Appendix A

Proofs of Chapter 3

A.1 Proofs of the Lemmas on Beliefs

Lemma 1 on Beliefs. *Consider* $(v,p),(v,p') \in \mathcal{L}^{\{1,2\}}$ *with* $p=(p_i,p_j)$, $p'=(p'_i,p_j)$ *and* $i \neq j \in \{1,2\}$. *If* $\varphi(v,p) \neq \varphi(v,p')$, *then*

$$\left[p'_i(\omega_1) > (<) p_i(\omega_1)\right] \Rightarrow \left[\begin{array}{ll} v_i(\varphi(v,p';\omega_1)) & > (<) v_i(\varphi(v,p;\omega_1)) \textit{ and} \\ v_i(\varphi(v,p';\omega_2)) & < (>) v_i(\varphi(v,p;\omega_2)) \end{array}\right].$$

Proof. Without loss of generality, consider $p'_1(\omega_1) > p_1(\omega_1)$. Assume also that $\varphi(v,p) \neq \varphi(v,p')$. The only interesting cases to consider are

1. $v_1(\varphi(v,p';\omega_1)) < v_1(\varphi(v,p;\omega_1))$ and $v_1(\varphi(v,p';\omega_2)) > v_1(\varphi(v,p;\omega_2))$;
2. $v_1(\varphi(v,p';\omega_1)) > v_1(\varphi(v,p;\omega_1))$ and $v_1(\varphi(v,p';\omega_2)) < v_1(\varphi(v,p;\omega_2))$.

These are the two cases in which at (v,p') the SCF selects in one state a strictly better outcome and in the other state a strictly worse outcome compared to the selection at (v,p). In the other cases strategyproofness is contradicted. Indeed, it is straightforward to show that one of the two acts selected by the SCF will make agent 1 strictly better off compared to the other act no matter p. This creates the possibility of a beneficial manipulation for agent 1 at one of the two profiles.

Consider now case 1. Without loss of generality assume that $v_1 : a \succ b \succ c$, $\varphi(v,p)=(b,b)$ and $\varphi(v,p')=(c,a)$. This is a case in which agent 1 receives a strictly worse outcome at state ω_1 when he increases his subjective belief that ω_1 is selected and a strictly better outcome at state ω_2 when he decreases this belief for ω_2.

In this case, either the marginal gain that the agent 1 gets at state 2 is higher to the marginal loss at state 1 or vice versa. Formally, with preferences (v_1,p_1):

1.1. either $p_1(\omega_2)v_1(a) - p_1(\omega_2)v_1(b) > p_1(\omega_1)v_1(b) - p_1(\omega_1)v_1(c)$;

1.2. or $p_1(\omega_2)v_1(a) - p_1(\omega_2)v_1(b) < p_1(\omega_1)v_1(b) - p_1(\omega_1)v_1(c)$.

Case 1.1 is excluded by strategyproofness. This axiom also implies that $p'_1(\omega_2)v_1(a) - p'_1(\omega_2)v_1(b) > p'_1(\omega_1)v_1(b) - p'_1(\omega_1)v_1(c)$. Putting this together with 1.2 leads to

$$p'_1(\omega_2)(v_1(a)-v_1(b)) > p'_1(\omega_1)(v_1(b)-v_1(c)) > p_1(\omega_1)(v_1(b)-v_1(c)) > p_1(\omega_2)(v_1(a)-v_1(b)),$$

which implies that $p'_1(\omega_2) > p_1(\omega_2)$. This contradicts the initial assumption that $p'_1(\omega_1) > p_1(\omega_1)$. Therefore, case 1 is also excluded by strategyproofness.
Consider now case 1, which is the case in the lemma. We want to show that this does not necessarily contradicts strategyproofness. We show that if some boundary conditions on valuations and beliefs are respected, this case is allowed by strategyproofness.
Without loss of generality, take the valuation function for agent 1 and $\varphi(v,p)$ as before and consider $\varphi(v,p') = (a,c)$, instead. By strategyproofness,

$$p_1(\omega_1)v_1(a) - p_1(\omega_1)v_1(b) < p_1(\omega_2)v_1(b) - p_1(\omega_2)v_1(c) \tag{2.1}$$

and

$$p'_1v_1(\omega_1)v_1(a) - p'_1(\omega_1)v_1(b) > p'_1(\omega_2)v_1(b) - p'_1(\omega_2)v_1(c) \tag{2.2}$$

We can rewrite the previous two inequalities as

$$\frac{p_1(\omega_2)}{p_1(\omega_1)} > \frac{v_1(a) - v_1(b)}{v_1(b) - v_1(c)} \tag{2.1}$$

and

$$\frac{p'_1(\omega_2)}{p'_1(\omega_1)} < \frac{v_1(a) - v_1(b)}{v_1(b) - v_1(c)}.$$

Which become, after some manipulation

$$p_1(\omega_1) < \frac{1}{1 + \frac{v(a)-v(b)}{v(b)-v(c)}} < p'_1(\omega_1).$$

Notice that $0 < \frac{1}{1+\frac{v(a)-v(b)}{v(b)-v(c)}} < 1$. Therefore, there are always $p_1(\omega_1)$ and $p'_1(\omega_1)$ such that the above condition is respected. Whenever this boundary condition is respected, case 2 does not contradict strategyproofness, proving Lemma 1.

□

Before proceeding to the proof of the other two lemmas we want to stress two facts about SEU preferences that are necessary to understand to fully grasp the proofs.
The first is that an *order over outcomes* and *order over beliefs on states of the world* is sufficient to determine the ranking of most of the couples of acts, without the need to specify the exact value of the valuation of the middle alternatives and of the subjective beliefs. In particular, with 3 alternatives and 2 states of nature, there are only 4 couples of acts whose ranking is left unknown when we have ordinal, but not cardinal information on the preferences over outcomes and on the subjective beliefs.

