

Faculté des sciences

Design of a reserving and pricing tool based on Thiele's differential equation

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Introduction

In the world of uncertainty that we are living, the global life insurance sector is facing a time of monumental vulnerability pressure. On one hand, the interest rates in the financial market are continuously declining to an unprecedented radical low and even negative rates level across vast swathes of the global markets, which is placing the industry under a massive challenge. On the other hand, the rise of other challengers such as insurtechs' or GAFAM (Google, Apple, Facebook, Amazon and Microsoft) selling insurance products strengthen the competition. The combination of these elements among others lay the insurance industry into a new reality. Eventually the life insurance product should reflect the uprising risk and uncertainties. This pushes the life insurance companies to create more complex products. As a consequence, pricing and reserving tools have to evolve in order to embrace this moving environment .

Classical pricing and reserving methods, however, showed their limitation : while Thiele's differential equation expresses the dynamism of the insurance reserve and while Cantelli's theorem allows every insurance contract to have a fictitious homonym under which Thiele's equation has a unique solution, some products such as typical reserve-dependent payments require the use of numerical approaches to solve them. One simple numerical approach that can be taken is based on the Euler's scheme, which consists into taking infinitesimal time steps and calculating the value of the insurance reserve and the value of the premiums backwardly.

The principal purpose of this paper is to investigate and assess the Euler numerical approach to Thiele's differential equation in order to propose a pricing and reserving tool for life insurance products under multiple decrements. To address the research objectives of this thesis, the structure of this document is organized into four chapters.

The first chapter discusses the pricing and reserving context of life insurance products under multiple decrements. It elaborates on the various techniques used, starting from Thiele's differential equation to other methods such as its approximation under Euler's scheme or a more common practice within the insurance industry, being the discrete Fourret analysis.

Chapter two suggests an application of Euler's scheme to common life insurance products within the industry. This consists into providing a potential design for a pricing and reserving tool in R by highlighting potential issues encountered with the application

of Euler's scheme to particular products and by suggesting approaches to solve it. Chapter three assesses the reliability of the results obtained by the pricing and reserving tool by comparing with the results obtained under Thiele's differential equation for a set of classical life insurance products. Moreover it also looks at particular cases such as reserve-dependent benefits, for both unique and periodic premiums. The last chapter assesses the reliability of the tool to extreme situations by the means of stress testing. This chapter also explores the evolution of pricing and reserving under different potential scenarios that can be foreseen in a near future such as under a pandemic or under a booming economy scenario's.

Chapter 1

Towards the modeling of the life insurance reserve

1.1 Life insurance contract

Let one quickly remind the structure of a classical life insurance contract. The purpose of a life insurance contract is to cover the policyholder against the risk of mortality by delivering either a capital C (or C^+) named the sum insured at the termination of the insurance contract, or insurance benefits along the contract. The termination of the contract is function of two elements : the lifetime of the policyholder and the duration of the contract. The sum insured is function of the previous payments $\Pi_t = \sum_{i=1}^t \pi_i$ made by the policyholder. Under its simplest version the life insurance contract is composed of a single unique payment π_0 . Other elements are also embedded along the contract : one can mention the interest rate and the guaranteed rate of the contract, but other elements can also be part of it upon the creativity of the actuary.

Now consider a life insurance policy financed by periodic premiums π_t . At the inception of the contract, the expected present value of the future insurance benefits(i.e., the sum insured) should be equivalent to the expected present value of the future premiums :

$$\mathbb{E} \left[\int_0^t v(0, \tau) \mathbb{I}[T_x > \tau] dB(\tau) + v(0, T_x) c(T_x) \mathbb{I}[T_x < t] \right] = \int_0^t v(0, \tau) \mathbb{E} [\mathbb{I}[T_x > \tau]] d\Pi(\tau) \quad (1.1)$$

This equivalence between future payments and future benefits does not exist along the development of the contract due to various parameters, such as variations between the nominal interest rate and the guaranteed rate, the non-linear impact of the mortality. The equivalence principle, however, is still applicable. More precisely, this defines the insurance premium reserve :

$$V(t) = \mathbb{E} \left[\int_t^n v(t, \tau) (\mathbb{I}[T_x > \tau] dB_\tau + c_\tau d\mathbb{I}[T_x \leq \tau] - \mathbb{I}[T_x > \tau] d\Pi_\tau) | T_x > t \right] \quad (1.2)$$

Equation (1.2) defines the reserve via the equivalence principle by considering the

future insurance benefits as the sum of the benefits paid dB_τ during the duration of the contract (i.e., when the remaining lifetime $T_x > \tau$), with an eventual capital c_τ which is delivered in case of the death of the policyholder, i.e., when $T_x < \tau$. On the other hand, equation (1.2) considers the future insurance premiums as defined above in this section.

The dynamics of the reserve $V(t)$ has been studied by numerous researchers : such as Hoem (1969), Wolthuis(1990) and Norberg (1991) who investigated the reserves in Life and Pension Insurance, in particular the early application of Thiele's differential equation, to Kalashnikov (2003) who established a sensitivity analysis or Denuit *et al.*(2014) who investigated the modeling feasibility of the reserve under multiple decrements. Multiple decrements will be further investigated in this chapter and will be considered as the central assumption for the rest of this document.

1.2 Multiple decrements

Multiple decrements suggests that the insurance contract includes several causes of exit, as illustrated in the figure below :

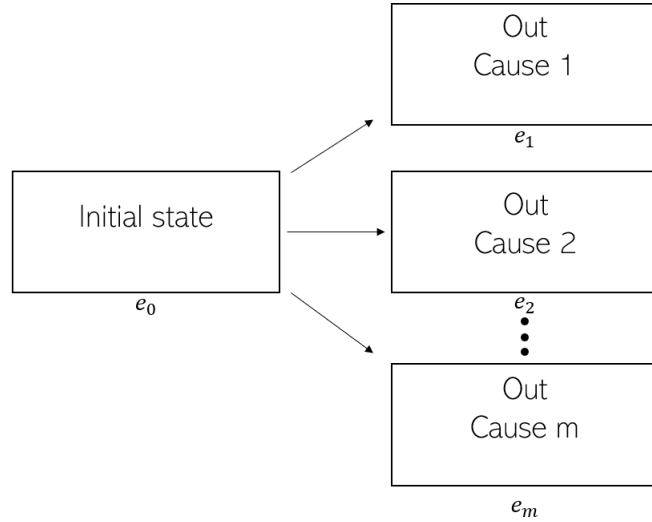


Figure 1.1 – Multiple Decrecments

Let one considers a policyholder of age x that is willing to take a life insurance contract. The life insurance contract considers m possible states described by a semi-Markov process represented by the random variable $X = [X_t, t \geq 0]$, with state space $e_S = \{e_0, e_1, \dots, e_m\}$ defined on the probability space $(\Omega, \mathcal{F}, P)$. The policyholder enters the contract in the initial state $e_S = e_0$. If the state of the policyholder changes from the initial state during the lifetime of the contract, the contract will terminate. Hence every state which is different than the initial state e_0 is considered as a cause of exit. Moreover,

we make the assumption that every cause of termination of the contract are mutually exclusive and cover every possible case of termination.

Other assumptions are taken into consideration about semi-Markov processes :

(i) The process X has continuous trajectories to the right but limited to the left. One may call the behavior of X càdlàg :

$$\lim_{\Delta t \rightarrow 0} \equiv X_{t+\Delta t} = X_t \text{ and } \lim_{\Delta t \rightarrow 0} X_{t-\Delta t} \equiv X_{t-}$$

(ii) There are a finite number of transitions that the process X can make.

(iii) The limit $\lim_{\Delta t \rightarrow 0} \frac{P(X_{t+\Delta t}=j|X_t=i)}{\Delta t}$ exists and is bounded, i.e $\forall t \in [0, T], i \neq j \in e_S$.

In fact (iii) defines the transition intensities μ :

$$\begin{aligned} \mu_{0i}(t) &= \lim_{\Delta \rightarrow 0} \frac{P[X_{t+\Delta t} = e_0 | X_t = e_i]}{\Delta t} \\ \mu(x) &= \sum_{i=1}^m \mu_{0i}(t) \end{aligned} \tag{1.3}$$

Two comments about equality (1.3) :

- We make the assumption that transition intensities may vary in function of the age of the policyholder, but the same rate is applied to all individuals of the same integer age. This may be summarized by saying that transitions intensities are assumed to be constant by periods.
- The last relation in equality (1.3) is only possible due to the mutual exclusivity assumption.

Next to the possible decrements embedded in the life insurance contract, the interest rate process $\delta(t)_{t \in [0, T]}$ is used to define the discount factor $v(s, t)$:

$$v(s, t) = \exp \left(- \int_s^t \delta(\xi) d\xi \right)$$

Therefore one may express the equivalence between future payments and future benefits at the inception of the contract as described with equation (1.1) as the following:

$$\begin{aligned} \mathbb{E} \left[\int_0^t v(\tau) \mathbb{I}[T_x > \tau] dB(\tau) + v(T_x) \sum_{i=1}^m c_i(T_x) \mathbb{I}[T_x \leq t, e_S = i] \right] &= \int_0^t v(\tau) \tau p_x d\Pi(\tau) \\ &\Leftrightarrow \\ \int_0^t v(\tau) \tau p_x dB(\tau) + \sum_{i=1}^m \int_0^t v(\tau) c_i(\tau) \tau p_x \mu_{0i}(x + \tau) d\tau &= \int_0^t \exp \left(- \int_0^\tau (\delta(\xi) + \mu(x + \xi)) d\xi \right) d\Pi(\tau) \end{aligned} \tag{1.4}$$

Some comments on equivalence (1.4) :

- $\Pi(t)$ is differentiable assumed left continuous, i.e :

$$\Pi(t) = \lim_{\Delta t \rightarrow 0} \Pi(t - \Delta t)$$

Also $\Pi(0) = 0$, but

$$\Pi(0^+) = \lim_{\Delta t \rightarrow 0} \Pi(\Delta t)$$

- $B(t)$ is differentiable and assumed left-continuous.
- c_i represents the eventual capital delivered to the policyholder in case of early termination of the contract due to a cause of exit.

By taking into account the features mentioned above, one can appropriately redefine the reserve (1.2) via the equivalence principle. The pure mathematical reserve at time t follows Denuit(2014) and takes the following form :

$$V(t) = \int_t^n \tau - t p_{x+t} v(t, \tau) (dB(\tau) - d\Pi(\tau)) + \sum_{i=1}^m \int_t^n \tau - t p_{x+t} (t, \tau) v(t, \tau) \mu_{0i}(x + \tau) c_i(\tau) d\tau \quad (1.5)$$

Equation (1.5) is also referred as the prospective form of the reserve. The prospective reserve looks at the expected present value of the future benefits to be paid minus the expected present value of the future premiums that are still to be paid as defined above in this text.

The dynamics of equation (1.5) are shown by Thiele's differential equation. Thiele's differential equation gives the value of the prospective reserve, under multiple decrements, as the solution of :

$$dV(t) = \delta(t)V(t)dt + d(\Pi(t) - B(t)) - \sum_{i=1}^m \mu_{0i}(x + t)(c_i(t) - V(t))dt \quad (1.6)$$

According to Wolthuis and Hoem(1990), equation (1.6) has a unique solution (i.e. equation 1.5) under the initial condition $V(0) = 0$ and if the insurance benefits are independent of $V(t)$. Nevertheless $V(0^+) = \Delta\Pi(0)$. However regarding the independence of the insurance benefits, Christiansen, Dhaene and Denuit(2014) developed an explicit expression for the accumulated reserve dynamics under typical reserve-dependent insurance payments. By typical reserve-dependent insurance payments, one means

- Surrender payments, defined as a linear combination of the insurance reserve.
- Capital management fees defined as a linear combination of the insurance reserve.

- Profit sharing, where the surplus is defined as a linear combination of the insurance reserve.

The paper emphasizes on the applicability of Cantelli's theorem to derive an explicit expression of the accumulated insurance reserve provided the potential dependencies in the structure of the payments as enumerated above.

Theorem I (Cantelli's Theorem) *The value of the reserve $V(t) = \tilde{V}(t)$ for all states $i = 1, \dots, k$ and all $t \in [0, n]$ if, and only if,*

$$\begin{aligned} V(t)\delta(t) + d\Pi(t) - dB(t) - \sum_{i=1}^m \mu_{0i}(t)(c_i(t) - V(t))dt \\ = \tilde{V}(t)\tilde{\delta}(t) + d\tilde{\Pi}(t) - d\tilde{B}(t) + \sum_{i=1}^m \tilde{\mu}_{0i}(t)(\tilde{c}_i(t) - \tilde{V}(t))dt \end{aligned} \quad (1.7)$$

for all $i = 1, \dots, m$ and all $t \in [0, n]$.

Proof. The proof follows Christiansen *et al.* (2014). If the value of the insurance reserve $V(t) = \tilde{V}(t)$ at all times t then both $V(t)$ and $\tilde{V}(t)$ are equivalent therefore (1.7) is valid. Also, if $V(t) = \tilde{V}(t)$ then the reserves have equivalent Thiele's equation and therefore the solutions $V(t)$ and $\tilde{V}(t)$ are equal. Moreover, note that $V(t)$ and $\tilde{V}(t)$ have the same initial/terminal condition. This holds for both prospective and retrospective expression of the insurance reserve, regardless of the choice of the payment functions and intensity functions. This ends the proof.

1.3 Numerical approximation

The initial condition suggested above by Wolthuis and Hoem suggests that the termination of the contract also means the release of the insurance capital C or c_i at the exact same time. Therefore one may say that the termination of the contract is either driven by its time length or by a change from the initial state, considering all states except e_0 to be terminal states. In both cases Thiele's differential equation is applicable, as shown by Christiansen, Denuit, Dhaene (2014). Based on this, and the equivalence principle, by taking infinitesimal time steps Δt , it appears obvious that the insurance reserve equals the insurance capital at time $t = T - \Delta t$, i.e. :

$$V(T - \Delta t) = C$$

This is also valid under the assumption that no periodic premium is paid between $T - \Delta t$ and T as it is commonly the case for most insurance contracts namely in the Belgian market. Following this, the same affirmation can be made at time $t = 0 + \Delta t$ to obtain the value of the premium to be paid. This can be generalized into a purely numerical and recursive *backward* approach which allows to approximate the exact value of the premium to be paid.

This paragraph will develop the backward approach. This approach is inspired on Djehiche and Löfdahl (2016) however it will rely on Euler's methodology to approximate first degree differential equations. Let one considers the prospective form of Thiele's differential equation, equation (1.6). By taking an infinitesimal time step Δt and by considering that equation (1.6) is constant by part, the Euler method applied to the reserve is expressed as :

$$\begin{aligned}\frac{\partial V}{\partial t} &= \lim_{\Delta t \rightarrow 0} \frac{V(t + \Delta t) - V(t)}{\Delta t} \\ \frac{\partial V}{\partial t} &\approx \frac{V(t + \Delta t) - V(t)}{\Delta t}\end{aligned}\tag{1.8}$$

Lemme I (Application of Euler method to Thiele's differential equation).

Let $V(t)$ be the reserve of a life insurance product as defined in equation (1.6). By applying Euler's Method as stated in equation (1.8), one obtains :

$$V(t - 1) = \frac{V(t) + \Pi(t) - v(t - 1, t)_t q_{x+t} B(t + 1)}{v(t - 1, t)_t p_{x+t}}\tag{1.9}$$

Intuitively, this equation can be split between gains(+) and losses (-) of the insurer. This section follows Willame *et al.* :

Gains (+)

As the insured is still alive at time t , $t \geq \Delta t$, this implies that he was certainly alive at time $t - \Delta t$. This results in a decrease of the benefit paid in case of death ,as a result of the insured survival:

$$-\frac{v(t - \Delta t, t)_{\Delta t} q_{x+\Delta t} B_{t+\Delta t}}{v(t - \Delta t, t)_{\Delta t} p_{x+\Delta t}}$$

Losses (-)

Similarly, while moving from time t , $t \geq \Delta t$, to $t - \Delta t$, the insurer loses the capitalization effect of the interest earned and the premium paid:

$$\frac{V(t) + \Pi(t)}{v(t - \Delta t, t)_{\Delta t} p_{x+\Delta t}}$$

By its incremental approach, one may consider this methodology as a discretization of Thiele's differential equation. This allows to facilitate numerical applications such as an automatic approach of premium pricing in life insurance, as suggested by Willame *et al.*(2017). Moreover, Willame(2017) investigated the unicity of the solution of Thiele's differential equation regarding non-linear reserve-dependent payments under the backward methodology based on Euler's approximation.

Theorem II (Unicity of Thiele's differential equation under non-linear reserve-dependent benefits) *let consider Thiele's differential equation as expressed in (1.6), with reserve-dependent benefits and premiums being independent of the reserve. This allows a unique solution.*

Proof. The proof as stated in the form below contains the general concepts. We refer to Willame(2017) for a complete developed proof. Let one considers that the magnitude of the jumps k referring to the premium payment is known. Therefore, it is possible to obtain the unique solution of (1.6) recursively :

- Let one begin on the interval $]t_k, T]$. As the premiums are considered to be constant on the interval, Thiele's differential equation admits a single value $V(t_k)$ on this interval for every $t \in]t_k, T]$.

- Taking into account the first point, the value $V(t_{k+})$ is known. Therefore as the magnitude of the jump at time t_k is defined by $\Delta_k = d\Pi(t_k) - dB(t_k)$, it is possible to obtain the value of the reserve on the interval $]t_{k-1}, t_k]$ by setting the initial condition (i.e., the value of the reserve $V(t_k)$) to be equal to $V(t_k) = V(t_{k+}) - \Delta_k$. This results into a single value of the reserve $V(t_{(k-1)})$ on this interval for each $t \in]t_{k-1}, t_k]$.

By iterating this process until one reaches the interval $[t_0, t_{k-n}]$, whereas the value of $V(0)$ is unique and known. This ends the proof.

