

Faculté de philosophie, arts et lettres

The system R of relevant implication

A philosophical and logical presentation

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Introduction

I.1 CarL

To introduce our subject matter, we invite our reader to the following fictional situation. We use the « you » pronoun for narrative immersion. As you are visiting the Silicon Valley, enjoying the warm summer of California, you stumble upon an old friend. You recognise each other, and after a brief moment of conversation, you learn that she is now an IT graduate. She works for a well-established company that is developing various systems of artificial intelligence (AI). She starts to expose her most recent project, called CarL. CarL is not merely an AI, with which the only way to communicate would be through the input of, say, a keyboard. CarL has been embodied in a functional, humanoid-like form. He has sensors (he can gather visual, auditive and direct-touch information), is connected to internet and is able to walk and talk.

In fact, your friend explains to you that CarL's main functionality is to express true propositions and valid reasonings, according to the standards of propositional classical logic, abbreviated **CL**. The name of the AI is a reminder of that important fact. Your friend adds : « CarL is basically never wrong. All the sentences that are verbally expressed by CarL are true, and all the arguments he produces are valid. This doesn't mean that CarL is omniscient, for he surely doesn't know all true sentences and all valid arguments ». You show curiosity, and you are soon invited to meet CarL.

The meeting goes rather unexpectedly. CarL seems at first to express true sentences. He says sentences like : « if Garfield the cat is on a table and playing with a toy, then Garfield the cat is on a table »¹ or « if it is either sunny or cloudy, then if it's not cloudy, it is sunny ». But quickly, among the various things expressed by CarL, you hear what sounds like weird, if not false, sentences. CarL says : « either if it's raining, then I'm a cat, or if I'm a cat, then it's raining » and later : « some humans like chocolate, therefore, there is no integer n greater than or equal to 3 such that for any non-zero integers x, y, z , $x^n = y^n + z^n$ »². Those are not the only doubtful sentences that CarL is capable of producing, but this is enough for you to go back to your friend and express your puzzlement to her. Your friend gets your point and adds : « maybe we could do something to CarL so he doesn't sound so strange ».

¹ Example adapted from RIVENC, F., *Introduction à la logique pertinente*, p.1 (now cited RIVENC, F., *ILP*).

² Example adapted from MARES, E.D., *Relevant Logic : A Philosophical Interpretation*, p. 3 (now cited MARES, E.D., *RLPI*).

I.2 The role of logic

Before we can understand what is going wrong with CarL, we must understand the role and the nature of his code. That is, we must discuss an important question : what is a logic and what is it expected to accomplish ? As Rivenc reminds us, « at the heart – or at the horizon, if one prefers – of logical theory is, since its birth, the concept of consequence »³. Indeed, he observes that Aristotle had such consideration in mind when he wrote in his *Prior Analytics* that « a deduction is a discourse (*logos*) in which, certain things having been supposed, something different from the things supposed results of necessity because these things are so »⁴. Since logical systems⁵ are interested in a notion of consequence that doesn't depend on the content of what is said or assumed, it should be no surprise that they do not use natural languages, like French or English, but instead use formal languages.

I.2.1 Syntax and semantics

Logical systems can be described from a *syntactic point of view*, or from a *semantical point of view*. The former is focused on notions like connectives, well-formed formulae (wffs), axioms, theorems and the like. The connectives have intended natural language counterparts, and logical systems are not usually produced without any philosophical goal in sight. The basic notions related to those systems are presented for the interested reader in an appendix to the present work, Appendix A. This being said, syntactic systems are nevertheless sometimes felt wanting. What they develop are no more, some say, than sets of strings of symbols without any meaning. If we are claiming that our syntactic system is representing, say, pieces of information, we need extra grounds. Now, it could be uncontroversial to say that the system represents « argumentative elements » or an expression of the kind. But this last expression is in need of analysis, that could be, indeed, claimed to be informational. And again we would be asked for a serious analysis of that notion to be linked with our logical system, that is, we would be asked for a *semantics* : a way to interpret our language with the help of intuitive notions that are nevertheless given mathematical treatment. The notion could be that of information, but it could be that of truth, or of possible worlds, or any other notion we are acquainted with.

Copeland, following Plantinga, draws a distinction between *pure semantics* which are just mathematical structures sometimes called « models », associated with the syntactic system considered ; and *applied semantics* which also yield an intuitive notion

³ RIVENC, F., *ILP*, p. XIII. All the quotes from this source are translated by us.

