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Category Theory and Risk Measure

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Preface.

This master thesis has been completed at the University catholique de Louvain (UCL) in the fulfillment of the grade of Master in Actuarial Sciences. The subject is an application of the category theory in Actuarial Sciences and more specifically in the context of risk measure theory. First, the classical monetary value measure theory is described and related problematics are introduced: Knightian uncertainty and the determination of an adequate set of minimal axioms for such a risk measure. A generalization of this theory is described in the language of categories and shows how to partially answer the two problematics. The last part is dedicated to provide some of the possible concrete applications of this categorical view of probability spaces, conditional expectation and monetary value measure.

Key words: Probability Measure, Risk Measure, Risk and Knightian Uncertainty, Coherent Measure, Category Theory, Functor, Category **Prob**, Conditional Expectation, Monetary Value Measure.

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Finally, I would like to thank the Mathematics which will always bring us to unknown and surprizing places.

Contents

Introduction.

Category theory has been studied for over 70 years. It was originally introduced in the mid-20th century by Samuel Eilenberg and Saunders Mac Lane who were working in algebraic topology. Since the first formal definition of a category in [EML45] and the first published book [ML71] on the theory, new applications have been found everyday. The strength of this theory is the abstraction it gives on mathematical structures and how it explains relations between them.

Due to its abstract point of view, category theory applies in almost all area of Mathematics. One can also find applications in physics such as described in [BA09].

But what may be more surprising is the existence of applications in other areas aside from math. There are applications in biology (e.g. [Tuy18a] and [Tuy18b]), as well as in chemistry, computing and social networks (as partially described in [BCC$^+$22]). There exists also applications in language studies and semantics (see [LPRMRZ99].

Therefore the question if category theory applies in Actuarial Science, statistics, finances,... raises.

Parallelly, risk measures have been studied for many years in Actuarial Sciences. Classical theory has been defined and used in practice by academics, insurance companies and regulators to understand, compute and regulate the risk an insurance company is facing. Tools such as $V@R$(Definition 1.2.2) or $TV@R$(Definition 1.2.3) are widely used and understood nowadays. Nevertheless, the theory is still facing some impediments:

- First, there is the issue of risk versus uncertainty following the work of Knight in [HK21]. Even if a proper risk measure can be defined (and one should define "proper" which is related to the point below), historical events (such as 2008 crisis) and theoritical considerations from the past decades confirm that there is not one objective probability measure.

  On one hand, depending on the beliefs the observer of an event has, he/she may consider different probability measures. For example, if one believes that a coin is loaded on one side, this person will not believe in a 50/50 percentage chance of having tails. On the contrary, someone who believes the coin is fair, may give the usual probability of the heads or tails game. In this case, depending on the beliefs of the person, two probability measures coexist.

  On the other hand, finance practitioners believed for a long time there was only one risk free probability measure. Then, events (such as the Lehman shock) happened and suggested this "uniqueness and objectiviness" assumption of the probability measure was wrong. Indeed, the uniqueness of risk neutral probability measure (also called martingale measure) is a necessary and sufficient condition to have a "complete" market[1]. This last theorem (Theorem 1.40 in [FS11] for more details) implies that if the market is not complete (which is usually the case), several risk free probability measures exist.

To conlude this first point, several probability measures coexist and the choice of one probability measure over another stays a subjective choice. Therefore, even if two experts have the same risk measure, if they choose a different probability measure they will end up with different results and will be both right under their own paradigms. This fact translates the notion of Knightian uncertainty and could be differentiated from the "risk" that can be measured (thanks to risk measure).

- Secondly, even if an abstraction is made of this uncertainty, the definition of a "proper" risk measure (and therefore the choice of the risk measure itself) is based on a subjective choice of axioms. As it is explained in Chapter 1, this notion of "proper" or more accuratly "coherent" risk measure in the classical theory may differ from one author to another. Practitioners (or regulators) choose subjectively a minimal and good set of axioms that their risk measure should fulfill. For example, even though SII regulation[2] is using $V@R$ (which is not subadditive) it would still be perfectly understandable that a risk manager prefers to use $TV@R$ to allow some diversification effect due to its subadditive property.

- Finally, sometimes it may be complicated to properly evaluate the probability measure associated to an event and therefore the associated risk. This is the case for some products in credit insurance where the insurance company does not know beforehand how many transactions will be insured within the total exposure and if the probability of default of those transactions will not change. This is also the case in the modelization of extreme and rare climatic events, such as the flood in Belgium during July 2021. The issue with this kind of events is the high severity for a really low frequency. Estimating the future frequency of those events (and therefore their probability to happen) is a difficult task. The only certainty one may have, is that this frequency will increase in the future, for example due to global warming.

The goal of this master thesis is to study an applications of the category theory on risk measure theory. In order to tackle this recent application of category theory, the framework settled by Adachi (in [Ada14], [AR16], [AR17] and [ANR20]) will be followed[3] and many theoretical consideration will need to be taken and explained. This theoritical analysis will be

[1] A complete market is defined by a an arbitrage free market where every contingent claim is attainable, e.g. where all derivatives admit a perfect hedge (Section 1.4 [FS11]). In theory, we often start by considering complete market but our real world suggests this is rather exceptional.

[2] Solvency II, often abreviated by SII, is a directive that prescribe a harmonised prudential framework for european insurer/reinsurer, more informations are available on the official websites of the European Commission and the European Insurance and Occupational Pensions Authority - EIOPA

[3] Other author such as P.Perrone, T.Fritz, P.Panangaden, E.Patterson or B.Jacobs could have been also mentionned for their contribution in the domain of the application of category theory in probability theory.

the first goal of this paper.

After this introduction, Chapter 1 is going to focus on classical risk measure and try to describe some of the issues explained above. The other purpose of the first chapter will be to set a common ground of notations and state the classical definition of monetary value measure including properties that may or may not follow.

Chapter 2 will be the pivot point of this master paper. The begining of the chapter will be dedicated to the language of the category, and could be almost read by itself as it constitutes a general introduction. Next, the category **Prob** will be introduced where its objects will be given by probability spaces and its arrows will be given by suitable applications. This will allow moves through "time" and "space" within probability spaces. Then the conditional expectation will be seen as a functor between categories. This new definition will constitute a generalization of the classical conditional expectation. Finally, the generalization of the monetary value measure as a functor will also be given. Thanks to all of those definitions and theoritical considerations, some partial answers will be given to the questions raised in the introduction and in Chapter 1

Last but not least, Chapter 3 will mention some possible concrete applications. Of course, this stays theoritical but it may open the door to (hopefully promising) future research. Three possible concrete applications will be discussed: credit insurance, modeling of rare and extreme events and the use of telematic data in car insurance pricing. The example regarding credit insurance will constitute the first trivial one but will be futher extended to consider several transactions. For that example, a numerical algorithm will be described in order to treat the issue related to a "fix" probability measure. The last section of that chapter is the most theoretical one. It introduces a way to see the theory developed above as a possible path to an answer to the other questions about having an adequate set of axioms to define a "good" risk measure but also, how to put some rationality behind such a subjective choice. This will be done considering the monetary value measure as pre-sheave and thanks to the Grothendieck topology as a sheave.

The ultimate hope behind this paper is to give the reader the feeling that taking the time to do some abstract and rigorous Mathematics leads to interesting and useful results in Actuarial Sciences.

Chapter 1

Classical Risk Monetary Measure

In this chapter, the basic definitions used in a classical theory of risk measure are presented. The different notions layed down here can be found in standard reference such as [ADEH99] or [FS11].

A categorical generalization of those notions will be presented in Chapter 2.

1.1 Introduction and General Settings

The general setting of our classical risk measure theory is introduced in this section. All the notions (σ-algebra, measurable function and Borel set) in this section can be considered as common knowledge by actuaries or matematicians. Nevertheless, the reader who might not be familiar with one of the concepts may consult the appendix (cfr. Section 5.1). The goal of this section is only to agree on the basic notations used within this paper.

Our objects (of our category, see Chapter 2) of interest are probablity spaces $\bar{\Omega} = (\Omega, \mathcal{G}, \mathbb{P})$ with:

- a universe/set Ω representing all possible scenarios of an event.
- a σ-algebra $\mathcal{G}$.
- a probability measure $\mathbb{P}$.

Those probability spaces are in the heart of the foundation of probability and risk measure theory. If only $(\Omega, \mathcal{G})$ is considered, the word "measurable space" is sometimes used.

Now, on a given probability space, a random variable is defined and may represent, for example, possible loss related to an underlying event or a possible gain in finance.

Definition 1.1.1 (Random Variable)**.** A *random variable* is a function $X : \Omega \longrightarrow \mathbb{R}$ which is measurable with respect to $\bar{\Omega} = (\Omega, \mathcal{G}, \mathbb{P})$.

In the definition above, the notion of "measurable" can be interpreted as every pre-image of a Borel set is a measurable subset of Ω. In other words, if X is a random variable then for any borel set $B \in \mathcal{B}(\mathbb{R})$

$$X^{-1}(B) = \{\omega \in \Omega \mid X(\omega) \in B\} \in \mathcal{G}$$

So a random variable $X : \Omega \longrightarrow \mathbb{R}$ can be seen as a $\mathcal{G}/\mathcal{B}(\mathbb{R})$-measurable function.

For a σ-algebra $\mathcal{F} \subseteq \mathcal{G}$, the space of all bounded $\mathbb{R}$-valued functions is denoted by $\mathcal{L}^\infty(\Omega, \mathcal{F})$. As usual, $L^\infty(\Omega, \mathcal{F}, \mathbb{P} \mid_\mathcal{F})$ also denotes the quotient space of $\mathcal{L}^\infty(\Omega, \mathcal{F})$ by the equivalence relation $\sim_\mathbb{P}$ defined by $X \sim_\mathbb{P} Y$ if and only if $X = Y$ $\mathbb{P}$-almost surely. Also it is a well established fact that L^∞ is a Banach space[1] with $\|.\|_\infty$.

Similarly, for any σ-algebra $\mathcal{F} \subseteq \mathcal{G}$, the real valued function which is $\mathbb{P}$-integrable is denoted by $\mathcal{L}^1(\Omega, \mathcal{F})$. $L^1(\Omega, \mathcal{F}, \mathbb{P} \mid_\mathcal{F})$ denotes again our quotient space by the same equivalence relation and defines also a Banach space but with the $\|.\|_1$ this time.

Starting from now, unless specified otherwise, random variables are seen as functions from $\mathcal{L}^\infty(\Omega, \mathcal{F})$. This assumption means that infinite losses (or gains) are not considered. This seems reasonable in Actuarial Sciences because the set of assets and liabilities existing on earth is bounded.

Unconditional and conditional expectations are important notions since they describe the value that the random variable will take in "average" (unconditional) with eventually some preliminary conditions (conditional). The definition below is usually given under a "theorem form". Nevertheless, one of the main results of this paper will be the generalization (categorical definition) of this notion (see Chapter 2). Rigorous proof will also be given in that chapter. As usual, the notation $F_X(x) = \mathbb{P}[X \leq x]$ is used to describe the cumulative distribution function (cdf) of a probability measure.

Definition 1.1.2 (Unconditional Expectation)**.** Let $X : \Omega \longrightarrow I \subset \mathbb{R}$ be a random variable and $g : I \longrightarrow \mathbb{R}$ a function then $g(X)$ is again a random variable and its (unconditional) *expectation* is defined by:

$$\mathbb{E}^\mathbb{P}[g(X)] = \int_I g(x) dF_X(x)$$

In particular, the following equality holds:

$$\mathbb{E}^\mathbb{P}[X] = \int_I X d\mathbb{P}$$

Here (and for the rest of this paper), the integral used has to be understood in the Riemann-Stieltjes sense.

Definition 1.1.3 (Conditional Expectation)**.** Let $\bar{\Omega} = (\Omega, \mathcal{G}, \mathbb{P})$ be a probability space and $\mathcal{H} \subset \mathcal{G}$ is a sub-σ-algebra. Let $X : \Omega \longrightarrow \mathbb{R}$ be an integrable random variable and $\mathcal{G}$-measurable. There exists an integrable $\mathcal{H}$-measurable random variable denoted $\mathbb{E}^\mathbb{P}[X \mid \mathcal{H}]$ called the *conditional expectation* with respect to $\mathcal{H}$ such that for any $H \in \mathcal{H}$:

$$\mathbb{E}^\mathbb{P}[\mathbb{E}^\mathbb{P}[X \mid \mathcal{H}] \mathbb{1}_H] = \int_H \mathbb{E}^\mathbb{P}[X \mid \mathcal{H}] d\mathbb{P} = \int_H X d\mathbb{P} = \mathbb{E}^\mathbb{P}[X \mathbb{1}_H]$$

Moreover, this random variable is almost surely unique (which means, that if another random variable satisifies the condition above, then it must be equal to $\mathbb{E}^\mathbb{P}[X \mid \mathcal{H}]$ except possibly on

[1]The definition of Banach Space can be found in appendix.

a set of measure zero).

In particular, if Z is another random variable and $\mathcal{H}$ is the σ-algebra generated by Z then the following notation is used:

$$\mathbb{E}^{\mathbb{P}}[X \mid Z] := \mathbb{E}^{\mathbb{P}}[X \mid \mathcal{H}]$$

For the classical conditional expectation, the following well established theorem is also satisfied.

Theorem 1.1.4. *If Y is a $\mathcal{G}$-measurable random variable, then for all X integrable random variables:*

$$\mathbb{E}^{\mathbb{P}}[YX \mid \mathcal{G}] \sim_{\mathbb{P}} Y\mathbb{E}^{\mathbb{P}}[X \mid \mathcal{G}]$$

A general version (and a proof) of this elementary property will be given by the Theorem 2.3.11.

1.2 1-Period Monetary Value Measure

Risk measures are essential for insurance companies. Besides the fact that they help to understand the risk a company is facing, they are also used in solvability assessment context.

For example, the Solvability Capital Requirement (SCR) under SII regulation (and the corresponding solvability ratio) is directly based on a $V@R$ measure of the liabilities of the company. SII regulation prescribes to any insurance company (which are subject to European Union regulations) to detain a capital which is a $V@R$ (therefore a quantile) at 99.5% of the projected liabilities. In other words, $K[X]$, capital requirement under SII is given by :

$$K[X] = V@R_{99.5\%}[X] - V$$

Where X is the random variable representing the future loss/liabilities and V is a technical provision (an estimation of X). Basel III prescribes something similar for the bank but is using a $TV@R$ measure instead.

Without losing any generalities, here is the first definition of a monetary value measure, one of a central notion of this master thesis:

Definition 1.2.1 (1-Period Monetary Value Measure)**.** A *1-period monetary value measure* is a function $\varphi : L^\infty(\bar{\Omega}) = L^\infty(\Omega, \mathcal{G}, \mathbb{P}) \longrightarrow \mathbb{R}$ satisfying the following properties:

- *Cash Invariance* : $\forall X \in L^\infty(\bar{\Omega}), \forall a \in \mathbb{R}$:

$$\varphi[X + a] = \varphi[X] + a$$

- *Monotonicity* : $\forall X, Y \in L^\infty(\bar{\Omega})$:

$$X \leq Y \Rightarrow \varphi[X] \leq \varphi[Y]$$

- *Normalization*: $\varphi[0] = 0$

Depending on the papers one considers, the above definition could be slightly modified by the authors. For example, one could drop the normalization assumption or reverse the sign in the cash invariance.

However, for the rest of this paper, only the above mentioned properties will need to be satisfied in order to be called "1-period monetary value measure".

This definition will already be generalized under the form of the multiperiod monetary measure in Section 1.3. The categorical version will be given in Chapter 2.

While a risk measure is usually defined only as a function $\varphi : L^\infty(\bar{\Omega}) \longrightarrow \mathbb{R}$, here, a monetary value measure is defined as a risk measure satisfying the three properties (cash invariance, monotonicity, normalization).

In addition to those three properties, there are also other properties that a monetary value measure may or may not satisfy. Those are, for example:

- *Convexity* : $\forall X, Y \in L^\infty(\bar{\Omega}), \forall \lambda \in [0,1]$:

$$\varphi[\lambda X + (1-\lambda)Y] \leq \lambda\varphi[X] + (1-\lambda)\varphi[Y]$$

- *Concavity* : $\forall X, Y \in L^\infty(\bar{\Omega}), \forall \lambda \in \mathbb{R}$:

$$\varphi[\lambda X + (1-\lambda)Y] \geq \lambda\varphi[X] + (1-\lambda)\varphi[Y]$$

- *Subadditivity* : $\forall X, Y \in L^\infty(\bar{\Omega})$:

$$\varphi[X+Y] \leq \varphi[X] + \varphi[Y]$$

- *Positive Homogenity* : $\forall X \in L^\infty(\bar{\Omega}), \forall a > 0$:

$$\varphi[aX] = a\varphi[X]$$

- *Law Invariance* : $\forall X, Y \in L^\infty(\bar{\Omega})$:

$$X \stackrel{d}{=} Y \Rightarrow \varphi[X] = \varphi[Y]$$

  Here, $X \stackrel{d}{=} Y$ means that X and Y have the same distribution so that $F_X(t) = \mathbb{P}[X \leq t] = \mathbb{P}[Y \leq t] = F_Y(t)$.

According to many authors, a monetary measure should satisfy some properties, deemed "good", introducing the notion of *coherent monetary measure*. This "coherence" also has a meaning when only talking about risk measure.

For example, while referring to [FS11], a monetary measure will be coherent if it is convexe, subadditive and positive homogene. It only requires subaddiditivity and positive homogenity (in addition to the three properties included in the definition of monetary measure) to be coherent in the framework of [ADEH99].

Then, on the other hand, some regulations do not even require all of those properties. For example, Solvency II standard formula for insurance companies is based on a Value-at-Risk measure (see Example 1.2.2) while Basel III regulation[2] is based on Tail-Value-at-Risk measure (see Example 1.2.3).

Those remarks inevitably raise the following questions:

- What is a standard or minimal set of axiom for a "coherent" monetary measure?
- What are the properties that could be deduced from this minimal set of axiom?
- What is the rationale behind choosing a set of axioms as opposed to another? What is the formality behind this choice?
- Is the choice of risk measure only a subjective choice made by an expert?

Those questions will be partially discussed later on.

To further outline the subject, here are some examples of monetary value measures[3].

Example 1.2.2 (V@R)**.** The *Value-at-Risk* with a confidence level α for a random variable $X \in L^\infty(\bar{\Omega})$ is defined by:

$$V@R_\alpha[X] = \inf\{x \in \mathbb{R} \mid F_X[x] \geq \alpha\}$$

where $F_X[x] = \mathbb{P}(X \leq x)$ is the cumulative distribution function of the random variable X.

The $V@R$ satisfies the definition of 1-period value measure as well as the positive homogenity and the law invariance, but it is not subadditive nor convex.

The $V@R$ may also be seen as a quantile: $V@R_\alpha[X] = F_X^{-1}(\alpha)$.

Example 1.2.3 (TV@R)**.** The *Tail Value-at-Risk* for confidence level α for a random variable $X \in L^\infty(\bar{\Omega})$ by:

$$TV@R_\alpha[X] = \frac{1}{1-\alpha}\int_\alpha^1 V@R_q[X]dq$$

The $TV@R$ satisfies the definition of 1-period value measure as well as the positive homogenity and the law invariance. It is also subadditive and convex.

Some practionner may prefer the $TV@R$ over the $V@R$ because it is subadditive. Usually subadditivity is included in the definition of "coherent" risk measure. Therefore, $TV@R$ is considered as coherent risk measure while $V@R$ is not.

The $TV@R$ is actually an 'average' of $V@R$, it takes a "sum" (the integrale) of the $V@R$ above the quantile (α) considered and devided by $1-\alpha$.

Another monetary measure is the entropic risk measure (described more in depth in [FK11]). This measure will be used and generalized (see Section 2.4):

[2]Basel III regulations is the counterpart of SII but for banks, more information can be found online on the European Banking Authority website

[3]The proof that the $V@R$ and $TV@R$ satisfies or not some properties can be found in any course in quantitative risk management

Example 1.2.4 (Entropic Risk Measure)**.** For $\lambda \in \mathbb{R}$ and $X \in L^\infty(\bar{\Omega})$ the classical *entropic risk measure* is defined by :

$$\varphi(X) = \lambda^{-1} \log \mathbb{E}^{\mathbb{P}}[e^{\lambda X}]$$

This entropic measure is a convex monetary value measure.

So far, all monetary value measures considered are fixed within the one period framework. To be more realistic, a dynamic version of this notion will be more suitable for our future consideration.

1.3 Dynamic Monetary Value Measure

The random variable X is measured several times between the present (time 0) and the future (time 1). The upcoming notion is presented in order to introduce a dynamic concept of our monetary value measure, where the probability space $\bar{\Omega} = (\Omega, \mathcal{G}, \mathbb{P})$ is still fixed, but the σ-algebra will move through "time".

In order to take into account this time dimension, the notion of filtration will be needed. Furthermore and for informative purposes only, a categorical generalization of this filtration will also be given (Section 2.5).

