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Zeta functions of graphs

Auteur·es : Thomas Lemmens

Promoteur·rices : Pierre-Emmanuel Caprace, François Thilmany

Lecteur·rices : Timothée Marquis, Enrico Vitale

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Thomas Lemmens
Advisors: Pierre-Emmanuel Caprace, François Thilmany
Readers: Timothée Marquis, Enrico Vitale

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Introduction

The Riemann zeta function, one of the most studied objects in mathematics, was first introduced by Euler in the first half of the 18th century. It took a century before a link between that function and prime numbers was brought to light, by none other than Riemann himself. Since then, it is at the center of one of the most important unsolved problem in mathematics, recognized as one of the seven so-called "Millenium Prize Problems", which still eludes some of the greatest minds in mathematics.

It shouldn't come as a surprise therefore that comparable functions, sharing similarities with the Riemann zeta, have been studied in other areas in mathematics. These are often also called zeta functions. Our aim today is to introduce two such zeta functions: the edge zeta function and vertex zeta function (also called Ihara zeta function), as well as provide additional results.

Historically, the Ihara zeta function came from work on p-adic groups from Yasutaka Ihara in the 1960s. Jean-Pierre Serre later remarked that this zeta function could be purely interpreted in terms of graphs. Work on the Ihara zeta function quickly followed based on that interpretation (see [1] for a list of main authors as well as additional references on the subject), it is this perspective we'll be adopting today. We'll introduce and proof main results for this vertex zeta, as well as provide results we found regarding the zeta function of inflated graphs, and products of graphs.

The Ihara zeta function was generalized in 1996 by M. Stark and A. Terras [2] into the edge zeta function, and used in 2006 in [1] to prove an important result previously obtained by Bass in 1992 [3]. Other than that, it hasn't been much studied. We use the idea of edge deletion found in [4] and ultimately find stronger results regarding the edge zeta: we claim that an md2 graph can be entirely retrieved from its edge zeta.

While historically the Ihara zeta function precedes the edge zeta, we choose to introduce them in the opposite order, starting with the more general edge zeta and finishing with the vertex zeta, one of its specializations.

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Chapter 1

Edge zeta function

1.1 Primes and graphs

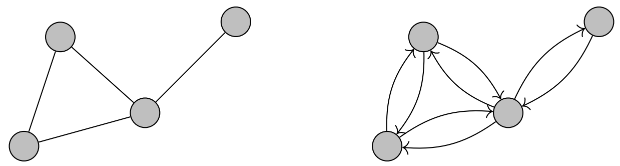
We introduce the objects we will be working with, starting with graphs. There are quite a few different definitions for graphs, we will be working with the following definition:

Definition 1.1. *An* ***oriented graph*** X *is given by two finite sets* $V(X)$ *and* $\vec{E}(X)$*, respectively called the set of* ***vertices*** *and the set of* ***oriented edges****, as well as two mappings:*

- *the initial map* $i : \vec{E}(X) \longrightarrow V(X)$
- *the terminal map* $t : \vec{E}(X) \longrightarrow V(X)$

This induces an orientation on edges. We say that edge e *goes from* $i(e)$ *to* $t(e)$*, starts at* $i(e)$ *and ends at* $t(e)$*.*

From this we get all of the information needed for an oriented graph, often represented using diagrams. Whenever we fail to specify edge directions in diagrams, we implicitly assume each edge is composed of two opposite oriented edges e and e^{-1} such that $i(e) = t(e^{-1})$ and $t(e) = i(e^{-1})$. As such, we have the following equivalent representations of the same graph:



This essentially enables us to extend our definition of oriented graphs to their unoriented versions. To us **unoriented graphs** are going to be such that each edge is actually a pair of opposite oriented edges (e, e^{-1}). The set of (unoriented) edges of such a graph is denoted $E(X)$, while its set of oriented edges is denoted $\vec{E}(X)$. If we don't specify a graph is oriented, we assume it is unoriented. As usual, we compare two oriented graphs through isomorphisms:

Definition 1.2. *We say two oriented graphs X and Y are **isomorphic** if there exists two bijective maps $f : \vec{E}(X) \to \vec{E}(Y)$ and $g : V(X) \to V(Y)$ such that:*

$$i(e) = a \Leftrightarrow i \circ f(e) = g(a)$$
$$t(e) = b \Leftrightarrow t \circ f(e) = g(b)$$

The pair (f, g) constitutes an oriented graph isomorphism.

This notion of isomorphism easily extends to unoriented graph isomorphisms, since we consider them as special cases of oriented graphs. If we consider a subset of oriented edges in a graph, we can consider the edge induced subgraph of that subset:

Definition 1.3. *Let X be an oriented graph and $\vec{E}' \subseteq \vec{E}(X)$. We can consider the following oriented graph Y:*

$$\vec{E}(Y) = \vec{E}' \text{ and } V(Y) = i(\vec{E}') \cup t(\vec{E}')$$

*With initial and terminal maps induced by those of X. This graph is called the subgraph **induced** by $\vec{E}'$.*

We can also extend this definition to unoriented graphs to get the graph induced by a subset of edges $E' \subseteq E(X)$: simply take the graph induced by the oriented edges of the elements of E'.

We now introduce constructions made from edges of oriented graphs, the simplest ones being of course, paths.

Definition 1.4. *A **path** in a graph is a finite sequence $e_1 e_2 \dots e_n$ of oriented edges such that:*

$$t(e_i) = i(e_{i+1})$$

*It is **backtrackless** whenever $e_{i+1} \neq e_i^{-1}$, and **tailless** whenever $e_n \neq e_1^{-1}$.*

Our goal is to study a certain kind of paths, more specifically closed, backtrackless, tailless, primitive paths.

Definition 1.5. *A **cycle** is a closed path, i.e a path $e_1 e_2 \dots e_n$ such that:*

$$t(e_n) = i(e_1)$$

*It is a **primitive** cycle whenever it can't be written as the repetition of another cycle.*

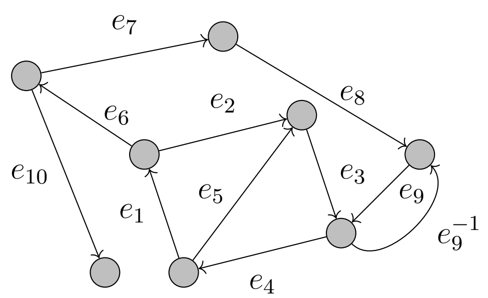
What we are going to be interested in is a version of "primes" in graphs, which we'll define as equivalence classes of those backtrackless, tailless, primitive cycles. We'll later study how much information these primes give on the graph they come from.

Definition 1.6. *A **prime** of a graph X is the equivalence class of a **backtrackless**,**tailless**, **primitive** cycle P. Let $P = e_1 e_2 \dots e_n$ then this equivalence class is given by:*

$$[P] = \{e_1 \dots e_n,\ e_2 \dots e_n e_1,\ \dots\ ,\ e_n e_1 \dots e_{n-1}\}$$

*Elements of this equivalence class will be called **prime cycles**.*

The purpose of this equivalence class is to ignore the starting (and therefore end) vertex of a cycle and focus on its edges. We'll take a look at examples and non-examples of primes in the following graph to familiarize ourselves with this new concept. For the sake of clarity, we won't be representing every edge inverse:



Following our definition, the cycle $e_1e_2e_3e_4$ would be considered prime. The cycle $(e_1e_2e_3e_4)^5$ however, isn't (powers are used to indicate we repeat a sequence a few times).

The cycles $e_9(e_4e_1e_2e_3)e_9^{-1}$ and $e_1e_2e_3e_9^{-1}e_9e_4$ are also both not prime: one has a tail while the other backtracks (we have $\left(e_9^{-1}\right)^{-1} = e_9$). Notice they both represent the same equivalence class. Indeed, the presence of a tail leads to having a backtrack in all of the other cycles of the equivalence class.

An important observation to make is that there is an infinite number of primes in this graph: the cycle $(e_1e_2e_3e_4)(e_1e_6e_7e_8e_9e_4)^n(e_1e_2e_3e_4)$ is prime regardless of the value of $n \in \mathbb{N}$. This trick of traveling one prime cycle then another one an arbitrary number of times and coming back to the first can be used whenever a graph has two cycles in the same connected component that don't coincide. In general, we'll have an infinite number of prime cycles in our graphs. For connected graphs, the only exceptions are cycle graphs (2 prime cycles, one going clockwise, the other going counter-clockwise), and trees (0 cycles).

1.2 Edge matrix and edge norm

We now look at ways of assigning quantities to those prime cycles. This will be done first by assigning weights to pairs of oriented edges. We encode those weights in an edge matrix:

Definition 1.7. *Let X be a graph composed of m edges, with oriented edges indexed from 1 to $2m$, then the $2m \times 2m$ **edge matrix** is given by:*

$$W_{ij} = \begin{cases} w_{e_i,e_j} \text{ if } t(e_i) = i(e_j) \text{ and } e_j \neq e_i^{-1} \\ 0 \text{ otherwise.} \end{cases}$$

With w_{e_i,e_j} complex number.

We'll often index our oriented edges as follows: we choose 1 element from each pair of oriented edges and index those from 1 to m, we then index their inverses in the same order, from $m+1$ to $2m$. We therefore get $e_{m+i} = e_i^{-1}$. We won't use this indexation yet but it will be necessary later on. We have everything needed to assign a complex number to a path via the edge norm:

Definition 1.8. *Let W be an edge matrix and $p = e_1 \dots e_n$ a sequence of edges. Then we define the **edge norm** as:*

$$N_E(W,p) = w_{e_1,e_2} w_{e_2,e_3} \dots w_{e_{n-1},e_n} w_{e_n,e_1}$$

Notice this definition of the edge norm only gives us non-zero values for paths that are cycles, as we have $w_{e_i,e_{i+1}} = 0$ if e_i doesn't feed into e_{i+1}, and $w_{e_n,e_1} = 0$ if e_n doesn't feed into e_1. It also gives us 0 for cycles that backtrack or have tails. This justifies the definition of the edge matrix: it enables us to conveniently ignore sequences that aren't **proper cycles**, cycles that don't backtrack nor have tails.

We'll end this section by proving a link between the trace of W^n and the edge norm of proper cycles of a certain length. This will be useful later on in the study of the edge zeta function.

Lemma 1.9. *Let W be an edge matrix of X and $\mathcal{P}(n,X)$ the set of proper cycles of length n in X. Then:*

$$\mathrm{Tr}(W^n) = \sum_{P \in \mathcal{P}(n,X)} N_E(W,P)$$

Proof. To show this result, we have to prove the following observation:

$$(W^n)_{ij} = \sum_{e_i e_{k_1} \dots e_{k_{n-1}}} w_{e_i,e_{k_1}} w_{e_{k_1},e_{k_2}} \dots w_{e_{k_{n-2}},e_{k_{n-1}}} w_{e_{k_{n-1}},e_j}$$

Essentially a sum over the sequences of length n starting with edge e_i. When $j = i$ this corresponds to looking at the sum of edge norms of sequences of length n starting with edge e_i. By our previous remark, it can be reduced to a sum of edge norms of proper cycles of length n starting with e_i. Taking the trace then concludes the proof, as we are considering the sum of edge norms of all proper cycles of length n, regardless of the starting edge.

