Louvain School of Management

Robust portfolio optimization: past performance versus future performance

Do portfolio combinations improve the performance compared to individual portfolio strategies?

Author(s): Jasmine Martin
Supervisor(s): Pr. Nathan Lassance
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Abstract

The selection of optimal portfolio strategies is crucial in active asset management, particularly when balancing the trade-off between risk and return. This thesis investigates whether combining different portfolio optimization strategies can enhance performance. Both the sample covariance matrix and linear shrinkage covariance matrix were employed under a rolling window of 120 months and 180 months to compare various individual portfolio strategies (mean-variance, equally weighted, minimum-variance) and their combinations. This analysis spans an out-of-sample period from January 1970 to December 2023. In theory, portfolio combinations maximizing the expected out-of-sample utility should outperform individual strategies. The findings confirm this theory when applying the Monte Carlo simulations approach. However, when applying the rolling-window method on real-world data, the individual global minimum-variance and equally weighted portfolio strategies outperformed all portfolio combinations when the sample covariance matrix is computed. It is only when a large rolling window and the linear shrinkage covariance matrix are used that portfolio combinations outperform for all levels of risk aversion. This comprehensive evaluation highlights the importance of strategy selection based on data sample length and shrinkage methods, providing valuable insights for constructing diversified and robust portfolios.

Keywords: Portfolio optimization, Portfolio rules, Portfolio combinations, Mean-Variance, Shrinkage, Rolling window, In-sample, Out-of-sample, Optimal Weights, Equally Weighted, Monte Carlo Simulations, Error Maximization, Covariance Matrix, Expected returns, Robustness, Performance, Turnover, Expected Out-of-Sample Utility.

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List of abbreviations

Abbreviation	Full Name	Description
EW	Equally Weighted portfolio	See subpart 1.3, page 15.
EWRF	Combination of the EW portfolio and the risk-free asset.	See subpart 2.2.1, page 17.
GMV	Global Minimum-Variance portfolio	See equation 9, page 7.
GMVRF	Combination of the GMV portfolio and the risk-free asset	See subpart 2.2.2, page 17.
KWZ	Combination of the GMV and the MVc portfolios	See subpart 2.2.4, page 19.
KZ2F	Combination of the MVunc portfolio and the risk-free asset	See subpart 2.2.3, page 18.
KZ3F	Combination of the MVunc and the GMV portfolios with the risk-free asset	See subpart 2.3.1, page 21.
LA	Combination of the GMV and EW portfolios with the risk-free asset	See subpart 2.3.3, page 22.
LMS	Combination of the MVunc, GMV and EW portfolios	See subpart 2.3.4, page 23.
MSR	Maximum Sharpe Ratio portfolio	See equation 5, page 5.
MV	Mean-Variance portfolio	See subpart 1.1, page 4.
MVc	Constrained Mean-Variance portfolio	See equation 7, page 5.
MVunc	Unconstrained Mean-variance portfolio	See equation 3, page 4.
OOSU	Out-of-sample Utility	See subpart 2.1, page 16.
SR(s)	Sharpe Ratio(s)	See equation 4, page 5.
TZ	Combination of the MVunc and EW portfolios with the risk-free asset	See subpart 2.3.2, page 22.

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Introduction

In 2021, Assets Under Management reached 112.3 trillion U.S. dollars globally (Statista Research Department, 2023). Managing portfolios goes beyond stock selection, it requires balancing risk and return to build a robust portfolio. Traditional strategies like mean-variance (MV) optimization often face real-world forecasting and estimation errors, whereas simpler approaches like equally weighted portfolios offer stability but may miss potential gains. By combining various portfolio optimization techniques, the research aims to discover a smarter investment approach that avoids the pitfalls of individual methods and enhances overall out-of-sample performance. By systematically analyzing and optimizing these portfolio combinations, considering factors like sample length and investor risk tolerance, this research aims to provide valuable insights for developing robust and efficient investment strategies for asset managers.

The renowned Markowitz framework, widely used in portfolio optimization, faces criticism for producing unstable solutions (Michaud, 1989; DeMiguel, Garlappi, & Uppal, 2009). The primary issue lies in the difficulty of accurately estimating the algorithm's inputs, the covariance matrix and the expected return vector, which often contain significant errors. Michaud (1989) addressed this through the concept of error maximization. When the covariance matrix, especially one with a large condition number, is inverted during the optimization process, minor changes in the input can lead to substantial variations in results due to the ill-conditioned matrix. Several studies (Kan & Zhou, 2007; Ao, Li & Zheng, 2019; Lassance, Martin-Utrera, & Simaan, 2023) found that plug-in approaches, which replace population parameters with sample estimates, tend to yield poor out-of-sample performance, especially in high-dimensional settings with limited sample data and high risk tolerance. DeMiguel et al. (2009) emphasize the need for long time series of data to accurately estimate expected returns while Chopra (1993) states the impact of errors in mean, variance, and covariance on the optimal MV rule.

To address these issues, Best & Grauer (1991) and DeMiguel et al. (2009) suggest imposing budget and short-selling constraints to improve performance. Ma and Jagannathan (2003) show that short-selling constraints can regularize the large elements of the covariance matrix. Additionally, James and Stein (1961) suggest the shrinkage estimator to reduce estimation errors in the mean, while Ledoit and Wolf (2004b) introduced the linear shrinkage for the covariance matrix, improving estimation accuracy by combining it with a structured estimator.

As an alternative, the global minimum-variance (GMV) portfolio emerges as a robust solution by keeping the mean constant and not relying on estimation errors. The equally weighted (EW) strategy is also favoured for its simplicity and robustness, as it does not depend on estimates and often outperforms MV portfolios in terms of Sharpe ratio (SR) and turnover (DeMiguel et al., 2009). Consequently, these individual strategies are expected to outperform.

To strike the optimal balance between diversification and risk management, a solution is to blend various portfolio strategies to maximize the expected out-of-sample utility (OOSU), as outlined by Kan and Zhou (2007). In this thesis, four two-fund portfolio combinations are evaluated: the risk-free asset with the EW portfolio (Kan & Lassance, 2024), the risk-free asset with the GMV portfolio (Lassance, Vanderveken & Vrins, 2024), the risk-free asset with the mean-variance portfolio (Kan & Zhou, 2007), and the combination of the MV and GMV rules (Kan, Wang & Zhou, 2022). To further enhance performance, three-fund combinations are explored: the risk-free asset with MV and GMV rules (Kan & Zhou, 2007), the risk-free asset with MV and EW portfolios (Tu & Zhou, 2011), the risk-free asset with GMV and EW portfolios, and a combination of EW, GMV, and MV rules (Lassance et al., 2023). As portfolio combination rules are designed to maximize expected OOSU, they should outperform single strategies.

The thesis is structured into five main chapters, separated in two main parts. The three below chapters report the first part, focusing on literature.

- Chapter 1 provides an overview of individual portfolio optimization methods. It introduces the Markowitz optimization theory, explores error maximization in mean and covariance matrix estimates, and discusses shrinkage methods.
- Chapter 2 explains how to determine the optimal weights for portfolio combinations aiming to maximize expected OOSU and details the computation process for both two-fund and three-fund rules mixing single rules (MVs, MSR, EW, GMV, risk-free asset).
- Chapter 3 describes the methodology used in the empirical analysis, including the steps for data collection, preprocessing, portfolio weight computation, and evaluation criteria.

The second part of the thesis presents the empirical analysis:

- Chapter 1 analyses out-of-sample performance of portfolio rules obtained from rolling-window method and Monte Carlo simulations, using criteria such as the SR and utility. This chapter aims to assess the accuracy of results and their alignment with literature.
- Chapter 2 addresses the research questions by comparing single portfolio rules with combined rules, identifying the best-performing ones under different conditions.

First part of body text: Research questions and empirical methodology

The first part of the body text establishes the research foundation by reviewing existing literature, portfolio combinations, and methodology. Chapter 1 examines renowned individual portfolio optimization methods, contrasting the advantages and drawbacks of MV rules against the EW strategy. Then, while Chapter 2 details the computation of optimal weights for two-fund and three-fund rules, exploring different portfolio combinations, Chapter 3 outlines the empirical methodology implemented in Python.

Chapter 1: Different portfolio optimization methods and their problems

This chapter reveals issues with Markowitz's optimal portfolio, showing that maximizing errors in estimating means and covariances reduces its out-of-sample performance, especially with higher risk tolerance. Alternatively, shrinkage methods can be used to obtain better estimators. It is proven that shrinkage techniques such as Ledoit and Wolf (2004b) for the covariance matrix and James and Stein for the sample mean improve estimation accuracy and outperform sample estimators that give extreme performance values in portfolios. However, in situations with short estimation windows or high number of assets, the EW portfolio outperforms due to reduced idiosyncratic volatility and enhanced wealth distribution. The last section of the chapter focuses on the benefits of EW portfolios.

1.1 Introduction to the Markowitz portfolio optimization theory

Before going into details Markowitz optimization, it is essential to revisit its key variables. In financial markets, portfolios consist of N distinct assets, denoted as $i = 1, \dots, N$. These assets include financial instruments such as common stocks, bonds, or cash (Jiao, 2003; Evstigneev, Hens, & Schenk-Hopp, 2016). The goal of portfolio optimization is to determine the optimal allocation of investable wealth $w = (w_1, \dots, w_N)'$ across assets based on the investor's preferences (Michaud, 1989, Maiti, 2021). The portfolio's overall return P is given by $P = w'R = \sum_{i=1}^{N} w_i R_i$, with weight w_i being the proportion of money invested in an individual asset i and R_i is the arithmetic random return of each asset:

$$R_i = \frac{S_{t+1}^i - S_t^i}{S_t^i}.$$

In this formula, $S_{\mathrm{t}}^i > 0$ represents the price of asset i at time t and $S_{\mathrm{t}+1}^i > 0$ represents the price of the asset at time $t + 1$ (Evstigneev et al., 2016; Lassance, 2022). Considering observations across the N assets over a time-period $t = 1, \dots, T$, the expected return of the portfolio is the sum of all the weighted returns of each asset composing the portfolio, with $\mu_i = E[R_i]$ representing the mean of random variables R_i (Jiao, 2003, Chapter 1, 2013):

$$\mu_p = \sum_{i=1}^{N} w_i \mu_i \,.$$

Additionally, the portfolio's risk, or volatility, is measured by the standard deviation σ_p:

$$\sigma_p^2 = \sum_{i=1}^{N} w_i \mathrm{w_j} \sigma_i \sigma_j \rho_{i,j} \,.$$

The covariance matrix of expected returns Σ, the portfolio weights w and the expected returns μ are written in a matrix form such as:

$$\Sigma = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1N} \\ \vdots & \ddots & \vdots \\ \sigma_{N1} & \cdots & \sigma_{NN} \end{bmatrix}, w = \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix}, \mu = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_N \end{bmatrix}.$$

The covariance matrix Σ is a square matrix of size $N * N$, encompassing the covariances $\sigma_{ij} = Cov(R_i, R_j)$ for all pairs of asset returns R_i, R_j. Each element σ_{ij} is located at the intersection of the i^{th} row and j^{th} column (Jiao, 2003; Evstigneev et al., 2016). Hence, the portfolio risk σ_p^2 and the portfolio return μ_p are calculated from:

$$\sigma_p^2 = w'\Sigma w \;\; and \;\; \mu_p = w'\mu. \tag{1}$$

After reviewing these key variables relevant to portfolio optimization, the thesis will focus primarily on the MV Markowitz optimal portfolio. The MV theory of Markowitz (1952) suggests solving the following quadratic Utility function $U(w)$:

$$\mu_p - \tfrac{\gamma}{2}\sigma_p^2 = w'\mu - \tfrac{\gamma}{2} w'\Sigma \mathrm{w}. \tag{2}$$

By maximizing this function, investors aim to find the optimal allocation of portfolio weights w that balances between return and risk, while considering the investor's risk aversion represented by the parameter $\gamma \geq 0$ (Michaud, 1989; Best & Grauer, 1991; Dr. Kempthorne, 2013; Holgersson & Singull, 2020; Lassance, 2022). The portfolio maximizing the utility function can be subject to the following linear constraints such as no short-selling, also called non-negativity constrained $w \geq 0$, and a budget constraint $\sum_{i=1}^{N} w_i = 1$. Additional linear constraints, such as those related to trading costs, may also be introduced (Michaud, 1989). The MV portfolio with risk-free asset (R_f), when there is no budget constraint and $1 - w'1$ is put in the risk-free asset can be easily found by taking the derivative problem:

$$\frac{\partial}{\partial \mathrm{w}}\left(w'\mu - \tfrac{\gamma}{2} w'\Sigma w\right)' = 0 \Leftrightarrow \mu - \lambda \Sigma w = 0 \Leftrightarrow$$

$$w_{MV,unc} = \tfrac{1}{\gamma}\Sigma^{-1}\mu. \tag{3}$$

The maximum Sharpe ratio portfolio (MSR) is a portfolio maximizing the SR:

$$SR(w) = \frac{\mu_p}{\sigma_p} = \frac{w'\mu}{\sqrt{w'\Sigma w}}, \tag{4}$$

with higher values indicating better risk-adjusted performance (Constantinides & Malliaris, 1995; Evstigneev, et al., 2016). Following Markowitz's approach, investors seek portfolios with the best risk-return trade-off by maximizing expected portfolio return μ_p and minimizing portfolio risk σ_p^2 (Michaud, 1989; Chapter 1, 2013). The preference for maximizing the SR over utility prompt questioning, as both SR and utility offer a mean-variance tradeoff. The Capital Market Line (CML), the black line on figure 1, is the efficient frontier when a risk-free rate R_f is introduced (Fabozzi & Grant, 2001; Maiti, 2021; Lassance, 2022). The CML is a positively sloped line with the maximum SR starting at the point $(\mu_p, \sigma_p) = (0, R_f)$ obtained from the set of pairs (μ_p, σ_p) representing portfolios with maximum Sharpe ratios (SRs) (Constantinides & Malliaris, 1995; Evstigneev et al., 2016). Then, a specific portfolio on the CML is identified by maximizing the utility based on an individual's risk aversion. The utility is maximized to find an individual MSR portfolio (the red tangency points in figure 1) on this line depending on risk aversion. The optimal weights of this MSR portfolio are determined by the following formula, with $\mathbf{1}'$ being the column vector of ones of dimension N. To ensure that the weights of MV_{unc} sum up to 1 as imposed by the budget constraint, $w_{MV,unc}$ must be normalized by dividing each element by the sum of all portfolio weights $w_{MV,unc}$, with the risk aversion (γ) cancelling out (Michaud & Michaud, 2008; DeMiguel et al., 2009; Holgersson & Singull, 2020):

$$w_{MSR} = \underset{w}{\operatorname{argmax}}\, SR(w) = \underset{w}{\operatorname{argmax}}\, \frac{w'\mu}{\sqrt{w'\Sigma w}} = \frac{w_{MV,unc}}{1' w_{MV,unc}} = \frac{\Sigma^{-1}\mu}{1'\Sigma^{-1}\mu}. \tag{5}$$

The maximum SR of this portfolio obtained by substituting w with w_{MSR} into the SR formula (4) is given by:

$$SR(w_{MSR}) = \sqrt{\mu'\Sigma^{-1}\mu}. \tag{6}$$

In the case without risk-free asset, the budget constraint is added, and the solution is given by (Kan, Wang & Zhou, 2022; Lassance et al., 2023):

$$w_{MV,c} = \underset{w}{\operatorname{argmax}}\, U(w) \text{ subject to } 1'w = 1.$$

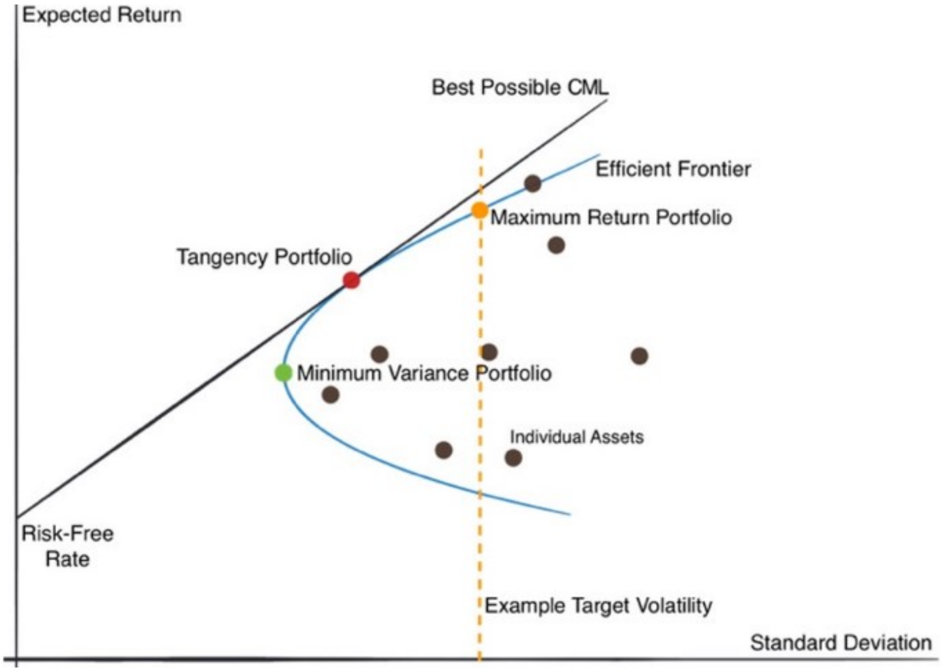
Using the method of Lagrange $L(w, \lambda) = U(w) - \lambda(1'w - 1)$, the obtained solution is:

$$w_{MV,c} = w_g + \frac{1}{\gamma} w_z\,, \tag{7}$$

with $w_g = \frac{\Sigma^{-1}1}{1'\Sigma^{-1}1}$, the GMV portfolio described in equation (9) below, and $w_z = \Sigma^{-1}(\mu - \mu_g' 1)$ a zero-investment portfolio ($w_z' 1 = 0$), where $\mu_g = w_g'\mu = \frac{1'\Sigma^{-1}\mu}{1'\Sigma^{-1}1}$ are the expected returns of w_g.

Consequently, the MV efficient frontier represented with the blue curve in figure 1 illustrates the mean and standard deviation (σ_p, μ_p) achieved by the Markowitz efficient set of portfolios. It maximizes return for a given level of risk ($\gamma > 0$) with the absolute risk aversion parameter (γ) being fluctuated to trace out the MV efficient frontier, helping investors in their decision-making process (Fabozzi & Grant, 2001; Chapter 1, 2013; Maiti, 2021).

Figure 1: Minimum-Variance portfolio and CML on the MV frontier (Ping, 2023)



A similar portfolio strategy is the global minimum-variance (GMV) portfolio, a specific point on the efficient frontier determined by the Markowitz optimization, represented by the green dot on Figure 1. It is a Markowitz portfolio solution that is deemed optimal across simulated efficient frontiers for risk averse investors willing to minimize their risk across assets (Best & Grauer, 1991; Michaud & Michaud, 2008; Evstigneev, Hens, & Schenk-Hopp, 2016). It aims to reduce the overall variance σ_p^2 across all combinations of assets (Constantinides & Malliaris, 1995; Jagannathan & Ma, 2003):

$$\min_{w} w'\Sigma w \, , \tag{8}$$

subject to a budget constraint $\sum_{i=1}^{N} w_i = 1$. This process yields the GMV optimal weights, characterized by the lowest possible variance across all portfolios (Michaud & Michaud, 2008; Holgersson & Singull, 2020; Lassance, 2022):

$$w_g = \frac{\Sigma^{-1}1}{1'\Sigma^{-1}1}. \tag{9}$$

The GMV portfolio only uses the covariance matrix of asset returns (Σ) and assumes investors are risk averse ($\gamma \to \infty$). However, it ignores the expected returns (μ) or restricts them so that they are identical across all assets regardless of their individual characteristics or market conditions: $\mu_t \propto \mathbf{1}_N$ (DeMiguel et al., 2009). Indeed, plugging μ proportional to 1 into the MSR portfolio in equation (5) yields to w_g.

Portfolios below the GMV portfolio on the efficient frontier are inefficient because they offer lower expected returns for the same level of risk of portfolios on the upper half of the frontier. Moreover, portfolios above the efficient frontier are unreachable as no investment can outperform a point on the frontier and portfolios below it are outperformed by the efficient frontier (Constantinides & Malliaris, 1995; Fabozzi & Grant, 2001; Hull, 2018).

The classical MV optimal portfolios of Markowitz are considered as ideal, as it has the highest SR, offering the highest return for a certain level of risk. This strategy is practical, as it relies on a single positive number for the risk tolerance parameter(γ), with only the expectations μ_p and covariance Σ of asset returns required (Evstigneev et al., 2016). Despite its significant contribution to financial economics through the efficient frontier, this optimization method is subject to estimation errors, particularly in mean and sample covariance matrix inputs, as explored in the following subsection.

1.2 Error maximization in mean and covariance matrix

Efficient MV portfolios represent a fundamental aspect of portfolio theory, providing a theoretically optimal method for balancing returns and risks. Despite their theoretical appeal, implementing them in practice encounters significant challenges. This section will delve into the complexities of estimating expected returns and covariance matrices, essential components for constructing MV portfolios. Methods for estimating these parameters will be explored, followed by the criteria for further evaluation of the portfolio performance out-of-sample. Finally, errors inherent in estimating means and covariance matrices will be examined, explaining why MV portfolios often underperform in real-world scenarios.

1.2.1 In-sample and out-of-sample estimates using a rolling window

Assessing the performance of $\widehat{w}$ on future out-of-sample data using a rolling window becomes essential for robust evaluation. A rolling window is a statistical technique to find out-of-sample returns with dataset containing $T + L$ observations where T is the length of the out-of-sample period and L is chosen fixed-length segment of months that divides the window. The L-month data is used to estimate in-sample parameters for a particular portfolio strategy. The sample mean is determined by:

$$\hat{\mu}_p = \sum_{i=1}^{N} w_i \hat{\mu}_i \, , \tag{10}$$

with $\hat{\mu}_i = \frac{1}{\mathrm{L}} \sum_{\mathrm{t}=1}^{\mathrm{L}} R_{\mathrm{it}}$ taking the average of L-month historical returns, while the estimate of the covariance matrix in-sample, the sample covariance, is:

$$\hat{\sigma}_p^2 = w' \hat{\Sigma} w, \tag{11}$$

with $\hat{\Sigma} = \frac{1}{\mathrm{L}-1} \sum_{\mathrm{t}=1}^{\mathrm{L}} (R_{\mathrm{it}} - \hat{\mu}_i)(R_{\mathrm{t}} - \hat{\mu}_i)'$. The sample estimates are maximum-likelihood estimators (MLE) under normality of historical data used as proxies to estimate future expected returns and ensure a well-suited fit of the model to historical data. These estimates are employed to compute sample-based MV portfolios $\widehat{w}$ using the "plug-in" approach, replacing the mean and covariance matrix with their respective counterparts for each window (DeMiguel et al., 2009; Kirby & Ostdiek, 2012; Ao et al., 2019; Vanderveken, 2019; Lassance, 2022). These weights $\widehat{w}$ computed from in-sample estimates for each portfolio strategy are then applied to compute returns for the following period $t + 1$. The out-of-sample returns r_{t+1} of a portfolio are obtained by computing the portfolio weights $\widehat{w}_t$ for one month and evaluate it on the next-month vector of returns $R_{t+1} \in \mathbb{R}^N$ at time $t + 1$, yielding in: $r_{t+1} = \widehat{w}_t' R_{t+1}$ (DeMiguel, Martin-Utrera & Nogales, 2013). The process is repeated until the dataset ends, continually incorporating the next period's return and removing the oldest one. This results weight vectors w_{it} containing weights of strategy i at time $t = L + 1, \dots, L + T$ used to generate a serie $\{R_t\}_{t=L+1}^{T+L}$ of T monthly out-of-sample returns of a portfolio strategy (DeMiguel et al., 2009, Kirby & Ostdiek, 2012, Vanderveken, 2019). Similar to in-sample data, the out-of-sample estimates of the mean and the covariance matrix found using the rolling window are respectively $\hat{\mu}_{p_{OOS}} = \sum_{i=1}^{N} w_i \hat{\mu}_{i_{OOS}}$ with $\hat{\mu}_{i_{OOS}} = \frac{1}{\mathrm{T}} \sum_{\mathrm{t}=\mathrm{L}+1}^{\mathrm{T}+\mathrm{L}} R_{\mathrm{it}}$ and $\hat{\sigma}_{p_{OOS}}^2 = w' \hat{\Sigma}_{OOS} w$ with $\hat{\Sigma}_{\mathrm{OOS}} = \frac{1}{\mathrm{T}-1} \sum_{\mathrm{t}=\mathrm{L}+1}^{\mathrm{T}+\mathrm{L}} (R_{\mathrm{it}} - \hat{\mu}_i)(R_{\mathrm{it}} - \hat{\mu}_i)'$. Testing the model on unseen data aims to determine if it is robust, delivering strong out-of-sample performance. As a result, low correlation between in-sample (past) data and out-of-sample (future) data indicates potential over-optimization, affecting portfolio performance. To ensure robustness in portfolio selection it is essential to incorporate more observations for efficiency gains while mitigating forecast precision loss from less relevant data. In general, the MV portfolios which work well in-sample encounter challenges because of their high variance estimation and model complexity, leading to poor out-of-sample performance. It emphasizes the importance of bias-variance trade-off to find the right level of complexity that avoids overfitting, occurring when a model fits historical data too closely and captures his noise but fails to generalize to new data (DeMiguel et al., 2009; Matteson & Ruppert, 2015; Lassance, 2022). The bias-variance trade-off is explained in definition 2 of the glossary (see Appendix A).

1.2.2 Performance evaluation criteria for out of sample estimates

Before proceeding to the next stage of the thesis, defining performance criteria is crucial for a comprehensive understanding of the literature. The portfolio performance is evaluated using variance $\hat{\sigma}_p^2$, Sharpe Ratio $SR(\widehat{w})$, turnover TO and utility U_p.