1.4 Discretization

While one may consider Euler's approach as a discretization of Thiele's differential equation, Dhaene *et al.*(2001) consider the approach of De Vylder(1997) on the "time-capital" for deterministic savings contracts with a focus on the reserve time capital. This paragraph will describe the approach taken and will highlight the applications regarding the classical life insurance. The authors define "time-capital" Q the couple which describes a capital and its time of payment. For instance, a time capital can take the form of a set of couples composed by an amount c to be paid at a certain time t : (c_k, t_k) , $(k = 1, \dots, n)$ is therefore a deterministic time-capital. Time capitals may also be stochastic, e.g.

$$\begin{aligned} Q &= (c_1(X), t_1(X)) + (c_2(X), t_2(X)) + \dots + (c_n(X), t_n(X)) \\ &= \sum_{k=1}^n (c_k(X), t_k(X)) \end{aligned} \tag{1.10}$$

where $c_k(X)$ and $t_k(X)$ are functions of a stochastic variable.

This time-capital can be associated with the insurance capital $B(t)$ or the premiums $\Pi(t)$ without losing sense of generality. Notice here that the insurance capital is represented by $B(t)$ instead of C or $C^\dagger$ for generality reasons.

Time capital payments have the following properties :

(i) At time capital level, there is no difference between the time capital obtained from Q starting at s on a defined interval $[s, s + t[$ where every other payments are set to be equal to 0 and the time capital obtained from Q on the interval $[s, s + t[$, where s is the new origin of time (defined below by $\bullet$) :

$${}_{s|t}Q = {}_{\bullet s|t}Q$$

(ii) At present value, property (i) is still applicable with the following :

$${}_{s|t}Q = v(s, t) {}_{\bullet s|t}Q$$

Theorem III For any time-capital Q , the following expressions hold:

(a) Decomposition formula , for $t > 0$:

$$Q = {}_{0|t}Q + v(0, t) {}_{\bullet 0|t}Q \quad (1.11)$$

(b) Generalized decomposition formula:

$${}_{\bullet s|}Q = {}_{\bullet s|t-s}Q + v(s, t) {}_{\bullet t|}Q \quad (1.12)$$

(c) Fouret's formula (at present value level) for $k \in [0, T]$:

$${}_{\bullet k+1|}Q = ({}_{\bullet k|}Q - {}_{\bullet k|1}Q)v(k, k + 1)^{-1} \quad (1.13)$$

Proof. The proof follows exactly Dhaene, Denuit, Goovaerts and Wolthuis (2001) :

(a) At time-capital level we have that $Q = {}_{0|t}Q + {}_{\bullet t|}Q$, from which (1.11) arises.

(b) At time-capital level, the equality ${}_{\bullet s|}Q = {}_{\bullet s|t-s}Q + {}_{\bullet t|}Q$ is valid and leads to (1.12) under (ii).

(c) Taking $s = k$ and $t = k + 1$ in (b) leads to (1.13). This ends the proof.

Notice that (a), (b) and (c) also hold at expected value level. When adapting to insurance products and consequently taking into account the impact of mortality, the time capital is expressed as the following :

$${}_{\bullet t|}Q = \int_0^t \mathbb{E}[{}_{\bullet t|}Q | X = s] d_s q_x + \mathbb{E}[{}_{\bullet t|}Q | X > t]_t p_x \quad (1.14)$$

Where X is the remaining lifetime of an individual.

Therefore based on those principles the notion of deterministic contract can be defined :

Definition 1 : *A deterministic saving contract is a pair of two deterministic time-capitals $(B(t), \Pi(t))$ where $B(t)$ is the time-capital describing the commitments of the insurer and $\Pi(t)$ is the time capital describing the commitments of the policyholder.*

At this point the insurance reserve can be derived from the equivalence principle. Note that, in this case, the reserve is called *reserve time-capital* of an insurance deterministic savings contract :

$$V(t) = B(t) - \Pi(t)$$

According to Dhaene *et al.*, the reserve time capital is defined as the present value, evaluated at time s , of all payments of the time-capital $V(s)$ in the interval $[t, \infty[$, where $s \in [t, \infty[$.

Theorem IV *For any reserve of a deterministic saving contract, the following expressions hold:*

(a) *Prospective expression of the reserves for $t \geq 0$:*

$${}_{\bullet|t}V = {}_{\bullet|t}B - {}_{\bullet|t}\Pi \quad (1.15)$$

(b) *Retrospective expression of the reserves for $t \geq 0$:*

$${}_{\bullet|t}V = (V + {}_{0|t}P - {}_{0|t}C)v(0, t)^{-1} \quad (1.16)$$

(c) *Fouret's formula for $k \in [0, T]$:*

$${}_{\bullet k+1|}V = ({}_{\bullet k|}V + {}_{\bullet k|}\Pi - {}_{\bullet k|}B)v(k, k+1)^{-1} \quad (1.17)$$

Proof. The proof follows exactly Dhaene, Denuit, Goovaerts and Wolthuis (2001) :

(a) Follows from the definition of $V(t)$

(b) From decomposition formula (1.11), one find

$$\begin{aligned} V(t) &= {}_{0|t}V + {}_{\bullet|t}Vv(0, t) \\ &= ({}_{0|t}\Pi - {}_{0|t}B) + {}_{\bullet|t}Vv(0, t) \end{aligned}$$

From which (1.16) is derived.

(c) From Fouret's formula (1.13), we find

$${}_{\bullet k+1|}V = ({}_{\bullet k|}V - {}_{\bullet k-1|}V)v(k-1, k)^{-1}$$

which leads to (1.17). This ends the proof.

When adapting to insurance product and therefore by taking into account the impact of mortality, Fouret's formula is expressed as :

$${}_{\bullet|t}V = \int_0^t {}_{\bullet|t}V(s) d_s q_x + {}_{\bullet|t}V_t p_x$$

Finally, the expression of the reserve for a life insurance contract can be established, following De Vylder(1997).

Theorem V *Let the couple (B, Π) be a life insurance contract of a policyholder at age x , then*

(a) *Prospective expression of reserves:*

$${}_{\bullet|t}V = {}_{\bullet|t}B - {}_{\bullet|t}\Pi, (t \geq 0), \quad (1.18)$$

(b) *Retrospective expression of reserves for $t \geq 0$:*

$${}_{\bullet|t}V = (V + {}_{0|t}\Pi - {}_{0|t}B)_t E_x^{-1} \quad (1.19)$$

(c) *Fouret's formula for $k \in [0, T]$:*

$${}_{\bullet|k+1}V = ({}_{\bullet|k}V + {}_{\bullet|k+1}\Pi - {}_{\bullet|k+1}B)_1 E_{x+k}^{-1} \quad (1.20)$$

As this proof follows theorems I and II, we refer the reader to Dhaene *et al.* (2001) for more details.

Equation (1.20) gives a proper discrete expression of the insurance reserve. The value of the insurance reserve at time $k + 1$ is a function of three principal components :

- (i) The value of the reserve at time k ,
- (ii) The difference between two flows being the present value at time k of the premiums paid over the period $[k, k + 1]$ and the present value at time k of the insurance benefits accumulated over the period $[k, k + 1]$,
- (iii) The impact of both mortality and interest rate ${}_1E_{x+k}^{-1}$.

Due to the various yearly reporting required for European insurances companies, Fouret's formula is mainly used in practice compared to Thiele's differential equation or the Euler's method. Therefore it is crucial to mention and describe Fouret's formula. Nevertheless one may remind that the purpose of this document is to describe the design of a pricing tool for classical life insurance products based on Thiele's equation approximated through the Euler's methodology. As a consequence of this, Fouret's formula will not be mentioned in the further chapters of this document.

1.5 Summary

The table below summarizes the main characteristics of the three expressions describing the dynamics of the insurance reserve :

Dynamics of $V(t)$ Characteristics	Thiele	Euler's method	Fouret's formula
Type	Continuous	Discrete	Discrete
Transition intensity	Assumed to be constant by part	Assumed to be constant by part	Either constant or nonconstant
Typical insurance payments	To some extent	Yes	To some extent
Premiums	Assumed to be paid constantly	Assumed to be paid constantly	Omissions can happen

We will elaborate on each of the four characteristics.

- Type : As shown with equation (1.6) and also by its name, Thiele's differential equation is applicable with a continuous reserve function whereas Fouret's formula is only applicable with a discrete representation of the reserve. The case of Euler's method is considered as discrete due to the fact that it approximates the differential equation however one may consider that by taking Δt sufficiently small the approximation should be in line with Thiele's differential equation value at a pre-defined time t .

- Transition intensity : Transition intensities are considered as constant by part for Thiele's differential equation and Euler's method. Whereby this assumption is usually taken for Thiele's differential equation, one may argue that Euler's method does not require such a strong assumption. Nevertheless and for the rest of this thesis we will consider Euler's method as constant by part. We believe that the size of the time steps Δt combined to the nature of Euler's method (i.e. the approximation of first degree differential equations) justify this choice. Regarding Fouret's formula, due to its discrete form, the expression allows irregularity between two defined discrete periods t .

- Typical insurance payments : As shown under Cantelli's theorem in equation (1.7), the dynamics of the insurance reserve expressed with Thiele's differential equation allows typical insurance payments to be a linear combination of $V(t)$. Moreover, the unicity of Thiele's differential equation under typical insurance payments being non linear combination of $V(t)$ has been investigated by Willame(2017) via an approximation of the differential equation performed under Euler's method. Fouret's formula also provides convenience to deal with non-linear insurance payments since those are under discrete assumptions.

- Premiums : Regarding the dynamics of Thiele's differential equation, the premium function $\Pi(t)$ is defined as an increasing function of cumulative cash-flows that can be either uniquely paid or periodically paid until the termination of the contract. Therefore by definition this implies constant payments. Explanations given for the transition intensity are also valid for Euler's method and Fouret's formula regarding the premiums.

Chapter 2

Backward pricing tool design for classical life insurance contracts

2.1 Architecture of the tool

The pricing tool is based on several sub-modules which tend to cover the backward pricing method but also features that have been chosen by the authors. The structure of the pricing tool contains 3 main modules : a backward pricing engine designed to price the unique premium, a backward pricing engine designed to price periodic premiums and an optimization engine designed to optimize the duration of the payments. This combination of engines allows the final user to have a better view on all the possibilities that can be chosen during the design of the desired life insurance product. This also allows the pricing tool to optimize its computation time by partitioning the calculation within the three engines : the utilization of the different engines depend on the design of the life insurance product chosen by the final user.

The design of the pricing engine mainly relies on Thiele's differential equation adapted to the recursive nature of Euler's scheme. In order to allow the backward pricing engine to benefit from the recursive nature of Euler's scheme, an incremental application of Thiele's differential equation to equation (1.8) is proposed. This starts with Thiele's differential equation as described in equation (1.6) and incrementally adds features which aim to reflect the current Belgian insurance industry and that are further developed in the design of the code. Euler's approximation is then applied to each of the steps, to finally obtain the form implemented within the design of the unique premium engine and the periodic premium engine, respectively.

In this chapter and more particularly regarding the design of the code itself, coding issues are highlighted and potential solutions are suggested. Particular segments of the code are described for illustrative purposes however these are reviewed for a clear understanding. Nevertheless as the language used to design the pricing engine is R, these segments follow the same coding logic.

2.2 Data preparation

Inputs parameters are in the center of the pricing engine. Classical life insurance contracts allow covering flexibility and are of many uses : from pure investments contract such as pure endowment, differed capital with reimbursement of the insurance reserve, mixed insurance and others to outstanding loan insurance. As a consequence every parameters that define the insurance contract suggested by the tool need to be carefully selected and defined.

2.2.1 Defined Insurance Benefits

By definition, life insurance contracts provide coverage against particular events linked to the life or the death of the insured. However the occurrence of such events is uncertain and the consequences can be dramatic. Therefore the policyholder delineates the insurance benefits desired to cover the occurrence of a particular event and personal information required by the insurer to provide adequate coverage :

- Age of the insured person when signing the contract.
- Duration of the contract, expressed in years.
- The desired capital in case of life at the end the contract. This can take two forms:
 - A fixed life capital obtained at the end of the contract.
 - An income that the insured would like to obtain for a certain period at end of the contract. This can be obtained by converting the flat life capital amount into an income by an annuity price. To do so, the following criteria are defined:
 - * The duration of the payment.
 - * The recurrence of payment per year.
 - * Income behavior, whether the requested income stays constant, increases or decreases during the defined period.
 - * The nature of the income behavior, whether the decrease or the increase follows a geometric or as an arithmetic pattern.
 - * If increasing or decreasing behavior, the desired pace. Note that the pace refers to the growth rate for the geometric behavior, whereas the pace refers to growth amount for arithmetic behavior.
- The desired capital to cover the occurrence of death. This can take different forms :
 - A fixed capital.
 - A function of a predefined fixed capital, such as for outstanding loan insurance.

- A function of other elements in the contract, such as the life capital or the insurance reserve. This can be either linear or non-linear.

The insurer can also define constraints to the defined insurance benefits of the policyholder :

- In case of surrender, a surrender penalty.
- In case of non-covered death, a partial to a non death coverage.

Every defined insurance benefits are chosen for a defined period within the total duration of the contract. In other words, this means that the type of defined insurance benefits may vary through the lifetime of the contract so the contract is tailor-made for the policyholder.

2.2.2 Actuarial Assumptions

The following actuarial assumptions are taken into account in the pricing tool :

- The mortality rate. Two methods are used in the pricing tool :
 - Through the mortality transition intensity. Therefore the survival probability and the probability of death are defined as follows :
The survival probability for n year:

$${}_np_x = \exp\left(-\int_0^n \mu^\dagger(x+t) dt\right) \quad (2.1)$$

The probability of death within a year:

$$q_x = 1 - p_x = 1 - \exp\left(-\int_0^1 \mu^\dagger(x+t) dt\right) \quad (2.2)$$

Note that $\mu^\dagger$ expresses the instantaneous mortality rate, defined as follows :

$$\mu^\dagger(x+t) = \lim_{\Delta t \rightarrow 0} \frac{P[t < T_x < t + \Delta t | T_x > t]}{\Delta t} \quad (2.3)$$

The instantaneous mortality rates defined in equation (2.3) above have the same behavior as the transition intensities μ defined with equality (1.3). This method is mainly used when the force of mortality is given. However the Belgian regulation imposes the use of the Belgian mortality tables to price any life insurance contract on the Belgian territory, following the Belgian Law 2014/04/04 on insurance. Those mortality table have been calculated following the Makeham model.

- The Makeham model establishes the survival of a population sample at a defined time x based on four parameters :

$$l_x = ks^x g^{c^x} \quad (2.4)$$

Once this population sample is calculated for each year the probability of survival ,probability of death, and the force of mortality for each year is obtained as follows:

$$\begin{aligned}
p_x &= \frac{l_{x+1}}{l_x} \\
{}_k p_x &= p_x \times p_{x+1} \times \dots \times p_{x+k} \\
\mu(x) &= -\frac{\partial}{\partial x} \ln l_x \\
&\Leftrightarrow \\
\mu(x) &= -\frac{l'_x}{l_x} \\
{}_n p_x &= \exp\left(-\int_0^n \mu(x+s)ds\right)
\end{aligned} \tag{2.5}$$

- A guaranteed interest rate in which the contract will be priced under the Branch 21. This rate is estimated based on the economic conjuncture and the financial markets.
- Policyholder surrender rates which reflect the possibility of the policyholder to terminate his insurance contract before the initially defined termination date.
- Estimation of the portfolio portion which results into a partially to non-covered death.

Every actuarial assumptions are chosen for a defined period within the total duration of the contract and therefore can vary through its lifetime. Moreover, the model includes only multiple decrements : each transition from the original state e_0 to another state $e_1, e_2, \dots, e_m$ causes the termination of the contract, as defined in chapter 1.

2.3 Life annuities

Life annuities play a major role in life insurance. Therefore the following annuity prices are defined and used within the pricing tool :

- n-Year temporary life annuity :

$$\bar{a}_{x:\overline{n}|} = \int_0^n v^t {}_t p_x dt \tag{2.6}$$

- Arithmetically increasing annuity :

$$(\bar{IA})_{x:\overline{n}|} = \int_0^n t v^t {}_t p_x \mu^\dagger(x+t) dt \tag{2.7}$$

where t is the rate of payment.

- Arithmetically decreasing annuity :

$$(D\bar{A})_{x:\overline{n}|} = \int_0^n (n-t)v^t {}_t p_x \mu^\dagger(x+t)dt \quad (2.8)$$

- Geometrically increasing annuity :

$$(G\bar{A})_{x:\overline{n}|} = \int_0^n \frac{1}{\delta(t)-g} \left(1 - \left(\frac{1+g}{1+\delta(t)}\right)^t\right) {}_t p_x \mu^\dagger(x+t)dt \quad (2.9)$$

Where g is the growth rate.

Moreover equations mentioned in this section may also be expressed with different periodicity which can be found in every good life insurance mathematics book.

2.4 Specification matrix

As the code is part of a pricing tool there is an incentive to make it user friendly. A direct consequence of this is on the structure of the tool which is designed following the three principles mentioned above in this document. Therefore this section aims to describe the code which is used by the user to provide every element of the insurance contract.

```
specification_matrix <- function(periode, other_parameters){
  if (missing(periode)){
    break
  }
  if (missing(other_parameters)==TRUE){
    other_parameters = rep(0, length(periode) )
  }
}
```

The so-called specification matrix includes 6 rows. While the first row describes the boundaries between each period, the others (here mentioned by `other_parameters`) rows give the underlying features of the insurance contract, as described in table 2.1.

The design of the first part of the function is made to allow the user to only define the parameters discussed with the future policyholder. Once the user defined the other parameters, the next part of the function aggregates them into a $6 \times n$ matrix where the number of columns n follows the number of periods in the insurance contract :

```
specifications <- rbind(rev(periode), rev(other_parameters))
```

This choice is motivated by the assumptions taken when using Euler's method : the reserve is assumed to be constant by part due to the assumptions taken in chapter 1. Therefore one has only to define the boundaries between each period, with the changes in assumptions associated with.