We therefore proceed by induction, our base case being $n = 1$:

$$(W)_{ij} = w_{e_i,e_j}$$

This corresponds trivially to our formula as only one sequence of length one starts with e_i. Remember also that $w_{e_i,e_j} = 0$ if $t(e_i) \neq i(e_j)$ or $e_j \neq e_i^{-1}$. We now move on to the more general case, we have:

$$\sum_{e_i e_{k_1} \dots e_{k_{n-1}}} w_{e_i,e_{k_1}} w_{e_{k_1},e_{k_2}} \dots w_{e_{k_{n-2}},e_{k_{n-1}}} w_{e_{k_{n-1}},e_j}$$

Which, by reassembling things properly, gives:

$$\sum_{e_{k_{n-1}}} w_{e_{k_{n-1}},e_j} \left(\sum_{e_i e_{k_1} \dots e_{k_{n-2}}} w_{e_i,e_{k_1}} w_{e_{k_1},e_{k_2}} \dots w_{e_{k_{n-2}},e_{k_{n-1}}} \right)$$

And by our induction hypothesis this equals:

$$\begin{aligned} \sum_{e_{k_{n-1}}} w_{e_{k_{n-1}},e_j} (W^{n-1})_{ik_{n-1}} &= \sum_{k=1}^{2m} w_{e_k,e_j} (W^{n-1})_{i,k} \\ &= \sum_{k=1}^{2m} W_{kj} (W^{n-1})_{ik} = (W^n)_{ij} \end{aligned}$$

This proves the formula we mentioned, and so:

$$\begin{aligned} \mathrm{Tr}(W^n) &= \sum_{i=1}^{2m} \sum_{e_i e_{k_1} \dots e_{k_{n-1}}} w_{e_i,e_{k_1}} w_{e_{k_1},e_{k_2}} \dots w_{e_{k_{n-2}},e_{k_{n-1}}} w_{e_{k_{n-1}},e_i} \\ &= \sum_{i=1}^{2m} \sum_{e_i e_{k_1} \dots e_{k_{n-1}}} N_E(W, e_i e_{k_1} \dots e_{k_{n-1}}) \\ &= \sum_{P \in \mathcal{P}(n,X)} N_E(W, P) \end{aligned}$$

The last equality coming from the fact that the edge norm of sequences that aren't proper cycles is zero. $\square$

1.3 Edge zeta function

We are now ready to introduce the edge zeta function of a graph, described by Stark and Terras in [2]. As a reminder, the oriented edges in our graphs come in pairs, and translate the fact that we are able to travel in both directions through an edge of an otherwise unoriented graph. We will index our oriented edges as mentioned previously: choose one edge out of each pair and index them from 1 to m, then index the others in the same order from $m+1$ to $2m$. We then introduce the edge zeta function (also called multi-edge zeta function in [5]) as a product over primes:

Definition 1.10. *The* ***edge zeta function*** *of a graph X is given by:*

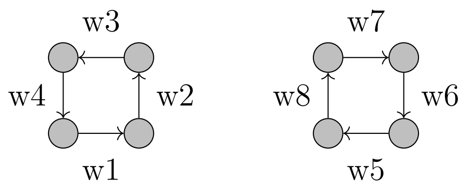
$$\zeta_E(W, X) = \prod_{[P]} (1 - N_E(W, P))^{-1}$$

Converging when the W_{ij} are sufficiently close to 0*.*

We will often consider this zeta function as a formal power series and study it algebraically rather than analytically. A discussion on the poles of these zeta functions can be found in [5]. Let's start with a simple example computation of this function in the case of a cycle graph.

Definition 1.11. *An* ***oriented cycle graph*** *is an oriented graph induced by oriented edges of a cycle, such that $|\vec{E}'| = |V'|$. If you add the inverse of each oriented edge of the cycle to the induced subgraph, you get a* ***cycle graph****.*

Cycle graphs contain 2 prime cycles, one going clockwise and the other going counter-clockwise:



In the case of the square graph C_4, we get the following edge zeta function:

$$\zeta_E(W, C_4) = \frac{1}{1 - w_1 w_2 w_3 w_4} \frac{1}{1 - w_5 w_6 w_7 w_8}$$

But how do we compute the edge zeta function for more complicated graphs? Having 2 different unoriented cycles in a connected graph leads to an infinite number of primes! One of the big results concerning the edge zeta function is that it can be written as the inverse of a polynomial. We'll state the theorem then provide a prove completing a sketch provided in [6]:

Theorem 1.12 (Determinant formula for the edge zeta)**.** *Let W be an edge matrix of a graph, then:*

$$\zeta_E(W, X) = \det(I - W)^{-1}$$

Proof. We first start by taking the logarithm of our edge zeta function, we get:

$$\log \zeta_E(W, X) = \sum_{[P]} -\log(1 - N_E(W, P))$$

Now, the power series for $\log(1 - x)$ is $\sum_{n=1}^{\infty} -\frac{x^n}{n}$ therefore:

$$\log \zeta_E(W, X) = \sum_{[P]} \sum_{n=1}^{\infty} \frac{N_E(W, P)^n}{n}$$

Instead of summing over primes we will be summing over prime cycles. The cardinality of $[P]$ being the length $l(P)$ of P, we'll need to compensate those repetitions of terms by dividing by $l(P)$. We get:

$$\begin{aligned}
\log \zeta_E(W, X) &= \sum_{P \text{ prime}} \frac{1}{l(P)} \sum_{n=1}^{\infty} \frac{N_E(W, P)^n}{n} \\
&= \sum_{P \text{ prime}} \sum_{n=1}^{\infty} \frac{N_E(W, P)^n}{nl(P)}
\end{aligned}$$

Notice that $N_E(W, P)^n = N_E(W, P^n)$ and $nl(P) = l(P^n)$: we are suddenly summing over proper cycles instead of prime cycles!

$$\log \zeta_E(W, X) = \sum_{C \text{ proper}} \frac{N_E(W, C)}{l(C)}$$

And this is where we use Lemma 1.9 of the previous section to introduce the trace of W. Summing over the length of proper cycles we get:

$$\begin{aligned}
\log \zeta_E(W, X) &= \sum_{l=1}^{\infty} \frac{\operatorname{Tr}(W^l)}{l} \\
&= \operatorname{Tr}\left(\sum_{l=1}^{\infty} \frac{W^l}{l}\right) \\
&= \operatorname{Tr}(\log(I - W)^{-1})
\end{aligned}$$

We therefore have:

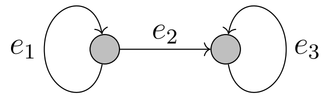
$$\log \zeta_E(W, X) = \log \exp \operatorname{Tr}(\log(I - W)^{-1})$$

We finally use the fact that $\exp \operatorname{Tr} A = \det \exp A$ to get this last equation:

$$\log \zeta_E(W, X) = \log \det(I - W)^{-1}$$

This concludes the proof. $\square$

This enables us to compute the edge zeta functions for more elaborate graphs, containing an infinite number of cycles! Let's take a look at the simplest example of such a graph: the dumbbell graph.

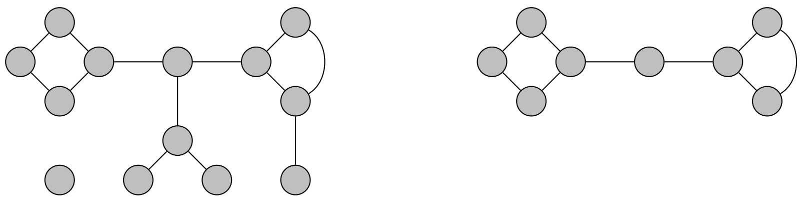


The inverses of each edge aren't represented, but they corresponded to $e_4 = e_1^{-1}$, $e_5 = e_2^{-1}$, $e_6 = e_3^{-1}$. The edge matrix we get for the dumbbell graph is:

$$W = \begin{pmatrix} w_{e_1e_1} & w_{e_1e_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & w_{e_2e_3} & 0 & 0 & w_{e_2e_6} \\ 0 & 0 & w_{e_3e_3} & 0 & w_{e_3e_5} & 0 \\ 0 & w_{e_4e_2} & 0 & w_{e_4e_4} & 0 & 0 \\ w_{e_5e_1} & 0 & 0 & w_{e_5e_4} & 0 & 0 \\ 0 & 0 & 0 & 0 & w_{e_6e_5} & w_{e_6e_6} \end{pmatrix}$$

As you can see what we would then need to do is compute the determinant of a 6×6 matrix, which is quite cumbersome, but feasible!

Notice some edges don't really intervene in the edge zeta. Indeed, define the **degree** of a vertex as the number of oriented edges starting with that vertex. We can observe that nodes of degree 0 can't appear in prime cycles, as well as nodes of degree 1. This comes from the fact that having a cycle go through one of those degree 1 vertices forces it to have either a tail or a backtrack, and degree 0 vertices naturally don't intervene in paths. We can therefore remove degree 0 vertices and iteratively remove degree 1 vertices (and the edges attached to them) out of a graph without modifying its edge zeta, we will call this process "pruning" a graph:

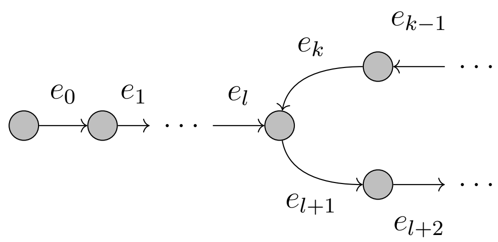


The resulting pruned graph will have vertices of minimum degree 2, we say such a graph is **md2**. Studying the zeta function of a graph is therefore equivalent to studying the zeta function of this md2 subgraph. We will now justify the uniqueness of the graph obtained by iteratively pruning out those degree 0 and 1 vertices via this proposition (which we'll also use later on):

Proposition 1.13. *Let X be a graph with $E(X) \neq \emptyset$. Then:*

$$X \text{ is md2} \iff \text{every oriented edge is part of a prime cycle.}$$

Proof. First of all, if every oriented edge is part of a prime cycle, this naturally excludes both degree 0 and degree 1 vertices, as those don't occur in prime cycles. This guarantees the graph is md2. Now suppose that the graph is md2, and take an oriented edge $e = e_0 \in \vec{E}(X)$. Since the graph is md2, there is an oriented edge e_1 starting with $t(e_0)$ and different to e_0^{-1}. Repeat the argument with e_1 to construct e_2, then again to construct e_3 etc. until the constructed path $e_0 \dots e_k$ ends with a vertex already visited, it suffices now to go back part of the path constructed to get a cycle that starts and ends with $i(e)$.



We therefore get a cycle $C_1 = (e_0 \dots e_l)(e_{l+1} \dots e_k)e_l^{-1} \dots e_0^{-1}$ starting with $i(e)$ and without backtracks. If $t(e_k) = i(e_0)$ (see diagram above), our work is done: C_1 is a prime cycle containing e. If not, we'll do the following: construct another cycle C_2 starting from $i(e)$ but whose first edge isn't $e = e_0$. This is possible as $i(e)$ is of degree at least 2. Joining those two cycles together we get the prime cycle C_1C_2, containing e. Since e was arbitrarily chosen, every edge is part of a prime cycle in an md2 graph. $\square$

Pruning a graph therefore removes all edges and nodes that aren't part of prime cycles. It turns out that this process can be seen as a manipulation of the edge zeta. Indeed notice it is quite easy to remove an edge from a graph and get the resulting edge zeta:

Proposition 1.14. *Let e be an unoriented edge of a graph X composed of $f, f^{-1} \in \vec{E}(X)$ and $X \setminus e$ be the graph obtained by removing that edge, with edge matrix W. Then:*

$$\zeta_E(W, X \setminus e) = \zeta_E(W_e, X)$$

Where W_e is the same as W but with additional columns and lines filled with zeros for the oriented edges f and f^{-1}.