To compare the performance of portfolio rules, the in-sample ($\hat{\sigma}_p^2$) and out-of-sample (${\hat{\sigma}_p^2}_{OOS}$) variances can be computed. Additionally, the expected returns obtained in-sample and out-of-sample can be plugged into the SR from equation (4) and results in $SR(\widehat{w}) = \frac{\hat{\mu}_p}{\hat{\sigma}_p}$. The SR reflects the risk-return trade-off in portfolio selection, with investors favoring strategies with high SR (Vanderveken, 2019; Füss, Glück, Koeppel & Miebs, 2023). Moreover, the utility function $U_p = \left(\hat{\mu}_p - \frac{\gamma}{2}\hat{\sigma}_p^2\right)$ given by equation (2) can be useful to compare for portfolio strategies both in-sample and out-of-sample (Lassance, Vanderveken & Vrins, 2024). High utility score means the portfolio strikes an optimal balance between return and risk, depending on the investor's preferences. Therefore, the more utility score is high, the more weights are considered as optimal.

Finally, to understand the trading activity needed for each portfolio strategy, the portfolio turnover is used. It is the sum of absolute values of rebalancing movements across all assets throughout the rolling window trading dates, normalized by the number of trading dates $T-1$:

$$TO_i = \frac{1}{T-1}\sum_{t=L+1}^{L+T}\sum_{j=1}^{N}\left(\left|\widehat{w}_{i,t+1} - \widehat{w}_{i,t}\right|\right). \tag{12}$$

The turnover of a strategy i is given by the above formula, in which $w_{i,t}$ is the portfolio weight in asset i at time t before rebalancing at $t+1$ and $\widehat{w}_{i,t+1}$ is the portfolio weight at time $t+1$, after rebalancing. Investors prefer strategies with low turnover to avoid transaction costs that can erode net returns (DeMiguel et al., 2009; Vanderveken, 2019). Then, integrating c as the proportional transaction costs, the total cost of conducting such a trade and rebalancing across all assets at a given month is (DeMiguel et al, 2009) :

$$c * \sum_{j=1}^{N} \left| \widehat{w}_{j,t+1} - \widehat{w}_{j,t} \right| . \tag{13}$$

The portfolio return, when corrected by the cost of rebalancing the portfolio, gives the portfolio return net of proportional transaction costs (DeMiguel et al., 2009; DeMiguel et al., 2013):

$$r_{t+1}^{i} = (1 + R_{t+1}' w_t)(1 - c \sum_{j=1}^{N} \left| \widehat{w}_{j,t+1} - \widehat{w}_{j,t} \right|) - 1. \tag{14}$$

1.2.3 Error in means and covariance matrix

Estimating optimal MV portfolio weights $\widehat{w}_{MSR}$ using expected returns and covariance matrix may result in estimation errors, addressed by Michaud (1989) as the concept of error maximization. It highlights how the process amplifies estimation errors arising from incomplete information, as estimated parameters ($\hat{\mu}$ and $\hat{\Sigma}$) only approximate the true values (μ, Σ). Consequently, MV optimization, sensitive to errors in input estimates, leads to biased outcomes and overweighting of certain stocks as small changes in input estimates cause significant changes in the optimal portfolio composition. Consequently, the final portfolio may not accurately reflect the true portfolio characteristics. For example, the true level of risk associated with the optimal portfolio could be underestimated. Going further, Jobson and Korkie (as cited in Michaud, 1989) measured error maximization in MV optimizers by comparing the SRs. Such as Ao, Li, and Zheng (2019), they noticed a significant performance gap between the plug-in MSR portfolio, and the true one obtained from a known distribution due to errors maximization coming from inaccuracies or biases in input data. Additionally, Kan and Zhou (2007) introduced the concept of expected out-of-sample utility (OOSU) losses for MV portfolios, while Lassance et al. (2023) theoretically characterized the OOSU variance of sample portfolios for performance evaluation. Both studies found that plug-in approaches, replacing population parameters with sample estimates, can result in poor out-of-sample performance, particularly in high-dimensional settings with limited sample data. Overall, these findings underscore the importance of considering out-of-sample performance when assessing strategies.

Besides errors maximization, the length of the estimation window also affects the performance of the sample-based MV strategy. A long time series of data is needed to estimate expected returns accurately (Merton, 1980 as cited in DeMiguel et al., 2007). However, in simulated data, the plug-in portfolio often employs a too-short estimation window compared to the number of assets, which induces systematic bias and diminishes its performance. Since these portfolios are generally estimated using only 60 or 120 months of data, there is a strong influence of estimation errors in covariance matrix and expected returns. Therefore, the gains promised by optimal portfolio choice cannot be realized out-of-sample (Michaud, 1989; DeMiguel et al, 2009; Ao, Li & Zheng, 2019).

Additionally, Chopra (1993) demonstrated that maximization errors in mean, variance, and covariance on the optimal MV strategy happen when investors use limited, non-stationary, or time-varying data to estimate model parameters, and occur regardless of the investor's risk tolerance level. In this context, non-stationary is defined as time series with changing means, variances, and covariances over time, making them unpredictable and difficult to model accurately, on the contrary to stationary data, where the mean and variance remain constant over time (Levendis, 2023). Moreover, the more the level of risk aversion (γ) increases, the more the error maximization of the means decreases. Finally, errors in means have the greatest impact, followed by errors in variance, which are more significant than errors in covariances.

To enhance portfolio performance, precise estimates of expected returns are crucial, prioritizing mean estimates over variance and covariance. Since portfolio optimization relies heavily on these inputs, accurate forecasts are essential to reflect asset returns and maintain MV performance. Firstly, setting all expected returns to zero when computing the GMV portfolio can improve performance by avoiding the "error in means" problem, but it often results in portfolio concentration. Another solution is to use the Bayesian framework, defined in point 3 in the glossary (See Appendix A), which considers suitable priors to address estimation risk when constructing the portfolio such as Bayesian estimator of Jorion (Bodnar, Okhrin & Parolya, 2023). Another way to obtain better estimates is classical shrinkage methods, which will be discussed in the following sections for mean and covariance matrix estimation.

Finally, implementing constraints could help to improve the performance (Best & Grauer, 1991). When only the budget constraint $\sum_{i=1}^{N} w_i = 1$ is added to the Markowitz portfolio, the weights, the expected returns, and the variance are acutely sensitive to modifications in asset means. On top of this constraint, incorporating short-selling constraints ($w \geq 0$) into MV strategy enhances its performance, surpassing unconstrained models globally. This improvement is reflected in a slightly higher SR and strong decrease in turnover (DeMiguel et al., 2009). This is because when short-selling constraints are added, changes in asset means solely affect the weights of the MSR portfolio, leaving its expected return and its standard deviation unchanged. However, as small increase in the mean leads to considerable changes in the allocation of weights of the optimal portfolio, especially in portfolios with highly correlated securities. For example, in a MSR portfolio composed of 100 assets, 11.6% rise in the mean of one specific asset is sufficient to remove half of the other assets, while the expected return and standard deviation of the portfolio only change by about 2% (Best & Grauer, 1991). This sensitivity to mean changes is amplified in larger portfolios; a 100-asset portfolio is 4000 times more responsive to changes in portfolio weights compared to a portfolio with only 10 assets. Additionally, the portfolio weights exhibit approximately 14,000 times greater elasticity than the average elasticity of individual portfolio return variables (Best & Grauer, 1991). In this context, the elasticity is a measure of a variable's sensitivity to a change in other variables.

1.2.4 Estimation of the sample mean

As sample means have the biggest impact on error maximization, it is crucial to understand why they perform poorly. Their use in estimating expected returns within the MV portfolio is suboptimal, leading to error maximization. Indeed, the Maximum Likelihood Estimator (MLE), aiming to maximize the probability of observations, lacks robustness, relying solely on observed data and assumed distribution, making it unreliable for new, unobserved data. In this context, Stein (1956) demonstrated that under standard conditions, sample means are inadmissible estimators of expected returns, as there exists another estimator that outperforms them considering a specific risk function.

As a solution, James and Stein (1961) proposed a Bayesian shrinkage estimator of the sample mean, introduced in definition 4 in the glossary (See Appendix A), which is suboptimal to estimate expected returns in the MV portfolio (Stein, 1956). Later, Jorion (1986) developed a Bayesian estimator by adjusting the covariance matrix for estimation using Bayesian methods to address parameter uncertainty in the James and Stein shrinkage intensity δ_{JS}. These shrinkage methods showed improvement over sample mean estimators in out-of-sample evaluations (Stein, 1956; James & Stein, 1961; Jorion, 1986; Michaud, 1989; DeMiguel et al., 2009; Hitaj, Martellini, & Zambruno, 2012).

1.2.5 Estimation of the sample covariance matrix

Estimating the covariance matrix of stock returns has always been one of the stickiest points. Regrettably, it entails the error maximization phenomenon resulting in estimation errors and extreme coefficients in the matrix, particularly noticeable when the number of stocks exceeds the available historical return observations $(N > T)$. The presence of errors in covariance estimates, compounded by the ill-conditioning of the covariance matrix, is challenging.

Ill-conditioned matrices refer to a matrix in which minor parameter changes in input data cause significant alterations in output results, creating unstable solutions. This instability in solutions often arises when the coefficient matrix is singular, lacking an exact inverse. Mathematically, the characterization of a matrix as ill-conditioned relies on its condition number. For symmetric matrix, the condition number is given by $\kappa(A) = \left|\frac{\lambda_{max}}{\lambda_{min}}\right|$ where λ_{max} and λ_{min} denote the largest and smallest eigenvalues of A. A small condition number indicates better results, with a minimum value of 1 ensuring that the solution can achieve precision equivalent to that of the data without amplifying any noise present in the data. Conversely, as the condition number surpasses one, indicating substantial variation between eigenvalues, issues intensify, rendering the matrix ill-conditioned and unstable. If $N > T$, the last $N - T$ eigenvalues of Σ are zero $(\lambda_{min} = 0)$ resulting in an infinite value for the condition number since it is divided by 0 (Böhm & von Sachs, 2009; Condition number, 2020; Manglik, 2021; Lassance, 2022).

Consequently, ill-conditioned matrices, as defined by Hadamard (1902), lack solution existence, uniqueness, and stability, contrasting with well-posed problems. It leads to errors in estimated model parameters, affecting calculations involving its inverse or determinant. Unless the number of observations significantly exceeds the number of assets $(T > N)$, the eigenvalues of the sample covariance matrix $\hat{\Sigma}$ undergo strong distortion compared to those of the true covariance matrix Σ. In fact, the initial eigenvalues tend to overestimate their population counterparts, while the final eigenvalues tend to underestimate them. Therefore, the inversion of the estimated covariance matrix $\hat{\Sigma}$ necessary to compute the optimal weight formula of the MSR portfolio $\hat{w}_{MSR}$ induces errors and performs poorly out-of-sample.

A straightforward way to enhance portfolio stability and reduce risk suggested by Jagannathan and Ma (2003) is to integrate short-sales constraints into MV optimization, acting as a type of shrinkage in the sample covariance matrix. Going further for improved covariance matrix estimations, Ledoit & Wolf (2003) introduced a linear shrinkage estimator, like James and

Stein's (1961) mean estimator. This estimator $\hat{\Sigma}_{\mathrm{LW}}$ is a linear combination balancing between bias and variance: $\hat{\Sigma}_{\mathrm{LW}} = (1-\delta)\hat{\Sigma} + \delta\hat{T}$. The sample covariance matrix $\hat{\Sigma}$ is unbiased but has high variance, resulting in difficulties when handling out-of-sample data. Conversely, a structured estimator $\hat{T}$ incorporating prior knowledge to improve estimation accuracy by imposing constraints on the matrix structure, is biased but has lower variance and so less estimation errors (Ledoit & Wolf, 2004a; Wiesel & Zhang, 2015). A common choice for $\hat{T}$ is assuming uncorrelated asset returns having the same variance $\hat{T} = \hat{\sigma}^2\boldsymbol{I}_N$, called the constant correlation covariance matrix. Additionally, Ledoit & Wolf (2004b) suggested to use two other respective target matrices $\hat{T}$, assuming asset independence or uniform distribution within the portfolio. The weight of the structured estimator is decided by the shrinkage intensity δ, ranging from 0 to 1; higher values of δ indicate stronger structural influence. It is computed as an estimate of the minimizer of the Frobenius norm of $\hat{\Sigma}_{\mathrm{LW}} - \Sigma$:

$$\delta = \underset{\delta \epsilon [0,1]}{\operatorname{argmin}} \left\| \hat{\Sigma}_{\mathrm{LW}} - \Sigma \right\|_F \text{, where } \|A\|_F = \sqrt{\textstyle\sum_{i=1}^{N} \sum_{j=1}^{N} a_{i,j}^2} \text{ .}$$

Determining the optimal shrinkage intensity, balancing between $\hat{\Sigma}$ and $\hat{T}$, is challenging due to the infinite possibilities within the range of 0 to 1. By minimizing the expected distance between the shrinkage estimator and the true covariance matrix, the optimal shrinkage constant serves to establish the shrinkage estimator of $\hat{\Sigma}_{\mathrm{LW}}$. Therefore, any shrinkage estimator consists of three parameters: the sample covariance matrix, a highly structured target estimator, and the shrinkage intensity, which demands careful consideration. As a result, implementing shrinkage on sample covariance matrices can offer a successful compromise estimator in practice (Ledoit & Wolf, 2004b; DeMiguel et al., 2013; Lassance, 2022).

To conclude, this section introduced error maximization in means and covariances, demonstrating that the MV portfolio poses challenges. One of the difficulties lies in accurately estimating expected returns and covariance matrices, resulting in biased outcomes and suboptimal portfolio compositions. Errors in mean estimation result in the overweighting of certain stocks and may underestimate portfolio risk. Moreover, covariance matrix estimation presents difficulties, especially with limited historical data, leading to ill-conditioned matrices and unstable solutions. As a result, MV rules tend to underperform in out-of-sample testing.

1.3 Naive strategy: the equally weighted portfolio 1/N

In contrast to the Markowitz portfolio, which relies on complex mathematical models, the Equally Weighted (EW) strategy, also known as the naive $1/N$ rule, emerges as a significant optimization technique. This portfolio rule allocates an equal weight of: $w_{EW} = 1/N$ to each of the N risky assets within the portfolio (Lassance, 2022). The formula ensures that the sum of weights across all assets in the portfolio sum up to 1. In contrast to the MV portfolios, this simplistic approach is easy to implement but may entail higher risk (De Miguel et al., 2009; Bloch, 2022). Indeed, it does not estimate moments (μ_t and Σ_t) but instead applies a condition $\mu_t \propto \Sigma_t \mathbf{1}_N$ for all t whereby expected returns (μ_t) are proportional to overall risk (Σ_t) of the portfolio, rather than systematic market risk. In this case where $\mu = c * \Sigma * 1$ with c being a constant, substituting μ into the expression of w_{MSR} (5), results in:

$$w_{MSR,\mu_t \propto \Sigma_t \mathbf{1}_N} = \frac{\Sigma^{-1}(c * \Sigma * 1)}{1'\Sigma^{-1}(c * \Sigma * 1)} = \frac{c * 1}{1'(c * 1)} = \frac{c * 1}{c * 1'1} = \frac{c * 1}{c * N} = \frac{1}{N}.$$

DeMiguel et al. (2009) found that the MSR portfolio underperforms the EW rule in terms of SR and turnover due to estimation errors. Indeed, strategies with high turnover face a disproportionate impact from transaction costs because portfolios with estimated moments produce extreme weights that oscillate greatly through time and underperform out-of-sample. Even if the short-selling and budget-constrained strategy outperformed the unconstrained MSR strategy, it does not improve performance sufficiently to outperform the EW strategy and to be statistically significant. Hence, sample-based MV strategy and its extensions surpass $1/N$ benchmark only with a long estimation window ($T > N$), high ex-ante SR, a small number of assets to reduce estimation errors and high level of idiosyncratic risks of individual assets (Jannagathan & Ma, 2003; DeMiguel et al., 2009). The definition of idiosyncratic volatility can be found in the first point of Appendix A. For example, for a MSR portfolio with 25 assets, an estimation window of around 3000 months is required to outperform the EW portfolio, and about 6000 months for a portfolio with 50 assets (DeMiguel et al., 2009).

Therefore, the EW portfolio, a simple biased but low-variance estimator, is preferred in predictions for its simplicity and robustness as it do not rely on expected return estimation. It outperforms MSR portfolios despite potential inadequacies in capturing data nuances. However, it may result in a highly constrained risk diversification when idiosyncratic risks of assets vary substantially (Michaud, 1989; Hastie, James, Tibshirani & Witten, 2013; Holgersson & Singull, 2020).

Chapter 2 : Portfolio combinations

The first chapter analysed two extreme approaches: the MV strategy, theoretically optimal but sensitive to estimation errors, and the EW rule, suboptimal yet robust against estimation errors. Combining rules these rules along with the GMV or the risk-free asset R_f is an appealing strategy. Indeed, it could unlock diversification benefits since estimation errors are not perfectly correlated between assets, resulting in portfolio weights less prone to errors and achieving improved out-of-sample performance (Füss et al., 2023). Therefore, this chapter examines two-fund and three-fund rules to achieve rules that balance diversification and risk management.

2.1. Computing optimal weights combining different strategies

Before computing strategy combinations, it's essential to understand how their optimal weights are determined. A first way to measure performance is by evaluating portfolios through a loss function $L(w^*, \widehat{w}) = U(w^*) - \widetilde{U}(\widehat{w})$, measuring the disparity between the utility of the optimal portfolio (w^*) and the actual portfolio $(\widehat{w})$. The loss function is strictly positive as w^* is different from $\widehat{w}$ and relies on historical returns data ϕ_T. However, for better decision-making, averaging across the various outcomes of ϕ_T is recommended. Therefore, the authors proposed using the expected loss function: $\rho(w^*, \widehat{w}) = E[L(w^*, \widehat{w})] = U(w^*) - E[\widetilde{U}(\widehat{w})]$ taken with respect to the true distribution of historical returns data. The portfolio rule having the lowest score is the most preferred. However, as $U(w^*)$ is the same in every loss function, portfolio rules can be ranked only by their expected out-of-sample performance. Therefore, Kan and Zhou (2007) elaborated an analytical expression to assess the expected OOSU: $E\left[\widetilde{U}(\widehat{w})\right] = E[\widehat{w}'\mu - \frac{\gamma}{2}\widehat{w}'\Sigma\widehat{w}]$, which is the expected out-of-sample performance an investor can achieve on average under parameter uncertainty with portfolio rule $\widehat{w}$. The estimator of $\widehat{w}$ can be any function $f(\hat{\mu}, \hat{\Sigma})$ of $\hat{\mu}$ and $\hat{\Sigma}$, and the goal of investor is to find a function $f(\hat{\mu}, \hat{\Sigma})$ maximizing $E\left[\widetilde{U}(\widehat{w})\right]$. Maximizing this utility allows the development of combination rules reducing error maximization and compare portfolio rules based on performance. Several theoretical assumptions were made to perform these calculations. Firstly, asset returns R_t at time t, follow a “multivariate Gaussian distribution with a mean vector μ and covariance matrix Σ" (Lassance et al., 2023, p. 9). All returns R_t are assumed independent and identically distributed (i.i.d) through time to calculate in-sample estimates. Finally, the condition $T > N + 4$ must be satisfied to ensure that $\hat{\Sigma}$ is invertible and that the second moment of $\hat{\Sigma}^{-1}$ exists to avoid division by zero in the denominator (Kan and Zhou, 2007, p.8).

2.2. Two-fund rules

The four two-fund rules explored in this section are those portfolio combinations: the risk-free asset with the EW portfolio, the risk-free asset with the GMV portfolio, the risk-free asset with the MV portfolio and finally the combination of the MV portfolio and GMV portfolio. Like the three-fund rules, these portfolio strategies can be reviewed in Kan, Lassance, and Wang (2024, Section 3).

2.2.1. Combination of the risk-free asset and the EW portfolio (EWRF)

This portfolio, detailed in Kan and Lassance (2024, p.26, equation 68) combines the risk-free asset with the EW portfolio. By doing so, the equal weights of the EW strategy maximize mean-variance utility, resulting in the combined portfolio weights:

$$w^*_{EWRF} = \frac{\hat{\mu}_{ew}}{\gamma \hat{\sigma}^2_{ew}} w_{ew}, \tag{15}$$

where w^*_{EWRF} is the optimal weight of our first portfolio combination. It is composed of the risk aversion parameter γ of the investor and the parameters of the EW portfolio such as its optimal weight $w_{ew} = \frac{1}{N}$, its mean $\hat{\mu}_{ew}$and its variance $\hat{\sigma}^2_{ew}$:

$$\hat{\mu}_{ew} = w'_{ew}\hat{\mu}, \;\; \hat{\sigma}^2_{ew} = w'_{ew}\hat{\Sigma}w_{ew} \,. \tag{15.1}$$

It is inspired from the combination of the MV, the EW and the risk-free rate from Tu and Zhou (2011): $\hat{w}(\kappa) = \kappa_1 \hat{w}^* + \kappa_2 w_{ew}$ in which $(\kappa_1, \kappa_2) = (0, \frac{\gamma_{ew}}{\gamma})$ are coefficients that maximize the expected OOSU, with $\gamma_{ew} = \frac{\hat{\mu}_{ew}}{\hat{\sigma}^2_{ew}}$ and $\kappa_1 = 0$ as the weight on the MV portfolio is close to 0 (Lassance, Vanderveken & Vrins, 2024).

2.2.2. Combination of the risk-free asset and the GMV portfolio (GMVRF)

The portfolio of this section combines the risk-free asset R_f with the GMV portfolio. The optimal weights of this portfolio can be determined using the weights identified by Kan and Zhou (2007) for combining the risk-free asset with the MV portfolio. Since the GMV portfolio is an MV portfolio considering $\mu_t \propto \mathbf{1}_N$, it follows that:

$$\hat{w} = \frac{c}{\gamma}\hat{\Sigma}^{-1}1,$$

the optimal weights of this combined portfolio, are obtained by finding the value of the optimal combination coefficient c maximizing the expected OOSU. Knowing that T is the number of observations, N is the number of assets, γ is the risk aversion parameter, $\hat{\Sigma}^{-1}$ is the estimated sample covariance matrix of the portfolio and $\hat{\mu}_g$ is the mean of the GMV portfolio:

$$\hat{\mu}_g = \hat{w}_g' \mu = \frac{1' \hat{\Sigma}^{-1} \hat{\mu}}{1' \hat{\Sigma}^{-1} 1}. \tag{16.1}$$

Its optimal weights are given by (Lassance, Vanderveken & Vrins, 2024, p.25):

$$w_{GMVRF}^{*} = \frac{\hat{\mu}_g}{\gamma} \hat{\Sigma}^{-1} 1 \, \frac{(T-N-1)(T-N-4)}{T(T-2)}. \tag{16}$$

2.2.3. Combination of the risk-free asset and the MV portfolio (KZ2F)

The optimal two-fund strategy suggested by Kan and Zhou (2007) integrates the risk-free asset with the sample MV portfolio. Even if the plug-in and Bayesian rules recommend holding the risk-free asset and the MV portfolio, their weights for the sample tangency portfolio may not be optimal for maximizing expected out-of-sample performance. Instead, it is better to consider the combination of two portfolio with weights:

$$\hat{w} = \frac{c}{\gamma} \hat{\Sigma}^{-1} \hat{\mu},$$

where c is a constant scalar, γ is the risk aversion coefficient, and $\hat{\Sigma}$ is the sample covariance matrix. The previous mentioned rules are particular cases of this class as the first plug-in and Bayesian rules set $c_1 = 1$ and $c_2 = \frac{T-N-2}{T+1}$, respectively. The expected out-of-sample performance of the two-fund rule is higher than both the classic plug-in and Bayesian rules. The value of c maximizing the expected OOSU of this portfolio rule is $c^* = c_3 \left(\frac{\theta^2}{\theta^2 + \frac{N}{T}} \right)$ with:

$$c_3 = \frac{(T-N-1)(T-N-4)}{T(T-2)}. \tag{17.1}$$

However, even if c^* is optimal, a feasible portfolio rule using c^* does not exist since θ is unknown in practice. Therefore, it is essential to find a good estimate of θ^2, the maximum squared SR in equation (6), that will allow the implementation of an approximate optimal two-fund rule. A natural estimator of θ^2 is its sample counterpart $\hat{\theta}^2 = \hat{\mu}' \hat{\Sigma}^{-1} \hat{\mu}$. However, the plug-in estimator $\hat{\theta}^2$ can be heavily biased when T is small. Therefore, they derived an unbiased estimator $\hat{\theta}_u^2$ of $\hat{\theta}^2$ but as it could yield negative value, it is an undesirable estimator of θ^2. Finally, they opted to use an adjusted estimator θ_a^2 of $\hat{\theta}^2$, inspired by Kubokawa, Robert, and Saleh (1993). This estimator is superior to the unbiased estimator in minimizing the quadratic loss function, which measures the difference between predicted and actual values. It can be computed using the following formula:

$$\hat{\theta}_a^2 = \frac{(T-N-2)\hat{\theta}^2 - N}{T} + \frac{2\left(\hat{\theta}^2\right)^{\frac{N}{2}}\left(1+\hat{\theta}^2\right)^{-\frac{T-2}{2}}}{T B_{\frac{\hat{\theta}^2}{1+\hat{\theta}^2}}\left(\frac{N}{2},\frac{T-N}{2}\right)},$$

where $B_x(a,b)$ is the incomplete beta function expressed as:

$$B_x(a,b) = \int_0^x y^{a-1}(1-y)^{b-1}dy.$$

The advantage of this estimator θ_a^2 lies in its components: the first part is an unbiased estimator of θ^2, while the second part adjusts this unbiased estimator to ensure its positivity when the estimator is too small. With the adjusted estimator of θ^2, the optimal c^*, which is random and data-dependent in comparison with c_3 coming from equation (17.1), can be estimated using:

$$\hat{c}^* = c_3\left(\frac{\hat{\theta}_a^2}{\hat{\theta}_a^2 + \frac{N}{T}}\right).$$

Finally, the optimal weights for the optimal two-fund portfolio are:

$$\hat{w}_{KZ2F}^* = \frac{\hat{c}^*}{\gamma}\hat{\Sigma}^{-1}\hat{\mu}. \tag{17}$$