⁴ ARISTOTLE, *Prior Analytics* (translation by Smith, R.), I.2, 24b18–20.

⁵ In the present work, we use the expressions « logic », « logical system », « logical theory » and the like interchangeably, without drawing any distinction between them.

fitting the mathematical entities of the structure⁶. The philosophically satisfying semantics required to answer the criticism formulated in the previous paragraph against syntactic systems is that of applied semantics. Apart from extra enlightening, such semantics elegantly reinforces the feeling that the notion of consequence defined really « [is] stable, that we can find it from different points of view »⁷; it can help settle particular philosophical questions related to the syntactic system considered, in particular the meaning of their connectives; and it is an important tool that can help prove new theorems in easier and more intuitive ways. The exact characteristics that should be expected from an applied semantics will be discussed later⁸.

Semantics also enable one to start talking about the universe, or domain, of discourse. Since they are interpretations of the logical system, they constitute the place where one is fully allowed to start talking about the entities represented in the formal language. So it is at a semantical level that one, it seems, starts interpreting the formulae of **CL** as representing *propositions* of natural languages⁹.

1.2.2 Pure and applied logics

Not all logics are used to merely link propositions. Some, like first-order (classical) logic, are more fine-grained, and enable one to analyse the internal structure of propositions. Here, the analysis can represent predicates and relations; and is capable of quantifying variables over individuals. Some others are not directly concerned with the structure of our arguments and so, unlike **CL** or its predicative extension, are not focused on developing a notion of validity. Instead, they provide an analysis of particular notions like beliefs, temporality, knowledge, duties and so on. We call logics that are primarily concerned with reasoning and validity *pure logics* and those that are destined for an analysis of particular notions like beliefs *applied logics*¹⁰.

1.2.3 Pure logics and reasoning

Two important philosophical questions might emerge from the idea of providing a theory of human reasoning through the construction of a logical system. First: what is the status of such theory? That is: how does it provide an analysis distinct from those of other fields that are concerned with similar phenomenon, like, say, psychology,

⁶ See COPELAND, B.J., « What is a semantics for classical negation? » in *Mind*, 95 (1986), p. 480.

⁷ RIVENC, F., *ILP*, p. XIII.

⁸ See section 5.2.

⁹ See BOUQUIAUX, L., and LECLERCQ, B., *Logique formelle et argumentation*, pp. 66-70 (now cited BOUQUIAUX, L., and LECLERCQ, B., *LFA*). And for first-order classical logic, the universe of discourse is enhanced to contain predicates and relations, with appropriate specifications.

¹⁰ We follow the terminology of RIVENC, F., *ILP*, p. XV. Usually, pure logics are called formal logics, but we avoided this term to circumvent any sense of confusion, since both pure logics and applied logics use formal languages.

sociology, neurobiology,... ? And, other subquestion, has logic a prescriptive role to play in our ways of reasoning, that is, should we feel compelled to make our reasonings fit the structure put to light by a pure logic ; or is the role of such logic purely descriptive ? Second question : is logic adapted to capture the whole of human reasoning, or is it confined to some particular kind, like deductive reasoning ? Here it is the range of applicability, the limits, of such theory that are questioned.

Let us start by saying this : those are difficult questions, for which a complete answer goes well beyond the scope of the present work. But since our general approach and a lot of what will be discussed is at stake in those questions, we shall risk the formulation of possible lines of thought and position ourselves on those matters.

1.2.3.1 The status of pure logics

The first question to examine is related to the status of pure logic. What is the role of logic and what should we expect it to provide us ? Edwin Mares' view on this matter can be seen through the prism of semantics. Once we have developed a syntactic logical system, providing an account of validity, we can attach it to a semantics :

what a semantics should explain [...] are certain sorts of linguistic intuitions held by native speakers. [...] Semanticists should judge the success of their project against the fit between entailments, synonymies, contradictions, and so on, that it forces and native speakers' intuitions about these phenomena¹¹.

Here the explanatory role of semantics is enabled by the syntactic systems, which constitute the basis for semantical analysis. And that analysis must be confronted with speakers' practice. Our account of validity is then justified by the fact that the appropriate semantics attached to it is shown fitting people's inferential practice.