Remark 1. *With 3 outcomes and 2 states of nature, there are only four couples of acts whose ranking is not fully determined by an order over outcomes an order order over beliefs on states of the world.*

Without loss of generality, consider $v_i : a \succ_i b \succ_i c$ and $p_i(\omega_1) > 0.5$. It is possible to show that the only four couples of acts whose ranking cannot be deduced by the above information are: {{(b,a), (a,c)},{(c,a), (b,c)},{(b,b), (a,c)},{(b,b), (c,a)}}. To see this, take instead the

two acts (a, c) and (b, c). Since $a \succ_i b$, it must be that $(a, c) \succ_i (b, c)$, because the agent will receive in one state the same outcome and in the other state a better outcome under (a, c) compared to (b, c). To take another example, consider the couple (a, b) and (c, a). Passing from (a, b) to (c, a) means a loss in state ω_1 and a gain in state ω_2, with the loss bigger than the gain. Since the agent attributes an higher subjective probability to state ω_1 relative to ω_2, this implies that $(a, b) \succ_i (c, a)$.
Instead, take now the two acts (b, c) and (c, a). In this case too, passing from (c, a) to (b, c) implies a loss in state ω_1 and a gain in state ω_2, but the loss is now smaller than the gain. Because $p_i(\omega_1) > 0.5$ we need the cardinal information about $v_i(b)$ and $p_i(\omega_1)$ to deduce the ranking for these two acts. It is straightforward to show that Remark 1 is true, by proceeding in this vain for each couple of acts.
To this we can add the following remark:

Remark 2. *Take any preference* (v_i, p_i) *and consider the four couples of acts mentioned in Remark 1 that are associated to* (v_i, p_i)*. For each one of these couples and for each* p'_i *that generates the same order over states as* p_i*, it is always possible to find a* v'_i *with* $v'_i \simeq v_i$ *and such that* (v'_i, p_i) *and* (v'_i, p'_i) *generates the same ranking over the two acts, for each one of the two possible rankings.*

Remark 2 comes from the fact that the ranking over the pairs identified in Remark 1 depend on the cardinal information on valuations and subjective probabilities. Then, fixing the subjective belief to any value, it is always possible to make the value of the middle alternative big or small enough so to change the preferences over this pair of acts. We can see this through an example.
Consider without loss of generality the case where $v_i : a \succ b \succ c$ and $p_i(\omega_1) > 0.5$. Take, for instance, the couple of acts (a, c) and (b, a). This couple is in the set described in Remark 1 for (v_i, p_i). Now, $E^{p_i}_{v_i}(a, c) = p_i(\omega_1)$ and $E^{p_i}_{v_i}(b, a) = p_i(\omega_1)v_i(b) + p_i(\omega_2)$. Therefore, if $v(b) > \frac{p_i(\omega_1)-p_i(\omega_2))}{p_i(\omega_1)}$ then $(b, a) \succ_i (a, c)$; if $v(b) < \frac{p_i(\omega_1)-p_i(\omega_2)}{p_i(\omega_1)}$ then $(a, c) \succ_i (b, a)$. Since $0 < \frac{p_i(\omega_1)-p_i(\omega_2))}{p_i(\omega_1)} < 1$, it is always possible to find v'_i such that (v'_i, p_i) and (v'_i, p'_i) generates the same order over acts for the two possible ranking over acts. This reasoning can be applied to all pairs of acts in the set defined in Remark 1.

Lemma 2 on beliefs. *Consider* $(v, p), (v, p') \in \mathcal{L}^{\{1,2\}}$ *with* $p = (p_i, p_j)$*,* $p' = (p'_i, p_j)$ *and* $i \neq j \in \{1, 2\}$*. If* $\varphi(v, p) \neq \varphi(v, p')$*, then either* $p_i(\omega_1) > 0.5$ *and* $p'(\omega_1) < 0.5$ *or* $p_i(\omega_1) < 0.5$ *and* $p'(\omega_1) > 0.5$*.*

Proof. Without loss of generality consider the pairs (v_1, p_1) and (v_1, p'_1), with $v_1 : a \succ b \succ c$, $p_1(\omega_1) > 0.5$ and $p'_1(\omega_1) > 0.5$. Call $(v, p) = ((v_1, v_2), (p_1, p_2)) \in \mathcal{L}^{\{1,2\}}$ and $(v, p') = ((v_1, v_2), (p'_1, p_2)) \in \mathcal{L}^{\{1,2\}}$ and assume that $\varphi(v, p) \neq \varphi(v, p')$.
By Remark 1, there are only four pairs of acts whose ranking is not determined by the above ordinal information on valuations and beliefs. This is the set

$$A = \{\{(b, a), (a, c)\}, \{(c, a), (b, c)\}, \{(b, b), (a, c)\}, \{(b, b), (c, a)\}\}$$

By definition of A, and since the order over states generated by p_1 and p'_1 is the same,

$$\big[\{\varphi(v, p), \varphi(v, p')\} \notin A\big] \Rightarrow \begin{bmatrix} E^{p_1}_{v_1}(\varphi(v, p)) > E^{p_1}_{v_1}(\varphi(v, p')) \Leftrightarrow E^{p'_1}_{v_1}(\varphi(v, p)) > E^{p'_1}_{v_1}(\varphi(v, p')) \\ \text{and} \\ E^{p_1}_{v_1}(\varphi(v, p)) < E^{p_1}_{v_1}(\varphi(v, p')) \Leftrightarrow E^{p'_1}_{v_1}(\varphi(v, p)) < E^{p'_1}_{v_1}(\varphi(v, p')) \end{bmatrix}$$