Resiliation	Elaborates about the surrender rate estimated by the insurer, per period.
Penalite	Elaborates about the penalty applied to the policyholder if a surrender occurs. The penalty is expressed as a percentage of the insurance reserve.
Accident	Elaborates about the expected percentage of non-covered accident in the insurance portfolio.
Increments	Elaborates about the type of increments underlying the death benefits. Increments takes values -1 for decreasing, 0 for constant and 1 for increasing.
Death_capital	Elaborates about the linearity of the death coverage. Death_capital takes value 0 when the death benefits is either a constant amount or a linear combination of the insurance reserve. If the death benefits are expressed as a non-linear combination of the insurance reserve, the value of Death_capital is 1.
Penalite_death	Elaborates about the penalty applied to the death capital in case of partial reimbursement

Table 2.1 – Parameters of the specification matrix

2.5 Unique premium engine design

2.5.1 Theoretical aspects

The unique premium engine is based on Euler's method applied to Thiele's differential equation as mentioned in the theoretical part, more specifically with equation (1.6). However if one considers the development of the tool to include real-life aspects of the business as observed within the insurance industry, the expression of the reserve as defined by equation (1.6) should therefore include more elements. In the life insurance industry, common contracts comprised events or elements such as a policy surrender, a non-covered or partially covered death (for instance due to an accident when practicing extreme sport) or linear and non-linear death benefits. One may argue that the list of elements is as extensible as the imagination of the actuary is limitless. However we decided only to take into account the aforementioned characteristics. Therefore and if one wants to design a life insurance contract containing every element mentioned above, Thiele's differential equation takes the following expression :

$$dV(t) = \delta(t)V(t) + d(\Pi(t) - B(t)) - (1 - \blacklozenge)\mu^\dagger(C_t^\dagger - V(t))dt - \mu^\chi(\chi V(t) - V(t))dt - \blacklozenge\mu^\dagger(\blacklozenge C_t^\dagger - V(t))dt \quad (2.10)$$

Equation (2.10) can be intuitively interpreted:

$\delta(t)V(t)$ and $d(\Pi(t) - B(t))$ did not change in comparison to Thiele's differential equation developed in chapter 2. While the first term represents the gain of interest

rate realized on $V(t)$, the second term expresses the increase of $V(t)$ which is represented by the difference between the amount of accumulated premium payments of the policyholder $\Pi(t)$ and the insurance benefits paid during the contract by the insurer $B(t)$. Regarding the eventual capital c_i delivered to the policyholder in case of early termination of the contract (due to a cause of exit), these are part of the following elements:

- The part $(1 - \blacklozenge)\mu^+(C_t^+ - V(t))dt$ expresses the amount required by the insurer in surplus of $V(t)$ to cover the death of the policyholder, where C_t^+ is the defined death coverage capital.
- The part $\mu^\chi(\chi V(t) - V(t))dt$ expresses the penalty applied to the policyholder in case of surrender. μ^χ is the surrender rate and $\chi V(t)$ is a function representing the surrender penalty applied to $V(t)$. Note that in this case the reserve is considered as a base for the surrender payment however other types of surrender benefits are also applicable.
- The part $\blacklozenge\mu^+(\blacklozenge C_t^+ - V(t))dt$ expresses the non coverage or the partial coverage of particular death, such as accidents due to extreme sports practice. $\blacklozenge$ represents the proportion of the death in the portfolio estimated by the insurer as non-covered or partially covered. $\blacklozenge C_t^+$ is a function representing the death coverage in case of partial coverage.

Note that the representation of the expressions $(1 - \blacklozenge)\mu^+$ and $\blacklozenge\mu^+$ are only possible because of the mutual exclusivity assumption defined in chapter 2.

Therefore, by applying Euler's methodology as defined in equation (1.8), one may obtain a discrete expression of the reserve including industry-specific features which can be used in the design of the pricing tool.

Euler scheme is a discrete recursive method used to price a policy based on an approximation of the reserve. The Euler scheme starts with an initial value at the end of contract, which is the value of the life capital as it was mentioned earlier, and then the reserve at time t will be estimated by taking a step backward in time, based on the reserve as estimated at time $t + \Delta t$. This iteration will keep on until it reaches the value of the reserve at time $t = 0^+$, which is the value of the first premium paid as described in chapter 1. Regarding this section, the value returned will be the unique premium.

Number of steps As a first step, one which should be done is to define the number of increments Δt that will be used. As the running time of the engine depends on the number of increments chosen, this value should be as small as possible in order to have the smallest the running time. On the other hand, as Δt gets smaller, a more accurate estimation will be obtained. Therefore the optimal number of steps should be carefully chosen in order to ensure an accurate approximation of the premium with a reasonable running time.

Let one considers a contract with a life capital of 100,000EUR with a duration of 30 years

the contract will be split into 2 periods the first one is for 10 years with a mortality rate of 0.01 and a death capital of 90% of the total reserve and for the second period of 20 years a death capital of 75% of the total reserve with a mortality rate of 0.05. Figure 2.9 shows the evolution of the unique premium by the number of steps chosen. Considering the contract mentioned above , the optimal number of steps should be at least 800 number of steps.

Once the number of steps is selected, Δt is expressed as follows :

$$\Delta t = \frac{\text{contract duration}}{\text{number of steps}} \quad (2.11)$$

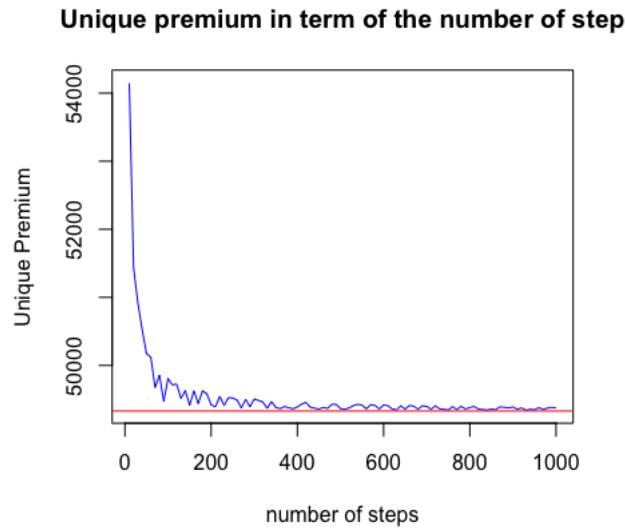


Figure 2.1 – Unique Premium value in terms of number of steps

Death capital The first contract that is considered is a simple life insurance with a fixed death capital, C_t^+ , that is paid to the insured in case of his death during the life span of the contract. In case the insured survived throughout the duration of the contract, he will receive a life capital, C , at the end of the contract. Hence the Euler methodology follows the reasoning below :

- If the policyholder is still alive at time t , then the policyholder was alive at time $t - \Delta t$. Hence he survived Δt . Therefore, the policyholder earned an interest rate $I_{\Delta t}$ on his reserve due to his survival.
- On the other hand, the policyholder was covered against the risk of death during Δt . Therefore a risk capital should be laid out from the reserve $RC_{\Delta t}$.

This results in the following changes :

Interest Earned:

During the period Δt , the reserve has earned an interest rate based on the interest rate guaranteed under branch 21 life insurance contracts.

$$I_{t-\Delta t} = \ln(1 + i_{t-\Delta t}) \times \Delta t \times (V(t - \Delta t)) \quad (2.12)$$

Risk Capital:

Due to the policyholder survival, the risk of death was covered by the following risk capital RC:

$$RC_{t-\Delta t} = \mu^+(x + t - \Delta t) \times \Delta t \times (C_{t-\Delta t}^+ - V(t - \Delta t))$$

Reserve value:

The Value of the reserve at time $t - \Delta t$ is the value of the reserve at a step ahead, at time t , minus the interest earned during Δt , plus the capital on risk that is covered during Δt .

$$\begin{aligned} V(t - \Delta t) &= V(t) - I_{t-\Delta t} + RC_{t-\Delta t} \\ &= \frac{V(t) + \mu^+(x + t - \Delta t) \times \Delta t \times C_{t-\Delta t}^+}{1 + (\ln(1 + i_{t-\Delta t}) + \mu^+(x + t - \Delta t)) \times \Delta t} \end{aligned} \quad (2.13)$$

Equation (2.13) is then driven by (i) the interest earned and (ii) the risk capital. Therefore, if the interest rate is higher than the risk capital, the value of the insurance reserve tends to increase over time. Inversely, if the risk capital is higher than the interest earned, the value of the reserve tends to decrease over time. These behaviors are represented in the figure 2.2 below :

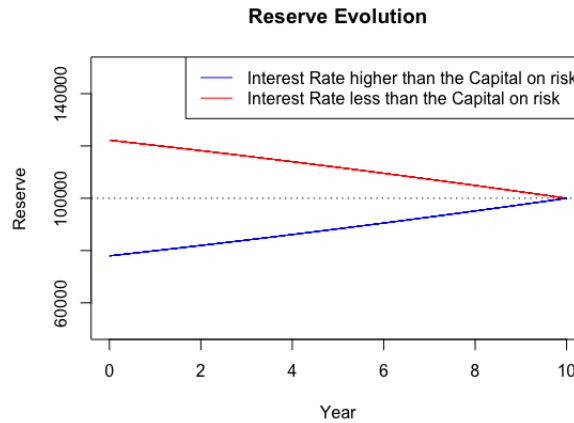


Figure 2.2 – Reserve growth over time

2.5.2 Death capital

In this section, we will consider the death capital as a percentage of the reserve instead of a fixed capital. Moreover the interest earned remains the same as the previous case. However the risk capital will be a percentage, $p_{t-\Delta t}$ of the current reserve as it is illustrated in the following equation:

Risk Capital :

$$\begin{aligned} RC_{t-\Delta t} &= \mu^+(x+t-\Delta t) \times \Delta t \times (p_{t-\Delta t} \times V(t-\Delta t) - V(t-\Delta t)) \\ &= \mu^+(x+t-\Delta t) \times \Delta t \times V(t-\Delta t) \times (p_{t-\Delta t} - 1) \end{aligned} \quad (2.14)$$

Therefore based on equation (2.12) and equation (2.14) the reserve $V(t-\Delta t)$ can be estimated as follows:

$$V(t-\Delta t) = \frac{V(t)}{1 + (\ln(1+i_{t-\Delta t}) + \mu^+(x+t-\Delta t)) \times (1-p_{t-\Delta t}) \times \Delta t} \quad (2.15)$$

On the other hand, Thiele equation can be applied thought an equivalent contract that has an equal reserve. This can be done by transforming this contract into a simple mixed life insurance with a fictive mortality rate as follows:

$$\frac{dV(t)}{dt} = \ln((1+i))V(t) - \mu^+((1-p)V(t) - V(t)) \quad (2.16)$$

As a result, for this main case the new fictitious mortality rate is

$$\mu_{\text{fictitious}}^+ = \mu_{\text{real}}^+(p)$$

Mortality Neutralization

A death capital which is equal to the reserve is a common insurance product insurance in the Belgium market, known as differed capital with reimbursement of the reserve ("CDRR"). Where in case of the death, the policyholder will receive the insurance reserve. Hence $p_{t-\Delta t} = 1$, the mortality will not have any effect on the pricing. This product is a purely financial product, hence there are no mortality risk, the risk will narrow to only a financial risk.

$$\begin{aligned} V(t-\Delta t) &= \frac{V(t)}{1 + (\ln(1+i_{t-\Delta t}) + \mu^+(x+t-\Delta t)) \times (1-1) \times \Delta t} \\ &= \frac{V(t)}{1 + (\ln(1+i_{t-\Delta t}))\Delta t} \end{aligned} \quad (2.17)$$

As it is illustrated in the figure below, two policies were taken with the same guaranteed interest rate, but with two different mortality rates of $\mu_1^+ = 0.01$ and $\mu_2^+ = 0.001$. Therefore one can observe that two curves overlap. Hence that proves that the mortality rate has no effect on the reserve.

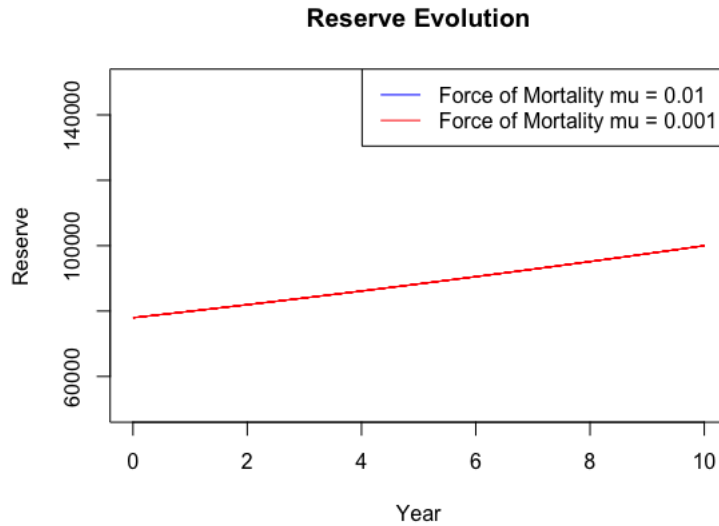


Figure 2.3 – Reserve evolution over time no Mortality effect

The following equation is the equivalent Thiele's differential equation of this particular case. It can be noticed that the mortality rate is neutralized.

$$\begin{aligned}\frac{dV(t)}{dt} &= \ln(1+i)V(t) - \mu(V(t) - V(t)) \\ &= \ln(1+i)V(t)\end{aligned}\tag{2.18}$$

2.5.3 Partial reimbursement and surrender options

In this section, the death cause is segregated into two categories, a death can be either be caused due to a natural cause or due to an accident. The insurer estimates that the proportion of death which is caused by an accident within the policyholder similar profile portfolio to be $\diamond$. Whether the policyholder dies due to an accident or due to a natural cause, he will receive a capital of $\diamond C_t^+$ and C_t^+ respectively. Next to the accident the policyholder has the option of surrender his contract, with a probability intensity of μ^χ . If the policyholder surrenders his contract therefore he will receive back a part of the insurance reserve. Hence he will get penalized on the value of his reserve, i.e the policyholder will receive $\chi V(t)$. Following the same procedure as previously, the reserve will evolve through earning an interest and paying out a risk capital on the reserve.

Although the interest earned remain the same as the previous case, the risk capital will be as follows:

Risk Capital :

Due to the policyholder survival ,his risk of death whether due to an accident cause or due to a natural cause and the risk of surrender were covered by the following risk capital RC:

$$\begin{aligned} RC_{\Delta t} = & \mu^+(x+t-\Delta t) \times \Delta t \times (1-\diamond) \times (C_{t-\Delta t}^+ - V(t-\Delta t)) \\ & + \mu^+(x+t-\Delta t) \times \Delta t \times \diamond \times (\diamond C_{t-\Delta t}^+ - V(t-\Delta t)) \\ & - \mu^x(t-\Delta t) \times \Delta t \times V(t-\Delta t) \times (1-\chi) \end{aligned} \quad (2.19)$$

Therefore based on equation (2.12) and equation (2.19) the Reserve $V(t-\Delta t)$ can be estimated as follows:

$$V(t-\Delta t) = \frac{V(t) + (C_{t-\Delta t}^+ \times (1-\diamond) + \diamond C_{t-\Delta t}^+ \times \diamond - \mu^x(t-\Delta t) \times (1-\chi)) \times \Delta t}{1 + (\ln(1+i_{t-\Delta t}) + \mu^+(x+t-\Delta t) + \mu^x(t-\Delta t) \times (1-\chi)) \times \Delta t} \quad (2.20)$$

2.5.4 Minimum guaranteed death capital

As the reserve generally grows throughout time (through the interest earned on the reserve and through the periodic premium paid), many policyholders desire to obtain their reserve as a death capital with a minimum guaranteed fixed death capital, C_t^+ . In other words, while the reserve of the policyholder is less than the minimum guaranteed fixed death capital C_t^+ , the death capital will be the minimum guaranteed capital. As the reserve passes the minimum guaranteed capital value, the death capital will be his current reserve. Thus the death capital can be expressed as the maximum between the reserve and C_t^+ . This results into a non-linear expression of the insurance reserve which as a consequence cannot be solved under Thiele's differential equation. Nevertheless there exists a unique solution due to Euler's method. As a result the reserve will evolve through earning interest on the reserve, as it is defined in the equation (2.12) and will pay out a risk capital on the reserve as follows:

Risk Capital :

The death capital in case of death during Δt is defined by:

$$RC_{t-\Delta t} = \mu^+(x+t-\Delta t) \times \Delta t \times (\max(C_t^+, V(t-\Delta t)) - V(t-\Delta t)) \quad (2.21)$$

However this risk capital can be split into two components :

- A fixed death capital before the reserve attained C_t^+ .
- A refund on the reserve as the reserve exceeds C_t^+ , as it is illustrated in the equation (2.14).

The risk capital can then be expressed as :

$$RC_{t-\Delta t} = \mu^+(x+t-\Delta t) \times \Delta t \times ((C_t^+ - V(t-\Delta t)) \times \mathbb{I}_{C_t^+ > V(t-\Delta t)} + (V(t-\Delta t) - C_t^+) \times \mathbb{I}_{V(t-\Delta t) > C_t^+}) \quad (2.22)$$

Where :

$$\mathbb{I}_{C_t^+ > V(t-\Delta t)} = \begin{cases} 1 & \text{if } V(t-\Delta t) < C_t^+ \\ 0 & \text{else} \end{cases}$$

and

$$\mathbb{I}_{V(t-\Delta t) > C_t^+} = \begin{cases} 1 & \text{if } V(t-\Delta t) > C_t^+ \\ 0 & \text{else} \end{cases}$$

Therefore based on equation (2.21) and equation (2.22) the Reserve $V(t-\Delta t)$ can be estimated as follows:

$$V(t-\Delta t) = \frac{V(t) + \mu^+(x+t-\Delta t) \times \Delta t \times C_t^+ \times \mathbb{I}_{V(t-\Delta t) > C_t^+}}{1 + (\ln(1 + i_{t-\Delta t}) + \mu^+(x+t-\Delta t) \times \mathbb{I}_{C_t^+ > V(t-\Delta t)}) \times \Delta t} \quad (2.23)$$

As it is illustrated in the figures 2.4 and 2.5 below, if one considers a life insurance contract with death capital equals to the reserve with a minimum death capital which is equal to 97000 EUR, the evolution of the reserve compared to the evolution of the death capital shows as follows :

- During the first period the insurance contract is considered to be as a simple life insurance with a death capital equal to 97000 EUR.
- As the value of the reserve exceeds the minimum death capital, the contract changes the death capital will be replaced by the reserve.

