Economics School of Louvain - ESL

Asian option pricing with comonotonic bounds

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Academic Year: 2021-2022

Master in Economics – 120 credits – Econometrics Focus

Erasmus Mundus Joint Master Degree in Models and Methods of Quantitative Economics

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Abstract

The goal of this master thesis is to implement and test the quality of Asian option price intervals derived from comonotonic bounds. The quality of these bounds has been evaluated in a Black-Scholes framework, and the benchmark prices have been obtained by Monte-Carlo simulations with variance reduction. Also, the quality of these bounds has been challenged in the context of stochastic volatility, when the stochastic volatility is modeled as the exponential of a fast mean-reverting Ornstein-Uhlenbeck process. In the case of stochastic volatility, the benchmark was given by Monte-Carlo simulations with variance reduction, while the bounds are computed by taking a deterministic time-dependent volatility set equal to the expectation of the volatility process.

Table of contents

1 Introduction

Options constitute a very important instrument on the financial market that is usually used for hedging, speculation, and arbitrage GRZELAK (2022). Also for these objectives, forward and futures contracts could be used. However, there is a significant difference between forward, futures contracts, and options: the holder of the option has the right to use the instrument but it is not an obligation, whereas the future or forward holder has to do a certain action at the expiration time. As a consequence, the price of forward or futures contracts can be zero, whereas the price of an option is always non-negative.

Generally, there are the most standard options on stock: call options and put options. The call option gives the right to buy an underlying asset or instrument at the prespecified date (expiration date), whereas the put option gives the right to sell under the same conditions. In terms of the underlying asset, options can be defined as vanilla options and exotic options. The vanilla option consists of European-style options and American-style options. The main difference is that the American option could be exercised within the period from the selling/buying contract to the expiration date, whereas the European option could be exercised exclusively at the expiration date. An exotic option is the type of option that has more complex features. One of the prominent examples is an Asian option. An Asian option is an option in which the underlying is an average of a financial variable, for example, the price. In this thesis, we consider a European-style Asian option that depends on the arithmetic average of the price of the underlying asset.

Pricing an arithmetic Asian option is particularly challenging even in the most simple pricing models HORVATH & MEDVEGYEV (2016). The two main reasons for the mathematical complexity are that the Asian options are path-dependent and the probability distribution of averages of prices is usually not available in closed form. For instance, the analytical expression of the distribution of the average of log-normal variables is unknown. Pricing of arithmetic Asian options even in the Black-Scholes settings is complex. A closed-form formula for pricing arithmetic Asian options has not been derived yet. Therefore, one of the most common approaches to pricing arithmetic-average Asian options is using Monte Carlo simulation, which gives relatively good results, but the Monte Carlo method demands computational power. It is also possible to use characteristic functions if the number of average points is relatively small. Another approach to price the Asian bound is through lower and upper bounds. If the difference between these bounds is sufficiently small, it means that the correct price is derived with some error, that is the distance between lower and upper bounds. CURRAN (1994) and ROGERS & SHI (1995) derived the lower bound using the conditioning approach. Later NIELSEN & SANDMANN (2003) derived the comonotonic upper bound and VANMAELE et al. (2006) improved the upper bound using the conditioning approach and proposed the partially exact comonotonic upper bound. All the above-mentioned models used the geometric Brownian motion model. Moreover, the results were presented for constant volatility only. The upper and lower bounds with different stochastic volatility models were presented, where the asset price is described by Lévy models FUSAI & KYRIAKOU (2016),ALBRECHER et al. (2005). The Lévy processes capture the common feature of the price processes which is stochastic volatility. Later CARR & WU (2004) proposed the time-changed Lévy processes that nest some popular stochastic volatility models. However, the time-changed Lévy processes pose a high level of mathematical challenge, therefore, it is difficult to use them. Moreover, recently it was shown in FALLAHGOUL & NAM (2019) that all the models that were proposed in CARR & WU (2004) fail to satisfy the core assumption that is used in the paper CARR & WU (2004).

The results for a set of bounds derived in VANMAELE et al. (2006) were presented for constant volatility. However, it is possible to extend the result to deterministic time-dependent volatility. The main aim of this thesis is to test the quality of the bounds in the context of stochastic volatility. This will fill a gap in the literature regarding the quality of the bounds of Asian options for stochastic volatility models. Moreover, the assumption of constant volatility seriously limits the results as the volatility in the real market is not constant.

In this thesis, as a benchmark price, we used the Monte Carlo simulations with variance reduction. We compared the plain Monte Carlo simulations with the Monte Carlo simulations with variance reduction and showed that the later method produces results with lower standard error. Then we computed the bounds presented in VANMAELE et al. (2006) for constant volatility. In order to introduce the stochastic volatility, we used the exponential of a fast mean-reverting Ornstein-Uhlenbeck process. Also, we computed the bounds for stochastic volatility and compared the results.

Overall, in this thesis, we focus on Asian option prices when the underlying asset follows geometric Brownian motion with volatility taking the form of a fast mean-reverting exponential Ornstein–Uhlenbeck (OU) process. Generally, our contribution is the following. First of all, we implement a Monte Carlo method to price

Asian options with stochastic volatility. In order to improve convergence, we use the set of European option prices as a control variate. Secondly, as the bounds given in VANMAELE et al. (2006) cannot accommodate the stochastic volatility, we apply the method where volatility is modeled as an expectation of the exponential of the OU process. The expectation of the exponential of the OU process is known in closed form in the case of the stochastic volatility model that we use in this thesis. Finally, we discuss the quality of the computed bounds by comparing them with the Monte Carlo approach.

The results of our work show that the dynamics of the quality of the bounds is mixed after introducing the stochastic volatility. For a certain set of bounds, the quality strictly decreases. For some strike prices, the difference between the bound and the Monte Carlo price increases by more than 34 times. For another set of bounds, the difference decreases, however, for some strike prices the difference becomes negative. Therefore, the quality of the bounds increases, but the operational range for strike price shrinks.

The structure of the thesis is the following: in Chapter 2 we compare the price of the Asin option computed using plain Monte Carlo simulations and Monte Carlo with control variate simulations; in Chapter 3 we present the bounds for the price of discrete arithmetic Asian options; in Chapter 4 we introduce the exponential Ornstein–Uhlenbeck process (expOU), then the process is used to compute the Monte Carlo price and bounds; in Chapter 5 we present the results of testing the bounds and compare their quality.

1.1 Features of the Asian option

In this section, we will elaborate on the description of features of the Asian options. The Asian option belongs to the class of the "lookback options". The payoff function does not depend exclusively on the maturity date but also depends on the time before the expiry date. Asian options are often used for protection against unexpected changes in the prices of an underlying asset. If the profit of a certain company strongly depends on one commodity price over a time period, an Asian option could be used to hedge against price changes. The Asian options are usually traded on currencies, interest rate markets, and other markets that have low trading volume which can lead to an unexpected change in asset prices. One of the most used examples for underlying assets for an Asian option is the exchange rate between currencies. As the volatility of the average price of an underlying asset is lower than the volatility of the price of the underlying asset, Asian options are cheaper than

European or American options. There are two major ways to get the average value in the framework of a discreet approach. For the arithmetic case:

$$\frac{1}{k}\sum_{i=1}^{k} S_{t_i} \tag{1}$$

and for the geometric case:

$$\sqrt[k]{S_{t_1}S_{t_2}...S_{t_k}} \tag{2}$$

where dates are fixed: $t_1, t_2, t_3, ..., t_k = T$.
In the Black and Scholes model F. BLACK (1973) that we use in sections 1.1 and 2, the price of underlying asset under the risk-neutral measure follows a geometric Brownian motion process with volatility σ and with drift equal to risk-freee rate:

$$\frac{dS(t)}{S(t)} = rdt + \sigma dB(t), \quad t \geq 0 \tag{3}$$

where $B(t), t \geq 0$ is a standart Brownian motion process under risk-neutral measure, r is risk-neutral interest rate, and σ is volatility of underlying asset. Therefore, the random variable $\frac{dS(t)}{S(t)}$ follows normal distribution with mean $(r - \frac{\sigma^2}{2})$ and variance $t\sigma^2$.

We will use the following function (4) to define the payoffs of an option:

$$\psi(X) = (X - K)^+ \tag{4}$$

Therefore, after combining (1) and (4), we have the payoff function of the Asian option with arithmetic averaging:

$$\psi\left(\frac{1}{k}\sum_{i=1}^{k} S_{t_i}\right) = \left(\frac{1}{k}\sum_{i=1}^{k} S_{t_i} - K\right)^+ \tag{5}$$

The no-arbitrage price at time 0 is the risk-neutral expectation of the payoff function given in (5) discounted back from the payment date T to the time of the valuation $t = 0$:

$$V_0 = \widetilde{\mathbb{E}}\left[\psi(\frac{1}{k}\sum_{i=1}^{k} S_{t_i})\right] e^{-rT} \tag{6}$$

where $\widetilde{\mathbb{E}}$ denotes a risk-neutral expectation and r is a risk-neutral interest rate.

The function $\psi(X)$ in (5) depends not only on the current value of the stock but on the values of the underlying asset that occurred before the expiry date. This way, the Asian options are path-dependent.

In order to price the European options we should compute $\widetilde{\mathbb{E}}[\psi(S_T)]e^{-rT}$, where S_T follows the log-normal distribution as it can be seen from (3). One way to do this is to use the Black-Scholes-Merton formula F. BLACK (1973). However, in the case of the Asian option, as it can be seen in (6), the argument of the function is a sum of log-normal dependent variables. Therefore, we cannot use the Black-Scholes-Merton formula to find the price of the Asian option. There are many numerical approaches to price Asian options such as moment matching, Monte Carlo simulations, etc. In this work, we will use the Monte Carlo simulations with variance reduction VRINS (2020).

2 Monte Carlo simulations

2.1 Plain Monte Carlo

In this section, we will present the basis of the Black-Scholes setup before moving into the stochastic volatility case. First of all, we will compute the price of the Asian option using naive Monte Carlo simulations [VRINS (2020), HULL (2003), JOSHI et al. (2003), GLASSERMAN (2004)]. The standard Monte Carlo estimator using n scenarios is the following:

$$\hat{V}_{0,n} = \frac{e^{-rT}}{n} \sum_{i=1}^{n} \psi\left(\frac{1}{k}\sum_{j=1}^{k} s_{i,j}\right) \tag{7}$$

where $s_{i,0} = S_0$ is initial price of the stock, $s_{i,j} = s_{i,j-1}e^{(r-\frac{\sigma^2}{2})+\sigma\sqrt{t_j-t_{j-1}}z_{i,j}}$ is value of the stock at time t_j on i^{th} scenario and $z_{i,j}$ is $n \times k$ samples of i.i.d. realization of standard Normal random variable.