Definition 1.3.1 (Filtration)**.** Let $T > 0$ be a fixed time (the *horizon*), we call a *filtration* of $\mathcal{G}$ a familly of σ-sub-algebra $\{\mathcal{G}_t\}_{t\in[0,T]}$ such that :

$$\mathcal{G}_s \subset \mathcal{G}_t \subset \mathcal{G}_T \subset \mathcal{G} \text{ if and only if } s \leq t \leq T$$

The random variable which is $\mathcal{G}_T$-measurable will naturally be $\mathcal{G}_t$-measurable at any time $0 \leq t \leq T$.

In this paper, the following convention is followed: if a σ-sub-algebra $\mathcal{F} \subset \mathcal{G}$, then the notation $L(\mathcal{F}) := L^\infty(\Omega, \mathcal{F}, \mathbb{P} \mid_{\mathcal{F}})$ is used.

Definition 1.3.2 (Dynamic Monetary Value Measure)**.** Let $\{\mathcal{G}_t\}_{t\in[0,T]}$ be a filtration of $\mathcal{G}$. A *dynamic monetary value measure* is a collection of functions:

$$\varphi = \{\varphi_t : L(\mathcal{G}_T) \longrightarrow L(\mathcal{G}_t)\}_{t\in[0,T]}$$

That satisfies the following set of equations:

- *Cash Invariance* : $\forall X \in L(\mathcal{G}_T), \forall Z \in L(\mathcal{G}_t)$ we have :

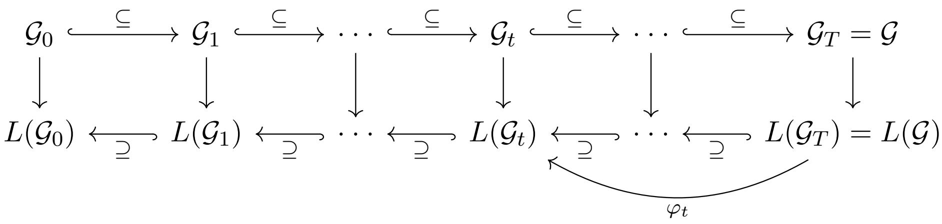
$$\varphi_t[X + Z] = \varphi_t[X] + Z$$

- *Monotonicity* : $\forall X, Y \in L(\mathcal{G}_T)$ we have:

$$X \leq Y \Rightarrow \varphi_t[X] \leq \varphi_t[Y]$$

- *Normalization*: $\varphi_t[0] = 0$

To illustrate the notion above, one can keep the following diagram in mind:



Two additional properties that a dynamic monetary measure may or may not satisfy are described below:

- *Dynamic Programing Principle* : for $0 \leq s \leq t \leq T$ and $\forall X \in L(\mathcal{G}_T)$ we have :
$$\varphi_s(X) = \varphi_s(\varphi_t(X))$$
- *Time Consistency* : for $0 \leq s \leq t \leq T$, $\forall X, Y \in L(\mathcal{G}_T)$ we have :
$$\varphi_t(X) \leq \varphi_t(Y) \Rightarrow \varphi_s(X) \leq \varphi_s(Y)$$

As mentioned by Adachi (in [Ada14]), those two extra axioms are popular in dynamic risk measure theory. Nevertheless, the fact that our monetary have or have not to satisfy this axiom is a legitimate question, anew.

The questions about the 1-period monetary measure raised above are also valid in this setting.

1.4 Change of Measure

Heretofore, all the definitions and notions explained above are dependent of an important factor: the underlying probability measure $\mathbb{P}$. Indeed, the probability space $\bar{\Omega} = (\Omega, \mathcal{G}, \mathbb{P})$ is fixed and all the described notions were based on it.

It is common knowledge that there is not one unique objective probability measure. Typically, several valid probability measures (and therefore probability spaces) may coexist. Ultimately, the choice of the probability space is subjective.

In a broader setting, a unique probability measure that cannot be determined is a situation to consider. This is called *uncertainty* and has been introduced by Knight in [HK21]. Back in classical risk measure theory, this uncertainty cannot be integrated and a more general definition has to be made to take it into account.

The next part of this chapter is dedicated to remind the reader of some tools used to handle multiple probability measures.

Definition 1.4.1 (Absolute Continuity)**.** Let $\mathbb{P}$ and $\mathbb{Q}$ be two probability measures on a space $(\Omega, \mathcal{G})$, then $\mathbb{Q}$ is *absolutely continuous* with respect to $\mathbb{P}$, if $\forall G \in \mathcal{G}$:

$$\mathbb{P}[G] = 0 \Rightarrow \mathbb{Q}[G] = 0 \tag{1.4.1}$$

it is noted by $\mathbb{Q} \ll \mathbb{P}$.

Also, $\mathbb{Q}$ is called equivalent to $\mathbb{P}$, notation $\mathbb{Q} \sim \mathbb{P}$ if both $\mathbb{Q} \ll \mathbb{P}$ and $\mathbb{P} \ll \mathbb{Q}$ hold simulatenously.

Another way to understand the definition above is to say that $\mathbb{Q}$ is *absolutely continuous* with respect to $\mathbb{P}$ if it preserves sets having null measure.

Finally, this chapter is concluded with one of the most famous theorem used in measure theory:

Theorem 1.4.2 (Radon Nikodym Theorem). *IF $\mathbb{Q} \ll \mathbb{P}$ then there exists a unique positive $\mathbb{P}$-integrable random variable noted $\frac{d\mathbb{Q}}{d\mathbb{P}} : \Omega \longrightarrow [0, \infty]$ such that for all $G \in \mathcal{G}$ we have :*

$$\mathbb{Q}[G] = \int_G \frac{d\mathbb{Q}}{d\mathbb{P}} d\mathbb{P} \tag{1.4.2}$$

The random variable $\frac{d\mathbb{Q}}{d\mathbb{P}}$ is called the Radon Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$.

For all random variable $X : \Omega \longrightarrow \mathbb{R}$ integrable with respect to $\mathbb{Q}$

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[X] &= \int_\Omega X d\mathbb{Q} \\ &= \int_\Omega X \frac{d\mathbb{Q}}{d\mathbb{P}} d\mathbb{P} \\ &= \mathbb{E}^{\mathbb{P}}[\frac{d\mathbb{Q}}{d\mathbb{P}} X] \end{aligned} \tag{1.4.3}$$

For further details of this theorem (and proof), it is suggested to check for example [Bau01], as well as any other standard book in measure and integration theory.

Chapter 2

Categorical Risk Monetary Measure

The aim of this chapter is to generalize the notions introduced in Chapter 1 in the category theory framework.

First, general definitions of category theory will be introduced. Then, a definition of the category used in the rest of the paper will be given in order to apprehend the generalization of the conditional and unconditional expectations. Finally, a categorical generalization of the monetary value measure will be given as well as the direct properties and consequences of it.

2.1 Introduction to Category Theory

All the notions in this section can be found more in depth in [Bor94a], as well as in most category theory books. Even though, this reference (together with [Bor94b] and [Bor94c]) is considered by many[1] as the "Bible" of category theory.

To understand and study category theory, the first step is to define what is a category. Roughly speaking, a category is a collection of objects that are connected (or not) by arrows (map). So, defining a category consists of defining what the objects are as well as what the arrows of the category are. Below, a formal definition is given.

Definition 2.1.1 (Category)**.** A *category* $\mathbf{C}$ consists of:

- A class $\mathcal{O}_{\mathbf{C}}$ called *objects of the category*;
- $\forall A, B \in \mathcal{O}_{\mathbf{C}}$, a set $\mathbf{C}(A, B)$ whose elements will be called *arrows* or *morphisms* from A to B. In this context, if $f \in \mathbf{C}(A, B)$, then the following notation may be used $f : A \longrightarrow B$ and A is called the *domain* of f and B, the *codomain*;
- $\forall A, B, C \in \mathcal{O}_{\mathbf{C}}$, we have a *composition* law:

  $$\mathbf{C}(A, B) \times \mathbf{C}(B, C) \longrightarrow \mathbf{C}(A, C)$$

  the composite of a pair (f, g) will be written $g \circ f$ or just gf;
- $\forall A \in \mathcal{O}_{\mathbf{C}}$, we have an arrow $id_A \in \mathbf{C}(A, A)$ called the identity on A;

[1] Including the humble author of this master paper

Such that the following axioms are satisfied:

1. *Associativity*: given the arrows $f \in \mathbf{C}(A, B)$, $g \in \mathbf{C}(B, C)$ and $h \in \mathbf{C}(C, D)$ the following is satisfied :
$$h \circ (g \circ f) = (h \circ g) \circ f$$

2. *Identity*: given the arrows $f \in \mathbf{C}(A, B)$ and $g \in \mathbf{C}(B, C)$ the following is satisfied:
$$id_B \circ f = f$$
and
$$g \circ id_B = g$$

Example 2.1.2. Here are some classical examples of categories:

- $\mathbf{Vect}_{\mathbb{K}}$: $\mathbb{K}$ - vector spaces and linear applications between them.
- **Set**: Sets and their corresponding mappings (this category will be used later on in this paper).
- Sets and bijective mappings (sub category of **Set**).
- **Meas**: Measurable spaces and arrows are given by measurable functions.
- **Prob**: This category of probability space will be defined in Section 2.2.
- An ordered set $(T, \leq)$ where $T \subset \mathbb{R}$. The objects are the element of T and we have an arrow between $x, y \in T : x \longrightarrow y$ if $x \leq y$.
- **Top**: Topological spaces and their corresponding continuous functions.
- $\mathbf{Ban}_{\infty}$: Banach space with bounded linear mappings.

More classical examples can be found in [Bor94a] but can also be created from any mathematical theory. A reader less familiar with this theory could see that category theory covers a large panel of different mathematical concepts by its simplicity and its abstractivity. In section 2.2, the focus will be given on the categories $\mathcal{X}$ and more generally **Prob**.

Two basic concepts from category theory are the functors and the natural transformations. A functor can be seen as an "arrow" or a map between categories and a natural transformation as an "arrow" between functors[2]. In section 2.3 and in section 2.4, the conditional expectation and the monetary value measure will be seen as special cases of functors.

Definition 2.1.3 (Covariant Functor)**.** A *covariant functor* from a category **C** to a category **D** is given by

- a mapping between the object of **C** and **D**:
$$F_{\mathcal{O}} : \mathcal{O}_{\mathbf{C}} \longrightarrow \mathcal{O}_{\mathbf{D}}$$
the image of $C \in \mathbf{C}$ is noted $F(C)$ or FC

[2]We are not going to talk about the category of categories and functors because we will not use it and because it brings some set theoretical issues that we do not want to consider here.

- For every pair of objects C, C' of $\mathbf{C}$ we have a mapping:

$$\mathbf{C}(C, C') \longrightarrow \mathbf{D}(FC, FC')$$

  the image of $f \in \mathbf{C}(C, C')$ is noted $F(f)$ or Ff

such that the following conditions are satisfied:

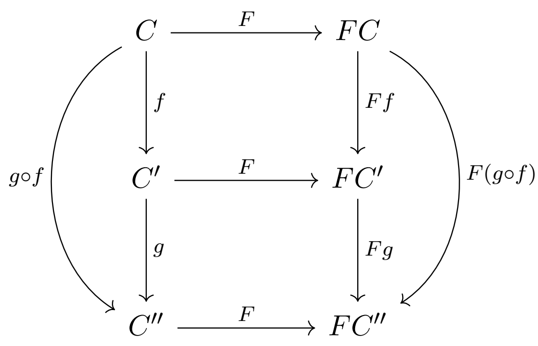
1. *Identity Arrow* $\forall C \in \mathbf{C}$ we have

$$F(Id_C) = Id_{F(C)}$$

2. *Composition Law*: $\forall f \in \mathbf{C}(C, C'), \forall g \in \mathbf{C}(C', C'')$ we have :

$$F(g \circ f) = F(g) \circ F(f)$$

Using a diagram representation can be useful to visualize properties in category theory. In the example hereafter, the axiom regarding the composition law is represented by the following commutative diagram:



(2.1.1)

Example 2.1.4. Some examples to illustrate the definition of covariant functor are given.

- The forgetful functor $U : \mathbf{Meas} \longrightarrow \mathbf{Set}$ from the category of measurable spaces to the category of sets. It maps a measurable space $(\Omega, \mathcal{G})$ to the underlying set Ω and an arrow (measurable function) f to the corresponding set mapping f. This functor "forgets" the σ-algebra structure on the underlying set.

- $\mathcal{P} : \mathbf{Set} \longrightarrow \mathbf{Set}$ from the category of sets to itself by mapping a set A to its power set $\mathcal{P}(A)$ and a mapping $f : A \longrightarrow B$ to the direct image mapping from $\mathcal{P}(A) \longrightarrow \mathcal{P}(B)$.

- For any category $\mathbf{C}$, the identity functor is defined : $Id_{\mathbf{C}} : \mathbf{C} \longrightarrow \mathbf{C}$ with $Id_{\mathbf{C}}(C) = C$ and $Id_{\mathbf{C}}(f) = f$ for an object C and arrow f.

- In Section 2.3, the covariant functor $L : \mathbf{Prob} \longrightarrow \mathbf{Set}$ which associates a proability space to an equivalence class of random variable will be defined.

- Another important class of functors are the ones called "representable functors". Given a category $\mathbf{C}$, for any object $C \in \mathbf{C}$, one can define a functor as:

$$\mathbf{C}(C,-) : \mathbf{C} \longrightarrow \mathbf{Set} : A \longrightarrow \mathbf{C}(C,A)$$

So for any given object C, a morphism is associated to any object A with the set of arrows from C to A. Now, if $f : A \longrightarrow B$ is an arrow in $\mathbf{C}$, the corresponding mapping is given by:

$$\mathbf{C}(C,-)(f) := \mathbf{C}(C,f) : \mathbf{C}(C,A) \longrightarrow \mathbf{C}(C,B)$$

defined by:

$$\mathbf{C}(C,f)(g) = f \circ g$$

In the last example, the reader could notice that the second component of $\mathbf{C}(C,-)$ is "free". This example could be easily dualized by inverting the "free" component and thus considering the following example :

Example 2.1.5. Given a category $\mathbf{C}$, for any object $C \in \mathbf{C}$, we can define :

$$\mathbf{C}(-,C) : \mathbf{C} \longrightarrow \mathbf{Set} : A \longrightarrow \mathbf{C}(A,C)$$

and for $f : A \longrightarrow B$ and $g : B \longrightarrow C$

$$\mathbf{C}(-,C)(f) : \mathbf{C}(A,C) \longrightarrow \mathbf{C}(B,C) : g \longrightarrow \mathbf{C}(-,C)(f)(g) = g \circ f$$

The above example may seem to look like the previous one, however it "inverses" the direction of the arrow. This is an example of a *contravariant* functor. Here is the definition of this notion:

Definition 2.1.6 (Contravariant Functor)**.** A *Contravariant Functor* from a category $\mathbf{C}$ to a category $\mathbf{D}$ is given by

- a mapping between the objects of $\mathbf{C}$ and $\mathbf{D}$:

$$F_{\mathcal{O}} : \mathcal{O}_{\mathbf{C}} \longrightarrow \mathcal{O}_{\mathbf{D}}$$

the image of $C \in \mathbf{C}$ is noted $F(C)$ or FC

- For every pair of objects C, C' of $\mathbf{C}$ we have a mapping:

$$\mathbf{C}(C,C') \longrightarrow \mathbf{D}(FC',FC)$$

the image of $f \in \mathbf{C}(C,C')$ is noted $F(f)$ or Ff

such that the following conditions are satisfied:

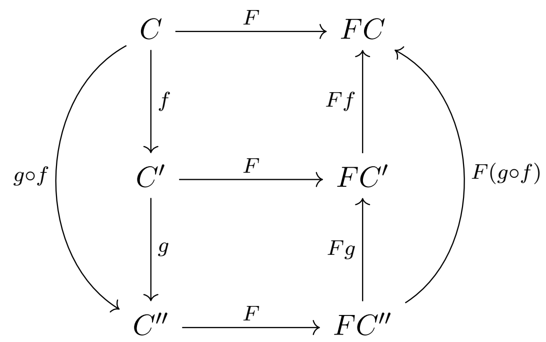
1. *Identity Arrow* $\forall C \in \mathbf{C}$, we have

$$F(Id_C) = Id_{F(C)}$$

2. *Composition Law*: $\forall f \in \mathbf{C}(C, C'), \forall g \in \mathbf{C}(C', C'')$, we have :

$$F(g \circ f) = F(f) \circ F(g)$$

The diagram expressing the composition law axiom (see 2.1.3 in the covariant case) becomes:



(2.1.2)

The reader may feel the duality we have between the notions. It is understandable that anything that could be defined or proved for a covariant functor would have its dual counterpart for a contravariant functor.

Following this idea, a new example of category is now introduced, based on an existing category:

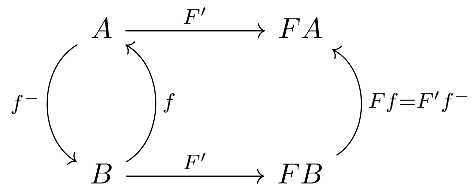
Example 2.1.7. Let $\mathbf{C}$ be a category. The category $\mathbf{C}^{op}$ (sometimes called the "dual category") is defined as follows:

- Both categories have the same class of objects: $\mathcal{O}_{\mathbf{C}} = \mathcal{O}_{\mathbf{C^{op}}}$
- $\forall A, B \in \mathcal{O}_{\mathbf{C^{op}}}$, $\mathbf{C^{op}}(A, B) = \mathbf{C}(B, A)$. The arrows of this opposite category are those written in the opposite direction. The notation $f^- : A \longrightarrow B$ is used for the arrow in $\mathbf{C^{op}}$ corresponding to $f : B \longrightarrow A$ in $\mathbf{C}$
- The composition law of $\mathbf{C^{op}}$ is given by:

$$f^- \circ g^- = (g \circ f)^-$$

From any covariant functor on a category $\mathbf{C}$, a contravariant functor on the category $\mathbf{C}^{op}$ can be constructed. If $F : \mathbf{C} \longrightarrow \mathbf{D}$ is a covariant functor, then the contravariant functor $F' : \mathbf{C}^{op} \longrightarrow \mathbf{D}$ is defined by:

- $F'C = FC$ for any object $C \in \mathcal{O}_{\mathbf{C^{op}}} = \mathcal{O}_{\mathbf{C}}$ (the equality between our objects holds by definition).
- $F'f^- = Ff$ for any $f^- \in \mathbf{C^{op}}(A, B)$, or in diagram representation:



(2.1.3)

The conditional expectation and the monetary measure (see Section 2.3 and Section 2.4) are also examples of contravariant functors.

Below is defined two properties that functors may or may not satisify. Those notions are only used in section 3.6 but are part of the fundamental notions in category theory.

Definition 2.1.8. Let $F : \mathbf{C} \longrightarrow \mathbf{D}$ be a functor and for any object $C, C' \in \mathbf{C}$ the following mapping is considered:

$$\mathbf{C}(C, C') \longrightarrow \mathbf{D}(FC, FC') : f \longrightarrow Ff$$

- The functor F is called *faithful* if the mappings above are injectives for C, C'
- The functor F is called *full* if the mappings above are surjectives for C, C'

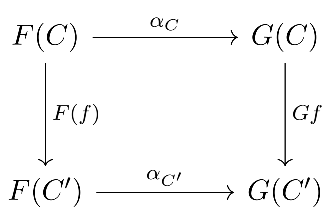
Example 2.1.9. The forgetful functor U : **Meas** $\longrightarrow$ **Set** is faithful. Indeed, a necessary condition for two measurable functions with the same domain and codomain to be equal is the equality of their underlying set.

Another important notion in category theory is the natural transformation. It plays a homologue role between functors as an homotopy between continuous functions in topology.

Definition 2.1.10. Let $F, G : \mathbf{C} \longrightarrow \mathbf{D}$ be two functors between the category $\mathbf{C}$ and the category $\mathbf{D}$. A natural transformation $\alpha : F \Rightarrow G$ from F to G is a class of arrows $\{\alpha_C : F(C) \longrightarrow G(C)\}_{C \in \mathbf{C}}$ of $\mathbf{D}$ such that for any arrows $f : C \longrightarrow C'$ in $\mathbf{C}$:

$$\alpha_{C'} \circ F(f) = G(f) \circ \alpha_C$$

Another way to understand the previous definition is to assume that the following diagram is commutative:



In a category theory there exists also what is called "limit object". As the general notion will not be used in the rest of this paper, only a particular case is defined below (and only used once in Chapter 3.6). If the reader wants to know more about those, he/she can consult [Bor94a].