2.2.4. Combination of the GMV portfolio and the MV portfolio (KWZ)

Kan, Wang & Zhou (2022) developed a strategy to minimize estimation risk in MV portfolio selection without a risk-free asset. To do so, they combined the GMV portfolio with the MV strategy such as in $w_{MV,c}$ from equation (7) using the plug-in sample estimators. When using estimated parameters instead of true parameters, it is widely recognized that the plug-in portfolio underestimates the true optimal portfolio due to estimation errors. To mitigate estimation risk, the authors concentrated on one portfolio rule adjusting exposure to the sample zero-investment portfolio $\hat{w}_{z,t}$ with weights:

$$\hat{w}_t(\tilde{c}_t) = \hat{w}_{g,t} + \frac{\tilde{c}_t}{\gamma}\hat{w}_{z,t},$$

where $\tilde{c}$ is the combining coefficient. The plug-in estimators are the sample weights of the GMV rule $\hat{w}_g = \frac{\hat{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\hat{\Sigma}^{-1}\mathbf{1}}$ and the efficient zero-investment portfolio $\hat{w}_z = \hat{\Sigma}^{-1}\left(\hat{\mu} - 1'\hat{\mu}_g\right)$ satisfying $(1_N' w_z = 0)$ and using the sample mean of the GMV rule $\hat{\mu}_g$ from equation (16.1). The value of $\tilde{c}$ maximizing the expected OOSU is: $\tilde{c}^* = \frac{k\psi^2}{\psi^2 + \frac{N-1}{T}}$ where $k = \frac{(T-N)(T-N-3)}{T(T-2)}$, and the squared slope of the asymptote to the ex-ante minimum-variance frontier is:

$$\psi^2 = \left(\mu - \mu_g 1\right)'\Sigma^{-1}\left(\mu - \mu_g 1\right). \tag{18.1}$$

The optimal combining coefficient $0 < \tilde{c}^* < 1$ is derived to mitigate the effect of estimation errors due to less exposure to $\hat{w}_{z,t}$ which has more estimation errors than $\hat{w}_{g,t}$ which only depends on $\hat{\Sigma}_t$ while $\hat{w}_{z,t}$ also depends on $\hat{\mu}_t$, a very noisy estimator of μ. Moreover, when $c = 1$ it results in the plug-in $w_{MV,c}$ portfolio. However, as $\tilde{c}^*$ depends on ψ^2 which is unknown to investors in practice, $\hat{w}_t(\tilde{c}^*)$ is not implementable. Therefore, a way to implement it is to use $\hat{\psi}_t^2$, the estimated squared slope of the asymptote to the ex-ante minimum-variance frontier:

$$\hat{\psi}_t^2 = \hat{\mu}_t'\hat{\Sigma}_t^{-1}\hat{\mu}_t - \frac{\left(1'\hat{\Sigma}_t^{-1}\hat{\mu}_t\right)^2}{\left(1'\hat{\Sigma}_t^{-1}1\right)} = \left(\hat{\mu} - \hat{\mu}_g 1\right)'\hat{\Sigma}^{-1}\left(\hat{\mu} - \hat{\mu}_g 1\right). \tag{18.2}$$

However, such as $\hat{\theta}^2$, $\hat{\psi}_t^2$ is a heavily biased estimator when T is small. Therefore the authors considered $\hat{\psi}_{a,t}^2$, the adjusted estimator of ψ^2, which is the difference between the maximum squared SR $\mu'\Sigma^{-1}\mu$ and that of GMV $\frac{\mu_g^2}{\sigma_g^2}$, formulated as:

$$\hat{\psi}_a^2 = \frac{(T-N-1)\hat{\psi}_t^2 - (N-1)}{T} + \frac{2\left(\hat{\psi}_t^2\right)^{\frac{N-1}{2}}\left(1+\hat{\psi}_t^2\right)^{-\frac{T-2}{2}}}{TB_{\frac{\hat{\psi}_t^2}{1+\hat{\psi}_t^2}}\left(\frac{N-1}{2}, \frac{T-N+1}{2}\right)}, \tag{18.3}$$

with T being the number of periods in the estimation window of historical return data, with $T > N$ and with $B_x(a, b)$ being the incomplete beta function as shown in Kan and Zhou (2007). Therefore, a realizable version of the optimal combining coefficient $\tilde{c}^*$ can be computed:

$$\hat{c}_t = \frac{k\hat{\psi}_{a,t}^2}{\hat{\psi}_{a,t}^2 + \frac{N-1}{T}},$$

trading off between estimation risk $\frac{N-1}{T}$ and the inefficiency of the GMV portfolio. This coefficient is then used to compute the optimal portfolio weights:

$$\hat{w}_{KWZ}^* = \hat{w}_t(\hat{c}_t) = \hat{w}_{g,t} + \frac{\hat{c}_t}{\gamma}\hat{w}_{z,t}. \tag{18}$$

This optimal combining portfolio outperforms other estimated portfolio rules both theoretically and empirically, showing the highest expected utility, the highest SR and the lowest turnover in all datasets examined.

2.3. Three-fund rules

To go deeper into the portfolio combination strategies and try to enhance the expected out-of-sample utility, four three-fund rules will be explored.

2.3.1. Combination of the risk-free asset, the MV and the GMV portfolios (KZ3F)

Kan and Zhou (2007) proposed a three-fund portfolio combining the riskless asset, the MV, and the GMV portfolios. It is well known that in the presence of unknown parameters, the tangency portfolio has estimation errors, so additional portfolios, such as the GMV portfolio, can help diversify estimation risk by relying solely on the covariance matrix $\hat{\Sigma}$. This three-fund portfolio rule may outperform previous two-fund rules. Consider the combined portfolio rule:

$$\hat{w} = \hat{w}(c^*, d^*) = \frac{1}{\gamma}\,(c^*\,\hat{\Sigma}^{-1}\hat{\mu} + d^*\,\hat{\Sigma}^{-1}1),$$

where c and d are optimally chosen constants. The portfolio rule $\hat{w}(c^*, d^*)$ invests in the MV and the GMV portfolios with weights proportional to $\hat{\Sigma}^{-1}\hat{\mu}$ and $\hat{\Sigma}^{-1}1$ respectively, as well as the risk-free asset. Maximizing the expected out-of-sample utility allows to find the optimal combining coefficients c^{**} and d^{**} :

$$c^{**} = c_3\left(\frac{\psi^2}{\psi^2+\frac{N}{T}}\right) \text{ and } d^{**} = c_3\left(\frac{\frac{N}{T}}{\psi^2+\frac{N}{T}}\right)\mu_g,$$

where c_3 is expressed in formula (17.1), ψ^2 described in (18.1) is the squared slope of the asymptote to the ex-ante minimum-variance frontier and μ_g from (16.1) is the expected return of the GMV portfolio. The higher the ratio of N to T, the more challenging it becomes to estimate the weights of the MV portfolio, requiring a larger investment in the GMV portfolio. Higher ψ^2 prompts more investment in the MV portfolio, yielding higher performance than the sample GMV. However, this rule is unachievable because ψ^2 and μ_g are unknown and must be estimated by their sample counterparts to implement this combination. The mean of the GMV portfolio $\hat{\mu}_g$can be computed from formula (16.1) and $\hat{\psi}_t^2$ is described in formula (18.2). As for $\hat{\theta}^2$, $\hat{\psi}_t^2$ is a heavily biased estimator when T is small. Therefore, similarly to θ_a^2, an adjusted estimator of ψ is suggested by Kan, Wang & Zhou (2022) in expression (18.3), resulting in adjusted combining coefficients:

$$c^* = c_3\left(\frac{\hat{\psi}_a^2}{\hat{\psi}_a^2+\frac{N}{T}}\right) \text{ and } d^* = c_3\left(\frac{\frac{N}{T}}{\hat{\psi}_a^2+\frac{N}{T}}\right)\hat{\mu}_g.$$

Therefore, with $\hat{\psi}_a^2$, $\hat{\psi}_t^2$ and $\hat{\mu}_g$ being the same parameter than the one used to estimate $\hat{w}^{*4}$, the optimal weights of the first three-fund strategy are:

$$\hat{w}^*_{KZ3F} = \frac{c_3}{\gamma}\left[\left(\frac{\hat{\psi}_a^2}{\hat{\psi}_a^2 + \frac{N}{T}}\right)\hat{\Sigma}^{-1}\hat{\mu} + \left(\frac{\frac{N}{T}}{\hat{\psi}_a^2 + \frac{N}{T}}\right)\hat{\mu}_g\, \hat{\Sigma}^{-1}1\right]. \tag{19}$$

2.3.2. Combination of the risk-free asset, the MV and the EW portfolios (TZ)

Tu and Zhou (2011) combined the Markowitz and the EW strategies with the risk-free asset. The Maximum Likelihood (ML) rule is given by $\hat{w}^{ML} = \frac{\hat{\Sigma}^{-1}\hat{\mu}}{\gamma}$. Instead of using $\hat{w}^{ML}$, they used a scaled one, $\bar{w}$, which is unbiased and performs better than $\hat{w}^{ML}$:

$$\bar{w} = \frac{1}{\gamma}\,\tilde{\Sigma}^{-1}\hat{\mu},$$

where $\tilde{\Sigma} = \left(\frac{T}{T-N-2}\right)\hat{\Sigma}$. The combination rule is: $\hat{w}_c = \left(1-\hat{\delta}\right)w_e + \hat{\delta}\bar{w}$ where $w_e = \frac{1}{N}$ is the EW rule. The optimal choice of the combination coefficient $\hat{\delta}$, which balances bias and the variance, is determined by $\hat{\delta}^* = \frac{\hat{\pi}_1}{\hat{\pi}_1 + \hat{\pi}_2}$. It is optimized such that the optimal combination rule $\hat{w}_c$ strictly dominates both the 1/N rule and $\bar{w}$. To estimate δ^*, π_1 and π_2 must be estimated as:

$$\hat{\pi}_1 = w_e'\hat{\Sigma}w_e - \frac{2}{\gamma}w_e'\hat{\mu} + \frac{1}{\gamma^2}\tilde{\theta}^2 \quad \mathit{and} \quad \hat{\pi}_2 = \frac{1}{\gamma^2}(c_1 - 1)\tilde{\theta}^2 + \frac{c_1}{\gamma^2}\frac{N}{T},$$

where $c_1 = \frac{(T-2)(T-N-2)}{(T-N-1)(T-N-4)}$ and $\tilde{\theta}^2$ is an estimator of $\left(\frac{T}{T-N-2}\right)\hat{\theta}^2 = \left(\frac{T}{T-N-2}\right)\hat{\mu}'\,\hat{\Sigma}^{-1}\hat{\mu}$, given by Kan and Zhou (2007) to compute $\hat{w}^*_{KZ2F}$. It allows for the optimal combination of the $1/N$ rule with the unbiased ML rule $\bar{w}$. This combination rule is easy to conduct in practice since it is only a given function of the data. Moreover, errors in estimating δ^* are generally small across different scenarios, resulting in $\hat{w}^{CML}$ improving upon $\bar{w}$ and sometimes outperforming the $1/N$ rule. Therefore, the optimal combining weight is given by:

$$w^*_{TZ} = \left(1 - \hat{\delta}^*\right)w_e + \hat{\delta}^*\bar{w}. \tag{20}$$

2.3.3. Combination of the risk-free asset, the GMV and the EW weighted portfolios (LA)

This combination, developed by my supervisor for this thesis, combines the GMV, the EW portfolio, and the risk-free asset:

$$\hat{w} = \frac{c_1}{\gamma}\hat{\Sigma}^{-1}1 + \frac{c_2}{\gamma}w_{ew}.$$

Optimizing the combination coefficients (c_1, c_2) to maximize the expected OOSU of $\hat{w}$, yields optimal combination coefficients (c_1^*, c_2^*) given by:

$$c_1^* = \frac{(T-N-1)(T-N-4)}{T(T-2)}\,\frac{\left(\mu_g\sigma_{ew}^2 - \mu_{ew}\sigma_g^2\right)}{\sigma_{ew}^2 - \frac{(T-N-1)(T-N-4)}{(T-2)(T-N-2)}\sigma_g^2}$$

$$c_2^* = \frac{\mu_{ew} - \frac{(T-N-1)(T-N-4)}{(T-2)(T-N-2)}\mu_g}{\sigma_{ew}^2 - \frac{(T-N-1)(T-N-4)}{(T-2)(T-N-2)}\sigma_g^2},$$

where $\mu_g = \frac{1'\Sigma^{-1}\mu}{1'\Sigma^{-1}1}$ and $\sigma_g^2 = \mathrm{w}_g'\Sigma w_g = \frac{1}{1'\Sigma^{-1}1}$ are the mean return and variance of the GMV portfolio and $\mu_{ew}$and σ_{ew}^2 are the mean return and the variance of the $1/N$ portfolio formulated in (15.1). Finally, the optimal combining weights are given by:

$$\widehat{w}_{\mathrm{LA}}^* = \frac{c_1^*}{\gamma}\widehat{\Sigma}^{-1}1 + \frac{c_2^*}{\gamma}w_{ew}. \tag{21}$$

2.3.4. Combination of the EW, the GMV and the MV portfolios (LMS)

Finally, Lassance et al. (2023) published a methodology for the shrinkage portfolio combining the sample MV portfolio, the sample GMV portfolio and the EW portfolios without the risk-free asset. The EW portfolio is included because it has been shown to outperform MV portfolios out-of-sample, as noted earlier by DeMiguel et al. (2009). Therefore, the three-fund shrinkage portfolio that combines the MV, EW and GMV portfolios is:

$$\widehat{w}^*(\pi,\kappa) = (1-\pi)w_{ew} + \pi\widehat{w}^*(\kappa) = (1-\pi)w_{ew} + \pi\left((1-\kappa)\widehat{w}_g + \kappa\widehat{w}^*\right),$$

where $\widehat{w}^* = \widehat{w}_{MV,c} = \widehat{w}_g + \frac{1}{\gamma}\,\widehat{w}_z$ from equation (7) and $\pi, \kappa \in [0,1]$. Then, the OOSU mean of the three-fund shrinkage portfolio $\widehat{w}^*(\pi,\kappa)$ is expressed as:

$$\begin{aligned}
&\mathbb{E}[U(\widehat{w}^*(\pi,\kappa))] \\
&\quad = (1-\pi)\mu_{ew} + \pi\left(\mu_{g^*} + \frac{\kappa}{\gamma}\frac{T}{T-N-1}\,\psi^2\right) \\
&\quad -\frac{\gamma}{2}\Bigg((1-\pi)^2\sigma_{ew}^2 \\
&\quad + \pi^2\left(\frac{T-2}{T-N-1}\sigma_g^2 + \frac{\kappa^2}{\gamma^2}\frac{T(T-2)(T\psi^2+N-1)}{(T-N)(T-N-1)(T-N-3)}\right) \\
&\quad + 2\pi(1-\pi)\left(\sigma_g^2 + \frac{\kappa}{\gamma}\frac{T}{T-N-1}(\mu_{ew}-\mu_{g^*})\right)\Bigg).
\end{aligned}$$

This expected OOSU depends on the following plug-in estimators of the distributional parameters of stock returns: $\mu_{g^*}, \sigma_g^2, \psi^2, \mu_{ew}$ and σ_{ew}^2. The mean μ_{ew} and variance σ_{ew}^2 of the EW portfolio are given by the formula (15.1). Then, the estimation of the return variance of the GMV portfolio (σ_g^2) is based on the shrinkage portfolio estimator proposed by Frahm and Memmel (2010, theorem 2), which results in a smaller mean out-of-sample variance than the GMV portfolio. For instance, σ_g^2 is estimated as:

$$\widehat{\sigma}_g^2 = \widehat{w}_{fm}'\widehat{\Sigma}\,\widehat{w}_{fm}\,,$$

where $\widehat{\mathrm{w}}_{\mathrm{fm}}$ combines the EW portfolio and the sample GMV portfolio such as:

$$\widehat{\mathrm{w}}_{\mathrm{fm}} = \widehat{\pi}_{\mathrm{fm}}\mathrm{w}_{\mathrm{ew}} + (1 - \widehat{\pi}_{\mathrm{fm}})\widehat{\mathrm{w}}_{\mathrm{g}},$$

with a shrinkage intensity:

$$\widehat{\pi}_{\mathrm{fm}} = \min\left(1, \frac{\mathrm{N} - 3}{\mathrm{T} - \mathrm{N} + 2} \; \frac{\widehat{\mathrm{w}}_{\mathrm{g}}'\widehat{\Sigma}\widehat{\mathrm{w}}_{\mathrm{g}}}{\mathrm{w}_{\mathrm{ew}}'\widehat{\Sigma}\mathrm{w}_{\mathrm{ew}} - \widehat{\mathrm{w}}_{\mathrm{g}}'\widehat{\Sigma}\widehat{\mathrm{w}}_{\mathrm{g}}}\right).$$

Then, ψ^2 is estimated using the adjusted estimator $\hat{\psi}_a^2$ introduced by Kan and Zhou (2007) and expressed in formula (18.3) while $\hat{\mu}_{ew}$ and $\hat{\sigma}_{ew}^2$ are the plug-in estimators of Tu and Zhou (2011) from (15.1). Finally, μ_g is estimated as the mean return of $\widehat{w}(\nu) = \nu w_{ew} + (1 - \nu)\widehat{w}_g$ which is the portfolio combining the EW and the GMV portfolios, avoiding estimation errors from the plug-in estimator of μ_g. In this portfolio, the shrinkage intensity ν is selected to minimize the mean squared error of the out-of-sample mean $\mathbb{E}\left[\left(\widehat{w}(\nu)'\mu - \mu_g\right)^2\right]$ and is obtained by:

$$\nu = \frac{\sigma_g^2\psi^2}{\sigma_g^2\psi^2 + \left(\mu_{ew} - \mu_g\right)(T - N - 1)}.$$

Therefore, μ_g is estimated as $\hat{\mu}_{g^*} = \widehat{w}(\hat{\nu})'\hat{\mu}$, with $\hat{\nu}$ can be computed this way:

$$\hat{\nu} = \frac{\hat{\sigma}_g^2\hat{\psi}_a^2}{\hat{\sigma}_g^2\hat{\psi}_a^2 + (\hat{\mu}_{ew} - \widehat{w}_g'\hat{\mu})(T - N - 1)}.$$

Finally, the optimal shrinkage intensities $(\hat{\pi}_E^*, \hat{\kappa}_E^*)$ are obtained numerically by maximizing the expected OOSU. To numerically find the optimal values of $\hat{\pi}_E^*$ and $\hat{\kappa}_E^*$, an objective function representing the negative expected OOSU is defined, which depends on π and κ. Initial guesses for π and κ are set at 0.5 each are constrained to remain between bonds 0 and 1. Using the Sequential Least Squares Quadratic Programming (SLSQP) method, the minimize function from the SciPy library is used to perform the optimization. This function iteratively adjusted the values of π and κ to minimize the negative OOSU, effectively maximizing the OOSU. Extracting the shrinkage intensities $(\hat{\pi}_E^*, \hat{\kappa}_E^*)$ from the minimize function allows to find the out-of-sample performance of the shrinkage portfolio $\widehat{w}^*(\pi, \kappa)$ which combines the GMV, MV and EW portfolios. The optimized combining weights for this three-fund strategy is expressed as:

$$\begin{aligned} \widehat{w}_{LMS}^* &= \widehat{w}^*(\pi, \kappa) = (1 - \hat{\pi}_E^*)w_{ew} + \hat{\pi}_E^*\widehat{w}^*(\hat{\kappa}_E^*) \\ &= (1 - \hat{\pi}_E^*)w_{ew} + \hat{\pi}_E^*\left((1 - \hat{\kappa}_E^*)\widehat{w}_g + \hat{\kappa}_E^*\widehat{w}^*\right). \end{aligned} \tag{22}$$

Chapter 3: Methodology of empirical part in Python

This chapter aims to develop a methodology for empirically analysing the theory from the first two chapters. It outlines steps to compute single portfolio rules and their combinations, using parameters from existing literature for comparative purposes.

3.1. Research questions

The research questions focus on robust portfolio optimization, specifically comparing past performance with future performance to determine if combining portfolios enhances overall performance. To answer fully to this question, the study is guided by two key questions:

1) *Question 1*: Is a combination of portfolios better than an individual portfolio in terms of annualized SR and annualized utility?
2) *Question 2*: Which individual or combined portfolio strategy works best overall?

These questions fundamentally shape and direct the methodology of the study. Each question requires specific analytical approaches to ensure comprehensive answers.

3.2. Data Collection and preprocessing

The data for U.S. returns was obtained from Kenneth French's website (French, 2024). The selected datasets included the 10 industry portfolios (IP10), the 48 industry portfolios (IP48), consisting of returns from 10 and 48 different industries, including manufacturing, high technology, energy, health, retail, and more. Additionally, the 6 portfolios formed on size and book-to-market (SBTM6), the 25 portfolios formed on size and book-to-market (SBTM25), and the 100 portfolios formed on size and book-to-market (SBTM100) were selected. The SBTM portfolios are categorized based on company size (market capitalization) and the book-to-market ratio, a valuation measure that compares a company's book value with its market value. The acronyms into parenthesis are the ones that will be used in the code. Then, monthly value weighted returns were extracted from the respective CSV files. Additionally, the risk-free rate was retrieved from the Fama/French three-factor dataset, specifically from the "RF" column. To calculate the excess returns, the risk-free rate was subtracted from the selected portfolio returns. To avoid missing data across all portfolios, represented by the value "−99.99" in the files, the starting period was adjusted to run from January 1970 to December 2023.

3.3. Portfolio Weight Computation

The computation of portfolio weights involves both in-sample and out-of-sample returns. The in-sample period spans from January 1970 to December 2023. Using in-sample data from a rolling-window of size $T = 120$ months, weights $\hat{w}_t$ are estimated and then multiplied by next monthly excess returns R_{t+1} to generate one monthly out-of-sample return for each portfolio strategy. The window is then rolled by one month and the process is repeated iteratively, giving at the end a series of out-of-sample portfolio returns on which the performance is computed. The same process is applied with a 180-month window, following Kan and Zhou (2007), to evaluate the impact of larger estimation windows. Finally, in addition to the results obtained using the sample covariance matrix for $T = 120$ and $T = 180$, computations are also realized using the linear shrinkage covariance matrix (Ledoit & Wolf, 2004b).

3.4. Portfolio Strategies

Table 1: List of portfolio strategies used in the empirical part

Name	**Funds**	**Description**
EW	EW	Equally weighted strategy. See Section 1.3.
GMV	GMV	Global Minimum Variance strategy. See Section 1.1, equation (9).
MSR	MSR	Maximum Sharpe Ratio strategy. See Section 1.1, equation (5).
Mvunc	Mvunc	Unconstrained Mean-Variance strategy. See Section 1.1, equation (3).
MVc	MVc	Constrained Mean-Variance strategy. See Section 1.1, equation (7).
EWRF	EW-Rf	C1 - Combination of the EW portfolio and the risk-free asset. See subpart 2.2.1.
GMVRF	GMV-Rf	C2 - Combination of the GMV portfolio and the risk-free asset. See subpart 2.2.2.
KZ2F	MVunc-Rf	C3 - Combination of the MVunc portfolio and the risk-free asset. See subpart 2.2.3.
KWZ	MVc-GMV	C4 - Combination of the GMV and the MVc portfolios. See subpart 2.2.4.
KZ3F	Mvunc-GMV-Rf	C5 - Combination of the unconstrained MV and the GMV portfolios with the risk-free asset. See subpart 2.3.1.
TZ	Mvunc-EW-Rf	C6 - Combination of the MVunc and EW portfolios with the risk-free asset. See subpart 2.3.2.
LA	GMV-EW-Rf	C7 - Combination of the GMV and EW portfolios with the risk-free asset. See subpart 2.3.3.
LMS	MVc-GMV-EW	C8 - Combination of the MVunc, GMV and EW portfolios. See subpart 2.3.4.

Note: this table lists the portfolio strategies described in "Chapter 2: Portfolio Combinations" of the literature part and used in the empirical part.

The portfolio strategies outlined in "Chapter 2: Portfolio Combinations" in the literature review have been applied in the empirical analysis. The empirical section examined five individual portfolio strategies and eight combinations of these strategies, as detailed in the Table 1. The objective of the analysis was to determine the optimal weights for various individual portfolio strategies, including w_{ew}, w_{MSR}, w_g, $w_{MV,unc}$ and $w_{MV,c}$. Additionally, two-fund portfolio combinations such as w^*_{GRF}, $w^*_{EWRF,}$ $\widehat{w}^*_{KZ2F}$ and $\widehat{w}^*_{KWZ}$ and three-fund portfolio combinations including $\widehat{w}^*_{KZ3F}$, $\widehat{w}^*_{TZ}$, $\widehat{w}^*_{LA}$, and $\widehat{w}^*_{LMS}$ were calculated. The goal of using different portfolio strategies is to analyse which one performs the best in-sample, out-of-sample and in the context of transaction costs. Typically, combining multiple portfolio rules should leverage the strengths of each individual strategy, leading to better overall performance.

3.5. Performance Evaluation

To assess and compare the performance of the portfolio strategies mentioned in the previous paragraph, performance criteria have been selected. Firstly, performance evaluation includes the consideration of a risk aversion parameter, evaluated at $\gamma = 1$ and $\gamma = 3$ used in the literature by Tu and Zhou (2011) and Lassance, Vanderveken and Vrins (2024). Transaction costs were considered at $c = 20$ basis points as in Kan, Wang and Zhou (2022) with turnover and performance net of transaction costs calculated accordingly. The performance metrics included annualized standard deviation $\hat{\sigma}_p * \sqrt{12}$ and the annualized mean $12 * \hat{\mu}_p$ useful to determinate the annualized SR $\frac{12*\hat{\mu}_p}{\hat{\sigma}_p*\sqrt{12}}$. To compare the performance of mean-variance portfolios that deliver the same SR, the annualized utility function $U_p = 12 * \left(\hat{\mu}_p - \frac{\gamma}{2}\hat{\sigma}_p^2\right)$ can be used as another performance criterion. This function identifies portfolios with optimal risk-return trade-offs based on investor preferences. In this thesis, it is the most important objective function for performance analysis of strategies, as it is the one that portfolio combinations maximize. These metrics were applied both in-sample and out-of-sample to evaluate the impact of estimation errors and the robustness of the model on unseen data. Then, excess portfolio returns were also compared to the portfolio returns net of transaction costs to evaluate the impact of transaction costs on performance. The comparative analysis ensured that both in-sample and out-of-sample returns, represented by IS and OOS in Datasets period in Table 2, were computed for the same period (1980 to 2023 for $T = 120$ and 1985 to 2023 for $T = 180$), to isolate the effect of estimation errors and impacts from the market trends. Finally, the parameters and metrics used to compute this analysis are summarized in the following table.

