg,alpha_neg,x0,L)
c= x(1);
x_p = x(2);
c2 = x(3);
phi = evalPhi(a,b,c_0,alpha,c,mu,sigma);
T1 = distorsionNegativeCDF(c,mu,sigma,delta_neg,alpha_neg);
T2 = distorsionNegativeCDF(c2,mu,sigma,delta_neg,alpha_neg);
e = expectationRho(c2,mu,sigma);
e2 = expectationRho2(c,c2,mu,sigma);
gain = phi.^(1-alpha).*x_p.^alpha;
loss1 = k_loss .* (T1 - T2) .* (x_p - x0 - L*e).^alpha./e2.^alpha;
loss2 = k_loss * L.^alpha .* T2;
y = gain - loss1 - loss2;
```

```

end

function [c,ceq] = nonlinConstraints(x,mu,sigma,x0,L)
c1 = x(1);
x_p = x(2);
c2 = x(3);
e1 = expectationRho(c1,mu,sigma);
e2 = expectationRho(c2,mu,sigma);
c(1) = max(max(x0,0), x0 + L*e2) - x_p;
c(2) = x_p - x0 - L*e1;
c(3) = c1-c2;
ceq = [];
end
```

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