The plain Monte Carlo simulations provide results with relatively large error GALDA (2008):

$$error = 1.96\sqrt{\frac{\sigma^2}{n}} \tag{8}$$

where variance σ^2 is computed the following way:

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}\left((\psi(\frac{1}{k}\sum_{j=1}^{k} s_{i,j})e^{-rT}\right)^2 - \left(\frac{1}{n}\sum_{i=1}^{n}(\psi(\frac{1}{k}\sum_{j=1}^{k} s_{i,j})e^{-rT}\right)^2 \tag{9}$$

We computed the price of the Asian option with the following parameters: $T = 1$,$S_0 = 100$, $K = [60, 140]$, $r = 0.05$, $\sigma = 0.3$, step between two consecutive times: $t_j - t_{j-1} = \delta = 1/12$, $n = 10000$. In order to compute error and variance we used (8) and (9) respectively.

Strike K	**Option price**	**Error**	**Variance**	**Coefficient of variation**
60	40.7543	0.3779	339.2753	0.9%
70	31.4116	0.3762	323.8897	1.2%
80	22.4028	0.3622	308.0586	1.6%
90	14.5304	0.3248	245.0	2.2%
100	8.5685	0.2675	164.1860	3.1%
110	4.5471	0.2020	94.0347	4.4%
120	2.3662	0.1449	47.7868	6.1%
130	1.0491	0.0989	21.9130	9.4%
140	0.4269	0.0653	10.3215	15.3%

Table 1: Plain Monte Carlo method

As it can be seen in Table 1, the error and variance decrease with strike K. However, the coefficient of variation increases with strike K and gains a maximum value of 15.3% which is significant. The error could be reduced if we increase the number of Monte Carlo simulations, but it will take more computation time and this way of improvement is labor-intensive. In order to have a small error, the number of Monte Carlo simulations should be around 20 000 (see Supplementary Figure 4 in Appendix A).

In this work, we will not use the plain Monte Carlo method due to the above-mentioned reasons. Instead, we will use Monte Carlo simulations with variance reduction.

2.2 Monte Carlo simulations with control variate

One way to improve the Plain Monte Carlo method is the control variate technique GALDA (2008), VRINS (2020). In this technique instead of estimating the price of options, one considers the difference between a problem and an analytical model:

$$X' = X - Y + \mathbb{E}(Y) \tag{10}$$

As the expectation of X' coincides with the expectation of X we can use $\mathbb{E}(X')$ as an estimator for computing $\mathbb{E}(X)$:

$$\mathbb{E}(X') = \mathbb{E}(X - Y) + \mathbb{E}(Y) \tag{11}$$

If $\mathbb{E}(Y)$ can be computed analytically, computing $\mathbb{E}(X')$ or $\mathbb{E}(X)$ requires the same computation efforts. However, variance of $\mathbb{E}(X')$ is lower. A special function Y should be found for which the $\mathrm{Cov}(X, Y) > 0$ and it will reduce the variance of X' in (10).

$$\mathrm{Var}(X') = \mathrm{Var}(X) + \mathrm{Var}(Y) - 2\mathrm{Cov}(X, Y) \tag{12}$$

As it can be seen in (12), functions X and Y should be positively correlated. A possible choice of Y was proposed in VRINS (2020):

$$Y = \frac{1}{k}\sum_{i=1}^{k}(S_{t_i} - K)^+ e^{-rt_i} = \frac{1}{k}\sum_{i=1}^{k}\psi(S_{t_i}e^{-rt_i}) \tag{13}$$

Each term in the sum in (13) is the discounted payoff of a European option with expiry t_i and strike price K. Indeed, they are both increasing functions of the value of underlying asset S. Therefore, we can easily compute the risk-neutral expectation of function Y:

$$\widetilde{\mathbb{E}}(Y) = \frac{1}{k}\sum_{i=1}^{k}\widetilde{\mathbb{E}}[(S_{t_i} - K)^+ e^{-rt_i}] = \frac{1}{k}\sum_{i=1}^{k}BS(r, S_0, K, t_i) \tag{14}$$

Generally, we have a new estimator which is unbiased:

$$\widetilde{\mathbb{E}}[\psi(\frac{1}{k}\sum_{i=1}^{k}S_{t_i})e^{-rT} - Y + \widetilde{\mathbb{E}}(Y)] = \widetilde{\mathbb{E}}[\psi(\frac{1}{k}\sum_{i=1}^{k}S_{t_i})]e^{-rT} = V_0 \tag{15}$$

We will use the following estimator of V_0 for Monte Carlo simulations:

$$\tilde{V}_{0,n} = \frac{1}{n}\sum_{i=1}^{n}\left(\psi(\frac{1}{k}\sum_{j=1}^{k}s_{i,j})e^{-rT} - \psi(\frac{1}{k}\sum_{j=1}^{k}s_{i,j}e^{-rt_j})\right) + \frac{1}{k}\sum_{j=1}^{k}BS(r, S_0, K, t_j) \tag{16}$$

We computed the prices of the Asian option with the following parameters: $T = 1$,$S_0 = 100$, $K = [60, 140]$, $r = 0.05$, $\sigma = 0.3$, step between two consecutive times: $t_j - t_{j-1} = \delta = 1/12$, $n = 10000$.

Strike K	**Option price**	**Error**	**Variance**	**Coefficient of variation**
60	40.6625	0.0480	6.0149	0.1%
70	31.1774	0.0521	7.0645	0.1%
80	22.1012	0.0557	8.0885	0.2%
90	14.2083	0.0539	7.5904	0.3%
100	8.4758	0.0449	5.2482	0.5%
110	4.2506	0.0378	3.7279	0.8%
120	1.9788	0.0327	2.7861	1.6%
130	0.8247	0.0303	2.3955	3.6%
140	0.3463	0.0302	2.3859	8.7%

Table 2: Monte Carlo with control variate

Table 2 demonstrates that the error and variance are significantly reduced in the Monte Carlo method with control variate. Moreover, the coefficient of variation is smaller for almost all strike prices compared to the plain Monte Carlo method.

In Supplementary Figure (5) in Appendix A, it can be seen that the error is smaller for the smaller strike prices. The error when using the Monte Carlo method with control variate is smaller than the error when using the plain Monte Carlo method for the corresponding number of simulations. Therefore, in terms of accuracy and speed of computation, it is better to use the Monte Carlo method with control variate.

2.3 Comparison

In Figure 1 we can see the convergence of the prices that were computed using the two methods with the above-given parameters. As it was mentioned above, the plain Monte Carlo method gives the price with significant error. Moreover, the error does not decrease fast with a number of simulations. Conversely, the Monte Carlo method with the control variate gives more accurate results with relatively small errors. Additionally, the error is negligible for a smaller number of simulations. Therefore, the later method converges faster than plain Monte Carlo.

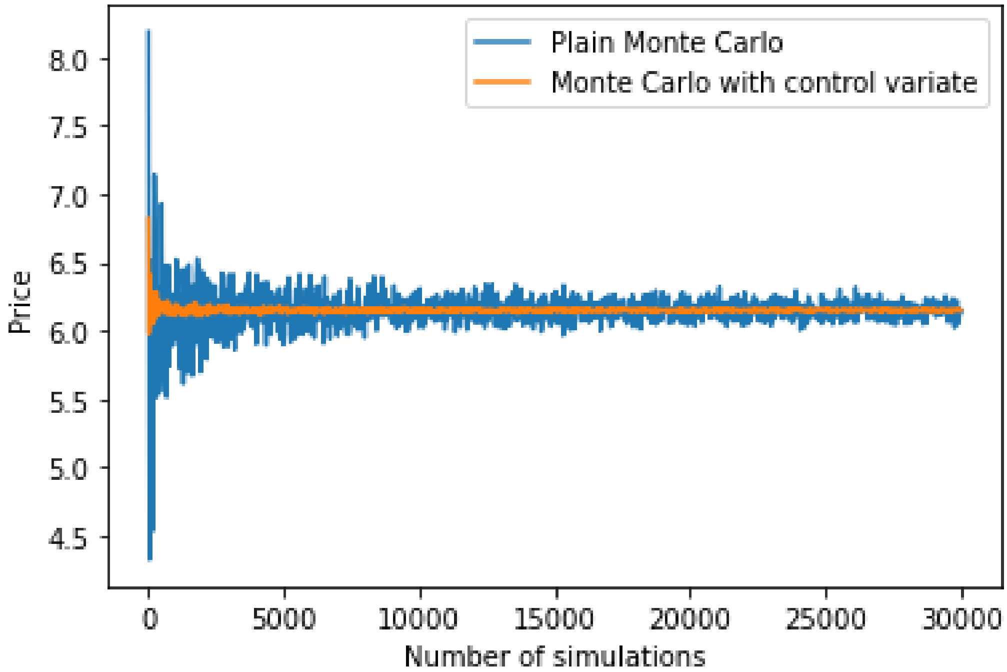


Figure 1: Dependence of the price on the number of the Monte Carlo simulations.

3 Bounds for the price of discrete arithmetic Asian options

In the article VANMAELE et al. (2006), the authors derived the lower and upper bounds for discrete arithmetic Asian options in the case of comonotonic bounds. The bounds were derived for call European-style discrete arithmetic Asian option. The authors obtained analytical and easily computable bounds. However, the results of the paper were presented for constant volatility. Therefore, the results could be extended for deterministic time-dependent volatility. In order to compare option prices with bounds, we will use Monte Carlo simulations with control variates. The stochastic volatility in Monte Carlo simulations will be modeled as an exponential of the fast mean-reverting Ornstein-Uhlenbeck process.

In order to be consistent with the paper VANMAELE et al. (2006), we will follow the same style of definitions. Given the price of risky underlying asset $S(T-t_i)$ at times $T-t_i, i=0,1,..,n-1, T=t_n$, a call European-style discrete Asian options exercised at time T with n averaging dates and with fixed strike price generates the following payoff function:

$$\left(\frac{1}{n}\sum_{i=1}^{n} S(T-t_i) - K\right)_+ \tag{17}$$

where $x_+ = \max(x, 0)$.

The price of the Asian call option at the initial time $t = 0$ is the risk-neutral expectation of the payoff function discounted back from the payment date T to the time of the valuation:

$$AC(n, K, T) = \frac{e^{-rT}}{n}\widetilde{\mathbb{E}}[(\sum_{i=0}^{n-1} S(T-t_i) - nK)_+] \tag{18}$$

where $\widetilde{\mathbb{E}}$ denotes a risk-neutral expectation and r is a risk-neutral interest rate. Within the Black and Scholes model, it is not possible to have a closed-form solution for Asian options with discrete arithmetic averaging. However, in the case of geometric averaging the closed-form solution is available. It is possible to have analytical solutions for arithmetic averaging in the form of bounds. In the paper VANMAELE et al. (2006), the authors focus on analytical methods based on comonotonic bounds through conditioning on a random variable.