Table 2 : Data and parameters considered for the empirical method

Datasets	10 industry portfolios: IP10
	48 industry portfolios: IP48
	6 portfolios formed on size and book-to-market: SBTM6
	25 portfolios formed on size and book-to-market: SBTM25
	100 portfolios formed on size and book-to-market: SBTM100
Datasets period	January 1970 to December 2023 IS and OOS returns from Jan. 1980 - Dec. 2023 when $T = 120$ IS and OOS returns from Jan. 1985 - Dec. 2023 when $T = 180$
Frequence	Monthly
Transaction costs	$c = 20$ basis points
Window length T	120 & 180
Risk aversion parameter γ	1 & 3
Single portfolio rules	w_{ew}, w_{MSR}, w_g, $w_{MV,unc}$ and $w_{MV,c}$
Two-fund portfolio rules	w^*_{EWRF},$w^*_{GMVRF,}$ $\widehat{w}^*_{KZ2F}$ and $\widehat{w}^*_{KWZ}$
Three-fund portfolio rules	$\widehat{w}^*_{KZ3F}$, $\widehat{w}^*_{TZ}$, $\widehat{w}^*_{LA}$, and $\widehat{w}^*_{LMS}$
Performance criteria	Annualized $\hat{\mu}_p$, annualized $\hat{\sigma}_p$, annualized SR, annualized utility with $\gamma = 1$ *and* $\gamma = 3$, and turnover.

To conclude, following a series of meticulously steps, including data collection, portfolio weight computation, and the assessment of various portfolio strategies, ensures a robust methodological framework. Parameters like those used in existing literature were selected to facilitate meaningful comparisons, and performance metrics were carefully chosen to evaluate both in-sample and out-of-sample results. The objective was to understand the impact of estimation errors, transaction costs, risk aversion, and the length of time series T on portfolio strategies, to identify the most robust strategy based on these parameters.

3.6. Monte Carlo Simulations

Monte Carlo simulations provide a robust alternative to traditional rolling window forecasting methods, which often suffer from uncertainties and overfitting. This method uses historical return data to introduce random variables and average worst and best simulated outcomes to generate expected outcome. By altering historical patterns, it provides a comprehensive risk assessment in financial forecasting. However, expected outcomes cannot be guaranteed in real life as this method assumes efficient markets (Kenton, 2024).

In this context, Monte Carlo simulations aim to estimate the expected OOSU of a portfolio using simulated returns, helping to assess how portfolios constructed from sample estimates ($\hat{\mu}$ and $\hat{\Sigma}$) would perform with true population parameters. The methodology for conducting simulations and evaluate the expected OOSU of portfolio rules involves several steps. First, for a given dataset (e.g., SBTM6), the population parameters (μ and Σ) are calculated from monthly excess returns over the entire available period (January 1970 to December 2023). The theoretical assumptions for Monte Carlo simulations, which are used to compute the expected OOSU, are detailed in subpart 2.1 of "Chapter 2: Portfolio Combinations". In these simulations, returns are assumed to be i.i.d. to avoid correlated returns and to follow a normal distribution with a specified population mean and variance. However, the assumptions are said to be simplified as assets are often correlated during stress periods, and transaction costs are relevant in real-world scenarios. Then, the following steps are repeated iteratively for each simulation. The first step is to compute $10{,}000$ simulations of T monthly returns ($T = 120$) for each of the N assets. Subsequently, sample estimates of μ and Σ are computed based on the simulated sample of size T, and are then used to determine the optimal portfolio weights for each strategy. Finally, the OOSU of the estimated portfolio is calculated by plugging the true population parameters in Equation (2). The expected OOSU derived by Kan and Zhou (2007) averages the OOSUs obtained from all simulations:

$$EU[w] = \frac{1}{M} \sum_{m=1}^{M} \widehat{w}_m^{*\prime} \mu - \frac{\gamma}{2} \widehat{w}_m^{*\prime} \Sigma \widehat{w}_m^{*},$$

using optimal portfolio weights of each individual and combined strategies (Lassance, Vanderveken & Vrins, 2024). This provides an estimate of the expected out-of-sample performance an investor can achieve on average under parameter uncertainty for each portfolio strategy. These steps are conducted for both $\gamma = 1$ and $\gamma = 3$, under $T = 120$, and also under $T = 180$ for all datasets. The aim is to offer insights into the robustness and reliability of different portfolio construction strategies in the face of parameter uncertainty.

Second part of body text: Results analysis and answers to the research questions

The second part of the body text consists of two chapters. The first chapter focuses on describing the results and assessing their coherence with existing literature. Thanks to these empirical findings, the second chapter addresses two research questions and draws conclusions about the best individual and combined portfolio strategies.

Chapter 1: Performance evaluation of portfolio strategies based on evaluation criteria

This chapter examines the results of applying single and combined portfolio strategies (see Table 1) to the five datasets shown in Table 2, excluding SBTM25 to avoid redundancy as its results fall between SBTM6 and SBTM100. The analysis focuses on in-sample and out-of-sample performance for $T = 120$ and $T = 180$, highlighting the impact of longer time series on estimations. Comparing the sample covariance matrix and the linear shrinkage covariance matrix shows the positive impact of shrinkage on out-of-sample performance. However, to shorten the analysis, the impact of the shrinkage covariance matrix under $T = 120$ is not detailed further as this is already covered for $T = 180$. Finally, evaluation criteria will be employed to assess their effectiveness, including the annualized mean, standard deviation, SR, utility for $\gamma = 1$ and $\gamma = 3$, and the turnover, as listed in Table 2. The goal of this evaluation is to identify the most effective strategies and provide insights into their relative performance across different scenarios.

1.1 In-sample results

The in-sample analysis examines the five single strategies to assess their performance before combining them into portfolio rules. This subpart excludes portfolio combinations, which are intended to maximize expected OOSU rather than in-sample performance. To clarify results presented in Tables, figures highlighted in red indicate lower values, while those in green denote higher values for each performance metric. Furthermore, results including the SBTM25 and the linear shrinkage covariance matrix are available in Table 8 of Appendix B.

The results presented in Table 3 are derived from the in-sample performance of the five single strategies. The first observation is that in-sample results differ whereas the rolling-window size, only used to compute out-of-sample return, should not impact in-sample performance. In this case, it is because the starting data varies (1980 for $T = 120$ and 1985 for $T = 180$) to maintain the same number of monthly returns for both in-sample and out-of-sample periods. Moreover, the linear shrinkage covariance matrix reduces the standard deviation as the shrinkage estimator aims to reduce noise, especially in high-dimensional matrices like the SBTM100 under MV strategies. Overall, the three different results ($T = 120$, $T = 180$ and $T = 180$ with linear shrinkage covariance matrix), yield similar values, leading to the same conclusions.

Table 3: Annualized in-sample performance of single strategies with T=120 & T=180

			T=120 (Sample CM*)					T=180 (Sample CM)				T=180 (Linear Shrinkage CM)			
	Portfolios		SBTM6	SBTM25	SBTM100	IP10	IP48	SBTM6	SBTM100	IP10	IP48	SBTM6	SBTM100	IP10	IP48
Mean		**EW**	0,09	0,10	0,07	0,09	0,09	0,10	0,07	0,09	0,09	0,10	0,07	0,09	0,09
		MSR	0,22	0,39	0,90	0,10	0,28	0,22	0,93	0,10	0,31	0,20	0,88	0,10	0,29
		GMV	0,12	0,12	0,12	0,08	0,07	0,12	0,12	0,09	0,07	0,12	0,12	0,09	0,07
	γ = 1	**Mvunc**	1,12	2,62	7,98	0,48	1,39	1,08	8,25	0,48	1,50	0,95	7,41	0,48	1,46
		MVc	0,65	1,92	7,02	0,17	1,11	0,59	7,27	0,16	1,21	0,51	6,49	0,16	1,16
	γ = 3	**Mvunc**	0,37	0,87	2,66	0,16	0,46	0,36	2,75	0,16	0,50	0,32	2,47	0,16	0,49
		MVc	0,29	0,72	2,42	0,11	0,42	0,28	2,51	0,11	0,45	0,25	2,25	0,11	0,44
Std		**EW**	0,17	0,18	0,18	0,15	0,17	0,17	0,18	0,15	0,17	0,17	0,18	0,15	0,17
		MSR	0,20	0,22	0,30	0,13	0,24	0,19	0,31	0,13	0,25	0,19	0,30	0,13	0,23
		GMV	0,14	0,12	0,10	0,12	0,11	0,14	0,11	0,12	0,11	0,14	0,11	0,12	0,11
	γ = 1	**Mvunc**	0,98	1,48	2,71	0,62	1,20	0,97	2,76	0,63	1,20	0,88	2,52	0,62	1,17
		MVc	0,71	1,27	2,59	0,29	1,08	0,67	2,64	0,30	1,09	0,59	2,38	0,28	1,04
	γ = 3	**MVunc**	0,33	0,49	0,90	0,21	0,40	0,32	0,92	0,21	0,40	0,29	0,84	0,21	0,39
		MVc	0,27	0,43	0,87	0,15	0,37	0,26	0,88	0,15	0,38	0,24	0,80	0,15	0,36
SR		**EW**	0,54	0,54	0,39	0,61	0,54	0,55	0,38	0,63	0,57	0,55	0,38	0,63	0,57
		MSR	1,15	1,76	2,94	0,77	1,16	1,11	2,99	0,77	1,24	1,08	2,94	0,77	1,25
		GMV	0,85	0,99	1,15	0,70	0,66	0,87	1,17	0,70	0,66	0,84	1,17	0,70	0,68
	γ = 1	**Mvunc**	1,15	1,76	2,94	0,77	1,16	1,11	2,99	0,77	1,24	1,08	2,94	0,77	1,25
		MVc	0,91	1,51	2,71	0,57	1,03	0,87	2,76	0,55	1,12	0,86	2,72	0,57	1,12
	γ = 3	**Mvunc**	1,15	1,76	2,94	0,77	1,16	1,11	2,99	0,77	1,24	1,08	2,94	0,77	1,25
		MVc	1,09	1,66	2,80	0,76	1,12	1,06	2,84	0,75	1,20	1,05	2,82	0,76	1,21
Utility γ=1		**EW**	0,08	0,08	0,06	0,08	0,08	0,08	0,05	0,08	0,08	0,08	0,05	0,08	0,08
		MSR	0,20	0,36	0,85	0,09	0,25	0,20	0,88	0,09	0,27	0,18	0,84	0,09	0,26
		GMV	0,11	0,11	0,12	0,08	0,07	0,11	0,12	0,08	0,07	0,11	0,12	0,08	0,07
		Mvunc	0,64	1,52	4,30	0,29	0,67	0,61	4,44	0,28	0,77	0,56	4,24	0,29	0,77
		MVc	0,39	1,12	3,66	0,13	0,53	0,36	3,79	0,12	0,62	0,33	3,65	0,12	0,62
Utility γ=3		**EW**	0,05	0,05	0,02	0,06	0,05	0,05	0,02	0,06	0,05	0,05	0,02	0,06	0,05
		MSR	0,17	0,31	0,76	0,08	0,19	0,16	0,78	0,08	0,21	0,15	0,75	0,08	0,21
		GMV	0,09	0,10	0,10	0,06	0,05	0,09	0,11	0,06	0,05	0,09	0,11	0,06	0,06
		Mvunc	0,21	0,51	1,43	0,10	0,22	0,20	1,48	0,09	0,26	0,19	1,41	0,10	0,26
		MVc	0,19	0,44	1,30	0,08	0,21	0,17	1,34	0,08	0,24	0,16	1,29	0,08	0,24

*CM stands for "Covariance Matrix". The Linear Shrinkage Covariance Matrix is from Ledoit and Wolf (2004b).

Firstly, the EW portfolio exhibits the lowest mean return and an average low standard deviation of 0.10, resulting in the smallest SR ranging between 0.38 and 0.63 depending on the dataset. It also has the lowest utility values, ranging from 0.02 to 0.08 for both levels of risk aversion ($\gamma = 1,\ \gamma = 3$). It is notable that strategies using $\gamma = 3$ involve taking less risk than those using $\gamma = 1$, leading to smaller performance differences in between all the rules. Secondly, the GMV rule demonstrated the lowest standard deviation, ranging from 0.10 for the SBTM100 to 0.14 for the SBTM6 dataset. This result was expected since its goal is to achieve minimum variance. Following the EW portfolio, it has the second-lowest mean return on average, resulting in the second-worst performance in terms of SR, standing between 0.66 to 1.17, while utilities are ranging from 0.05 to 0.12. Finally, the MVc strategy has the lowest SR among MV rules, ranging between 0.76 for IP10 to 2.80 for SBTM100 when $\gamma = 3$ when $T = 120$. This is because this strategy does not include the risk-free asset and is subject to budget constraints compared to the two other rules (MVunc, MSR). Even though the MVunc strategy with $\gamma = 1$ and $\gamma = 3$ has more extreme values for mean and standard deviation compared to the MSR rule, they exhibit the same SR. This is because portfolios on the efficient frontier all have the maximum SR but differ in means and volatilities and MSR depend on budget constraint. The SR's lowest values are obtained by the IP10 dataset, while its highest values are obtained by the SBTM100, reaching 2.94 for $T = 120$. Finally, on top of having the highest SR, the MVunc strategy also demonstrates the highest utilities for both $\gamma = 1$ and $\gamma = 3$. This can be attributed to its lack of budget constraints, allowing for significant changes in the portfolio weights. Therefore, it is the best strategy based on historical data, with its superior performance more evident in the SBTM datasets as the number of assets increases.

1.2 Out-of-sample results

As outlined in Section 1.2.1, the primary focus for assessing portfolio robustness is to see if it delivers strong out-of-sample performance on new data rather than considering its in-sample performance. Therefore, this analysis prioritizes evaluating the net out-of-sample performance of single and combined strategies using a rolling window of 120 and 180 months. For further analysis, Tables 13-17 in Appendix B include the net out-of-sample performance for the SBTM25 dataset and the linear shrinkage covariance matrix.

Table 4: Turnover with T=120 and T=180

			T=120 (Sample CM*)				T=180 (Sample CM)				T=180 (Linear Shrinkage CM)			
	Portfolios		SBTM6	SBTM100	IP10	IP48	SBTM6	SBTM100	IP10	IP48	SBTM6	SBTM100	IP10	IP48
Turnover		EW	0,02	0,03	0,03	0,05	0,02	0,03	0,03	0,05	0,02	0,03	0,03	0,05
		MSR	5,95	514,61	0,97	170,66	0,49	12,60	0,49	4,94	0,32	2,15	0,37	1,94
		GMV	0,18	6,77	0,14	0,74	0,12	1,79	0,09	0,40	0,06	0,52	0,07	0,24
	γ = 1	Mvunc	5,61	2206,08	4,94	42,79	3,51	210,12	3,03	18,36	1,63	33,82	2,32	11,13
		MVc	5,11	1824,65	4,64	40,75	3,13	179,47	2,90	17,96	1,52	31,59	2,21	10,93
		EWRF	0,21	0,20	0,25	0,21	0,13	0,12	0,16	0,14	0,14	0,13	0,16	0,15
		GMVRF	2,07	706,07	1,34	8,21	1,30	54,59	0,75	2,96	0,56	6,74	0,58	1,77
		KZ2F	5,21	1407,20	4,02	34,32	3,23	117,57	2,25	13,83	1,46	16,45	1,68	8,19
		KWZ	4,70	1161,42	3,53	27,93	2,80	98,28	1,74	9,46	1,30	14,60	1,17	4,94
		KZ3F	5,14	1409,07	3,77	29,63	3,14	117,54	1,87	9,85	1,39	16,20	1,29	5,19
		TZ	5,60	1780,30	4,93	38,78	3,51	159,15	3,02	16,05	1,63	24,13	2,31	9,59
		LA	3,04	721,77	1,49	8,11	1,94	57,85	0,88	2,75	0,85	7,47	0,70	1,63
		LMS	1,57	387,05	1,19	9,33	0,94	32,92	0,59	3,17	0,44	13,62	0,42	2,17
	γ = 3	Mvunc	1,87	735,36	1,65	14,26	1,17	70,04	1,01	6,12	0,54	11,27	0,77	3,71
		MVc	1,71	608,02	1,55	13,59	1,05	59,94	0,97	5,99	0,52	30,45	0,80	4,72
		EWRF	0,07	0,07	0,08	0,07	0,04	0,04	0,05	0,05	0,05	0,04	0,05	0,05
		GMVRF	0,69	235,36	0,45	2,74	0,43	18,20	0,25	0,99	0,19	2,25	0,19	0,59
		KZ2F	1,74	469,07	1,34	9,89	1,08	39,19	0,75	3,36	0,49	5,48	0,56	1,82
		KWZ	1,58	386,96	1,18	9,33	0,94	32,90	0,59	3,17	0,44	4,90	0,40	1,66
		KZ3F	1,71	469,69	1,26	9,88	1,05	39,18	0,62	3,28	0,46	5,40	0,43	1,73
		TZ	1,87	633,24	1,65	14,00	1,17	58,13	1,01	5,98	0,54	9,09	0,77	3,62
		LA	1,01	240,59	0,50	2,70	0,65	19,28	0,29	0,92	0,28	2,49	0,23	0,54
		LMS	1,58	386,99	1,18	9,33	0,96	32,91	0,60	3,17	0,45	13,62	0,43	2,17

*CM stands for "Covariance Matrix". The Linear Shrinkage Covariance Matrix is from Ledoit and Wolf (2004b).

The first point of the out-of-sample analysis focuses on turnover, a key factor in transitioning from gross to net out-of-sample performance, representing the cost of rebalancing portfolio weights. Turnover results are presented in Table 4 for single and combined strategies across four datasets, comparing two rolling window lengths ($T = 120, 180$) and examining the impact of the Ledoit and Wolf (2004b) linear shrinkage covariance matrix for $T = 180$.

Starting with $T = 120$, the EW strategy has the lowest turnover, peaking at 0.05 for the IP48 dataset. The weights $w_{ew} = 1/N$ (see Section 1.3) do not require parameter estimation, avoiding estimation errors and ensuring consistent performance across in-sample and out-of-sample periods. The EWRF strategy follows with a maximum turnover of 0.25 for $\gamma = 1$. The GMV portfolio, unaffected by expected return estimation errors, shows the third lowest turnover, ranging from 0.14 to 6.77. Although turnover values for $\gamma = 1$ are higher than for $\gamma = 3$, the overall conclusions remain the same, aligning with low out-of-sample standard deviation. Subsequently, LMS, GMVRF, and LA rules, which incorporate GMV or EW strategies, exhibit low turnover for $\gamma = 1$. These findings are consistent with Lassance, Vanderveken and Vrins (2024), who found that EWRF and GMVRF have the lowest turnover for $\gamma = 1$ and $T = 120$. The EWRF strategy also reported turnover values of 0.17 for SBTM25 and 0.21 for IP48 for $\gamma = 1$. Incorporating the shrinkage covariance matrix generally improves results, except for the LSM rule, which excels due to its reliance on the weakest strategy, MVc.

Conversely, the strategies with the highest turnover are the MVunc and MVc, with values of 2206 and 1824 for $\gamma = 1$, and 735 and 608 for $\gamma = 3$, respectively. The MVunc rule requires higher rebalancing because it uses the sample estimates of μ and Σ without any adjustment or constraint, resulting in extremely volatile weights that fluctuate significantly over time. Finally, the MSR portfolio obtained a lower turnover due to its more stable weights as it enforces the budget constraint and includes the risk-free rate. The results appear consistent as the dataset with the highest volatility in-sample (see Table 3) and gross out-of-sample (see Table 10 of Appendix B), namely SBTM100 under $\gamma = 1$, has the highest turnover. It is due to the large $N = 100$ relative to T, leading to significant estimation risk and extreme weights. This supports Best and Grauer's (1991) finding that a 100-asset portfolio is more sensitive to changes in portfolio weights than a 10-asset one. However, using a longer forecasting period ($T = 180$), turnover values significantly decrease. Indeed, a long time series incorporating more observations allows to estimate expected returns more precisely by reducing impact from less relevant data (De Miguel et al, 2007). For instance, for the MVunc rule drops from 2206 to 210. Introducing a shrinkage covariance matrix further reduces turnover, with the maximum decreasing to 33 for the MVunc when $\gamma = 1$, due to noise reduction and portfolio stability provided by shrinkage estimators.

Table 5: Annualized net out-of-sample SR with T=120 and T=180

			T=120 (Sample CM*)				T=180 (Sample CM)				T=180 (Linear Shrinkage CM)			
		Portfolios	SBTM6	SBTM100	IP10	IP48	SBTM6	SBTM100	IP10	IP48	SBTM6	SBTM100	IP10	IP48
SR		EW	0,52	0,51	0,60	0,53	0,53	0,51	0,61	0,55	0,53	0,51	0,61	0,55
		MSR	0,28	0,09	0,17	0,17	0,87	0,09	0,28	-0,08	0,82	1,56	0,33	0,05
		GMV	0,94	-0,11	0,63	0,26	0,97	0,60	0,69	0,38	0,85	0,95	0,69	0,53
	γ = 1	Mvunc	0,88	-0,56	0,20	0,06	0,86	0,46	0,24	0,06	0,81	1,56	0,28	0,10
		MVc	0,55	-0,37	-0,08	-0,01	0,46	0,35	-0,16	-0,04	0,47	1,27	-0,14	-0,05
		EWRF	0,34	0,16	0,41	0,35	0,46	0,26	0,53	0,47	0,46	0,25	0,52	0,47
		GMVRF	0,79	-0,48	0,47	0,14	0,90	0,46	0,60	0,31	0,77	0,89	0,60	0,42
		KZ2F	0,87	-0,61	0,19	0,07	0,85	0,62	0,19	0,06	0,80	1,56	0,23	0,10
		KWZ	0,55	-0,42	-0,09	0,01	0,47	0,42	-0,12	-0,02	0,48	1,30	-0,07	-0,02
		KZ3F	0,88	-0,61	0,25	0,09	0,88	0,62	0,38	0,11	0,82	1,57	0,43	0,20
		TZ	0,88	-0,58	0,20	0,07	0,86	0,56	0,24	0,06	0,81	1,57	0,28	0,10
		LA	0,74	-0,46	0,42	0,16	0,85	0,44	0,58	0,36	0,77	0,94	0,58	0,44
		LMS	0,74	-0,76	0,09	0,04	0,69	0,49	0,17	0,04	0,69	1,13	0,20	0,05
	γ = 3	Mvunc	0,88	-0,73	0,20	0,07	0,86	0,67	0,24	0,06	0,81	1,57	0,28	0,10
		MVc	0,72	-0,60	0,05	0,02	0,66	0,46	0,03	0,00	0,62	1,00	0,00	0,00
		EWRF	0,34	0,16	0,41	0,35	0,46	0,26	0,53	0,47	0,46	0,25	0,52	0,47
		GMVRF	0,79	-0,62	0,47	0,15	0,90	0,49	0,60	0,31	0,77	0,89	0,60	0,42
		KZ2F	0,87	-0,84	0,19	0,08	0,85	0,69	0,19	0,06	0,80	1,56	0,23	0,08
		KWZ	0,74	-0,76	0,09	0,04	0,69	0,49	0,17	0,04	0,76	1,42	0,26	0,10
		KZ3F	0,88	-0,84	0,25	0,10	0,88	0,70	0,38	0,11	0,82	1,57	0,43	0,20
		TZ	0,88	-0,77	0,20	0,08	0,86	0,68	0,24	0,07	0,81	1,57	0,28	0,10
		LA	0,74	-0,63	0,42	0,16	0,85	0,48	0,58	0,36	0,77	0,94	0,58	0,44
		LMS	0,74	-0,76	0,10	0,04	0,68	0,49	0,17	0,04	0,70	1,13	0,19	0,05

*CM stands for "Covariance Matrix". The Linear Shrinkage Covariance Matrix is from Ledoit and Wolf (2004b).

The annualized SR in Table 5 is calculated using the mean and standard deviation from Tables 14-15 in Appendix B. The gross out-of-sample performance (Table 11, Appendix B) shows that the best strategies for $T = 120$ are GMV, KZ3F, and TZ/MVunc/KZ2F with average SRs of 0.72, 0.66, and 0.64, respectively, for $\gamma = 1$. Conversely, the MSR strategy is the worst, with an average SR of 0.21 across four datasets. EWRF, MVc, and KWZ strategies follow with values between 0.33 to 0.43. Considering transaction costs for $T = 120$, the EW strategy emerges as the best with an average SR of 0.54, despite its in-sample performance. The GMV portfolio outperforms the EW rule for SBTM6, SBTM25, and IP10 with SRs of 0.94, 0.89, and 0.63, respectively, for $T = 120$. Michaud's (1989) error maximization problem supports these results, suggesting that since errors in means are significant, maintaining them constant improves performance in real-world scenarios. It also aligns with De Miguel et al. (2009), stating that the EW rule strongly outperforms MSR and MV rules in terms of SR and turnover due to estimation errors. These rules only outperform the EW rule with a long estimation window and a small number of assets. Overall, underperforming strategies include the MVc, KWZ and LMS rules, both including the MVc.

Using the shrinkage covariance matrix, MVc, KWZ, and EWRF strategies still show low average values for both risk aversions. However, the KZ3F rule improves, becoming one of the best alongside GMV, with values ranging from 0.2 to 1.57 for KZ3F and 0.53 to 0.95 for GMV. This improvement is logical as KZ3F combines the risk-free asset with the MVunc and GMV rules, both of which perform well. These rules are closely followed by the MSR, MVunc, TZ and LA with values averaging 0.69.