2.5.5 Euler Equation

Combining equation 2.15 , equation 2.20 and equation 2.23 we obtain the following equation:

$$V(t-\Delta t) = \frac{V(t) + A \times \Delta t}{1 + B \times \Delta t} \quad (2.24)$$

with :

$$A = C_{t-\Delta t}^+ \times (1 - \Diamond) + \Diamond C_{t-\Delta t}^+ \times \Diamond - \mu^\chi(t-\Delta t) \times (1 - \chi) + \mu^+(x+t-\Delta t) \times C_t^+ \times \mathbb{I}_{V(t-\Delta t) > C_t^+}$$

and

$$B = \ln(1 + i_{t-\Delta t}) + \mu^+(x+t-\Delta t) \times (1 - p_{t-\Delta t}) + \mu^+(x+t-\Delta t) \times \mathbb{I}_{C_t^+ > V(t-\Delta t)} + \mu^\chi(t-\Delta t) \times (1 - \chi)$$

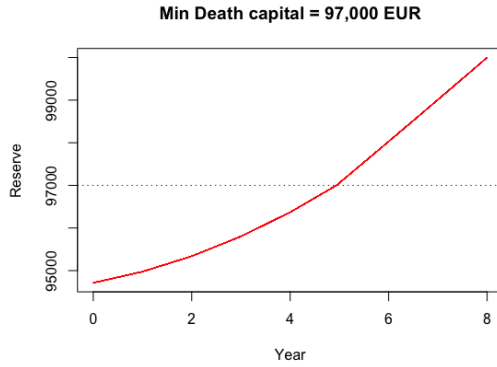


Figure 2.4 – Reserve evolution over the duration of the contract

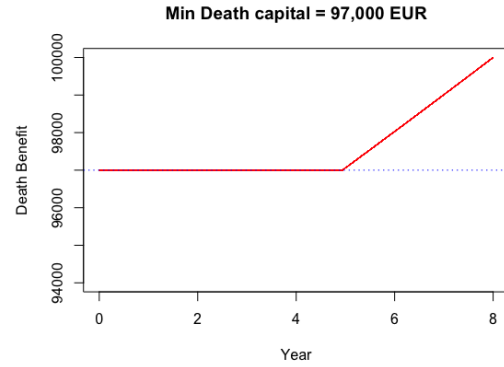


Figure 2.5 – Death capital evolution over the duration of the contract

2.5.6 Implementation within the code

Equation 2.24 is the core element of the numerical approach used to calculate the value of the insurance reserve $V(t)$ as defined by Thiele's differential equation (2.10). This section will explain the implementation of 2.24 as a part of the code to calculate the unique premium of classical life insurance contracts.

```
Backward_thiele <- function(cv,cd,ca,age,n,interest_rate,num_of_step,
specifications){
  PU_vector <- c()
  V<- c()
  reserve <- cv
  V[1] <- x
  increment=1
  delta <- n/num_of_step
  temps <- n
  for (i in 1:ncol(specifications)){
    time <- specifications[1,i]
    other_parameters <- specifications[2:6,i]
```

The first part of the unique premium backward pricing tool consists into a **for** loop which picks all the necessary parameters from the specification matrix to perform the backward iteration further in the function. In the first part of the function is also defined the insurance reserve vector **V** which will capture the value of the insurance reserve throughout each step of the iteration until $t - \Delta t = 0$. Other parameters explained earlier in this chapter are also defined.

```

while(temps > time + delta){
  temps <- temps - delta
  ratio <- (0.5*(ceiling(temps)*(1+increments)
    +(n-floor(temps))*(1-increments)))^abs(increments)
  mu <- mortality_table[age+floor(temps)+1,2]
  if((reserve/(1+(log(1+interest_rate)+penalite*resiliation)*delta)) > cd
    && max_cd == 1) {dummy = 0} else {dummy = 1}
  Reserve <- (reserve+cd*mu*ratio*(1-accident)*delta*(as.integer(cd!=0))*dummy
    +(accident)*mu*ca*delta*(as.integer(cd!=0)))
    /(1+(log(1+interest_rate)+penalite*resiliation
    +penalite_death*mu+mu*dummy*(1-accident)*(as.integer(cd!=0))
    +mu*(accident))*delta)
  V[increment+1] <- reserve
  increment <- increment + 1
}

```

The second part of the unique premium engine hosts the backward numerical approach defined by Euler's method in equation (2.24). However, some preliminary steps are necessary before calculating the value of the reserve at $T - \Delta t$ and therefore achieving one iteration. The while loop starts one increment before the termination of the contract, i.e., at $V(T^-)$. At that point, the value of the reserve equals the life capital of the contract. The blocking condition stops the loop one increment before the transition between two periods defined within the specification matrix via the vector `time`. Obviously, the blocking condition which triggers the last transition is at $t = \Delta t$, i.e., where one assumes that the premium is paid.

The iteration begins by decreasing the starting time by one increment. Then, the adequate mortality rate is extracted from the mortality table : to do so, one uses the function `floor` which rounds down the current time of the corresponding increment number, and sums its up accordingly to the age of the policyholder.

In the meantime the objects `ratio` and `dummy` are defined. The `ratio` and the `dummy` objects refer to functions of death coverage :

- The object `ratio` refers to the function of the death coverage associated with arithmetic increasing or arithmetic decreasing benefits. The object `ratio` can be interpreted as a function of the duration of the contract and the variable `increments` which take the values -1, 0 and 1. If `increments` takes the value 1, then `0.5*(ceiling(temps)*(1+increments) + (n-floor(temps))*(1-increments))` takes the value 0. On the contrary, if `increments` takes the value -1, then `0.5*(ceiling(temps)*(1+increments) + (n-floor(temps))*(1-increments))` returns 0 while `(n-floor(temps))*(1-increments)` returns the difference between the length of the contract and the lower rounded value of the current time of the corresponding increment number. Finally, if `increments` takes the value 0, `ratio` takes the value 1 due to `exp(abs(increments))`.

- The object `dummy` is an index that can take a value of 0 or 1. In which `dummy` refers to the function of the death coverage associated with a non linear death benefits. In that case, the non-linear benefit associated is of the form $\max(V(t), C_t^\dagger)$. `dummy` therefore takes the value 0 if the value of the reserve as expressed by equation 2.22 and the condition `max_cd` previously defined in the matrix of specifications is equal to one. Otherwise, the value of `dummy` takes 1.

The next step for the function is the application of Euler's method (2.24). The logical condition `cd` $\neq 0$ allows to identify the presence of a death capital : if the condition returns `TRUE`, this means that the contract includes a coverage in case of death and therefore the function `as.integer` returns a 1. Accordingly, if the condition returns `FALSE`, this means that the contract does not include a coverage in case of death and as a consequence the function `as.integer` returns a 0.

Once the value $V(T - \Delta t)$ is calculated, the vector V is updated with the latter and the iteration continues until $T - \Delta t = 0^+$.

2.6 Periodic premium engine design

Most insurance contracts are paid on a periodic basis rather than on a unique premium. Similar to unique premium, the periodic premium can be obtained by applying Euler's methodology. As it was applied with unique premium, the Euler scheme starts with an initial value at the end of contract, which is the value of the life capital, and then the reserve at time t will be estimated by taking a step backward in time Δt , based on the reserve as estimated at time $t + \Delta t$. This iteration will keep on until it reaches the value of the reserve at time $t = 0^+$, where it is the value of the first premium paid which is the first periodic premium paid as described in chapter 1.

2.6.1 Rationale

The Reserve, similar to the unique premium, evolves through laying out the risk capital RC and through earning interest on the reserve. In addition the reserve increases throughout the periodic payments which are paid by the policyholder whenever the payments are due. As it is illustrated in the figures below, the same contract (in this case : a simple mixed capital with a death capital equal to the life capital) was taken for both cases and it can be noticed that the reserve with an unique premium continuously evolves through the interest earned and the risk capital laid at every Δt . However for the reserve it can be noticed that the reserve evolves, in an addition to the interest earned and the risk capital laid out, through jumps which corresponds to the periodic premium that is paid at the beginning of each year (in that case).

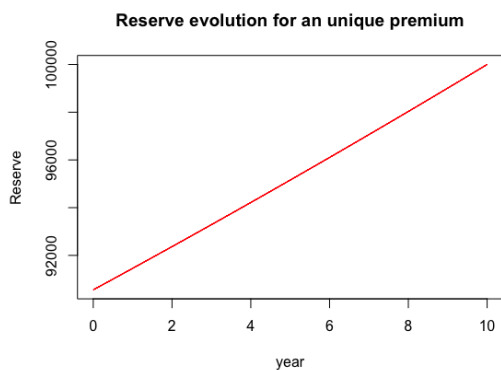


Figure 2.6 – Reserve evolution for a unique premium

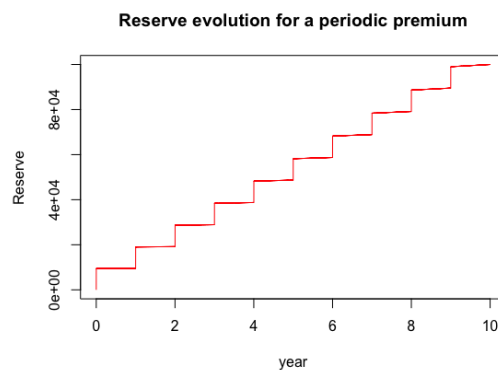


Figure 2.7 – Reserve evolution for a periodic premium

If the policyholder is still alive at time t during his contract, that is implicit that he was certainly alive at time $t - \Delta t$. This implies that the policyholder has earned an income and paid a risk capital on his reserve as follows:

Income Earned :

During the period Δt , the reserve has earned an interest based on the interest rate guaranteed under branch 21 life insurance contracts. Moreover the policyholder will pay his periodic premium when the payment is due :

$$I_{t-\Delta t} = \ln(1 + i_{t-\Delta t}) \times \Delta t \times (V(t - \Delta t, PP)) + PP \times \mathbb{I}_{\text{payment due}}$$

Where

$$\mathbb{I}_{\text{payment due}} = \begin{cases} 1 & \text{if the increment equals the payment date} \\ 0 & \text{else} \end{cases}$$

Risk Capital :

Due to the policyholder survival, the risk of death was covered by the following risk capital RC:

$$RC_{t-\Delta t} = \mu^+(x + t - \Delta t) \times \Delta t \times (C_{t-\Delta t}^+ - V(t - \Delta t))$$

Reserve value:

The value of the reserve at time $t - \Delta t$ is the value of the reserve at a step ahead, at time t , minus the interest rate earned during Δt , plus the capital on risk that is covered during Δt .

$$\begin{aligned} V(t - \Delta t, PP) &= V(t, PP) - I_{\Delta t} + RC_{\Delta t} \\ &= \frac{V(t) + \mu^+(x + t - \Delta t) \times \Delta t \times C_{t-\Delta t}^+ - PP \times \mathbb{I}_{\text{payment due}}}{1 + (\ln(1 + i_{\Delta t}) + \mu^+(x + t - \Delta t)) \times \Delta t} \end{aligned} \quad (2.25)$$

Equation (2.25) is then driven by (i) the interest rate, (ii) the risk capital and (iii) the periodic premium paid when it is due.

2.6.2 Partial reimbursement and surrender options

Let one considers a contract with a mixed life insurance including a death accident and a surrender options. The insurer estimates that the proportion of death which is caused by an accident within the policyholder similar profile portfolio to be $\diamond$. Whether the policyholder dies due to an accident cause or due to a natural cause, he will received a fixed capital of $\diamond C_t^+$ and C_t^+ respectively.

Beside to the accident option the policyholder has the option to surrender his contract, with a probability intensity of μ^χ . If the policyholder surrenders and therefore receives back a portion of his reserve, he will get penalized on the value of his reserve, the policyholder will receive $\chi V(t)$. Following the same procedure as previously, the reserve will grow through earning an interest and annual premium paid and the reserve will shrink through laying out a risk capital.

Income Earned :

During the period Δt , the reserve grows due to the interest earned and the premium paid if payment is due during this period Δt

$$I_{t-\Delta t} = \ln(1 + i_{t-\Delta t}) \times \Delta t \times (V(t - \Delta t, PP)) + PP \times \mathbb{I}_{\text{payment due}} \quad (2.26)$$

Risk Capital :

Due to the policyholder survival, his risk of death whether by an accident or due to a natural caused and the risk of surrender were covered by the following risk capital RC:

$$\begin{aligned} RC_{t-\Delta t} = & \mu^+(x+t-\Delta t) \times \Delta t \times (1-\diamond) \times (C_{t-\Delta t}^+ - V(t-\Delta t)) \\ & + \mu^+(x+t-\Delta t) \times \Delta t \times \diamond \times (\diamond C_{t-\Delta t}^+ - V(t-\Delta t)) \\ & - \mu^x(t-\Delta t) \times \Delta t \times V(t-\Delta t) \times (1-\chi) \end{aligned} \quad (2.27)$$

Therefore based on equation (2.26) and equation (2.27) the reserve $V(t-\Delta t)$ can be estimated as follows:

$$V(t-\Delta t, PP) = \frac{V(t) + (C_{t-\Delta t}^+ \times (1-\diamond) + \diamond C_{t-\Delta t}^+ \times \diamond - \mu^x(t-\Delta t) \times (1-\chi)) \times \Delta t - PP \times \mathbb{I}_{\text{payment due}}}{1 + (\ln(1+i_{t-\Delta t}) + \mu(x+t-\Delta t)^+ + \mu^x(t-\Delta t) \times (1-\chi)) \times \Delta t} \quad (2.28)$$

2.6.3 Euler equation for periodic premium

Based on the cases explained above, the Euler approximation for periodic premium takes the following form :

$$V(t-\Delta t) = \frac{V(t) + A \times \Delta t - PP \times \mathbb{I}_{\text{payment due}}}{1 + B \times \Delta t} \quad (2.29)$$

where :

$$A = C_{t-\Delta t}^+ \times (1-\diamond) + \diamond C_{t-\Delta t}^+ \times \diamond - \mu^x(t-\Delta t) \times (1-\chi) + \mu^+(x+t-\Delta t) \times C_t^+ \times \mathbb{I}_{V(t-\Delta t) > C_t^+}$$

and

$$B = \ln(1+i_{t-\Delta t}) + \mu^+(x+t-\Delta t) \times (1-p_{t-\Delta t}) + \mu^+(x+t-\Delta t) \times \mathbb{I}_{C_t^+ > V(t-\Delta t)} + \mu^x(t-\Delta t) \times (1-\chi)$$

Although the due date of the periodic premium payment is known, the periodic premium itself remains unknown. However the reserve at time $t = 0^+$ is equal to the first periodic premium paid and the reserve at time $t = 0$ is equal to zero.

Assumption Let one assumes that the annual premium is the unique premium divided by the annuity due $\ddot{a}_{x:\overline{n}|}$ based on the mortality rate μ^+

$$\text{Annual Premium} = \frac{\text{Unique Premium}}{\text{Annuity due}} = \frac{UP}{\sum_{j=0}^T v^j p_{x+j}} \quad (2.30)$$

where $v = \frac{1}{(1+i)}$

As it is illustrated in figure 2.8, the reserve $V(0, PP)$ is not equal to zero. The contract is under-financed hence the annual period is not valid, since the periodic premium which is paid on an annual basis does not take into account the surrender and the accident options.

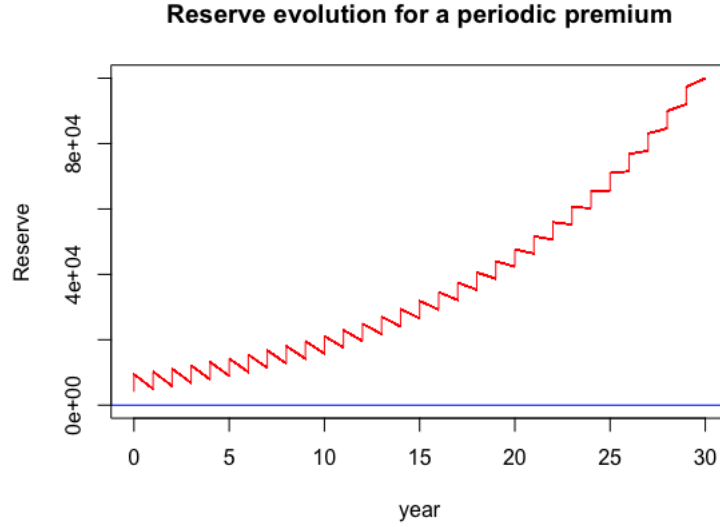


Figure 2.8 – Periodic premium which does not take into account the impact of surrender rates and accident options

The difference between the annuity price taken as assumption in equation (2.30) and the real annuity price to reach the initial condition $V(0) = 0$ can be explained via Cantelli's theorem. As the contract described above contains multiple decrements, the mortality rate and other elements depending on the contract, such as the death capital, of the fictitious contract will differ from the original one.