3.1 Bounds based on comonotonicity reason

In the financial science random variables of the type $\mathcal{S} = \sum_{i=1}^{n} X_i$ are often used, where the variables X_i are not mutually independent, however, the multivariate distribution function of the random vector $(X_1, X_2, ..., X_n)$ is not known because only marginal distributions of random variables X_i are known. In order to be able to infer some results, one wants to have a lower bound of the form $\underline{\mathcal{S}} = \sum_{i=1}^{n} \underline{X_i}$ and an upper bound of the form $\overline{\mathcal{S}} = \sum_{i=1}^{n} \overline{X_i}$ for the sum $\mathcal{S} = \sum_{i=1}^{n} X_i$. The proposed bounds should satisfy the following conditions: $i)$ marginal distribution functions of X_i, $\underline{X_i}$, $\overline{X_i}$ $(i = 1, 2, ..., n)$ are equal $ii)$ $\underline{\mathcal{S}} \preceq_{cx} \mathcal{S} \preceq_{cx} \overline{\mathcal{S}}$, where $\preceq_{cx}$ denotes the convex order, which means that $\mathbb{E}[\underline{\mathcal{S}}] = \mathbb{E}[\mathcal{S}] = \mathbb{E}[\overline{\mathcal{S}}]$ and $\mathbb{E}[(\underline{\mathcal{S}} - d)_+] = \mathbb{E}[(\mathcal{S} - d)_+] = \mathbb{E}[(\overline{\mathcal{S}} - d)_+]$ for all $d \in \mathbb{R}$.

One of the possible choices for upper bound $\overline{\mathcal{S}}$ is based on an assumption that the additional random variable Λ with given distribution exists and it provides some additional information about the random vector $(X_1, X_2, ..., X_n)$ VANMAELE et al. (2006).

In the paper KAAS et al. (2000), the following choice was proposed: $\overline{\mathcal{S}} = \mathcal{S}^u$ with

$$\mathcal{S}^u = F^{-1}_{X_1|\Lambda}(U) + F^{-1}_{X_2|\Lambda}(U) + ... + F^{-1}_{X_n|\Lambda}(U) \tag{19}$$

where $F^{-1}_X(U)$ is the inverse of a distribution function that is non-decreasing and left-continuous, and U being a (0,1)-uniform random variable independent of Λ.

$$F^{-1}_X(U) = \inf\{x \in \mathcal{R}|\ F_X(x) \leq\ U\},\ U \in [0,1]$$

For the lower bound, it has been chosen that $\underline{\mathcal{S}} = \mathcal{S}^l$, also proposed in the paper KAAS et al. (2000), where $\mathcal{S}^l$ is a conditional expectation given random variable Λ:

$$\mathcal{S}^l = \mathbb{E}[\mathcal{S}|\Lambda] \tag{20}$$

Alternatively, the elements of the random vector $(X_1, X_2, ..., X_n)$ are chosen in the following way: $\underline{X_i} = \mathbb{E}[X_i|\Lambda]$

In summary, in order to achieve the lower and upper bounds for an Asian option, we will substitute the above-given expression for $\mathcal{S}^l$ and $\mathcal{S}^u$ into the formula of the price of European-style discrete arithmetic Asian call option with strike price K, maturity date T and averaging over n prices of the underlying asset:

$$AC(n, K, T) = \frac{e^{-rT}}{n}\widetilde{\mathbb{E}}[(\mathcal{S} - nK)_+] \tag{21}$$

3.2 Lower bound

In this section, we will present the lower bound for different random variables Λ. The lower bound is derived by substituting $\mathcal{S}^l$ for $\mathcal{S}$ on the right-hand side of the (21), where $\mathcal{S}^l$ is the following:

$$\mathcal{S}^l = \sum_{i=0}^{n-1} \mathbb{E}[X_i|\Lambda]$$

Henceforth, we assume that the conditioning variable Λ is normally distributed such that $(\sigma(t_i)B(T - t_i), \Lambda)$ are bivariate normally distributed for all i and $\sigma(t)$ is deterministic time-dependent that will be presented in section 4. Under this assumption, the lower bound which was presented in VANMAELE et al. (2006) is the following:

$$\begin{aligned} \mathrm{LB}\Lambda &= \frac{e^{-rT}}{n}\widetilde{\mathbb{E}}[(\sum_{i=0}^{n-1}\mathcal{S}^l - nK)_+] \\ &= \frac{S(0)}{n}\sum_{i=0}^{n-1} e^{-rt_i}\Phi\left[\sigma(t_i)\rho_{T-t_i}\sqrt{T-t_i} - \Phi^{-1}(F_{\mathcal{S}^l}(nK))\right] - e^{-rT}K(1-(F_{\mathcal{S}^l}(nK))) \end{aligned} \tag{22}$$

where $\rho_{T-t_i} = corr(\sigma(t_i)B(T-t_i), \Lambda) \geq 0$ and $F_{\mathcal{S}^l}(nK)$ is a solution to:

$$S(0)\sum_{i=0}^{n-1} exp\left[(r - \frac{\sigma^2(t_i)}{2}\rho^2_{T-t_i})(T-t_i) + \sigma(t_i)\rho_{T-t_i}\sqrt{T-t_i}\Phi^{-1}(F_{\mathcal{S}^l}(nK))\right] = nK \tag{23}$$

where $\Phi(\cdot)$ is the cumulative distribution function (cdf) of a standard normal variable and $F_{\mathcal{S}^l}(\cdot)$ represents the cdf of $\mathcal{S}^l$.

As it can be seen, the conditional variable Λ intervenes in the lowest bound only through the correlation terms ρ_{T-t_i}. The choice of ρ_{T-t_i} is up to the researcher. However, the choice of Λ in the paper VANMAELE et al. (2006) was made in order to have a closed-form solution for the lower bound. With this aim in mind, the general form of Λ is:

$$\Lambda = \sum_{i=0}^{n-1}\beta_i B(T-t_i), \quad \beta_i \in \mathcal{R}^+$$

The variance of Λ for positive β_i is:

$$\sigma^2_\Lambda = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\beta_i\beta_j \min(T-t_i, T-t_j)$$

The ρ_{T-t_i} in this setting has the following general form:

$$\rho_{T-t_i} = corr(\sigma(t_i)B(T-t_i),\Lambda) = \frac{cov(B(T-t_i),\Lambda)}{\sqrt{T-t_i}\sigma_\Lambda} = \frac{\sum_{j=0}^{n-1}\beta_i \min(T-t_i, T-t_j)}{\sqrt{T-t_i}\sigma_\Lambda} \geq 0 \tag{24}$$

The choice of β_i is motivated by the quality of lower bound (20) and defined by its variance. The higher quality should have lower variance:

$$\widetilde{\mathbb{E}}[\widetilde{var}[\mathcal{S}|\Lambda]] = \widetilde{var}[\mathcal{S}] - \widetilde{var}[\widetilde{\mathbb{E}}[\mathcal{S}|\Lambda]]$$

Naturally, in order to get the best lower bound for (21), Λ and $\mathcal{S}$ should be as alike as possible. In the paper VANMAELE et al. (2006) proposed three possible choices for Λ which result in low variance:

1. authors show that the following conditional variable leads to sharp upper bound:
$$\Lambda = \sum_{k=1}^{T} W_k = B(T),$$
with W_k i.i.d. realization of standard normal variable.
The correlation terms are:
$$r_i = \rho_{T-t_i} = \frac{cov(B(T-t_i),\Lambda)}{\sqrt{T-t_i}\sigma_\Lambda} = \frac{T-t_i}{\sqrt{T-t_i}\sqrt{T}} = \frac{\sqrt{T-t_i}}{\sqrt{T}},\ i = 0, 1, ..., n-1 \tag{25}$$
It will be shown later that this choice fits better than others for sum $\mathcal{S}^u$.

2. A linear transformation of first order approximation to $\sum_{i=0}^{n-1} B(T-t_i)$ in $\mathcal{S}$. It was also proposed in papers KAAS et al. (2000), DHAENE et al. (2002):
$$\Lambda = \sum_{i=0}^{n-1} e^{(r-\frac{\sigma^2(t_i)}{2})(T-t_i)} B(T-t_i)$$
The correlation and variance terms are:
$$\begin{aligned} \rho_{T-t_i} &= \frac{\sum_{j=0}^{n-1} e^{(r-\frac{\sigma^2(t_j)}{2})(T-t_j)} \min(T-t_i, T-t_j)}{\sqrt{T-t_i}\sigma_\Lambda} \\ \sigma_\Lambda^2 &= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} e^{(r-\frac{\sigma^2(t_i)}{2})(T-t_i)+(r-\frac{\sigma^2(t_j)}{2})(T-t_j)} \min(T-t_i, T-t_j) \end{aligned} \tag{26}$$

3. The logarithm of the geometric average used as a conditional variable was shown to give good results NIELSEN & SANDMANN (2003), ROGERS & SHI (1995). The conditional variable has the following form:
$$\Lambda = \frac{\ln\mathcal{G} - \widetilde{\mathbb{E}}[\ln\mathcal{G}]}{\sqrt{\widetilde{var}[\ln\mathcal{G}]}} = \frac{\sum_{i=0}^{n-1} B(T-t_i)}{\sqrt{\widetilde{var}[\sum_{i=0}^{n-1} B(T-t_i)]}}$$
where
$$\widetilde{var}[\sum_{i=0}^{n-1} B(T-t_i)] = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} min(T-t_i, T-t_j)$$

ρ_{T-t_i} has the following form:

$$\rho_{T-t_i} = \frac{\sum_{j=0}^{n-1} min(T-t_i, T-t_j)}{\sqrt{\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} min(T-t_i, T-t_j)}} \tag{27}$$

This conditional variable is used because geometric average (2) $\mathcal{G} = \sqrt[n]{S(T-t_0)S(T-t_2)...S(T-t_{n-1})}$ is less or equal than arithmetic average:

$$\frac{1}{k}\sum_{i=1}^{k} S_{t_i} \geq \sqrt[k]{S_{t_1}S_{t_2}...S_{t_k}}$$

The formulae for lower bound (22),(20) can be computed for all ρ_{T-t_i}, but conditioning variable Λ should be normally distributed. Moreover, the quality of the lower bound depends on the choice of Λ.

3.3 Improved comonotonic upper bound

Following the paper VANMAELE et al. (2006), in this section the conditioning normal random variable Λ was used in order to derive the upper bound for Asian option price (21). An improved comonotonic upper bound for Asian option price is derived using the inequality:

$$AC(n,K,T) = \frac{e^{-rT}}{n}\widetilde{\mathbb{E}}\left[((\mathcal{S}-nK)_+)\right] \leq \frac{e^{-rT}}{n}\widetilde{\mathbb{E}}\left[(\mathcal{S}^u-nK)_+\right] \tag{28}$$

where $\mathcal{S}^u$ was defined in (19) as $\mathcal{S}^u = \sum_{i=0}^{n-1} F^{-1}_{X_i|\Lambda}(U)$, where U stands for (0,1)-uniform random variable independent of Λ. The analytical formula for upper bound was derived in the paper VANMAELE et al. (2006):

$$\begin{aligned}
\text{ICUB}\Lambda &= \frac{e^{-rT}}{n}\widetilde{\mathbb{E}}\left[(\mathcal{S}^u-nK)_+\right] \\
&= \frac{e^{-rT}}{n}\sum_{i=0}^{n-1} S(0)e^{-r(T-t_i)}e^{-\frac{\sigma^2(t_i)}{2}\rho^2_{T-t_i}(T-t_i)} \\
&\times \int_0^1 e^{\rho_{T-t_i}\sigma(t_i)\sqrt{T-t_i}\Phi^{-1}(v)}\Phi\left(\sqrt{1-\rho^2_{T-t_i}}\sigma(t_i)\sqrt{T-t_i} - \Phi^{-1}\left(F_{\mathcal{S}^u|V=v}(nK)\right)\right)dv \\
&- e^{-rT}K(1-F_{\mathcal{S}^u}(nK))
\end{aligned} \tag{29}$$

where

$$V = \Phi\left(\frac{\Lambda - \mathbb{E}[\Lambda]}{\sigma_\Lambda}\right)$$

is a uniform (0,1) random variable, $\rho_{T-t_i} = corr(\sigma(t_i)B(T-t_i), \Lambda)$, and

$$F_{\mathcal{S}^u}(nK) = \int_0^1 F_{\mathcal{S}^u|V=\upsilon}(nK)d\upsilon$$

where $F_{\mathcal{S}^u|V=\upsilon}(nK)$ is a solution to the following equation:

$$\begin{aligned} &S(0)\sum_{i=0}^{n-1}\exp[(r-\frac{\sigma^2(t_i)}{2})(T-t_i)] \\ &\times \exp\left[\rho_{T-t_i}\sigma(t_i)\sqrt{T-t_i}\Phi^{-1}(\upsilon) + \sqrt{1-\rho^2_{T-t_i}}\sigma(t_i)\sqrt{T-t_i}\Phi^{-1}(F_{\mathcal{S}^l}(nK))\right] = nK \end{aligned} \tag{30}$$

VANMAELE et al. (2006) found that the conditioning variable (25) $\Lambda = \sum_{k=1}^{T} W_k$ leads to the sharpest ICUBΛ.