Proof. This comes from the fact that by putting all weights of oriented edges of e to 0 we force edge norms of paths containing them to also be equal to 0. This is therefore equivalent to ignoring e in our zeta function and studying the edge zeta of $X \setminus e$. $\square$

We can use this to "remove" edges from a graph:

Corollary 1.15. *Let X be an unoriented graph, we can prune it using its edge zeta function.*

Proof. By Proposition 1.13 all we need to do is find the weights of W that don't intervene in the edge zeta: they correspond to oriented edges that aren't part of the pruned version of our graph. Pruning is therefore equivalent to removing those edges. All of this can be done without actually having an idea of what the graph looks like, and comes solely from manipulating the edge zeta. □

Another thing we can do with the edge zeta is extract information on the connectivity of our graph, but first notice that we have the following:

Proposition 1.16. *Let X be a a graph whose connected components are $X_1, \ldots, X_n$, and let $W_1, \ldots, W_n$ be the associated edge matrices. Then we have:*

$$\zeta_E(W, X) = \zeta_E(W_1, X_1) \ldots \zeta_E(W_n, X_n)$$

Proof. The formula relating the edge zeta of a graph to the edge zetas of its connected components is relatively straightforward, we can simply separate the product of the edge zeta along its connected components:

$$\begin{aligned}\zeta_E(W, X) &= \prod_{[P]} (1 - N_E(P))^{-1} \\ &= \prod_{i=1}^{n} \prod_{[P] \subset X_i} (1 - N_E(P))^{-1} = \prod_{i=1}^{n} \zeta_E(W_i, X_i)\end{aligned}$$

Where $[P] \subset X_i$ means P is contained in X_i. □

This gives us the following useful lemma:

Lemma 1.17. *Let X_1 and X_2 be two connected md2 subgraphs of X induced by sets of edges E_1 and E_2. We are able to detect whether they are in the same connected component by knowing the content of E_1 and E_2 and by manipulating $\zeta_E(W)$.*

Proof. If $E_1 \cap E_2 \neq \emptyset$ (which we can check by looking at the content of E_1 and E_2) then we know that both of these graphs are connected, since X_1 and X_2 are connected.
Suppose now $E_1 \cap E_2 = \emptyset$. If X_1 and X_2 are in the same connected component, there exists a shortest path p going from a vertex v of X_1 to a vertex w of X_2. We can make a few observations regarding this shortest path. First of all, since it is a shortest path between both subgraphs, the vertices along this path (that aren't starting vertices nor end vertices) don't belong to X_1 or X_2. This same minimality condition also guarantees

this path doesn't intersect itself.
Now let $p = e_1 \dots e_n$, then the graph induced by $\vec{E}_i \cup \{e_i^{\pm 1} | 1 \leq i \leq n\}$ doesn't contain more prime cycles than X_i individually: p can be seen as a long tail added to X_i (which we would usually have removed by pruning).
The graph induced by $\vec{E}_1 \cup \vec{E}_2 \cup \{e_i^{\pm 1} | 1 \leq i \leq n\}$ though contains more prime cycles than X_1 and X_2 separately! Indeed, take a prime cycle C_1 in X_1 starting at v (this path exists as X_1 is md2, and by Proposition 1.13), and do the same for w in X_2 to get C_2. We can then construct the prime cycle $C_1 e_1 \dots e_n C_2 e_n^{-1} \dots e_1^{-1}$. The presence of such a prime cycle can be detected by computing the edge zeta of the graph induced by $\vec{E}_1 \cup \vec{E}_2 \cup \{e_i^{\pm 1} | 1 \leq i \leq n\}$ and realizing it is different from the product $\zeta_E(W_1, X_1)\zeta_E(W_2, X_2)$.
To check whether two edge induced md2 subgraphs are in the same connected component we therefore have to find a subset of oriented edges $\vec{F} = \{e_1^{\pm 1}, \dots, e_n^{\pm 1}\}$ such that the edge zeta of the graph induced by $\vec{E}_i \cup F$ is equal to the edge zeta of X_i and such that the edge zeta of $\vec{E}_1 \cup \vec{E}_2 \cup \vec{F}$ is different to $\zeta_E(W_1, X_1)\zeta_E(W_2, X_2)$.
If both X_1 and X_2 are in the same connected component, such a subset exist as we've just shown with p, and if they aren't connected, a change in the zeta function of the graph induced by $\vec{E}_1 \cup \vec{E}_2 \cup \vec{F}$ compared to the one induced by $\vec{E}_1 \cup \vec{E}_2$ is due to extra prime cycles added in one of the X_i: the zeta of the graph induced by $\vec{E}_i \cup \vec{F}$ will be different to the zeta of the graph induced by $\vec{E}_i$ for a certain i. $\square$

We've introduced the main tools we'll need to prove the main result of the next section, it turns out we can reconstruct a weighted md2 graph from its edge zeta.

1.4 Weighted graphs

In this section, we look at weighted graphs, and edge zetas of weighted graphs. We introduce a few new concepts then use all the tools at our disposal to prove that the edge zeta is very close to a total invariant of weighted graphs: we are able to reconstruct a weighted graph from its edge zeta as well as put back weights up to permutation along lines and closed lines. This extends results found in [4].

We start by defining what we mean by a weighted graph:

Definition 1.18. *Let X be an unoriented graph whose set of edges is $E(X)$. We say X is a **weighted graph** if we have a weight function:*

$$\omega : E(X) \to \mathbb{C}$$

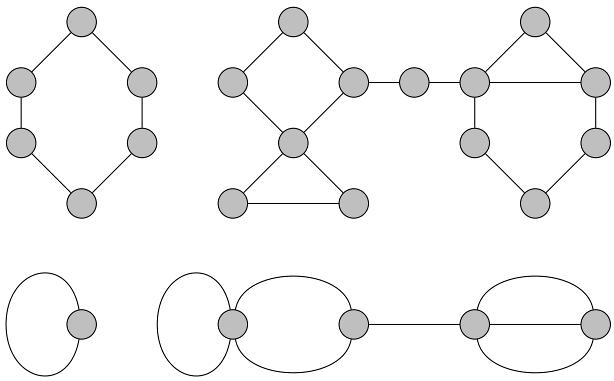
*In that case the **weight** of an edge $e \in E(X)$ is $\omega(e)$. We can use it to define the weight of the oriented edges making up e: we get a weight function defined on $\vec{E}(x)$, also denoted ω (small abuse of notation).*

We can therefore study edge zetas of weighted graphs: we consider edge matrices where $w_{e_i,e_j} = \omega(e_i)$ when $t(e_i) = i(e_j)$ and $e_j \neq e_i^{-1}$ (we also have $\omega(e_i) = \omega(e_i^{-1})$). The resulting edge zeta is the inverse of a polynomial made from edge weights. Suppose now we denote those weights $w_1, \ldots, w_m$, we'll show that we are able to find quite a bit of information from $\zeta_E(w_1, \ldots, w_m)$.

First, we'll see there is a way of further simplifying md2 graphs while essentially preserving the same information contained in its zeta function.

Definition 1.19. *Let $l = e_0 \ldots e_n$ be a path in an md2 graph. If every node lying on l, that is to say every $i(e_k)$ and $t(e_k)$, is of degree 2, but not the starting node $i(e_0)$ nor the end node $t(e_n)$ then we say the graph induced by l is an* ***oriented line****. If $i(e_0) = t(e_n)$, without conditions on the degree of that node, we say the graph induced by l is a* ***closed oriented line****. Adding the inverses of the e_i's, we get* ***lines*** *and* ***closed lines****. A closed line with one edge is a* ***loop****.*

Edges on a same line or closed line of a weighted graph can't be distinguished using the edge zeta. Indeed, once a prime cycle goes through the edge of one of those it is forced to visit all of it, as no backtrack nor tail is allowed: we deduce that permuting weights along those shapes doesn't modify the edge zeta function. This will motivate the following further simplification: md2 graphs can be simplified into graphs where connected components are md3 (minimum degree 3) or isolated loops. The edge zeta of the initial graph is simply the edge zeta of this **contracted graph** with appropriate edge weights (a line or closed line with weights w_1, w_2, w_3 is transformed into an edge of weight $w_1w_2w_3$):



We'll see we can get the edge zeta of this contracted graph from the zeta of X. We'll first need to prove a detection theorem for cycle graphs.

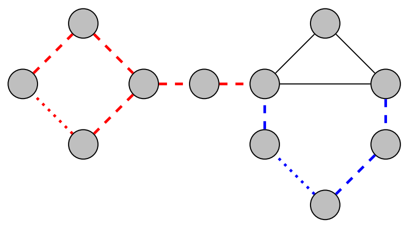
Proposition 1.20. *Suppose we have an edge induced md2 subgraph of a graph X, we can deduce whether it is isomorphic to a cycle graph using its zeta function.*

Proof. We'll prove that a cycle graph is equivalent to a graph whose edge zeta follows the following condition: removing any unoriented edge makes the edge zeta trivial. Indeed, suppose we have a cycle graph, it is made of n vertices and n edges. Removing one edge leaves us with a connected subgraph with n vertices and $n-1$ edges: it is therefore a tree and its edge zeta is trivial.
Suppose now we have an md2 graph satisfying this minimality condition and take a vertex v (therefore of degree at least 2). We can construct a path in a similar fashion as in the proof of Proposition 1.13, starting with an oriented edge $e_0 = e$ such that $i(e) = v$ and without backtracks, until it intersects itself at some $t(e_k)$. This gives us an edge induced oriented subgraph with a non trivial edge zeta, as it contains a cycle without backtrack. Adding the inverses of those edges, we get an induced unoriented subgraph with a non trivial edge zeta. Our minimality condition implies that this subgraph must contain all edges of our graph, otherwise we are able to remove edges without making the edge zeta trivial. This subgraph is therefore our graph! It also intersected itself at v, v is otherwise of degree 1 - prohibited in an md2 graph. We deduce all vertices are on a cycle, with as many edges as vertices: indeed, the first (and only) intersection is done at the initial vertex. Our graph is therefore a cycle graph.
We deduce that cycle graphs are entirely described by how their zeta function evolves upon edge removal. This is detectable using the edge zeta function of X: we can detect whether a subgraph induced by edges is isomorphic to a cycle graph.

□

Proposition 1.21. *Let X be an md2 graph, we are able to find all lines and closed lines of X using its edge zeta.*

Proof. For each edge e, we'll find the line or closed line containing that edge. We first remove e and prune the graph. This is done by finding edges that can be further removed without modifying the edge zeta. We illustrate this process in this example:



Two behaviors are possible, either we remove an edge contained in a closed line and the pruning process leads us to finding edges that form a cycle graph with (eventually) a tail (here in red), either we obtain a line (here in blue). As we are able to detect cycle subgraphs (Proposition 1.20) we can distinguish both situations: we can find whether an edge is part of a line or a closed line and if so, deduce in which line it is contained, or which closed line it is contained. If it was part of a closed line, simply find which edges obtained through the pruning process form a cycle graph. If none of those form a cycle graph, those edges form a line. □

Proposition 1.22. *Let X be an md2 weighted graph. We are able to obtain the zeta function of its contracted graph, as well as the edge weights associated to each edges of the contracted graph.*

Proof. We can use Proposition 1.21 to find the edge weights of lines and closed lines making up our graph. These always appear together in the edge zeta: suppose $w_1, \dots, w_n$ forms a line or closed line then whenever we see one of these terms in the edge zeta we actually find their product $w_1 \dots w_n$. The edge zeta of the contracted graph is found by replacing lines and closed lines with single edges. The product $w_1 \dots w_n$ would therefore be replaced by an edge weight w. Doing this for each line and closed line, we get the edge zeta of a new graph: the contracted graph. If we want to get back the initial edge zeta, simply put the product of the weights of each line/closed line on their respective edge. □

Theorem 1.23 (Reconstruction of weighted md2 graphs via the edge zetas)**.** *Let X be an md2 weighted graph and $\zeta_E(W, X)$ its edge zeta function. We are able to reconstruct X from $\zeta_E(W, X)$ and put back its edge weights, up to permutations along lines and closed lines.*

Proof. We will show this by induction on the number of edges.