The approach taken to counter the issue mentioned above is to approximate the periodic premium value based on an iterative algorithm which ensures that the reserve at time $t = 0$ equals to zero. Therefore as the Euler method suggests the value of the insurance reserve at $t = \Delta t$ is the first periodic premium paid $V(\Delta t, PP) = PP$. However, when the value of the reserve does not reach zero at $t = 0$, this implicitly means that the features of the real insurance contract are different than the features of the fictitious contract. If the value of the reserve at time $t = 0$ is positive, then one may say that the features of the real insurance contract used in equation (2.30) leads the reserve to be under financed compared to the features of the fictitious contract. This difference in features is due to a difference in mortality rates μ^+ and in the death capital C_t^+ .

If one uses assumption (2.30) on the value of the reserve at time $t = 0$ as returned by the pricing tool when the value of the periodic premium is defined by (2.30) and adds it to the value of the periodic premium, a new periodic premium is therefore obtained. By using this new periodic premium as an input for the pricing tool, the value of the reserve at time $t = 0$ as returned by the pricing tool will get closer to the expected value. Therefore by iterating this process a certain amount of time, one reaches $V(t = 0t, PP) = 0$, as shown in figure 2.9.

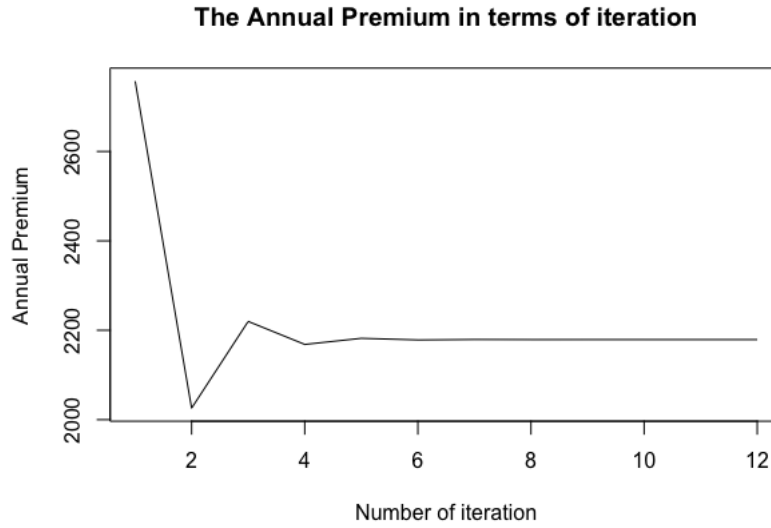


Figure 2.9 – Number of iteration needed by the algorithm to reach the periodic premium

2.6.4 Implementation within the code

Therefore one can use the general form of the annual premium as expressed under Euler's approximation 2.29 in the periodic premium engine. As the design of the code relies on the approximation of Euler, the code for periodic premium therefore follow the same behavior than the unique premium engine. The two elements that differs from the periodic premium engine are therefore the iterative algorithm mentioned above and the Euler's approximation 2.29.

As a first step, the `for()` loop at the beginning of the function extracts the parameters from the specification matrix. Then the iterative algorithm is set in place.

Speaking of code, the iterative algorithm takes place around the backward pricing algorithm. As the algorithm aims to reach the null value at $V(0) = 0$ the iterative process then takes the form of a `while()` loop. The remaining part of the algorithm takes place after the backward pricing tool. Basically, this part consists into redefining assumption (2.30) by applying an annuity price to the value of the reserve obtained in the pricing tool at $t = 0$, and add this value to the previous periodic premium. The code can be summarized as follows :

```
while(Reserve_0 > 0){
  Backward_Pricing_Tool_Periodic_Premium(premium)
  Reserve_0 <- Reserve[1]
  premium <- Annuity_price(Reserve_0) + premium
}
```

Where `Backward_Pricing_Tool_Periodic_Premium()` defines the backward pricing tool based on equation (2.29) and `Annuity_price()` defines the function which returns the

annuity price based as defined above in this chapter.

The core design backward pricing tool engine for periodic premiums is similar than the core design backward pricing engine for unique premium. The algorithm starts with a **while** loop which starts one increment before the termination of the contract. At that point, the value of the reserve equals the life capital of the contract. The blocking condition stops the loop one increment before the transition between two periods. Then, previously described elements such as **ratio** and **dummy** are defined.

Then the loop reaches the calculation of the reserve step. As mentioned above in this section one of the main differences in equation (2.29) compared to the unique premium adapted to Euler in equation (1.8) is the treatment of the periodic premium to be released at particular increments along the backward computation of the insurance reserve. The approach taken to isolate the increments that corresponds to the release of a periodic premium rely mainly on the desired number of payments. The difference between the initial number of increments and the increment number of the current iteration should therefore equal a multiple of the periodicity for the periodic premium to be release. Therefore the design of this condition within the periodic premium engine is expressed as :

```
if((floor((current_increment*PP_release))
-floor(PP_release*(next_increment)))==periodicity
&& current_increment < Number_of_PP)
{PP=premium[prime]; prime<-(prime-1) } else {PP=0}
```

We will elaborate on the part of the code written above.

The code is designed under an **if** statement which allows to apply defined modification if a certain conditions are met. In the case of the periodicity of the premiums, the **if** statement contains 2 conditions. The first one refers to the treatment of the release of the periodic premium. The code uses the difference between the lower rounding of the current increment and the lower rounding of the increment following the current one to determine the release of the periodic premium. The second condition ensures that the release of the periodic premium is performed within the initial conditions of the contract. **Number_of_PP** defines then the number of premiums to be paid during the insurance contract, and the object **periodicity** defines the number of periodic premiums to be released during a year.

One major difference with the backward engine for unique premium is the length of the vector which defines the insurance reserve. While the unique premium engine stops the backward process one increment before the insurance reserves reaches $t = 0$, the periodic premium engine stops the backward process when the value of the insurance reserve equals zero. This difference is motivated by the design of the periodic premium engine, and more particularly the iterative algorithm which approximate the value of the periodic premium. Moreover, as the unique premium returns a much higher value at $V(\Delta t)$ than the periodic premium for the same contract (i.e. all other things being equal), this choice is therefore also motivated for visual convenience.

2.7 Optimal number of payments

2.7.1 Rationale

From a business point of view, the reserve should be positive at any stages of the insurance contract. If the reserve happens to be negative at one point in time, the contract is considered to be over-financed. Therefore this implicitly means that the policyholder is creditor of the insurer of a surplus which is equivalent of the absolute negative amount at $V(0)$ over the agreed coverage. However, even with an accurate periodic premium, the reserve can be negative in some cases. When the death capital is much greater than life capital the reserve evolution is a convex curved and can be negative at some points. For an outstanding loan insurance, the death coverage is linked to the outstanding amount of the loan to be paid. As this outstanding amount decreases at each payment to the credit company, the death coverage of the outstanding loan insurance decreases in such a way that it is always equal to the remaining outstanding amount. As a consequence the death capital of the insurance contract adopts a decreasing behavior along its duration. The figure 2.10 below shows the evolution of the insurance reserve for an outstanding loan insurance contract lambda, if the number of periodic payments equals its duration:

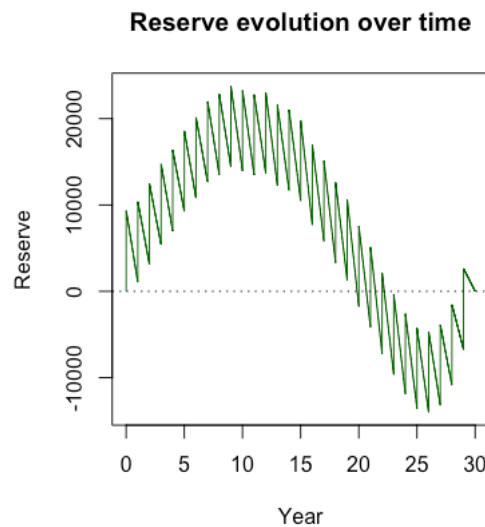


Figure 2.10 – Evolution of the reserve for an outstanding loan insurance with the number of periodic payments equaling the length of the contract

The evolution of the reserve in figure 2.10 is negative at some points of the contract, making therefore the debt of the insurer towards the policyholder higher than what was expected by the contract.

To avoid such a scenario, the policyholder can pay his premium through a decreasing

periodic premium. Yet this method is technically viable, however, this is maybe not the most efficient from a marketing point of view. First, the first premium to be paid will be expensive for a policyholder who just take a loan. Also for management aspect, the insurer will have more difficulties to manage the contract. Another method consists in reducing the number of payments of the premium, in order to ensure sufficient reserve to cover the capital at risk at any time during the lifespan of the contract. From a marketing point of view however, the policyholder prefers to maximize the number of payments as he prefers to pay less premium per payment whereas the insurer recommends fewer premiums for management aspects.

Therefore and ultimately for a business point of view, the optimal number of payment for a life insurance contract is then defined as the maximum number of payments possible that does not lead the insurance reserve to be negative at any point in time during the duration of the contract and until termination.

2.7.2 Implementation within the code

The optimal number of payments as defined above can be implemented within the code through an iterative process. This process takes the original number of payments, M , and the value of the reserve at any time during the lifespan of the contract as input. Then, the iterative process will decrease the original number of payments by 1 payment until the value of the reserve is positive at any time of the contract, i.e $\forall t$. Note that the optimal number of payments only shows the maximum number of payments possible for the insurance reserve not to be negative at any point of the contract, according to the definition mentioned above. This also suggests that any number of payments below this limit is also viable for the insurer and therefore can serve as input for the periodic premium engine.

A potential solution regarding the design of the code is given below :

```
while (any(V[i]<0)){
  Number_of_payments <- Number_of_payments - 1
  if (Number_of_payments==0) {break}
  Optimal_Backward_Pricing_Tool_Periodic_Premium(premium)
}
```

Where `Optimal_Backward_Pricing_Tool_Periodic_Premium` refers to the periodic premium engine as described in section 2.6.

Chapter 3

Numerical analysis

This section aims to verify the correctness and the accuracy of the backward pricing tool explained above. The verification is based on numerical examples which can be elaborated towards Thiele's differential equation. The first part of this section is considered as the calibration part : this consists into simple numerical examples which will be run throughout a backward pricing tool and where the accuracy and correctness can be verified via Thiele's differential equation. Different scenarios will be considered : contracts including surrender option, linear death benefits and accidental segmentation. Both unique premium and annual premiums will be calculated. The second part of this section will have a specific focus on particular cases that classical mathematical method cannot solve and where numerical reasoning takes all its significance : one speaks about linear dependencies.

3.1 Surrender option

This section will develop a classical life insurance product with a surrender option. Unique premium is considered first then two cases will develop annual premium. Let one considers a mixed life insurance on an individual aged of 30 years. The interest rate and mortality rate are set to be 2% and 0.005, respectively. At the end of the contract, the insured will receive a capital $C = 100000\text{€}$ in case of his survival and a capital of $C = 200000\text{€}$ will be given to his inheritance in case of death. The length of the contract is 15 years which are divided into two an uneven period:

- The first period considers the inception of the contract until and including the ninth year. For this period, the insurer expects surrenders to be around 3%. Considering this, he decided to apply a surrender penalty equals to 5% on the value of the insured reserve at time of the surrender.

- The second period covers from year 10 to the end of the contract. On this period the insurer considers the surrender rates to be 1% and therefore sets the surrender penalty to be equal to 20% of the value of the insured reserve at time of the surrender.

The surrender penalty makes sense and is aimed to reflect in fact what is known by the actuaries as the triple penalty : the more advanced the life insurance contract, the more impact has the capitalization on the accumulated reserve amount and the less impact has the mortality. On top of that and from a business point of view, surrendering means for the insurer the loss of a customer for a potential competitor.

Therefore, following Thiele's differential equation (1.6), the expression of the reserve for this particular insurance contract takes the following form :

$$\frac{dV(t)}{dt} = \begin{cases} \ln(1.02)V(t) + 0.015V(t) - 0.005(200,000 - V(t)) & \text{for } 0 < t < 10 \\ \ln(1.02)V(t) + 0.002V(t) - 0.005(200,000 - V(t)) & \text{for } 10 < t < 15 \end{cases} \quad (3.1)$$

By developing Cantelli's theorem (1.7), one can express equation (3.1) as the equivalent fictitious contract :

$$\frac{dV(t)}{dt} = \begin{cases} \ln(1.02)V(t) - 0.02(50,000 - V(t)) & \text{for } 0 < t < 10 \\ \ln(1.02)V(t) - 0.007(142,857.14 - V(t)) & \text{for } 10 < t < 15 \end{cases} \quad (3.2)$$

The new insurance contract has a new death capital regarding the first period (i.e. between 0 and 10 years) which equals 50,000 € with a mortality rate of 0.02, while the death capital for the second period is assumed to be 142,857.14EUR with a new mortality rate which is equals to 0.007. Moreover the life capital remains the same 100,000 EUR. Therefore the unique premium can be extracted and takes the following value :

$$\begin{aligned} PU &= 50,000 \int_0^{10} \exp(-0.02t) 0.02 (1.02)^{-t} dt \\ &+ 142,857.14 \exp(-10 \times 0.02) \int_0^5 \exp(-0.007t) 0.007 (1.02)^{-t-10} dt \\ &+ 100,000 \exp(-10 \times 0.02 - 5 \times 0.007) (1.02)^{-15} \\ &= 70133,1 \end{aligned} \quad (3.3)$$

Using the backward pricing tool the value of the unique premium is estimated to be 70133.52€. The evolution of the insurance reserve over time as designed by the tool is shown in the figure 3.1. The slope describing the evolution of the reserve in figure (3.1) is steeper for the first period of the contract. This is motivated by a higher mortality rate due to Cantelli's theorem than in the second period in one hand, and the intensity of the interest rate which is more important at later stages on a second hand. Overall, the reserve starts at time $t = 0^+$ at the unique premium value and end at time $t = 15$ when the contract with the exact value of the life capital (100,000 EUR).

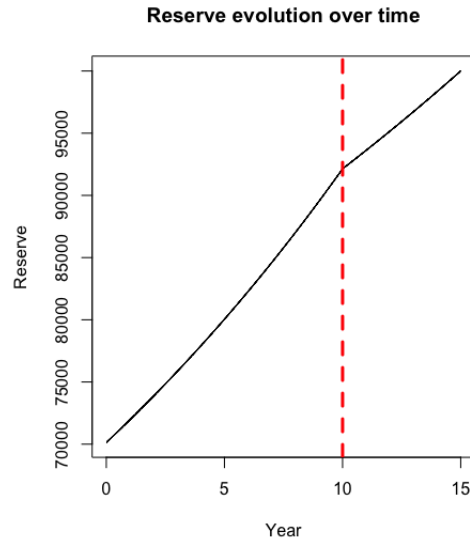


Figure 3.1 – Reserve evolution over the duration of the contract with a unique premium

The following figures describe the evolution of the insurance reserve over time for the contract described above with annual payments :

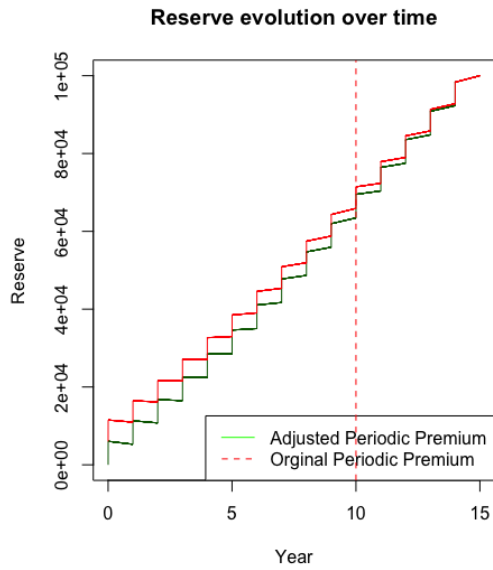


Figure 3.2 – Reserve Evolution over the duration of the contract with an Annual Premium for 15 years

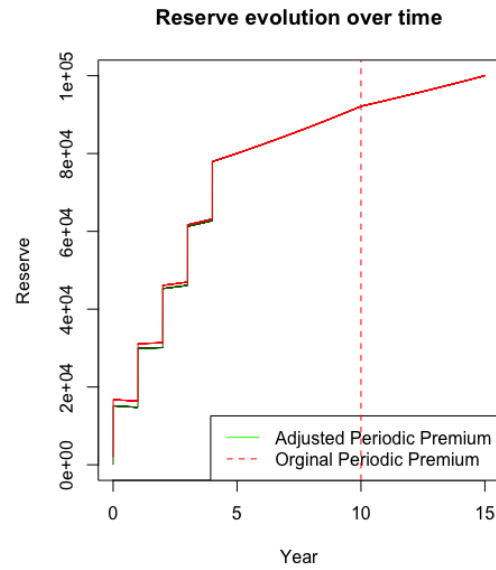


Figure 3.3 – Reserve Evolution over the duration of the contract with an Annual Premium for 5 years

The evolution of the insurance reserve driven by periodic premium payments as computed by the backward pricing tool in figures 3.2 and 3.3 show a similar behavior than the evolution of the reserve as calculated under classical life insurance mathematics. However, while the annual premium calculated and used to calibrate the periodic premium engine returns 5530.906 EUR, the annual premium computed by the backward pricing tool returns an annual premium of $PA = 6046.315$ EUR for figure 3.2. The inclusion of surrenders lead the pricing tool to a more conservative result. The difference between the initial calibration and the annual premium returned by the tool increases with the number of payments. This is justified by the inclusion of surrender rates or other insurance-industry characteristics which equals the insurance contract to another fictitious contract with different features among which a fictitious mortality rate under Cantelli's theorem (1.7). Nevertheless one can calculate the exact value of the periodic premium by using the mortality rate of the fictitious contract as described in equation (3.2), as the equality of the evolution of the reserve holds under Cantelli's theorem:

$$PA = \frac{PU}{\ddot{a}_{x:\overline{10}|} + \ddot{a}_{x+10:\overline{5}|} \times {}_{10}E_x} = 6053.65 \text{ EUR} \quad (3.4)$$

Therefore the difference between the exact periodic premium under the fictitious contract expressed in equation 3.4 and the pricing engine is of 0.121%.