We see clearly here that logical systems must « confront reality », and that they cannot be seen as a set of undeniable truths everyone should endorse. A subtle distinction must be introduced here, between a necessary proposition or entity and an undeniable proposition. Consider numbers. In particular, consider the number « 7 ». One may try to show the particularity of such entity in the following way. One may start by writing « 7 » down on a blackboard. And then one may ask the audience, or a particular member of her audience : « what is this ? » pointing at the symbol « 7 » on the blackboard. The common answer would be « this is the number 7 ». But then one may mark another symbol « 7 » on the blackboard, and ask the same question. The answer may come out identical. Then one can go on to ask : « which one of those is 7 » ? A natural answer would be « both ».

¹¹ MARES, E.D., *RLPI*, p. 15. Mares quotes a citation from other supporters of this view in exposing his own (DOWTY, D., PETERS, S., and WALL, R., *Introduction to Montague Semantics*, p. 2).

But it seems that our 7 is weirdly non-local, it is at two places at the same time. One may erase the symbols « 7 » of the blackboard and ask : « what is 7 added to 3 » ? A typical answer would be : « 10 ». But then one may ask : how can 7 added to 3 result to 10 if 7 has been erased from the surface of the Earth (or at least from the surface of the blackboard) ? Clearly, the answer is that our « 7 »'s on the blackboard were *representations* of some abstract entity, the number 7. We can say a lot of things about 7 that stay true once we have erased its representations (it is a prime number, it is the only integer strictly between 6 and 8, it is the result obtained by subtracting 3 from 10, etc.) : 7 is an abstract entity that relates to other numbers in a way that consequently does not seem to depend on empirical entities like blackboards, schools, cats, chairs and so on ; since 7 is not an entity of the empirical world. Those particular entities have been pinpointed as being *necessary*.

Now, one may want to say that « $7+3=10$ » is a necessary truth without committing herself to the existence of the corresponding entities. And the existential status of necessary entities can be argued to be quite different from those of empirical entities. There have been a lot of metaphysical discussions around those issues in the history of philosophy all the way up to our days. Mathematical entities are primary targets for the rank of necessary entities, but there have been other candidates : think of the concept of essence, of an absolute space and time, or, more important for us, think of logical truths.

Our goal here is not to throw ourselves in discussions that go way beyond the scope of the present work. The important question here concerns the *fallible* character of truths concerning necessary entities. Put bluntly : how can it be that logical truths are necessary even though the success, and so the acceptance, of a logical theory rests on empirical data, such as its capacity to fit speakers' practice ? The answer rests on a distinction between a claim to knowledge and a claim to truth.

Fallibilism, as presented by Read, is the thesis according to which « any statement, whether universal or particular, is defeasible »¹². Read makes the point :

For the fact that a proof, if correct, will establish its conclusion as necessarily true cannot preclude the possibility that the supposed 'proof' is incorrect. Mistake is as likely in constructing proofs as it is in carrying out observation and experiment. That a discipline deals in necessities does not mean that it and it alone can preclude error¹³.

This does not condemn us to scepticism, for Read notes that the sceptic's traditional argument against the possibility of knowledge rests on a conflation of justification and

¹² READ, S., *Relevant Logic : A Philosophical Examination of Inference*, p.17 (now cited READ, S., *PEI*).

¹³ *Ibidem*, p. 13.

truth : « [the] mistake is, in essence, to suppose that if knowledge requires interalia the truth of the claim, then in order to justify one's claim to knowledge, one must justify the claim that it is true »¹⁴. Our claims to knowledge can be justified or not justified depending on the proofs or evidence available. And furthermore, those claims will be true if the so-believed-to-be-acknowledged theories or states of affairs are so, and false if they are not. As Read states : « we can give grounds for justifiably making a claim to knowledge, but we cannot establish it categorically. Later discovery may show it to have been mistaken – it is fallible »¹⁵.

If indeed $3+7=10$, we can claim that it is *necessarily* so. But we can never preclude any risk of error altogether. So, we can never be *totally certain that* what we take to be knowledge really is knowledge, and that what we think is true indeed is true. But we can have very *good grounds* to justify our claims to knowledge, and so to back up our claims to *know the truth*. And when we have very good grounds to assert « $3+7=10$ », we can have equally good grounds to assert « necessarily, $3+7=10$ ». Now, Read does not believe in metaphysical necessity, to him « what is necessary is [just] what we decide, at any given time, not to give up come what may »¹⁶ ; so he denies the idea expressed earlier that numbers, logical truths, etc. indeed have a special metaphysical status and potentially a special « mode of existence ».