3.4 Bounds based on the Rogers and Shi approach

In the paper ROGERS & SHI (1995), the authors used the following general inequality for any random variables Y and Z:

$$0 \leq \mathbb{E}[\mathbb{E}[Y_+|Z] - \mathbb{E}[Y|Z]_+] \leq \frac{1}{2}\mathbb{E}\left[\sqrt{var(Y|Z)}\right] \tag{31}$$

VANMAELE et al. (2006) propose to apply (31) to the case when Y is $[\sum_{i=0}^{n-1} S(T-t_i) - nK]$ and Z is conditioning variable Λ. The following error bound for differences between option price and lower bound was derived:

$$0 \leq \widetilde{\mathbb{E}}[\widetilde{\mathbb{E}}[(\mathcal{S}-nK)_+|\Lambda] - \widetilde{\mathbb{E}}(\mathcal{S}^l - nK)_+] \leq \frac{1}{2}\mathbb{E}\left[\sqrt{\widetilde{var}(\mathcal{S}|\Lambda)}\right] \tag{32}$$

Form the equation (32), the following upper bound for Asian option price was derived in the paper VANMAELE et al. (2006):

$$\text{UB}\Lambda = \frac{e^{-rT}}{n}\left[\widetilde{\mathbb{E}}\left[(\mathcal{S}^l - nK)_+\right] + \epsilon\right] \tag{33}$$

where the error bound ϵ is:

$$\begin{aligned}\epsilon &= \frac{1}{2}\mathbb{E}\left[\sqrt{\widetilde{var}(\mathcal{S}|\Lambda)}\right]\\ &= \frac{1}{2}\int_0^1 \Bigg[\sum_{j=0}^{n-1}\sum_{i=0}^{n-1}\alpha_i\alpha_j e^{r_{ij}\sigma(t_i)\sigma_{ij}\Phi^{-1}(\upsilon)+\frac{1}{2}(1-r_{ij}^2)\sigma(t_i)\sigma(t_j)\sigma_{ij}^2}\\ &- \left(\sum_{i=0}^{n-1} S(0)e^{(r-\frac{1}{2}\sigma^2(t_i)\rho_{T-t_i}^2)(T-t_i)+\rho_{T-t_i}\sigma(t_i)\sqrt{T-t_i}\Phi^{-1}(\upsilon)}\right)^2\Bigg]^{\frac{1}{2}} d\upsilon\end{aligned} \tag{34}$$

where

$$\alpha_i\alpha_j = S(0)^2 exp\big[(r-\frac{\sigma^2(t_i)}{2})(T-t_i)+(r-\frac{\sigma^2(t_j)}{2})(T-t_j)\big]$$

$$\sigma_{ij} = \sqrt{(T-t_i)+(T-t_i)+2min(T-t_i,T-t_j)}$$

$$r_{ij} = \frac{\sqrt{T-t_i}}{\sigma_{ij}}\rho_{T-t_i} + \frac{\sqrt{T-t_j}}{\sigma_{ij}}\rho_{T-t_j}$$

As it can be seen from (34), the error bound does not depend on K. It is possible to make the upper bound sharper by introducing the constant d_Λ such that $\Lambda \geq d_\Lambda$. Therefore, it implies that $\mathcal{S} \geq nK$. If it is possible to find constant d_Λ, the following relation that allows strengthening the upped bound holds:

$$\text{on the set } \{\Lambda \geq d_\Lambda\}: \ \widetilde{\mathbb{E}}[(\mathcal{S}-nK)_+|\Lambda] = \widetilde{\mathbb{E}}[(\mathcal{S}-nK)|\Lambda] = (\mathcal{S}^l - nK)_+$$

NIELSEN & SANDMANN (2003) derived the upped bounds for Λ given by (27), whereas in VANMAELE et al. (2006) the result was extended for any normally distributed conditioning random variable Λ.

Suppose there exists a d_Λ in $\mathbb{R}$ such that $\Lambda \geq d_\Lambda$. It implies that $\mathcal{S} \geq nK$. Then the upped bound to the Asian option price (21) is the following:

$$\text{UB}\Lambda_d = \text{LB}\Lambda + \frac{e^{-rT}}{n}\epsilon(d_\Lambda) \tag{35}$$

where $\epsilon(d_\Lambda)$ is given by:

$$\begin{aligned}\epsilon(d_\Lambda) &= \frac{S(0)}{2}\{\Phi(d_\Lambda^*)\}^{\frac{1}{2}} \times \Bigg[\sum_{j=0}^{n-1}\sum_{i=0}^{n-1} e^{r(2T-t_i-t_j)+\sigma(t_i)\rho_{T-t_i}\sigma(t_j)\rho_{T-t_j}\sqrt{T-t_i}\sqrt{T-t_j}}\\ &\times \Phi\left(d_\Lambda^* - (\sigma(t_i)\rho_{T-t_i}\sqrt{T-t_i}+\sigma(t_j)\rho_{T-t_j}\sqrt{T-t_j})\right)\\ &\times \left(e^{\sigma(t_i)\sigma(t_j)(min(T-t_i,T-t_j)-\rho_{T-t_i}\rho_{T-t_j}\sqrt{T-t_i}\sqrt{T-t_i})}-1\right)\Bigg]^{\frac{1}{2}}\end{aligned} \tag{36}$$

where $d_\Lambda^* = \frac{d_\Lambda - \widetilde{\mathbb{E}}[\Lambda]}{\sigma_\Lambda}$,$\Phi(\cdot)$ is standard normal cdf and $\rho_{T-t_i} = corr(\sigma(t_i)B(T-t_i), \Lambda) \geq 0$. The error bound (36) holds for conditioning random variable Λ that satisfies the following assumption:

Assumption 1

- *The conditioning variable Λ is normally distributed such that $(\sigma(t_i)B(T-t_i), \Lambda)$ are bivariate normally distributed for all i.*
- *There exists an integration bound d_Λ such that $\Lambda \geq d_\Lambda$ and it implies $\mathcal{S} \geq nK$.*

In the case of Λ given by (27), NIELSEN & SANDMANN (2003) derived the following d_Λ^*:

$$d_{GA} = \frac{n \ln \frac{K}{S(0)} - \sum_{i=0}^{n-1}(r - \frac{\sigma^2(t_i)}{2})(T - t_i)}{\sigma(T) \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} min(T - t_i, T - t_j)} \tag{37}$$

In the case of Λ given by (26), VANMAELE et al. (2006) derived the following d_Λ^*:

$$d_{FA} = \frac{nK - \sum_{i=0}^{n-1} S(0)e^{(r - \frac{\sigma^2(t_i)}{2})(T - t_i)}}{S(0)\sigma(T)} \tag{38}$$

For both cases, d_{GA}^* and d_{FA}^* depend on strike price K and this helps to strengthen the error bound compared to (34).

3.5 Partially exact/comonotonic upper bound

In this section, we will describe the improvement that was made in the paper VANMAELE et al. (2006) based on the results of CURRAN (1994). This method is based on a combination of two techniques. The first one is deriving an improved comonotonic upper bound by conditioning on a normally distributed random variable Λ. The second is an idea of calculation of the bound that consists of an exact part and an approximation part. The exact part is the option price and the approximation is an improved comonotonic upper bound. Under the assumption 1, the partially exact/comonotonic upper bound to the option price (21) is given by:

$$
\begin{aligned}
\text{PECUB}\Lambda &= \frac{S(0)}{n}\sum_{i=0}^{n-1} e^{-ri}\Phi\left(\rho_{T-t_i}\sigma(t_i)\sqrt{T-t_i}-d_\Lambda^*\right) - e^{-rT}K\Phi(-d_\Lambda^*) \\
&+ \frac{S(0)}{n}\sum_{i=0}^{n-1} e^{-ri}e^{-\frac{-\sigma^2(t_i)}{2}\rho_{T-t_i}(T-t_i)} \\
&\times \int_0^{\Phi(d_\Lambda^*)} e^{\rho_{T-t_i}\sigma(t_i)\sqrt{T-t_i}\Phi^{-1}(\upsilon)}\Phi\left(\sqrt{1-\rho_{T-t_i}^2}\sigma(t_i)\sqrt{T-t_i}-\Phi^{-1}(F_{\mathcal{S}^u|V=\upsilon}(nK))\right)d\upsilon \\
&- e^{-rT}K\left(\Phi(d_\Lambda^*) - \int_0^{\Phi(d_\Lambda^*)} F_{\mathcal{S}^u|V=\upsilon}(nK))d\upsilon\right)
\end{aligned}
\tag{39}
$$

where $d_\Lambda^* = \frac{d_\Lambda - \widetilde{\mathbb{E}}[\Lambda]}{\sigma_\Lambda}$ and $F_{\mathcal{S}^u|V=\upsilon}$ is a solution to equation (30) and $\rho_{T-t_i} = corr(\sigma(t_i)B(T-t_i),\Lambda)$.

4 Stochastic volatility

In the section 2, we assumed constant volatility of the stock price. As it was shown in HULL & WHITE (1987) constant volatility approach frequently overprices options particularly when pricing is performed in Black-Scholes settings. Moreover, after 1987 geometric Brownian motion model and the Black-Scholes formula were unable to give the option price that reproduces the real markets. One way to overcome the problem related to constant volatility was through stochastic volatility models. There are two diffusion processes when the pricing of options is performed with the stochastic volatility model: one describes the asset price dynamics and the second one governs the volatility evolution [HULL & WHITE (1987),HESTON (1993)].

In this work we will model volatility as an exponential Ornstein–Uhlenbeck (expOU) process [PERELLÓ et al. (2008),KUCHUK-IATSENKO & MISHURA (2015)] as it is one of the simplest and classic stochastic volatility models. Despite its simplicity expOU process is widely used and adequately describes the real volatility in markets FOUQUE et al. (2000).