By comparing figures 3.2 and figure 3.3, the longer the number of payments, the higher the impact of the fictitious mortality rate on the periodic premium. When the periodicity is lower than the length of the contract, the value of the periodic payment converges towards the exact value. The value of the insurance contract for this figure with the initial calibration returns 14,731.87 EUR whereas the annual premium estimated by the code is equal to $PA = 15,166$ EUR, which is 2.95% higher. The two periods comprised in the contract are clearly observable in figure 3.3 as the payments are completed at year 5. This can be also deduced from figure 3.2 : beyond year 10, the steep of the steps between two payments is higher however this effect is partially offset by the impact of the capitalization at later stages of the insurance contract. Moreover, as years of payment of the premium decrease by 80% (from 15 to 3 years) the difference with the premium decrease by 68.38%. Hence as it is illustrated in the 3.3 the two plots are more clustered than those of figure 3.2.

3.2 Death capital linked with the reserve

Insurance contracts delivering a proportion of the reserve as death capital are commonly used in Belgium. Differed capital with reimbursement of the reserve, CDARR, is one of the most common examples of this type of life insurance product and is widely distributed within the Belgian market. For this particular case where the death capital delivered is equal to the reserve (i.e., by setting the proportion to be 100%), the mortality effect is negated, equaling the product to a purely investment contract. This can be proven by

Thiele's differential equation :

$$\begin{aligned}\frac{dV(t)}{dt} &= (\ln(1+i))V(t) - \mu(V(t) - V(t)) \\ &= (\ln(1+i))V(t)\end{aligned}\tag{3.5}$$

Nevertheless, this particular scenario is designed to assess the case where the death capital is proportional to the reserve and not a constant amount as it was assumed in the previous scenario. The mortality is then still to be considered as an effective component of the contract. Hence the particular case described with equation (3.5) will not be further developed.

Let one considers an insurance contract of 30 years' duration, delivering a life capital of 100,000EUR at the end of the period. The interest rate is set to 1.5% and the contract is split towards two uneven periods of 10 and 20 years respectively. Moreover, the contract also delivers a capital in case of death of the insured which is equivalent of a proportion of the accumulated insurance reserve. During the first period of 10 years, the death capital is considered to be 90% of the total reserve with a mortality rate of 0.01. The death capital during the second period (as from years 11 to 30) is set to be equal to 75% of the total reserve with a mortality rate of 0.05.

Following Thiele's differential equation (1.6), the expression of the reserve for this particular insurance contract takes the following form :

$$\begin{aligned}\frac{dV(t)}{dt} &= \begin{cases} \ln(1.015)V(t) - 0.01(0.9V(t) - V(t)) & \text{for } 0 < t < 10 \\ \ln(1.015)V(t) - 0.05(0.75V(t) - V(t)) & \text{for } 10 < t < 30 \end{cases} \\ \frac{dV(t)}{dt} &= \begin{cases} \ln(1.015)V(t) - 0.001(V(t)) & \text{for } 0 < t < 10 \\ \ln(1.015)V(t) - 0.0125(V(t)) & \text{for } 10 < t < 30 \end{cases}\end{aligned}\tag{3.6}$$

Hence the fictitious contract expressed under Cantelli's theorem is set to be a simple life insurance with a mortality rate of the first and second period of 0.001 and 0.0125 respectively with a death capital equal to zero. Therefore the unique premium can be calculated and takes the following value :

$$\begin{aligned}PU &= 100,000 \exp(-10 \times 0.001 - 20 \times 0.0125) \times (1.015)^{-30} \\ &= 49,328.98 \text{ EUR}\end{aligned}\tag{3.7}$$

Giving that in this case as the death capital is considered to be a portion of the reserve, the surrender rate and the death rate have the same mechanism: the death capital is set to 0 and the surrender rate is replaced by the mortality rate. In this case the penalty rate equals 1 - the death capital ratio. Using the code, the unique premium value is 49,329.01 EUR. The premium returned by the backward pricing tool has a higher value than the exact value as calculated by the means of Thiele's differential equation. Hence the Euler method in this case is considered to be a more conservative approach.

The next figures highlight the evolution of the insurance reserve through different scenarios. As the mortality has already been considered comparable with the previous insurance product, the main impact expected with this contract is the length being set at 30 years.

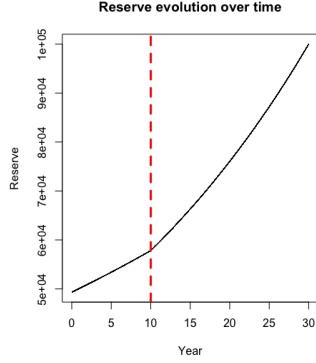


Figure 3.4 – Reserve Evolution over the duration of the contract with a Unique Premium

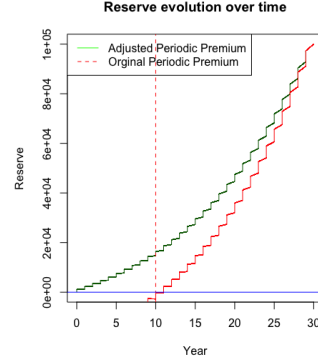


Figure 3.5 – Reserve Evolution over the duration of the contract with an Annual Premium for 30 years

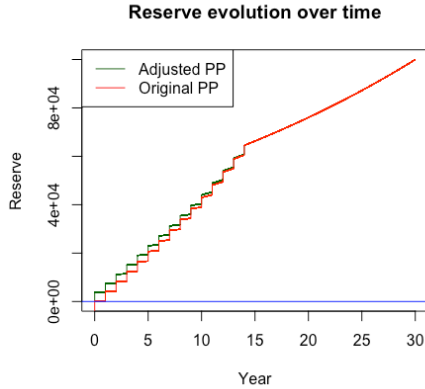


Figure 3.6 – Reserve Evolution over the duration of the contract where the annual premiums are paid during the first 15 years of the contract

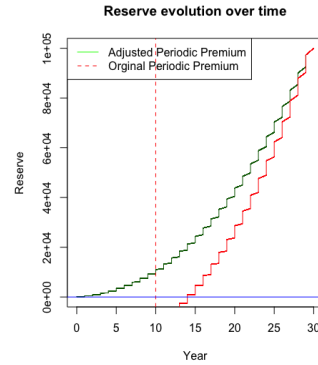


Figure 3.7 – Reserve Evolution over the duration with an increasing arithmetic annuity

The evolution of the reserve over time for the unique premium in figure 3.4 confirms that both contract with either a surrender option included or a death capital share the same mechanism when paid with a single premium. In this case, as the mortality and the penalty rates both increase in the second period of the contract compared to the first one, the slope describing the evolution of the reserve get steeper.

Note that the evolution of the insurance reserve illustrated by the red line in the

figures above showed the first iteration of the calibration of the annual premium engine. Considering this particular case, it takes into account the unique premium calculated by the pricing tool being 49,328.98 EUR and divides it by an annuity price. In the case of figure 3.5 the value of the unique premium divided by the annuity price is equal to 2757.179 EUR, leading the insurance contract to be loss-making for the insurer before year 10. This behavior is expected as the number of payments equals the length of the contract which is not common in practice due to this particular issue. Moreover, the annuity price used to calculate the first iteration did not take into account the death benefits being a function of the reserve. By using the mortality rate of the fictitious insurance contract developed under Cantelli's theorem in (1.7), the exact value of the annual premium for this particular contract is equal to 2179.98 EUR. The pricing tool, however, returned an annual premium value of 2,179.117 EUR which allows the insurance contract to be correctly financed over the period as shown by the evolution of the reserve in figure 3.5. Also, this differs from the value returned by the tool by less than 0.04% which gives confidence about the pricing tool. The comparison of the evolution of the reserve for this contract can be observed in figure 3.8 below.

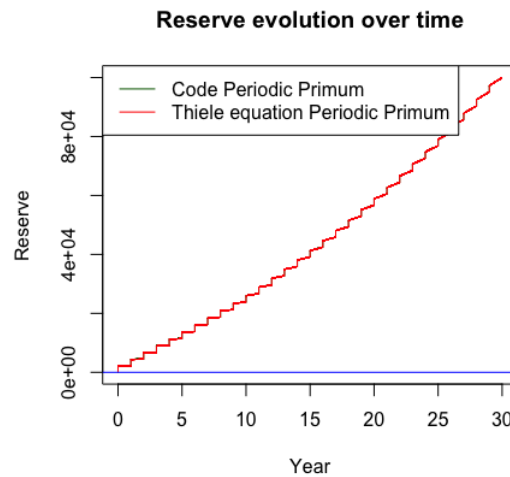


Figure 3.8 – Comparison between the tool value and the exact value of the evolution of the reserve over the duration of the contract with 30 years of annual payments

When setting the number of payments $m = 15$ years, one can observe that the first value used to calibrate the periodic premium still makes the insurance reserve negative. However the difference between the value returned by the pricing tool and the later are getting closer : the initial calibration of the periodic payments and the numerically-computed payments dropped from 28,6670% when $m = 30$ to 8.53% when $m = 15$ as shown in figure 3.6. If the insured has the opportunity to opt from arithmetically increasing periodic premiums, the evolution of the reserve described in figure 3.7 by

the pricing tool makes the contract still technically viable for the insurer. However the gap between the periodic premium used for the calibration of the engine and the value returned by the engine reaches 32%.

3.3 Death caused by an accident

In this scenario the insurer does not cover death due to an accident. The insurer estimates the proportion of death due to accident in his portfolio to be 15%. In other words, this means that if the insured is considered to be in this proportion when dying, his family will then receive nothing for death capital. If not, the death capital delivered is of 100,000 EUR.

Moreover the interest rate is assumed to be constant over the total length of the contract and a life capital of 100,000EUR will be delivered to the insurer when reaching the maturity. The surrender rate is assumed to be 5% while the surrender penalty is set at 20% for the first 5 years of the contract. The next 5 years include surrender rate and surrender penalties assumed to be equal to 10% for both rates respectively. The mortality rate is considered to be constant along the 10 years of the life of the contract ($\mu = 0.001$). Following Thiele's differential equation (1.6), the expression of the reserve for this particular insurance contract takes the following form :

$$\begin{aligned} \frac{dV(t)}{dt} &= \begin{cases} \ln(1.01)V(t) - 0.001 \times 0.85(100,000 - V(t)) \\ -0.001 \times 0.15(0 - V(t)) - 0.05(0.8V(t) - V(t)) \text{ for } 0 < t < 5 \\ \ln(1.01)V(t) - 0.001 \times 0.85(100,000 - V(t)) \\ -0.001 \times 0.15(0 - V(t)) - 0.1(0.9V(t) - V(t)) \text{ for } 5 < t < 10 \end{cases} \\ \frac{dV(t)}{dt} &= \begin{cases} \ln(1.01)V(t) - 0.011(7,727,27 - V(t)) \text{ for } 0 < t < 5 \\ \ln(1.01)V(t) - 0.011(7,727,27 - V(t)) \text{ for } 5 < t < 10 \end{cases} \end{aligned} \quad (3.8)$$

The fictitious contract under Cantelli's theorem has a new death capital for both periods equaling 7,727,27 EUR, with a mortality rate of 0.011. Therefore the contract has constant parameters over its duration. The unique premium can be extracted and takes the following expression :

$$\begin{aligned} PU &= 7,727,27 \int_0^{10} \exp(-0.011t) 0.011(1.01)^{-t} dt + 100,000 \exp(-10 \times 0.0011) \times (1.01)^{-10} \\ &= 81,865.56 \text{ EUR} \end{aligned} \quad (3.9)$$

Using the code, the unique premium value returned equals 81,865.54 EUR . The evolution of the reserve is homogeneously impacted by the interest rate and mortality rate over the whole duration of the contract.

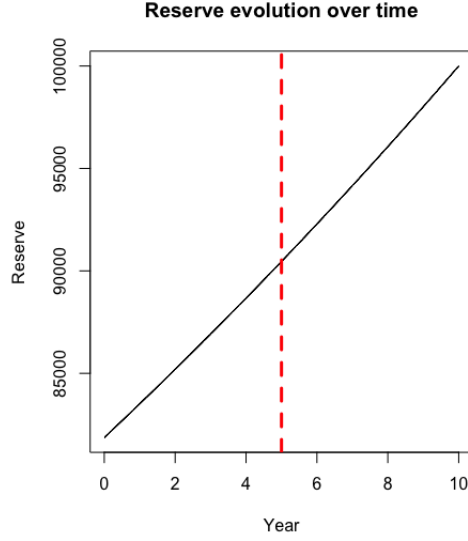


Figure 3.9 – Reserve evolution over the duration of the contract

3.4 Non-Linear Benefit

Although some technical drawbacks are noticed and explained further in this text, non-linear death benefits are common products within the insurance industry. For instance, a death benefit designed as the maximum between a pre-defined capital and the level of the insurance reserve for this product has found its place within the Belgian insurance market. Nevertheless, the estimation of the insurance reserve for non-linear benefit case is more intricate than the linear case. As a matter of fact, using Thiele's differential equation to solve the value of the reserve for a particular time t is impossible. This also motivates the use of Fouret's formula (1.20) when it comes to represent the movements in the insurance reserve between two discrete periods.

However and as shown by Christiansens et al. (2014), Thiele's differential equation is still admissible and, moreover, has a unique solution, even with non-linear insurance benefits. According to the article proposed by Djehiche and Löfdahl (2016), if one assumes that the benefits and the premiums are continuous functions over time (or, at least, continuous by parts) such as

$$dB_i(t) = b_i(t)dt \text{ and } d\Pi_i(t) = \pi_i(t)dt$$

Then under the fact that Cantelli's theorem is still applicable provided the benefits payments and the modification of the technical basis in the fictitious contract, Thiele's differential equation is valid and admits a unique solution.

3.4.1 Numerical non-linear application

Let one considers a policy with an individual of 30 years old, who signed up for a life insurance contract for 8 years' duration, with a life capital of 100,000 EUR and a death capital equivalent to the maximum between the reserve and a fixed death capital. The mortality and interest rates are assumed to be constant over the period of the contract and are assumed to be 0,001 and 0,01 respectively.

This application considers 4 different scenarios with 4 different minimum death capital (80,000 EUR ; 97,000 EUR ; 240,272 EUR; 500,000 EUR). By taking into account the fact that exact premium has already been considered for comparison into the previous scenarios, and that the duration of the contract had also already been analyzed, this scenario focuses on the potential impact of non-linearity. From a theoretical perspective, one already knows that the reserve of a life insurance contract is driven by several components which are all function of the time t . As a matter of fact, this comes back to the triple penalty mentioned in the surrender case. In one hand, the longer the duration of the contract, the more likely the policyholder has to die within this duration. On the other hand, the longer the duration of the contract, the higher the effect of the interest capitalization will be at latest stages.

The combination of these three effects on the evolution of the insurance reserve over the time duration of the contract will be foreseen in the slope of the function : one expects the slope to be flatter when the mortality has a more consequent impact than the interest capitalization and one expects the slope to be steeper when the interest capitalization has a more consequent impact than the mortality on the reserve. Therefore the evolution of the reserve over time with a unique premium for the different death benefits mentioned above take the following shape as described in figure 3.10.

The unique premium for the each reserve evolution over time shown in figure 3.10 are respectively 92,348.33 EUR, 94,713.01 EUR, 100,000 EUR and 109,066.73 EUR. As a first analysis, one can notice that the marginal impact of the death capital over the unique premium decreases with the increase of the death capital. This is naturally justified by the mortality rate which is independent of the death capital. However this stays positive: the more death capital the insurer wants, the higher the premium he will pay.

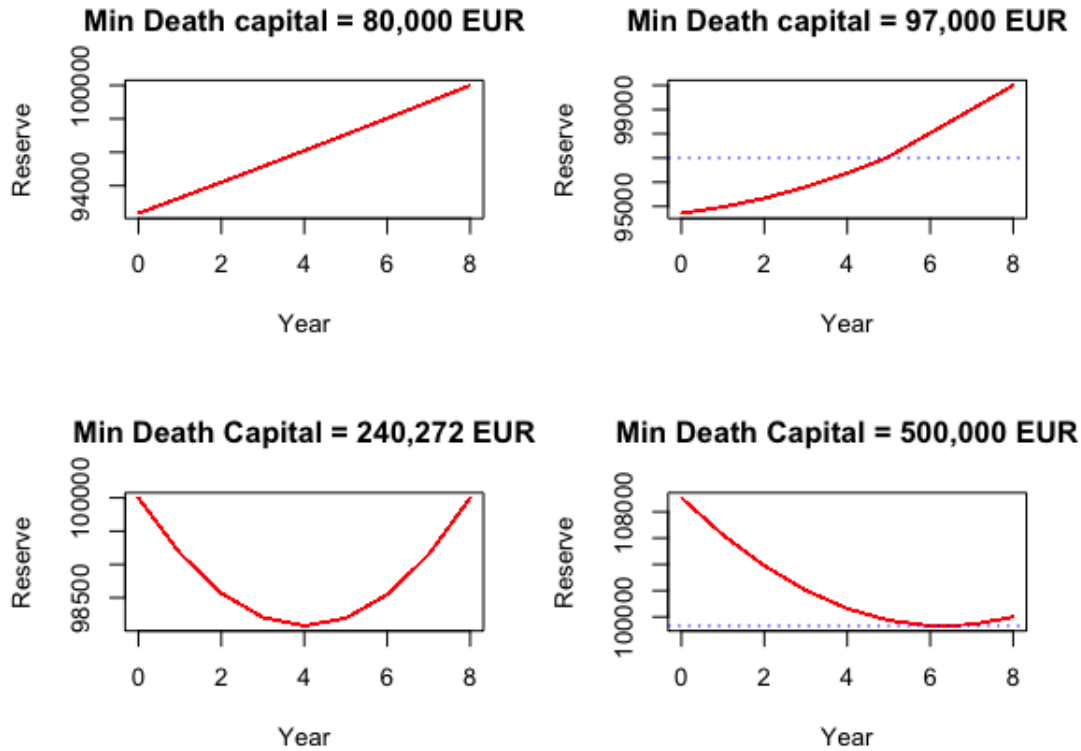


Figure 3.10 – Reserve evolution over the duration of the contract for non-linear unique premium

When the minimum death capital is set at 80,000 EUR (which is less than the unique premium), the reserve is always greater than the minimum death capital. This is due by the amount of the life capital that the individual decided to obtain at the end of its contract within a short period of type. In this case the policy is considered to be a pure CDARR policy, hence the non-linear death benefit has no effects on the premium pricing.