Another philosopher, Quine, has famously advocated a position very close to Read's. Quine has developed a holistic epistemology¹⁷, according to which our beliefs constitute a web, a structure where each element is susceptible to revision, some with more impact, when modified, on the rest of the structure than others. At the center of the web stand the beliefs we are least susceptible to change, because doing so can have dramatic impact on the rest of the web : those are mathematical and logical beliefs. In Quine's view, necessary truths (studied in mathematics, logic, etc.) are also defeasible, they just happen to be very widely connected to our other beliefs, so it is not economical, when counter-evidence appears to a particular belief, to change our beliefs deeply to account for the new evidence. We try to change our beliefs « locally ». This gives a way to understand the way science works : as Kuhn noted¹⁸, when scientists encounter new evidence that contradict their theories (Kuhn calls such evidence an « anomaly »), they do not throw away the whole theory. Instead, they try to add an *ad hoc* explanation to account for the anomaly. They are very local and economical in their belief changes, until so much

¹⁴ READ, S., *PEI*, p. 11.

¹⁵ *Ibidem*, p. 16.

¹⁶ *Ibidem*, p. 17. The expressions between « [] » in citations are added by us.

¹⁷ See QUINE, W.V.O., « Two dogmas of empiricism » in *The Philosophical Review*, 60 (1951), pp. 20-43.

¹⁸ For this part, see KUHN, T.S., *The Structure of Scientific Revolution*.

persistent anomalies reluctant to their explanatory attempts have gathered that members of the scientific community start looking for more general solutions : they try to fix the web from « deeper » inside in order to explain away the anomalies. This is what leads to scientific revolutions. Since Quine considers our beliefs as constituting a web, theories must be accepted, on his view, holistically : we have to consider them as « packages » of beliefs that may interfere greatly with our current web. A logical theory, in this view, is accepted not only for, say, its axioms and rules of inference, but also for its theorems, the view of negation it provides, the semantics that may be associated to it, the theory of validity that can be recovered from it,... ; and we need to see all those elements in the light of the way logic may be linked to science, to our inferential practices, to the general view of the world it may ground, and in general to the philosophical ideas that may justify endorsing it.

Quine, just like Read, has rejected the idea that logical truths had a special, necessary status¹⁹. But as Gillian K. Russell has argued²⁰, Quine's holistic epistemology is perfectly compatible with a view of logical truths as having the status of necessities. Such status may be important for metaphysical reasons, we said it, but also due to the way logicians conceive their work²¹, and it can be hard to relinquish because « it is embedded so deeply in logical practice. For example, [the] rule of necessitation in the proofs systems for standard modal logics [...] allows one to place logical truths within the scope of a '□' »²².

Russell holds that accepting a logical theory as a whole, holistically, doesn't prevent that theory from establishing atomic proofs or proofs based on claims about meaning²³. There is a twofold movement here, reminiscent of that exposed by Read²⁴ between theory and evidence. According to him, theories are inductively (and we could add, holistically) accepted based on their explanatory success for a class of phenomena, but the relation between the theory and the evidence is deductive. Alike, here we accept our logical theory based on its success on providing an account of (our uses of) consequence and validity. But the nature of the relations so established in our theory once we have accepted it could be that of necessity.

¹⁹ See RUSSELL, G.K., « Metaphysical analyticity and the epistemology of logic » in *Philosophical Studies*, 171 (2014), pp. 168-170.

²⁰ See *ibidem*, pp. 170-174.

²¹ See *ibidem*, p. 170.

²² *Ibidem*, p. 169. We will see that standard modal logics are sometimes called « normal modal logics », see section 5.2.

²³ See *ibidem*, p. 173.

²⁴ See READ, S., *PEI*, p. 12.