Table 6: Annualized net out-of-sample utility with T=120 and T=180

		T=120 (Sample CM*)				T=180 (Sample CM)				T=180 (Linear Shrinkage CM)			
	Portfolios	**SBTM6**	**SBTM100**	**IP10**	**IP48**	**SBTM6**	**SBTM100**	**IP10**	**IP48**	**SBTM6**	**SBTM100**	**IP10**	**IP48**
Utility γ=1	**EW**	0,07	0,08	0,08	0,07	0,08	0,07	0,08	0,08	0,08	0,07	0,08	0,08
	MSR	0,01	-9919	0,01	-4376	0,16	-0,15	0,03	-0,24	0,13	0,36	0,04	-0,06
	GMV	0,12	-0,06	0,07	0,03	0,13	0,08	0,08	0,04	0,11	0,11	0,08	0,06
	Mvunc	-0,03	-126296	-0,91	-13,42	0,05	-49	-0,44	-5,22	0,28	-3,49	-0,28	-2,96
	MVc	-0,08	-63126	-0,80	-12,35	-0,10	-39,54	-0,57	-5,16	0,08	-2,96	-0,41	-2,95
	EWRF	0,01	-0,06	0,04	0,03	0,11	0,03	0,14	0,11	0,10	0,02	0,14	0,11
	GMVRF	0,21	-1379	0,00	-0,99	0,38	-7,47	0,16	-0,09	0,29	0,17	0,16	0,03
	KZ2F	0,09	-18802	-0,58	-8,95	0,16	-20,16	-0,21	-3,03	0,31	0,94	-0,12	-1,61
	KWZ	0,01	-9052	-0,45	-6,17	0,00	-15,18	-0,21	-1,62	0,12	0,69	-0,12	-0,67
	KZ3F	0,10	-18621	-0,51	-6,84	0,20	-20,23	-0,08	-1,61	0,32	0,92	0,00	-0,63
	TZ	-0,03	-50965	-0,90	-11,24	0,05	-32,73	-0,43	-4,01	0,28	-0,38	-0,28	-2,18
	LA	0,06	-1517	-0,08	-0,96	0,26	-7,85	0,15	-0,05	0,29	0,28	0,15	0,03
	LMS	0,21	-69,67	-0,02	-0,71	0,18	-1,48	0,01	-0,16	0,13	-219	0,02	-0,22
Utility γ=3	**EW**	0,05	0,04	0,06	0,05	0,05	0,04	0,06	0,05	0,05	0,04	0,06	0,05
	MSR	-0,25	-29784	-0,05	-13159	0,12	-0,58	0,00	-0,63	0,10	0,30	0,01	-0,20
	GMV	0,10	-0,13	0,06	0,01	0,11	0,06	0,06	0,02	0,09	0,10	0,06	0,04
	Mvunc	-0,01	-3011	-0,31	-5,13	0,01	-29,16	-0,15	-1,84	0,09	-1,37	-0,09	-1,02
	MVc	0,03	-1380	-0,24	-4,65	0,02	-21,62	-0,15	-1,78	0,05	-3262	-0,21	-3,19
	EWRF	0,00	-0,02	0,01	0,01	0,04	0,01	0,05	0,04	0,03	0,01	0,05	0,04
	GMVRF	0,07	-58,99	0,00	-0,35	0,13	-2,99	0,05	-0,03	0,10	0,05	0,05	0,01
	KZ2F	0,03	-413,68	-0,20	-2,51	0,05	-9,59	-0,07	-0,58	0,10	0,30	-0,04	-0,25
	KWZ	0,06	-192,42	-0,12	-2,23	0,06	-6,67	-0,03	-0,53	0,09	0,30	0,00	-0,20
	KZ3F	0,03	-409,52	-0,17	-2,52	0,07	-9,63	-0,03	-0,55	0,11	0,30	0,00	-0,21
	TZ	-0,01	-1478	-0,31	-4,95	0,01	-20,62	-0,15	-1,75	0,09	-0,51	-0,09	-0,96
	LA	0,02	-61,10	-0,03	-0,34	0,09	-3,17	0,05	-0,02	0,10	0,09	0,05	0,01
	LMS	0,06	-192,51	-0,12	-2,23	0,06	-6,67	-0,03	-0,53	0,08	-707	-0,03	-0,73

*CM stands for "Covariance Matrix". The Linear Shrinkage Covariance Matrix is from Ledoit and Wolf (2004).

The net annualized OOSU reflects investor satisfaction based on their risk tolerance. Investors with a higher risk appetite opt for a low risk aversion parameter ($\gamma = 1$). The gross annualized OOSU in Table 12 of Appendix B supports the conclusions below and won't be detailed further, and average utility values from Table 6 are in Table 18 in Appendix.

Like for the SR, the EW strategy has positive utility values across all datasets and risk aversion parameters, averaging 0.08 for $\gamma = 1$ and 0.05 for $\gamma = 3$. For $T = 120$, the GMV rule follows with an average of 0.04, and EWRF ranks third. For $\gamma = 1$, the MVunc rule performs the worst, followed by MVc, TZ, KZ2F, and KZ3F, all showing highly negative results due to their reliance on single MV rules. The MSR strategy also performs poorly, with values of -9919 for SBTM100 and -4376 for IP48. This is due to unreliable out-of-sample performance when the number of assets is large relative to the estimation window ($N > T$). For $\gamma = 3$, the MSR rule performs even worse, with extreme negative values of -29784 for SBTM100 and -13159 for IP48. With $T = 180$, extreme negative utility values are greatly reduced for $\gamma = 1$, but the ranking remains the same, with EW, GMV, and EWRF outperforming others.

Using the shrinkage covariance matrix, rankings become more complex. KZ2F, KZ3F, and KWZ demonstrate the highest utilities of 0.94, 0.69, and 0.92, respectively, for $\gamma = 1$ for SBTM100. However, IP48 shows only negative values, indicating that these combined rules do not apply to all high-asset portfolios. On average, LA, KZ3F, and GMVRF outperform others (see Table 18 in Appendix B), while other rules show generally stable results.

1.3 Monte Carlo Simulation

Table 7: Expected out-of-sample utility under T=120 using Monte Carlo simulations

		Expected OOSU with T=120						Expected OOSU with T=180					
Portfolios		**SBTM6**	**SBTM25**	**SBTM100**	**IP10**	**IP48**	**Mean**	**SBTM6**	**SBTM25**	**SBTM100**	**IP10**	**IP48**	**Mean**
γ = 1	**EW**	0,58	0,59	0,35	0,54	0,52	0,52	0,58	0,59	0,35	0,54	0,52	0,52
	MSR	-137	-751	-47427	-120	-15642	-12815	-15,05	-38,77	-13095	-2,87	-566	-2744
	GMV	0,74	0,78	0,56	0,53	0,42	0,61	0,75	0,79	0,76	0,53	0,43	0,65
	Mvunc	0,80	-17	-22146	-4,14	-110,02	-4455	2,05	-4,22	-577	-1,82	-34,16	-123
	MVc	0,22	-16	-17591	-4,11	-98,60	-3542	1,14	-4,77	-524	-2,15	-32,25	-112
	EWRF	0,51	0,50	0	0,59	0,37	0,39	0,66	0,65	0	0,75	0,55	0,55
	GMVRF	1,58	1,92	0,34	0,87	0,36	1,01	1,74	2,26	1,55	1,05	0,64	1,45
	KZ2F	2,07	3,02	1,15	0,18	0,40	1,37	2,57	4,32	6,71	0,42	1,06	3,02
	KWZ	1,43	2,42	1,57	0,20	0,56	1,24	1,72	3,25	6,30	0,34	1,10	2,54
	KZ3F	2,24	3,45	0,94	0,58	0,42	1,53	2,69	4,67	6,86	0,89	1,27	3,28
	TZ	0,82	-8,78	-399	-3,95	-28,85	-87,99	2,05	-1,06	-60,43	-1,76	-11,65	-14,57
	LA	1,25	1,60	0,17	0,56	0,30	0,78	1,56	2,02	1,34	0,84	0,63	1,28
	LMS	1,42	2,38	1,15	0,21	0,54	1,14	1,71	3,26	6,10	0,35	1,10	2,51
γ = 3	**EW**	0,32	0,31	0,05	0,35	0,27	0,26	0,32	0,31	0,05	0,35	0,27	0,26
	MSR	-398	-12034	-39906	-170	-195637	-49629	-24	-24	-1230	-321	-772	-474
	GMV	0,57	0,62	-0,04	0,39	0,25	0,36	0,58	0,64	0,53	0,40	0,29	0,49
	Mvunc	0,26	-5,83	-7346	-1,37	-36,27	-1478	0,68	-1,40	-193	-0,58	-11,44	-41,07
	MVc	0,39	-5,09	-5880	-1,15	-32,87	-1184	0,71	-1,20	-174	-0,51	-10,58	-37,06
	EWRF	0,17	0,16	-0,01	0,19	0,13	0,13	0,22	0,22	0,04	0,25	0,18	0,18
	GMVRF	0,53	0,64	0,11	0,29	0,12	0,34	0,58	0,75	0,52	0,35	0,21	0,48
	KZ2F	0,69	1,01	0,40	0,06	0,13	0,46	0,86	1,43	2,24	0,14	0,35	1,00
	KWZ	0,80	1,15	0,30	0,29	0,30	0,57	0,89	1,47	2,38	0,34	0,51	1,12
	KZ3F	0,75	1,15	0,34	0,19	0,16	0,52	0,90	1,56	2,28	0,30	0,42	1,09
	TZ	0,27	-5,24	-151	-1,35	-22,56	-36,04	0,68	-1,19	-29,93	-0,59	-8,70	-7,95
	LA	0,42	0,53	0,05	0,19	0,10	0,26	0,51	0,68	0,45	0,28	0,21	0,43
	LMS	0,80	1,16	0,42	0,28	0,29	0,59	0,90	1,47	2,40	0,34	0,52	1,12

Note: This table reports the Expected OOSU in percentage points.

Following the subsection 3.6 of the methodology, the expected OOSU is obtained in Table 7 for all individual and combined portfolio rules for simulations periods of $T = 120$ and $T = 180$. The two more colourful columns show the average of values obtained across all datasets, making it easier to see which portfolio strategies perform the best.

Firstly, the EW rule values for $T = 120$ and $T = 180$ being equal proves sufficient iterations, leading to convergence and stable results. With $T = 180$, results improve as longer timeframes enhance mean and variance accuracy, improving simulation quality and risk diversification. For $\gamma = 1$, the top portfolios are KZ3F, KZ2F, KWZ, LMS, and GMVRF with values of 3.28, 3.02, 2.54, 2.51 and 1.45. For $\gamma = 3$, KWZ and LMS lead with 1.12, followed by KZ3F (1.09) and KZ2F (1.00). MSR performs very bad and shows extremely low values (-49629 for $T = 120$, -474 for $T = 180$). However, MSR, MVunc and MVc remain the worst strategies, showing negative average expected OOSU.

Therefore, in Monte Carlo simulations, portfolio combinations outperform single rules due to simplified assumptions like i.i.d. returns and normal distribution, reducing data noise and complexity. This makes sense as portfolio combinations are designed to maximize expected OOSU. These combinations better leverage the complementarity of single rules and balance risk-return trade-offs. Thus, by modelling a wider range of outcomes, Monte Carlo simulations offer a way to test the performance of portfolio strategies under varying market conditions.

Chapter 2: Research results

This part aims to summarize the out-of-sample findings from the previous chapter in relation to the research questions outlined in subpart "3.1. Research Questions" from the methodology.

2.1 Question 1: Is a combination of portfolios better than an individual portfolio in terms of annualized SR and annualized utility ?

For this research question, the performance is evaluated using rolling window lengths of 120 and 180 months with both the sample covariance matrix and the shrinkage covariance matrix. To help decide, the average annualized SRs and utilities are put in Table 18 of Appendix.

With $T = 180$ and the linear shrinkage covariance matrix used to reduce turnover in volatile portfolios, the EW rule, with an average turnover around 0.03, outperformed all other rules. It is followed by EWRF and GMV rules. Portfolio combination strategies using the EW rule consistently underperformed it, while those relying on MVc and MVunc rules outperformed them due to the low turnover of the GMV or EW rules. This is logical since the EW rule avoids estimation errors arising from sample mean and sample covariance matrix, while the GMV rule reduces turnover by not relying on expected return estimates compared to MV single rules.

Under $T = 120$, the EW strategy consistently showed the best and most stable SR, averaging between 0.51 and 0.61. The GMV rule was second overall despite a negative value for the SBTM100 dataset. With $T = 180$, the GMV rule improves and becomes the best strategy with an average SR of 0.66. The GMVRF rule follows (0.57), while the EW rule ranks third (0.55). When using the shrinkage covariance matrix, even if SRs improve significantly and portfolio combinations perform better, the GMV rule performs the best, with KZ3F nearly matching it. Additionally, the MSR, MVunc single strategies, and TZ and LA combination all achieve an average value of 0.69.

Then, the average OOSU values show that the EW, GMV, and EWRF rules have the highest utility values up to 0.09 for both $T = 120$ and $T = 180$. However, with the linear shrinkage covariance matrix for $T = 180$ estimation period and only when $\gamma = 1$, the LA rule leads, followed closely by GMVRF and KZ3F, with averages of 0.19, 0.16 and 0.15 respectively. The MSR strategy follows, while the GMV and EWRF rules both achieve an average utility of 0.09, slightly surpassing the EW strategy. However, when $\gamma = 3$, the single portfolio rules MSR, GMV and EW outperform the other ones in that order.

Monte Carlo simulations simplify real-world complexities by assuming i.i.d. returns and a normal distribution with specific mean and covariance matrix. This reduces data noise and variability, resulting in more robust performance evaluations that mitigate uncertainty and balance risk-return trade-offs. As a result, these assumptions impact the optimal portfolio combinations, with the best strategies for $\gamma = 1$ being KZ3F, KZ2F, KWZ, and LMS, and for $\gamma = 3$, LMS and KWZ take the lead, followed by KZ3F and KZ2F.

In conclusion, both the EW and GMV individual strategies show strong net out-of-sample performance. The EW rule excels with low turnover, providing stability even with a high number of assets and a short rolling window ($T = 120$), while the GMV rule performs best as sample size increases ($T = 180$), reducing estimation errors in sample covariance matrix. Theoretically, Monte Carlo simulations shows that portfolio combinations like KZ3F and LMS outperform. However, in real-world data, it is only with the shrinkage covariance matrix under $\gamma = 1$ that KZ3F achieves the highest SR and the LA rule delivers the greatest utility.

2.2 Question 2: Which individual or combined portfolio strategy works best overall?

Choosing one strategy that works best is not an easy task as it depends on the sample data length and investor's risk appetite. As portfolio combinations are built to maximize the expected OOSU, this metric will be the primary metric for selection, with the SR as a tiebreaker.

In theory, Monte Carlo simulations have shown that KZ3F achieves the best expected OOSU with $\gamma = 1$, while LMS outperforms when $\gamma = 3$. However, in practice, using the historical performance from the rolling window method leads to different results.

With the sample covariance matrix for short data samples ($T = 120$) or easy computation of optimal weights, the EW rule is the best. Indeed, it obtains robust performance in high-dimensional settings with limited sample data ($N > T$), due to its low turnover and avoidance of estimation risk by not relying on sample estimates. However, when $T = 180$, the GMV rule outperforms, minimizing the impact of estimation errors in expected returns.

Using the linear shrinkage covariance matrix with $T = 180$, the ranking shifts. For risk adverse investors ($\gamma = 3$), the MSR strategy delivers the best average OOSU, followed by GMV and EW. It is only for $\gamma = 1$, that portfolio combinations outperform. The LA rule demonstrates strong performance both in terms of annualized OOSU (0.19) and SR (0.68) proving its good risk-return profile and consistency. This rule excels in high-dimensional portfolios with significant turnover.

Conclusion

The goal of this thesis is to identify the most effective investment strategy by comparing individual and combined portfolio optimization techniques. To achieve this, the research evaluates the out-of-sample performance of various portfolio rules and their combinations across five selected datasets. The analysis considers different rolling window sizes, risk aversion parameters, and the use of a linear shrinkage covariance matrix.

The findings offer valuable insights into portfolio performance and align well with existing literature. Individual strategies that perform well in-sample, such as the MSR and MV approaches, often struggle in out-of-sample scenarios due to errors in predictive estimates. Portfolios with a higher number of assets tend to exhibit greater volatility and reduced net out-of-sample performance due to increased turnover and transaction costs. Extending the rolling window length further improves performance by providing more accurate sample estimates and reducing extreme values, as evidenced by the SBTM100 results. Additionally, the use of a linear shrinkage covariance matrix enhances outcomes by reducing noise and preventing overfitting.

Overall, the analysis shows that individual portfolio strategies frequently outperform the combined approaches analysed. The EW portfolio, in particular, proves to be highly robust and stable in out-of-sample scenarios, benefiting from low turnover, no reliance on sample estimates, and offering substantial diversification advantages. Increasing the rolling window length, the GMV portfolio achieves the highest annualized SR and utility due to better covariance matrix estimation. Ultimately, it was only with the combination of a shrinkage covariance matrix and a long rolling window that the LA combined portfolio outperformed, followed closely by the GMVRF and the KZ3F strategies.

In contrast, portfolio combinations like KZ3F and LMS outperform in Monte Carlo simulations, depending on the risk-aversion parameter. It highlights the robustness of portfolio combinations over single rules due to simplified assumptions like i.i.d. returns and normal distribution, which reduce data noise and complexity. These simulations demonstrate that portfolio combinations made to maximize the expected OOSU adapt more flexibly to varying market conditions, and leverage the complementarity of single rules to balance risk-return trade-offs effectively.

To conclude, this research highlights the advantages of combined portfolio rules, which show enhanced expected OOSU with simulated returns, making them preferred in theory. However, when assessing real-world data, individual strategies like EW and GMV demonstrates empirical outperformance and robustness.

Limitations

The completion of this thesis has uncovered several significant limitations that impacted the results and sometimes made the writing process difficult. A primary challenge is the impact of the COVID-19 pandemic, as it has influenced financial markets in unprecedented ways. This made it difficult to compare current results with historical analyses from the literature, potentially affecting the reliability of comparisons with previous studies. Additionally, evaluating the LA results is challenging because this rule has not been extensively tested or documented in the existing literature. Finally, the broad scope of literature in portfolio optimization, which spans a wide range of concepts, mathematical and statistical knowledge, and various optimization methods also presents a limitation. This extensive breadth complicates efforts to address every aspect thoroughly and to summarize the literature in a way that is both concise and accessible to every reader.

Recommendations

Lastly, two recommendations could provide valuable directions for future research. Firstly, while this research focused on the shrinkage covariance matrix methods of Ledoit and Wolf (2004b), incorporating shrinkage of the mean and the new nonlinear shrinkage covariance matrix by Ledoit and Wolf (2017, 2020) could provide further insights and potential improvements in portfolio performance. Moreover, there is fewer exploration of the impact on performance of the shrinkage estimator of sample mean introduced by James and Stein (1961). Although significant, this aspect was not addressed in this study due to its comparatively lesser relevance to covariance matrix estimates, particularly in high-dimensional settings. Secondly, to simplify the portfolio optimization process and its application for people being less familiar with the topic, a suggestion would be to rely on the EW rule, preferred due to its simplicity, robustness, and stability across various portfolios analysed. However, when long sample data lengths are available, the GMV portfolio remains a strong recommendation.

By addressing these limitations and considering the recommendations, future research could further refine portfolio optimization strategies and enhance their applicability and robustness.

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Appendices

Appendix A - Glossary:

1. Idiosyncratic and systematic risk:

Markowitz’s modern portfolio theory has enabled a breakthrough in financial economics, such as the Capital Asset Pricing Model (CAPM) and the distinction between systematic and idiosyncratic risks (Michaud, 1989). To define the idiosyncratic risk, the CAPM can be formulated as a factor model (Evstigneev, Hens, & Schenk-Hopp, 2016, p. 59-63; Fama & French, 2015):

$$R_i = E(R_i) + \beta_i\big(R_M - E(R_M)\big) + \varepsilon_i$$

The factor $F = R_M - E(R_M)$ is interpreted as the basic source of randomness, common to all asset returns and is used to compute the systematic risk. The residual term ε_i represents the specific source of randomness used to calculate the idiosyncratic risk. Additionally, $E(R_i)$ is the expected value of each random variable R_i and $E(\varepsilon_i) = Cov(R_M, \varepsilon_i) = 0$. This yields to:

$$Var(R_i) = \beta_i^2 Var(R_M) + Var(\varepsilon_i)$$

We can recall that the variance of the sum of uncorrelated random variables is equal to the sum of their variances. In this formula, the term $\beta_i^2 Var\boldsymbol{R}^M$ is the systematic risk, which reflects uncertainty associated with the market. This risk cannot be reduced by portfolio diversification since every asset with non-zero beta contains this risk. The second part, $Var\ \varepsilon_i$, which is defined as the variance of residual term ε_i, is the idiosyncratic risk (also called non-systematic risk, or specific risk) of the portfolio x with $\varepsilon_x = \sum_i \varepsilon_i x_i$ (Evstigneev, Hens & Schenk-Hopp, 2016). Idiosyncratic volatility is asset-specific, quantifying the portion of the asset's return total volatility uncorrelated with the market and thus not explained by market returns. It thus represents the difference between a stock's overall risk and its systematic risk (Chen, Kliger & Qadan, 2019; Drew, Naughton, & Veeraraghavan, 2004 in Di Iorio & Liu, 2012; Aabo, Pantzalis & Park, 2017). As per the CAPM, since idiosyncratic risk is unrewarded for its failure to explain returns, the only risk factor eligible for pricing and corresponding compensation is the systematic risk. (Evstigneev, Hens, & Schenk-Hopp, 2016; Chen, Kliger & Qadan, 2019; Liu & Mo, 2023). Despite a portfolio with many diversified assets, Merton claims that in a market with incomplete information, investors struggle to eliminate idiosyncratic risk, suggesting a need for a risk premium on individual stocks' idiosyncratic risk (Merton 1987 in Liu & Mo, 2023).

2. Bias-variance trade-off:

The use of a rolling window underscores the multifaceted nature of risk in portfolio management (DeMiguel et al., 2009). In the domain of portfolio optimization, risk is commonly associated with the variance of portfolio returns, denoted as R_i. Nevertheless, in the context of statistics, "risk" relates to the sampling variance and bias of a parameter estimator, exemplified by $\widehat{W}_{MSR}$ for example. Therefore, the portfolio return involves risk in both senses represented as $P = \boldsymbol{\widehat{w}'R}$ (Holgersson & Singull, 2020).

The variance measures the dispersion of returns of a portfolio, capturing how prediction errors vary across diverse samples or datasets. Increased variance may result in instability and suboptimal performance when applied to new and unobserved data. Where $E[\hat{\theta}]$ is the expected value of the estimator and θ is the true parameter value, the variance of an estimator $Var(\hat{\theta})$ is given by (Bishop, 2006, p.20):

$$Var(\hat{\theta}) = E\left[\left(\hat{\theta} - E[\hat{\theta}]\right)^2\right]$$

Shifting to bias, it gauges the accuracy of the portfolio mean prediction, reflecting the extent to which these estimators consistently deviate from the true but unknown values. The bias of an estimator is given by (Friedman, Hastie, & Tibshirani, 2009; Lassance, 2022; Matteson & Ruppert, 2015, p.132):

$$Bias(\hat{\theta}) \ = \ E[\hat{\theta}] - \theta$$

A bias of zero indicates an unbiased estimator, meaning that the estimator provides accurate predictions of the true parameter value, resulting in $E[\hat{\theta}] \ = \ \theta.$ Hence, the Mean Squared Error (MSE):

$$MSE = Biais(\hat{\theta})^2 + Var(\hat{\theta})$$

is expressed as the sum of its squared bias, variance. (Lassance, 2022; Fabozzi, Kolm & Tütüncü, 2014). The MSE of an estimator serves to measure its accuracy, capturing the squared difference between the estimator and the actual value of a parameter. It highlights the bias-variance trade-off's importance in portfolio optimization. Balancing bias and variance is essential in predictive modelling to find the right level of complexity that avoids overfitting and underfitting, ensuring robustness in portfolio selection for both in-sample and out-of-sample data (DeMiguel et al., 2009; Lassance, 2022; Matteson & Ruppert, 2015).

3. Bayesian prior method:

Bayes theorem allows for updating the prior distribution with the likelihood function:

$$\mathcal{L}(\theta|\boldsymbol{R}) = f(\boldsymbol{R}|\theta) = \prod_{t=1}^{T} f(R_{iT}|\theta)$$

resulting in the posterior distribution $f(\theta)$ for the parameters θ:

$$Posterior \; = \; Prior \; * \; Likelihood \Leftrightarrow f(\theta|\boldsymbol{R}) \; = \frac{f(\theta)}{f(\boldsymbol{R})} \; * \; f(\boldsymbol{R}|\theta)$$

With $f(x) \; \neq \; 0$.

The prior represents knowledge or uncertainty before observing data, while the posterior $f(\theta|\boldsymbol{R})$ is the conditional probability distribution indicating likely parameters after data observation. A prior commonly used for W_{MSR} is the EW portfolio $W_{EW} = \frac{1}{N}$. Finally, the likelihood is the probability of belonging to a specific category or class (Lassance, 2022; University of Cambridge, n.d.). Bayesian prior methods offer a more robust approach by incorporating prior knowledge, reducing the sensitivity of estimators to observed data and improving prediction accuracy.