This implies that either $E_{v_i}^{p_i}(\varphi(v,p')) > E_{v_i}^{p_i}(\varphi(v,p))$ or $E_{v_i}^{p'_i}(\varphi(v,p)) > E_{v_i}^{p'_i}(\varphi(v,p'))$, contradicting strategyproofness.
Consider now the case $\{\varphi(v,p),\varphi(v,p')\} \in A$. By Remark 2, there exists $v'_1 \simeq v_1$ such that $E_{v'_1}^{p_1}(\varphi(v,p)) > (<) E_{v'_1}^{p_1}(\varphi(v,p'))$ and $E_{v'_1}^{p'_1}(\varphi(v,p)) > (<) E_{v'_1}^{p'_1}(\varphi(v,p'))$. Call $(v',p) = ((v'_1,v_2),(p_1,p_2))$ and $(v',p') = ((v'_1,v_2),(p'_1,p_2))$. Then, by ordinality, $\varphi(v,p) = \varphi(v',p)$ and $\varphi(v,p') = \varphi(v',p')$. Therefore, either $E_{v'_1}^{p_1}(\varphi(v',p')) > E_{v'_1}^{p_1}(\varphi(v',p))$ or $E_{v'_1}^{p'_1}(\varphi(v',p)) > E_{v'_1}^{p'_1}(\varphi(v',p'))$, contradicting strategyproofness and proving lemma 2. □

Lemma 3 on beliefs. *Consider* $(v,p),(v,p') \in \mathcal{L}^{\{1,2\}}$ *with* $p = (p_i,p_j)$, $p' = (p'_i,p_j)$ *and* $i \neq j \in \{1,2\}$. *If* $\varphi(v,p) \neq \varphi(v,p')$, *then* $\varphi(v,p') = (\varphi(v,p;\omega_2),\varphi(v,p;\omega_1))$.

Proof. Consider the profiles of preferences $(v,p) = ((v_1,v_2),(p_1,p_2)) \in \mathcal{L}^{\{1,2\}}$ and $(v,p') = ((v_1,v_2),(p'_1,p_2)) \in \mathcal{L}^{\{1,2\}}$. Assume that $\varphi(v,p) \neq \varphi(v,p')$. Without loss of generality consider $v_1 : a \succ b \succ c$ and $p_1(\omega_1) > 0.5$.
By Lemma 1, $p'_1(\omega_1) < 0.5$. As we did previously with (v_1,p_1) (set A in the previous proof), we can define for the preference (v_1,p'_1) the set of four pairs of acts described in Remark 1:

$$B = \{\{(a,b),(c,a)\},\{(a,c),(c,b)\},\{(b,b),(a,c)\},\{(b,b),(c,a)\}\}.$$

When $\{\varphi(v,p),\varphi(v,p')\} \in A \cap B = \{\{(b,b),(a,c)\},\{(b,b),(c,a)\}\}$, with A as in the proof of Lemma 2. By Remark 2, it is always possible to find $v'_1 \simeq v_1$ such that the preference over these two acts does not change passing from (v_1,p_1) to (v_1,p'_1). By applying a similar reasoning to the one used in for Lemma 2, this contradicts strategyproofness. Together with Lemma 1, this implies that if $\varphi(v,p) \neq \varphi(v,p')$, then $\varphi(v,p) \neq (b,b)$.
Consider $\{\varphi(v,p),\varphi(v,p')\} \notin A(B)$ and $\{\varphi(v,p),\varphi(v,p')\} \in B(A)$. By the first condition and by Remark 1, for all $(v'_1,p_1)((v'_1,p'_1))$ with $v'_1 \simeq v_1$ and $p_1(\omega_1) > 0.5(p'_1(\omega_1) < 0.5)$ the ranking at of the two act at $(v'_1,p_1)((v'_1,p'_1))$ must be the same. By the second condition and by Remark 2, we can always find $v'_1 \simeq v_1$ such that the ranking over the two acts at $(v'_1,p'_1)((v'_1,p_1))$ will be the same than at $(v'_1,p_1)((v'_1,p'_1))$. This contradicts strategyproofness. With this we have shown that when $\varphi(v,p) \neq \varphi(v,p')$ and $p_1(\omega_1) > 0.5$, $p'_1(\omega_1) < 0.5$, then $\{\varphi(v,p),\varphi(v,p')\} \notin A \cup B$.
By Lemma 1, it must be that $\varphi(v,p')$ gives a strictly better alternative at state ω_2 and a strictly worse alternative at state ω_1 compared to $\varphi(v,p)$. As a consequence $\varphi(v,p) \notin \{(a,a),(b,a),(c,a),(c,b),(c,c)\}$. We have also previously shown that $\varphi(v,p) \neq (b,b)$. Therefore $\varphi(v,p) \in \{(a,b),(b,c),(a,c)\}$. By Lemma 1,

- if $\varphi(v,p) = (a,b)$ then $\varphi(v,p') = (b,a)$ or $\varphi(v,p') = (c,a)$. Since $\{(a,b),(c,a)\} \in B$, then $\varphi(v,p') = (b,a)$;
- if $\varphi(v,p) = (b,c)$, then $\varphi(v,p') = (c,b)$ or $\varphi(v,p') = (c,a)$. Since $\{(b,c),(c,a)\} \in A$ then $\varphi(v,p') = (c,b)$;
- if $\varphi(v,p) = (a,c)$, then $\varphi(v,p') \in \{(b,b),(b,a),(c,b),(c,a)\}$. Since $\{(b,b),(a,c)\} \in A \cap B$, $\{(a,c),(b,a)\} \in A$ and $\{(a,c),(c,b)\} \in B$, then $\varphi(v,p') = (c,a)$.