Definition 2.1.11. Let $f : A \longrightarrow C$ and $g : B \longrightarrow C$ be two arrows in a category $\mathbf{C}$. A *pullback* of (f, g) is defined by a triple (P, f', g') where :

1. P is an object of $\mathbf{C}$

2. $f' : P \longrightarrow B$ and $g' : P \longrightarrow A$ are arrows of $\mathbf{C}$ such that $f \circ g' = g \circ f'$

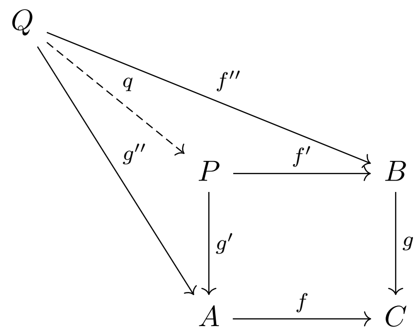
and the triple is universal in the sense that if any other triple (Q, f'', g'') exists with

1. Q is an object of $\mathbf{C}$

2. $f'' : Q \longrightarrow B$ and $g'' : Q \longrightarrow A$ are arrows of $\mathbf{C}$ such that $f \circ g'' = g \circ f''$

then there exists a unique arrow $q : Q \longrightarrow P$ such that $f'' = f' \circ Q$ and $g'' = g' \circ q$

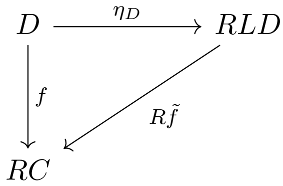
Another way to see the definition above is to understand the following commutative diagram:



Another general notion of category theory (also only used once in this paper in Chapter 3.6) is the notion of adjunction.

Definition 2.1.12. Let $L : \mathbf{C} \longrightarrow \mathbf{D}$ and $R : \mathbf{C} \longrightarrow \mathbf{D}$ two functors such that for all $D \in \mathbf{D}$ there exists $\eta_D : D \longrightarrow RLD$ such that:

1. For all $C \in \mathbf{C}$ and for all $f : D \longrightarrow RC$, there exists a unique $\tilde{f} : LD \longrightarrow C$ such that the following diagram commutes:



2. $\eta = \{\eta_D \mid D \in \mathbf{D}\}$ is a natural transformation $\mathbb{1}_{\mathbf{D}} \Rightarrow RL$

In this case, L is called the left adjoint of R.

Usually the first property in the definition above is called the universality and the second one the naturality.

Finally, the last notion strictly from category theory that will be used is:

Definition 2.1.13. An object F of a category is a final object when every object C has exactly one arrow from C to F.

An object I of a category is an initial object when every object C has exactly one arrow from I to C.

Example 2.1.14. In **Set**, the empty set $\emptyset$ is an initial object and the singleton $\{*\}$ is a terminal object.

All of the above section was dedicated to introducing, or reminding, about notions of the category theory. All of them were general and not specific to risk measure. Those notions could be used in other areas of Actuarial Sciences as they are general and abstract. The goal from now on will be to focus on one application of this theory in the setting of monetary value measure theory.

2.2 Category Prob

In this chapter, the category **Prob** of probality spaces will be defined. Its objects will be the probability spaces and its arrows will satisfy some properties linked to the absolute continuity of the underlying probability measure.

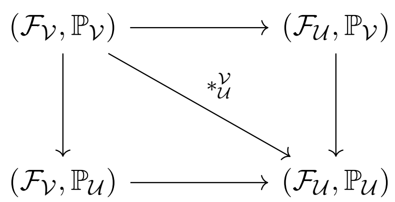
In [Ada14], a first definition of a category with probality spaces is given. It is actually a particular case of the category **Prob** defined in [AR16] and in [AR17].

Definition 2.2.1 (Category $\mathcal{X}$)**.** Let $\mathcal{X} = \mathcal{X}(\Omega, \mathcal{G})$ be the set of all pairs of the form $(\mathcal{F}, \mathbb{P})$ where $\mathcal{F}$ is a σ-sub-algebra of $\mathcal{G}$ and $\mathbb{P}$ is a probability measure on $\mathcal{G}$. For an element $\mathcal{U} \in \mathcal{X}$, its σ-algebra and probability measure is denoted by $\mathcal{F}_\mathcal{U}$ and $\mathbb{P}_\mathcal{U}$, respectively. That is, $\mathcal{U} = (\mathcal{F}_\mathcal{U}, \mathbb{P}_\mathcal{U})$. A binary relation $\leq_\mathcal{X}$ is introduced on $\mathcal{X}$ such that for $\mathcal{U}$ and $\mathcal{V}$ in $\mathcal{X}$,

$$\mathcal{V} \leq_\mathcal{X} \mathcal{U} \text{ iff } \mathcal{F}_\mathcal{V} \subset \mathcal{F}_\mathcal{U} \text{ and } \mathbb{P}_\mathcal{U} \ll \mathbb{P}_\mathcal{V}$$

Hence, $\mathcal{X}$ can be seen as a category having exactly one arrow $*^\mathcal{V}_\mathcal{U} : \mathcal{V} \longrightarrow \mathcal{U}$ in $\mathcal{X}$ if and only if $\mathcal{V} \leq_\mathcal{X} \mathcal{U}$

In this definition, the category $\mathcal{X}$ has in some way two dimensions. One is the temporal/risk dimension, which is represented by the inclusion of our sub-algebra (horizontal line in the diagram below), and the second one is a spacial/uncertainty, which is represented by the difference in the probability measure (vertical line in the diagram below). Following the notation in the definition, the diagram represents our situation:



Before advancing more in general category **Prob**, let us set the definition of a null-preserving function.

Definition 2.2.2 (null-preserving function)**.** A measurable function $f : (\Omega, \mathcal{G}, \mathbb{P}) \longrightarrow (\Omega', \mathcal{G}', \mathbb{P}')$ is called $\mathbf{null-preserving}$ if $\mathbb{P} \circ f^{-1} \ll \mathbb{P}'$

Example 2.2.3. A more complex example of null-preserving function will be given in the numerical application developped in Section 3.3. Two trivial examples are given here:

- Let us consider $\Omega = \{Tails, Heads\}$ a game of coins. The probability usually defined is given by $\mathbb{P}[Tails] = \mathbb{P}[Heads] = 1/2$. In this case, the function

$$Id_\Omega : (\Omega, 2^\Omega, \mathbb{P}) \longrightarrow (\Omega, 2^\Omega, \mathbb{P})$$

is an example of a null-preserving function.

- Now, let us consider $(\Omega, 2^\Omega, \mathbb{P})$ as above and $(\Omega', 2^{\Omega'}, \mathbb{P}')$ with $\Omega' = \Omega$ (also the game of coins) but now with a tricked coin such that $\mathbb{P}'[Tails] = 1$ and $\mathbb{P}'[Heads] = 0$. In this case the application:

$$f : (\Omega, 2^\Omega, \mathbb{P}) \longrightarrow (\Omega', 2^{\Omega'}, \mathbb{P}')$$

with $f(Tails) = Tails$ and $f(Heads) = Heads$ is not null-preserving. Indeed, on one hand we have $\mathbb{P}'[Heads] = 0$ but on the other $\mathbb{P}[f^{-1}(Heads)] = \mathbb{P}[(Heads)] = 1/2$.

For the rest of this paper, the notation $\bar{\Omega} = (\Omega, \mathcal{G}, \mathbb{P})$ is used to represent a probability space with its σ-algebra and its probability measure.

Following the convention above, the category **Prob** is defined below.

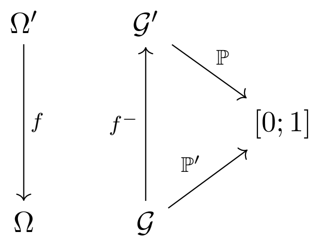
Definition 2.2.4 (Category **Prob**)**.** The category **Prob** is the category whose objects are all probability spaces and the set of arrows between them are defined by

$$\mathbf{Prob}(\bar{\Omega}, \bar{\Omega}') := \{f^- \mid f : \bar{\Omega}' \longrightarrow \bar{\Omega} \text{ is a null-preserving function.}\}$$

So $f^- : \bar{\Omega} \longrightarrow \bar{\Omega}'$ if there exist $f : \bar{\Omega}' \longrightarrow \bar{\Omega}$ null-preserving. The reader should not confuse it with

$$f^{-1} : \bar{\Omega} \longrightarrow \bar{\Omega}' : \omega \longrightarrow f^{-1}(\omega) = \{\omega' \in \Omega' \mid f(\omega') = \omega\}$$

Here is a diagram representation (that does not commute in general):



The definition above will only make sense if the axioms in Definition 2.1.1 are respected. The unit arrow is trivial in **Prob**. It still leads one to check that the composition law is well defined. This is the object of the next proposition:

Proposition 2.2.5. *Let f and g be two measurable functions:*

$$\bar{\Omega}'' \xrightarrow{g} \bar{\Omega}' \xrightarrow{f} \bar{\Omega}$$

If $\mathbb{P}' \circ f^{-1} \ll \mathbb{P}$ and $\mathbb{P}'' \circ g^{-1} \ll \mathbb{P}'$ then we have $\mathbb{P}'' \circ (f \circ g)^{-1} \ll \mathbb{P}$

Proof. we have :

$$\mathbb{P}'' \circ (f \circ g)^{-1} = (\mathbb{P}'' \circ g^{-1}) \circ f^{-1} \ll \mathbb{P}' \circ f^{-1} \ll \mathbb{P}$$

□

In the particular case where $f = Id_\Omega$ then:

$$Id_\Omega^- : (\Omega, \mathcal{G}, \mathbb{P}) \longrightarrow (\Omega, \mathcal{G}', \mathbb{P}')$$

with $\mathcal{G} \subset \mathcal{G}'$ and $\mathbb{P}' = \mathbb{P}' \circ Id_\Omega^{-1} \ll \mathbb{P}$. In this situation, the information the actuary has is evolving through time while the support of the probability measure is decreasing at the same time.

This could happen if the actuary believed that an event may happen, but has since acquired information that leads him to believe this event will not occur.

Another point, already mentioned at the beginning of this section, the category $\mathcal{X}$ defined above is a particular case of the category **Prob**. Indeed, a natural embedding $\iota : \mathcal{X} \hookrightarrow$ **Prob** is defined:

$$\begin{array}{ccc}
\mathcal{X}(\Omega,\mathcal{G}) & \xrightarrow{\quad\iota\quad} & \mathbf{Prob} \\
(\mathcal{F}_\mathcal{V}, \mathbb{P}_\mathcal{V}) & \xrightarrow{\quad\iota\quad} & (\Omega, \mathcal{F}_\mathcal{V}, \mathbb{P}_\mathcal{V}) \\
\Big\downarrow & & \Big\downarrow Id_\Omega^- \\
(\mathcal{F}_\mathcal{U}, \mathbb{P}_\mathcal{U}) & \xrightarrow{\quad\iota\quad} & (\Omega, \mathcal{F}_\mathcal{U}, \mathbb{P}_\mathcal{U})
\end{array}$$

The following property of our category (see 2.1.13 for the definition of an initial object) is given before introducing functors on **Prob**.

Proposition 2.2.6. *The probability space* $\bar{0} = (\{*\}, \{\{*\}, \emptyset\}, \mathbb{P}_0)$ *with*

$$\mathbb{P}_0[\emptyset] = 0 \text{ and } \mathbb{P}_0[\{*\}] = 1$$

is an initial object for **Prob**.
Moreover, for any probability space $\bar{\Omega} = (\Omega, \mathcal{G}, \mathbb{P})$ *the arrow* $!_\Omega^- : \bar{0} \longrightarrow \bar{\Omega}$ *with*

$$!_\Omega : X \longrightarrow \{*\} : x \longrightarrow *$$

is unique.

Proof. First, $!_\Omega^-$ is an arrow in **Prob** since $\mathbb{P}\circ !_\Omega^{-1} \ll \mathbb{P}_0$. Indeed, if we suppose that for some $G \in \mathcal{G} = \{\{*\}, \emptyset\}$ with $\mathbb{P}_0[G] = 0$ then we have $G = \emptyset$, so we directly get :

$$\mathbb{P}\circ !_\Omega^{-1}(G) = \mathbb{P}[!_\Omega^{-1}(\emptyset)] = \mathbb{P}[\emptyset] = 0$$

The uniqueness of $!_\Omega^-$ comes directly from the fact that the arrow $!_\Omega$ from Ω to $\{*\}$ is unique. □

2.3 Conditional Expectation as a Functor

As partially explained in Section 1.1 and in the introduction, conditional and unconditional expectation are basic tools used in probability theory and in Actuarial Sciences. They allow us to compute the fair value (or best estimate) of random losses (random variables). They provide the value that the random variable gives in average with or without conditions.

In the previous section, the category **Prob** has been defined and now a functor L on it will be introduced in order to associate random variables in this categorical context. The next step will be to define the conditional expectation.

In order to have this functor L well defined, the following proposition is needed:

Proposition 2.3.1. *Let $X, Y \in \mathcal{L}^\infty(\Omega, \mathcal{G})$ (can be seen as random variables, but this proposition is true in general) and f^- be an arrow in $\mathbf{Prob}(\bar{\Omega}, \bar{\Omega}')$. If $X \sim_{\mathbb{P}} Y$ then $X \circ f \sim_{\mathbb{P}'} Y \circ f$*

Proof. Let us assume that we have $X \sim_{\mathbb{P}} Y$, so

$$\mathbb{P}[X \neq Y] = 1 - \mathbb{P}[X = Y] = 1 - 1 = 0$$

If we take $\omega' \in \{\omega' \in \Omega' \mid X \circ f(\omega') \neq Y \circ f(\omega')\}$ then $X(f(\omega')) \neq Y(f(\omega'))$. Therefore, we have $f(\omega') \in \{X \neq Y\} := \{\omega \in \Omega \mid X(\omega) \neq Y(\omega)\}$. We can now conclude that

$$f(\{\omega' \in \Omega' \mid X \circ f(\omega') \neq Y \circ f(\omega')\}) \subseteq \{\omega \in \Omega \mid X(\omega) \neq Y(\omega)\}$$

The above statement is equivalent to:

$$\{\omega' \in \Omega' \mid X \circ f(\omega') \neq Y \circ f(\omega')\}) \subseteq f^{-1}(\{\omega \in \Omega \mid X(\omega) \neq Y(\omega)\}) \tag{2.3.1}$$

Since $\mathbb{P}' \circ f^{-1} \ll \mathbb{P}$ and $\mathbb{P}[X \neq Y] = 0$ we get :

$$(\mathbb{P}' \circ f^{-1})[X \neq Y] = \mathbb{P}'(f^{-1}[X \neq Y]) = 0$$

and so we have (thanks to the statement 2.3.1) $\mathbb{P}'[X \circ f \neq Y \circ f] = 0$ which gives by definition $X \circ f \sim_{\mathbb{P}'} Y \circ f$ □

Definition 2.3.2. We have a covariant functor $L : \mathbf{Prob} \longrightarrow \mathbf{Set}$ which is defined by:

$$\begin{array}{ccccccc}
\Omega & \bar{\Omega} & \xrightarrow{L} & L(\bar{\Omega}) = L^\infty(\bar{\Omega}) & \ni & [X]_{\sim_{\mathbb{P}}} \\
{\scriptstyle f}\Big\uparrow & \Big\downarrow{\scriptstyle f^-} & & \Big\downarrow{\scriptstyle Lf^-} & & \Big\downarrow{\scriptstyle Lf^-} \\
\Omega' & \bar{\Omega}' & \xrightarrow{L} & L(\bar{\Omega}') = L^\infty(\bar{\Omega}') & \ni & [X \circ f]_{\sim_{\mathbb{P}'}}
\end{array}$$

with $Lf^-([X]_{\sim_{\mathbb{P}}}) = [X \circ f]_{\sim_{\mathbb{P}'}}$

The definition above defines a covariant functor that verifies the three axioms (see Definition 2.1.3):

- *Covariant:* If $f^- \in \mathbf{Prob}[\bar{\Omega}, \bar{\Omega}']$ then for any $[X]_{\sim_{\mathbb{P}}} \in L^\infty(\bar{\Omega})$, we have

  $$Lf^-([X]_{\sim_{\mathbb{P}}}) = [X \circ f]_{\sim_{\mathbb{P} \circ f}} = [X \circ f]_{\sim_{\mathbb{P}'}}$$

  since we have $\mathbb{P} \circ f \ll \mathbb{P}'$. Therefore we have $Lf^- \in \mathbf{Set}[L(\bar{\Omega}), L(\bar{\Omega}')]$.

- *Identity arrow:* We have $Id^-_{\bar{\Omega}} : \bar{\Omega} \longrightarrow \bar{\Omega}$ which is a **Prob** arrow. Then for any $[X]_{\sim_{\mathbb{P}}} \in L^\infty(\bar{\Omega})$ we have :

  $$LId^-_{\bar{\Omega}}([X]_{\sim_{\mathbb{P}}}) = [X \circ Id_{\bar{\Omega}}]_{\sim_{\mathbb{P}}} = [X]_{\sim_{\mathbb{P}}}$$

  So we can conclude $LId_{\bar{\Omega}} = Id_{L(\bar{\Omega})} = Id_{L^\infty(\bar{\Omega})}$.

- *Composition law:* Let $f^- \in \mathbf{Prob}[\bar{\Omega}, \bar{\Omega}']$ and $g^- \in \mathbf{Prob}[\bar{\Omega}', \bar{\Omega}'']$. First we see $g^- \circ f^- \in \mathbf{Prob}[\bar{\Omega}, \bar{\Omega}'']$ so $f \circ g : \Omega'' \longrightarrow \Omega$. Then for any $[X]_{\sim_\mathbb{P}} \in L^\infty(\bar{\Omega})$:

  $L(g^- \circ f^-)([X]_{\sim_\mathbb{P}}) = [X \circ f \circ g]_{\sim_{\mathbb{P}''}} = Lg^-([X \circ f]_{\sim_{\mathbb{P}'}}) = Lg^-(Lf^-([X]_{\sim_\mathbb{P}})) = Lg^- \circ Lf^-([X]_{\sim_\mathbb{P}})$

  we can deduce $L(g^- \circ f^-) = Lg^- \circ Lf^-$

In order to get the generalized definition of the conditional expectation as a functor, an additional theorem, definition and proposition are needed:

Theorem 2.3.3. *For $f^- \in \mathbf{Prob}[\bar{\Omega}, \bar{\Omega}']$, $\forall Y \in \mathcal{L}^1(\bar{\Omega}')$, there exists $X \in \mathcal{L}(\bar{\Omega})$ such that for any $G \in \mathcal{G}$ we have :*

$$\int_G X d\mathbb{P} = \int_{f^{-1}(G)} Y d\mathbb{P}' \tag{2.3.2}$$

Moreover, X is unique up to $\mathbb{P}$-null sets. Therefore, if we have $X_1, X_2 \in \mathcal{L}(\bar{\Omega})$ satisfying 2.3.2 then we have $\mathbb{P}[X_1 \neq X_2] = 0$ or equivalently we can say that $X_1 \sim_\mathbb{P} X_2$.

Proof. First, let us prove the almost-surely uniqueness. Let us take $X_1, X_2 \in \mathcal{L}(\bar{\Omega})$ satisfying 2.3.2 meaning that $\forall G \in \mathcal{G}$:

$$\int_G X_1 d\mathbb{P} = \int_G X_2 d\mathbb{P} = \int_{f^{-1}(G)} Y d\mathbb{P}'$$

In particular, we have $\forall G \in \mathcal{G}$:

$$\int_G X_1 - X_2 d\mathbb{P} = 0 \tag{2.3.3}$$

By contradiction, let us assume that $P[X_1 \neq X_2] > 0$. Or in other words, if $W = X_1 - X_2$, then $0 < \mathbb{P}[W \neq 0] = \mathbb{P}[W > 0] + \mathbb{P}[W < 0]$ which implies that $\mathbb{P}[W > 0] > 0$ or $\mathbb{P}[W < 0] > 0$.

In the last case, since $\{W < 0\} = \lim_{n \to \infty} \{W < \frac{1}{n}\}$, there exists $n \in \mathbb{N}$ such that $\mathbb{P}[W < \frac{1}{n}] > 0$, by the monotone convergence theorem of measure and by 2.3.3 , we now have:

$$0 = \int_{\{W < \frac{1}{n}\}} W d\mathbb{P} \geq \frac{1}{n}\mathbb{P}[W < \frac{1}{n}] > 0$$

and we therefore obtain our contradiction.

The other case works similarly and is proved in [AR16].

We can conclude that $P[X_1 \neq X_2] = 0$.

Let $Y \in \mathcal{L}^1(\bar{\Omega}')$, we can define $Y^* : \mathcal{G}' \longrightarrow \mathbb{R}$ by

$$Y^*(H) := \int_H Y d\mathbb{P}' \text{ for } H \in \mathcal{G}'$$

Y^* is a measure satisfying $Y^* \ll \mathbb{P}'$. Therefore, $Y^* \circ f^{-1} \ll \mathbb{P}' \circ f^{-1} \ll \mathbb{P}$, since f^- is a **Prob** arrow. Now, by the the Radon-Nykodim theorem (cfr 1.4.2), there exists $X \in \mathcal{L}(\bar{\Omega})$ such that $\forall G \in \mathcal{G}$:

$$(Y^* \circ f^{-1})(G) = \int_G d(Y^* \circ f^{-1}) = \int_G X d\mathbb{P}$$

Moreover, this random variable X is unique up to $\mathbb{P}$-null set. We can finally conclude that:

$$\int_G X d\mathbb{P} = (Y^* \circ f^{-1})(G) = Y^*(f^{-1}(G)) = \int_{f^{-1}(G)} Y d\mathbb{P}'$$

which gives us 2.3.2. □

Definition 2.3.4. The random variable X, defined almost uniquely (Theorem 2.3.3), is called a *conditional expectaction of* Y *along* f^- and it is denoted $\mathbb{E}^{f^-}[Y]$. Therefore, we have :

$$\int_G \mathbb{E}^{f^-}[Y] d\mathbb{P} = \int_{f^{-1}(G)} Y d\mathbb{P}' \tag{2.3.4}$$

$\forall G \in \mathcal{G}$.