4. James & Stein and Jorion's shrinkage estimator of sample mean

James and Stein (1961) introduced the a shrinkage estimator of sample mean represented by : $\hat{\mu}_{JS} = (1 - \delta_{JS})\hat{\mu}_{MLE} + \delta_{JS}\mu_g\mathbf{1}$. This estimator always achieves a lower risk and reduces estimation error (MSE), compared to the Maximum Likelihood Estimator $\hat{\mu}_{MLE}$. Therefore, the goal of the shrinkage method is to blend a high-dimensional model with low bias and high variance ($\hat{\mu}_{MLE}$), and a lower-dimensional model with larger bias but smaller variance (μ_g). The shrinkage intensity δ_{JS} adjusts $\hat{\mu}_{MLE}$ towards a constant mean μ_g, being the sample mean of the GMV portfolio:

$$\delta_{JS} \; = \frac{N+2}{(N+2) + T\left(\hat{\mu}_{t,MLE} - \mu_{t,g}\mathbf{1}\right)' \Sigma^{-\mathbf{1}}(\hat{\mu}_{t,MLE} - \mu_{t,g}\mathbf{1})} \; \epsilon \; [0,1]$$

Going deeper, Jorion developed a Bayesian estimator of δ_{JS} to address parameter uncertainty. It does not only reduce the estimates of the mean but also addresses estimation errors in the covariance matrix. The Jorion covariance matrix is given by (DeMiguel, Garlappi, Uppal, 2009; Jorion, 1986; Lassance, 2022):

$$\hat{\Sigma}_{Jorion} = \Sigma\left(1+\frac{1}{T+\tau}\right)+\frac{\tau}{T(T+1+\tau)}\frac{\mathbf{1}\mathbf{1}'}{\mathbf{1}'\Sigma\mathbf{1}}$$

Where:

$$\tau = \frac{N+2}{(\hat{\mu}_{MLE}-\mu_0\mathbf{1})'\Sigma^{-\mathbf{1}}(\hat{\mu}_{MLE}-\mu_0\mathbf{1})}$$

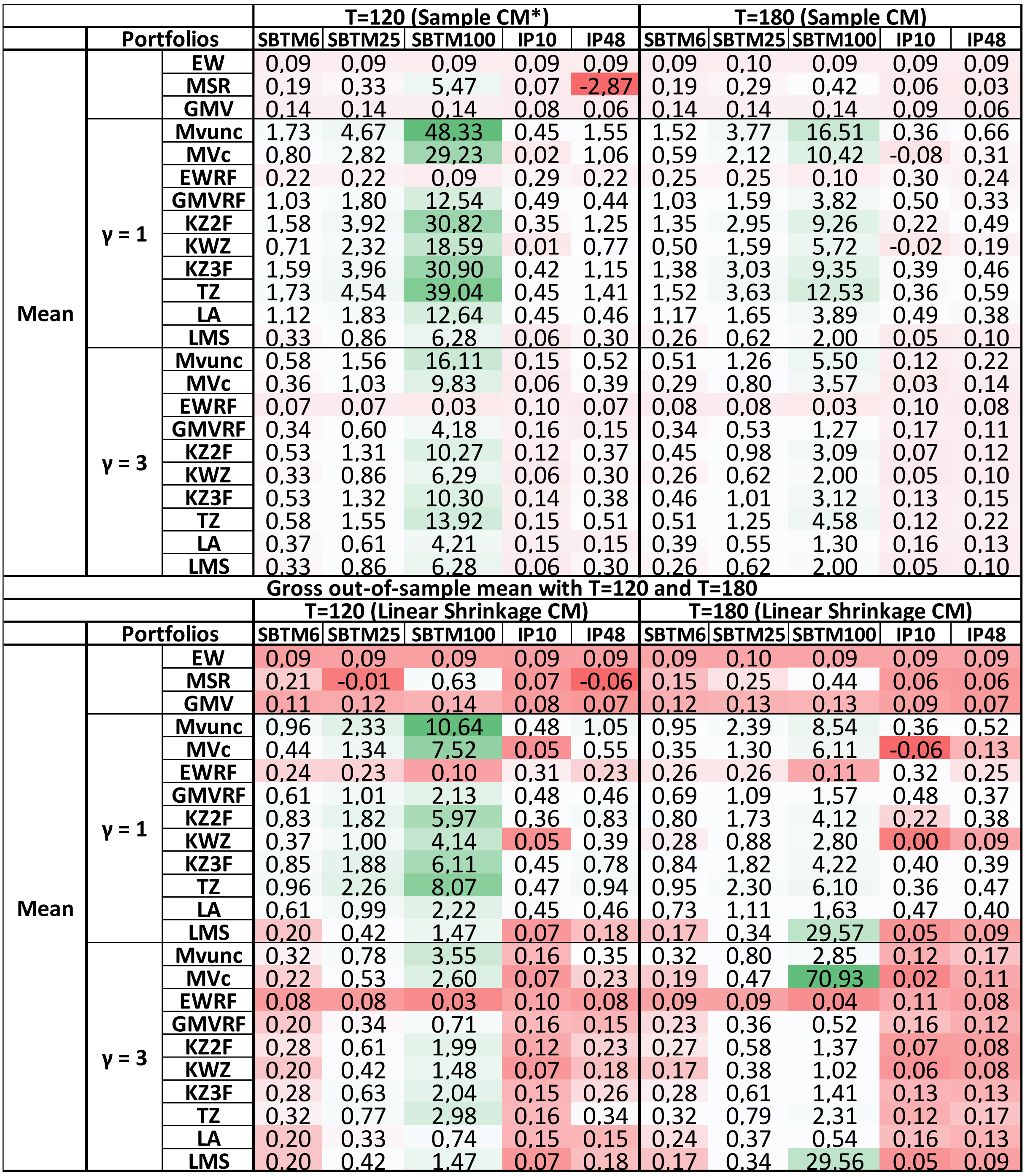
Replace Σ in the δ_{JS} formula by $\frac{T-1}{T-N-2}\,\hat{\Sigma}$ which has the property $\mathbb{E}\left[\left(\frac{T-1}{T-N-2}\,\hat{\Sigma}\right)^{-1}\right] = \Sigma^{-\mathbf{1}}$. In simpler terms, it suggests that under certain conditions, such as when the sample size T is sufficiently large relative to the number of variables N, the expected value of the inverse of the estimated covariance matrix approaches the true inverse covariance matrix. This Bayesian estimator, derived from the posterior predictive distribution of returns, showed significant improvement over classical sample mean estimators in out-of-sample evaluations (Stein, 1956; James & Stein, 1961; Michaud, 1989, p.34; DeMiguel, Garlappi, & Uppal, 2009; Hitaj, Martellini, & Zambruno, 2012; Jorion, 1986).

Appendix B – Tables:

Table 8: In-sample performance of single strategies

			T=120 (Sample CM)					T=180 (Sample CM)				
	Portfolios		SBTM6	SBTM25	SBTM100	IP10	IP48	SBTM6	SBTM25	SBTM100	IP10	IP48
Mean		EW	0,09	0,10	0,07	0,09	0,09	0,10	0,10	0,07	0,09	0,09
		MSR	0,22	0,39	0,90	0,10	0,28	0,22	0,38	0,93	0,10	0,31
		GMV	0,12	0,12	0,12	0,08	0,07	0,12	0,12	0,12	0,09	0,07
	γ = 1	Mvunc	1,12	2,62	7,98	0,48	1,39	1,08	2,60	8,25	0,48	1,50
		MVc	0,65	1,92	7,02	0,17	1,11	0,59	1,89	7,27	0,16	1,21
	γ = 3	Mvunc	0,37	0,87	2,66	0,16	0,46	0,36	0,87	2,75	0,16	0,50
		MVc	0,29	0,72	2,42	0,11	0,42	0,28	0,71	2,51	0,11	0,45
Std		EW	0,17	0,18	0,18	0,15	0,17	0,17	0,18	0,18	0,15	0,17
		MSR	0,20	0,22	0,30	0,13	0,24	0,19	0,22	0,31	0,13	0,25
		GMV	0,14	0,12	0,10	0,12	0,11	0,14	0,12	0,11	0,12	0,11
	γ = 1	Mvunc	0,98	1,48	2,71	0,62	1,20	0,97	1,49	2,76	0,63	1,20
		MVc	0,71	1,27	2,59	0,29	1,08	0,67	1,25	2,64	0,30	1,09
	γ = 3	MVunc	0,33	0,49	0,90	0,21	0,40	0,32	0,50	0,92	0,21	0,40
		MVc	0,27	0,43	0,87	0,15	0,37	0,26	0,43	0,88	0,15	0,38
SR		EW	0,54	0,54	0,39	0,61	0,54	0,55	0,55	0,38	0,63	0,57
		MSR	1,15	1,76	2,94	0,77	1,16	1,11	1,75	2,99	0,77	1,24
		GMV	0,85	0,99	1,15	0,70	0,66	0,87	1,00	1,17	0,70	0,66
	γ = 1	Mvunc	1,15	1,76	2,94	0,77	1,16	1,11	1,75	2,99	0,77	1,24
		MVc	0,91	1,51	2,71	0,57	1,03	0,87	1,51	2,76	0,55	1,12
	γ = 3	Mvunc	1,15	1,76	2,94	0,77	1,16	1,11	1,75	2,99	0,77	1,24
		MVc	1,09	1,66	2,80	0,76	1,12	1,06	1,65	2,84	0,75	1,20
Utility γ=1		EW	0,08	0,08	0,06	0,08	0,08	0,08	0,08	0,05	0,08	0,08
		MSR	0,20	0,36	0,85	0,09	0,25	0,20	0,36	0,88	0,09	0,27
		GMV	0,11	0,11	0,12	0,08	0,07	0,11	0,12	0,12	0,08	0,07
	γ = 1	Mvunc	0,64	1,52	4,30	0,29	0,67	0,61	1,49	4,44	0,28	0,77
		MVc	0,39	1,12	3,66	0,13	0,53	0,36	1,10	3,79	0,12	0,62
Utility γ=3		EW	0,05	0,05	0,02	0,06	0,05	0,05	0,05	0,02	0,06	0,05
		MSR	0,17	0,31	0,76	0,08	0,19	0,16	0,31	0,78	0,08	0,21
		GMV	0,09	0,10	0,10	0,06	0,05	0,09	0,10	0,11	0,06	0,05
	γ = 3	Mvunc	0,21	0,51	1,43	0,10	0,22	0,20	0,50	1,48	0,09	0,26
		MVc	0,19	0,44	1,30	0,08	0,21	0,17	0,43	1,34	0,08	0,24
			T=120 (Linear Shrinkage CM)					T=180 (Linear Shrinkage CM)				
	Portfolios		SBTM6	SBTM25	SBTM100	IP10	IP48	SBTM6	SBTM25	SBTM100	IP10	IP48
Mean		EW	0,09	0,10	0,07	0,09	0,09	0,10	0,10	0,07	0,09	0,09
		MSR	0,21	0,37	0,85	0,10	0,27	0,20	0,36	0,88	0,10	0,29
		GMV	0,11	0,12	0,12	0,08	0,07	0,12	0,12	0,12	0,09	0,07
	γ = 1	Mvunc	0,99	2,40	7,12	0,47	1,36	0,95	2,37	7,41	0,48	1,46
		MVc	0,56	1,74	6,23	0,16	1,07	0,51	1,70	6,49	0,16	1,16
	γ = 3	Mvunc	0,33	0,80	2,37	0,16	0,45	0,32	0,79	2,47	0,16	0,49
		MVc	0,26	0,66	2,16	0,11	0,40	0,25	0,65	2,25	0,11	0,44
Std		EW	0,17	0,18	0,18	0,15	0,17	0,17	0,18	0,18	0,15	0,17
		MSR	0,19	0,21	0,29	0,13	0,23	0,19	0,21	0,30	0,13	0,23
		GMV	0,14	0,12	0,11	0,12	0,11	0,14	0,12	0,11	0,12	0,11
	γ = 1	Mvunc	0,88	1,36	2,46	0,61	1,16	0,88	1,37	2,52	0,62	1,17
		MVc	0,62	1,15	2,33	0,28	1,03	0,59	1,13	2,38	0,28	1,04
	γ = 3	MVunc	0,29	0,45	0,82	0,20	0,39	0,29	0,46	0,84	0,21	0,39
		MVc	0,24	0,40	0,78	0,15	0,36	0,24	0,39	0,80	0,15	0,36
SR		EW	0,54	0,54	0,39	0,61	0,54	0,55	0,55	0,38	0,63	0,57
		MSR	1,12	1,76	2,90	0,77	1,17	1,08	1,74	2,94	0,77	1,25
		GMV	0,81	0,98	1,15	0,70	0,68	0,84	0,99	1,17	0,70	0,68
	γ = 1	Mvunc	1,12	1,76	2,90	0,77	1,17	1,08	1,74	2,94	0,77	1,25
		MVc	0,91	1,51	2,67	0,59	1,03	0,86	1,50	2,72	0,57	1,12
	γ = 3	Mvunc	1,12	1,76	2,90	0,77	1,17	1,08	1,74	2,94	0,77	1,25
		MVc	1,09	1,66	2,77	0,76	1,12	1,05	1,65	2,82	0,76	1,21
Utility γ=1		EW	0,08	0,08	0,06	0,08	0,08	0,08	0,08	0,05	0,08	0,08
		MSR	0,19	0,35	0,81	0,09	0,25	0,18	0,34	0,84	0,09	0,26
		GMV	0,10	0,11	0,12	0,08	0,07	0,11	0,11	0,12	0,08	0,07
	γ = 1	Mvunc	0,60	1,47	4,10	0,29	0,68	0,56	1,44	4,24	0,29	0,77
		MVc	0,37	1,08	3,52	0,12	0,53	0,33	1,06	3,65	0,12	0,62
Utility γ=3		EW	0,05	0,05	0,02	0,06	0,05	0,05	0,05	0,02	0,06	0,05
		MSR	0,16	0,30	0,72	0,08	0,19	0,15	0,30	0,75	0,08	0,21
		GMV	0,08	0,10	0,10	0,06	0,06	0,09	0,10	0,11	0,06	0,06
	γ = 3	Mvunc	0,20	0,49	1,37	0,10	0,23	0,19	0,48	1,41	0,10	0,26
		MVc	0,18	0,42	1,24	0,08	0,21	0,16	0,42	1,29	0,08	0,24

Table 9: Annualized gross out-of-sample mean with T = 120 and T=180



			T=120 (Sample CM*)					T=180 (Sample CM)				
	Portfolios		SBTM6	SBTM25	SBTM100	IP10	IP48	SBTM6	SBTM25	SBTM100	IP10	IP48
Mean		EW	0,09	0,09	0,09	0,09	0,09	0,09	0,10	0,09	0,09	0,09
		MSR	0,19	0,33	5,47	0,07	-2,87	0,19	0,29	0,42	0,06	0,03
		GMV	0,14	0,14	0,14	0,08	0,06	0,14	0,14	0,14	0,09	0,06
	γ = 1	Mvunc	1,73	4,67	48,33	0,45	1,55	1,52	3,77	16,51	0,36	0,66
		MVc	0,80	2,82	29,23	0,02	1,06	0,59	2,12	10,42	-0,08	0,31
		EWRF	0,22	0,22	0,09	0,29	0,22	0,25	0,25	0,10	0,30	0,24
		GMVRF	1,03	1,80	12,54	0,49	0,44	1,03	1,59	3,82	0,50	0,33
		KZ2F	1,58	3,92	30,82	0,35	1,25	1,35	2,95	9,26	0,22	0,49
		KWZ	0,71	2,32	18,59	0,01	0,77	0,50	1,59	5,72	-0,02	0,19
		KZ3F	1,59	3,96	30,90	0,42	1,15	1,38	3,03	9,35	0,39	0,46
		TZ	1,73	4,54	39,04	0,45	1,41	1,52	3,63	12,53	0,36	0,59
		LA	1,12	1,83	12,64	0,45	0,46	1,17	1,65	3,89	0,49	0,38
		LMS	0,33	0,86	6,28	0,06	0,30	0,26	0,62	2,00	0,05	0,10
	γ = 3	Mvunc	0,58	1,56	16,11	0,15	0,52	0,51	1,26	5,50	0,12	0,22
		MVc	0,36	1,03	9,83	0,06	0,39	0,29	0,80	3,57	0,03	0,14
		EWRF	0,07	0,07	0,03	0,10	0,07	0,08	0,08	0,03	0,10	0,08
		GMVRF	0,34	0,60	4,18	0,16	0,15	0,34	0,53	1,27	0,17	0,11
		KZ2F	0,53	1,31	10,27	0,12	0,37	0,45	0,98	3,09	0,07	0,12
		KWZ	0,33	0,86	6,29	0,06	0,30	0,26	0,62	2,00	0,05	0,10
		KZ3F	0,53	1,32	10,30	0,14	0,38	0,46	1,01	3,12	0,13	0,15
		TZ	0,58	1,55	13,92	0,15	0,51	0,51	1,25	4,58	0,12	0,22
		LA	0,37	0,61	4,21	0,15	0,15	0,39	0,55	1,30	0,16	0,13
		LMS	0,33	0,86	6,28	0,06	0,30	0,26	0,62	2,00	0,05	0,10
Gross out-of-sample mean with T=120 and T=180												
			T=120 (Linear Shrinkage CM)					T=180 (Linear Shrinkage CM)				
	Portfolios		SBTM6	SBTM25	SBTM100	IP10	IP48	SBTM6	SBTM25	SBTM100	IP10	IP48
Mean		EW	0,09	0,09	0,09	0,09	0,09	0,09	0,10	0,09	0,09	0,09
		MSR	0,21	-0,01	0,63	0,07	-0,06	0,15	0,25	0,44	0,06	0,06
		GMV	0,11	0,12	0,14	0,08	0,07	0,12	0,13	0,13	0,09	0,07
	γ = 1	Mvunc	0,96	2,33	10,64	0,48	1,05	0,95	2,39	8,54	0,36	0,52
		MVc	0,44	1,34	7,52	0,05	0,55	0,35	1,30	6,11	-0,06	0,13
		EWRF	0,24	0,23	0,10	0,31	0,23	0,26	0,26	0,11	0,32	0,25
		GMVRF	0,61	1,01	2,13	0,48	0,46	0,69	1,09	1,57	0,48	0,37
		KZ2F	0,83	1,82	5,97	0,36	0,83	0,80	1,73	4,12	0,22	0,38
		KWZ	0,37	1,00	4,14	0,05	0,39	0,28	0,88	2,80	0,00	0,09
		KZ3F	0,85	1,88	6,11	0,45	0,78	0,84	1,82	4,22	0,40	0,39
		TZ	0,96	2,26	8,07	0,47	0,94	0,95	2,30	6,10	0,36	0,47
		LA	0,61	0,99	2,22	0,45	0,46	0,73	1,11	1,63	0,47	0,40
		LMS	0,20	0,42	1,47	0,07	0,18	0,17	0,34	29,57	0,05	0,09
	γ = 3	Mvunc	0,32	0,78	3,55	0,16	0,35	0,32	0,80	2,85	0,12	0,17
		MVc	0,22	0,53	2,60	0,07	0,23	0,19	0,47	70,93	0,02	0,11
		EWRF	0,08	0,08	0,03	0,10	0,08	0,09	0,09	0,04	0,11	0,08
		GMVRF	0,20	0,34	0,71	0,16	0,15	0,23	0,36	0,52	0,16	0,12
		KZ2F	0,28	0,61	1,99	0,12	0,23	0,27	0,58	1,37	0,07	0,08
		KWZ	0,20	0,42	1,48	0,07	0,18	0,17	0,38	1,02	0,06	0,08
		KZ3F	0,28	0,63	2,04	0,15	0,26	0,28	0,61	1,41	0,13	0,13
		TZ	0,32	0,77	2,98	0,16	0,34	0,32	0,79	2,31	0,12	0,17
		LA	0,20	0,33	0,74	0,15	0,15	0,24	0,37	0,54	0,16	0,13
		LMS	0,20	0,42	1,47	0,07	0,18	0,17	0,34	29,56	0,05	0,09

*CM stands for "Covariance Matrix". The Linear Shrinkage Covariance Matrix is from Ledoit and Wolf (2004).

Table 10: Annualized gross out-of-sample standard deviation, T = 120 and T=180

			T=120 (Sample CM*)					T=180 (Sample CM)				
	Portfolios		SBTM6	SBTM25	SBTM100	IP10	IP48	SBTM6	SBTM25	SBTM100	IP10	IP48
Std		EW	0,17	0,18	0,18	0,15	0,17	0,17	0,18	0,18	0,15	0,16
		MSR	0,66	0,58	36,64	0,25	13,25	0,21	0,25	0,57	0,19	0,67
		GMV	0,14	0,13	0,27	0,12	0,14	0,14	0,13	0,16	0,12	0,13
	γ = 1	Mvunc	1,81	4,12	83,52	1,58	5,82	1,67	3,32	16,09	1,21	3,43
		MVc	1,23	3,34	68,92	1,20	5,47	1,12	2,69	13,26	0,93	3,30
		EWRF	0,64	0,63	0,54	0,70	0,60	0,53	0,53	0,38	0,56	0,50
		GMVRF	1,25	1,99	30,23	0,96	1,62	1,12	1,61	4,99	0,80	0,84
		KZ2F	1,66	3,44	53,34	1,30	4,68	1,49	2,65	9,02	0,88	2,60
		KWZ	1,08	2,71	43,89	0,88	3,78	0,94	2,04	7,23	0,54	1,82
		KZ3F	1,66	3,44	53,38	1,30	4,09	1,49	2,65	9,05	0,93	1,96
		TZ	1,81	4,00	67,45	1,58	5,29	1,67	3,20	12,19	1,21	3,00
		LA	1,41	2,09	30,81	1,00	1,62	1,31	1,71	5,11	0,80	0,86
		LMS	0,39	0,91	14,62	0,32	1,26	0,35	0,70	2,42	0,21	0,62
	γ = 3	Mvunc	0,60	1,37	27,84	0,53	1,94	0,56	1,11	5,36	0,40	1,14
		MVc	0,43	1,12	22,96	0,42	1,83	0,41	0,91	4,43	0,33	1,10
		EWRF	0,21	0,21	0,18	0,23	0,20	0,18	0,18	0,13	0,19	0,17
		GMVRF	0,42	0,66	10,08	0,32	0,54	0,37	0,54	1,66	0,27	0,28
		KZ2F	0,55	1,15	17,78	0,43	1,36	0,50	0,88	3,01	0,29	0,65
		KWZ	0,39	0,91	14,61	0,32	1,26	0,35	0,70	2,42	0,21	0,62
		KZ3F	0,55	1,15	17,79	0,43	1,36	0,50	0,88	3,02	0,31	0,65
		TZ	0,60	1,37	23,98	0,53	1,91	0,56	1,10	4,45	0,40	1,12
		LA	0,47	0,70	10,27	0,33	0,54	0,44	0,57	1,70	0,27	0,29
		LMS	0,39	0,91	14,61	0,32	1,26	0,35	0,70	2,42	0,21	0,62

			T=120 (Linear Shrinkage CM)					T=180 (Linear Shrinkage CM)				
	Portfolios		SBTM6	SBTM25	SBTM100	IP10	IP48	SBTM6	SBTM25	SBTM100	IP10	IP48
Std		EW	0,17	0,18	0,18	0,15	0,17	0,17	0,18	0,18	0,15	0,16
		MSR	0,26	3,40	4,00	0,23	2,58	0,18	0,20	0,26	0,17	0,39
		GMV	0,14	0,13	0,13	0,12	0,12	0,14	0,13	0,13	0,12	0,12
	γ = 1	Mvunc	1,14	2,13	6,48	1,39	3,52	1,13	1,98	4,94	1,09	2,60
		MVc	0,68	1,66	5,50	1,00	3,23	0,70	1,53	4,23	0,79	2,44
		EWRF	0,68	0,67	0,58	0,74	0,65	0,55	0,56	0,40	0,59	0,53
		GMVRF	0,99	1,28	2,40	0,92	1,17	0,87	1,16	1,59	0,78	0,77
		KZ2F	1,01	1,66	3,69	1,12	2,79	0,96	1,46	2,39	0,77	1,93
		KWZ	0,54	1,20	3,01	0,69	2,01	0,52	1,01	1,89	0,41	1,15
		KZ3F	1,05	1,70	3,75	1,14	2,29	0,99	1,51	2,44	0,85	1,36
		TZ	1,14	2,07	4,95	1,39	3,15	1,13	1,91	3,52	1,09	2,24
		LA	1,00	1,28	2,39	0,96	1,20	0,91	1,17	1,54	0,78	0,81
		LMS	0,22	0,41	1,00	0,26	0,68	0,23	0,43	26,43	0,22	0,72
	γ = 3	Mvunc	0,38	0,71	2,16	0,46	1,17	0,38	0,66	1,65	0,36	0,87
		MVc	0,25	0,56	1,83	0,36	1,08	0,29	0,63	62,34	0,38	1,47
		EWRF	0,23	0,22	0,19	0,25	0,22	0,18	0,19	0,13	0,20	0,18
		GMVRF	0,33	0,43	0,80	0,31	0,39	0,29	0,39	0,53	0,26	0,26
		KZ2F	0,34	0,55	1,23	0,37	0,75	0,32	0,49	0,80	0,26	0,44
		KWZ	0,22	0,41	1,00	0,26	0,68	0,22	0,36	0,64	0,18	0,40
		KZ3F	0,35	0,57	1,25	0,38	0,76	0,33	0,50	0,81	0,28	0,45
		TZ	0,38	0,71	1,82	0,46	1,15	0,38	0,66	1,33	0,36	0,84
		LA	0,33	0,43	0,80	0,32	0,40	0,30	0,39	0,51	0,26	0,27
		LMS	0,22	0,41	1,00	0,26	0,68	0,23	0,43	26,43	0,22	0,72

*CM stands for "Covariance Matrix". The Linear Shrinkage Covariance Matrix is from Ledoit and Wolf (2004).