This proves Lemma 3 on beliefs. □

A.2 Proof of the Characterization Theorem

Theorem 1 ***(Characterization)*** *An SCF $\varphi : \mathcal{L}^{\{1,2\}} \to X^{\Omega}$ is strategyproof and it compromises if and only if it is one of the following four kinds of functions:*

1. *(b, b) is selected at any profile of preferences;*
2. *if at least one agent prefers b to a(b to c), then the SCF selects (b, b); when both agents prefer outcome a to b (c to b), then the SCF will select (b, a) or (a, b) (respectively (b, c) or $(c.b)$) in a way that can depend on agents beliefs (see Remark 3.3.1 for details);*
3. *if at least one agent prefers b to a(b to c), then the SCF selects (b, b); when both agents prefer outcome a to b (c over b), then the SCF will select (a, a) $((c, c))$;*
4. *if both agents prefer a to b and at least one agent prefers b to c, then (b,a)((a,b)) is selected; if both agents prefer c to b and at least one agent prefers b to a then (c,b)((b,c)) is selected; if they both prefer a to b and c to b, then (c,a)((a,c)) is selected; otherwise (b,b) is selected.*

We first focus on the sufficiency side of the statement, namely we first prove that if the SCF is strategyproof and it compromises, then it is one of the functions described in the theorem. It is not difficult to show that all these functions respect these two axioms; we will do that in the next section.

A.2.1 Proof of sufficiency

Throughout this section we write $\varphi : \mathcal{L}^{\{1,2\}} \to X^{\Omega}$ to represent a SCF that respects both strategyproofness and the compromise axiom. In addition, for most section A.2.1 we **fix the value of** $p \in \mathcal{P}^{\{1,2\}}$. This assumption will be relaxed at the end of the section, when we consider the four cases that are associated with the four kind of SCFs of Theorem 1.

The first step of the proof consists in deriving Lemma 4 and Lemma 5.

Lemma 4. *When b is the top alternative of at least one agent, then φ must select (b, b).*

Proof. We first consider all the cases in which at least one agent ranks b as top outcome while the other agent ranks this outcome in first or second place. All the possible profiles of this kind are represented in Table A.2.1 together with the two profiles in which φ compromises.

φ	$v_1 : a \succ b \succ c$	$v_1' : b \succ a \succ c$	$v_1'' : b \succ c \succ a$	$v_1''' : c \succ b \succ a$
$v_2 : c \succ b \succ a$	(b, b)	$\cdots$		▨
$v_2' : b \succ c \succ a$	$\vdots$	$\ddots$		
$v_2'' : b \succ a \succ c$			$\ddots$	$\vdots$
$v_2''' : a \succ b \succ c$	▨		$\cdots$	(b, b)

Table A.2.1

Consider the profile $(v, p) = ((v_1, v_2), (p_1, p_2))$. By compromise $\varphi(v, p) = (b, b)$. Any deviation of agent 1 from v_1 to v_1' (all else equal) cannot change the outcome by monotonicity. By permutation invariance and monotonicity, if then agent 1 deviates from v_1' to v_1'' the outcome cannot change. As similar reasoning applies if we start from $(v''', p) = ((v_1''', v_2'''), (p_1, p_2))$ and we allow agent 1 to deviate to v_1'' and to v_1'. Therefore, by ordinality, for all the profiles in the first row and in the last row of Table A.2.1 the SCF must select (b, b).

For any of the profiles in the first row we can make agent 2 change his valuation from v_2 first to v_2' and then to v_2''. This by monotonicity and permutation invariance cannot change the selection at those profiles. A similar reason can be applied to all the profiles in the last row, by allowing agent 2 to change his valuation from v_2''' to v_2'' and to v_2'. By ordinality, we have shown that, given p, at any profile in Table A.2.1, the SCF will select (b, b).

To prove the lemma it remains to show that (b, b) is selected when one agent ranks outcome b as first while the other ranks it third.

Without loss of generality take a profile at which agent 1 ranks outcome b as his best outcome whereas agent 2 ranks this profile as his worst. These profiles of preferences are shown in the second and third rows of the first column of Table A.2.2.

φ	$v_1 : b \succ a \succ c$	$v_1' : b \succ c \succ a$
$v_2 : c \succ b \succ a$	(b, b)	(b, b)
$v_2' : c \succ a \succ b$	Assume (a, a), (a, b) or (b, a)	...
$v_2'' : a \succ c \succ b$	$\varphi((v_1, v_2'), p)$	...
$v_2''' : a \succ b \succ c$	!	...

Table A.2.2

By contradiction, assume that $\varphi(v_1, v_2', p) \neq (b, b)$. This implies by monotonicity that $\varphi(v_1, v_2', p) \in \{(a, a), (a, b), (b, a)\}$. By permutation invariance and monotonicity, then $\varphi(v_1, v_2'', p) = \varphi(v_1, v_2', p)$. By permutation invariance $\varphi(v_1, v_2''', p) \neq (b, b)$. This contradicts the previous result that requires $\varphi(v_1, v_2''', p) = (b, b)$. Therefore, $\varphi(v_1, v_2', p) = (b, b)$. Then, by applying monotonicity and permutation invariance, the SCF must choose (b, b) at any profile in Table A.2.2. This proves Lemma 4, by ordinality.

□

Lemma 5. *For each* $v = (v_1, v_2) \in \mathcal{V}^{\{1,2\}}$ *and* $w = (w_1, w_2)$ *with* $v_1 = w_2$ *and* $v_2 = w_1$*, then* $\varphi(v, p) = \varphi(w, p)$*.*

Proof. By Lemma 4, we know that Lemma 5 is true for all v such that at least one agent ranks b as first. By ordinality, it is obviously true whenever $v_1 \simeq v_2$. Then, by compromise it also applies whenever b is ranked in second position by both agents. The only cases remaining are two: when one agent is ranking b second and the other is ranking b third; when both agents are ranking b third. Let us begin with the first case.

Consider Table A.2.3.