For the base case, suppose $|E| = 1$. Only one type of md2 graph contains a single edge: an isolated loop/cycle graph of length 1. We therefore know how to reconstruct X, with its weights.

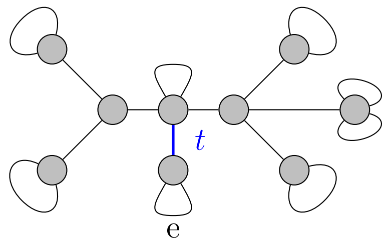
Suppose now $|E| > 1$. Use Proposition 1.22 to get the edge zeta of its contracted graph: the zeta function of an md3 graph with isolated loops. We keep in mind the edges/weights that were associated to each closed line and line (i.e the change of variables that was performed).

We then remove each isolated loop using Proposition 1.20 and Lemma 1.17 to detect them: they are cycle subgraphs made of one edge that aren't connected to any other $md2$ subgraph. We also keep this operation in mind. We are left with an md3 graph X'.

The rest of the proof focuses on reconstructing this md3 graph X': after this is done we will put back the isolated loops and put back the edges that formed lines and closed lines to get a construction of the weighted graph X, up to permutation of weights along lines and closed lines. Remember the whole proof is based on an induction argument.

What we'll first do is look at all subgraphs of X' that are isomorphic to a cycle graph which we can do using Proposition 1.20. Find one of those cycle subgraphs with maximal length L.

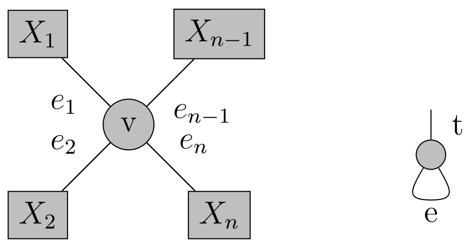
We'll treat different values of L separately. First suppose $L = 1$, then our graph is md3 with cycle subgraphs of size 1: it is a tree to which we appended loops at the leaves and on some vertices. We'll remove an edge e forming a loop, i.e a cycle subgraph of size 1 (which we can detect using Proposition 1.20). We don't have a guarantee the resulting graph is still md2, as we are decreasing the degree of the node to which e was attached by 2. We therefore prune the graph after having removed e, which may give us up to one edge forming the tail t to which e was attached as in this figure (in blue):



Of course there might be situations where pruning won't give us an edge t, in that case $X' \setminus e$ was already md2. We will continue the proof under the assumption t exists, the same arguments can be used when it doesn't (just ignore t whenever we mention it). Our goal now is to remove e (and t if it exists) then to find where to reattach it after having used our induction hypothesis.

We therefore apply our induction hypothesis on the subgraph induced by $E(X') \setminus \{t, e\}$ to reconstruct it. At this point we know the number of vertices of this graph and how they are connected. Our goal is now to put back the edges we've removed in the graph. To accomplish this, iterate through every node v of this graph as such: if a loop l is attached to v, check whether e is in the same connected component as l (using Lemma 1.17) in the graph induced by $\{l, t, e\}$, whose zeta function is found by only keeping the weights w_l, w_t, w_e from the zeta function of X. If l and e are in the same connected component, we know we need to reattach at v.

If no loop is at v, we'll do the following: first isolate the connected components $X_1, \ldots, X_n$ that are adjacent to v. We can illustrate this in the following figure (remember we are in a tree with loops):

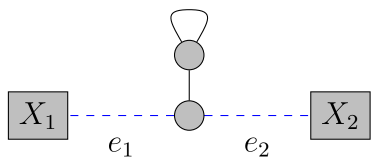


Notice these connected components are md2, we can therefore use Lemma 1.17 to check whether e is in the same connected component as X_i in the graph induced by $E(X_i)$ and the edges $\{e_i, e, t\}$. If e lies in the same connected component for two such X_i, we know we need to reattach at v!

We therefore iterate through every vertex until we end up finding which one we need to reattach e to (or its tail t, if it exists). We are guaranteed to find that vertex as our graph is md2 after having removed t and e: a vertex therefore has to either be connected to a loop or to 2 different edges (we can therefore rely on one of the 2 previous methods to check whether we need to connect to that vertex).

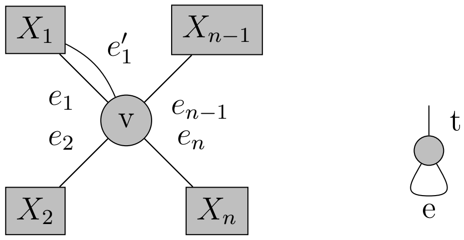
What is left to do is to make sure we are able to put back weights properly. Indeed, we need to make sure we can still locate them up to permutation along lines **after** having put back our edge e (and t), an operation which can sometimes modify lines. This happens whenever we append an edge to a degree 2 vertex: it increases its degree and can therefore split a line in 2 (or a closed line but none of them are of length > 1 here since $L = 1$).

Since our graph becomes md3 again after reattaching, lines reduce to edges. We therefore need to be able to find the weight of each edge. We have the following situation:



The indeterminacy on weights along the line created by the edges e_1, e_2 now translates into an indeterminacy on the weight of e_1 and e_2. Starting from now we will put dashed edges in our diagrams for edges whose weights aren't perfectly determined. Here though, the indetermination can quite easily be removed: suppose the weights associated to edges e_1, e_2 were $\{w_1, w_2\}$ then simply put one of those to 0 then see whether e is in the same connected component as X_1 or X_2: this tells us where each weight is located. If for example we realize e is in the same connected component as X_1 after having put w_1 to 0, we know the weight of e_1 was w_2. We are therefore able to reconstruct graphs for $L = 1$, using our induction hypothesis.

Suppose now $L = 2$. Then the only thing that changes from the previous case is that some edges are double edges. There are still loops at the leaves of the tree. We find a loop in the graph, using Proposition 1.21, remove it then reconstruct our graph. If a vertex is of degree 2 after reconstruction, we know we need to reattach at that vertex, since our graph X' was originally md3. Otherwise, use pretty much the same methods as before to check if $\{t, e\}$ is connected to a vertex v. There is but a slight difference when no loop is at v, we need to take into account that some edges incident to v are doubled:

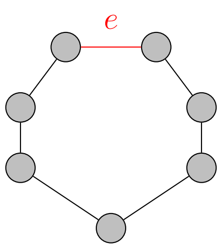


The components X_i are therefore not guaranteed to be md2 anymore, but they do contain loops, as there are loops at the end of the leaves of our tree. We therefore need to use Lemma 1.17 to see whether e is in the same connected component as one of the loops of X_i, in the graph induced by $E(X_i)$ and the edges $\{e_i, e, t\}$. This enables us to check whether e is connected to that X_i as we did previously for $L = 1$. Using the same argument as before we can check whether e (or t if it exists) is connected to v.

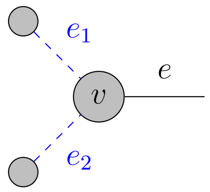
As for the weight indeterminacies, they are solved exactly as before, except when the degree two vertex responsible for the indeterminacy is connected to overlapping edges. In that case it doesn't matter how we put back our weights: the resulting weighted graphs are isomorphic (since the 2 weights overlap).

Finally suppose $L \geq 3$ and remove an edge e from a maximal cycle subgraph. Doing so leaves the graph md2 at the very least (it used to be md3), and our induction hypothesis enables us to reconstruct $X' \setminus e$. We continue our discussion based on the number of degree 2 vertices in $X' \setminus e$.

If it is still md3, we were able to reconstruct $X' \setminus e$ as well as put back weights on each edge (closed lines and lines are edges in that case: there is no indeterminacy). Since e was part of a subgraph isomorphic to a cycle graph of which we know the edge weights (and therefore the edges), we are able to find where to put e back: simply find the border vertices of the incomplete cycle subgraph and connect the two using e.



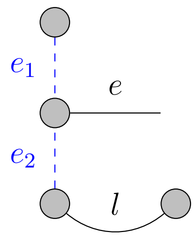
Suppose now we have an indeterminacy on one vertex, meaning only one vertex v is of degree 2 after removing e, with 2 different edges coming out of it. We know we need to reattach e to that vertex as degree 2 vertices didn't exist in X'. This gives us the following diagram:



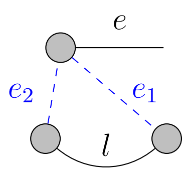
We know e_1 and e_2 are associated to weights w_1, w_2, what we need to do is find which edge these weights correspond to.

First of all remember overlaps solve indeterminations: if e_1 and e_2 overlap it doesn't really matter where we put w_1 and w_2, the two weighted graphs we would get are isomorphic to one another. In that case we solve our indetermination and revert to our previous discussion (when there are no indeterminations) to put e back in its place.

Suppose they don't overlap. We know the maximal cyclic subgraph e is part of (through its weights), and can therefore partially find it again in our graph. Indeed, we are guaranteed to be able to locate at least one edge of this cyclic subgraph as $L \geq 3$. Denote this partial cycle subgraph l. If one of the edges e_i (say e_1) isn't connected to the border vertices of l, we get the following situation:

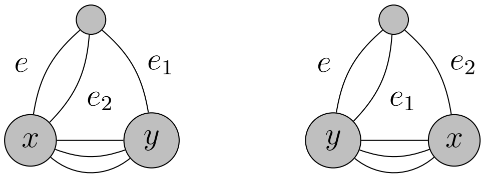


Knowing then that weight w_2 (without loss of generality) was part of the cycle subgraph, we are able to deduce that w_2 corresponded to edge e_2, and therefore w_1 to e_1. This solves the weight indeterminacy. We then have a unique way of attaching the other end of e to close the cyclic subgraph.
The indeterminate situation we have to solve is therefore when e_1 and e_2 don't overlap and are both connected to the border vertices of this partially reconstructed cyclic subgraph:



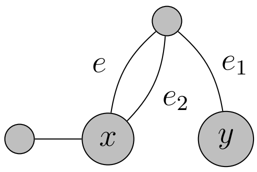
Denote the two border vertices as x and y. What we need to do is find which of those vertices e is connected to. Our edge e also overlaps with one of the e_i. The only thing we know from the edge zeta though is that w_e overlaps with, say, w_2, and that w_1 is part of the cycle graph.