Table 11: Annualized gross out-of-sample SR with T=120 and T=180

			T=120 (Sample CM*)					T=180 (Sample CM)				
	Portfolios		SBTM6	SBTM25	SBTM100	IP10	IP48	SBTM6	SBTM25	SBTM100	IP10	IP48
SR		**EW**	0,54	0,53	0,52	0,61	0,54	0,54	0,54	0,52	0,63	0,57
		MSR	0,29	0,56	0,15	0,28	-0,22	0,92	1,18	0,74	0,34	0,04
		GMV	0,98	1,03	0,51	0,66	0,40	0,99	1,06	0,88	0,71	0,45
	γ = 1	**Mvunc**	0,96	1,13	0,58	0,29	0,27	0,91	1,13	1,03	0,30	0,19
		MVc	0,65	0,84	0,42	0,02	0,19	0,52	0,79	0,79	-0,09	0,09
		EWRF	0,35	0,34	0,17	0,41	0,36	0,48	0,48	0,27	0,54	0,48
		GMVRF	0,83	0,90	0,41	0,51	0,27	0,93	0,98	0,77	0,63	0,39
		KZ2F	0,95	1,14	0,58	0,27	0,27	0,90	1,11	1,03	0,25	0,19
		KWZ	0,66	0,86	0,42	0,02	0,20	0,53	0,78	0,79	-0,04	0,10
		KZ3F	0,96	1,15	0,58	0,32	0,28	0,93	1,14	1,03	0,42	0,23
		TZ	0,96	1,13	0,58	0,29	0,27	0,91	1,13	1,03	0,30	0,20
		LA	0,79	0,87	0,41	0,45	0,28	0,89	0,96	0,76	0,61	0,44
		LMS	0,84	0,95	0,43	0,19	0,23	0,75	0,90	0,82	0,24	0,17
	γ = 3	**Mvunc**	0,96	1,13	0,58	0,29	0,27	0,91	1,13	1,03	0,30	0,19
		MVc	0,82	0,92	0,43	0,15	0,21	0,71	0,88	0,81	0,10	0,13
		EWRF	0,35	0,34	0,17	0,41	0,36	0,48	0,48	0,27	0,54	0,48
		GMVRF	0,83	0,90	0,41	0,51	0,27	0,93	0,98	0,77	0,63	0,39
		KZ2F	0,95	1,14	0,58	0,27	0,27	0,90	1,11	1,03	0,25	0,18
		KWZ	0,84	0,95	0,43	0,19	0,23	0,75	0,90	0,83	0,24	0,17
		KZ3F	0,96	1,15	0,58	0,32	0,28	0,93	1,14	1,03	0,42	0,23
		TZ	0,96	1,13	0,58	0,29	0,27	0,91	1,13	1,03	0,30	0,19
		LA	0,79	0,87	0,41	0,45	0,28	0,89	0,96	0,76	0,61	0,44
		LMS	0,84	0,95	0,43	0,19	0,23	0,75	0,90	0,83	0,24	0,17

			T=120 (Linear Shrinkage CM)					T=180 (Linear Shrinkage CM)				
	Portfolios		SBTM6	SBTM25	SBTM100	IP10	IP48	SBTM6	SBTM25	SBTM100	IP10	IP48
SR		**EW**	0,54	0,53	0,52	0,61	0,54	0,54	0,54	0,52	0,63	0,57
		MSR	0,81	0,00	0,16	0,29	-0,02	0,85	1,24	1,69	0,38	0,16
		GMV	0,83	0,97	1,08	0,68	0,56	0,87	1,01	1,06	0,70	0,58
	γ = 1	**Mvunc**	0,84	1,09	1,64	0,34	0,30	0,84	1,21	1,73	0,33	0,20
		MVc	0,66	0,81	1,37	0,05	0,17	0,51	0,85	1,44	-0,07	0,05
		EWRF	0,35	0,35	0,17	0,42	0,36	0,47	0,47	0,26	0,53	0,48
		GMVRF	0,62	0,79	0,89	0,52	0,40	0,79	0,94	0,99	0,62	0,48
		KZ2F	0,82	1,09	1,62	0,32	0,30	0,83	1,19	1,73	0,28	0,20
		KWZ	0,67	0,84	1,37	0,07	0,19	0,53	0,87	1,48	-0,01	0,08
		KZ3F	0,81	1,10	1,63	0,39	0,34	0,85	1,21	1,73	0,47	0,29
		TZ	0,84	1,09	1,63	0,34	0,30	0,84	1,21	1,73	0,33	0,21
		LA	0,61	0,78	0,93	0,47	0,38	0,80	0,95	1,06	0,60	0,49
		LMS	0,92	1,01	1,47	0,27	0,26	0,73	0,81	1,12	0,24	0,12
	γ = 3	**Mvunc**	0,84	1,09	1,64	0,34	0,30	0,84	1,21	1,73	0,33	0,20
		MVc	0,89	0,94	1,42	0,21	0,21	0,66	0,74	1,14	0,05	0,07
		EWRF	0,35	0,35	0,17	0,42	0,36	0,47	0,47	0,26	0,53	0,48
		GMVRF	0,62	0,79	0,89	0,52	0,40	0,79	0,94	0,99	0,62	0,48
		KZ2F	0,82	1,09	1,62	0,32	0,31	0,83	1,19	1,73	0,28	0,18
		KWZ	0,92	1,01	1,47	0,27	0,26	0,80	1,06	1,60	0,32	0,19
		KZ3F	0,81	1,10	1,63	0,39	0,34	0,85	1,21	1,73	0,47	0,29
		TZ	0,84	1,09	1,63	0,34	0,30	0,84	1,21	1,74	0,33	0,20
		LA	0,61	0,78	0,93	0,47	0,38	0,80	0,95	1,06	0,60	0,49
		LMS	0,92	1,01	1,47	0,26	0,26	0,73	0,81	1,12	0,24	0,12

*CM stands for "Covariance Matrix". The Linear Shrinkage Covariance Matrix is from Ledoit and Wolf (2004).

Table 12: Annualized gross out-of-sample utilities with T=120 and T=180

		T=120 (Sample CM*)					T=180 (Sample CM)				
		SBTM6	SBTM25	SBTM100	IP10	IP48	SBTM6	SBTM25	SBTM100	IP10	IP48
Utility γ=1	EW	0,08	0,08	0,08	0,08	0,08	0,08	0,08	0,08	0,08	0,08
	MSR	-0,02	0,16	-666	0,04	-90,67	0,17	0,26	0,26	0,05	-0,19
	GMV	0,13	0,13	0,10	0,07	0,05	0,13	0,13	0,13	0,08	0,05
	Mvunc	0,09	-3,82	-3440	-0,80	-15,41	0,12	-1,76	-112,93	-0,37	-5,22
	MVc	0,04	-2,76	-2346	-0,69	-13,89	-0,04	-1,50	-77,47	-0,51	-5,15
	EWRF	0,02	0,02	-0,06	0,04	0,04	0,11	0,11	0,03	0,14	0,12
	GMVRF	0,26	-0,18	-444	0,03	-0,87	0,41	0,29	-8,61	0,18	-0,03
	KZ2F	0,20	-1,99	-1392	-0,49	-9,69	0,23	-0,56	-31,40	-0,16	-2,89
	KWZ	0,12	-1,34	-944	-0,37	-6,36	0,06	-0,48	-20,41	-0,17	-1,47
	KZ3F	0,21	-1,96	-1394	-0,43	-7,19	0,27	-0,48	-31,61	-0,04	-1,46
	TZ	0,09	-3,48	-2236	-0,80	-12,56	0,12	-1,50	-61,77	-0,37	-3,92
	LA	0,13	-0,36	-462,02	-0,05	-0,85	0,31	0,18	-9,17	0,17	0,01
	LMS	0,25	0,45	-100,52	0,01	-0,50	0,20	0,38	-0,93	0,03	-0,09
Utility γ=3	EW	0,05	0,05	0,05	0,06	0,05	0,05	0,05	0,05	0,06	0,05
	MSR	-0,46	-0,18	-2008	-0,02	-266	0,13	0,20	-0,07	0,01	-0,64
	GMV	0,11	0,11	0,03	0,06	0,03	0,11	0,11	0,10	0,06	0,03
	Mvunc	0,03	-1,27	-1147	-0,27	-5,14	0,04	-0,59	-37,64	-0,12	-1,74
	MVc	0,07	-0,85	-781	-0,20	-4,62	0,04	-0,44	-25,83	-0,13	-1,69
	EWRF	0,01	0,01	-0,02	0,01	0,01	0,04	0,04	0,01	0,05	0,04
	GMVRF	0,09	-0,06	-148	0,01	-0,29	0,14	0,10	-2,87	0,06	-0,01
	KZ2F	0,07	-0,66	-464	-0,16	-2,39	0,08	-0,19	-10,47	-0,05	-0,51
	KWZ	0,10	-0,38	-314	-0,09	-2,10	0,08	-0,10	-6,78	-0,02	-0,47
	KZ3F	0,07	-0,65	-465	-0,14	-2,40	0,09	-0,16	-10,54	-0,01	-0,49
	TZ	0,03	-1,26	-849	-0,27	-4,94	0,04	-0,57	-25,07	-0,12	-1,66
	LA	0,04	-0,12	-154,01	-0,02	-0,28	0,10	0,06	-3,06	0,06	0,00
	LMS	0,10	-0,38	-314	-0,09	-2,10	0,08	-0,10	-6,78	-0,02	-0,47
		T=120 (Linear Shrinkage CM)					T=180 (Linear Shrinkage CM)				
		SBTM6	SBTM25	SBTM100	IP10	IP48	SBTM6	SBTM25	SBTM100	IP10	IP48
Utility γ=1	EW	0,08	0,08	0,08	0,08	0,08	0,08	0,08	0,08	0,08	0,08
	MSR	0,18	-5,78	-7,36	0,04	-3,38	0,14	0,23	0,41	0,05	-0,01
	GMV	0,11	0,11	0,13	0,08	0,06	0,11	0,12	0,13	0,08	0,06
	Mvunc	0,30	0,06	-10,37	-0,49	-5,16	0,31	0,43	-3,64	-0,23	-2,85
	MVc	0,22	-0,03	-7,63	-0,45	-4,68	0,11	0,13	-2,86	-0,37	-2,83
	EWRF	0,01	0,01	-0,07	0,03	0,03	0,11	0,11	0,03	0,14	0,11
	GMVRF	0,12	0,19	-0,74	0,06	-0,22	0,31	0,42	0,31	0,18	0,07
	KZ2F	0,32	0,44	-0,85	-0,27	-3,06	0,34	0,67	1,27	-0,08	-1,48
	KWZ	0,22	0,29	-0,40	-0,19	-1,63	0,14	0,36	1,01	-0,09	-0,57
	KZ3F	0,30	0,43	-0,93	-0,20	-1,85	0,35	0,69	1,25	0,04	-0,53
	TZ	0,30	0,12	-4,17	-0,49	-4,01	0,31	0,48	-0,09	-0,23	-2,05
	LA	0,11	0,18	-0,64	-0,01	-0,27	0,31	0,42	0,44	0,16	0,07
	LMS	0,18	0,33	0,97	0,04	-0,06	0,14	0,25	-320	0,03	-0,17
Utility γ=3	EW	0,05	0,05	0,05	0,06	0,05	0,05	0,05	0,05	0,06	0,05
	MSR	0,11	-17,32	-23,33	-0,01	-10,03	0,11	0,19	0,34	0,02	-0,16
	GMV	0,09	0,10	0,12	0,06	0,05	0,09	0,10	0,11	0,06	0,05
	Mvunc	0,10	0,02	-3,46	-0,16	-1,72	0,10	0,14	-1,21	-0,08	-0,95
	MVc	0,13	0,06	-2,42	-0,12	-1,53	0,06	-0,13	-5759	-0,19	-3,15
	EWRF	0,00	0,00	-0,02	0,01	0,01	0,04	0,04	0,01	0,05	0,04
	GMVRF	0,04	0,06	-0,25	0,02	-0,07	0,10	0,14	0,10	0,06	0,02
	KZ2F	0,11	0,15	-0,28	-0,09	-0,62	0,11	0,22	0,42	-0,03	-0,21
	KWZ	0,13	0,16	-0,03	-0,03	-0,52	0,10	0,19	0,41	0,01	-0,16
	KZ3F	0,10	0,14	-0,31	-0,07	-0,62	0,12	0,23	0,42	0,01	-0,18
	TZ	0,10	0,02	-2,01	-0,16	-1,64	0,10	0,14	-0,35	-0,08	-0,90
	LA	0,04	0,06	-0,21	0,00	-0,09	0,10	0,14	0,15	0,05	0,02
	LMS	0,13	0,16	-0,04	-0,03	-0,52	0,09	0,07	-1018	-0,02	-0,69

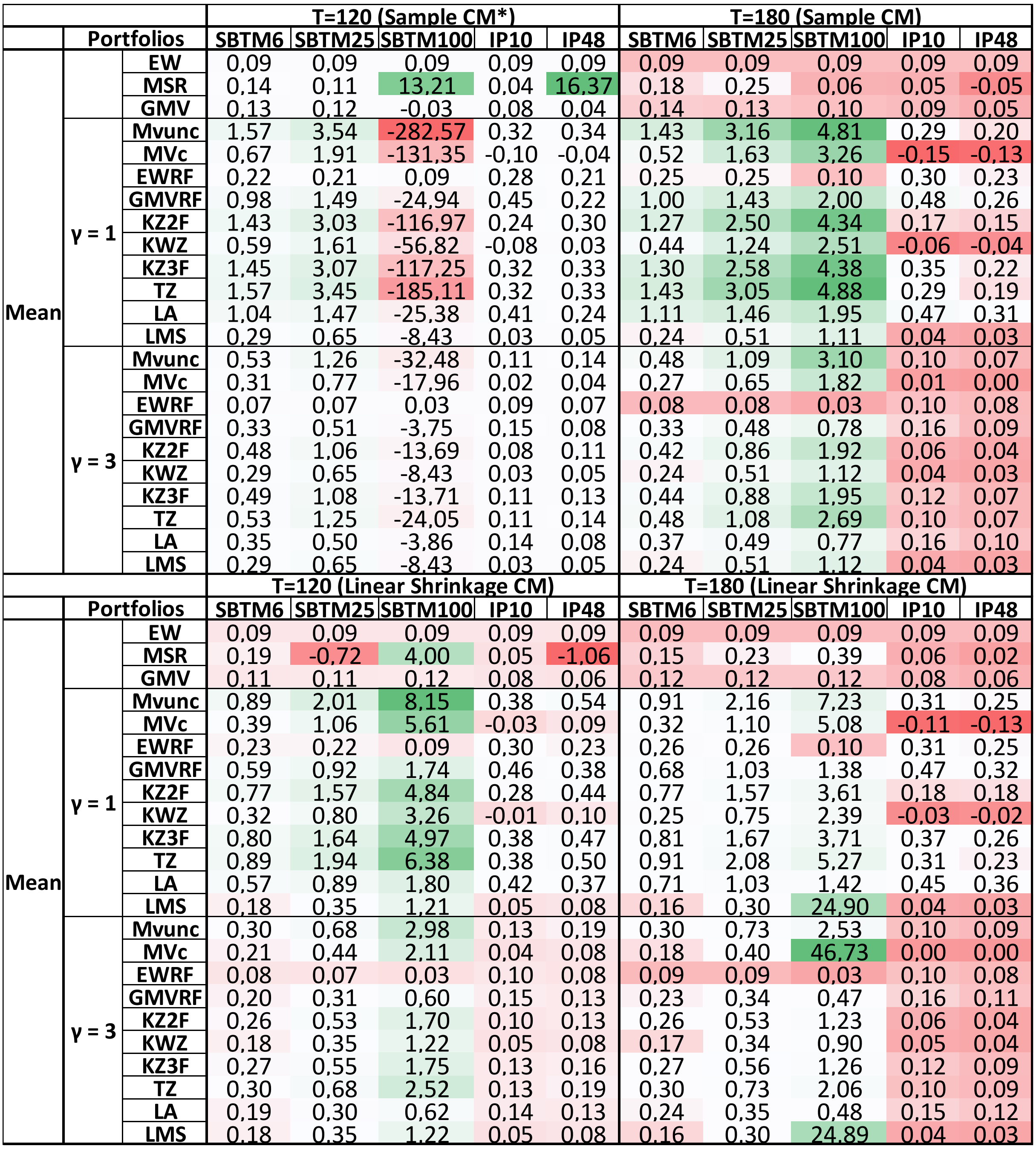
*CM stands for "Covariance Matrix". The Linear Shrinkage Covariance Matrix is from Ledoit and Wolf (2004).

Table 13: Turnover including SBTM25

			T=120 (Sample CM*)					T=180 (Sample CM)				
	Portfolios		SBTM6	SBTM25	SBTM100	IP10	IP48	SBTM6	SBTM25	SBTM100	IP10	IP48
Turnover		EW	0,02	0,02	0,03	0,03	0,05	0,02	0,02	0,03	0,03	0,05
		MSR	5,95	13,96	514,61	0,97	170,66	0,49	1,87	12,60	0,49	4,94
		GMV	0,18	0,74	6,77	0,14	0,74	0,12	0,44	1,79	0,09	0,40
	γ = 1	Mvunc	5,61	33,57	2206,08	4,94	42,79	3,51	18,81	210,12	3,03	18,36
		MVc	5,11	30,66	1824,65	4,64	40,75	3,13	17,47	179,47	2,90	17,96
		EWRF	0,21	0,20	0,20	0,25	0,21	0,13	0,13	0,12	0,16	0,14
		GMVRF	2,07	11,10	706,07	1,34	8,21	1,30	5,59	54,59	0,75	2,96
		KZ2F	5,21	27,66	1407,20	4,02	34,32	3,23	14,64	117,57	2,25	13,83
		KWZ	4,70	24,70	1161,42	3,53	27,93	2,80	13,01	98,28	1,74	9,46
		KZ3F	5,14	27,36	1409,07	3,77	29,63	3,14	14,24	117,54	1,87	9,85
		TZ	5,60	32,57	1780,30	4,93	38,78	3,51	18,12	159,15	3,02	16,05
		LA	3,04	12,81	721,77	1,49	8,11	1,94	6,65	57,85	0,88	2,75
		LMS	1,57	8,25	387,05	1,19	9,33	0,94	4,37	32,92	0,59	3,17
	γ = 3	Mvunc	1,87	11,19	735,36	1,65	14,26	1,17	6,27	70,04	1,01	6,12
		MVc	1,71	10,22	608,02	1,55	13,59	1,05	5,85	59,94	0,97	5,99
		EWRF	0,07	0,07	0,07	0,08	0,07	0,04	0,04	0,04	0,05	0,05
		GMVRF	0,69	3,70	235,36	0,45	2,74	0,43	1,86	18,20	0,25	0,99
		KZ2F	1,74	9,22	469,07	1,34	9,89	1,08	4,88	39,19	0,75	3,36
		KWZ	1,58	8,24	386,96	1,18	9,33	0,94	4,36	32,90	0,59	3,17
		KZ3F	1,71	9,12	469,69	1,26	9,88	1,05	4,75	39,18	0,62	3,28
		TZ	1,87	11,14	633,24	1,65	14,00	1,17	6,24	58,13	1,01	5,98
		LA	1,01	4,27	240,59	0,50	2,70	0,65	2,22	19,28	0,29	0,92
		LMS	1,58	8,25	386,99	1,18	9,33	0,96	4,36	32,91	0,60	3,17
			T=120 (Linear Shrinkage CM)					T=180 (Linear Shrinkage CM)				
	Portfolios		SBTM6	SBTM25	SBTM100	IP10	IP48	SBTM6	SBTM25	SBTM100	IP10	IP48
Turnover		EW	0,02	0,02	0,03	0,03	0,05	0,02	0,02	0,03	0,03	0,05
		MSR	0,89	68,44	90,69	0,68	26,33	0,32	1,02	2,15	0,37	1,94
		GMV	0,07	0,29	0,70	0,09	0,34	0,06	0,22	0,52	0,07	0,24
	γ = 1	Mvunc	2,23	11,61	54,81	3,37	17,71	1,63	8,30	33,82	2,32	11,13
		MVc	2,07	10,93	51,04	3,19	17,16	1,52	7,91	31,59	2,21	10,93
		EWRF	0,22	0,21	0,21	0,26	0,23	0,14	0,14	0,13	0,16	0,15
		GMVRF	0,81	3,38	13,27	0,93	3,21	0,56	2,19	6,74	0,58	1,77
		KZ2F	2,00	8,92	31,06	2,67	14,00	1,46	6,01	16,45	1,68	8,19
		KWZ	1,78	7,93	27,98	2,26	10,54	1,30	5,26	14,60	1,17	4,94
		KZ3F	1,91	8,57	30,80	2,41	11,05	1,39	5,64	16,20	1,29	5,19
		TZ	2,23	11,23	41,69	3,36	15,78	1,63	7,97	24,13	2,31	9,59
		LA	1,18	4,00	14,20	1,07	3,06	0,85	2,68	7,47	0,70	1,63
		LMS	0,60	2,65	9,35	0,76	3,52	0,44	1,90	13,62	0,42	2,17
	γ = 3	Mvunc	0,74	3,87	18,27	1,12	5,90	0,54	2,77	11,27	0,77	3,71
		MVc	0,69	3,65	16,99	1,07	5,72	0,52	2,87	30,45	0,80	4,72
		EWRF	0,07	0,07	0,07	0,09	0,08	0,05	0,05	0,04	0,05	0,05
		GMVRF	0,27	1,13	4,42	0,31	1,07	0,19	0,73	2,25	0,19	0,59
		KZ2F	0,67	2,97	10,35	0,89	3,74	0,49	2,00	5,48	0,56	1,82
		KWZ	0,60	2,65	9,31	0,76	3,52	0,44	1,77	4,90	0,40	1,66
		KZ3F	0,64	2,86	10,27	0,80	3,68	0,46	1,88	5,40	0,43	1,73
		TZ	0,74	3,85	15,36	1,12	5,78	0,54	2,75	9,09	0,77	3,62
		LA	0,39	1,33	4,73	0,36	1,02	0,28	0,89	2,49	0,23	0,54
		LMS	0,60	2,65	9,33	0,78	3,52	0,45	1,90	13,62	0,43	2,17

*CM stands for "Covariance Matrix". The Linear Shrinkage Covariance Matrix is from Ledoit and Wolf (2004).

Table 14: Annualized net out-of-sample mean including SBTM25



			T=120 (Sample CM*)					T=180 (Sample CM)				
		Portfolios	SBTM6	SBTM25	SBTM100	IP10	IP48	SBTM6	SBTM25	SBTM100	IP10	IP48
Mean		EW	0,09	0,09	0,09	0,09	0,09	0,09	0,09	0,09	0,09	0,09
		MSR	0,14	0,11	13,21	0,04	16,37	0,18	0,25	0,06	0,05	-0,05
		GMV	0,13	0,12	-0,03	0,08	0,04	0,14	0,13	0,10	0,09	0,05
	γ = 1	Mvunc	1,57	3,54	-282,57	0,32	0,34	1,43	3,16	4,81	0,29	0,20
		MVc	0,67	1,91	-131,35	-0,10	-0,04	0,52	1,63	3,26	-0,15	-0,13
		EWRF	0,22	0,21	0,09	0,28	0,21	0,25	0,25	0,10	0,30	0,23
		GMVRF	0,98	1,49	-24,94	0,45	0,22	1,00	1,43	2,00	0,48	0,26
		KZ2F	1,43	3,03	-116,97	0,24	0,30	1,27	2,50	4,34	0,17	0,15
		KWZ	0,59	1,61	-56,82	-0,08	0,03	0,44	1,24	2,51	-0,06	-0,04
		KZ3F	1,45	3,07	-117,25	0,32	0,33	1,30	2,58	4,38	0,35	0,22
		TZ	1,57	3,45	-185,11	0,32	0,33	1,43	3,05	4,88	0,29	0,19
		LA	1,04	1,47	-25,38	0,41	0,24	1,11	1,46	1,95	0,47	0,31
		LMS	0,29	0,65	-8,43	0,03	0,05	0,24	0,51	1,11	0,04	0,03
	γ = 3	Mvunc	0,53	1,26	-32,48	0,11	0,14	0,48	1,09	3,10	0,10	0,07
		MVc	0,31	0,77	-17,96	0,02	0,04	0,27	0,65	1,82	0,01	0,00
		EWRF	0,07	0,07	0,03	0,09	0,07	0,08	0,08	0,03	0,10	0,08
		GMVRF	0,33	0,51	-3,75	0,15	0,08	0,33	0,48	0,78	0,16	0,09
		KZ2F	0,48	1,06	-13,69	0,08	0,11	0,42	0,86	1,92	0,06	0,04
		KWZ	0,29	0,65	-8,43	0,03	0,05	0,24	0,51	1,12	0,04	0,03
		KZ3F	0,49	1,08	-13,71	0,11	0,13	0,44	0,88	1,95	0,12	0,07
		TZ	0,53	1,25	-24,05	0,11	0,14	0,48	1,08	2,69	0,10	0,07
		LA	0,35	0,50	-3,86	0,14	0,08	0,37	0,49	0,77	0,16	0,10
		LMS	0,29	0,65	-8,43	0,03	0,05	0,24	0,51	1,12	0,04	0,03

			T=120 (Linear Shrinkage CM)					T=180 (Linear Shrinkage CM)				
		Portfolios	SBTM6	SBTM25	SBTM100	IP10	IP48	SBTM6	SBTM25	SBTM100	IP10	IP48
Mean		EW	0,09	0,09	0,09	0,09	0,09	0,09	0,09	0,09	0,09	0,09
		MSR	0,19	-0,72	4,00	0,05	-1,06	0,15	0,23	0,39	0,06	0,02
		GMV	0,11	0,11	0,12	0,08	0,06	0,12	0,12	0,12	0,08	0,06
	γ = 1	Mvunc	0,89	2,01	8,15	0,38	0,54	0,91	2,16	7,23	0,31	0,25
		MVc	0,39	1,06	5,61	-0,03	0,09	0,32	1,10	5,08	-0,11	-0,13
		EWRF	0,23	0,22	0,09	0,30	0,23	0,26	0,26	0,10	0,31	0,25
		GMVRF	0,59	0,92	1,74	0,46	0,38	0,68	1,03	1,38	0,47	0,32
		KZ2F	0,77	1,57	4,84	0,28	0,44	0,77	1,57	3,61	0,18	0,18
		KWZ	0,32	0,80	3,26	-0,01	0,10	0,25	0,75	2,39	-0,03	-0,02
		KZ3F	0,80	1,64	4,97	0,38	0,47	0,81	1,67	3,71	0,37	0,26
		TZ	0,89	1,94	6,38	0,38	0,50	0,91	2,08	5,27	0,31	0,23
		LA	0,57	0,89	1,80	0,42	0,37	0,71	1,03	1,42	0,45	0,36
		LMS	0,18	0,35	1,21	0,05	0,08	0,16	0,30	24,90	0,04	0,03
	γ = 3	Mvunc	0,30	0,68	2,98	0,13	0,19	0,30	0,73	2,53	0,10	0,09
		MVc	0,21	0,44	2,11	0,04	0,08	0,18	0,40	46,73	0,00	0,00
		EWRF	0,08	0,07	0,03	0,10	0,08	0,09	0,09	0,03	0,10	0,08
		GMVRF	0,20	0,31	0,60	0,15	0,13	0,23	0,34	0,47	0,16	0,11
		KZ2F	0,26	0,53	1,70	0,10	0,13	0,26	0,53	1,23	0,06	0,04
		KWZ	0,18	0,35	1,22	0,05	0,08	0,17	0,34	0,90	0,05	0,04
		KZ3F	0,27	0,55	1,75	0,13	0,16	0,27	0,56	1,26	0,12	0,09
		TZ	0,30	0,68	2,52	0,13	0,19	0,30	0,73	2,06	0,10	0,09
		LA	0,19	0,30	0,62	0,14	0,13	0,24	0,35	0,48	0,15	0,12
		LMS	0,18	0,35	1,22	0,05	0,08	0,16	0,30	24,89	0,04	0,03

*CM stands for "Covariance Matrix". The Linear Shrinkage Covariance Matrix is from Ledoit and Wolf (2004).