4.1 Ornstein–Uhlenbeck process

As it was mentioned in the previous sections, we will use a modified Ornstein–Uhlenbeck (OU) process to model the stochastic volatility. The OU process has the following form:

$$d\upsilon_t = \theta(\mu - \upsilon_t)dt + \eta dW_t \tag{40}$$

Now we will describe the roles of parameters of the OU process. The process describes υ_t as an instantaneous variance rate at time t. The first parameter is μ which is the variance υ_t in a long run. It can be seen as an expectation of υ_t when t tends to infinity. The parameter θ represents the rate at which υ_t reverts to its long-run value μ, which is why the OU process is one of the mean-reverting stochastic volatility models. The parameter η represents the volatility of the volatility and determines the variance of υ_t. W stands for a standard one-dimensional Brownian motion.

It can be shown that $\upsilon_t|\upsilon_s$ for $t \geq s$ is a normally distributed random variable with time dependent mean $a(t,s)$ and variance $b^2(t,s)$, $\upsilon_t|\upsilon_s \sim \mathcal{N}(a(t,s),\, b^2(t,s))$. Where $a(t,s)$ and $b^2(t,s)$ are given by:

$$\begin{aligned} a(t,s) &\coloneqq \upsilon_s e^{-\theta(t-s)} + \mu(1 - e^{-\theta(t-s)}) \\ b^2(t,s) &\coloneqq \frac{\eta^2}{2\theta}(1 - e^{-2\theta(t-s)}) \end{aligned} \tag{41}$$

The OU process has one significant drawback: the variance rate could be negative. As the volatility of the underlying asset cannot be negative, we will use the exponential Ornstein–Uhlenbeck (expOU) process to model the volatility:

$$\sigma_t = m e^{\upsilon_t} \tag{42}$$

where υ_t, $t \geq 0$ is the OU process (40) and the parameter m is a scaling parameter that is $m = \frac{\mu}{e^{\mu}}$ PERELLÓ et al. (2008).

All bounds that were presented in section 3 were extended to the case of deterministic volatility function $\sigma = \sigma(t)$. In this thesis we will use the time-average of the expected value of the stochastic volatility (42) as deterministic time-varying volatility:

$$\sigma(t) = \int_0^t \mathbb{E}[\sigma_s]ds = \int_0^t \mathbb{E}[me^{\upsilon_s}]ds \tag{43}$$

The OU process is normally distributed with known mean and variance (41). The moment generating function of normally distributed variable X with mean μ and variance σ^2 has the following form:

$$M_{X(u)} = \mathbb{E}[e^{uX}] = e^{\mu u + \frac{\sigma^2}{2}u}$$

Therefore, the expectation $\mathbb{E}[\sigma_t]$ is a moment generating function of normally distributed variable at point $u = 1$:

$$\mathbb{E}[\sigma_t] = me^{a(t,0)+\frac{b^2(t,0)}{2}} \tag{44}$$

where $a(t, 0)$ and $b^2(t, 0)$ are given by (41).

4.2 Monte Carlo simulations

In order to compute the price of the Asian option, we should perform a discretization of the price and volatility processes. We will use the following form for the volatility process based on the OU process (40):

$$\upsilon_{t+\delta} = a(t, t+\delta) + b(t, t+\delta)Z \tag{45}$$

For the price process:

$$S_{t+\delta} = S_t e^{\left(r*\delta - \frac{\hat{\sigma}(t)}{2}\right) + \sqrt{\hat{\sigma}(t)}Z^{\perp}} \tag{46}$$

where

$$\hat{\sigma}(t) = \int_t^{t+\delta} \sigma_s^2 ds$$

$Z, Z^{\perp} \backsim \mathcal{N}(0, 1), \delta = t_{i+1} - t_i$ and σ_s is given in (42) and υ_s in given in (45). We will use price discretization (46) in order to perform the Monte Carlo simulations with control variate (16).

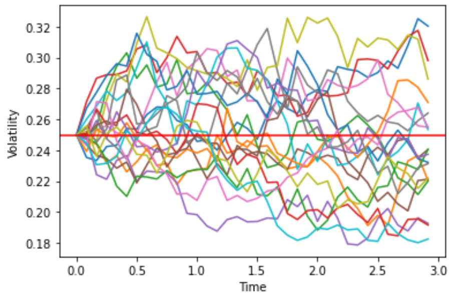
5 Results

5.1 Simulations of volatility

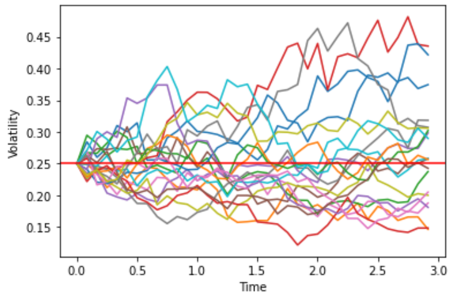
In order to simulate exponential Ornstein–Uhlenbeck process, the following parameters were chosen: $\theta = 0.3, \mu = 0.25, \eta = \{0.15, 0.25\}, \upsilon_0 = \mu = 0.25, T = t_n = 3$ years, $\delta = t_{i+1} - t_i = \frac{1}{12}$. As it can be seen in Figure 2[a,b], indeed the volatility does not become negative because it was modeled by the exponential Ornstein–Uhlenbeck process. Also, increasing the parameter η leads to an increase in amplitude in both stochastic and deterministic volatilities.

5.2 Computation of bounds and price of an Asian option

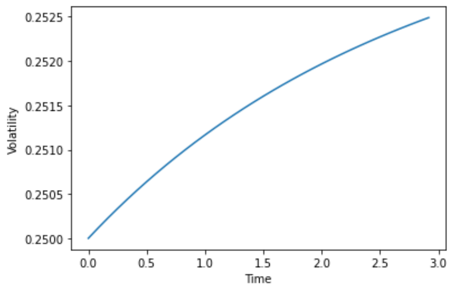
In this section, we will discuss the results of computations. In the supplementary tables [3, 4, 5] in Appendix B we tabulated the results of Monte Carlo simulations with control variate (MC) and bounds computed using the above given formulae.



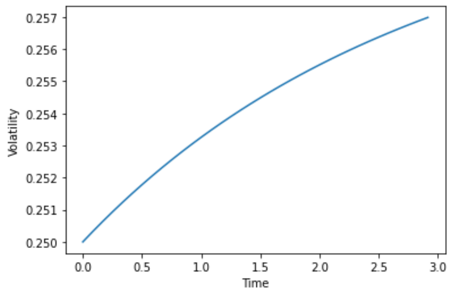
(a) Exponential Ornstein–Uhlenbeck process, $\eta = 0.15$



(b) Exponential Ornstein–Uhlenbeck process, $\eta = 0.25$



(c) Deterministic time-varying volatility, $\eta = 0.15$



(d) Deterministic time-varying volatility, $\eta = 0.25$

Figure 2: Simulations of exponential Ornstein–Uhlenbeck process and deterministic time-varying volatility

In table 3 we present the results of the exponential OU process for $\eta = 0$ when it becomes deterministic. Even though the volatility is deterministic, the results of Monte Carlo simulations in table 3 are different from the results of Monte Carlo simulations presented in the paper VANMAELE et al. (2006) because of the volatility averaging that we introduce in this thesis. In tables [3, 4, 5] we used the notations where Λ can be B_T, FA, GA which are given in [25, 26, 27] respectively. For bounds: LBΛ are lower bounds given in 22, UBΛ are upper bounds given in (35) and (33), PECUBAΛ are partially exact/comonotonic upper bounds given in (39), ICUBΛ are improved comonotonic upper bounds given in (29).

In the supplementary figures [6,7,8] in Appendix C we plotted: the differences between lower bounds and Monte Carlo prices, the differences between upper bounds and Monte Carlo prices to visualize the dynamics with respect to strike price and parameter η of OU process (40). Generally, it is clear that the quality of the bounds (inversely proportional to the difference between the bound and Monte Carlo price) is decreasing with η, but there are some exceptions that should be emphasized. In order to test the bound, we plotted each bound for different η on one figure.

5.3 Comparison

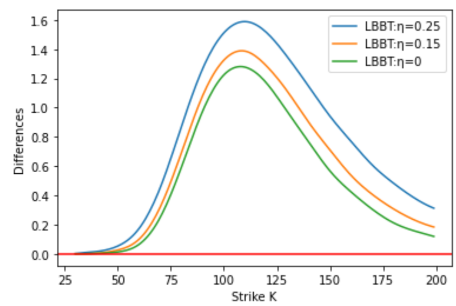
In this section, we will compare the bounds for the different parameters of the OU process $\eta = \{0, 0.15, 0.25\}$. As we can see from the figures [3a,3b,3c,3d,3e], the difference between the bound and the MC price strictly increases for the following bounds: LBB_T, LBFA, LBGA, UBFA, UBGA. Therefore, the quality of the above-mentioned bounds is decreasing with η. Especially the decrease in quality can be seen for bounds LBFA and LBGA. For constant volatility, the difference is the smallest for strikes from 30 to 200, whereas the difference increases more than 5 times when $\eta = 0.25$ for K=170. The potential reason is that the formulae for bounds LBFA and LBGA do not depend on strike K, hence, the quality of the bounds strictly decreases.

The differences for the rest of the bounds are not as significant as for LBFA and LBGA. The differences for bounds UBGA_d and UBFA_d decrease, but reach the negative values for certain strike prices. Therefore, the MC price exceeds the upper bound. However, the bounds UBGA_d and UBFA_d show better quality when increasing the parameter η for strikes K that are larger than 150 when $\eta = 0.25$, and larger than 90 when $\eta = 0.15$. The differences for bounds PECUBFA and PECUBGA decrease with η, however, they are reaching negative values at the lowest and the highest strike prices. Therefore, the operational range of bounds PECUBFA and PECUBGA shrinks with the increase of parameter η. For the strike price 170, the quality of bound PECUBGA increases by more than 34 times when $\eta = 0.25$ compared to $\eta = 0$. The quality of the bounds ICUBΛ, ICUBFA, and ICUBGA changes approximately the same as for PECUBFA and PECUBGA.

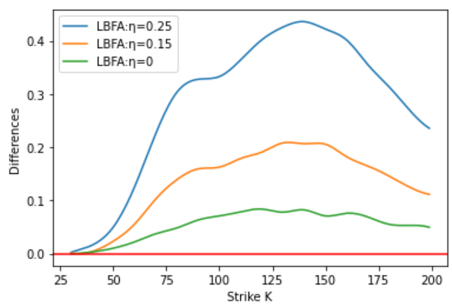
As it was shown in VANMAELE et al. (2006), the best quality for bound (29) is reached for conditioning variable (25). This result holds also for stochastic volatility as we show in this thesis. Moreover, the quality of the bound ICUBΛ increases with η. However, the differences reach negative values when parameter η increases. Therefore, the dynamics of quality of the bounds ICUBΛ, ICUBFA, and ICUBGA replicates the dynamics of bounds PECUBFA and PECUBGA. For the strike price 150, the difference for ICUBΛ is around 6 times lower when $\eta = 0.25$ compared to the case when $\eta = 0$. The case of $\eta = 0.15$ gives the result which is an intermediate stage between $\eta = 0$ and $\eta = 0.25$, so the quality of the bounds decreases with the increase of the parameter η.