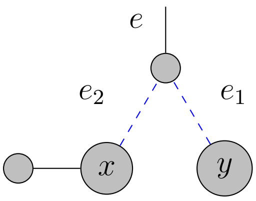
If x and y share all incident edges (except for e, e_1, and e_2), it doesn't matter to which vertex e is connected, as long as we make sure w_2 overlaps with w_e we get isomorphic weighted graphs:



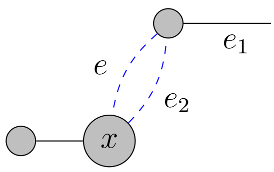
If there exists an edge attached to one of the two but not the other, it creates an asymmetry we can exploit to figure out whether to connect e to x or y. First notice this asymmetry is noticeable during the reconstruction (we simply notice the existence of an edge different from e, e_1, e_2 that is adjacent to x or y but not the other). We now do a **second reconstruction** where instead of removing e (through its weight w_e) we remove w_1, the other weight that is part of the cycle subgraph. It then becomes apparent whether e is connected to x or y. To illustrate this we represent the situation where this protruding edge is connected to the end of e, and e overlaps e_2:

Actual (sub)graph





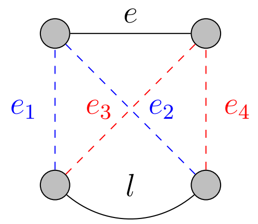
First reconstruction



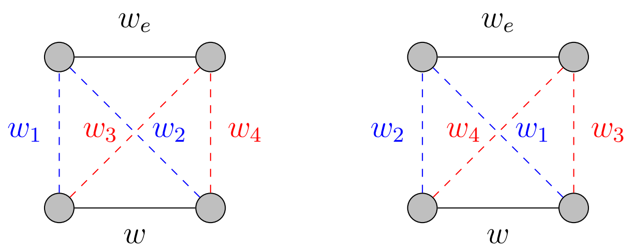
Second reconstruction

In the second reconstruction it becomes apparent that e (through its weight) was connected to x, while it wasn't in the first reconstruction. Using this reconstruction we can therefore verify if e is connected to the vertex from which there is a protruding edge. We can therefore find on which side e was connected and with which edge it overlaps (here, e_2). Knowing that the weight w_e overlaps with weight w_2, we know on which edge we should put w_2 (here, e_2). This solves our indeterminacy. We can therefore reconstruct X' when there is one indeterminacy.

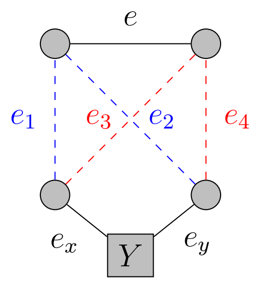
Finally, discuss how to reconstruct X' when we have 2 indeterminacies. The situation becomes somewhat simpler. Indeed, with 2 indeterminacies, we already know where to glue back our edge e (we connect the two degree 2 vertices), all we need to do is find where to put back our weights. Once one weight pair is solved, the other comes automatically: we essentially come back to a particular case where we only have one indeterminacy. As before, some situations are easy to solve, the "real" indeterminate situation is:



We suppose every edge of the maximal cyclic subgraph leads to 2 indeterminacies upon removal. If not, just choose one that gives a lower amount of indeterminacies and refer to our previous reconstructions. If each give 2 indeterminacies, we deduce every vertex of this cycle subgraph is of degree exactly three in X'. We therefore have an edge e_x adjacent to x and another e_y adjacent to y, both different from e, e_1, e_2, e_3, e_4. If $e_x = e_y$, $L = 4$ and it doesn't matter how we assign weights between e_1 and e_2 (which then enables us to find the weights of e_3 and e_4), as we get isomorphic weighted graphs. Indeed, suppose the weights on the maximal cyclic subgraph are w_e, w, w_1, w_4, and the indeterminacies are between the pairs w_1, w_2 and w_3, w_4, then there is no difference between those two weighted graphs:



Now as before, suppose e_x is sticking out. Then so is e_y (vertices of the cyclic subgraph need to be of degree exactly 3). We have:



Where Y is the connected component attached to both e_x and e_y (same connected component as they are both part of the same maximal cycle subgraph). This connected component has non-trivial zeta function since X' is md3. We can put the weight of e_y to

0 then check if the the triangle made by the weight w_e, w_1, w_3 is in the same connected component as Y in the graph induced by $\{w_e, w_1, w_3\} \cup \{w_{e_x}\}$ and the edges of Y, if so, the triangle induced by $\{w_e, w_1, w_3\}$ is connected to x: it corresponded to edges e, e_1, e_3. Doing the same argument by exchanging the roles of x and y, we can deduce to which vertex w_1 and w_3 where connected: this solves our indeterminacies.

We are therefore able to reconstruct X' in all situations. Putting back the loops we've removed at the beginning and replacing edges by the lines and closed lines they represented, we get our initial graph X, with weights put back up to permutation along these lines and closed lines. $\square$

We see that the edge zeta is a total invariant of graphs and a very good invariant of weighted graphs (we only fail to locate weights along lines and closed lines). There is another zeta function of graphs called the path zeta that specializes to the edge zeta: we can therefore recover at least as much information on weighted graphs from it. We won't go into further details however as it goes beyond the scope of this thesis.

Chapter 2

Ihara zeta function

2.1 Vertex zeta function

The rest of our dissertation won't actually be on the edge zeta function, but on one of its specialization: the case when all non-zero weights are set to be equal to a complex value u. We will call this function the Ihara zeta function.

Definition 2.1. *The **Ihara zeta function** of a graph X is given by:*

$$\zeta_V(u, X) = \prod_{[P]} \left(1 - u^{l(P)}\right)^{-1}$$

Where $l(P)$ is the length of P. This expression converges when u is small.

We've seen in Theorem 1.12 that we have a concise formula for the edge zeta of a graph. It can therefore be applied to the Ihara zeta function, and we will see how doing so leads to a determinant formula in terms of the adjacency matrix and the degree of the nodes in the graph. This description in term of properties of vertices (their adjacency and their degrees) justifies an alternative name sometimes used for the Ihara zeta function: the **vertex zeta function**.

We introduce the concept of adjacency matrix.

Definition 2.2. *Let X be an unoriented graph with n vertices, then the $n \times n$ **adjacency matrix** is given by:*

$$A_{ij} = \begin{cases} \# \left\{ e \in \vec{E}(X) \text{ such that } v_i = i(e) \text{ and } v_j = t(e) \right\} \\ 0 \text{ otherwise.} \end{cases}$$

We say two nodes are adjacent if they form the initial and terminal nodes of an edge.

We do still use the convention that edges in unoriented graphs are made with two opposing oriented edges. Remember the **degree** of a vertex is simply the number of edges starting with that vertex. We could also have defined it has the number of edges ending with that vertex: there is no distinction in unoriented graphs. Putting those degrees into a diagonal matrix gives us the degree matrix:

Definition 2.3. *Let X be an unoriented graph with n vertices, then the $n \times n$* ***degree matrix*** *is given by:*

$$D = \operatorname{diag}(d_1, \ldots, d_n)$$

Where d_i is the degree of vertex v_i.

We will give a version of Bass's proof of the Ihara determinant formula, found in [1]. To do so we'll introduce a few matrices and justify some properties regarding their multiplication:

Definition 2.4. *Let X be an unoriented graph with n nodes and $2m$ oriented edges, then the $n \times 2m$* ***start matrix*** *T and* ***terminal matrix*** *T are given by:*

$$S_{ij} = \begin{cases} 1 \text{ if } v_i = i(e_j) \\ 0 \text{ otherwise.} \end{cases} \quad T_{ij} = \begin{cases} 1 \text{ if } v_i = t(e_j) \\ 0 \text{ otherwise.} \end{cases}$$

Now let $J = \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}$ with I_m the $m \times m$ identity matrix, then:

Lemma 2.5. *We have the following identities between the starting matrix S, terminal matrix T, adjacency matrix A, degree matrix D and matrix J as defined previously:*

1. *$SJ = T$ and $TJ = S$*
2. *$ST^t = A$ and $D = SS^t = TT^t$*
3. *$W_1 + J = T^tS$*

Where the matrix W_1 is linked to the edge matrix by $W = uW_1$. The matrices S^t and T^t correspond to the transpose of S and T respectively.

Proof. For the first set of equalities we have:

$$SJ = S \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix} \text{ and } TJ = T \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}$$

Suppose $S = \begin{pmatrix} S_1 & S_2 \end{pmatrix}$ and $T = \begin{pmatrix} T_1 & T_2 \end{pmatrix}$ with S_i, T_j $n \times m$ matrices corresponding to both halves of S and T. We have:

$$SJ = S \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix} = \begin{pmatrix} S_1 & S_2 \end{pmatrix} \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix} = \begin{pmatrix} S_2 & S_1 \end{pmatrix}$$
$$TJ = T \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix} = \begin{pmatrix} T_1 & T_2 \end{pmatrix} \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix} = \begin{pmatrix} T_2 & T_1 \end{pmatrix}$$

Effectively swapping both halves of S and T. Now, if you are the initial vertex of e_i, you are the terminal vertex of $e_i^{-1} = e_{i+m}$ if $i \leq m$ or $e_i^{-1} = e_{i-m}$ if $i > m$, due to the way we've indexed our edges. This means that having a non-zero value in S_1 for example, translates to having a non zero value at the same place in T_2: we have $T_2 = S_1$, and by the same argument $S_2 = T_1$. This proves the first two sets of equalities.
We now move on to compute:

$$(ST^t)_{ij} = \sum_{k=1}^{2m} S_{ik} T_{jk}$$

The terms of this sum are different from 0 whenever we find an edge e_k such that $v_i = i(e_k)$ and $v_j = t(e_k)$, in which case the corresponding term contributes 1 to the sum. This is essentially equivalent to counting the number of edges of that type, as in our definition of the adjacency matrix. We conclude $ST^t = A$.
Moreover, we have:

$$(SS^t)_{ij} = \sum_{k=1}^{2m} S_{ik} S_{jk}$$

Here, we are counting the number of edges e_k such that $i(e_k) = v_i = v_j$. Naturally the second equality in this chain forces us to have a diagonal matrix: we are essentially counting the number of edges starting with v_i, which was our definition of degree of a vertex. We conclude $SS^t = D$. The same can be done for TT^t, by remembering we could have defined the degree of a vertex in terms of the number of edges ending with that vertex: as we are in an unoriented graph, there is no distinction between the two (in oriented graphs however, there is a need to define the in-degree and out-degree of a vertex).
We will now proof the last set of equalities, linking these matrices to W, or rather W_1, the edge matrix when all weights are put to 1 (instead of a complex variable u). We have:

$$(T^t S)_{ij} = \sum_{k=1}^{n} T_{ki} S_{kj}$$

Notice $T_{ki} S_{kj}$ is different than 0 if edge e_i ends with vertex v_k and edge e_j starts with vertex v_k. It is therefore different than 0 if e_i feeds into e_j. In that case, the sum is equal to 1 as only one node can serve as terminal node for e_i and initial node for e_j. This gives us the W_1 matrix with a few extra non-zero entries: we don't disregard edges that are inverses of each other as we did in Definition 1.7. We therefore get the equality:

$$T^t S = W_1 + J$$

Where the identity blocks of J correspond to edges inverses to one another and therefore also feeding into each other. $\square$

We now have all the tools to prove an important determinant formula regarding the Ihara zeta function. It was first proved by Hashimoto and Bass, we will give a version of Bass's proof, found in [1].

Theorem 2.6 (Determinant formula for the vertex zeta). *Let X be an unoriented graph with n nodes and m edges, with adjacency matrix A and degree matrix D. Then:*

$$\zeta_V(u, X)^{-1} = (1-u^2)^{m-n} \det(I_n - Au + Qu^2)$$

Where $Q = D - I_n$.