Regarding the second case, the minimum death capital lies between the life capital and the premium calculated. The evolution of the insurance reserve over time is therefore dependent of the amount already reserved, in opposition with the different rates as seen in the previous cases. At time $t = 0^+$, the insurance reserve equals 94'713.01 EUR, i.e the unique premium. Then the evolution of the reserves represented by the slope is increasing. However, when the reserve reaches 97'000 EUR which corresponds to the minimum death capital, the slope of the reserve becomes steeper. This confirms that the reserve is driven by the minimum death capital until the capitalization of the unique premium reaches a higher level than 97'000 EUR. Moreover this justifies that the mortality impact has a tendency to flatten the reserve as mentioned above. One also

notes that the first part of the reserve can be considered as a mixed insurance product with a death capital $C = 97000\text{EUR}$. Therefore as previously seen the second part of the reserve can be considered as a differed capital with reimbursement of the reserve.

The third case highlights the unique premium being equal to the life capital, i.e., $C = 100000\text{EUR}$. For this to happen, the death capital should equal $C = 240272\text{EUR}$. As a consequence, the behavior regarding the evolution of the reserve over time takes a parabolic shape : at $t = 0^+$ the reserve equals the unique premium. As the contract increases in time, the impact of the mortality starts to decrease. Therefore the probability for the policyholder to die within the duration of the contract decreases as well. Consequently, the insurer is not required to have a level of reserve of this amplitude as the interest capitalization on the premium increases with time : as at year 4, the effect of the capitalization overrides the impact of the mortality, letting the slope of the reserve increase. This can also be illustrated by the relation of the capital at risk compared to the amount reserved : one may express the capital at risk as the difference between the value of the insurance benefits and the reserve, which is multiplied by the probability of the insurer not to live, i.e:

$$\text{Capital at Risk} = (B(t) - V(t))_t q_x$$

Therefore, the more the time increase, the less impact the term $_t q_x$ will have on the reserve, the higher the level of the reserve $V(t)$ will be and as a consequence the lower the capital at risk it will remain.

The last scenario includes a death capital being five times the life capital. As it was the case for the death capital $C = 240272\text{EUR}$, the evolution of the reserve is in a first time driven by the decrease in death probability. However as the contract reaches its maturity the interest capitalization effect takes over. Hence covering death risk is more expensive for the insurer than life capital. Therefore the contract is considered to be as a mixed insurance product until $t = 6^+$. The evolution of the insurance reserve has then a convex shape.

The following figures describe the evolution of the insurance reserve over time for the same contracts as described above, with annual payments :

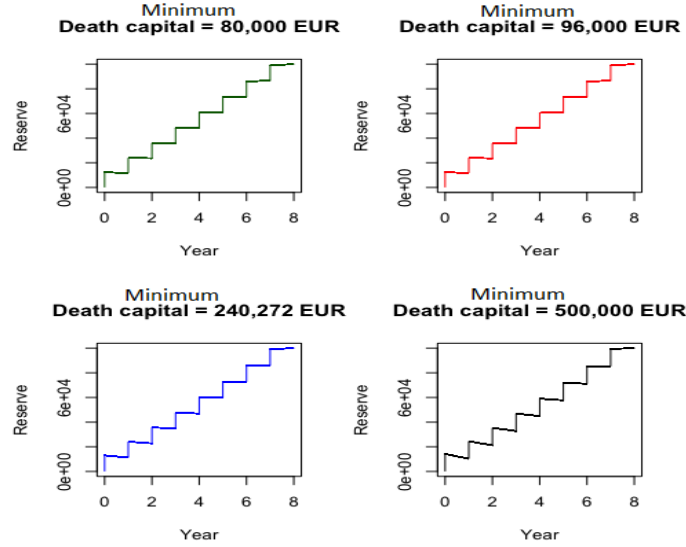


Figure 3.11 – Reserve evolution over the duration of the contract for non-linear unique premium with the corresponding minimum death capital for each year

The impact of the mortality rate and the interest rate capitalization is also noticeable on the annual premium. One can observe the latter between each premium payment: when the contract is considered by Cantelli's theorem as a mixed insurance (i.e., when the minimum death capital is not reached by the insurance reserve), the value of the reserve after receiving a payment starts to decrease until the next following payment. This behavior is clearly noticeable with cases where the minimum death capital is equal to $C = 240272$ EUR and $C = 500000$ EUR respectively. For the first one the value of the reserve decreases after each payment of the policyholder until $t = 4^+$. This point reflects then the achievement of the minimum death capital being behind the value of the insurance reserve. In addition to the first case, the second case clearly highlights both the decreasing impact of the mortality and the increasing impact of the interest capitalization on the insurance reserve : between the first periodic payments, the value of the reserve is pulled down by the impact of the minimum death capital. However, the more the contract decreases in duration, the less the value of the reserve between two periodic payments has a decreasing behavior.

3.5 Other applications: Increasing and decreasing Death Capital

Next to the minimum guaranteed death capital illustrated above, other insurance products including non-linear death benefits are also in the landscape of the insurance industry. For instance, increasing (respectively, decreasing) death capitals are commonly

sold products : depending of his age, the policyholder may want to get his death capital increase (decrease) without changing his insurance policy.

Increasing death benefit product: An increasing death benefit product is a very common product that is used for multiple reasons such as to protect against inflation, or to cover the risk of death for a growing business, or even to provide a coverage against the potential loss of the growing financial responsibilities of the policyholder.

Decreasing death benefit product: Decreasing death benefit product is another very common life insurance product. This product is mainly used within the banking sector, and more specifically to provide coverage along a long-term loan. As for example, if an individual wants to obtain a housing mortgage, the bank may require him to acquire an insurance in order to guarantee that the bank will have his money back in case of death. Therefore as the death benefit is set to be equivalent to the value of the asked credit, the capital insured will decrease accordingly as the borrower pays back his loan.

For this application the same life insurance contract is used with a constant, increasing and decreasing death capital. Therefore, as the death capital is not constant over time, one may speak about an average death capital of $C = 100000$ EUR. The contract has a duration equivalent to 10 years and constant interest rate and mortality rates of 0.01 and 0.001 respectively over the duration of the contract. In this case, no life capital will be delivered to the policyholder at the termination of the contract.

The first contract has an increasing arithmetic death capital benefit of 18181.82 EUR insured for the first year with an arithmetic increase of $C = 18181.82$ EUR at year until the contract reaches maturity. In opposition to the first contract, the second contract has a death capital benefit which is equivalent to $C = 181818.2$ EUR for the first year, with a decreasing arithmetic each year of $C = 18181.82$ until the contract reaches maturity. As a matter of comparison, the third contract cover the death along the duration of the contract up to $C = 100000$ EUR.

The results for the three aforementioned contracts as returned by the pricing tool are as follows : The unique premium for increasing, constant and decreasing death benefit are 669.33EUR, 618.19EUR and 567.05EUR respectively. Figure 3.12 illustrates the evolution of the reserve for these policies over the whole duration of the contract.

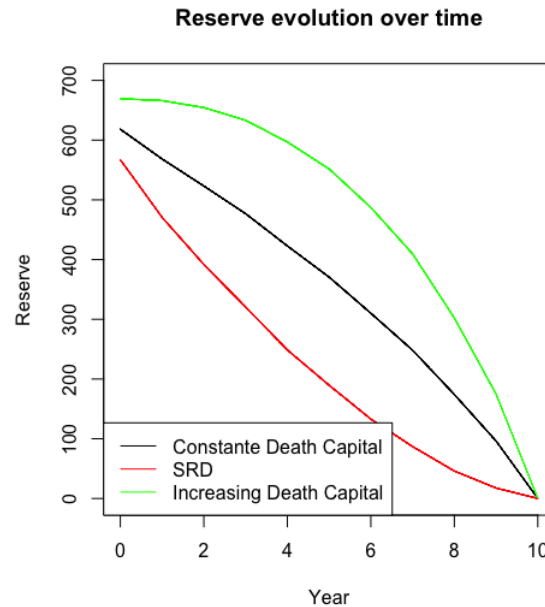


Figure 3.12 – Reserve evolution over the duration of the contract

Note for this case as the contract does not deliver a life capital at termination that the reserve is a purely decreasing function. The evolution of the reserve for the both contracts reflects the increasing (respectively, decreasing) option of the death capital : the progressive increase (decrease) of death capital offsets (adds- up to) the decreasing impact of the mortality on the reserve at early stages.

As most insurances are paid on an annual basis, the evolution of the reserve is also being simulated using annual premium paid at the beginning of each year as a basis, over the entire duration of the contract. Figure 3.13 illustrates the reserve evolution.

The Increasing death benefit has an insurance reserve which takes a concave shape since death coverage increases along with each period of the contract. In opposition to the increasing death capital, the shape of the reserve evolution for the decreasing death capital is convex. As previously observed with the linear constant death capital case, this results into a deficit for the insurer as the reserves becomes negative: the capital at risk in the early stages of the contract is greater than the premium paid by the insurer. This type of contract is not considered as a best business practice for the insurer and therefore is to be avoided.

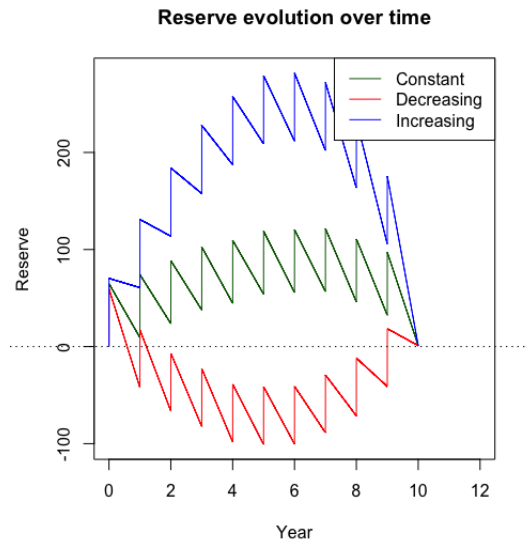


Figure 3.13 – Reserve evolution over the duration of the contract

As it is shown in the figure below, the optimal number of payments for this particular contract is 5, by considering the periodicity as annual. This means that the annual premium should be paid is within 5 years, otherwise the reserve becomes negative.

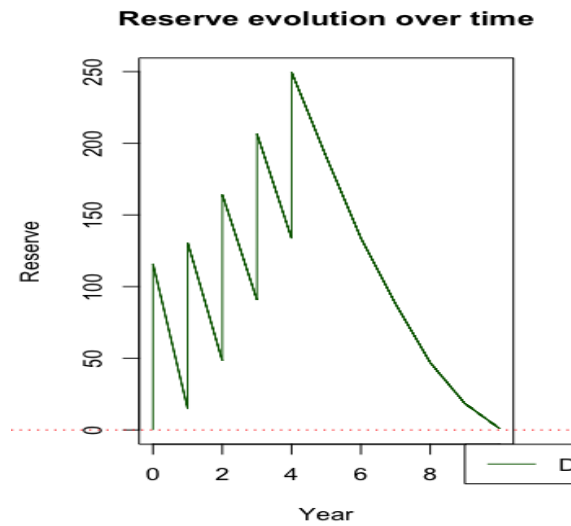


Figure 3.14 – Reserve Evolution over the duration of the contract with an optimal number of payments

Chapter 4

Advanced model analysis

After setting the model and calibrating the parameters, one should assess the performance in order to test the reliability of the assumptions and inputs that are taken into consideration in modeling. This section explores two common methodologies in the financial modeling : sensitivity and impact analysis. Both methodologies have importance in order to mitigate the impact of a certain investment decision that may have a peculiar uncertainty and weighed up it against potential benefits. In other words, we will reevaluate a policy unique premium by changing certain input parameter. This will allow to assess the impact of the input parameter of the policy on the volatility of the unique premium.

4.1 Sensitivity analysis

A sensitivity analysis is used to evaluate the impact of changing one variable on the premium. Moreover a sensitivity analysis serves to assess the effect of dependencies on a particular parameter.

In this section we will apply a sensitivity analysis via both the tool developed and Thiele's differential equation. Hence this will allow to deduce the discrepancy of the premium due to a parameter input change. On the other hand, a robustness assessment of the tool will be performed by comparing premiums obtained by different methods of calculation.

The aim of this analysis is to ensure credibility to any variation on the financial model by testing a single change in the interest rate and on the mortality rate. Moreover this will allow to test the magnitude of severe fluctuation of both rates boundaries that can happen in the future.

4.1.1 Numerical illustration

First case which therefore will be considered as a standard is a mixed life insurance contract for an individual who is 40 years old, for a contract duration of 10 years, with both equal life and death capitals of 100 000 EUR. The mortality rate of $\mu = 0,001$

and interest rate of $i = 1\%$. This case will be considered as the reference case where the following cases will be compared to it.

The second and third cases that will be taken into consideration will be the same as the standard case, but the interest rate will be increased to $i = 5\%$ for the second case whereas the third case considers a decrease of i by -3% . The differences in unique premiums will be calculated in order to estimate the extreme market risk.

The stress test regarding the fourth and fifth case is performed on the mortality rates, where the same scenario as the standard case will be used as a reference. The mortality rate will be set to $\mu = 0,01$ and $\mu = 0,0005$ respectively. The differences of unique premiums will be estimated in order to evaluate the effect of the mortality on the premium.

Therefore, due to this sensitivity analysis, one can deduce a concrete understanding of the impact that can be induced due to this particular model.

Case study	Manual Premium value	Tool Premium	Tool Error	ΔUP
Standard case	90575.11	90575.16	5.744991e-05 %	NA
Interest rate increase to $i = 5\%$	61568.16	61568.91	0.00121 %	32.03 %
Interest rate decrease to $i = -3\%$	135420.74	135421.4	0.00045 %	49.51 %
Mortality rate decrease to $\mu = 0,0005$	90551.94	90551.99	5.467909e-05 %	0.0256 %
Mortality rate increase to $\mu = 0,01$	90979.38	90979.47	9.998259e-05 %	0.446 %

Table 4.1 – Interest rate sensitivity analysis

Based on the results obtained in the table 4.1 above, one can deduce that the tool developed remains accurate throughout extreme fluctuations on both mortality rate and interest rate. Moreover, the premium is more volatile to the interest rate fluctuation than mortality rate fluctuation. More precisely a drop of 4% is more significant than an increase of 4%. Therefore if the interest rate drop to -3% this will lead to a great loss of 44846.24 EUR. On the other hand, if the interest rate increased to 5% this will lead to profit estimated of 29006.25 EUR. Regarding the mortality rate, we can deduce that it had a minor effect on the premium.

Interest rate analysis As it is shown in the figure below a different set of interest rates were simulated (from -3% to 5%) to a same policy. As a result, one can deduce as the interest rate increases the unique premium decrease, which is very logical. Moreover the average is higher than the median which explains the convex behavior of the curve. Therefore as the interest rate increases its impact on the unique premium increases. In addition, the interest rate is a very volatile input as the standard deviation is 21203.13, and the max unique premium obtained is more than twice than the min unique premium obtained.

Number of observations	Average	Median	Max	Min	Variance	Range
801	93200.37	90575.16	135421.4	61568.91	449572606	73852.45

Table 4.2 – Interest rate statics description

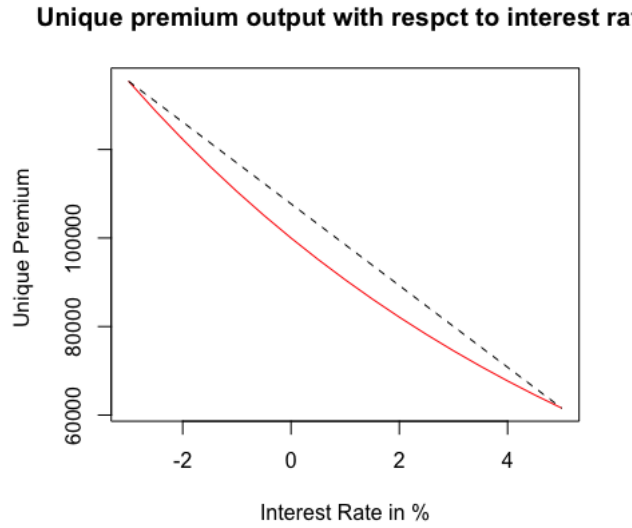


Figure 4.1 – Unique premium value in terms of different interest rate

Mortality rate analysis Similarly to the interest rate analysis, likewise analysis was done to the mortality rate, as it is shown in the figure below. A different set of mortality rate ranges were simulated (from $\mu = 0.0005$ till $\mu = 0.01$). One can deduce that the unique premium increases with the increase of mortality rate increases. Whereby the mortality rate increases the probability of death increases hence the unique premium increases too. In this particular case, the death capital has more significant impact on the premium than that of life capital. The mean and the median both have a close value, hence that explains the linear behavior of the curve in the graph below. As the mortality rate increases its impact on the unique premium remains the same. The mortality rate impact is less volatile than the interest rate impact on the unique premium since the standard deviation is significantly less (124.37). The maximum unique premium obtained is less than 0.5% higher than that of the minimum unique premium obtained.