In fact, the confusion may stem from the following modal analysis²⁵. Fallibilism holds that for any proposition p , p is defeasible. So we can write for any p « $\neg \Box p$ » which can be read « it is false that we know for sure that p ». This is equivalent to « $\Diamond \neg p$ ». How should it be read ? One may want to read it as « it is possible that $\neg p$ ». On the other hand, necessary truths would be written « $\Box p$ », to be read « it is necessary that p ». Which is in turn equivalent to « $\neg \Diamond \neg p$ », « it is not possible that $\neg p$ ». So it looks like we have two contradictions, and that we can't admit necessity and fallibilism. But the operators²⁶ are misleading. Fallibilism is only a thesis about knowledge, not about the actual metaphysical properties of the world. We can rewrite the fallibilist thesis as « $\neg \Box_E p$ » for any p , to be read as before. For its equivalent, « $\Diamond_E \neg p$ », it is unclear what the correct reading is. The operator says something about knowledge only. Something along the line : « it may be the case that we end up realizing that p was false ». And we can rewrite our metaphysical necessities as « $\Box_m p$ » and « $\neg \Diamond_m \neg p$ ». We would have then that we cannot know for sure that a necessary proposition holds. That is, we'd have « $\neg \Box_E (\Box_m p)$ ». But it still may be that any p holds (that it is true in our world), despite our epistemic impossibility to know it for sure. So, we can still have p , and in particular, if p is a necessity, $\Box_m p$. Hence it seems that we can endorse fallibilism and admit necessities after all.

Those different positions made clear, we can now clarify our philosophical stance. In the present work, we chose to opt for a rather pragmatic attitude. Concerning the status of pure logics, we could argue as follows : in this domain, the work of the logician constitutes a clarificatory task that studies the structure of our inferences. Her work is made distinct by the methodological choices that underly it : representing elements of the language with letters, linking them with symbols that have intended natural language counterparts, providing a list of plausible rules of inference, proving theorems to see how they can relate to our linguistic practice, giving an interpretation of the language obtained, and so on. This would justify pure logic as a particular tool for the understanding of inferential practice, the success of which should be evaluated by the enlightenment provided and its correspondence with the phenomenon at hand. Just like applied logics should be judged by their capacity to provide a satisfying analysis of the concept they try to capture. This is a fairly weak assumption on the nature of logic, and is in the spirit of Mares' position depicted earlier. Furthermore, we accept the fallibilist thesis and Quine's epistemological holism. It may be that we thought for a long time that « $\vdash A \supset (B \supset A)$ »

²⁵ If our reader is not at ease with modal talk, she can skip this paragraph and come back after reading section 5.3.

²⁶ An « operator is a mathematical entity which takes an element from a set and returns a unique element from the same set », MARES, E.D., *RLPI*, p. 78. Modal connectives take propositions and return propositions, so they are operators.

was a valid way of reasoning because we accepted a logical theory that validated it. So, we thought that the scheme could hold for any two propositions A and B. But after comparing it to our inferential practice, we may have found that it wasn't actually a valid mode of inference, that it did not fit our practice, and we may consequently have chosen to reject it and the logical theory that made us validate it in the first place. Should we follow Quine and Read by holding that there is no metaphysical necessity? We think not. We prefer to stay neutral on this particular matter, and leave open the possibility that logical truths really have a special, distinct modal status.

The prescriptive status of logic can be explained through two different, but, we think, compatible stories. First, it could just be a by-product of its metaphysical status. If necessary truths really are necessary, then there cannot be any deductive reasoning that does not fit their form. Much like there cannot be in our universe any state of affairs that goes against a natural law, there cannot be any state of affairs that goes against a logical truth. So we must make our inferences fit our logical theory. It may still be the case that we are wrong about the truths of necessary propositions or entities, and so on what counts as prescriptive. But this is without impact on the present story of the prescriptive status of logic.

Second, and this is our preferred explanation, since it does not rely on the metaphysical status of logical truths, and since it 'fits' better the general pragmatic approach we are trying to sketch here, one could explain the prescriptive status of logic as follows. Since what we are clarifying with those relations is nothing less than (part of) the structure of human reasoning, we expect, if our (pure) logical theory gets things right, that (some class of) valid arguments should have the structure we put to light. So, the particular 'prescriptive' nature of logical theory would come, it seems, from the general relationship between theories and evidence, and from its object of investigation. And this should not be overestimated: remember that the grounds for accepting or rejecting a logical theory are linked to the capacity of the theory to explain properly our reasonings. Given any reasoning that is incompatible with her theory, the logician has to weight the different elements at stake and opt for the best solution. She can amend her theory in an *ad hoc* manner, give up some parts of the logical theory she has developed (and so, technically, develop a new proof system/a new semantics), embrace some kind of pluralism, or reject the reasoning as fallacious. In any case she has to provide a justification²⁷. On this view, the prescriptive status of logic for individuals is restricted to