Generally, the dynamics of the quality of the bounds after the introduction of the stochastic volatility differs. For the first set of bounds, which consists of LBB_T, LBFA, LBGA, UBFA, UBGA, the introduction of the stochastic volatility worsens the quality of the bounds. For the subset of the first set: LBFA and LBGA, the decrease in quality is drastic. For the second set of bounds, which consists of UBGA_d, UBFA_d, PECUBFA, PECUBGA, ICUBA, ICUBFA, and ICUBGA, the introduction of the stochastic volatility affected the quality in two ways. Firstly, the differences decrease with the parameter η for all strike prices. Secondly, the differences reach negative values for low and high strike prices. Therefore, the bounds are applicable for a smaller set of strike prices.

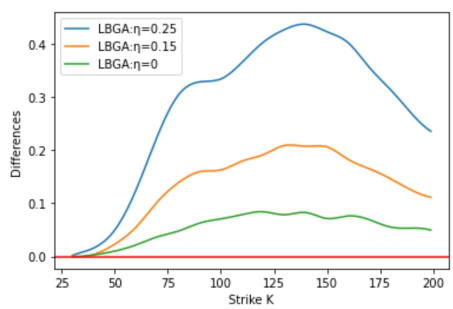
Finally, in order to compute the lower and upper bounds, the numerical methods in Python were used. The results that we obtained could be improved with better discretization which requires more time and computational power. Due to lack thereof, we were unable to reach better accuracy of the Monte Carlo prices and the bounds. Therefore, with better discretization one can compute Monte Carlo prices and bounds that differ slightly from our results. Also, the absolute difference between the Monte Carlo price and the bounds could be different. However, the general dynamics of quality will be the same. In future research, we will increase accuracy and implement the bounds with floating strikes.



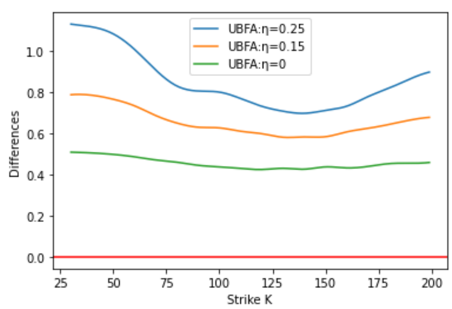
(a) LBB_T



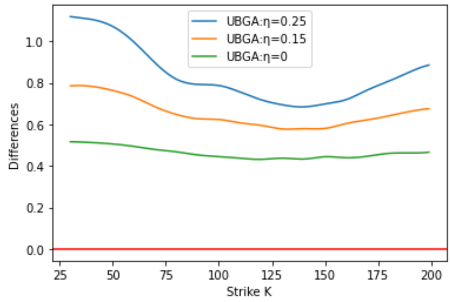
(b) LBFA



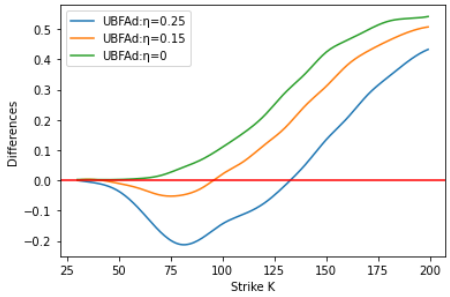
(c) LBGA



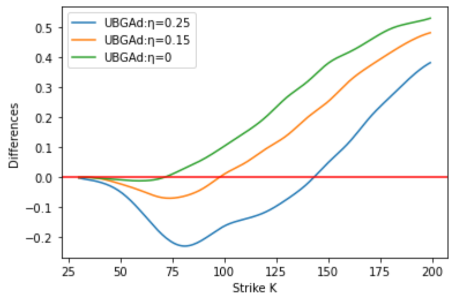
(d) UBFA



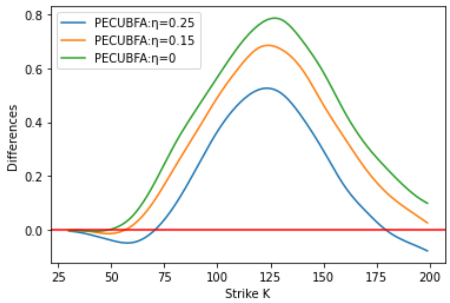
(e) UBGA



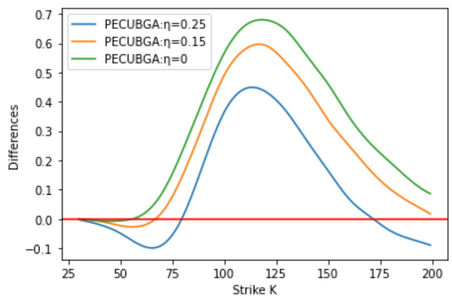
(f) UBFA_d



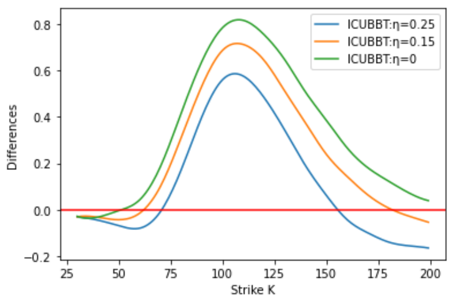
(g) UBGA_d



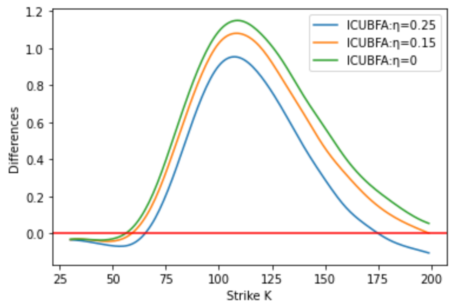
(h) PECUBFA



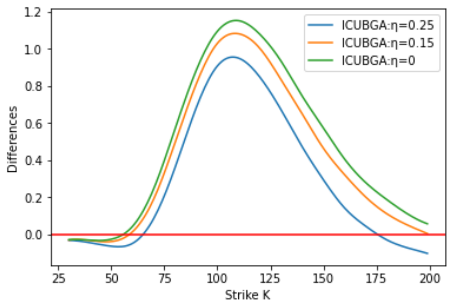
(i) PECUBGA



(j) ICUBB_T



(k) ICUBFA



(l) ICUBGA

Figure 3: Comparison of bounds for different η

6 Conclusions

Overall, the results of our work consist of two parts: Monte Carlo simulations with constant and stochastic volatility; and presentation of the bounds derived in VANMAELE et al. (2006) and their computation in the case of constant ($\eta = 0$) as well as stochastic volatility ($\eta = 0.15$, $\eta = 0.25$). In the first part, we showed that the Monte Carlo simulations with control variate (MC CV) give better results in terms of standard error. Moreover, the price computed using MC CV converges faster to its true price which saves time and computational power. In order to introduce the stochastic volatility, we used the exponential Ornstein–Uhlenbeck process because the unmodified OU process can reach negative values and volatility can be negative which is restricted by the exponent function. We computed the MC price in order to have the benchmark for boundaries.

In the second part, we computed the boundaries for different parameters of the OU process η. First of all, the MC prices with their confidence intervals were found to be between lower and upper bounds for all strike prices K and parameters η. Generally, the quality of bounds (in terms of the difference between the MC price and a certain bound) decreases with the parameter η. The following bounds show strictly worse quality with increasing parameter η: LBB_T, LBFA, LBGA, UBFA, UBGA. For some bounds, the difference increases more than 36 times with parameter η. The dynamics of quality for other bounds is mixed. The differences decrease, but the operational range of strike prices shrinks: in the case of ICUBB_T, the operational range shrinks. It was shown in numerous papers that the assumption of constant volatility does not describe the situation on the market sufficiently. Therefore, despite the decreasing quality of bounds in the case of stochastic volatility, the bounds which are studied in this thesis give a more realistic and widely applicable result.

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Appendix A

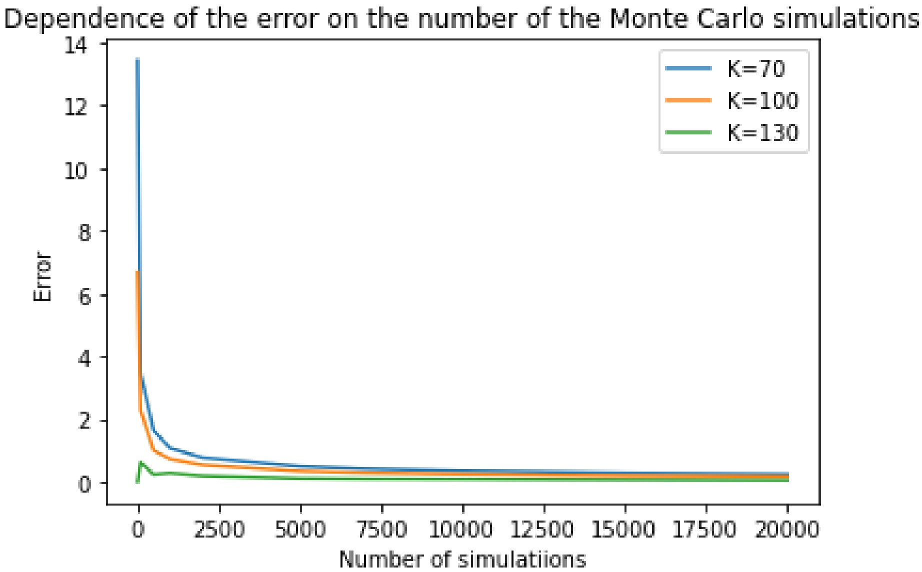

Figure 4: Dependence of the error on the number of the Monte Carlo simulations and Strike prices K=70,100,130.

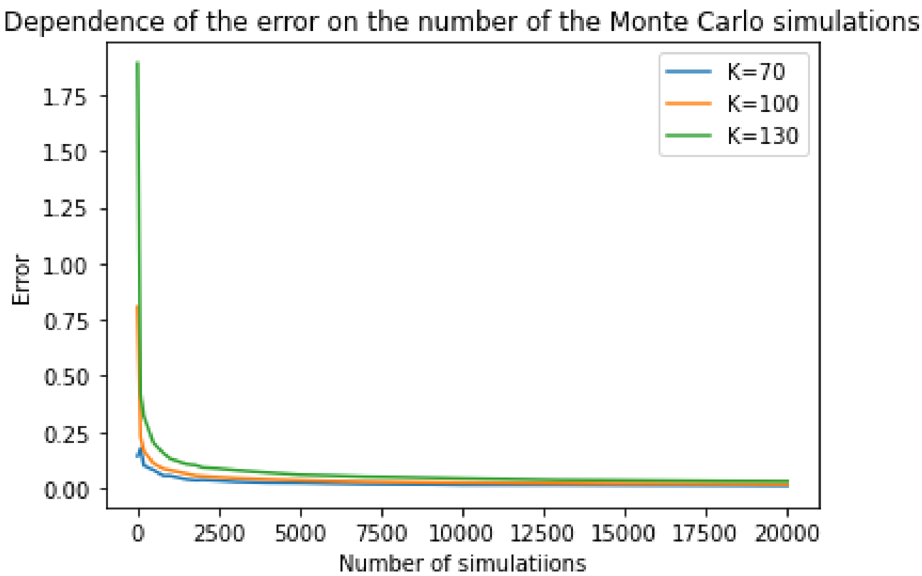

Figure 5: Dependence of the error on the number of the Monte Carlo simulations and Strike prices K=70,100,130.