The following properties are also satisfied:

Proposition 2.3.5. *We consider* f^- *and* g^- *two* **Prob***-arrows:*

$$\bar{\Omega} \xrightarrow{f^-} \bar{\Omega}' \xrightarrow{g^-} \bar{\Omega}''$$

The following properties are satified:

1. *For* $Y_1, Y_2 \in \mathcal{L}^1(\bar{\Omega}')$, $Y_1 \sim_{\mathbb{P}'} Y_2$ *implies* $\mathbb{E}^{f^-}[Y_1] \sim_{\mathbb{P}} \mathbb{E}^{f^-}[Y_2]$.
2. *For* $Z \in \mathcal{L}^1(\bar{\Omega}'')$, $\mathbb{E}^{f^-}(\mathbb{E}^{g^-}[Z]) \sim_{\mathbb{P}} \mathbb{E}^{g^- \circ f^-}[Z]$.
3. *For* $X \in \mathcal{L}^1(\bar{\Omega})$, $\mathbb{E}^{Id_{\bar{\Omega}}}[X] \sim_{\mathbb{P}} X$.

Proof. 1. If $Y_1 \sim_{\mathbb{P}'} Y_2$ then

$$Y_1^* = \int Y_1 d\mathbb{P}' = Y_2^* = \int Y_2 d\mathbb{P}'$$

As for Theorem 2.3.3, the uniqueness of the conditional expectation up to $\mathbb{P}$-null set is found. In other words : $\mathbb{E}^{f^-}[Y_1] \sim_{\mathbb{P}} \mathbb{E}^{f^-}[Y_2]$

2. It is enough to show that for any $G \in \mathcal{G}$:

$$\int_G \mathbb{E}^{f^-}(\mathbb{E}^{g^-}[Z]) d\mathbb{P} = \int_{(f \circ g)^{-1}(G)} Z d\mathbb{P}''$$

but by definition, applying 2.3.4 twice gets us :

$$\begin{aligned}\int_G \mathbb{E}^{f^-}(\mathbb{E}^{g^-}[Z]) d\mathbb{P} &= \int_{f^{-1}(G)} \mathbb{E}^{g^-}[Z] d\mathbb{P}' \\ &= \int_{g^{-1} \circ f^{-1}(G)} Z d\mathbb{P}'' \\ &= \int_{(f \circ g)^{-1}(G)} Z d\mathbb{P}''\end{aligned}$$

3. For $G \in \mathcal{G}$ and for $X \in \mathcal{L}^1(\bar{\Omega})$, the following is true:

$$\int_G \mathbb{E}^{Id^-_{\bar\Omega}}[X]d\mathbb{P} = \int_{Id^{-1}_{\bar\Omega}(G)} Xd\mathbb{P} = \int_{(G)} Xd\mathbb{P}$$

but by the unicity of $\mathbb{E}^{Id^-_{\bar\Omega}}[X]$ up to null-set, we know that $\mathbb{P}[\mathbb{E}^{Id^-_{\bar\Omega}}[X] \neq X] = 0$ which is equivalent to saying that $\mathbb{E}^{Id^-_{\bar\Omega}}[X] \sim_{\mathbb{P}} X$.

□

The conditional expectation can be seen as a functor owing to the properties and definitions described above.

Definition 2.3.6 (Functor $\mathcal{E}$)**.** The contravariant functor $\mathcal{E} : \mathbf{Prob}^{op} \longrightarrow \mathbf{Set}$ is defined as followed

$$\begin{array}{ccccccc}
\Omega & \bar\Omega & \xrightarrow{\mathcal{E}} & \mathcal{E}(\bar\Omega) := L^1(\bar\Omega) & \ni & \left[\mathbb{E}^{f^-}[Y]\right]_{\sim_{\mathbb{P}}} \\
f \uparrow & \downarrow f^- & & \mathcal{E}f^- \uparrow & & \mathcal{E}f^- \uparrow \\
\Omega' & \bar\Omega' & \xrightarrow{\mathcal{E}} & \mathcal{E}(\bar\Omega') := L^1(\bar\Omega') & \ni & [Y]_{\sim_{\mathbb{P}'}}
\end{array}$$

Due to the properties described above, the contravariant functor $\mathcal{E}$ is well defined and respects our three axioms:

- *Contravariant*: If $f^- \in \mathbf{Prob}[\bar\Omega, \bar\Omega']$ then for any $[Y]_{\sim_{\mathbb{P}'}} \in L^1(\bar\Omega')$:

$$\mathcal{E}f^-([Y]_{\sim_{\mathbb{P}'}}) = [\mathbb{E}^{f^-}[Y]]_{\sim_{\mathbb{P}}}$$

  and so the functor "reverses the sense of the arrows": $\mathcal{E}f^- : \mathcal{E}(\bar\Omega') \longrightarrow \mathcal{E}(\bar\Omega)$

- *Identity arrow*: We have $Id^-_{\bar\Omega} : \bar\Omega \longrightarrow \bar\Omega$ which is a **Prob** arrow. Then for any $[X]_{\sim_{\mathbb{P}}} \in L^1(\bar\Omega)$ we have :

$$\mathcal{E}Id^-_{\bar\Omega}([X]_{\sim_{\mathbb{P}}}) = [\mathbb{E}^{Id^-_{\bar\Omega}}[X]]_{\sim_{\mathbb{P}}} = [X]_{\sim_{\mathbb{P}}}$$

  The last equality is obtained through 2.3.5.

- *Composition law*: Let $f^- \in \mathbf{Prob}[\bar\Omega, \bar\Omega']$ and $g^- \in \mathbf{Prob}[\bar\Omega', \bar\Omega'']$ then $g^- \circ f^- \in \mathbf{Prob}[\bar\Omega, \bar\Omega'']$. So by definition and based on 2.3.5, for any $[Z]_{\sim_{\mathbb{P}''}} \in L^1(\bar\Omega'')$:

$$\begin{array}{rcl}
\mathcal{E}(g^- \circ f^-)([Z]_{\sim_{\mathbb{P}''}}) & = & [\mathbb{E}^{g^- \circ f^-}[Z]]_{\sim_{\mathbb{P}}} \\
& = & [\mathbb{E}^{f^-}(\mathbb{E}^{g^-}[Z])]_{\sim_{\mathbb{P}}} \\
& = & \mathcal{E}f^-[\mathbb{E}^{g^-}[Z])]_{\sim_{\mathbb{P}'}} \\
& = & \mathcal{E}f^- \circ \mathcal{E}g^-[Z]_{\sim_{\mathbb{P}''}}
\end{array}$$

The above definition is a generalization of the classical conditional expectation (Definition 1.1.3). Indeed, let $\bar\Omega = (\Omega, \mathcal{G}, \mathbb{P})$ and $\bar\Omega' = (\Omega, \mathcal{H}, \mathbb{P})$ with $\mathcal{H} \subseteq \mathcal{G}$. An arrow $\iota^- \in \mathbf{Prob}[\bar\Omega, \bar\Omega']$ is defined by the inclusion $\iota : \bar\Omega' \hookrightarrow \bar\Omega$.

The following diagram is commutative:

$$\begin{array}{ccccc} \bar{\Omega} & \bar{\Omega} & \xrightarrow{\mathcal{E}} & \mathcal{E}(\bar{\Omega}) \\ {\scriptstyle \iota}\big\uparrow & \big\downarrow{\scriptstyle \iota^-} & & \big\uparrow{\scriptstyle \mathcal{E}\iota^-} \\ \bar{\Omega}' & \bar{\Omega}' & \xrightarrow{\mathcal{E}} & \mathcal{E}(\bar{\Omega}') \end{array}$$

Let $Y : \bar{\Omega}' \longrightarrow \mathbb{R}$ be a random variable. Then :

$$\mathcal{E}\iota^-([Y]_{\sim_{\mathbb{P}}}) = [\mathbb{E}^{\iota^-}[Y])]_{\sim_{\mathbb{P}}}$$

but by definition for any $H \in \mathcal{H}$:

$$\int_H \mathbb{E}^{\iota^-}[Y]d\mathbb{P} = \int_{\iota^{-1}(H)} Y d\mathbb{P}$$

Though, the equation above can be rewritten using the classical definition of expectation and conditional expectation like so:

$$\mathbb{E}^{\mathbb{P}}[\mathbb{E}^{\iota^-}[Y]\mathbb{1}_H] = \mathbb{E}^{\mathbb{P}}[Y\mathbb{1}_{\iota^{-1}(H)}] = \mathbb{E}^{\mathbb{P}}[Y\mathbb{1}_H] = \mathbb{E}^{\mathbb{P}}[\mathbb{E}^{\mathbb{P}}[Y \mid \mathcal{H}]\mathbb{1}_H]$$

This is true for any H, therefore the conclusion is $\mathbb{E}^{\iota^-}[Y] \sim \mathbb{E}^{\mathbb{P}}[Y \mid \mathcal{H}]$ follows from the almost uniqueness of the conditional expectation.

The definition of the unconditional expectation can also be generalized:

Definition 2.3.7 (Unconditional Expectation)**.** For any $Y \in \mathcal{L}^1(\bar{\Omega}')$ the *unconditional expectation* of Y is given by $\mathbb{E}^{!^-_{\Omega'}}[Y]$ with $!^-_{\Omega'}$ defined in proposition 2.2.6.

The following proposition expresses how the unconditional expectation is a natural generalization of the classical one:

Proposition 2.3.8. *Let* $!^-_{\Omega'} : \bar{0} \longrightarrow \bar{\Omega}'$ *be the unique* **Prob** *arrow defined in proposition 2.2.6, then the following equality is satisfied:*

$$\mathbb{E}^{!^-_{\Omega'}}[Y](*) = \mathbb{E}^{\mathbb{P}'}[Y]$$

Proof. The following equalities are almost tautological:

$$\mathbb{E}^{!^-_{\Omega'}}[Y](*) = \int_{\{*\}} \mathbb{E}^{!^-_{\Omega'}}[Y]d\mathbb{P}_0 = \int_{!^{-1}_{\Omega'}(\{*\})} [Y]d\mathbb{P}' = \int_{\Omega'} [Y]d\mathbb{P}' = \mathbb{E}^{\mathbb{P}'}[Y]$$

□

This section will be concluded with some adaptations of usual properties and definitions for the conditional expectation.

Proposition 2.3.9. *Let* $f^- : \bar{\Omega} \longrightarrow \bar{\Omega}'$ *be a* **Prob** *arrow, for* $Y_1, Y_2 \in \mathcal{L}^1(\bar{\Omega}')$ *and any* $a, b \in \mathbb{R}$*:*

1. *Linearity:* $\mathbb{E}^{f^-}[aY_1 + bY_2] \sim_{\mathbb{P}} a\mathbb{E}^{f^-}[Y_1] + b\mathbb{E}^{f^-}[Y_2]$

2. *Positivity: if* $[Y_1] \gtrsim_{\mathbb{P}'} 0$ *then* $\mathbb{E}^{f^-}[Y_1] \gtrsim_{\mathbb{P}} 0$

Proof. 1. $\forall G \in \mathcal{G}$:

$$\begin{aligned}\textstyle\int_G \mathbb{E}^{f^-}[aY_1 + bY_2]d\mathbb{P} &= \int_{f^{-1}(G)} aY_1 + bY_2 d\mathbb{P}' \\ &= a\int_{f^{-1}(G)} Y_1 d\mathbb{P}' + b\int_{f^{-1}(G)} Y_2 d\mathbb{P}' \\ &= a\int_{f^{-1}(G)} \mathbb{E}^{f^-}[Y_1]d\mathbb{P} + b\int_{f^{-1}(G)} \mathbb{E}^{f^-}[Y_2]d\mathbb{P}\end{aligned}$$

Which proves the statement.

2. $\forall G \in \mathcal{G}$ since $Y_1 \gtrsim_{\mathbb{P}'} 0$:

$$\int_G \mathbb{E}^{f^-}[Y_1]d\mathbb{P} = \int_{f^{-1}(G)} Y_1 d\mathbb{P}' \geq 0$$

we can conclude that $\mathbb{E}^{f^-}[Y_1] \gtrsim_{\mathbb{P}} 0$

□

A generalization of the measurability of a random variable is now introduced.

Definition 2.3.10. Let $f^- : \bar{\Omega} \longrightarrow \bar{\Omega}'$ be a **Prob** arrow and $Y \in \mathcal{L}^\infty(\bar{\Omega}')$. Y is called f^--measurable if there exists $X \in \mathcal{L}^\infty(\bar{\Omega})$ such that $Y \sim_{\mathbb{P}'} X \circ f$.

and the following theorem is a generalization of the classical Theorem 1.1.4.

Theorem 2.3.11. *Let* $f^- : \bar{\Omega} \longrightarrow \bar{\Omega}$ *be a* **Prob** *arrow,* $Z \in \mathcal{L}^1(\bar{\Omega}')$ *and* $Y \in \mathcal{L}^\infty(\bar{\Omega}')$ *a* f^-*-measurable random variable, then we have*

$$\mathbb{E}^{f^-}[YZ] \sim_{\mathbb{P}'} X\mathbb{E}^{f^-}[Z]$$

with $X \in \mathcal{L}^\infty(\bar{\Omega})$ *a random variable satisfying* $Y \sim_{\mathbb{P}'} X \circ f$.

Proof. By definition of the conditional expectation along f^-, it is enough to prove that for any $G \in \mathcal{G}$:

$$\int_{f^{-1}(G)} YZd\mathbb{P}' = \int_G X\mathbb{E}^{f^-}(Z)d\mathbb{P} \tag{2.3.5}$$

First, we look at the specific case when $X = \mathbb{1}_H$ with some $H \in \mathcal{G}$. We note that $Y \sim_{\mathbb{P}'} X \circ f = \mathbb{1}_H \circ f = \mathbb{1}_{f^{-1}(H)}$ and then we obtain:

$$\begin{aligned}\int_{f^{-1}(G)} YZd\mathbb{P}' &= \int_{f^{-1}(G)} \mathbb{1}_{f^{-1}(H)} Zd\mathbb{P}' \\ &= \int_{f^{-1}(G)\cap f^{-1}(H)} Zd\mathbb{P}' \\ &= \int_{G\cap H} \mathbb{E}^{f^-}[Z]d\mathbb{P} \\ &= \int_G \mathbb{1}_H \mathbb{E}^{f^-}[Z]d\mathbb{P} \\ &= \int_G X\mathbb{E}^{f^-}[Z]d\mathbb{P}\end{aligned}$$

Secondly, we can check the case where $X = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{H_i}$ for some $n \in \mathbb{N}$, $\alpha_i \in \mathbb{R}$ and $H_i \in \mathcal{G}$. By noticing that $Y \sim_{\mathbb{P}'} X \circ f = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{H_i} \circ f = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{f^{-1}(H_i)}$ and thanks to what we have already proven, we have:

$$\begin{aligned}\int_{f^{-1}(G)} YZd\mathbb{P}' &= \sum_{i=1}^{n} \alpha_i \int_{f^{-1}(G)} \mathbb{1}_{H_i} Zd\mathbb{P}' \\ &= \sum_{i=1}^{n} \alpha_i \int_{G} \mathbb{1}_{H_i} \mathbb{E}^{f^-}[Z]d\mathbb{P} \\ &= \int_{G} X\mathbb{E}^{f^-}[Z]d\mathbb{P}\end{aligned}$$

To conclude with the case $X \in \mathcal{L}^\infty(\Omega, \mathcal{G})$, we know there exists a sequence of step functions $\{X_n\}_{n\in\mathbb{N}}$ such that $|X_n| \leq |X|$ and $\lim X_n = X$. Then we know that $\lim X_n \circ f = X \circ f \sim_{\mathbb{P}'} Y$ and we have :

$$\begin{aligned}\int_{f^{-1}(G)} YZd\mathbb{P}' &= \lim_{n\longrightarrow\infty} \int_{f^{-1}(G)} (X_n \circ f)Zd\mathbb{P}' \\ &= \lim_{n\longrightarrow\infty} \int_{G} X_n \mathbb{E}^{f^-}[Z]d\mathbb{P} \\ &= \int_{G} X\mathbb{E}^{f^-}[Z]d\mathbb{P}\end{aligned}$$

□

Additional properties of this conditional expectation are given in [AR16].

2.4 Monetary Value Measure as a Functor

Definition 2.4.1 (Monetary Value Measure)**.** A *monetary value measure* is a contravariant functor:

$$\Phi : \mathbf{Prob}^{op} \longrightarrow \mathbf{Set}$$

defined by

$$\begin{array}{ccccccc} \Omega & \bar{\Omega} & \xrightarrow{\Phi} & \Phi\bar{\Omega} := L^1(\bar{\Omega}) & \ni & \left[\varphi^{f^-}[Y]\right]_{\sim_{\mathbb{P}}} \\ f \uparrow & \downarrow f^- & & \uparrow \Phi f^- & & \uparrow \Phi f^- \\ \Omega' & \bar{\Omega}' & \xrightarrow{\Phi} & \Phi\bar{\Omega}' := L^1(\bar{\Omega}') & \ni & [Y]_{\sim_{\mathbb{P}'}} \end{array}$$

where φ^{f^-} satisfies the following properties:

1. *Cash Invariance*: $\forall Y \in \mathcal{L}^\infty(\bar{\Omega}')$ and $\forall X \in \mathcal{L}^\infty(\bar{\Omega})$

$$\varphi^{f^-}(Y + X \circ f) \sim_{\mathbb{P}} \varphi^{f^-}(Y) + X$$

2. *Monotonicity*: $\forall Y_1, Y_2 \in \mathcal{L}^\infty(\bar{\Omega}')$, if $Y_1 \lesssim_{\mathbb{P}'} Y_2$ then

$$\varphi^{f^-}(Y_1) \lesssim_{\mathbb{P}} \varphi^{f^-}(Y_2)$$

3. *Normalization*: $\varphi^{f^-}(0_{\Omega'}) \sim_{\mathbb{P}} 0_{\Omega}$ if f^- is measure preserving.

4. $Y \in \mathcal{L}^{\infty}(\bar{\Omega}')$ implies $\varphi^{f^-}(Y) \in \mathcal{L}^{\infty}(\bar{\Omega})$ if f^- is measure preserving.

Let us note that the notation $\Phi[\varphi^{\cdot}]$ is sometimes used to explicitly show that the arrows mapped via Φ are determined by $\varphi^{\cdot}$.

The first important point is that Φ defines a contravariant functor if the following properties are satisfied :

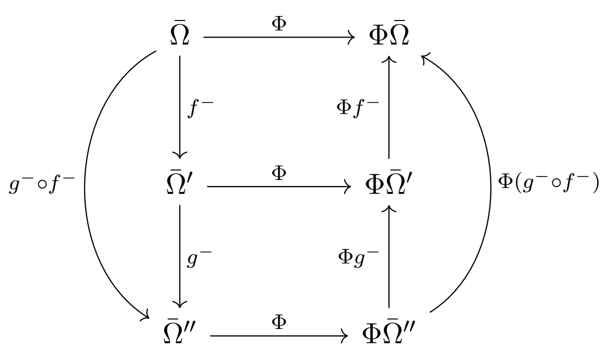
- *Contravariant* : For $f^- : \bar{\Omega} \longrightarrow \bar{\Omega}'$ a **Prob** arrow, then $\Phi f^- : \Phi(\bar{\Omega}') \longrightarrow \Phi(\bar{\Omega})$ with $\Phi f^-([Y]_{\sim_{\mathbb{P}'}}) = [\varphi^{f^-}(Y)]_{\sim_{\mathbb{P}}}$.

- *Identity Arrow*: For all $[Y]_{\sim_{\mathbb{P}}} \in L^1[\bar{\Omega}]$ the following is needed:

$$\Phi Id^-_{\bar{\Omega}}([Y]_{\sim_{\mathbb{P}}}) = [\varphi^{Id^-_{\bar{\Omega}}}(Y)]_{\sim_{\mathbb{P}}} = [Y]_{\sim_{\mathbb{P}}}$$

  in order to have $\Phi Id^-_{\bar{\Omega}} = Id_{L^1(\bar{\Omega})}$

- *Composition Law*: For all $f^- \in \mathbf{Prob}[\bar{\Omega}, \bar{\Omega}']$ and $g^- \in \mathbf{Prob}[\bar{\Omega}', \bar{\Omega}'']$, the following diagram has to be commutative:



  In other words, for all $[Z]_{\sim_{\mathbb{P}''}} \in \Phi\bar{\Omega}'' = L^1(\bar{\Omega}'')$, the following:

$$\begin{aligned} \Phi f^- \circ \Phi g^-([Z]_{\sim_{\mathbb{P}''}}) &= \Phi f^-([\varphi^{g^-}(Z)]_{\sim_{\mathbb{P}'}}) \\ &= [\varphi^{f^-}(\varphi^{g^-}(Z))]_{\sim_{\mathbb{P}}} \end{aligned}$$

  has to be equal to

$$\Phi(g^- \circ f^-)[Z]_{\sim_{\mathbb{P}''}} = [\varphi^{g^- \circ f^-}(Z)]_{\sim_{\mathbb{P}}}$$

  in order to have $\Phi f^- \circ \Phi g^- = \Phi(g^- \circ f^-)$

The second important point about our contravariant functor is that it moves internally over different several absolutely continuous probability measures.