φ	$v_1 : a \succ b \succ c$	$v_1' : c \succ a \succ b$	$v_1'' : a \succ c \succ b$
$v_2 : a \succ b \succ c$	!	Assume $\neq \varphi((v_1, v_2'), p)$	$= \varphi((v_1, v_2''), p)$
$v_2' : c \succ a \succ b$	Assume $\neq \varphi((v_1', v_2), p)$		
$v_2'' : a \succ c \succ b$	$= \varphi((v_1'', v_2), p)$		

Table A.2.3

φ	$v_1 : c \succ b \succ a$	$v_1' : a \succ c \succ b$	$v_1'' : c \succ a \succ b$
$v_2 : c \succ b \succ a$	!	Assume $\neq \varphi((v_1, v_2'), p)$	$= \varphi((v_1, v_2''), p)$
$v_2' : a \succ c \succ b$	Assume $\neq \varphi((v_1', v_2), p)$		
$v_2'' : c \succ a \succ b$	$= \varphi((v_1'', v_2), p)$		

Table A.2.4

Consider the profiles $((v_1, v_2'), p)$ and $((v_1', v_2), p)$ and assume by contradiction that $\varphi((v_1, v_2'), p) \neq \varphi((v_1', v_2), p)$. By compromise, monotonicity and permutation invariance, $\varphi((v_1, v_2'), p)$,$\varphi((v_1', v_2), p) \in \{(b, b), (a, b), (b, a), (a, a)\}$. Given this, permutation invariance and monotonicity imply that $\varphi((v_1, v_2), p) = \varphi((v_1, v_2'), p)$ and $\varphi((v_1, v_2), p) = \varphi((v_1', v_2), p)$. By ordinality, this contradicts the assumption that $\varphi((v_1, v_2'), p) \neq \varphi((v_1', v_2), p)$.
Since $\varphi((v_1, v_2'), p) \in \{(b, b), (a, b), (b, a), (a, a)\}$, by permutation invariance and monotonicity, $\varphi((v_1, v_2''), p) = \varphi((v_1, v_2'), p)$. For the same reason, $\varphi((v_1'', v_2), p) = \varphi((v_1', v_2), p)$. Since $\varphi((v_1', v_2), p) = \varphi((v_1, v_2'), p)$, then, by ordinality, $\varphi((v_1, v_2''), p) = \varphi((v_1'', v_2), p)$.
A similar reasoning can be applied to Table A.2.4, showing that Lemma 5 holds in those cases where one agent ranks b in second position and the other ranks it in third position. It remains to consider the case where both agents rank b in third position. To this end consider Table A.2.5.

φ	$v_1 : a \succ b \succ c$	$v_1' : a \succ c \succ b$	$v_1'' : c \succ a \succ b$	$v_1''' : c \succ b \succ a$
$v_2 : c \succ b \succ a$	(b, b)	$\{(c, c), (c, b), (b, c), (b, b)\}$		
$v_2' : c \succ a \succ b$	$\{(a, a), (a, b), (b, a), (b, b)\}$	$= \varphi((v_2'', v_1''), p)$		
$v_2'' : a \succ c \succ b$			$= \varphi((v_1', v_2'), p)$	$= \varphi((v_1', v_2), p)$
$v_2''' : a \succ b \succ c$			$= \varphi((v_1, v_2'), p)$	(b, b)

Table A.2.5

By compromise, $\varphi((v_1, v_2), p) = (b, b)$. By monotonicity and permutation invariance, $\varphi((v_1, v_2'), p) \in \{(a, a), (a, b), (b, a), (b, b)\}$and $\varphi((v_1', v_2), p) \in \{(c, c), (c, b), (b, c), (b, b)\}$.
Fixed $\varphi((v_1, v_2'), p)$ and $\varphi((v_1', v_2), p)$, there is at most one social choice that respects monotonicity and permutation invariance at profile $((v_1', v_2'), p)$. For instance, if we fix $\varphi((v_1, v_2'), p) = (b, a)$ and $\varphi((v_1', v_2), p) = (c, b)$, then the only selection that respects monotonicity and permutation invariance is $\varphi((v_1', v_2'), p) = (c, a)$.
By the previous result, $\varphi((v_1''', v_2''), p) = \varphi((v_1', v_2), p)$ and $\varphi((v_1'', v_2'''), p) = \varphi((v_1, v_2'), p)$. As before, fixed $\varphi((v_1'', v_2'''), p)$ and $\varphi((v_1''', v_2''), p)$, there is at most one social act that respects monotonicity and permutation invariance at profile $((v_1'', v_2''), p)$. As a consequence $\varphi((v_1'', v_2''), p) = \varphi((v_1', v_2'), p)$, proving Lemma 5. □

We continue the proof of sufficiency by exhaustion. Namely, we take two particular profiles of preferences and we consider all the possible selections allowed by strategyproofness at these profiles. We end our proof by following the implications of strategyproofness at for each one of these selections.
In particular, consider the the profiles $((v_1, v_2'), p)$ and $((v_1', v_2), p)$ in Table A.2.6.

φ	$v_1 : a \succ b \succ c$	$v'_1 : a \succ c \succ b$
$v_2 : c \succ b \succ a$	(b,b)	$\{(b,b),(c,b),(b,c),(c,c)\}$
$v'_2 : c \succ a \succ b$	$\{(b,b),(a,b),(b,a),(a,a)\}$	

Table A.2.6

First, by compromise, permutation invariance and monotonicity, $\varphi((v_1, v'_2), p) \in \{(b,b),(a,b),(b,a),(a,a)\}$ and $\varphi((v'_1, v_2), p) \in \{(b,b),(c,b),(b,c),(c,c)\}$. Strategyproofness also implies that:

- by monotonicity, if $\varphi(v'_1, v'_2) \neq \varphi(v_1, v'_2, p)$, then $p_1(\varphi^c((v'_1, v'_2), p)) > p_1(\varphi^c((v_1, v'_2), p))$, and if $\varphi((v'_1, v'_2), p) \neq \varphi((v'_1, v_2), p)$, then $p_2(\varphi^a((v'_1, v'_2), p)) > p_2(\varphi^a((v'_1, v_2), p))$;
- by permutation invariance, $\varphi^a((v'_1, v'_2), p) = \varphi^a((v_1, v'_2), p)$ and $\varphi^c((v'_1, v'_2), p) = \varphi^c((v'_1, v_2), p)$;
- by ordinality, $\varphi(w_1, w_2), p) = \varphi((v'_1, v'_2), p)$ for all $w_1 \simeq v'_1$ and $w_2 \simeq v'_2$.