Appendix B

K	MC(CI *10^3)	LBB_T	LBFA	LBGA	UBG_d	$UBFA_d$	UBFA	UBGA	PECUBGA	PECUBFA	$ICUBB_T$	ICUBFA	ICUBGA
30	67.7818 (2.5)	67.7825	67.7825	67.7825	67.7825	67.7842	68.2909	68.2978	67.7825	67.7789	67.7533	67.7516	67.7516
40	58.9164 (2.2)	58.9133	58.9133	58.9133	58.9134	58.9183	59.4218	59.4287	58.9097	58.9101	58.8844	58.8828	58.8828
50	50.0574 (1.9)	50.0443	50.0475	50.0473	50.0489	50.0599	50.5557	50.5626	50.0512	50.0590	50.0532	50.0295	50.0295
60	41.2507 (2.0)	41.1858	41.2312	41.2289	41.2392	41.2566	41.7372	41.7442	41.2629	41.3015	41.2925	41.2922	41.2925
70	32.6965 (2.8)	32.4464	32.6691	32.6596	32.6942	32.7124	33.1677	33.1749	32.7843	32.8649	32.8809	32.9408	32.9414
80	24.7939 (3.0)	24.2002	24.7683	24.7461	24.8211	24.8345	25.2541	25.2614	25.0292	25.1128	25.2089	25.3591	25.3602
90	17.9942 (2.7)	17.0172	17.9675	17.9314	18.0552	18.0634	18.4395	18.4467	18.4039	18.4412	18.6330	18.8751	18.8766
100	12.5467 (3.2)	11.3225	12.5218	12.4759	12.6507	12.6565	12.9843	12.9912	13.1140	13.1138	13.3313	13.6384	13.6402
110	8.4645 (4.0)	7.1878	8.4349	8.3857	8.6138	8.6203	8.8944	8.9010	9.1249	9.1461	9.2804	9.6131	9.6150
120	5.5654 (4.9)	4.3960	5.5279	5.4813	5.7667	5.7781	5.9902	5.9966	6.2444	6.3299	6.3193	6.6411	6.6429
130	3.5841 (5.0)	2.6140	3.5461	3.5056	3.8504	3.8721	4.0147	4.0209	4.2266	4.3659	4.2293	4.5152	4.5167
140	2.2883 (6.3)	1.5232	2.2384	2.2056	2.6071	2.6413	2.7146	2.7209	2.8411	2.9915	2.7937	3.0308	3.0321
150	1.4425 (6.7)	0.8754	1.3964	1.3711	1.8228	1.8660	1.8801	1.8864	1.9018	2.0277	1.8275	2.0132	2.0143
160	0.9216 (7.3)	0.4987	0.8641	0.8453	1.3384	1.3837	1.3542	1.3606	1.2697	1.3589	1.1872	1.3249	1.3257
170	0.5875 (5.8)	0.2827	0.5318	0.5183	1.0438	1.0842	1.0271	1.0336	0.8464	0.9034	0.7674	0.8636	0.8642
180	0.3721 (6.6)	0.1600	0.3264	0.3168	0.8667	0.8978	0.8255	0.8321	0.5637	0.5979	0.4943	0.5564	0.5569
190	0.2467 (6.1)	0.0906	0.2001	0.1934	0.7611	0.7815	0.7021	0.7087	0.3752	0.3951	0.3175	0.3527	0.3530
200	0.1664 (5.0)	0.0514	0.1227	0.1181	0.6984	0.7091	0.6267	0.6334	0.2497	0.2610	0.2034	0.2177	0.2180

K: strike price	UBFA: upper bound equal to lower bound LBFA plus constant $\epsilon\frac{e^{-rT}}{n}$
MC: Monte Carlo with control variate price with its 95% confidence interval (CI) based on 10000 pathes	UBGA: upper bound equal to lower bound LBGA plus constant $\epsilon\frac{e^{-rT}}{n}$
LBB_T:lower bound with $\Lambda = \sum_{k=1}^{T} W_k = B(T)$	PECUBGA: partially exact/comonotonic upper bound with $\Lambda = (\ln \mathcal{G} - \widetilde{\mathbb{E}}[\ln \mathcal{G}])/\sqrt{\widetilde{var}[\ln \mathcal{G}]}$
LBFA: lower bound with $\Lambda = \sum_{i=0}^{n-1} e^{(r-\frac{\sigma^2}{2})(T-t_i)} B(T-t_i)$	PECUBFA: partially exact/comonotonic upper bound with $\Lambda = \sum_{i=0}^{n-1} e^{(r-\frac{\sigma^2}{2})(T-t_i)} B(T-t_i)$
LBGA: lower bound with $\Lambda = (\ln \mathcal{G} - \widetilde{\mathbb{E}}[\ln \mathcal{G}])/\sqrt{\widetilde{var}[\ln \mathcal{G}]}$	ICUBBT: improved comonotonic upper bound with $\Lambda = \sum_{k=1}^{T} W_k = B(T)$
UBGAd: upper bound equal to lower bound LBGA plus $\frac{e^{-rT}}{n}\epsilon(d_{GA})$	ICUBFA: improved comonotonic upper bound with $\Lambda = \sum_{i=0}^{n-1} e^{(r-\frac{\sigma^2}{2})(T-t_i)} B(T-t_i)$
UBFAd: upper bound equal to lower bound LBGA plus $\frac{e^{-rT}}{n}\epsilon(d_{FA})$	ICUBGA: improved comonotonic upper bound with $\Lambda = (\ln \mathcal{G} - \widetilde{\mathbb{E}}[\ln \mathcal{G}])/\sqrt{\widetilde{var}[\ln \mathcal{G}]}$

Table 3: The result of Monte Carlo simulations and different bounds with the following parameters: $S(0) = 100, r = 0.04, T = 3$ years, $\delta = \frac{1}{12}, n = 36, \theta = 0.3, \mu = 0.25, v_0 = 0.25, \eta = 0$

K	MC(CI $*10^3$)	LBB_T	LBFA	LBGA	$UBGA_d$	$UBFA_d$	UBFA	UBGA	PECUBGA	PECUBFA	$ICUBB_T$	ICUBFA	ICUBGA
30	67.7837 (2.2)	67.7825	67.7825	67.7825	67.7825	67.7842	68.5722	68.5686	67.7825	67.7789	67.7531	67.7513	67.7514
40	58.9171 (2.4)	58.9133	58.9133	58.9133	58.9133	58.9183	59.7030	59.6995	58.9097	58.9102	58.8843	58.8827	58.8827
50	50.0708 (2.3)	50.0443	50.0475	50.0475	50.0492	50.0599	50.8372	50.8336	50.0478	50.0560	50.0276	50.0304	50.0304
60	41.2865 (2.5)	41.1864	41.2312	41.2312	41.2419	41.2566	42.0208	42.0174	41.2625	41.3023	41.2710	41.2987	41.2989
70	32.7717 (3.3)	32.4510	32.6691	32.6692	32.7043	32.7124	33.4588	33.4553	32.7944	32.8763	32.8709	32.9606	32.9610
80	24.9074 (3.4)	24.2147	24.7683	24.7684	24.8436	24.8345	25.5580	25.5545	25.0573	25.1411	25.2169	25.3984	25.3992
90	18.1267 (3.7)	17.0445	17.9675	17.9674	18.0908	18.0634	18.7572	18.7535	18.4517	18.4886	18.6597	18.9340	18.9352
100	12.6843 (4.4)	11.3594	12.5218	12.5214	12.6953	12.6565	13.3115	13.3076	13.1763	13.1759	13.3713	13.7115	13.7129
110	8.6136 (5.4)	7.2273	8.4349	8.4343	8.6619	8.6203	9.2245	9.2204	9.1931	13.1759	9.3255	9.6923	9.6937
120	5.7186 (5.3)	4.4323	5.5279	5.5271	5.8137	5.7781	6.3176	6.3133	6.3103	6.3967	6.3617	6.7187	6.7201
130	3.7541 (6.3)	2.6440	3.5461	3.5453	3.8932	3.8721	4.3357	4.3314	4.2848	4.4252	4.2637	4.5858	4.5870
140	2.4449 (6.3)	1.5460	2.2384	2.2376	2.6446	2.6413	3.0280	3.0237	2.8889	3.0406	2.8173	3.0918	3.0928
150	1.6019 (6.7)	0.8917	1.3964	1.3958	1.8549	1.8660	2.1861	2.1819	1.9383	2.0658	1.8397	2.0635	2.0643
160	1.0461 (6.8)	0.5099	0.8641	0.8635	1.3659	1.3837	1.6537	1.6497	1.2957	1.3864	1.1886	1.3651	1.3657
170	0.6965 (7.1)	0.2902	0.5318	0.5314	1.0677	1.0842	1.3215	1.3176	0.8631	0.9212	0.7594	0.8949	0.8954
180	0.4721 (6.8)	0.1649	0.3264	0.3261	0.8879	0.8978	1.1160	1.1122	0.5725	0.6077	0.4785	0.5803	0.5807
190	0.3243 (6.1)	0.0937	0.2001	0.1998	0.7804	0.7815	0.9897	0.9860	0.3776	0.3981	0.2955	0.3706	0.3709
200	0.2333 (5.8)	0.0534	0.1227	0.1225	0.7164	0.7091	0.9123	0.9086	0.2468	0.2526	0.1764	0.2311	0.2313

K: strike price	UBFA: upper bound equal to lower bound LBFA plus constant $\epsilon\frac{e^{-rT}}{n}$
MC: Monte Carlo with control variate price with its 95% confidence interval (CI) based on 10000 pathes	UBGA: upper bound equal to lower bound LBGA plus constant $\epsilon\frac{e^{-rT}}{n}$
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LBGA: lower bound with $\Lambda = (\ln \mathcal{G} - \widetilde{\mathbb{E}}[\ln \mathcal{G}])/\sqrt{\widetilde{var}[\ln \mathcal{G}]}$	ICUBBT: improved comonotonic upper bound with $\Lambda = \sum_{k=1}^{T} W_k = B(T)$
UBGAd: upper bound equal to lower bound LBGA plus $\frac{e^{-rT}}{n}\epsilon(d_{GA})$	ICUBFA: improved comonotonic upper bound with $\Lambda = \sum_{i=0}^{n-1} e^{(r-\frac{\sigma_i^2}{2})(T-t_i)} B(T-t_i)$
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Table 4: The result of Monte Carlo simulations and different bounds with the following parameters: $S(0) = 100, r = 0.04, T = 3$ years, $\delta = \frac{1}{12}, n = 36, \theta = 0.3, \mu = 0.25, \upsilon_0 = 0.25, \eta = 0.15$