Therefore, this set up will allow to have risk measure that includes the Knightian uncertainty dimension.

A useful lemma is introduced in order to define an example of monetary value measure in our more general context.

Lemma 2.4.2. *Let $f^- : \bar{\Omega} \longrightarrow \bar{\Omega}'$ be a measure-preserving arrow in* **Prob***, then $Y \in \mathcal{L}^\infty(\bar{\Omega}')$ implies $\mathbb{E}^{f^-}[Y] \in \mathcal{L}^\infty(\bar{\Omega})$.*

Proof. By assumption, $Y \in \mathcal{L}^\infty(\bar{\Omega}')$ so there exist $M \geq 0$ such that $-M \lesssim_{\mathbb{P}'} Y \lesssim_{\mathbb{P}'} M$. Then, by the positivity property of the conditional expectation (proposition 2.3.9):

$$0 \lesssim_{\mathbb{P}} \mathbb{E}^{f^-}[M - Y] \sim_{\mathbb{P}} \mathbb{E}^{f^-}[M] - \mathbb{E}^{f^-}[Y]$$

but since f is measure-preserving, we also have

$$\mathbb{E}^{f^-}[M] \sim_{\mathbb{P}} M\mathbb{E}^{f^-}[Id_{\bar{\Omega}'}] \sim_{\mathbb{P}} M$$

Finally, putting these two relations together we get:

$$\mathbb{E}^{f^-}[Y] \lesssim_{\mathbb{P}} \mathbb{E}^{f^-}[M] \lesssim_{\mathbb{P}} M$$

In a similar way, we can prove that $-M \lesssim_{\mathbb{P}} \mathbb{E}^{f^-}[Y]$ and the conclusion $\mathbb{E}^{f^-}[Y] \in \mathcal{L}^\infty(\bar{\Omega})$ follows. □

Here is a specific example of a monetary value measure.

Proposition 2.4.3 (Entropic Value Measures)**.** *Let $f^- \in$ **Prob**$[\bar{\Omega}, \bar{\Omega}']$, a* **Prob***-arrow and $\lambda \in \mathbb{R}$. We can define $\varphi^{f^-} : L^1(\bar{\Omega}') \longrightarrow L^1(\bar{\Omega})$ by*

$$\varphi^{f^-}[X] := \lambda^{-1} \log \mathbb{E}^{f^-}[e^{\lambda X}]$$

then, $\Phi := \Phi[\varphi^\cdot]$ is a monetary value measure. We call this Φ an entropic value measure.

Proof. 1. *Φ is a contravariant functor*

Let us consider the following arrows in **Prob**:

$$\bar{\Omega} \xrightarrow{f^-} \bar{\Omega}' \xrightarrow{g^-} \bar{\Omega}''$$

First, for $Z \in \mathcal{L}^1(\bar{\Omega}'')$ we have for the identity property :

$$\varphi^{Id^-_{\bar{\Omega}''}}(Z) = \lambda^{-1} \log \mathbb{E}^{Id^-_{\bar{\Omega}''}}[e^{\lambda Z}] \sim_{\mathbb{P}} \lambda^{-1} \log e^{\lambda Z} = \lambda^{-1}\lambda Z = Z$$

Where the equivalence is obtained thanks to Proposition 2.3.5.

Second, for the composition law, for $Z \in \mathcal{L}^1(\bar{\Omega}'')$ we have (again by Proposition 2.3.5) :

$$\begin{aligned}
\varphi^{g^- \circ f^-}(Z) &= \lambda^{-1} \log(\mathbb{E}^{g^- \circ f^-}[e^{\lambda Z}]) \\
&\sim_{\mathbb{P}} \lambda^{-1} \log(\mathbb{E}^{f^-}[\mathbb{E}^{g^-}[e^{\lambda Z}]]) \\
&= \lambda^{-1} \log(\mathbb{E}^{f^-}[e^{\log(\mathbb{E}^{g^-}[e^{\lambda Z}])}]) \\
&= \lambda^{-1} \log(\mathbb{E}^{f^-}[e^{\lambda\lambda^{-1}\log(\mathbb{E}^{g^-}[e^{\lambda Z}])}]) \\
&= \lambda^{-1} \log(\mathbb{E}^{f^-}[e^{\lambda\varphi^{g^-}(\lambda Z)}]) \\
&= \varphi^{f^-}(\varphi^{g^-}(\lambda Z))
\end{aligned}$$

Finally for $Z_1, Z_2 \in \mathcal{L}^1(\bar{\Omega}'')$, with $Z_1 \sim_{\mathbb{P}''} Z_2$ we have

$$\begin{aligned}
\varphi^{g^-}(Z_1) &= \lambda^{-1} \log(\mathbb{E}^{g^-}[e^{\lambda Z_1}]) \\
&\sim_{\mathbb{P}} \lambda^{-1} \log(\mathbb{E}^{g^-}[e^{\lambda Z_2}]) \\
&= \varphi^{g^-}(Z_2)
\end{aligned}$$

2. *Cash invariance* For $Y \in \mathcal{L}^\infty(\bar{\Omega}')$ and $X \in \mathcal{L}^\infty(\bar{\Omega})$, we have by Theorem 2.3.11:

$$\begin{array}{rcl} \varphi^{f^-}(Y + X \circ f) & = & \lambda^{-1}\log(\mathbb{E}^{f^-}[e^{\lambda(Y+X\circ f)}]) \\ & = & \lambda^{-1}\log(\mathbb{E}^{f^-}[e^{\lambda Y}(e^{\lambda X}\circ f)]) \\ & \sim_{\mathbb{P}} & \lambda^{-1}\log(e^{\lambda X}\mathbb{E}^{f^-}[e^{\lambda Y}]) \\ & = & X + \varphi^{f^-}(Y) \end{array}$$

3. *Monotonicity*

   If $Y_1 \lesssim_{\mathbb{P}'} Y_2$ then we have $0 \lesssim_{\mathbb{P}'} Y_2 - Y_1$ and $0 \lesssim_{\mathbb{P}} \mathbb{E}^{f^-}[Y_2 - Y_1]$. and $\mathbb{E}^{f^-}[Y_1] \lesssim_{\mathbb{P}} \mathbb{E}^{f^-}[Y_2]$ follows. With these result, we can conclude:

$$\begin{array}{rcl} \varphi^{f^-}(Y_1) & = & \lambda^{-1}\log(\mathbb{E}^{f^-}[e^{\lambda Y_1}]) \\ & \lesssim_{\mathbb{P}} & \lambda^{-1}\log(\mathbb{E}^{f^-}[e^{\lambda Y_2}]) \\ & = & \varphi^{f^-}(Y_2) \end{array}$$

4. *Normalization*

   if f^- is measure-preserving, then we have :

$$\begin{array}{rcl} \varphi^{f^-}(0_{\bar{\Omega}'}) & = & \lambda^{-1}\log(\mathbb{E}^{f^-}[e^{\lambda 0_{\bar{\Omega}'}}]) \\ & = & \lambda^{-1}\log(\mathbb{E}^{f^-}[1]) \\ & \sim_{\mathbb{P}} & \lambda^{-1}\log(1) \\ & = & 0_{\bar{\Omega}} \end{array}$$

5. Finally, the last property of the monetary value follows from Lemma 2.4.2

□

Now, a partial answer to the question about the properties that a monetary measure should satisfy can be given. Some properties are not axioms, anymore but direct consequences from the definition.

Theorem 2.4.4. *We consider $\Phi = \Phi[\varphi^\cdot]$: $\mathbf{Prob}^{op} \longrightarrow \mathbf{Set}$ a monetary measure. The following properties are satisfied:*

1. *If f^- is measure-preserving, then $\Phi f^- \circ Lf^- = Id_{L\bar{\Omega}}$.*

2. *Idempotence: If f^- is measure preserving:*

$$\Phi f^- \circ Lf^- \circ \Phi f^- = \Phi f^-$$

3. *Local property: for all $Y_1, Y_2 \in \mathcal{L}^\infty(\bar{\Omega}')$ and for all $G \in \mathcal{G}$:*

$$\Phi f^-[\mathbb{1}_{f^{-1}(G)}Y_1 + \mathbb{1}_{f^{-1}(G^c)}Y_2]_{\sim_{\mathbb{P}'}} = [\mathbb{1}_{f^{-1}(G)}]_{\sim_{\mathbb{P}'}}\Phi f^-[Y_1]_{\sim_{\mathbb{P}'}} + [\mathbb{1}_{f^{-1}(G^c)}]_{\sim_{\mathbb{P}}}\Phi f^-[Y_2]_{\sim_{\mathbb{P}'}}$$

4. *Dynamic programming principle: If g^- is measure-preserving, for $Z \in \mathcal{L}^\infty(\bar{\Omega}'')$*

$$\varphi^{g^- \circ f^-}(Z) = \varphi^{g^- \circ f^-}(\varphi^{g^-}(Z) \circ g)$$

5. *Time Consistency: For all $Z_1, Z_2 \in \mathcal{L}^\infty(\bar{\Omega}'')$, if $\varphi^{g^-}(Z_1) \lesssim_{\mathbb{P}'} \varphi^{g^-}(Z_2)$ then*

$$\varphi^{g^- \circ f^-}(Z_1) \lesssim_{\mathbb{P}} \varphi^{g^- \circ f^-}(Z_2)$$

Proof. 1. For $X \in \mathcal{L}(\bar{\Omega})$, we have

$$\Phi f^-(Lf^-[X]_{\sim_\mathbb{P}}) = [\varphi^{f^-}(X \circ f)]_{\sim_\mathbb{P}}$$

and by cash invariance and normalization we have

$$\varphi^{f^-}(X \circ f) = \varphi^{f^-}(0_{\Omega'} + (X \circ f)) \sim_\mathbb{P} \varphi^{f^-}(0_{\Omega'}) + X \sim_\mathbb{P} 0_\Omega + X = X$$

and we can conclude :

$$\Phi f^- \circ Lf^- = Id_{L\bar{\Omega}}$$

2. Now by using 1) we have :

$$\Phi f^- \circ Lf^- \circ \Phi f^- = Id_{L\bar{\Omega}} \circ \Phi f^- = \Phi f^-$$

3. First, we see that for any $Y \in \mathcal{L}^\infty(\bar{\Omega}')$ and for all $\omega' \in \Omega'$ we have

$$|Y(w')| \lesssim_{\mathbb{P}'} \|Y\|_{\mathcal{L}^\infty(\bar{\Omega}')}$$

Therefore for any $G \in \mathcal{G}$:

$$\mathbb{1}_{f^{-1}(G)}Y - \mathbb{1}_{f^{-1}(G^c)}\|Y\|_{\mathcal{L}^\infty(\bar{\Omega}')} \lesssim_{\mathbb{P}'} \mathbb{1}_{f^{-1}(G)}Y + \mathbb{1}_{f^{-1}(G^c)}Y \lesssim_{\mathbb{P}'} \mathbb{1}_{f^{-1}(G)}Y + \mathbb{1}_{f^{-1}(G^c)}\|Y\|_{\mathcal{L}^\infty(\bar{\Omega}')}$$

Since $\mathbb{1}_G \circ f = \mathbb{1}_{f^{-1}(G)}$, by cash invariance and monotonicity we have :

$$\begin{aligned}
\varphi^{f^-}(\mathbb{1}_{f^{-1}(G)}Y) - \|Y\|_{\mathcal{L}^\infty(\bar{\Omega}')}\mathbb{1}_{G^c} &\sim_\mathbb{P} \varphi^{f^-}(\mathbb{1}_{f^{-1}(G)}Y - (\|Y\|_{\mathcal{L}^\infty(\bar{\Omega}')}\mathbb{1}_{G^c}) \circ f) \\
&= \varphi^{f^-}(\mathbb{1}_{f^{-1}(G)}Y - \mathbb{1}_{G^c}\|Y\|_{\mathcal{L}^\infty(\bar{\Omega}')}) \\
&\lesssim_\mathbb{P} \varphi^{f^-}(Y) \\
&\lesssim_\mathbb{P} \varphi^{f^-}(\mathbb{1}_{f^{-1}(G)}Y + \mathbb{1}_{G^c}\|Y\|_{\mathcal{L}^\infty(\bar{\Omega}')}) \\
&= \varphi^{f^-}(\mathbb{1}_{f^{-1}(G)}Y + (\|Y\|_{\mathcal{L}^\infty(\bar{\Omega}')}\mathbb{1}_{G^c}) \circ f) \\
&\sim_\mathbb{P} \varphi^{f^-}(\mathbb{1}_{f^{-1}(G)}Y) + \|Y\|_{\mathcal{L}^\infty(\bar{\Omega}')}\mathbb{1}_{G^c}
\end{aligned}$$

which is equivalent to:

$$\varphi^{f^-}(\mathbb{1}_{f^{-1}(G)}Y) - \mathbb{1}_{G^c}\|Y\|_{\mathcal{L}^\infty(\bar{\Omega}')} \lesssim_\mathbb{P} \varphi^{f^-}(Y) \lesssim_\mathbb{P} \varphi^{f^-}(\mathbb{1}_{f^{-1}(G)}Y) + \mathbb{1}_{G^c}\|Y\|_{\mathcal{L}^\infty(\bar{\Omega}')}$$

By multiplying by $\mathbb{1}_G$ we have :

$$\mathbb{1}_G\varphi^{f^-}(\mathbb{1}_{f^{-1}(G)}Y) \lesssim_\mathbb{P} \mathbb{1}_G\varphi^{f^-}(Y) \lesssim_\mathbb{P} \mathbb{1}_G\varphi^{f^-}(\mathbb{1}_{f^{-1}(G)}Y)$$

and we can conclude that:

$$\mathbb{1}_G\varphi^{f^-}(Y) \sim_\mathbb{P} \mathbb{1}_G\varphi^{f^-}(\mathbb{1}_{f^{-1}(G)}Y) \tag{2.4.1}$$

Now using 2.4.1 twice, we have for $Y_1, Y_2 \in \mathcal{L}^\infty(\bar{\Omega}')$ and $G \in \mathcal{G}$:

$$\begin{array}{rcl} \varphi^{f^-}(\mathbb{1}_{f^{-1}(G)}Y_1 + \mathbb{1}_{f^{-1}(G^c)}Y_2) & = & \mathbb{1}_{f^{-1}(G)}\varphi^{f^-}(\mathbb{1}_{f^{-1}(G)}Y_1 + \mathbb{1}_{f^{-1}(G^c)}Y_2) \\ & & +\mathbb{1}_{f^{-1}(G^c)}\varphi^{f^-}(\mathbb{1}_{f^{-1}(G)}Y_1 + \mathbb{1}_{f^{-1}(G^c)}Y_2) \\ & \sim_{\mathbb{P}} & \mathbb{1}_{f^{-1}(G)}\varphi^{f^-}(\mathbb{1}_{f^{-1}(G)}(\mathbb{1}_{f^{-1}(G)}Y_1 + \mathbb{1}_{f^{-1}(G^c)}Y_2)) \\ & & +\mathbb{1}_{f^{-1}(G^c)}\varphi^{f^-}(\mathbb{1}_{f^{-1}(G^c)}(\mathbb{1}_{f^{-1}(G)}Y_1 + \mathbb{1}_{f^{-1}(G^c)}Y_2)) \\ & = & \mathbb{1}_{f^{-1}(G)}\varphi^{f^-}(\mathbb{1}_{f^{-1}(G)}Y_1) + \mathbb{1}_{f^{-1}(G^c)}\varphi^{f^-}(\mathbb{1}_{f^{-1}(G^c)}Y_2) \\ & \sim_{\mathbb{P}} & \mathbb{1}_{f^{-1}(G)}\varphi^{f^-}(Y_1) + \mathbb{1}_{f^{-1}(G^c)}\varphi^{f^-}(Y_2) \end{array}$$

This concludes the proof of the local property.

4. Using the idempotence property, we have for $Z \in \mathcal{L}^\infty(\bar{\Omega}'')$ that $\varphi^{g^-}(\varphi^{g^-}(Z) \circ g) \sim_{\mathbb{P}'} \varphi^{g^-}(Z)$. Since, $\varphi^\cdot$ is a contravariant functor we can also conclude :

$$\begin{array}{rcl} \varphi^{g^- \circ f^-}(Z) & \sim_{\mathbb{P}} & \varphi^{f^-}(\varphi^{g^-}(Z)) \\ & \sim_{\mathbb{P}} & \varphi^{f^-}(\varphi^{g^-}(\varphi^{g^-}(Z) \circ g)) \\ & = & (\varphi^{f^-} \circ \varphi^{g^-})(\varphi^{g^-}(Z) \circ g) \\ & \sim_{\mathbb{P}} & \varphi^{g^- \circ f^-}(\varphi^{g^-}(Z) \circ g) \end{array}$$

5. Finally, to prove 5), let us assume that we have $Z_1, Z_2 \in \mathcal{L}^\infty(\bar{\Omega}'')$ with $\varphi^{g^-}(Z_1) \lesssim_{\mathbb{P}'} \varphi^{g^-}(Z_2)$. Then, using the monotonicity property and the fact that $\varphi^\cdot$ is a contravariant functor:

$$\varphi^{g^- \circ f^-}(Z_1) \sim_{\mathbb{P}} \varphi^{f^-}(\varphi^{g^-}(Z_1)) \lesssim_{\mathbb{P}} \varphi^{f^-}(\varphi^{g^-}(Z_2)) \sim_{\mathbb{P}} \varphi^{g^- \circ f^-}(Z_2)$$

which proves our time consistency.

□

Finally, the Yoneda Lemma (major in category theory) is declined for our monetary value measure:

Theorem 2.4.5 (Yoneda Lemma). *For any monetary value measure $\Phi : \mathbf{Prob}^{op} \longrightarrow \mathbf{Set}$ and an object $\bar{\Omega}$ in* **Prob**, *there exists a bijective correspondance $y_{\Phi,\bar{\Omega}}$ specified by*

$$y_{\Phi,\bar{\Omega}} : \mathcal{NAT}(\mathbf{Prob}(-,\bar{\Omega}),\Phi) \xrightarrow{\simeq} \Phi(\bar{\Omega}) \quad : \quad \begin{array}{l} \alpha \longrightarrow \alpha_{\bar{\Omega}}(Id^-_{\bar{\Omega}}) \\ \tilde{X} \longleftarrow X \end{array}$$

where $\tilde{X}$ is the natural transformation defined for any $f^- : \bar{\Omega} \longrightarrow \bar{\Omega}'$ in **Prob** *by $\tilde{X}_{\bar{\Omega}'}(f^-) := \Phi f^- X$. Moreover, the correspondance is natural in both Φ and $\bar{\Omega}$.*

A natural transformation from $\mathbf{Prob}(-,\bar{\Omega})$ to Φ may be seen as a stochastic process somehow adapted to Φ. The corresponding $\mathcal{G}$-measurable random variable may represent a terminal value (payoff or possible loss) at a horizon. The result above can be translated by saying that those stochastic processes and those random variables are in bijection.

2.5 Generalized Filtration

In this final section of this chapter, generalized filtrations are defined. In [ANR20], those generalized filtrations are applied on a binomial pricing model. The purpose of this explanation is informative, as it will not be further explore in the upcoming chapters.

Definition 2.5.1 (Generalized filtration). A generalized filtration is a family of null-preserving functions

$$\{\bar{\Omega}_s \xleftarrow{f_{s,t}} \bar{\Omega}_t\}_{s \leq t}$$

satisfying

$$f_{t,t} = Id_{\bar{\Omega}_t} \text{ and } f_{s,t} \circ f_{t,u} = f_{s,u}$$

for all triples $s \leq t \leq u \in \mathcal{T}$ a time domain.

The next goal of this section is to see our filtration as a functor.

First, the time domain $\mathcal{T}$ can be seen as a category whose objects are its elements and we have an arrow from t to s when there is a relation $s \leq t$.

The filtration defined above can be defined as a functor $F : \mathcal{T} \longrightarrow \mathbf{Prob}$
We have the following diagram :

$$\begin{array}{ccccccccc}
\mathcal{T} & & t_0 & \xleftarrow[\leq]{} & t_1 & \xleftarrow[\leq]{} & t_2 & \xleftarrow[\leq]{} & \ldots \\
\Big\downarrow{\scriptstyle F} & & & & & & & & \\
\mathbf{Prob} & & \bar{\Omega}_{t_0} & \xleftarrow[f_{t_0,t_1}]{} & \bar{\Omega}_{t_1} & \xleftarrow[f_{t_0,t_1}]{} & \bar{\Omega}_{t_2} & \xleftarrow[f_{t_0,t_1}]{} & \ldots
\end{array}$$

Chapter 3

Examples/Possible Applications

The goal of this chapter is to introduce some examples and possible applications of the theory described above. First, an introductive example is given. Second, a trivial example from credit insurance will be given and will lead to a more complex and numerical example (see Section 3.3).