Proof. Consider the following product of $(n+2m) \times (n+2m)$ matrices, we have:

$$\begin{pmatrix} I_n & 0 \\ T^t & I_{2m} \end{pmatrix} \begin{pmatrix} I_n(1-u^2) & Su \\ 0 & I_{2m} - W_1 u \end{pmatrix} = \begin{pmatrix} I_n(1-u^2) & Su \\ T^t(1-u^2) & T^t S u + I_{2m} - W_1 u \end{pmatrix}$$

Which gives us:

$$\begin{pmatrix} I_n & 0 \\ T^t & I_{2m} \end{pmatrix} \begin{pmatrix} I_n(1-u^2) & Su \\ 0 & I_{2m} - W_1 u \end{pmatrix} = \begin{pmatrix} I_n(1-u^2) & Su \\ T^t(1-u^2) & I_{2m} + Ju \end{pmatrix} \tag{2.1}$$

Now, consider this product:

$$\begin{pmatrix} I_n - Au + Qu^2 & Su \\ 0 & I_{2n} + Ju \end{pmatrix} \begin{pmatrix} I_n & 0 \\ T^t - S^t u & I_{2m} \end{pmatrix} = \begin{pmatrix} I_n - Au + Qu^2 + ST^t u - SS^t u^2 & Su \\ T^t - S^t u + JT^t u - JS^t u^2 & I_{2m} + Ju \end{pmatrix}$$

Which further gives us:

$$\begin{aligned} \begin{pmatrix} I_n - Au + Qu^2 & Su \\ 0 & I_{2n} + Ju \end{pmatrix} \begin{pmatrix} I_n & 0 \\ T^t - S^t u & I_{2m} \end{pmatrix} &= \begin{pmatrix} I_n - Au + Qu^2 + Au - Du^2 & Su \\ T^t - S^t u + J^t u - T^t u^2 & I_{2m} + Ju \end{pmatrix} \\ &= \begin{pmatrix} I_n(1-u^2) & Su \\ T^t(1-u^2) & I_{2m} + Ju \end{pmatrix} \end{aligned}$$

Combining this with Eq. (2.1) we get:

$$\begin{pmatrix} I_n & 0 \\ T^t & I_{2m} \end{pmatrix} \begin{pmatrix} I_n(1-u^2) & Su \\ 0 & I_{2m} - W_1 u \end{pmatrix} = \begin{pmatrix} I_n - Au + Qu^2 & Su \\ 0 & I_{2n} + Ju \end{pmatrix} \begin{pmatrix} I_n & 0 \\ T^t - S^t u & I_{2m} \end{pmatrix}$$

Taking determinants we have:

$$1 \times (1-u^2)^n \det(I_{2m} - W_1 u) = \det(I_n - Au + Qu^2) \det(I_{2n} + Ju) \times 1 \tag{2.2}$$

Which is very close to what we want. Now to compute $\det(I_{2n} + Ju)$ notice:

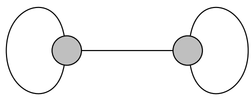
$$\begin{pmatrix} I_{2n} & 0 \\ -I_{2n}u & I_{2n} \end{pmatrix} (I_{2m} + Ju) = \begin{pmatrix} I_{2n} & 0 \\ -I_{2n}u & I_{2n} \end{pmatrix} \begin{pmatrix} I_m & I_m u \\ I_m u & I_m \end{pmatrix} = \begin{pmatrix} I_m & I_m u \\ 0 & I_m(1-u^2)I \end{pmatrix}$$

And so $\det(I_{2m} + Ju) = (1-u^2)^m$. Plugging this into Eq. (2.2) we finally have:

$$\det(I_{2m} - W_1 u) = (1-u^2)^{m-n} \det(I_n - Au + Qu^2)$$

We conclude by noticing that $\zeta_V(u, X) = \zeta_E(W_1 u, X)$ and that the left hand side of the equation corresponds to $\zeta_V(u, X)^{-1}$ by Theorem 1.12. □

This is a handy formula that enables us to quite easily compute the Ihara zeta function of a graph, in no small part due to the fact the dimension of the matrices appearing in the formula is relatively low compared to those of the edge zeta formula. As an example take the dumbbell graph we've seen before:



Our formula tells us that its vertex zeta is given by:

$$\begin{aligned} \zeta_V(u, X)^{-1} &= (1-u^2)^{3-2} \det \begin{pmatrix} 1 - 2u + (3-1)u^2 & -u \\ -u & 1 - 2u + (3-1)u^2 \end{pmatrix} \\ &= (1-u^2)(4u^4 - 8u^3 + 7u^2 - 4u + 1) \end{aligned}$$

Which is a degree 6 polynomial, twice the number of edges in our graph. This isn't a coincidence, in fact as long as the graph we study is md2, the degree of $\zeta_V(u, X)^{-1}$ will be equal to $2m$. This is a consequence of the following property, found independently in [5] and [7]:

Proposition 2.7. *Let X be a graph with $|E(X)| = m$ and $V(X) = n$ then the degree of $\zeta_V(u, X)^{-1}$ is less than or equal to $2m$. Moreover, the coefficient of u^{2m} is:*

$$c_{2m} = (-1)^{m-n} \prod_{i=1}^{n} (d_i - 1)$$

Where d_i is the degree of vertex v_i. In particular $c_{2m} \neq 0$ if the graph is md2.

Proof. By Theorem 2.6, the vertex zeta follows the formula:

$$\zeta_V(u,X)^{-1} = (1-u^2)^{m-n}\det(I - Au + Qu^2)$$

The highest degree coefficient in the determinant is therefore produced by the product of the u^2 on the diagonal of $I - Au + Qu^2$. This means:

$$\zeta_V(u,X)^{-1} = (1-u^2)^{m-n}\left(1+\ldots+\left(\prod_{i=1}^{n} Q_{ii}\right)u^2\right)$$

The highest power of u can then be obtained:

$$\zeta_V(u,X)^{-1} = 1+\ldots+\left((-1)^{m-n}\prod_{i=1}^{n}(d_i-1)\right)u^{2m-2n}u^{2n}$$

We deduce the degree of $\zeta_V(u,X)^{-1}$ doesn't exceed $2m$, and c_{2m} can be written as stated in the proposition. $\square$

A full discussion of the coefficients of $\zeta_V(u,X)^{-1}$ can be found in [8]. We can state a rather direct consequence of the previous proposition:

Corollary 2.8. *Let X be an md2 graph with m edges, then:*

$$m = \frac{\deg(\zeta_V(u,X)^{-1})}{2}$$

It turns out we can also deduce the number of vertices of an md2 graph when it is connected, by looking at the multiplicity of 1 as a root of $\zeta_V(u,X)^{-1}$. We will need to use a result found in [9], as suggested in [5].

Proposition 2.9. *Let X be a connected graph and k the multiplicity of 1 as a root of $\zeta_V(u,X)^{-1}$ then:*

$$|V| = \begin{cases} n & \textit{if } \zeta_V(u,X)^{-1} = (1-u^n)^2 \\ m-k+1 & \textit{otherwise} \end{cases}$$

With m being the number of edges of X.

Proof. By the determinant formula of Theorem 2.6 we have:

$$\zeta_V(u,X)^{-1} = (1-u^2)^{m-n}\det(I_n - Au + Qu^2)$$

What we want to do is find the multiplicity of 1 as a root of $\det(I_n - Au + Qu^2)$. If our graph isn't cyclic, theorem 5.26 of [9] essentially tells us that it is a simple root of that determinant and therefore:

$$k = m - n + 1$$

We deduce that $n = m - k + 1$. If X is md2 we can even compute this directly from the vertex zeta:

$$n = \frac{\deg(\zeta_V(u, X)^{-1})}{2} - k + 1$$

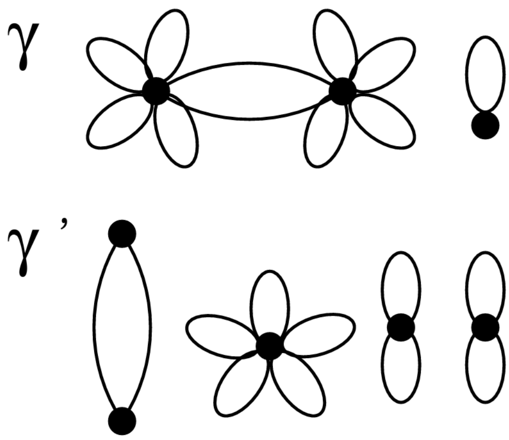
Now, suppose the graph is cyclic. We know therefore that its zeta function is of the form:

$$\zeta_V(u, X)^{-1} = (1 - u^n)^2$$

Where n is the number of vertices. It turns out that cycle graphs are the only ones with these types of edge zeta. Indeed, by Proposition 2.7 we know that an md2 graph associated to such a zeta has vertices of degree 2 (since $c_{2m} = 1$). It is therefore made of isolated cycles. It is then easy to see that our zeta function stems from one cycle of length n. It therefore has n vertices. $\square$

We are therefore able to find the number of edges from the vertex zeta function of an md2 graph. Moreover, if this graph is connected, we are also able to deduce its number of vertices.

The number of vertices cannot usually be retrieved using the vertex zeta, Czarneski gives the following example of two non-isomorphic graphs with the same zeta function ([7]):



Not only do we overlook the number of vertices, in most cases we are unable to find the number of connected components using the vertex zeta. We'll continue our exploration of the zeta function in the next two sections to retrieve more intricate properties of our graph.

2.2 Formal power series of the vertex zeta

We'll study the coefficients of the power series of $\zeta_V(u, X)$ and see how we can use them to better understand the structure of X.

Proposition 2.10. *Let $(a_n)_{n\in\mathbb{N}}$ be the coefficients of the power series of $\zeta_V(u, X)$ such that:*

$$\zeta_V(u, X) = \sum_{n=0}^{\infty} a_n u^n$$

Then $a_0 = 1$ and for $n > 0$:

$$a_n = \left| \left\{ (k_i)_{i\in\mathbb{N}} \left| \sum_{i=0}^{\infty} k_i l([P_i]) = n \text{ with } k_i \in \mathbb{N} \right.\right\} \right|$$

Where P_i is a representative of a prime cycle, with every prime cycle represented once.

The indexation of prime cycles can be done because they are countable: the set of prime cycles is the union of countably many countable sets (take the union of the sets of prime cycles of all lengths). We'll proof our proposition:

Proof. While the coefficients formula is quite hard to write down, it comes naturally when we write down the vertex zeta function and use the power series of $(1-x)^{-1}$:

$$\zeta_V(u, X) = \prod_{[P]} \left(1 - u^{l(P)}\right)^{-1} = \prod_{[P]} (1 + u^{l(P)} + u^{2l(P)} + \dots)$$

We deduce that $c_0 = 1$, the other coefficients a_n are simply equal to the number of ways we can get n as a sum of $k_i l([P_i])$. $\square$

We'll now use this to detect bipartite graphs:

Definition 2.11. *Let X be a graph and suppose there exists a partition of $V(X)$ into two sets V_1 and V_2 such that edges of $\vec{E}(X)$ only travel between those set. Then we say X is **bipartite**.*

There is a link between being bipartite and cycles, indeed:

Proposition 2.12. *Let X be a graph then:*

$$X \text{ is bipartite} \Leftrightarrow X \text{ does not contain an odd cycle.}$$

Proof. Suppose X is bipartite, partitioned into V_1 and V_2. If there exists a cycle $c = e_0 e_1 \dots e_n$ starting, without loss of generality, in V_1, then whenever an edge e_i leaves V_1 to go to V_2, e_{i+1} starts from V_2 and goes back to V_1. The cycle c therefore alternates between leaving V_1 and coming back to V_1. Since it is a cycle, we end up at the starting vertex: there are as many edges coming back to V_1 as leaving V_1. The length of c is therefore even.