²⁷ This complicates the pragmatic features displayed by the logician when constructing or evaluating a theory. Where the logician has the option to deny the existence of a good argument presented to her by holding, that, in fact, it is not a good argument at all, most scientists cannot just deny the existence of an explanandum presented to them.

the acceptance of certain language games : if a person accepts a certain view of what a correct argument is, that is, accepts a certain logical theory, then this person has to endorse whatever comes with that theory. Again, as we said, we like to think that those two stories about the prescriptive status of logic are compatible, and leave open the metaphysical question of the status of logic. We consequently endorse Russell's view that a holistic epistemology does not lead to the rejection of the necessary status of logic altogether.

I.2.3.2 The limitations of pure logics

Those processes of weighing and finding the best solution to a problematic new evidence are among the class of reasonings that are sometimes considered as involving something else than deductive arguments, those studied by pure logics. Here comes the second question we asked in the beginning of section I.2.3, concerning the limitations of pure logics in their ability to capture the whole of human reasoning. As is well exposed in the second part of *Logique formelle et argumentation*²⁸, there are indeed legitimate and serious doubts on the pretention of logic to do so. A lot of the arguments that we construct do not take the form of deductive inferences. This fact was already acknowledged by Aristotle²⁹, and number of more recent authors have criticized too « deductivist » approaches³⁰.

In a lot of cases indeed, the premises of an argument make its conclusion more likely, without validly inferring it. Such arguments are not inductive either, but possess a distinct structure : they are called « conductive arguments »³¹. Most of the time, the different

premises constitute, not a logical chain which proceeds from first premises all the way to the ultimate conclusions ; [but] acts as a web of interlinked arguments, the global solidity of which depends more on the cohesion of its fibres, on the mutual support they provide to each other, than on its weakest link, like in a demonstration³².

Another problem of trying to capture our reasonings with logical systems lies in the gap between the formal languages used to represent the inferences and the natural languages from which they originate. The logician methodologically ignores some peculiarities of natural languages when she translates expressions from it into her formal language, and an important work of interpretation must be accomplished in the process. But such

²⁸ BOUQUIAUX, L., and LECLERCQ, B., *LFA*, pp. 120-159. The content and references of section 1.2.3.2 stem from these pages.

²⁹ See ARISTOTLE, *Topics*, (translation by Pickard-Cambridge, W.A.), in particular I.1-2, 100a18-101a25 ; and BOUQUIAUX, L., and LECLERCQ, B., *LFA*, pp. 129 ; 137-138.

³⁰ See for example PERELMAN, C., *Justice et raison* or BLAIR, J.A., and JOHNSON, R., (eds.), *New essays in Informal Logic*.

³¹ See BLAIR, J.A., and JOHNSON, R., (eds.), *Conductive argument. An overlooked type of defeasible reasoning*.

³² BOUQUIAUX, L., and LECLERCQ, B., *LFA*, p. 128, our translation.

peculiarities may be of great importance, even essential to the activity of communication³³.

From those observations, some philosophers have developed a new way to analyse reasoning, developing « informal logics » which incorporate, among other things, the general and transdisciplinary conditions of acceptability of premises, the evaluation of defeasible schemes of argumentation, the dialectic dimension of communication and evaluation, and the necessity to consider multiple layers for the understanding of sophisms and reasoning mistakes³⁴.

We may defend the rather pragmatic attitude we have tried to sketch in the preceding subsection from such criticism in two ways. We may (1) restrict the logician's field of investigation to *deductive* reasoning only, and extend the features of the logician's work to include the interpretative components that are stake when translating a reasoning from natural languages into formal languages. This reduces the scope of the logician's work³⁵ and further elaborates the particular methodological features of the field of logic. We may alternatively (2) keep the logician's studied phenomenon constant, and extend her methodology, not only to include interpretative components, but also to integrate the kind of elements that have been studied by « informal » logics. Now we can also ponder such criticism by noting that entailment is not the only relation studied by logics. When syntactic systems study entailment, they do not provide a criterion to distinguish the cases in which $A \supset B$ holds from the cases in which it does not. They are primarily concerned with *logically* (and so *necessary*, whatever this might mean) valid implications or arguments. This relation of *entailment* relates to the theoremhood of the system. For instance, $A \wedge B$ entails A in **CL**, $\vdash (A \wedge B) \supset A$: if we have $\vdash A \wedge B$, then we have $\vdash A$ ³⁶.