Given the previous conditions, for all possible values of $\varphi(v_1, v'_2)$ we can establish the following relations:

- if $\varphi((v_1, v'_2), p) = (b,b)$, then $\varphi((v'_1, v_2), p) \in \{(b,b),(c,b),(b,c),(c,c)\}$;
- if $\varphi((v_1, v'_2), p) = (a,b)$, then $\varphi((v'_1, v_2), p) \in \{(b,b),(b,c)\}$;
- if $\varphi((v_1, v'_2), p) = (b,a)$, then $\varphi((v'_1, v_2), p) \in \{(b,b),(c,b)\}$;
- if $\varphi((v_1, v'_2), p) = (a,a)$, then $\varphi((v'_1, v_2), p) \in \{(b,b)\}$.

Without loss of generality, we can focus our analysis on the following four cases:

1. $\varphi((v_1, v'_2), p) = (b,b)$ and $\varphi((v'_1, v_2), p) = (b,b)$;
2. $\varphi((v_1, v'_2), p) = (b,b)$ and $\varphi((v'_1, v_2), p) = (c,b)$;
3. $\varphi((v_1, v'_2), p) = (b,b)$ and $\varphi((v'_1, v_2), p) = (c,c)$;
4. $\varphi((v_1, v'_2), p) = (a,b)$ and $\varphi((v'_1, v_2), p) = (b,c)$.

There is no loss in generality here because, as it will become clear throughout the proof, all the other cases can be derived from these four.

CASE 1 ($\varphi(v_1, v'_2) = (b,b)$ and $\varphi(v'_1, v_2) = (b,b)$).

Consider Table A.2.6. By monotonicity, permutation invariance and ordinality, $\varphi((v'_1, v'_2), p) = (b,b)$. This implies, by permutation invariance and ordinality, that at any profile of preferences such that b is ranked third by both players the SCF must select (b,b). If in any one of these profiles any one of the two agents raises his valuation of outcome b, then by monotonicity the outcome cannot change. By ordinality, this shows that (b,b) will be selected at any preference profile such that beliefs are p.
Consider now a change in p. By Lemma 1 and Lemma 3 on beliefs, the outcome cannot change with a change in beliefs.

φ	$v_1 : a \succ b \succ c$	$v_1' : a \succ c \succ b$	$v_1'' : c \succ a \succ b$
$v_2 : c \succ b \succ a$	(b, b)	by asm. (c, b)	
$v_2' : c \succ a \succ b$	by asm. (b, b)	(c, b)	(c, b)
$v_2'' : a \succ c \succ b$	(b, b)	if $\neq (c, b) \Rightarrow !$	by L5 (c, b)

Table A.2.7

CASE 2 ($\varphi(v_1, v_2') = (b, b)$ and $\varphi(v_1', v_2) = (c, b)$)

Claim A.2.1. *When both agents rank b as their worst outcome, then the SCF must select (c, b).*

Proof. Consider Table A.2.7.
By monotonicity, permutation invariance and ordinality, $\varphi((v_1', v_2'), p) = (c, b)$. By monotonicity and permutation invariance $\varphi((v_1'', v_2'), p) = (c, b)$. By Lemma 5 and ordinality, since $\varphi((v_1', v_2'), p) = (c, b)$, then $\varphi((v_1'', v_2''), p) = (c, b)$. To prove the claim, it remains to show that $((v_1', v_2''), p) = (c, b)$.
By contradiction, assume that $((v_1', v_2''), p) \neq (c, b)$. Then, by monotonicity and permutation invariance, since $\varphi((v_1', v_2'), p) = (c, b)$, it must be $\varphi((v_1', v_2''), p) = (a, b)$. We also know that, since we assumed $\varphi((v_1, v_2'), p) = (b, b)$, then $\varphi((v_1, v_2''), p) = (b, b)$ by permutation invariance. By monotonicity and permutation invariance $\varphi((v_1', v_2''), p) \in \{(c, c), (c, b), (b, c), (b, b)\}$. By ordinality, this contradicts $\varphi((v_1', v_2''), p) = (a, b)$, proving the claim. □

Corollary A.2.1.1. *As long as outcome c is preferred to b by both agents, then the SCF must select (c, b).*

Proof. Take $((v_1', v_2''), p)$ in Table A.2.7. We know from the previous claim that $\varphi((v_1', v_2''), p) = (c, b)$. Starting from $((v_1', v_2''), p)$, any other profile at which both agents prefer c to b can be obtained by decreasing accordingly the position of a. Permutation invariance, monotonicity and ordinality imply that this manipulation cannot change the social choice. □

Claim A.2.2. *As long as one the two agents prefer b over c, then the SCF must select (b, b).*

Proof. Consider the profile $((v_1, v_2''), p)$ in Table A.2.7. We know that $\varphi((v_1, v_2''), p) = (b, b)$. By decreasing the position of a in $((v_1, v_2''), p)$ accordingly, we can get any profile of valuations at which agents 1 prefers b over c and agent 2 prefers c over b. At those profiles, by permutation invariance, monotonicity and ordinality, the outcome must remain (b, b). By Lemma 5, (b, b) must be selected at any profile at which one agent prefers b over c and the other prefers c over b. At any of those profiles, the social choice cannot change by raising b over c, by monotonicity, proving the claim.