Suppose now that X does not contain an odd cycle. We define the distance between two vertices in a same connected component as the length of the shortest path between the two. Take a vertex v in our graph and look at the set of vertices V_1 at an odd distance to v and the set V_2 of vertices at an even distance to v. We observe vertices in a V_i aren't adjacent to each other: if two of those are adjacent it is quite easy to create an odd cycle. Indeed, if two such vertices are adjacent, take the shortest path from v to the first vertex, travel one edge to the other vertex, then take the shortest path back to v: the resulting cycle will be odd. We deduce that all oriented edges starting in V_1 go to V_2, and vice versa: the connected component v is part of is bipartite, partitioned into V_1 and V_2. We deduce that every connected component of X is bipartite, it is quite easy to see that this implies that X is bipartite. $\square$

This gives us strong hopes that the Ihare zeta function still captures the bipartite property of md2 graphs as it essentially encodes information regarding (prime) cycles. Indeed:

Proposition 2.13. *Let X be an md2 graph.*

$$X \text{ is bipartite} \Leftrightarrow \text{The coefficients } a_n \text{ are zero for odd values of } n.$$

Proof. If X is bipartite, every cycle is of even length, as well as prime cycles. Linear combinations of $l([P_i])$ are therefore also even: $a_n = 0$ for odd values of n.

Suppose now X is not bipartite: by our previous proposition there exists an odd cycle $C = e_0 \dots e_n$ in X. Our goal is to find an odd prime cycle from it. We first take care of backtracks by going through each edge one by one: if $e_{i+1} = e_i^{-1}$ remove both e_i and e_{i+1}. The resulting sequence is a cycle as $i(e_i) = t(e_{i+1})$ (we don't break the path), and is of odd length as the number of edges removed is even. It has the advantage of being backtrackless.

Now remove tails from this new cycle $f_0 \dots f_n$: if $f_n = f_0^{-1}$ then remove both f_0 and f_n to get a new cycle (indeed, $i(f_1) = t(f_0) = i(f_n) = t(f_{n-1})$). Repeat the operation as much as necessary to remove tails. This operation also doesn't change the parity of the cycle length as we remove an even number of edges. We get a proper cycle of odd length.

Finally, if the cycle isn't prime it means it is the multiple of another cycle: simply find the edges being repeated in its sequence to get a prime cycle. It is of odd length since its length has to divide the length of the odd proper cycle.

This means that if X is not bipartite we are able to find a prime cycle of odd length l: $a_l > 0$ and a_n is not zero for all odd values of n. $\square$

Corollary 2.14. *Let X be a graph. Then:*

$$\zeta_V(u,X)^{-1} = P(u^2) \text{ with } P \in \mathbb{C}[X] \Leftrightarrow X \text{ is bipartite.}$$

Proof. First of all the independent term of P is equal to 1 (see proof of Proposition 2.7), which is invertible in $\mathbb{C}$: P is therefore invertible in the ring of formal power series over $\mathbb{C}$. We therefore have a power series for P^{-1}. If $\zeta_V(u,X)^{-1} = P(u^2)$, the power series of $\zeta_V(u,X)^{-1}$ is the power series of $1/P(u^2)$: all powers of u in this power series are even and the graph is therefore bipartite.

If X is bipartite, then the length of all prime cycles is even and since we have:

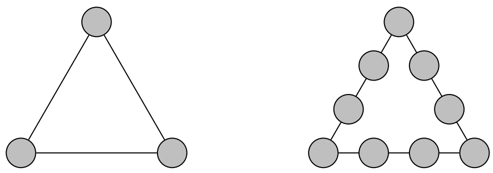
$$\zeta_V(u,X)^{-1} = \prod_{[P]} \left(1 - u^{l(P)}\right)$$

The powers of u in the right hand side of the equation will all be even: therefore $\zeta_V(u,X)^{-1} = P(u^2)$. $\square$

More generally, we might be wondering what we can deduce from X if $\zeta_V(u,X)^{-1} = P(u^k)$ for some $k \in \mathbb{N}$. It turns out we can quite easily construct graphs with those types of zeta functions if we **inflate** a graph:

Definition 2.15. *Let X be a graph, and $L \in \mathbb{N}$. Then the inflated graph X_L is obtained by replacing each edge of X with a line of length L.*

We can for example inflate a triangle with $L = 3$ to get:



Doing such an inflation multiplies the length of each cycle by L: we have $\zeta_V(u,X_L) = \zeta_V(u^L,X)$. Therefore $\zeta_V(u,X_L)^{-1} = P(u^L)$. It turns out this is the only way of getting such a vertex zeta!

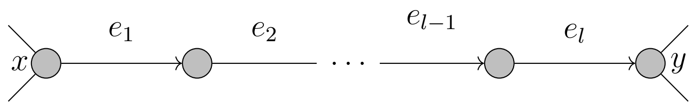
Proposition 2.16. *Let X be an md2 non-bipartite graph let and $L \in \mathbb{N}$ then:*

$$\zeta_V(u,X)^{-1} = P(u^L) \text{ with } P \in \mathbb{C}[X] \Leftrightarrow X \text{ is an inflated graph.}$$

Proof. Suppose $\zeta_V(u,X)^{-1} = P(u^L)$ then the coefficients of the power series of $\zeta_V(u,X)$ are non-zero only for degrees that are multiples of L. We deduce that the lengths of prime cycles in our graph are multiples of L. Indeed, were that not the case, then a prime cycle of length l such that $L \nmid l$ would exist and we would have $a_l > 0$ (which contradicts the assumption that $\zeta_V(u,X)^{-1} = P(u^L)$).

What we need to show is that the length of every line and closed line is a multiple of L. If that is the case, we can replace lines/closed lines of length l with l/L edges: our initial graph is the inflated version of this reduced graphs.

Suppose we have a line $e_1 \dots e_l$. Removing the unoriented edges associated to that line keeps the graph $md2$ since the degrees of the border vertices are greater or equal to 3:



These border vertices x and y are therefore still part of prime cycles after removing the unoriented edges associated to the line $e_1 \dots e_l$.

Take one such prime cycle starting at x and call it P_x, take another starting at y and call it P_y. Then the path $P_x e_1 \dots e_l P_y e_l^{-1} \dots e_1^{-1}$ is a prime cycle. Its length is therefore a multiple of L:

$$l(P_x) + l + l(P_y) + l \equiv 0 \mod L$$

And since the lengths of P_x and P_y are both multiples of L we deduce:

$$2l \equiv 0 \mod L$$

Now, we assume X is not bipartite, therefore by Corollary 2.14, L must odd. We therefore deduce that:

$$l \equiv 0 \mod L$$

Which ends our proof: the length of lines and closed lines are multiples of L. $\square$

In conclusion, if the degrees of u in $\zeta_V(u,X_L)^{-1}$ are all even, the graph is bipartite, if they are not all even but are multiples of a certain L, the graph is inflated.

2.3 Regular graphs

There is an interesting link between the Ihara zeta function and the eigenvalues of its adjacency matrix (its spectrum) when each of its connected components are regular. In this section we explore this link and see what additional information we can get in that case.

Definition 2.17. *Let X be a graph. It is d-regular if each of its vertices is of degree d.*

We first show that we are able to detect regularity, provided we know the number of vertices [10]:

Proposition 2.18. *Let X be a md2 graph with $m = |E|$, and suppose $n = |V|$ is known. We are able to deduce whether or not it is a regular graph, and if so, compute the degree of its vertices.*

Proof. Remember we have Theorem 2.6 telling us that:

$$\zeta_V(u, X)^{-1} = (1 - u^2)^{m-n} \det(I_n - Au + Qu^2)$$

We will focus on the elements on the diagonal of Q. Now, the arithmetic mean of these elements is simply:

$$S = \frac{\sum_{i=1}^{n} q_i}{n} = \frac{\sum_{i=1}^{n}(d_i - 1)}{n} = \frac{2m - n}{n}$$

And its geometric mean can be compute using Proposition 2.7:

$$G = \sqrt[n]{\prod_{i=1}^{n} q_i} = \sqrt[n]{\prod_{i=1}^{n}(d_i - 1)} = \sqrt[n]{|c_{2m}|}$$

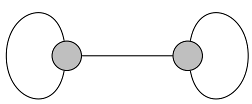
By the inequality of arithmetic and geometric means, those two are equal if and only if each of the terms q_i are equal to each other i.e if the graph is regular. This can be checked by computing the relevant quantities.

The vertex degrees are then obtained by computing:

$$d = \frac{dn}{n} = \frac{2m}{n}$$

This concludes the proof. $\square$

As an example take the dumbbell graph from before:



$$\zeta_V(u, X)^{-1} = (1 - u^2)(4u^4 - 8u^3 + 7u^2 - 4u + 1)$$

The degree of its zeta function is 6: we deduce our graph has $6/2 = 3$ edges. The root 1 has an algebraic multiplicity of 2, and since our graph is connected we can deduce that the number of nodes is $n = 3 - 2 + 1 = 2$. We compute $S = 2m/n - 1 = 3 - 1 = 2$ and $G = \sqrt{|c_{2m}|} = \sqrt{|-4|} = 2$. Since $S = G$ we conclude our graph is regular, of vertex degree $2m/n = 6/2 = 3$.

We are therefore able to detect regularity of md2 graphs for which we know the number of vertices! The number of vertices can be deduced when the graph is connected (see Proposition 2.9).

It turns out that we can exploit this regularity property to get more information out of our vertex zeta. We'll get to it in time but first, notice the following:

Proposition 2.19. *Let X be a graph, A its adjacency matrix and D its degree matrix. Then:*

$$A \text{ and } Q = D - I \text{ commute} \Leftrightarrow \text{The connected components of } X \text{ are regular.}$$

Proof. Order the connected components and index your vertices according to that ordering so that the adjacency matrix is a block diagonal matrix:

$$A = \begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_k \end{pmatrix}$$

If the connected components of X are regular with respective number of vertices n_i then the Q matrix is of the form:

$$Q = \begin{pmatrix} (d_1 - 1)I_{n_1} & 0 & \dots & 0 \\ 0 & (d_2 - 1)I_{n_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (d_k - 1)I_{n_k} \end{pmatrix}$$

We can see that A and Q commute.