Then, possible applications in telematic insurance pricing and rare event modeling are mentioned. Finally, a more theoretical idea related to the axioms chosen for risk measure will be sketched.

3.1 First introductive example

One possible situation would be to have two probability spaces $\bar{\Omega} = (\Omega, \mathcal{G}, \mathbb{P})$ and $\bar{\Omega}' = (\Omega', \mathcal{G}', \mathbb{P}')$. The first one represents a set of scenarios (or events) and their associated probabilities in the future. The second one represents the data (and probability measure) that are available today in the present. Then, the goal is to define :

$$f : \underbrace{(\Omega, \mathcal{G}, \mathbb{P})}_{\text{Future}} \longrightarrow \underbrace{(\Omega', \mathcal{G}', \mathbb{P}')}_{\text{Present}}$$

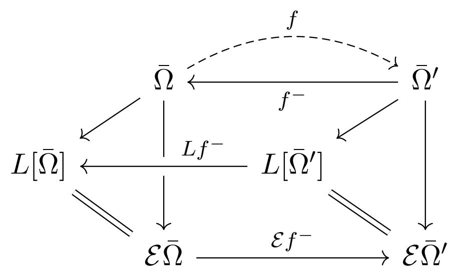
which is null-preserving. That is for all $G' \in \mathcal{G}'$ with $\mathbb{P}'[G'] = 0$ implies $\mathbb{P}[f^{-1}(G')] = 0$.

In other words, the theory needs a function that is taking elements from the future and bringing them back to today with the particularity that if an event has a probability of zero of happening today then it should have its pre-image with a probability of zero that happen tomorrow.

If one succeeds to define such a function, altough it is the knottiest part of all the applications, then the generalized conditional expectation follows. Indeed, Theorem 2.3.3 guarantees that for all random variables $X \in L^\infty(\Omega, \mathcal{G}, \mathbb{P})$ (so defined on the future, e.g. possible loss happening in the case of future scenario), there is $\mathbb{E}^{f^-}[X] \in L^\infty(\Omega', \mathcal{G}', \mathbb{P}')$ (so a random variable defines on the present), such that for all $G' \in \mathcal{G}'$:

$$\mathbb{E}^{\mathbb{P}'}[\mathbb{1}_{G'}\mathbb{E}^{f^-}[X]] = \int_{G'} \mathbb{E}^{f^-}[X] d\mathbb{P}' = \int_{f^{-1}(G')} X d\mathbb{P} = \mathbb{E}^{\mathbb{P}}[\mathbb{1}_{f^{-1}(G')}X]$$

in this setting, the following diagram is satified (notations are the same as in the previous chapter):



Where we have :

$$\mathcal{E}f^-([X]_{\mathbb{P}}) = \left[\mathbb{E}^{f^-}[X]\right]_{\mathbb{P}'}$$

The main difficulty is to define the null-preserving function. The (trivial) example developed in the next section shows that it is possible.

On the other hand, another possibility to extend this work would be to relax the assumptions made on this null-preserving function in order to be able to find such a function. This has been discussed, but nothing conclusive has been found at the moment[1]. This could constitute a possible path of improvement of the theory presented here.

3.2 An Elementary Application in Credit Insurance

Some products in credit insurance are difficult to evaluate in terms of underlying risk. For example, let us imagine that the client is a beer producer. He desires to sell some barrels to buyers accross the world. He wishes to insure his financial transactions in case his buyer cannot pay for external reasons[2].

The insurance company could propose an insurance on the value of the transaction for each barrel that he sells with a maximal exposure. At the begining of the insurance contract (when the insurance company estimates the premium), the problems are:

- The insurance company cannot anticipate how many barrels the client is going to sell
- The insurance company cannot really take into account that the probability of default may change because of future political event
- The amount per transaction may also change through time up to the maximal exposure
- It will not be commercial to compute a premium based only on the maximal exposure and really high probability of default

[1] From our email discussions with Adachi and personal research, I did not find such a reliable extension at the moment.

[2] Those reasons may be war, political instability, pandemy, bankruptcy... Recently, we have had a couple of examples in the world.

It is only at the end of the coverage period that the insurance company will know exactly the number and the amount of transactions covered (the real exposure) and the real probability of default it has been exposed to.

Keeping in mind this example, the goal of this section is to describe what is happening in a simpler setting. The "present" data considered is the following trivial space presented in proposition 2.2.6 (initial object of our category):

$$\bar{\Omega}' = \bar{0} = (\{*\}, \{\{*\}, \emptyset\}, \mathbb{P}_0)$$

with

$$\mathbb{P}_0[\emptyset] = 0 \text{ and } \mathbb{P}_0[\{*\}] = 1$$

Here, the singleton could represent the object of the transaction (for example one barrel of beer). Then, the second space which is the "future" data is the following :

- $\Omega = \{\omega_1, \omega_0\}$ where ω_1 is the scenario where the buyer succeed to pay the transaction and ω_0 is the scenario where he was in default.
- $\mathcal{G} = \{\emptyset, \Omega, \{\omega_1\}, \{\omega_0\}\}$ is the smallest σ-algebra defined on Ω
- $\mathbb{P}[\emptyset] = 0$,$\mathbb{P}[\omega_1] = \gamma$,$\mathbb{P}[\omega_0] = 1 - \gamma$ and $\mathbb{P}[\Omega_1] = 1$ for some $\gamma \in [0, 1]$

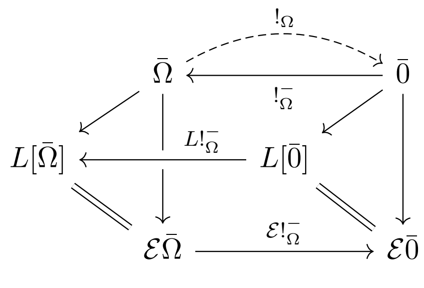
Now, the function defined in Proposition 2.2.6 is used:

$$f = !_\Omega : \Omega \longrightarrow \{*\} : \omega \in \Omega \longrightarrow \{*\}$$

This function associates the result of the transaction with the initial object of the transaction. One can check that this function is null-preserving:

if $G' \in \bar{\Omega}' = \bar{0}$ with $\mathbb{P}_0[G'] = 0$ then necessarily $G' = \emptyset$ and $\mathbb{P}[!_\Omega^{-1}(\emptyset)] = \mathbb{P}[\emptyset] = 0$ follows.

The diagram above becomes:



Let $X \in L[\bar{\Omega}]$ be a random variable defined by:

$$X : \Omega \longrightarrow \mathbb{R} : \begin{cases} \omega_1 \longrightarrow 0 & \text{No losses for the insurer} \\ \omega_0 \longrightarrow -K & \text{Value of the object of the transaction} \end{cases}$$

By definition of the conditional expectation, for all $G' \in \mathcal{G}' = \{\{*\}, \emptyset\}$ the following is satisfied:

$$\mathbb{E}^{\mathbb{P}_0}[\mathbb{1}_{G'}\mathbb{E}^{!_{\bar{\Omega}}}[X]] = \int_{G'} \mathbb{E}^{!_{\bar{\Omega}}}[X]d\mathbb{P}_0 = \int_{!_{\bar{\Omega}}^{-1}(G')} Xd\mathbb{P} = \mathbb{E}^{\mathbb{P}}[\mathbb{1}_{!_{\bar{\Omega}}^{-1}(G')}X]$$

But there are only two possibilities for G' which are $\{\{*\}, \emptyset\}$:

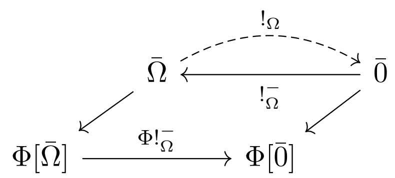
- $G' = \emptyset$ is trivial but has no interest since no transaction takes place.
- $G' = \{*\}$ then the equation above becomes:

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}_0}[\mathbb{1}_{\{*\}}\mathbb{E}^{!_{\bar{\Omega}}}[X]] &= \int_{\{*\}} \mathbb{E}^{!_{\bar{\Omega}}}[X]d\mathbb{P}_0 \\
&= \mathbb{E}^{\mathbb{P}}[\mathbb{1}_{!_{\bar{\Omega}}^{-1}(\{*\})}X] \\
&= \mathbb{E}^{\mathbb{P}}[\mathbb{1}_{\Omega}X] \\
&= \mathbb{E}^{\mathbb{P}}[X] \\
&= X(\omega_1)\mathbb{P}[\omega_1] + X(\omega_0)\mathbb{P}[\omega_0]
\end{aligned}$$

Therefore, the expectation under $\mathbb{P}_0$ of the conditional expectation of the random variable is simply the unconditional expectation of the random variable[3]:

$$\mathbb{E}^{\mathbb{P}_0}[\mathbb{1}_{\{*\}}\mathbb{E}^{!_{\bar{\Omega}}}[X]] = -K(1-\gamma)$$

The entropic measure can be also computed :



where

$$\Phi!_{\bar{\Omega}} : \Phi[\bar{\Omega}] = L^1[\Omega] \longrightarrow \Phi[\bar{0}] = L^1[\bar{0}] : [X]_{\mathbb{P}} \longrightarrow \left[\varphi^{!_{\bar{\Omega}}}[X]\right]_{\mathbb{P}_0}$$

with $\varphi^{!_{\bar{\Omega}}}[X] = \lambda^{-1}\log(\mathbb{E}^{!_{\bar{\Omega}}}[e^{\lambda X}])$ for $\lambda \in \mathbb{R}$. This is a random variable in $L^1[\bar{0}]$ which has only one element, the singleton. So, the following is also satisfied:

$$\begin{aligned}
\varphi^{!_{\bar{\Omega}}}[X]\{*\} &= \lambda^{-1}\log(\mathbb{E}^{!_{\bar{\Omega}}}[e^{\lambda X}]\{*\}) \\
&= \lambda^{-1}\log(e^{-\lambda K}(1-\gamma)) \\
&= \lambda^{-1}(\log(e^{-\lambda K}) + \log(1-\gamma)) \\
&= -K + \lambda^{-1}\log(1-\gamma)
\end{aligned}$$

which coincides with the classical entropic measure given by Definition 1.2.4:

$$\begin{aligned}
\varphi[X] &= \lambda^{-1}\log(\mathbb{E}^{\mathbb{P}}[e^{\lambda X}]) \\
&= \lambda^{-1}\log(e^{-\lambda K}(1-\gamma)) \\
&= -K + \lambda^{-1}\log(1-\gamma)
\end{aligned}$$

This trivial example shows that the null-preserving function can always be defined.

[3]This result was expected because of Definition 2.3.7 and Proposition 2.3.8

One natural extention of the example above could be to consider more than one transaction and see what will become of the conditional expectations. This will be the topic of the next section.

3.3 Credit Insurance and Multiple Transactions: A Numerical Example

The goal of this section is to describe a practical numerical application. This will be a fictive example, but will allow the reader to understand how one may apply the previous sections.

It is important to emphasize that this example shows only the technicity of the methodology in a simple case. Here, only one simple rule on the probabity spaces is introduced and could be in this case treated with other means such as a binomial process methodology. Nevertheless, the goal is to show how in practice one could apply the methodology. Of course, to use the full potential of the theory, several rules should be applied on the different spaces.

Let us imagine someone who is doing business with another country; for example, someone who is exporting beer barrels to another country. This person wishes to insure his transactions with that country against political risks (e.g. civil war, etc.).

The simplest and most naive way to compute the risk would be to check on average how many transactions (n) this person is usually making, the average value of each transaction (K) and what political risk the country is facing (1-γ). In this trivial model, the maximal exposure for the insurer may be given by nK and one could- assign a risk premium of $nK(1-\gamma)$.

They are, of course, several reasons why this estimation is not accurate or at least may be a lower bound:

1. The insurer is not sure that the client is going to perform n transactions again.
2. The average value K of transactions may differ from one exportation to another.
3. The probability that the transaction goes well γ related to the country is fixed at the begining of the contract and does not change.

One classical way to face those issue is the following:
First, the insurer computes a minimal premium (given by $nK(1-\gamma)$) at the begining of the coverage period.
Then, at the end of the contract, the insurer re-computes the real risk it has been exposed to during the lifetime of the contract and can ask for an adjustment of the original premium.
The issue here is that the insurer does not really know what is the risk is exposed to and only knows it at the end.

With the previous example in mind, and the general theory that has been developed above, the third issue described is going to be addressed.

If a transaction does not go well with a debtor because a political risk has occured (e.g. war started), it is more likely that the next transaction in the same country does not work as well. In other words, as soon as a transaction fails, the probability of default of the next one increases.

Another way to see the consideration above is to say that at $t = 0$ (at the begining of the contract). Depending if future transactions will be honored or not, several probability spaces co-exist. The more convenient one would be where all transactions suceed until the end and the probability does not change. The worst one to consider would be where a political risk occurs at the beginning, the first transaction fails then the next one and so on. In this case the probability of default is increasing at each transaction.

In general, let us consider the following probability space associated with the first transaction performed:

- $\Omega_1 = \{\omega_1, \omega_0\}$ where ω_1 is the scenario where the buyer succeeded in making the transaction and ω_0 is the scenario where he was in default.
- $\mathcal{G} = \{\emptyset, \Omega, \{\omega_1\}, \{\omega_0\}\}$ is the smallest σ-algebra defined on Ω
- $\mathbb{P}_1[\emptyset] = 0$,$\mathbb{P}[\omega_1] = \gamma$,$\mathbb{P}[\omega_0] = 1 - \gamma_1$ and $\mathbb{P}[\Omega_1] = 1$ for some $\gamma_1 \in [0, 1]$ associated with the orignial risk of the country the client is exporting to.

Now for each transaction (natural number) $t \in [2 : n]$, two probability spaces are associated:

1. the space $\bar{\Omega}_t^{i_1 \ldots i_{t-2}1}$ where the previous transaction worked :
   - $\Omega_t^{i_1 \ldots i_{t-2}1} = \{\omega_{i_1 \ldots i_{t-2}11}, \omega_{i_1 \ldots i_{t-2}10}\}$ where $\omega_{i_1 \ldots i_{t-2}11}$ is the scenario where the buyer succeeds in making the transaction t and $\omega_{i_1 \ldots i_{t-2}10}$ is the scenario where he is in default.
   - $\mathcal{G}_t^{i_1 \ldots i_{t-2}1} = \{\emptyset, \Omega, \{\omega_{i_1 \ldots i_{t-2}11}\}, \{\omega_{i_1 \ldots i_{t-2}10}\}\}$ is the smallest σ-algebra defined on $\Omega_t^{i_1 \ldots i_{t-2}1}$
   - For some $\gamma_t^{i_1 \ldots i_{t-2}1} \in [0, 1]$ (associated with the risk of the country the client is exporting to), the probabilities are given by:

$$\begin{array}{ll} \mathbb{P}_t^{i_1 \ldots i_{t-2}1} & [\emptyset] = 0 \\ \mathbb{P}_t^{i_1 \ldots i_{t-2}1} & [\omega_{i_1 \ldots i_{t-2}11}] = \gamma_t^{i_1 \ldots i_{t-2},1} \\ \mathbb{P}_t^{i_1 \ldots i_{t-2}1} & [\omega_{i_1 \ldots i_{t-2}10}] = 1 - \gamma_t^{i_1 \ldots i_{t-2},1} \\ \mathbb{P}_t^{i_1 \ldots i_{t-2}1} & [\Omega_t^{i_1 \ldots i_{t-2}1}] = 1 \end{array}$$

2. the space $\bar{\Omega}_t^{i_1 \ldots i_{t-2}0}$ where the previous transaction failed:
   - $\Omega_t^{i_1 \ldots i_{t-2}0} = \{\omega_{i_1 \ldots i_{t-2}01}, \omega_{i_1 \ldots i_{t-2}00}\}$ where $\omega_{i_1 \ldots i_{t-2}01}$ is the scenario where the buyer succeeds to pay the transaction t and $\omega_{i_1 \ldots i_{t-2}00}$ is the scenario where he is in default.
   - $\mathcal{G}_t^{i_1 \ldots i_{t-2}0} = \{\emptyset, \Omega, \{\omega_{i_1 \ldots i_{t-2}01}\}, \{\omega_{i_1 \ldots i_{t-2}00}\}\}$ is the smallest σ-algebra defined on $\Omega_t^{i_1 \ldots i_{t-2}0}$

- For some $\gamma_t^{i_1 \dots i_{t-2}0} \in [0,1]$ (associated with the risk of the country the client is exporting to), the probabilities are given by:

$$\begin{array}{ll} \mathbb{P}_t^{i_1 \dots i_{t-2}0} & [\emptyset] = 0 \\ \mathbb{P}_t^{i_1 \dots i_{t-2}0} & [\omega_{i_1 \dots i_{t-2}01}] = \gamma_t^{i_1 \dots i_{t-2}0} \\ \mathbb{P}_t^{i_1 \dots i_{t-2}0} & [\omega_{i_1 \dots i_{t-2}00}] = 1 - \gamma_t^{i_1 \dots i_{t-2}0} \\ \mathbb{P}_t^{i_1 \dots i_{t-2},0} & [\Omega_t^{i_1 \dots i_{t-2}0}] = 1 \end{array}$$

In this example, only the probability measure change depending on the previous transaction. In a more complexe example, one could imagine several transactions happening in different countries in the same times (so a space of scenarios with more than two elements at each iteration). Also, different conclusions on the spaces could be defined at each iterations, e.g. the transaction may stop if they are two transactions in a row that fail. In this paper, only this simple case is developped as the goal is to show the technicity of the methodology.

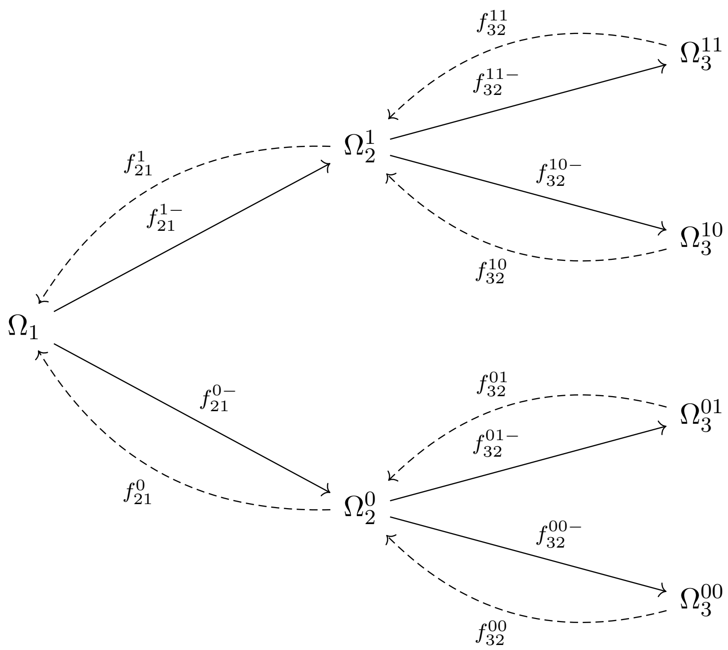
Now that the spaces $\bar{\Omega}_t^{i_1 \dots i_t}$ are defined for each transaction t, the next step is to define a function which is null-preserving between them. The projection on the first $t-1$ component is a natural candidate:

$$f_{tt-1}^{i_1 \dots i_{t-1}} : \bar{\Omega}_t^{i_1 \dots i_{t-1}} \longrightarrow \bar{\Omega}_{t-1}^{i_1 \dots i_{t-2}} : \omega_{i_1 \dots i_{t-1} i_t} \longrightarrow \omega_{i_1 \dots i_{t-1}}$$

This function is null-preserving. Indeed if $G_t^{i_1 \dots i_{t-1}} \in \mathcal{G}_t^{i_1 \dots i_{t-1}}$ with $\mathbb{P}_t^{i_1 \dots i_{t-1}}[G_t^{i_1 \dots i_{t-1}}] = 0$ then by definition of $\bar{\Omega}_{t-1}^{i_1 \dots i_{t-1}}$, necessarly $G_t^{i_1 \dots i_{t-1}} = \emptyset$ and the following is satsified:

$$\mathbb{P}_t^{i_1 \dots i_{t-1}}[(f_{tt-1}^{i_1 \dots i_t})^{-1}(\emptyset)] = \mathbb{P}_t^{i_1 \dots i_{t-1}}[\emptyset] = 0$$

For $t = 3$, the situation has the following diagram representation:



Now, let us define a random variable on a space $\bar{\Omega}_t^{i_1 \dots i_t}$ by

$$X_t^{i_1 \dots i_t} : \Omega_t^{i_1 \dots i_t} \longrightarrow -K \sum_{j=1}^{t} (1 - i_j)$$

In other words, at transaction t, the random variable is associating the total loss for all the failed transactions (past and present).