□

We can relax at this point the assumption that fixes beliefs to $p = (p_1, p_2) \in \mathcal{P}^{\{1,2\}}$, and allow them to vary. By Lemma 1 and Lemma 3 on beliefs, it must be that at all those profiles of valuations in which the SCF selects (b, b) under p, the social choice cannot vary. We have now to consider those profiles of valuations in which the SCF selects (c, b) under p. At all these profiles we know that c is preferred to b by both agents. This together with Lemma 1 and Lemma 3 implies that: (a) an increase in the subjective probability of state ω_1 cannot change the outcome; (b) a decrease in the subjective probability of state ω_1 by one

agent or both can change the social act to (b, c). Call $p' = (p'_1, p'_2) \in \mathcal{P}^{\{1,2\}}$. By Lemma 2, if $(v, p) = (c, b)$ and $\varphi(v, p') = (b, c)$, it must be the case that (c) $p'_i(\omega_1) > 0.5$ and $p_i(\omega_1) < 0.5$ for some $i \in \{1, 2\}$. By strategyproofness, this would then imply that $(w, p') = (b, c)$, for all $w \in \mathcal{V}^{\{1,2\}}$ such that both agents prefer c to b. By consistently applying (a), (b) and (c), we can generate all the rules that we talked about in section 3.3.
The reasoning for the functions that select (a, b) instead of (c, b) in some valuations proceed in similar way, when the valuations and beliefs are adequately adjusted and we do not present it here. With this we have identified the second type of functions in Theorem 1.

CASE 3($\varphi(v_1, v'_2) = (b, b)$ and $\varphi(v'_1, v_2) = (c, c)$)

The proof for Case 3 goes exactly as for Case 2 by substituting (c, c) to (b, c). The only thing that changes is that in this case beliefs cannot influence the outcome. This simply follows from Lemma 1 and Lemma 3 on beliefs.

CASE 4($\varphi(v_1, v'_2) = (a, b)$ and $\varphi(v'_1, v_2) = (b, c)$)

Let us begin by analyzing the case when both agents rank b as the worst outcome.

Claim A.2.3. *Whenever the outcome b is ranked third by both agents, then the SCF must select* (a, c).

Proof. Consider Table A.2.8.

φ	$v_1 : a \succ b \succ c$	$v'_1 : a \succ c \succ b$	$v''_1 : c \succ a \succ b$
$v_2 : c \succ b \succ a$	(b, b)	by asm. (b, c)	
$v'_2 : c \succ a \succ b$	by asm. (a, b)	(a, c)	
$v''_2 : a \succ c \succ b$	(a, b)	if $\neq (a, c) \Rightarrow !$	by L5 (a, c)

Table A.2.8

Since, by assumption, $\varphi((v_1, v'_2), p) = (a, b)$ and $\varphi((v'_1, v_2), p) = (b, c)$, then monotonicity and permutation invariance imply that $\varphi((v'_1, v'_2), p) = (a, c)$. By Lemma 5, $\varphi((v''_1, v''_2), p) = (a, c)$.
By contradiction assume that $\varphi((v'_1, v''_2), p) \neq (a, c)$. Then, by monotonicity and permutation invariance, since $\varphi((v'_1, v'_2), p) = (a, c)$ then $\varphi((v'_1, v''_2), p) \in \{(c, a), (a, a)\}$. We also know that by monotonicity and permutation invariance $\varphi((v_1, v''_2), p) = (a, b)$ because $\varphi((v_1, v'_2), p) = (a, b)$ by assumption. Given this, the previous two axioms imply $\varphi((v'_1, v''_2), p) \in \{(a, b), (a, c)\}$. By ordinality we have produced a contradiction. As a consequence $\varphi((v'_1, v''_2), p) = (a, c)$. A similar reasoning by contradiction can be made for $\varphi((v''_1, v'_2), p) = (a, c)$, proving the claim by ordinality.

□

We focus now on those valuations where both agents prefer c to b and at least one prefers b to a.

Claim A.2.4. *When both agents prefer c to b and at least one prefers b to a, then the SCF must select* (b, c).

φ	$v_1 : a \succ b \succ c$	$v_1' : a \succ c \succ b$	$v_1'' : c \succ a \succ b$	$v_1''' : c \succ b \succ a$
$v_2 : c \succ b \succ a$	(b, b)	by asm. (b, c)	(b, c)	(b, c)

Table A.2.9

Proof. Consider a situation as in the claim, and assume without loss of generality that agent 1 prefers a to b and that agent 2 prefers b to a. This can correspond to both profile $((v_1', v_2), p)$ and to profile $((v_1'', v_2), p)$ in Table A.2.8.
Starting by our assumption that $\varphi((v_1', v_2), p) = (b, c)$, permutation invariance and monotonicity implies first $\varphi((v_1'', v_2), p) = (b, c)$ and then $\varphi((v_1''', v_2), p) = (b, c)$. By lemma 5, the same would apply if we switch the name of the two agents, proving the claim by ordinality. □

An exactly specular claim can be made when both agents prefer a to b and at least one prefers b to a.

Claim A.2.5. *When both agents prefer a to b and at least one prefers b to c, then the SCF must select* (a, b)*.*

The proof of this claim goes in a similar vain than the previous claim considering Table A.2.10 instead of Table A.2.9.

φ	$v_1 : c \succ b \succ a$	$v_1' : c \succ a \succ b$	$v_1'' : a \succ c \succ b$	$v_1''' : a \succ b \succ c$
$v_2 : a \succ b \succ c$	(b, b)	by asm. (a, b)	(a, b)	(a, b)

Table A.2.10

By Claim A.2.3, Claim A.2.4, Claim A.2.5 and Lemma 4 we have identified the fourth function in Theorem 4. The only thing that remains to show is that a change in beliefs cannot the social choice in this case.