Now suppose A and Q commute. We can compute:

$$(AQ)_{ij} = \sum_{k=1}^{n} A_{ik}Q_{kj} = A_{ij}(d_j - 1)$$

As well as:

$$(QA)_{ij} = \sum_{k=1}^{n} QikA_{kj} = A_{ij}(d_i - 1)$$

Since A and Q commute, those two quantities are the same. If vertex i is adjacent to vertex j, we have $A_{ij} \neq 0$ and deduce $d_j - 1 = d_i - 1$ from the two previous equations. This implies $d_i = d_j$: we've just shown that vertex i and vertex j are of the same degree if they are adjacent. Expanding this to the whole connected component, we deduce it is regular. □

Why does the commutativity of A and Q interest us? Because in that case A and Q are simultaneously diagonalizable (in fact, two diagonalizable matrices are simultaneously diagonalizable if and only if they commute), which enables us to obtain the following result:

Proposition 2.20. *Let X be a graph with k connected components. If those connected components are regular of respective vertex degrees d_i:*

$$\zeta_V(u, X)^{-1} = (1 - u^2)^{m-n} \prod_{i=1}^{k} \prod_{\lambda \in spec(A_i)} (1 - \lambda u + q_i u^2)$$

Where $spec(A)$ is the spectrum of A, the multiset containing its eigenvalues, and where $q_i = d_i - 1$

Proof. If the connected components of X are regular, A and Q commute, by Proposition 2.19. More specifically, A_i commutes with $(d_i - 1)I_{n_i}$: they are therefore simultaneously diagonalizable (adjacency matrices of unoriented graphs are symmetric and therefore diagonalizable). Collecting those simultaneous eigenvectors for each values of i, we are able to construct simultaneous eigenvectors of A and Q (fill the remaining components with 0). If s is such an eigenvector, obtained from the i-th connected component and of eigenvalue λ, we can compute:

$$(I - Au + Qu^2)s = (1 - \lambda u + q_i u^2)s$$

It is an eigenvector for the matrix $I - Au + Qu^2$ of eigenvalue $1 - \lambda u + q_i u^2$. The determinant of $I - Au + Qu^2$ is the product of all these eigenvalues, we therefore deduce:

$$\begin{aligned}\zeta_V(u, X)^{-1} &= (1 - u^2)^{m-n} \det(I_n - Au + Qu^2) \\ &= (1 - u^2)^{m-n} \prod_{i=1}^{k} \prod_{\lambda \in spec(A_i)} (1 - \lambda u + q_i u^2)\end{aligned}$$

Which is what we wanted. □

Corollary 2.21. *Let X be a $(q+1)$-regular graph, then:*

$$\zeta_V(u, X)^{-1} = (1 - u^2)^{m-n} \prod_{\lambda \in spec(A)} (1 - \lambda u + qu^2)$$

Now, it turns out that we can retrieve all of the spectral information (information regarding the eigenvalues of the adjacency matrix) of an md2 regular graph from its vertex zeta and the number of vertices, an observation attributed to A. Mellein [11]:

Proposition 2.22. *Let X be an md2 regular graph with $m = |E|$ and $n = |V|$ known. Knowing its vertex zeta is equivalent to knowing its spectra.*

Proof. If we have the vertex zeta of X, knowing the values of m and n enables us to isolate:

$$\det(I_n - Au + Qu^2) = \prod_{\lambda \in spec(A)} (1 - \lambda u + qu^2)$$

We have n complex roots for this determinant (with multiplicity). Take one and denote it x. Our goal is to find the value of λ associated to x. We first find the value of $q = d-1$. The degree d corresponds to the sum of degrees divided by n, which we can compute as such:

$$d = \frac{2m}{n}$$

This then enables us to compute q. We therefore have a complex number x root of:

$$\frac{1}{q} - \frac{\lambda}{q}u + u^2$$

If we denote x' the other root of that polynomial we can solve:

$$\begin{cases} xx' = \frac{1}{q} \\ x + x' = \frac{\lambda}{q} \end{cases}$$

Which gives us $\lambda = qx + 1/x$ and $x' = 1/qx$. Knowing the roots of $\prod_{\lambda \in spec(A)}(1 - \lambda u + qu^2)$ then, enables us to find the root pairs as well as values of λ. We are therefore able to find the spectrum of X.

Suppose now we have the spectrum of X as well as the values of m and n: we find $q = 2m/n - 1$ and compute the vertex zeta using Corollary 2.21. □

Knowing the number of vertices as well as the vertex zeta of an md2 regular graph therefore allows us to introduce all of the machinery of spectral graph theory to get back information on X. We are for example able to tell the number of connected components of X: it corresponds to the multiplicity of $q+1$ as an eigenvalue of A.

A more advanced result links Ramanujan graphs (a regular graph with certain conditions on its eigenvalues) to the equivalent of the Riemann hypothesis for the vertex zeta. It turns out those two conditions are equivalent: being Ramanujan is equivalent to satisfying this Riemann hypothesis. See [1] for more information.

2.4 Products of graphs

We end this chapter by applying what we know to products of unoriented simple graphs. We'll explore two types of graph products: the tensor product and the cartesian product.

Definition 2.23. *A graph is said to be* ***simple*** *if there is no more than one edge connecting two adjacent vertices and if there are no loops.*

We first treat the tensor product of two matrices:

Definition 2.24. *Let X and Y be simple graphs. The* ***tensor product*** *$X \times Y$ is an unoriented graph with set of vertices $V = V(X) \times V(Y)$ such that:*

$$(a_1, b_1) \text{ adjacent to } (a_2, b_2) \Leftrightarrow a_1 \text{ adjacent to } a_2 \text{ and } b_1 \text{ adjacent to } b_2$$

We can quite easily compute a few important quantities for the tensor product of two graphs. Let n_X, n_Y be the number of vertices of X and Y respectively, and let m_X, m_Y be the number of edges of X and Y respectively. Then:

- $n = |V(X \times Y)| = n_X n_Y$
- $m = 2m_x m_y$ because each pair in $E(X) \times E(Y)$ corresponds to 2 edges of $X \times Y$. If a_1 is adjacent to a_2 and b_1 is adjacent to b_2: (a_1, b_1) adjacent to (a_2, b_2) and (a_1, b_2) adjacent to (a_2, b_1).
- The degree of (a, b) is $d(a)d(b)$

As for the adjacency matrix, we'll need the Kronecker product of two matrices:

Definition 2.25. *Let A and B be two matrices, of dimensions $k \times l$ and $m \times n$ then:*

$$A \otimes B = \begin{pmatrix} A_{11}B & A_{12}B & \dots & A_{1l}B \\ A_{21}B & A_{22}B & \dots & A_{2l}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{k1}B & A_{k2} & \dots & A_{kl}B \end{pmatrix}$$

Which is a $km \times ln$ matrix.

We can order the elements of $V(X) \times V(Y)$ using the index of the first component then the second, meaning $(a_i, b_j) < (a_k, b_l)$ if $i < k$ and $(a_i, b_j) < (a_i, b_l)$ if $j < l$. This ordering gives us an indexation of vertices we'll use to find the adjacency matrix. The adjacency condition of the tensor product enables us to deduce that the adjacency matrix A of the tensor product of X and Y, whose respective adjacency matrices are A_X and A_Y is:

$$A = A_X \otimes A_Y$$

We can use all of this information to compute the zeta function of tensor products of two graphs.

It is usually complicated to find the zeta function of the tensor product using the zeta functions of the two initial graphs. If both of them are regular however:

Proposition 2.26. *Let X and Y be two simple regular graphs with degrees d_X and d_Y:*

$$\zeta_V(u, X \times Y) = (1-u^2)^{2m_X m_Y - n_X n_Y} \prod_{\substack{\lambda_X \in spec(A_X) \\ \lambda_Y \in spec(A_Y)}} (1 - \lambda_X \lambda_Y u + (d_X d_Y - 1)u^2)$$

Where A_X, A_Y are the adjacency matrices, m_X, m_Y the number of edges, n_X, n_Y the number of vertices, of X and Y respectively.

Proof. From the computations we've done previously we deduce that the tensor product of two regular graphs is regular. We can therefore use Corollary 2.21, plugging in the values of m and n in terms of m_X, m_Y, n_X, and n_Y. Finally, the spectrum of the Kronecker product of two matrices corresponds to the products of the eigenvalues of those two matrices. Indeed, if v is an eigenvector of A and w is an eigenvector of B, then:

$$\begin{pmatrix} v_1 w \\ \vdots \\ v_k w \end{pmatrix} \text{ is an eigenvector of } A \otimes B$$

And a simple computation shows that its eigenvalue is the product of the eigenvalues of v and w. Those eigenvectors are also independent (can be shown by using the independence of the eigenvectors of A and the independence of the eigenvectors of B). $\square$

Corollary 2.27. *If X and Y are both regular, simple connected md2 graphs, the zeta function of their tensor product can be computed from their zeta functions.*

Proof. We can find the number of edges of an md2 graph using Proposition 2.7, its number of vertices by using Proposition 2.9 and the fact it is connected, the values of the degree of its vertices from Proposition 2.18, and its spectrum using Proposition 2.22. Do this for both X and Y. We then plug all of those values in the formula we've just found. $\square$

We'll now do the same for the cartesian product:

Definition 2.28. *Let X and Y be simple graphs. The **cartesian product** $X \square Y$ is an unoriented graph with set of vertices $V = V(X) \times V(Y)$ such that:*

$$(a_1,b_1) \textit{ adjacent to } (a_2,b_2) \Leftrightarrow \textit{one of the two: } \begin{cases} a_1 = a_2 \textit{ and } b_1 \textit{ adjacent to } b_2 \\ a_1 \textit{ adjacent to } a_2 \textit{ and } b_1 = b_2 \end{cases}$$

It is also sometimes called the "box product". A typical example would be the cartesian product of two edges, which gives us a square.

As before, we compute:

- $n = |V(X) \times V(Y)| = n_X n_Y$.
- For each vertex in X that stays equal in the adjacency we have n_Y edges in $X\Box Y$. This also holds if we swap X and Y. We deduce $m = m_X n_Y + m_Y n_X$.
- The degree of (a,b) is simply $d(a)+d(b)$ (fix a and move using the edges of b, or fix b and move using the edges of a).

As for the adjacency matrix, the two adjacency conditions translate into:

$$A = 1_{n_X} \otimes A_Y + A_X \otimes 1_{n_Y}$$

We have all the information needed to compute the zeta function of cartesian products. In the case of regular graphs:

Proposition 2.29. *Let X and Y be two simple regular graphs with degrees d_1 and d_2:*

$$\zeta_V(u, X\Box Y) = (1-u^2)^{m_X n_Y + m_Y n_X - n_X n_Y} \prod_{\substack{\lambda_X \in spec(A_X) \\ \lambda_Y \in spec(A_Y)}} (1-(\lambda_X+\lambda_Y)u + (d_1+d_2-1)u^2)$$

Where A_X, A_Y are the adjacency matrices, m_X, m_Y the number of edges, n_X, n_Y the number of vertices, of X and Y respectively.

Proof. As before, we use the result of our computations, which we plug into Corollary 2.21. The eigenvalues of A correspond to the sums of the eigenvalues of A_X and A_Y. This is proved by seeing that if v is an eigenvector of A_X and w is an eigenvector of A_Y:

$$\begin{pmatrix} v_1 w \\ \vdots \\ v_{n_x} \end{pmatrix} \text{ is an eigenvector of } A$$

And a computation shows that its eigenvalue is the sum of the eigenvalues of v and w. These constructed eigenvectors are independent, as explained previously. $\square$

Corollary 2.30. *If X and Y are both regular, simple connected md2 graphs, the zeta function of their cartesian product can be computed from their zeta functions.*

Proof. Same as for Corollary 2.27. $\square$

Conclusion

In conclusion, zeta functions such as the edge zeta function and the vertex zeta function provide different ways of understanding mathematical structures. They store quite a bit of information and often enable us to understand concepts in a new way.

We've seen how properties such as being bipartite, inflated, or regular, are linked to the vertex zeta function. As for the edge zeta, we've seen it holds much more information about a graph's structure than previously anticipated.

We've also explored ways of computing the edge zeta for more complicated graphs created through standard operations.

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UNIVERSITÉ CATHOLIQUE DE LOUVAIN
Faculté des sciences
Place des sciences, 2 bte L6.06.01, 1348 Louvain-la-Neuve, Belgique | **www.uclouvain.be/sc**