Therefore, $X_t^{i_1\dots i_{t-1}} \in L(\Omega_t^{i_1\dots i_t})$, and by Theorem 2.3.3, since $f_{tt-1}^{i_1\dots i_{t-1}-} \in \mathbf{Prob}[\bar{\Omega}_{t-1}^{i_1\dots i_{t-2}}, \bar{\Omega}_t^{i_1\dots i_{t-1}}]$, the conditional expectation $\mathbb{E}^{f_{tt-1}^{i_1\dots i_{t-1}-}}[X_t^{i_1\dots i_{t-1}}]$ is a well defined random variable on $L(\Omega_{t-1}^{i_1\dots i_{t-2}})$ such that for all $G \in \mathcal{G}_{t-1}^{i_1\dots i_{t-1}}$ the following is satisfied:

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}_{t-1}^{i_1\dots i_{t-2}}}\left[\mathbb{1}_G \mathbb{E}^{f_{tt-1}^{i_1\dots i_{t-1}-}}[X_t^{i_1\dots i_{t-1}}]\right] &= \int_G \mathbb{E}^{f_{tt-1}^{i_1\dots i_{t-1}-}}[X_t^{i_1\dots i_{t-1}}] d\mathbb{P}_{t-1}^{i_1\dots i_{t-2}} \\
&= \int_{(f_{tt-1}^{i_1\dots i_{t-1}})^{-1}(G)} X_t^{i_1\dots i_{t-1}} d\mathbb{P}_t^{i_1\dots i_{t-1}} \\
&= \mathbb{E}^{\mathbb{P}_t^{i_1\dots i_{t-1}}}[\mathbb{1}_{f_{tt-1}^{i_1\dots i_{t-1}-}(G)} X_t^{i_1\dots i_{t-1}}]
\end{aligned}$$

If $\mathcal{G}_{t-1}^{i_1\dots i_{t-1}} = \{\emptyset, \Omega, \{\omega_{i_1\dots i_{t-2}11}\}, \{\omega_{i_1\dots i_{t-2}10}\}\}$ (meaning previous transactions succeeded), then, the random variable $\mathbb{E}^{f_{tt-1}^{i_1\dots i_{t-1}-}}[X_t^{i_1\dots i_{t-1}}]$ can be explicitely computed:

$$\mathbb{E}^{f_{tt-1}^{i_1\dots i_{t-2}1-}}[X_t^{i_1\dots i_{t-1}}] : \Omega_{t-1}^{i_1\dots i_{t-2}} \longrightarrow \mathbb{R} : \begin{cases} \omega_{i_1\dots i_{t-2}0} \longrightarrow 0 \\ \omega_{i_1\dots i_{t-2}1} \longrightarrow -K\sum_{j=1}^t (1-i_j)(1-\gamma^{i_1\dots i_{t-2},1}) \end{cases}$$

In the first situation, the result is null because $(f_{tt-1}^{i_1\dots i_{t-2}1})^{-1}(\omega_{i_1\dots i_{t-2}0}) = \emptyset$.

Similarly, if $\mathcal{G}_{t-1}^{i_1\dots i_{t-1}} = \{\emptyset, \Omega, \{\omega_{i_1\dots i_{t-2}01}\}, \{\omega_{i_1\dots i_{t-2}00}\}\}$ (previous transaction failed) then the random variable gives:

$$\mathbb{E}^{f_{tt-1}^{i_1\dots i_{t-2}0-}}[X_t^{i_1\dots i_{t-1}}] : \Omega_{t-1}^{i_1\dots i_{t-2}} \longrightarrow \mathbb{R} : \begin{cases} \omega_{i_1\dots i_{t-2}1} \longrightarrow 0 \\ \omega_{i_1\dots i_{t-2}0} \longrightarrow -K\sum_{j=1}^t (1-i_j)(1-\gamma^{i_1\dots i_{t-2},0}) \end{cases}$$

For each transaction t, two random variables are defined $X_t^{i_1\dots i_{t-2}1}$ and $X_t^{i_1\dots i_{t-2}0}$, both give two random variables $\mathbb{E}^{f_{tt-1}^{i_1\dots i_{t-2}1-}}[X_t^{i_1\dots i_{t-2}1}]$ and $\mathbb{E}^{f_{tt-1}^{i_1\dots i_{t-2}1-}}[X_t^{i_1\dots i_{t-2}0}]$. In total, at transaction t there are 2^{t-1} random variables

$$X_t^{i_1\dots i_{t-1}} = -K\sum_{j=1}^t (1-i_j)$$

defined on the 2^{t-1} probability space existing for all types of scenarios.

For each of those random variables, the conditional expectation is defined and gives a new random variable on the previous space. So, on all of the $t-1$ transaction spaces, two new random variables are defined. Since the sum of random variables is again a random variable, the following recurring process can apply:

$$X_{t-1}^{i_1\dots i_{t-2}} := \mathbb{E}^{f_{tt-1}^{i_1\dots i_{t-2}1-}}[X_t^{i_1\dots i_{t-1}}] + \mathbb{E}^{f_{tt-1}^{i_1\dots i_{t-2}0-}}[X_t^{i_1\dots i_{t-1}}]$$

The goal is then to recompute the conditional expectation for this new random variable and apply the same process.

Before describing a numerical algorithm that peforms the process, let us explicitly solve the case of $n = 3$ transactions.

Example 3.3.1. Let us assume $\alpha \in [0, 1]$ with:

$$\begin{array}{rcl} \gamma_2^1 &=& \gamma_1 \\ \gamma_2^{11} &=& \gamma_1 \\ \gamma_2^{10} &=& \gamma_2 = \alpha\gamma_1 \\ \gamma_2^0 &=& \gamma_2 = \alpha\gamma_1 \\ \gamma_2^{01} &=& \gamma_2 \\ \gamma_2^{00} &=& \gamma_3 = \alpha\gamma_2 \end{array}$$

Each time a transaction fails the following transaction has a percentage α of the previous probability to succeed. This means that each time there is a falied transaction, the following transaction has a lesser probability to succeed.

The first random variable is given by:

$$X_3^{11} : \Omega_3^{11} \longrightarrow \mathbb{R} : \begin{cases} \omega_{111} \longrightarrow 0 \\ \omega_{110} \longrightarrow -K \end{cases}$$

The conditional expectation for $G \in \mathcal{G}_2^1$ becomes:

$$\begin{aligned} \mathbb{E}^{\mathbb{P}_2^1}\left[\mathbb{1}_G \mathbb{E}^{f_{32}^{11-}}[X_3^{11}]\right] &= \int_G \mathbb{E}^{f_{32}^{11-}}[X_3^{11}] d\mathbb{P}_2^1 \\ &= \int_{(f_{32}^{11})^{-1}(G)} X_3^{11} d\mathbb{P}_3^{11} \\ &= \mathbb{E}^{\mathbb{P}_3^{11}}[\mathbb{1}_{f_{32}^{11-}(G)} X_3^{11}] \end{aligned}$$

In particular, for $G = \{\omega_{11}\}$, this becomes:

$$\begin{array}{rcl} \mathbb{E}^{\mathbb{P}_2^1}\left[\mathbb{1}_{\{\omega_{11}\}} \mathbb{E}^{f_{32}^{11-}}[X_3^{11}]\right] &=& \mathbb{E}^{\mathbb{P}_3^{11}}[\mathbb{1}_{(f_{32}^{11})^{-1}(\omega_{11})} X_3^{11}] \\ &=& \mathbb{E}^{\mathbb{P}_3^{11}}[\mathbb{1}_{\{\omega_{111},\omega_{110}\}} X_3^{11}] \\ &=& X_3^{11}(\omega_{111})\mathbb{P}_3^{11}(\omega_{111}) + X_3^{11}(\omega_{112})\mathbb{P}_3^{11}(\omega_{110}) \\ &=& 0\gamma_1 + (-K)(1-\gamma_1) \end{array}$$

and for $G = \{\omega_{10}\}$

$$\begin{array}{rcl} \mathbb{E}^{\mathbb{P}_2^1}\left[\mathbb{1}_{\{\omega_{10}\}} \mathbb{E}^{f_{32}^{11-}}[X_3^{11}]\right] &=& \mathbb{E}^{\mathbb{P}_3^{11}}[\mathbb{1}_{(f_{32}^{11})^{-1}(\omega_{10})} X_3^{11}] \\ &=& \mathbb{E}^{\mathbb{P}_3^{11}}[\mathbb{1}_{\emptyset} X_3^{11}] \\ &=& 0 \end{array}$$

so the conditional expectation is given by:

$$\mathbb{E}^{f_{32}^{11-}}[X_3^{11}] : \Omega_2^1 \longrightarrow \mathbb{R} : \begin{cases} \omega_{11} \longrightarrow -K(1-\gamma_1) \\ \omega_{10} \longrightarrow 0 \end{cases}$$

Similarly, one can show that the other conditional expectations are given by:

$$\mathbb{E}^{f_{32}^{10-}}[X_3^{10}] : \Omega_2^1 \longrightarrow \mathbb{R} : \begin{cases} \omega_{11} \longrightarrow 0 \\ \omega_{10} \longrightarrow -K\gamma_2 - 2K(1-\gamma_2) = -2K + K\gamma_2 \end{cases}$$

$$\mathbb{E}^{f_{32}^{01-}}[X_3^{01}] : \Omega_2^0 \longrightarrow \mathbb{R} : \begin{cases} \omega_{01} \longrightarrow -K\gamma_2 - 2K(1-\gamma_2) \\ \omega_{00} \longrightarrow 0 \end{cases}$$

and

$$\mathbb{E}^{f_{32}^{00-}}[X_3^{00}] : \Omega_2^0 \longrightarrow \mathbb{R} : \begin{cases} \omega_{01} \longrightarrow 0 \\ \omega_{00} \longrightarrow -2K\gamma_3 + K\gamma_3 = -3K + K\gamma_3 \end{cases}$$

Now the following new random variables $X_2^1 \in \mathcal{L}(\Omega_2^1)$ and $X_2^0 \in \mathcal{L}(\Omega_2^0)$ are given by:

$$X_2^1 : \Omega_2^1 \longrightarrow \mathbb{R} : \begin{cases} \omega_{11} \longrightarrow -K(1-\gamma_1) \\ \omega_{10} \longrightarrow -2K + K\gamma_2 \end{cases}$$

and

$$X_2^0 : \Omega_2^1 \longrightarrow \mathbb{R} : \begin{cases} \omega_{00} \longrightarrow -3K + K\gamma_3 \\ \omega_{01} \longrightarrow -K\gamma_2 - 2K(1-\gamma_2) = -2K + K\gamma_2 \end{cases}$$

the same process as above is applied and the following results are satisified:

$$\mathbb{E}^{f_{21}^{1-}}[X_2^1] : \Omega_1 \longrightarrow \mathbb{R} : \begin{cases} \omega_0 \longrightarrow 0 \\ \omega_1 \longrightarrow -K(1-\gamma_1)\gamma_1 + (-2K + K\gamma_2)(1-\gamma_1) \end{cases}$$

$$\mathbb{E}^{f_{21}^{0-}}[X_2^0] : \Omega_1 \longrightarrow \mathbb{R} : \begin{cases} \omega_1 \longrightarrow 0 \\ \omega_0 \longrightarrow (-2K + K\gamma_2)\gamma_2 + (-3K + K\gamma_3)(1-\gamma_2) \end{cases}$$

Finally, combining the two random variables above gives the new random variable on Ω_1 defined by:

$$X_1 : \Omega_1 \longrightarrow \mathbb{R} : \begin{cases} \omega_1 \longrightarrow -K(1-\gamma_1)\gamma_1 + (-2K + K\gamma_2)(1-\gamma_1) \\ \omega_0 \longrightarrow (-2K + K\gamma_2)\gamma_2 + (-3K + K\gamma_3)(1-\gamma_2) \end{cases}$$

and the conlusion follows:

$$\mathbb{E}^{\mathbb{P}_1}[X_1] = (-K(1-\gamma_1)\gamma_1 + (-2K + K\gamma_2)(1-\gamma_1))\,\gamma_1 + ((-2K + K\gamma_2)\gamma_2 + (-3K + K\gamma_3)(1-\gamma_2))\,(1-\gamma_1)$$

Trying to compute this closed formula for larger values of n could be really tedious. Therefore, the current goal is to extend the process and describe a numerical algorithm (R code can be found in the Annexe 5.2) in order to treat a larger number of transactions. The process is the following:

- Step 1: Define a matrix with "0" and "1" describing all of the different scenarios. For example, if $n = 3$ you have:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

- Step 2: Compute the value of the random variable in each scenario
- Step 3: Associate the probabilities γ_t and $1 - \gamma_t$
- Step 4: Determine the conditional expectation for each space $t - 1$
- Step 5: Combine the two conditional expectations at each space $t - 1$ in one new random variable
- Step 6: Loop step 3, 4 and 5 until $t = 1$
- Step 7: Compute the final expectation of your new random variable

Below are the results for some value of depletion α, base probability γ_1 and number of transactions n. It is assumed that $K = 10000$. The first column gives the maximal exposure nK and the second gives the "classical" results $nK(1 - \gamma_1)$ without any degradation of the probability of default:

Result for $n = 3$							
γ_1	max exp.	$nK(1-\gamma_1)$	$\alpha = 1$	$\alpha = 0.95$	$\alpha = 0.75$	$\alpha = 0.50$	$\alpha = 0.25$
1	30000	0	0	0	0	0	0
0.999	30000	-30	-30	-31.53	-37.96	-46.23	-53.88
0.99	30000	-300	-300	-315.08	-378.78	-460.50	-535.98
0.95	30000	-1500	-1500	-1572.26	-1875.91	-2262.97	-2618.85
0.50	30000	-15000	-15000	-15374	-16855.47	-18593.75	-20097.66
0	30000	-30000	-30000	-30000	-30000	-30000	-30000

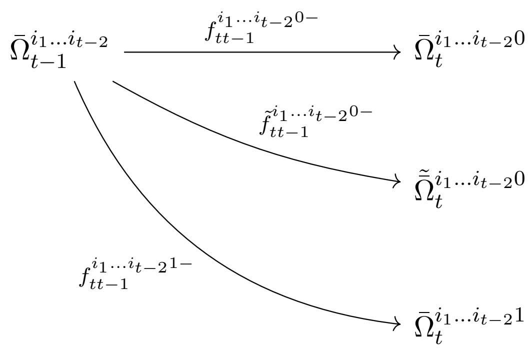
Result for $n = 5$							
γ_1	max exp.	$nK(1-\gamma_1)$	$\alpha = 1$	$\alpha = 0.95$	$\alpha = 0.75$	$\alpha = 0.50$	$\alpha = 0.25$
1	50000	0	0	0	0	0	0
0.999	50000	-50	-50	-55.24	-79.82	-111.54	-135.59
0.99	50000	-500	-500	-551.85	-794.28	-1105.74	-1341.17
0.95	50000	-2500	-2500	-2747.79	-3886.535	-5319.89	-6390.03
0.50	50000	-25000	-25000	-26247.68	-30981.02	-35656.43	-38650.62
0	50000	-50000	-50000	-50000	-50000	-50000	-50000

Result for $n = 25$							
γ_1	max exp.	$nK(1-\gamma_1)$	$\alpha = 1$	$\alpha = 0.95$	$\alpha = 0.75$	$\alpha = 0.50$	$\alpha = 0.25$
1	250000	0	0	0	0	0	0
0.999	250000	-250	-250	-468.72	-2045.76	-2872.85	-3126.927
0.99	250000	-2500	-2500	-4646.14	-19406.44	-26893.31	-29172.65
0.95	250000	-12500	-12500	-22365.70	-78118.37	-102701.20	-109947.60
0.50	250000	-125000	-125000	-160406.61	-220479.36	-233933.1	-238144.03
0	250000	-250000	-250000	-250000	-250000	-250000	-250000

The table shows that the average loss computed by our technique is always between the classical results and the maximum exposure. Moreover, if $\alpha = 1$, or in other words if there is no deterioration of risk, the classical result will coincides with the results of our method.

This example shows the technicity of the methodology but it allows to see how the uncertainty could be taken into account. At $t = 0$, several scenarios and spaces co-exist (with their associated probabilities). Here, the rules on each iterations have been defined mechanically with only two possibilities (transactions succeed or failed) in order to explicitly show the computations.

Nevertheless, one could also have defined even more co-existing spaces with different probability measures and then, could compute the final expectation in a similar way. For example, after a transaction $t-1$, different practionners may expect different behaviour depending on their beliefs after a transaction failed. So the following situation may be described:



With different probability measures for our spaces $\bar{\Omega}_t^{i_1 \ldots i_{t-2}0}$ and $\tilde{\bar{\Omega}}_t^{i_1 \ldots i_{t-2}0}$ and therefore different conditional expecations.

Also, in our example, only two states are possible in each $\bar{\Omega}_t$ because only one transaction in one country at a time has been considered. Another improvement could be to consider a space with more than two possiblities. For example, it could be a space with four states given the status of two transactions in two different countries. The spaces defined at each iterations could also change depending on the assumptions made (e.g. if two transactions failed in one country, then the client stop performing any transaction in that country) and therefore the number of events at each iteration may change as well.

This approach (adapted from Adachi's approach on binomial asset pricing [ANR20]) could be useful in Actuarial Sciences because it allows an actuary to take into account the incertainty in his computations. Indeed, besides a linking null-preserving function there is no restriction on the $\bar{\Omega}_t$ spaces themselves. Therefore, the probability and number of events possible are totally flexible (given the null-preserving function).

3.4 Modeling of Rare, Extreme Climatic Events

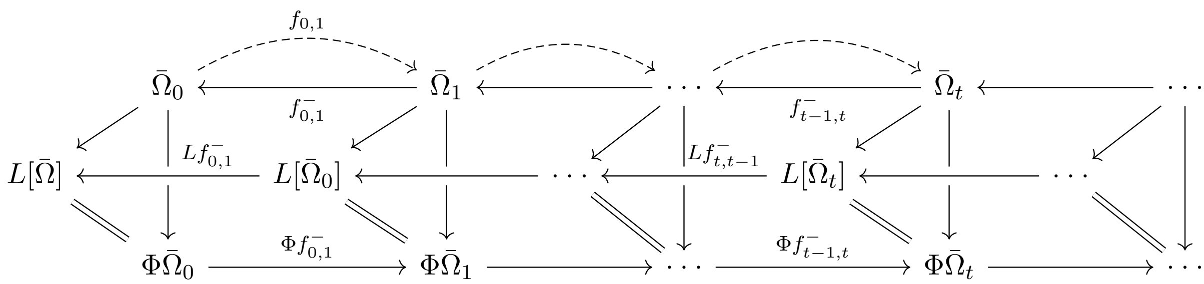
Modeling rare events, such as big floods (e.g. in Belgium during summer 2021) is a significant task. The main issue is the availibility of the data or more specifically how to describe the probability measure behind such events through time.

Besides the human tragedy this kind of event implies, modeling catnat is crucial for insurance and reinsurance companies because of the financial repercussions. Especially because they can vary significantly from one year to another. For example, in Beligum, the yearly report of Assuralia[4] shows that the cost of the flood went from 14.3 mios and 16.2 mios euros for 2019 and 2020 respectively, to 2.573.4 mios for 2021. Understanding and trying to plan for this low frequency, high severity type of event is as important for insurance/reinsurance companies (to avoid bankruptcy) as it is for human kind. This especially true for the insured people directly affected by those events.

Also with the consequences of global warming, everyone may expect that the probability of those rare events (which is small) will increase in the upcoming decades. One possible application of the theory presented here may help in this situation.

For example, let us imagine a probability space $\bar{\Omega}_t = (\Omega_t, \mathcal{G}_t, \mathbb{P}_t)$ that evolves through time. The scenarios in Ω_t would be the magnitude of such events (e.g. Richter or magnitude scale in case of earthquake). The probability $\mathbb{P}_t$ associated with those events could also evolve through time and become more significant in the future. The goal of the insurer would be to evaluate the risk associated with the loss related to those future events. In other words, the insurer may want to evaluate the risk associated with those future significant events which have an increasing probability of happening and an expending set of dramatic scenarios.

Another way to see the situation above is with the following diagram:



The difficulty in this example will once more be to define a general null-preserving function for all $t \in \mathbb{R}^+$:

[4] Assuralia Website

$$f_{t-1,t} : (\Omega_{t-1}, \mathcal{G}_{t-1}, \mathbb{P}_{t-1}) \longrightarrow (\Omega_t, \mathcal{G}_t, \mathbb{P}_t)$$

Once this difficulty is overcame, the theory will allow us to move through probability spaces at each time t with a set of events and their corresponding probability measure that changes continuously. Finally, with a suitable monetary value measure, the risk associated to those random variables (losses) could be computed.

3.5 Car Insurance Pricing and Telematic Data

Today, most of the traditional insurance companies derive their premium prices for car insurance from characteristics (known a priori) of the driver (age, postal code, ...) and the car (age, mileage, engine size, ...). Of course, those relate to the risk profile of the customer, but not directly.