Claim A.2.6. *Under Case 4, agents' beliefs do not matter.*

Proof. To show this, we relax the assumption that fix the beliefs to $p \in \mathcal{P}^{\{1,2\}}$.
By contradiction and without loss of generality, consider $p' = (p_1', p_2) \in \mathcal{P}^{\{1,2\}}$ such that $\varphi(v, p) \neq \varphi(v, p')$ with $\varphi(v, p) = (b, c)$. By the Three Lemmas on beliefs and by the fact that at v both agents prefer c over b, it must be that $\varphi(v, p') = (c, b)$ and $p_1'(\omega_1) > 0.5, p_1(\omega_1) < 0.5$. By strategyproofness, for all w such that both agents prefer b over c and at least one prefers b over a it must be that $\varphi(w, p') = (c, b)$. Indeed, for all profiles (w, p') the ranking over (c, b) and (b, c) for both agents does not change. Therefore, a change in the social choice from (c, b) to (b, c) would produce an opportunity for a beneficial deviation for agent 1. This implies that $\varphi((v_1', v_2), p') = (c, b)$ with v_1' and v_2 as in Table A.2.8.
Consider now the profile of valuations (v_1, v_2') in Table A.2.8. By Lemma 1 it must be $\varphi((v_1, v_2'), p') = \varphi((v_1, v_2'), p) = (a, b)$.
By strategyproofness, $\varphi((v_1', v_2), p') = (c, b)$ and $\varphi((v_1, v_2'), p) = (a, b)$ leads to a contradiction. Indeed, there is no social choice at $((v_1', v_2'), p')$ that respects monotonicity, permutation invariance and ordinality. This shows that at all those profiles of valuations at which the SCF selects (a, b) and (b, c) under p, it must also select these acts under p'. By following the implications plotted in Table A.2.8, it is possible to show that the same must happen

at all those profiles in which (a,c) is selected under p. Since we have chosen p' without loss of generality, this shows that a change in beliefs cannot change the social act in Case 4. $\square$

With this we have identified the four kinds of social choice functions in Theorem 1.
Returning back at Table A.2.6, it is easy to see at this point that we have consider all the possible cases. The one that we have not explicitly covered can be derived by fixing a value for $((v_1', v_2), p)$ and considering all the possible values for $\varphi((v_1, v_2'), p)$. Due to the symmetry of our proofs, these will not generate other functions.

A.2.2 Proof of necessity

That all the function in the Theorem 1 compromise is obvious. To show that those functions also respect strategyproofness, we consider each one at a time. The numbering is as in Theorem 1.

SCF 1

This is obviously strategyproof since there is only one possible social act, no matter agents' preferences.

SCFs 2

Without loss generality, consider the following SCF:

- if at least one agent prefers b to c, then (b,b) is selected;
- if both agents prefer c to b and at least one think that state ω_1 is more likely than ω_2, then (a,b) is the social choice;
- if both prefer c to b and both think that state ω_2 is more likely than state ω_1, then (b,a) is the social act.

Consider first a unilateral deviation in valuations. We know that, no matter p, as long as an agent i prefers outcome b to a, then $(b,b) \succ_i (a,b)$ and $(b,b) \succ_i (b,a)$, and vice-versa. Therefore, we can easily see that the function always give the preferred social act, fixed a belief profile p.
Consider now a unilateral deviation in beliefs. The only case in which a change in belief an change the social act is when both agents prefer c to b. In this case we know that $(c,b) \succ_i (b,c)$ whenever $p_i(\omega_1) > 0.5$, and vice-versa. Even in this case, we can see that the players always receive their best social act given the profile of valuations. This holds for all possible function of type 2.
When agents change both valuations and beliefs, for the valuation to have a role in the change in the social choice it must be that the this change does not contradicts strategyproofness by the previous reasoning. If the change in valuation does not have a role in determining the social act, then we know by the previous argument on beliefs that the choice cannot contradicts stategyproofness. Hence, we can conclude that agents cannot benefit by misreporting their preferences.

SCFs 3

The previous argument for the SCFs 2 also applies in this case.

SCFs 4

These functions do not depend on beliefs. It is then easy to prove that these functions do indeed respect strategyproofness by cases.
Without loss of generality, we take a particular SCF of type 4 and we consider all the possible orders over outcomes for agent 2. With the following list we show that for each of these orders, agent 1 cannot benefit from a manipulation of his preferences no matter p_1:

- if $v_2 : a \succ b \succ c$, then whenever agent 1 prefers a to b then (b, a) is the social act; otherwise (b, b) is selected. Hence, agent 1 always gets his preferred social act among the two available;
- if $v_2 : a \succ c \succ b$, then
  - if $v_1 : a \succ b \succ c$, the SCF selects (b, a);
  - if $v_1 : a \succ c \succ b$ or $v_1 : c \succ a \succ b$, then the SCF selects (c, a);
  - if $v_1 : c \succ b \succ a$, then the SCF selects (c, b);
  - otherwise the SCF selects (b, b).

  For each one of the v_1 it is possible to show that agent 1 is receiving his preferred act among the four possible;
- if $v_2 : c \succ b \succ a$, then whenever agent 1 prefers c to b then (c, b) is the social act; otherwise (b, b) is selected. Hence, agent 1 always gets his preferred social act among the two available;
- if $v_2 : c \succ a \succ b$, then
  - if $v_1 : a \succ b \succ c$, the SCF selects (b, a);
  - if $v_1 : a \succ c \succ b$ or $v_1 : c \succ a \succ b$, then the SCF selects (c, a);
  - if $v_1 : c \succ b \succ a$, then the SCF selects (c, b);
  - otherwise the SCF selects (b, b).

  For each one of the v_1 it is possible to show that agent 1 is receiving his preferred act among the four possible;
- when agent 2 ranks b is first position, the SCF will select (b, b) no matter agent 1's valuation.

With this we have proven that the SCFs presented in Theorem 1 do indeed respect strategyproofness, completing the proof of Theorem 1. □