K	MC(CI $*10^3$)	LBB_T	LBFA	LBGA	$UBGA_d$	$UBFA_d$	UBFA	UBGA	PECUBGA	PECUBFA	$ICUBB_T$	ICUBFA	ICUBGA
30	67.7848 (2.1)	67.7825	67.7825	67.7825	67.7825	67.7845	68.9165	68.9030	67.7825	67.7789	67.7527	67.7510	67.7510
40	58.9292 (2.2)	58.9133	58.9133	58.9133	58.9134	58.9189	60.0474	60.0339	58.9098	58.9103	58.8840	58.8824	58.8824
50	50.0980 (2.7)	50.0443	50.0480	50.0479	50.0498	50.0617	51.1820	51.1685	50.0489	50.0579	50.0289	50.0322	50.0322
60	41.3611 (3.0)	41.1876	41.2357	41.2357	41.2470	41.2651	42.3697	42.3562	41.2708	41.3129	41.2811	41.3108	41.3109
70	32.9126 (3.6)	32.4593	32.6870	32.6868	32.7229	32.7420	33.8210	33.8073	32.8233	32.9074	32.9034	32.9969	32.9971
80	25.1127 (4.9)	24.2409	24.8092	24.8088	24.8844	24.9004	25.9433	25.9293	25.1202	25.2043	25.2832	25.4697	25.4701
90	18.3607 (5.5)	17.0938	18.0330	18.0322	18.1546	18.1693	19.1671	19.1527	18.5512	18.5872	18.7609	19.0405	19.0411
100	12.9372 (6.1)	11.4256	12.6044	12.6032	12.7753	12.7936	13.7384	13.7238	13.3029	13.3023	13.4976	13.8432	13.8438
110	8.8869 (7.5)	7.2985	8.5228	8.5216	8.7483	8.7746	9.6569	9.6421	9.3316	9.3542	9.4624	9.8349	9.8355
120	6.0128 (6.5)	4.4979	5.6110	5.6097	5.8981	5.9370	6.7450	6.7302	6.4466	6.5346	6.4955	6.8588	6.8594
130	4.0437 (7.4)	2.6980	3.6179	3.6168	3.9704	4.0264	4.7519	4.7373	4.4090	4.5514	4.3848	4.7138	4.7143
140	2.7325 (9.3)	1.5873	2.2966	2.2956	2.7122	2.7855	3.4306	3.4161	2.9959	3.1502	2.9206	3.2025	3.2029
150	1.8633 (7.3)	0.9215	1.4413	1.4405	1.9130	1.9982	2.5753	2.5611	2.0270	2.1572	1.9241	2.1554	2.1558
160	1.2989 (7.4)	0.5305	0.8974	0.8968	1.4157	1.5042	2.0314	2.0174	1.3670	1.4602	1.2552	1.4389	1.4392
170	0.9111 (7.5)	0.3040	0.5559	0.5555	1.1110	1.1945	1.6900	1.6760	0.9191	0.9793	0.8106	0.9528	0.9530
180	0.6570 (7.8)	0.1740	0.3434	0.3431	0.9264	0.9995	1.4774	1.4636	0.6157	0.6524	0.5171	0.6248	0.6250
190	0.4801 (7.1)	0.0996	0.2119	0.2117	0.8155	0.8765	1.3459	1.3322	0.4105	0.4321	0.3240	0.4043	0.4044
200	0.3639 (7.4)	0.0572	0.1308	0.1306	0.7491	0.7990	1.2648	1.2512	0.2716	0.2806	0.1972	0.2562	0.2563

K: strike price	UBFA: upper bound equal to lower bound LBFA plus constant $\epsilon\frac{e^{-rT}}{n}$
MC: Monte Carlo with control variate price with its 95% confidence interval (CI) based on 10000 pathes	UBGA: upper bound equal to lower bound LBGA plus constant $\epsilon\frac{e^{-rT}}{n}$
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LBFA: lower bound with $\Lambda = \sum_{i=0}^{n-1} e^{(r-\frac{\sigma_i^2}{2})(T-t_i)} B(T-t_i)$	PECUBFA: partially exact/comonotonic upper bound with $\Lambda = \sum_{i=0}^{n-1} e^{(r-\frac{\sigma_i^2}{2})(T-t_i)} B(T-t_i)$
LBGA: lower bound with $\Lambda = (\ln \mathcal{G} - \widetilde{\mathbb{E}}[\ln \mathcal{G}])/\sqrt{\widetilde{var}[\ln \mathcal{G}]}$	ICUBBT: improved comonotonic upper bound with $\Lambda = \sum_{k=1}^{T} W_k = B(T)$
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UBFAd: upper bound equal to lower bound LBGA plus $\frac{e^{-rT}}{n}\epsilon(d_{FA})$	ICUBGA: improved comonotonic upper bound with $\Lambda = (\ln \mathcal{G} - \widetilde{\mathbb{E}}[\ln \mathcal{G}])/\sqrt{\widetilde{var}[\ln \mathcal{G}]}$

Table 5: The result of Monte Carlo simulations and different bounds with the following parameters: $S(0) = 100, r = 0.04, T = 3$ years, $\delta = \frac{1}{12}, n = 36, \theta = 0.3, \mu = 0.25, \upsilon_0 = 0.25, \eta = 0.25$

Appendix C

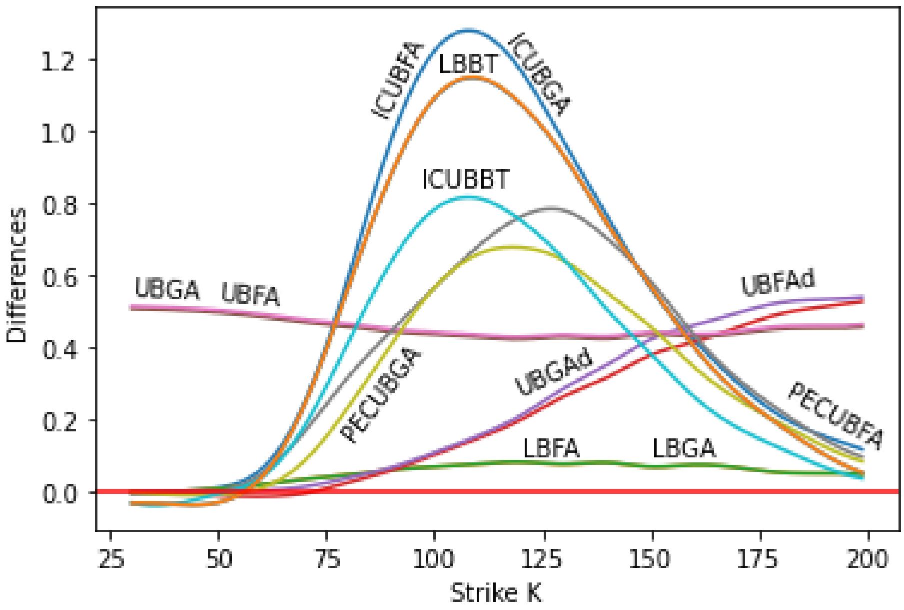

Figure 6: Differences between bounds and Monte Carlo price. Parameter of OU (40) process is $\eta = 0$.

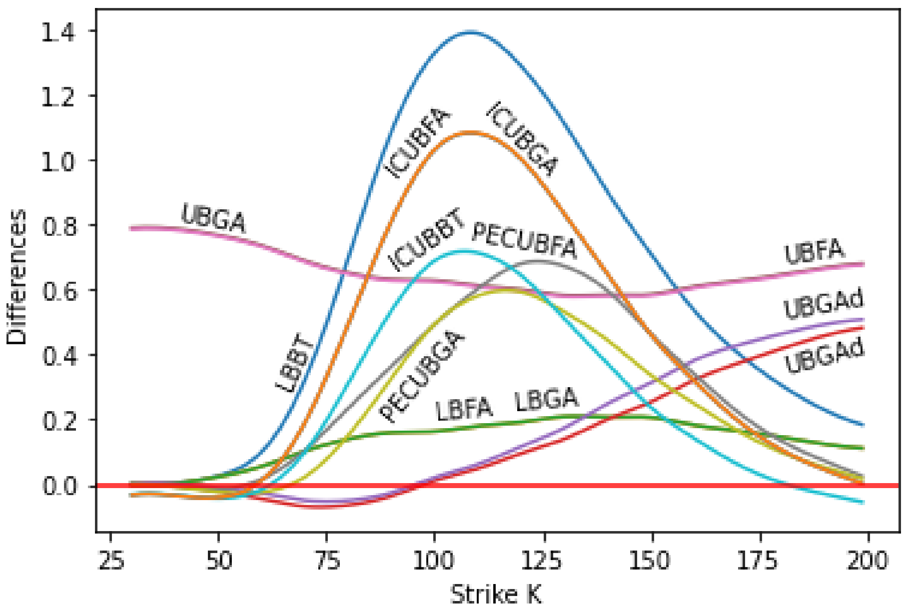

Figure 7: Differences between bounds and Monte Carlo price. Parameter of OU (40) process is $\eta = 0.15$.

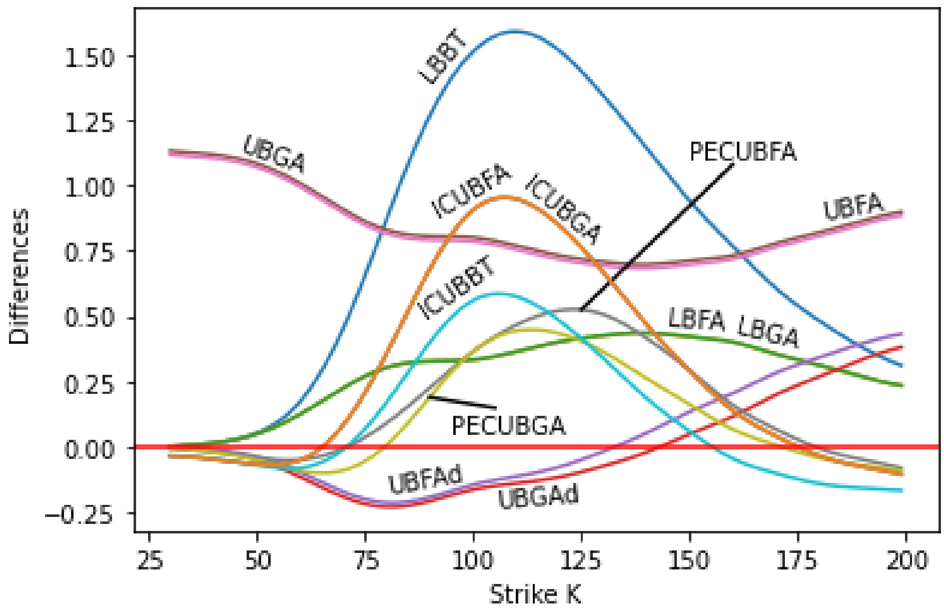

Figure 8: Differences between bounds and Monte Carlo price. Parameter of OU (40) process is $\eta = 0.25$.