Table 15: Annualized net out-of-sample standard deviation including SBTM25

			T=120 (Sample CM*)					T=180 (Sample CM)				
		Portfolios	SBTM6	SBTM25	SBTM100	IP10	IP48	SBTM6	SBTM25	SBTM100	IP10	IP48
Std		EW	0,17	0,18	0,18	0,15	0,17	0,17	0,18	0,18	0,15	0,16
		MSR	0,51	0,68	140,94	0,25	93,72	0,21	0,25	0,65	0,19	0,62
		GMV	0,14	0,13	0,27	0,12	0,14	0,14	0,13	0,16	0,12	0,13
	γ = 1	Mvunc	1,79	3,79	502,02	1,56	5,25	1,66	3,18	10,36	1,20	3,29
		MVc	1,22	3,09	354,95	1,18	4,96	1,12	2,59	9,25	0,92	3,17
		EWRF	0,64	0,63	0,54	0,70	0,60	0,53	0,53	0,38	0,56	0,50
		GMVRF	1,24	1,92	52,04	0,96	1,56	1,11	1,59	4,35	0,80	0,84
		KZ2F	1,64	3,21	193,31	1,28	4,30	1,48	2,55	7,00	0,88	2,52
		KWZ	1,07	2,54	134,13	0,87	3,52	0,94	1,97	5,95	0,54	1,78
		KZ3F	1,64	3,21	192,37	1,29	3,79	1,48	2,55	7,02	0,93	1,91
		TZ	1,79	3,69	318,68	1,56	4,81	1,66	3,07	8,67	1,20	2,90
		LA	1,40	2,01	54,62	0,99	1,55	1,30	1,68	4,43	0,80	0,85
		LMS	0,39	0,89	11,07	0,32	1,23	0,35	0,69	2,28	0,21	0,61
	γ = 3	Mvunc	0,60	1,33	44,56	0,53	1,87	0,56	1,09	4,64	0,40	1,13
		MVc	0,43	1,09	30,13	0,42	1,77	0,40	0,90	3,95	0,33	1,09
		EWRF	0,21	0,21	0,18	0,23	0,20	0,18	0,18	0,13	0,19	0,17
		GMVRF	0,41	0,65	6,07	0,32	0,53	0,37	0,53	1,59	0,27	0,28
		KZ2F	0,55	1,12	16,33	0,43	1,32	0,50	0,87	2,77	0,29	0,64
		KWZ	0,39	0,89	11,08	0,32	1,23	0,35	0,69	2,28	0,21	0,61
		KZ3F	0,55	1,12	16,24	0,43	1,33	0,50	0,87	2,78	0,31	0,65
		TZ	0,60	1,33	31,13	0,53	1,84	0,56	1,08	3,94	0,40	1,10
		LA	0,47	0,69	6,18	0,33	0,53	0,44	0,57	1,62	0,27	0,29
		LMS	0,39	0,89	11,08	0,32	1,23	0,35	0,69	2,28	0,21	0,61
			T=120 (Linear Shrinkage CM)					T=180 (Linear Shrinkage CM)				
		Portfolios	SBTM6	SBTM25	SBTM100	IP10	IP48	SBTM6	SBTM25	SBTM100	IP10	IP48
Std		EW	0,17	0,18	0,18	0,15	0,17	0,17	0,18	0,18	0,15	0,16
		MSR	0,26	13,36	68,28	0,23	9,95	0,18	0,20	0,25	0,17	0,38
		GMV	0,14	0,13	0,13	0,12	0,12	0,14	0,13	0,13	0,12	0,12
	γ = 1	Mvunc	1,14	2,08	5,57	1,38	3,37	1,12	1,95	4,63	1,09	2,53
		MVc	0,67	1,61	4,92	0,99	3,11	0,69	1,50	4,01	0,78	2,38
		EWRF	0,68	0,66	0,58	0,74	0,65	0,55	0,55	0,40	0,59	0,53
		GMVRF	0,98	1,27	2,25	0,92	1,15	0,87	1,15	1,56	0,78	0,77
		KZ2F	1,01	1,63	3,36	1,11	2,69	0,96	1,44	2,31	0,77	1,89
		KWZ	0,54	1,17	2,83	0,69	1,95	0,52	1,00	1,85	0,41	1,14
		KZ3F	1,04	1,66	3,41	1,13	2,22	0,98	1,49	2,36	0,85	1,34
		TZ	1,13	2,01	4,38	1,38	3,02	1,12	1,88	3,36	1,08	2,19
		LA	0,99	1,26	2,24	0,95	1,18	0,91	1,17	1,51	0,78	0,80
		LMS	0,21	0,41	0,98	0,26	0,67	0,23	0,42	22,10	0,22	0,71
	γ = 3	Mvunc	0,38	0,70	2,05	0,46	1,15	0,37	0,66	1,61	0,36	0,86
		MVc	0,25	0,56	1,77	0,35	1,07	0,29	0,63	46,97	0,38	1,46
		EWRF	0,23	0,22	0,19	0,25	0,22	0,18	0,18	0,13	0,20	0,18
		GMVRF	0,33	0,43	0,78	0,31	0,39	0,29	0,39	0,53	0,26	0,26
		KZ2F	0,34	0,55	1,19	0,37	0,74	0,32	0,48	0,79	0,26	0,44
		KWZ	0,21	0,41	0,98	0,26	0,67	0,22	0,36	0,63	0,18	0,40
		KZ3F	0,35	0,56	1,21	0,38	0,75	0,33	0,50	0,80	0,28	0,45
		TZ	0,38	0,70	1,74	0,46	1,13	0,37	0,65	1,31	0,36	0,84
		LA	0,33	0,42	0,78	0,32	0,40	0,30	0,39	0,51	0,26	0,27
		LMS	0,22	0,41	0,98	0,26	0,67	0,23	0,42	22,09	0,22	0,71

*CM stands for "Covariance Matrix". The Linear Shrinkage Covariance Matrix is from Ledoit and Wolf (2004).

Table 16: Annualized net out-of-sample SR including SBTM25

			T=120 (Sample CM*)					T=180 (Sample CM)				
	Portfolios		SBTM6	SBTM25	SBTM100	IP10	IP48	SBTM6	SBTM25	SBTM100	IP10	IP48
SR		**EW**	0,52	0,52	0,51	0,60	0,53	0,53	0,52	0,51	0,61	0,55
		MSR	0,28	0,16	0,09	0,17	0,17	0,87	1,02	0,09	0,28	-0,08
		GMV	0,94	0,89	-0,11	0,63	0,26	0,97	0,97	0,60	0,69	0,38
	γ = 1	**Mvunc**	0,88	0,93	-0,56	0,20	0,06	0,86	0,99	0,46	0,24	0,06
		MVc	0,55	0,62	-0,37	-0,08	-0,01	0,46	0,63	0,35	-0,16	-0,04
		EWRF	0,34	0,33	0,16	0,41	0,35	0,46	0,46	0,26	0,53	0,47
		GMVRF	0,79	0,77	-0,48	0,47	0,14	0,90	0,90	0,46	0,60	0,31
		KZ2F	0,87	0,94	-0,61	0,19	0,07	0,85	0,98	0,62	0,19	0,06
		KWZ	0,55	0,63	-0,42	-0,09	0,01	0,47	0,63	0,42	-0,12	-0,02
		KZ3F	0,88	0,96	-0,61	0,25	0,09	0,88	1,01	0,62	0,38	0,11
		TZ	0,88	0,93	-0,58	0,20	0,07	0,86	1,00	0,56	0,24	0,06
		LA	0,74	0,73	-0,46	0,42	0,16	0,85	0,87	0,44	0,58	0,36
		LMS	0,74	0,73	-0,76	0,09	0,04	0,69	0,75	0,49	0,17	0,04
	γ = 3	**Mvunc**	0,88	0,94	-0,73	0,20	0,07	0,86	1,00	0,67	0,24	0,06
		MVc	0,72	0,70	-0,60	0,05	0,02	0,66	0,73	0,46	0,03	0,00
		EWRF	0,34	0,33	0,16	0,41	0,35	0,46	0,46	0,26	0,53	0,47
		GMVRF	0,79	0,77	-0,62	0,47	0,15	0,90	0,90	0,49	0,60	0,31
		KZ2F	0,87	0,95	-0,84	0,19	0,08	0,85	0,98	0,69	0,19	0,06
		KWZ	0,74	0,73	-0,76	0,09	0,04	0,69	0,75	0,49	0,17	0,04
		KZ3F	0,88	0,96	-0,84	0,25	0,10	0,88	1,02	0,70	0,38	0,11
		TZ	0,88	0,94	-0,77	0,20	0,08	0,86	1,00	0,68	0,24	0,07
		LA	0,74	0,73	-0,63	0,42	0,16	0,85	0,87	0,48	0,58	0,36
		LMS	0,74	0,73	-0,76	0,10	0,04	0,68	0,75	0,49	0,17	0,04
			T=120 (Linear Shrinkage CM)					**T=180 (Linear Shrinkage CM)**				
	Portfolios		**SBTM6**	**SBTM25**	**SBTM100**	**IP10**	**IP48**	**SBTM6**	**SBTM25**	**SBTM100**	**IP10**	**IP48**
SR		**EW**	0,52	0,52	0,51	0,60	0,53	0,53	0,52	0,51	0,61	0,55
		MSR	0,73	-0,05	0,06	0,21	-0,11	0,82	1,14	1,56	0,33	0,05
		GMV	0,81	0,90	0,94	0,66	0,49	0,85	0,96	0,95	0,69	0,53
	γ = 1	**Mvunc**	0,79	0,97	1,46	0,28	0,16	0,81	1,11	1,56	0,28	0,10
		MVc	0,58	0,66	1,14	-0,03	0,03	0,47	0,73	1,27	-0,14	-0,05
		EWRF	0,34	0,34	0,16	0,41	0,35	0,46	0,46	0,25	0,52	0,47
		GMVRF	0,60	0,73	0,77	0,50	0,33	0,77	0,89	0,89	0,60	0,42
		KZ2F	0,77	0,97	1,44	0,26	0,16	0,80	1,09	1,56	0,23	0,10
		KWZ	0,59	0,68	1,15	-0,02	0,05	0,48	0,75	1,30	-0,07	-0,02
		KZ3F	0,76	0,99	1,46	0,34	0,21	0,82	1,12	1,57	0,43	0,20
		TZ	0,79	0,97	1,46	0,28	0,16	0,81	1,11	1,57	0,28	0,10
		LA	0,58	0,70	0,81	0,44	0,32	0,77	0,89	0,94	0,58	0,44
		LMS	0,85	0,86	1,24	0,19	0,12	0,69	0,71	1,13	0,20	0,05
	γ = 3	**Mvunc**	0,79	0,97	1,45	0,28	0,17	0,81	1,11	1,57	0,28	0,10
		MVc	0,81	0,79	1,19	0,12	0,07	0,62	0,64	1,00	0,00	0,00
		EWRF	0,34	0,34	0,16	0,41	0,35	0,46	0,46	0,25	0,52	0,47
		GMVRF	0,60	0,72	0,76	0,50	0,33	0,77	0,89	0,89	0,60	0,42
		KZ2F	0,77	0,97	1,42	0,26	0,18	0,80	1,09	1,56	0,23	0,08
		KWZ	0,85	0,86	1,24	0,19	0,12	0,76	0,94	1,42	0,26	0,10
		KZ3F	0,76	0,99	1,44	0,34	0,21	0,82	1,12	1,57	0,43	0,20
		TZ	0,79	0,97	1,44	0,28	0,17	0,81	1,11	1,57	0,28	0,10
		LA	0,58	0,70	0,79	0,44	0,32	0,77	0,89	0,94	0,58	0,44
		LMS	0,85	0,86	1,24	0,19	0,12	0,70	0,71	1,13	0,19	0,05

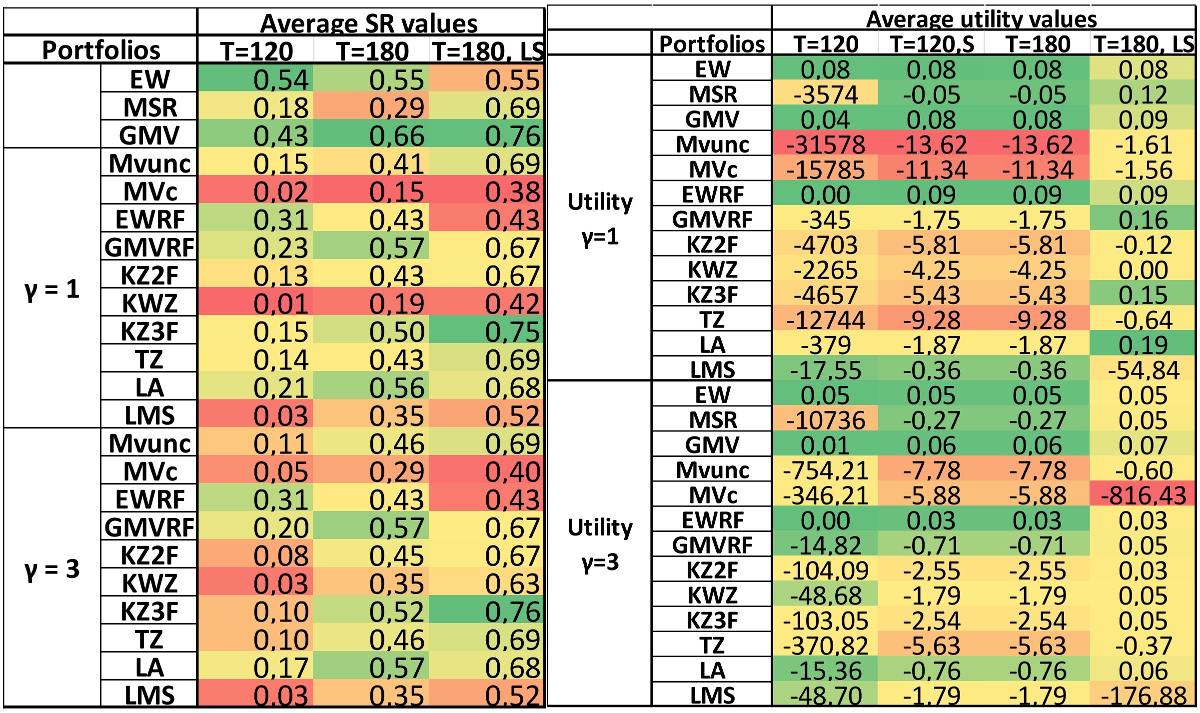
*CM stands for "Covariance Matrix". The Linear Shrinkage Covariance Matrix is from Ledoit and Wolf (2004).

Table 17: Annualized net out-of-sample utility including SBTM25

		T=120 (Sample CM*)					T=180 (Sample CM)				
		SBTM6	SBTM25	SBTM100	IP10	IP48	SBTM6	SBTM25	SBTM100	IP10	IP48
Utility γ=1	EW	0,07	0,08	0,08	0,08	0,07	0,08	0,08	0,07	0,08	0,08
	MSR	0,01	-0,12	-9919	0,01	-4376	0,16	0,22	-0,15	0,03	-0,24
	GMV	0,12	0,11	-0,06	0,07	0,03	0,13	0,12	0,08	0,08	0,04
	Mvunc	-0,03	-3,66	-126296	-0,91	-13,42	0,05	-1,89	-48,87	-0,44	-5,22
	MVc	-0,08	-2,87	-63126	-0,80	-12,35	-0,10	-1,71	-39,54	-0,57	-5,16
	EWRF	0,01	0,01	-0,06	0,04	0,03	0,11	0,10	0,03	0,14	0,11
	GMVRF	0,21	-0,36	-1379	0,00	-0,99	0,38	0,17	-7,47	0,16	-0,09
	KZ2F	0,09	-2	-18802	-0,58	-8,95	0,16	-0,75	-20,16	-0,21	-3,03
	KWZ	0,01	-2	-9052	-0,45	-6,17	0,00	-0,71	-15,18	-0,21	-1,62
	KZ3F	0,10	-2,09	-18621	-0,51	-6,84	0,20	-0,67	-20,23	-0,08	-1,61
	TZ	-0,03	-3,38	-50965	-0,90	-11,24	0,05	-1,65	-32,73	-0,43	-4,01
	LA	0,06	-0,55	-1517,00	-0,08	-0,96	0,26	0,05	-7,85	0,15	-0,05
	LMS	0,21	0,26	-69,67	-0,02	-0,71	0,18	0,28	-1,48	0,01	-0,16
Utility γ=3	EW	0,05	0,04	0,04	0,06	0,05	0,05	0,05	0,04	0,06	0,05
	MSR	-0,25	-0,58	-29784	-0,05	-13159	0,12	0,16	-0,58	0,00	-0,63
	GMV	0,10	0,09	-0,13	0,06	0,01	0,11	0,10	0,06	0,06	0,02
	Mvunc	-0,01	-1,41	-3011	-0,31	-5,13	0,01	-0,69	-29,16	-0,15	-1,84
	MVc	0,03	-1,02	-1380	-0,24	-4,65	0,02	-0,56	-21,62	-0,15	-1,78
	EWRF	0,00	0,00	-0,02	0,01	0,01	0,04	0,03	0,01	0,05	0,04
	GMVRF	0,07	-0,14	-58,99	0,00	-0,35	0,13	0,05	-2,99	0,05	-0,03
	KZ2F	0,03	-0,81	-413,68	-0,20	-2,51	0,05	-0,28	-9,59	-0,07	-0,58
	KWZ	0,06	-0,54	-192,42	-0,12	-2,23	0,06	-0,19	-6,67	-0,03	-0,53
	KZ3F	0,03	-0,80	-409,52	-0,17	-2,52	0,07	-0,25	-9,63	-0,03	-0,55
	TZ	-0,01	-1,40	-1478	-0,31	-4,95	0,01	-0,68	-20,62	-0,15	-1,75
	LA	0,02	-0,21	-61,10	-0,03	-0,34	0,09	0,01	-3,17	0,05	-0,02
	LMS	0,06	-0,54	-192,51	-0,12	-2,23	0,06	-0,19	-6,67	-0,03	-0,53
		T=120 (Linear Shrinkage CM)					T=180 (Linear Shrinkage CM)				
		SBTM6	SBTM25	SBTM100	IP10	IP48	SBTM6	SBTM25	SBTM100	IP10	IP48
Utility γ=1	EW	0,07	0,08	0,08	0,08	0,07	0,08	0,08	0,07	0,08	0,08
	MSR	0,15	-90,03	-2327	0,02	-50,58	0,13	0,21	0,36	0,04	-0,06
	GMV	0,10	0,11	0,11	0,07	0,05	0,11	0,11	0,11	0,08	0,06
	Mvunc	0,25	-0,15	-7,35	-0,57	-5,12	0,28	0,26	-3,49	-0,28	-2,96
	MVc	0,16	-0,24	-6,49	-0,53	-4,74	0,08	-0,03	-2,96	-0,41	-2,95
	EWRF	0,00	0,00	-0,07	0,03	0,02	0,10	0,10	0,02	0,14	0,11
	GMVRF	0,10	0,12	-0,79	0,04	-0,28	0,29	0,36	0,17	0,16	0,03
	KZ2F	0,27	0,25	-0,81	-0,33	-3,18	0,31	0,53	0,94	-0,12	-1,61
	KWZ	0,17	0,11	-0,74	-0,25	-1,80	0,12	0,25	0,69	-0,12	-0,67
	KZ3F	0,25	0,26	-0,84	-0,26	-2,00	0,32	0,56	0,92	0,00	-0,63
	TZ	0,25	-0,08	-3,22	-0,57	-4,07	0,28	0,32	-0,38	-0,28	-2,18
	LA	0,08	0,09	-0,71	-0,03	-0,32	0,29	0,36	0,28	0,15	0,03
	LMS	0,16	0,27	0,73	0,02	-0,14	0,13	0,21	-219	0,02	-0,22
Utility γ=3	EW	0,05	0,04	0,04	0,06	0,05	0,05	0,05	0,04	0,06	0,05
	MSR	0,09	-269	-6989	-0,03	-149,60	0,10	0,17	0,30	0,01	-0,20
	GMV	0,08	0,09	0,10	0,06	0,04	0,09	0,10	0,10	0,06	0,04
	Mvunc	0,08	-0,06	-3,35	-0,19	-1,81	0,09	0,08	-1,37	-0,09	-1,02
	MVc	0,11	-0,02	-2,58	-0,14	-1,63	0,05	-0,19	-3262	-0,21	-3,19
	EWRF	0,00	0,00	-0,02	0,01	0,01	0,03	0,03	0,01	0,05	0,04
	GMVRF	0,03	0,04	-0,32	0,01	-0,10	0,10	0,12	0,05	0,05	0,01
	KZ2F	0,09	0,08	-0,44	-0,11	-0,70	0,10	0,18	0,30	-0,04	-0,25
	KWZ	0,11	0,10	-0,23	-0,05	-0,59	0,09	0,15	0,30	0,00	-0,20
	KZ3F	0,08	0,08	-0,46	-0,09	-0,69	0,11	0,19	0,30	0,00	-0,21
	TZ	0,08	-0,06	-2,05	-0,19	-1,73	0,09	0,08	-0,51	-0,09	-0,96
	LA	0,03	0,03	-0,29	-0,01	-0,11	0,10	0,12	0,09	0,05	0,01
	LMS	0,11	0,10	-0,23	-0,05	-0,59	0,08	0,03	-706,83	-0,03	-0,73

*CM stands for "Covariance Matrix". The Linear Shrinkage Covariance Matrix is from Ledoit and Wolf (2004).

Table 18: Average Annualized SR and utility with T=120 and T=180



Portfolios		Average SR values		
		T=120	T=180	T=180, LS
	EW	0,54	0,55	0,55
	MSR	0,18	0,29	0,69
	GMV	0,43	0,66	0,76
γ = 1	Mvunc	0,15	0,41	0,69
	MVc	0,02	0,15	0,38
	EWRF	0,31	0,43	0,43
	GMVRF	0,23	0,57	0,67
	KZ2F	0,13	0,43	0,67
	KWZ	0,01	0,19	0,42
	KZ3F	0,15	0,50	0,75
	TZ	0,14	0,43	0,69
	LA	0,21	0,56	0,68
	LMS	0,03	0,35	0,52
γ = 3	Mvunc	0,11	0,46	0,69
	MVc	0,05	0,29	0,40
	EWRF	0,31	0,43	0,43
	GMVRF	0,20	0,57	0,67
	KZ2F	0,08	0,45	0,67
	KWZ	0,03	0,35	0,63
	KZ3F	0,10	0,52	0,76
	TZ	0,10	0,46	0,69
	LA	0,17	0,57	0,68
	LMS	0,03	0,35	0,52

		Average utility values			
	Portfolios	T=120	T=120,S	T=180	T=180, LS
Utility γ=1	EW	0,08	0,08	0,08	0,08
	MSR	-3574	-0,05	-0,05	0,12
	GMV	0,04	0,08	0,08	0,09
	Mvunc	-31578	-13,62	-13,62	-1,61
	MVc	-15785	-11,34	-11,34	-1,56
	EWRF	0,00	0,09	0,09	0,09
	GMVRF	-345	-1,75	-1,75	0,16
	KZ2F	-4703	-5,81	-5,81	-0,12
	KWZ	-2265	-4,25	-4,25	0,00
	KZ3F	-4657	-5,43	-5,43	0,15
	TZ	-12744	-9,28	-9,28	-0,64
	LA	-379	-1,87	-1,87	0,19
	LMS	-17,55	-0,36	-0,36	-54,84
Utility γ=3	EW	0,05	0,05	0,05	0,05
	MSR	-10736	-0,27	-0,27	0,05
	GMV	0,01	0,06	0,06	0,07
	Mvunc	-754,21	-7,78	-7,78	-0,60
	MVc	-346,21	-5,88	-5,88	-816,43
	EWRF	0,00	0,03	0,03	0,03
	GMVRF	-14,82	-0,71	-0,71	0,05
	KZ2F	-104,09	-2,55	-2,55	0,03
	KWZ	-48,68	-1,79	-1,79	0,05
	KZ3F	-103,05	-2,54	-2,54	0,05
	TZ	-370,82	-5,63	-5,63	-0,37
	LA	-15,36	-0,76	-0,76	0,06
	LMS	-48,70	-1,79	-1,79	-176,88

Notes: This table presents the average net out-of-sample values for the SBTM6, SBTM100, IP10, and IP48 datasets. The SBTM25 was excluded to ensure an equal number of datasets from each type (two SBTMs and two IPs). Including SBTM25, however, led to an overall increase in values, as it follows a similar trend to SBTM6, but the overall trends remain unchanged.

UNIVERSITÉ CATHOLIQUE DE LOUVAIN
Louvain School of Management
Chaussée de Binche 151, 7000 Mons, Belgique | **www.uclouvain.be/lsm**