Number of observations	Average	Median	Max	Min	Variance	Range
191	90767.95	90769.07	90979.47	90551.99	15468.43	427.48

Table 4.3 – Mortality rate statics description

4.2 Scenario analysis

A scenario analysis evaluates the impact of changing multiple inputs at the same time, rather than changing one input as it is the case in sensitive analysis. The variation of these inputs should be coherent to a specific scenario that might occur in the future. This



Figure 4.2 – Unique premium value in terms of different mortality rate

analysis is often referred to as what-if or simulation analysis, whereby this allows to assess the results on the output parameter (the premium) in case if other inputs assumption were taken into consideration. The objective of the scenario analysis is to simulate the model with a potential scenario that can occur in the future where it is assumed as a result to have a significant impact on the output.

As the results of the sensitivity analysis showed confident outcome, one may apply the tool to stressed scenarios in order to obtain a 'stressed' price of the premium. Here it is considered by using the term *stressed* that premium prices are determined under unusual situation. Moreover, the term *stressed* can be easily applied towards stressed scenarios as developed under the Solvency II framework. Therefore the outcome of the pricing tool can be considered as *stressed prices* that can be used by the insurer to identify potential gaps with the insurance price of the premium as determined under Belgian law. One consider here as the mathematical price of the premiums, pure premium, i.e., without any expenses or administration fees.

In this section, we will evaluate the flowing scenarios:

- base scenario – Refers to as it is planned scenario.
- A pandemic scenario - Refers to possibility of occurrence of a pandemic.
- A booming economy scenario - Refers to possibility of a sudden economic growth .

4.2.1 Initial Scenario

We will consider a mixed life insurance contract for male individual who is 25 years old, who would like to receive a life capital of 100 000 EUR at the end of the contract (at age of 65). In addition ,he would like to cover his death during his contract, with a death capital of 200 000 EUR in case of a natural cause death, and an accident capital of 500 000 EUR in case of death due to an accident. In these following scenarios, we will be based on an initial scenario where we will consider a MK table as a table of mortality and a guarantee interest rate of 2%. Moreover we will assume that cause of death due to an accident represents 10% of total death. The insured has the possibility to concede his contract, with a surrendered rate probability of 10% .Nevertheless he will receive 80% of his reserve.

4.2.2 Pandemic Scenario

Here we will consider a *Pandemic* scenario where we will assume that the rate of mortality will be increased in a crescendo pace, proportionally to the age, as it is shown in the figure below:

Age	0-30	30-44	45-64	65+
Mortality Increasing factor	+5%	+10%	+15%	+20%

Table 4.4 – Pandemic scenario Mortality increase

In this particular scenario, we will consider that the mortality rate will increase due to a pandemic and we will assume that it threats more the senior population than the youth population. The mortality rate μ will be augmented by a mortality increasing factor as it is shown in the table 4.5,

$$\text{i.e } \mu_{pandemic}^{50} = 1.15 \times \mu_0^{50} .$$

The Interest rate will be affected as well, since a pandemic is often associated with an economical depression.we will assume that the market will crush and then it will recover in a slow pace within 10 years.as it is shown in the table below:

Duration	0-2	2-4	4-6	6-8	8-10
Interest Rate	0.2%	0.5%	1%	1.5%	2%

Table 4.5 – Pandemic scenario Interest rate

Finally, we will assume that the death due to an accident will decrease by 30%.

4.2.3 Booming Economy Scenario

The second scenario that will be taken into consideration is a booming economy scenario. Where we will assume that the interest rate will be increased to 3%.As consequence the insured is more keen to cancel his contract, as he can obtain a cheaper premium with

another insurer. Hence we will assume that the surrendered rate will be increased to 20%

Scenario	based	Pandemic	Booming
Unique Premium	35587.02	39917.76	15198.83
Δ UP	NA	12.17%	57.29%

Table 4.6 – scenario Unique premium

As it is shown in the table 4.6, one can deduce that the pandemic will have a crucial loss impact on the insurer portfolio with an estimated loss of 20388.19 EUR on this contract, despite the extreme severity on the amplitude but the frequency of a major pandemic is very low. For a booming economy, although the surrendered rate increases, nevertheless the insurer can do a profit of 4330.74 EUR on the contract.

Conclusion

This thesis investigated on the applicability of Euler's scheme to approximate Thiele's differential equation with the development of a pricing and reserving tool for life insurance products under multiple decrements in R. While chapter one examined the context of life insurance products, by elaborating on the multiple approaches possible, chapter two suggested a potential design based on Euler's scheme for a pricing and reserving tool. Although pricing for a unique premium was a direct application of Euler's scheme, the periodic premium was more challenging to estimate, mainly because of the unknown value of the periodic payments. The recursive approach chosen, however, provided a great and relatively fast convergence to the final periodic premium amount. Another main obstacle that we have faced was the value of the reserve which has to remain positive along the entire duration of the contract .

Overall, the results given by the case study on typical life insurance products confirm that the tool is valid since the difference between both unique and periodic premiums obtained via the tool and premiums obtained via Thiele's differential equation was negligible. Therefore this allowed us to go further and to price products in which it cannot be computed through Thiele's differential equation, such as non-linear dependencies. The results were in line with what is expected.

The sensitivity analysis applied on both mortality rate and interest rate proved that the tool developed obtains consistent results over the extreme changes in parameters as the difference in the results provided by the tool remained negligible when compared with their exact value. Therefore based on the level of robustness given by the results different scenarios have been applied to illustrate the impact of potential changes of parameters on the evolution of the reserve and the premiums. While the impact of the pandemic scenario results in a crucial loss for the portfolio of the insurer, the booming scenario showed a potential profit.

However, this study has its limitations as the tool was only designed for life insurance contracts under multiple decrements. Moreover, only non-linear dependencies which aimed to represent industry specific cases have been investigated. The tool does not include further non-linear model such as a non-linear surrender. In addition other insurance products are not covered such as a death benefit which is equal to the value non-capitalized premiums. Also, chapter 3 does not cover the entire range of life insur-

ance products : one may mention for instance multi-life contract or advanced pension benefits which are not covered in the case study. Further analysis on sensitivity can be applied such as extreme surrenders rates. Regarding the scenarios analysis, other scenarios can be applied such as extreme longevity.

Finally, the annuities can be calculated through `lifecontingencies` r package, yet the annuities were computed within the tool. Regarding the approach taken, although Euler scheme is the central element of this paper and was chosen because of its intuitive application, other more advanced techniques can be used, such as the Runge-Kutta approach.

Appendix

Appendix A

Income convergence to life capital

```
Life_Capital <- function(PA, n, nr, age, geo, k=NULL, increase=NULL){
  if( missing(geo)){
    geo <- 0
  }
  if(missing(k)){
    k <- 0
  }
  progression <- 1:nr-1
  if (isTRUE(increase) == FALSE){
    progression <- rev(progression)
  }

  given <- matrix(nrow = 4, ncol = nr)

  #first row = what type annuity(geo, arithmetic, increasing, decreasing)
  given[1,] <- ((1+ geo)^progression) +k*progression
  #second row = mortality rate
  #third row = interest rate
  #second row = mortality
  #third row = interest rate
  given[2,1] <- 1
  given[3,1] <- 1

  for (l in 1:(nr-1)){
    given[2,l+1] <- MR[age+n+1,3]
    given[3,l+1] <- 1/(1+IR[l-1+n,2])
  }

  #fourth row = cumproduct
  given[4,] <-cumprod(given[2,]*given[3,])
  given[4,]<-given[4,]*given[1,]
  #transform PA into PU
  print(given)
  print(PA*sum(given[4,]))
}
```

```

    return(PA*sum(given[4,]))
}

```

Appendix B

Specification matrix

```

specification_matrix <- {function(periode, resiliation, penalite,
                                accident, increments,
                                death_capital, penalite_death){
}
  if (missing(periode)){
    print("Periode is missing")
    break
  }
  if (missing(resiliation)==TRUE){resiliation = rep(0, length(periode))}
  if (missing(penalite)==TRUE){penalite = rep(0, length(periode))}
  if (missing(accident)==TRUE){accident= rep(0, length(periode))}
  if (missing(increments)==TRUE){increments = rep(0, length(periode))}
  if (missing(death_capital)==TRUE){death_capital = rep(0, length(periode))}
  if (missing(penalite_death)==TRUE){penalite_death = rep(0, length(periode))}

  specifications <- rbind(rev(periode),
                          rev(resiliation),
                          rev(penalite),
                          rev(accident),
                          rev(increments),
                          rev(death_capital),
                          rev(penalite_death)
                          )
  row.names(specifications) <-{ c("Periode", "Resiliation",
                                "Penalty" , "Accident",
                                "Increments", "Death Reserve",
                                "Death Penalty")
  }
  return(specifications)
}

specifications <- {specification_matrix(periode = periode,
                                       resiliation = resiliation,
                                       penalite = penalite,
                                       accident = accident,
                                       penalite_death = penalite_death,
                                       death_capital=death_capital,
                                       increments=increments)
}

```

Appendix C

Unique premium engine design

```
library(readxl)

Backward_thiele <- function(cv,cd,ca,age,n,num_of_step,specifications){
  V<- c()
  x <- cv
  V[1] <- x
  j=1
  delta <- n/num_of_step
  #backward loop
  for (i in 1:ncol(specifications)){
    #parameters
    time <- specifications[1,i]
    resiliation <- specifications[2,i]
    penalite <- specifications[3,i]
    accident <- specifications[4,i]
    increments <- specifications[5,i]
    max_cd <- specifications[6,i]
    penalite_death <- specifications[7,i]

    while(temps >time+delta){
      #loop
      temps <- temps - delta
      ratio <- {(0.5*(ceiling(temps)
                        *(1+increments)
                        +(n-floor(temps))
                        *(1-increments)))^abs(increments)}
      interest_rate <- IR[floor(temps)+1,2]
      mu <- MR[age+floor(temps)+1,2]

      {if((x/(1+(log(1+interest_rate)+penalite
                    *resiliation)*delta))> cd
          && max_cd == 1){dummy = 0}else{dummy = 1}

      {x <- (x+cd*mu*ratio*(1-accident)*delta
              *(as.integer(cd!=0))*dummy+(accident)
              *mu*ca*delta*(as.integer(cd!=0)))/(1+(log(1+interest_rate)
              +penalite*resiliation+penalite_death*mu+mu*dummy
              *(1-accident)*(as.integer(cd!=0))+mu*(accident))*delta)

      #update of reserve
      V[j+1] <- x
      j <- j + 1
    }
  }
}
```

```

    PU <- x
    return(PU)
}

```

Appendix D

Unique premium convergence to periodic premium engine design

```

Annuity_due_weekly <- function(PU,age, M,m, geo=NULL, k=NULL, increase=NULL){
  if( missing(geo)){
    geo <- 0
  }
  if(missing(k)){
    k <- 0
  }
  progression <- sort(rep(1:M-1,m))
  if (isTRUE(increase) == FALSE){
    progression <- rev(progression)
  }
  given <- matrix(nrow = 4, ncol = M*m)
  i <- IR[1:M,2]
  mu <- MR[(age+1):(age+M),4]
  given[1,1] <- 1
  #first row = what type (geo, arithmetic, increasing, decreasing)
  given[1,] <- ((1+ geo)^progression) +k*progression
  #second row = mortality
  #third row = interest rate
  given[2,1] <- 1
  given[3,1] <- 1
  for (j in 1:(M*m-1)){
    given[2,j+1] <- exp(-mu[ceiling(j/m)]*(1/m))
    given[3,j+1] <- (1+i[ceiling(j/m)])^(-1/m)
  }
  given[4,]<-cumprod(given[2,]*given[3,])
  given[4,]<-given[4,]*given[1,]

  #fourth row = result
  a <- sum(given[4,])
  #transform PU to PA
  PA <-PU/a
  #print(given)
  premium <- given[1,]*PA
  return(premium)
}

```

Appendix E

Periodic premium engine design

```
library(readxl)

Backward_thiele <- function(premium,cv,cd,ca,age,n,num_of_step,specifications,m,M){

  x <- cv
  V<- c()

  delta <- n/num_of_step
  while (round(x)!=0){
    x      <- cv
    V      <- c()
    prime <- length(premium)

    V[1] <- 0
    j=0
    #backward loop
    for (i in 1:ncol(specifications)){
      #parameters
      time      <- specifications[1,i]
      resiliation <- specifications[2,i]
      penalite   <- specifications[3,i]
      accident   <- specifications[4,i]
      increments <- specifications[5,i]
      max_cd     <- specifications[6,i]
      penalite_death <- specifications[7,i]

      while(temps >time+delta){
        #loop
        V[j+1] <- x
        temps  <- temps - delta
        ratio  <- {(0.5*(ceiling(temps)*(1+increments)
                        +(n-floor(temps))
                        *(1-increments)))^abs(increments)}
        }
        interest_rate <- IR[floor(temps)+1,2]

        mu <- MR[age+floor(temps)+1,2]
        { if((x/(1+(log(1+interest_rate)+penalite*resiliation)
                    *delta))> cd
          && max_cd == 1){dummy = 0}else{dummy = 1}

          {if((floor((temps*m))-floor(m*(temps-delta)))==1
              && temps < M)
            {PAA=premium[prime]; prime<-(prime-1) } else {PAA=0}

          }

        x <- {(x+cd*mu*ratio*(1-accident)*delta*(as.integer(cd!=0))
```

```

        *dummy+(accident)*mu*ca*delta*(as.integer(cd!=0))-PAA)
      /(1+(log(1+interest_rate)+penalite*resiliation+penalite_death*mu
        +mu*dummy*(1-accident)*(as.integer(cd!=0))+mu*(accident))*delta)
    }

    #update of reserve

    j <- j + 1
    V[j+1]<-x

  }

}

premium = {( premium+Annuity_due_weekly(PU = as.double(x),age = age ,
      M = M, m=m , k=k, geo=geo ,increase = increase))
}
temps=n
}

return(premium)
}

```

Appendix F

Optimal number of payment engine design

```

library(readxl)

#m=period per year
geo <- 0
k <- 0
increase <- TRUE
#prm <- premium

#####
Annuity_due_weekly <- function(PU,age, M,m, geo=NULL, k=NULL, increase=NULL){
  if( missing(geo)){
    geo <- 0
  }
  if(missing(k)){
    k <- 0
  }
}

```

```

progression <- sort(rep(1:M-1,m))
if (isTRUE(increase) == FALSE){
  progression <- rev(progression)
}
given <- matrix(nrow = 4, ncol = M*m)
i <- IR[1:M,2]
mu <- MR[(age+1):(age+M),4]
given[1,1] <- 1
#first row = what type (geo, arithmetic, increasing, decreasing,...)
given[1,] <- ((1+ geo)^progression) +k*progression
#second row = mortality
#third row = interest rate
given[2,1] <- 1
given[3,1] <- 1
for (j in 1:(M*m-1)){
  given[2,j+1] <- exp(-mu[ceiling(j/m)]*(1/m))
  given[3,j+1] <- (1+i[ceiling(j/m)])^(-1/m)
}
given[4,]<-cumprod(given[2,]*given[3,])
given[4,]<-given[4,]*given[1,]
#fourth row = result
a <- sum(given[4,])
#transform PU to PA
PA <-PU/a
#print(given)
premium <- given[1,]*PA
#print(PU/a)
#print(premium)
return(premium)
}

Optimal_M <- function(premiumo,PUS,cv,cd,ca,age,n,num_of_step,specifications,
                      m,M,k,geo,increase){
  U=-1
  premiumo <- {Annuity_due_weekly(PU = PUS,age = age , M = M, m=m , k=k, geo=geo
                                ,increase = increase)}
  while (any(U<0)){
    delta <- n/num_of_step

    x <- cv
    U<- c()

    premium <- premiumo
    #SDR
    while (round(x)!=0){
      x <- cv
      U<- c()
      prime <- length(premium)

```

```

U[1] <- 0
j=0
#backward loop
for (i in 1:ncol(specifications)){
  #parameters
  time <- specifications[1,i]
  resiliation <- specifications[2,i]
  penalite <- specifications[3,i]
  accident <- specifications[4,i]
  increments <- -1
  max_cd <- specifications[6,i]
  penalite_death <- specifications[7,i]

  while(temps > time+delta){
    #loop
    U[j+1] <- x
    temps <- temps - delta
    ratio <- {(0.5*(ceiling(temps)*(1+increments)+(n-floor(temps))*
      (1-increments)))^abs(increments)}
    interest_rate <- IR[floor(temps)+1,2]

    mu <- MR[age+floor(temps)+1,2]
    {if((x/(1+(log(1+interest_rate)+penalite*resiliation)*delta))> cdo
      && max_cd == 1){dummy = 0}}

    else{dummy = 1}

    if((floor((temps*m))-floor(m*(temps-delta)))==1 && temps < M)
      {PAA=premium[prime]; prime<-(prime-1) } else {PAA=0}

    x <-{ (x+cdo*mu*ratio*(1-accident)*delta*(as.integer(cdo!=0))
      *dummy+(accident)*mu*ca*delta*(as.integer(cdo!=0))-PAA)
      /(1+(log(1+interest_rate)+penalite*resiliation+penalite_death*mu+mu*dummy
      *(1-accident)*(as.integer(cdo!=0))+mu*(accident))*delta)
      }

    #update of reserve

    j <- j + 1
    U[j+1]<-x

  }

}

premium = {( premium+Annuity_due_weekly(PU = as.double(x),age = age ,
  M = M, m=m , k=k, geo=geo ,increase = increase))}
temps=n

```

```

}
M= M-1
if (M==0){break}
premiumo <-{ Annuity_due_weekly(PU = PUS,age = age , M = M, m=m , k=k,
                                geo=geo ,increase = increase)}

}
par(xpd=TRUE)
legend(9,-10,legend=c("Decreasing_Death_Capital"), col=c("dark_green"),
      lty=c(1,2), ncol=1)
# abline(v =periode[-1], col="red", lwd=1, lty=2)
abline (h=0,col="red", lty=3, lwd=1)
return(M+1)
}

opt<- {Optimal_M(premiumo = premiumo,PUS=PUS,cv=1,cd=cd,ca=ca,age=age,n=n,
                num_of_step=num_of_step,specifications=specifications,m=m,
                M=M , k=k, geo=geo ,increase = increase)

}

```


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