The probability of having a car accident is more correlated with the type of driving behavior than the car iself or the driver. Some of those characteristics may translate partially (as significative variable) its behavior, but not necessarily. For example, if someone has a powerful car, one may expect that the driver to go fast. However, it is its driving behavior that will influence its speed and its tendencies to follow legal limitations.

With the technology evolving at an increasing speed, computers can collect more and more telematic data related to the driving style of people. The data collected can be speed, fuel consumption, breaking delay and so on. This massive amount of data could be collected and computed every second, and could accurately evaluates the driving profile of the driver and therefore their real risk profile..

In this setting, the first space of events could be the telematic data Ω'.

For example, it could be the speed of the car and its GPS location that would give the information to know if the car was above the speed limit. And with the data history[5], one could actually evaluate the probabilty of a driver to drive over the speed limit.

The second space of event could be a more classic pricing space with Ω describing the risk profile. The associated probability $\mathbb{P}$ would be the probability to have an accident with that risk profile. Having this risk profile based on telematic data would give a pricing model which is more fair and reliable.

If anyone wishes to continue the investigation in this direction, it would require defining a proper null-preserving function:

$$f : (\Omega, \mathcal{G}, \mathbb{P}) \longrightarrow (\Omega', \mathcal{G}', \mathbb{P}')$$

So a function, such that if a risk profile has a probability of zero of having an accident then the pre-image of this risk profile would be telematic data that has a probability of zero to happen (for example, probability of zero to drive above the speed limit).

[5]Even if they are somehow hidden or anonymized for personal privacy issues

3.6 Choice of Axioms for a Risk Measure

This last section will only be a glance of this possible application. Indeed, the mathematical background needed for this part is going far beyond what has been explained so far in this paper.

Nevertheless, the goal of this section is to try to give an idea of what is possible[6] in this more theorestical application. If the reader is not familiar with one of the categorical notions presented here, [Bor94a], [Bor94b] and especially [Bor94c] are good references to deep dive in those concept.

Let's recall that the category $\mathcal{X}$ (cfr Definition 2.2.1) is embedded in the category **Prob** (cfr the beginning of Section 2.2). Therefore, all the notions defined on **Prob** naturally apply in particular on $\mathcal{X}$.

In [Ada14], Adachi sees the monetary value measure $\Phi : \mathcal{X}^{op} \longrightarrow \mathbf{Set}$ as a presheaf:

Definition 3.6.1. If $\mathbf{C}$ is a (small) category, a presheaf on $\mathbf{C}$ is a functor $F : \mathbf{C}^{op} \longrightarrow \mathbf{Set}$. If T is a topological space, a presheaf on T is a presheaf on $O(T)$ the ordered set of open subsets of T sees as a category with at most one arrow between two objects.

Theory of sheaf and presheaf were originaly described in topology and complex geometry, but Grothendieck succeeded to extend it to category theory by introducing the topology of Grothendieck.

Definition 3.6.2. A *Grothendieck topology* on a (small) category $\mathbf{C}$ is defined as a specification for each object $C \in \mathbf{C}$ of a family $\mathcal{L}(C)$ of subfunctors of the representable functor $\mathbf{C}(-, C)$ such that this collection satisfies:

1. For all objects $C \in \mathbf{C}$, $\mathbf{C}(-, C) \in \mathcal{L}(C)$
2. For an arrow $f : D \longrightarrow C$, given the following pullback (cfr definition 2.1.11):

$$\begin{array}{ccc} R_f & \xrightarrow{\quad f_R \quad} & R \\ \big\downarrow{\scriptstyle r_f} & & \big\downarrow{\scriptstyle r} \\ \mathbf{C}(-, D) & \xrightarrow{\mathbf{C}(-,f)} & \mathbf{C}(-, C) \end{array}$$

   if $R \in \mathcal{L}(C)$ then $R_f \in \mathcal{L}(C)$

3. For an object $C \in \mathbf{C}$, an arbitrary subfunctor $R \longrightarrow \mathbf{C}(-, C)$ and $S \in \mathcal{L}(C)$, if for every object $D \in \mathbf{C}$ and every morphism $f \in S(D)$, following the notation above, $R_f \in \mathcal{L}(D)$ then $R \in \mathcal{L}(C)$

When $\mathcal{L}$ is a Grothendieck topology on a (small) category $\mathbf{C}$, the pair $(\mathbf{C}, \mathcal{L})$ is called a *site*.

[6] As for the other points of this chapter, I personally hope to follow the research in those areas

Thanks to this definition, a category can be seen as a topological space when a Grothendieck topology can be defined on it. Now, the general notions of sheaves[7] for a category with a Grothendieck topology is introduced:

Definition 3.6.3. Let $\mathbf{C}$ be a category and $\mathcal{L}$ a Grothendieck topology on $\mathbf{C}$.

1. A *presheaf* on $(\mathbf{C}, \mathcal{L})$ is a functor $F : \mathbf{C}^{op} \longrightarrow \mathbf{SET}$
2. A presheaf F on $(\mathbf{C}, \mathcal{L})$ will be called a sheaf if for $C \in \mathbf{C}$ and $R \in \mathcal{L}(C)$, any natural transformation $\alpha : R \Rightarrow F$ extends uniquely to $\mathbf{C}(-, C)$. In other words:

$$\begin{array}{ccc} R & \longrightarrow & \mathbf{C}(-,C) \\ \Big\downarrow \alpha & \swarrow_{\beta} & \\ F & & \end{array}$$

3. Natural transformations between sheaves or presheaves can be seen as arrows between them. Therefore, one can consider the category of presheaves on $\mathbf{C}$ denoted $\mathbf{Pr}(\mathbf{C})$ and the category of sheaves on $(\mathbf{C}, \mathcal{L})$ denoted by $\mathbf{Sh}(\mathbf{C}, \mathcal{L})$

The definitions above can apply on the presheaf $\Phi : \mathcal{X}^{op} \longrightarrow \mathbf{Set}$.

Given a set $\mathcal{A}$ of axioms for the monetary value measures, there exists the largest Grothendieck topology $\mathcal{L}_{\mathcal{A}}$ such that any Φ satifying $\mathcal{A}$ becomes a sheaf. The existence of the Grothendieck topology is given by the example 3.2.14d in [Bor94c].

It is clear that $\mathbf{Sh}(\mathcal{X}^{\mathbf{op}}, \mathcal{L})$ is a subcategory of $\mathbf{Pr}(\mathcal{X}^{\mathbf{op}})$ in the sense that there is an inclusion functor from one into the other. It is also known[8] that this inclusion functor as a left adjoint functor (see definition 2.1.12) is called a reflection that preserves finite limits:

$$\pi_{\mathcal{L}} : \mathbf{Pr}(\mathcal{X}^{\mathbf{op}}) \longrightarrow \mathbf{Sh}(\mathcal{X}^{\mathbf{op}}, \mathcal{L}) : \Phi \longrightarrow \pi_{\mathcal{L}}(\Phi)$$

This reflection functor allows one to get the closest monetary value measures $\pi_{\mathcal{L}_{\mathcal{A}}}(\Phi)$ that may or may not satisfy the set of axiom $\mathcal{A}$, from an arbitrary measure Φ. If all sheaves $\pi_{\mathcal{L}_{\mathcal{A}}}(\Phi)$ satisfy $\mathcal{A}$ then the set of axioms is called complete. Now, below is the formal definition of completeness of axioms given in [Ada14]:

Definition 3.6.4. $\mathcal{A}$ denotes a set of axioms for monetary value measures.

1. $\mathcal{M}[\mathcal{A}]$ is a subcategory of $\mathbf{Pr}(\mathcal{X}^{\mathbf{op}})$ whose objects are all monetary value measures satifying $\mathcal{A}$ and such that the inclusion functor is full and faithful (cfr defintion 2.1.8)
2. $\mathcal{M}_0 := \mathcal{M}[\emptyset]$ is the category of all monetary value measures[9].

[7] In the definition 3.2.2 in [Bor94c], the notions of presheaves and sheaves are first defined on $\mathcal{L}$ a localizing system but then after defining Grothendieck topology in the same book (definition 3.2.4), it is shown in detail why the Grothendieck topology on a small category is a special case of a localizing system. This is why the definition given is a bit more restrictive than the one from the reference, but is still coherent.

[8] For the details of these results, the reader can check [Bor94c]

[9] The differerence with $\mathbf{Pr}(\mathcal{X}^{\mathbf{op}})$ is that we have mentioned that all monetary value measures can be seen as presheaf but not the other way around. We do not know at the moment if this is true

3. $\mathcal{A}$ is called *complete* if there exists a functor $\eta_{\mathcal{A}} : \mathcal{M}_0 \longrightarrow \mathcal{M}[\mathcal{A}]$ such that

$$\begin{array}{ccc} \mathcal{M}_0 & \rightarrowtail & \mathbf{Pr}(\mathcal{X}^{\mathbf{op}}) \\ \downarrow{\scriptstyle \eta_{\mathcal{A}}} & & \downarrow{\scriptstyle \pi_{\mathcal{L}_{\mathcal{A}}}} \\ \mathcal{M}[\mathcal{A}] & \rightarrowtail & \mathbf{Sh}(\mathcal{X}^{\mathbf{op}}, \mathcal{L}_{\mathcal{A}}) \end{array}$$

commutes.

In the above diagram, the two horizontal arrows represent the inclusion functors. The next (and last) result will help to understand the importance of the construction above:

Theorem 3.6.5. *If $\mathcal{A}$ is a complete set of axioms and if $\Phi \in \mathcal{M}_0$ then $\pi_{\mathcal{L}_{\mathcal{A}}}(\Phi)$ is the best approximation*[10] *satifsying the set of axioms $\mathcal{A}$.*

Proof. The fact that $\mathcal{A}$ is complete since the diagram above commutes by definition, it implies that for any $\Phi \in \mathcal{M}_0$ the following equality is satified $\pi_{\mathcal{L}_{\mathcal{A}}}(\Phi) = \eta_{\mathcal{A}}(\Phi)$ which is in $\mathcal{M}[\mathcal{A}]$. The conclusion follows that $\pi_{\mathcal{L}_{\mathcal{A}}}(\Phi)$ is a monetary value measure satisfying $\mathcal{A}$. □

Adachi mentioned the importance of the theorem above for practitioners. It may be difficult to check if a monetary value measure satisfies a set of axioms. The theorem above gives us a way to determine the best approximation that satisfies the set of axioms. The difficulty here is to describe a set of axioms which is complete in these setting. This is at the moment an open point that could also be a subject for future studies.

Two possible additional applications/improvements of the above are also prognosticated. First, an improvement would be to describe in the general setting of the category **Prob** the result above. Indeed, $\mathcal{X}$ is a particular case of **Prob** so generalizing the constructions of sheaves on **Prob** would be a natural extension.

Last but not least, a possible application would be to use the result above to discuss another problematic mentioned a couple of times in this paper: to choose such or such set of axioms is somehow arbitrary and often the argument "Expert Judgment" is widely use to justify this choice. In Chapter 2, it has been proven that some properties were labeled as axioms but are in fact consequences of other axioms in a categorical world. Investigation in this way, and taking into account the notion of "completeness" above, may lead to a more formal background to justify the use of a set of axioms and therefore the corresponding chosen monetary value measure.

[10] As discussed via email with Adachi, the word "best approximation" or "closest" means here that the functor $\pi_{\mathcal{L}_{\mathcal{A}}}$ is a reflector.

Chapter 4

Conclusion

The goal of this master thesis was to study the category theory applied to risk measure theory. With that purpose in mind, first was introduced the classical risk measure theory. The conditional expectation, 1-period monetary and dynamic monetary value measure have been defined in this classical setting. While defining the above tools, we have raised two main underlying issues:

- The minimum set of axioms to define a "good" risk measure is a subjective choice.
- The risk vs Knightian uncertainty or more precisely the subjective choice of a probability measure behind each risk measure.

The second chapter was where the magic happened. First, category theory was introduced and then a generalization of the definition of all the notions previously seen. The category **Prob**, the functors L, $\mathcal{E}$ and finally Φ were defined as generalization of our probability spaces, random variables and monetary value measures. The two direct benefits of that generalization were the following:

- The category **Prob** allows us to move through spaces with different probability measures and therefore allows us to capture the uncertainty
- Some axioms that are usually required for dynamic monetary value measure became concequences of the basic one.

Finally, for informative purposes, a generalization of filtration as a functor has also been given

The last chapter gave an introduction to some applications that have been foreseen so far. Concrete applications in credit insurance (a trivial and a more extensive numerical one), car insurance pricing with telematic data, and modelization of natural rare and severe events have been discussed. The section ended with a more theoretical open road on the possibility to set up a concrete framework to define the expert jugdment behind the choice of a "proper" set of axioms for a risk measure.

Of course, many open points remain, as this paper was written to constitute the entrance into the sandbox within the playground of possible applications of category theory in Actuarial Sciences.

Chapter 5

Appendix

5.1 Reminder of Elementary Mathematics

For this section V is a vector space defined on $\mathbb{R}$.

5.1.1 Metric Spaces

Definitions presented here can be found for example in [Wil08].

Definition 5.1.1. A map $\|\cdot\| : V \longrightarrow \mathbb{R}$ is called a norm if it satisfies the following conditions:

- (Minkowsky Inequality) $\forall v, w \in V$, we have $\|v + w\| \leq \|v\| + \|w\|$
- (Absolute Homogenitiy) $\forall v \in V$ and $\forall a \in \mathbb{R}$ then $\|av\| =| a | \|v\|$
- (Positive Definitness): for all $v \in V$, if $\|v\| = 0$ then $v = 0$

A vector space equipped with a norm is called a norm space.

Example 5.1.2. On the space $L^1(\Omega, \mathcal{F}, \mathbb{P})$ of the real valued function (r.v) as defined in section 1.1, we have a norm defined by:

$$\|X\|_1 := \int |X| d\mathbb{P}$$

On the space $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ of the real valued bounded function (r.v.) as defined in section 1.1, we have a norm defined by:

$$\|X\|_\infty := \sup\left(\alpha \in \mathbb{R} : \mathbb{P}[\omega : |X(\omega)| > \alpha] > 0\right)$$

Definition 5.1.3. A normed space V is called complete if every Cauchy sequence of points in V has a limit that is also in V.

Definition 5.1.4. A Banach Space is a complete normed vector space.

Example 5.1.5.

- $\mathbb{R}$ with the absolute value is a Banach space
- $\mathbb{R}^n$ with the usual euclidian norm (for $x \in \mathbb{R}^n$ the euclidian norm is defined by $\|x\| = \sqrt{x_1 + ... + x_n^2}$) is a Banach space.
- L^∞ and L^1 with respectively $\|\cdot\|_\infty$ and $\|\cdot\|_1$ as defined in Chapter 1.1 are also examples of Banach spaces.

5.1.2 Topology and Sets

The notions presented here can be found in courses or books in topology and measure theory.

Definition 5.1.6. A set Ω with a collection $\mathcal{O}$ of subsets is said to be a topological space if the subsets in $\mathcal{O}$ satisfy:

- $\emptyset, \Omega \in \mathcal{O}$.
- if $U, V \in \mathcal{O}$ then $U \cap V \in \mathcal{O}$.
- all unions (countable or uncountable) of sets in $\mathcal{O}$ must be in $\mathcal{O}$.

The elements from $\mathcal{O}$ are called the open set of Ω.

Definition 5.1.7. A function $f : \Omega \longrightarrow \Omega'$ between two topolical spaces is said to be continuous if the pre-image of every open set from Ω' is an open set of Ω.

Definition 5.1.8. A σ-algebra $\mathcal{G}$ on a set Ω is defined as a nonempty collection of subsets of Ω such that:

- $\Omega \in \mathcal{G}$
- if $G \in \mathcal{G}$, then $G^c \in \mathcal{G}$
- if $(G_n)_{n\in N}$ is a sequence in G then $\cup G_n \in G$

Example 5.1.9.

- The power set of Ω noted 2^Ω or $\mathcal{P}(\Omega)$ is a σ-algebra defined by the set of all subsets of Ω. Therefore, any collection of subsets of Ω can always be contained in a σ-algebra which is this power set.
- If X is any collection of subsets of Ω, then we can always define a σ-algebra by the intresection of all σ-algebra containing X. This is the smallest σ-aglebra containing X.
- The Borel σ-algebra is defined as the σ-algebra generated by the open sets.

All topological spaces are not σ-algebra and all σ-algebra are not topological spaces even if their definitions look similar.

Definition 5.1.10. Let $(\Omega, \mathcal{G})$ and $(\Omega', \mathcal{G}')$ be two spaces with their corresponding σ-algebra, a function $f : \Omega \longrightarrow \Omega'$ is said to be measurable if for every $G' \in \mathcal{G}'$ we have

$$f^{-1} = \{\omega \in \Omega \mid f(\omega) \in G'\} \in \mathcal{G}$$

5.2 R code

The code used in Section 3.3 is the following:

```
library(stats)
library(tidyr)
library(hier.part)
```

```
library(dplyr)

### function that will compute the estimation of the variable X
###with a given number of transation n, value K, risk gamma and deterioration a
est_cat <- function(n, K, gamma1, alpha) {
  gamma <- data.frame(proba_value = matrix(1, n))
  gamma[1, ] <- gamma1
  for (t in c(2:n)) {
    gamma[t, ] = gamma[t - 1, ] * alpha

  }
  gamma$gamma <- as.character(c(1:n))

  ######prepa last even
  scenarios <- as.data.frame(expand.grid(rep(list(1:0), n)))
  names(scenarios) <- paste("event", c(1:n), sep = "_")
  names(scenarios)[length(names(scenarios))] <- "last_event"
  scenarios_prev <- as.data.frame(scenarios[, c(1:n - 1)])
  scenarios$counting_0_prev <-
    as.character(pmax(rowSums(scenarios_prev == 0) + 1, 1))
##counting how many fail before

  scenarios <-###associate rv
    merge(scenarios, gamma, by.x = "counting_0_prev", by.y = "gamma")

  scenarios$proba_scen <-
    scenarios$last_event * scenarios$proba_value +
(1 - scenarios$last_event) *
    (1 - scenarios$proba_value)
#gamma for success and 1-gamma for fail
  scenarios$rv <-
    -(as.numeric(scenarios$counting_0_prev) - 1) * K
 - (1 - scenarios$last_event) * K
 ###value of loss at the end
  scenarios$lost_proba <- scenarios$rv * scenarios$proba_scen


  #####now we loop to arrive at the end

  for (j in c(2:n - 1)) {
    #j<-2 #for test purpose
    k <- n - j

    namevar <-
      names(scenarios)[!names(scenarios) %in% c("counting_0_prev",
                                                 "last_event",
                                                 "proba_value",
```

```
                                                      "proba_scen",
                                                      "rv")]
    namevartogroup <-
      names(scenarios)[!names(scenarios) %in% c(
        "counting_0_prev",
        "last_event",
        "proba_value",
        "proba_scen",
        "rv",
        "lost_proba"
      )]
    new_scenarios <- scenarios[, namevar] %>%
      group_by_at(namevartogroup) %>%
      summarize_each(fun = sum)
    scenarios <- new_scenarios
    names(scenarios)[length(names(scenarios)) - 1] <- "last_event"
    scenarios_prev <- as.data.frame(scenarios[, c(1:k - 1)])
    scenarios$counting_0_prev <-
      as.character(pmax(rowSums(scenarios_prev == 0) + 1, 1))
##counting how many fail before
    scenarios <-
      merge(scenarios, gamma, by.x = "counting_0_prev", by.y = "gamma")
    scenarios$proba_scen <-
      scenarios$last_event * scenarios$proba_value + (1 - scenarios$last_event)
      (1 - scenarios$proba_value)
    scenarios$rv <- scenarios$lost_proba  ###value of loss at the end
    scenarios$lost_proba <- scenarios$rv * scenarios$proba_scen

  }

  ##finalization
  newsol <-
    scenarios[scenarios$last_event == 0, "lost_proba"] +
scenarios[scenarios$last_event ==1, "lost_proba"]

  sol1 <- -n * (1 - gamma1) * K
  maxexposure <- n * K

  result <- c(sol1, newsol, maxexposure)
  result
}

memory.limit(240000)
n <- 5
alpha <- 0.95
gamma1 <- 0.999
est_cat(n, K, gamma1, alpha)  ##restults for one set of parameters
```

```
##restults for several set of parameters
alpha_test <- c(1, 0.95, 0.75, 0.50, 0.25)
gamma1_test <- c(1, 0.999, 0.99, 0.95, 0.5, 0)
K <- 10000
n <- 5
for (alpha in alpha_test) {
  for (gamma in gamma1_test) {
    gamma1 <- gamma
    result <- est_cat(n, K, gamma1, alpha)
    print(c(gamma, alpha, result))
  }
}


###comparison with explicit close formula when n=3
  explisol3 <-
    (-K * (1 - gamma1) * gamma1 + (-2 * K +
K * gamma[2, "proba_value"]) *
       (1 - gamma1)) * gamma1 + ((-2 * K +
K * gamma[2, "proba_value"]) *
 gamma[2, "proba_value"] +
(-3 * K + K * gamma[3, "proba_value"]) *
(1 - gamma[2, "proba_value"])) *
    (1 - gamma1)
```

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