In contrast, the system can study the contingent relation that can hold between A and B , in which case the relation is pinned under the term « implication », and is weaker than entailment. Adding a semantics here comes handily : for example, **CL**'s truth-functional semantics presented in the next section tells us that an implication holds between A and B whenever either A is false or either B is true.

Anyways, even if we are to accept that the logical criteria for an implication to hold are too strong for encoding the validity of conductive arguments, we can still endorse that logical systems provide precious information on our inferential practices, and the

³³ See BOUQUIAUX, L., and LECLERCQ, B., *LFA*, pp. 128-129 ; 145-150.

³⁴ See *ibidem*, pp. 128-130.

³⁵ Note that if we are to believe Read's analysis on the relation between evidence and theory, deductive reasoning should be particularly important for the scientist's work. See subsection I.2.3.1, §5.

³⁶ Given the Deduction Equivalence, see section I.4.

structure put to light by pure logics can still be philosophically very valuable, in particular when associated with an interesting semantics. Widening the logician's scope of investigation doesn't mean that her previous tools of analyse are made useless, it mostly means that she should be more open minded when it comes to evaluating new tools of inquiry.

I.3 Truth-functionality of propositional classical logic

Having examined important critical aspects related to our subject matter, it is time for us to move on and get an understanding of classical logic, necessary to analyse CarL's problems. Classical logic is a logical system that can be semantically analysed as a purely *truth-functional* system. This is the main feature of the system, which enables us to bypass a deep axiomatic study of it steadily and still get a clear idea of how the system works. The logic is said to be truth-functional because the *truth value* of any wff of the system depends entirely on the truth values of its components. A truth value is the possible value that a wff can take. It is a notion that fits the concepts of truth and falsity, with which we are acquainted with by the common usage we make of it. In particular, **CL** is *bivalent*. Any wff of the system always takes exactly one truth value, either the value true or the value false.

More formally³⁷, we construct an *interpretation* denoted « $\mathcal{I}$ », for our formal language. An interpretation of a set of wffs associates to each of them either the value « true » or the value « false ». Here the wffs can be considered plainly as propositions in the case of **CL**. The interpretation $\mathcal{I}$ rests on a function v that associates to each atomic formula either the value true or the value false. Then $\mathcal{I}$ is defined on complex propositions by associating to them truth values that vary with the connective and that are functions of the truth values of the atoms. A formula A is a *tautology* if it is always true under $\mathcal{I}$, that is, it is true for all value assignments to its atomic formulae. We denote the tautology A with « $\models A$ ». « $\models$ » belongs to the metalanguage, just like its syntactic counterpart, « $\vdash$ ». Table I.1 presents the function $\mathcal{I}$ for **CL**'s usual connectives. We denote the value « true » with the numeral « 1 » and the value « false » with the numeral « 0 ».

Given the present interpretation and any axiomatic system equivalent to Lukasiewicz' (actually, any syntactic system sound and complete with respect to it)³⁸, it

³⁷ For this paragraph, see BOUQUIAUX, L., and LECLERCQ, B., *LFA*, p. 68.

³⁸ When both completeness and soundness of a system **A** are proven with respect to a system **B**, we obtain an equivalence property : a formula C is a theorem of **A** *if and only if* it is a theorem of **B**. The same goes when considering a semantics and a syntactic system, except that we are interested in tautologies, rather than theorems, in the former. But such equivalence properties are transitive : if we prove **A** to be sound and complete with respect to **B** and a semantics S to be sound and complete with respect to **A**, then S is sound and complete with respect to **B**.

can be proven that $\llmodels A\gg$ in our semantics if and only if $\ll\vdash A\gg$ in our syntactic system **CL**. That is, the set of tautologies of the semantics perfectly fits the set of theorems of the syntactic system considered. This metatheorem rests on a soundness and completeness proof of the syntactic system with respect to the semantics.

Table I.1³⁹ : CL's usual connectives and their truth value

(1) Unary connective : negation, denoted $\ll\neg\gg$ or $\ll\sim\gg$

p	$\neg p$ or $\sim p$
1	0
0	1

(2) Binary connectives : conjunction, $\ll\wedge\gg$; disjunction $\ll\vee\gg$; and material implication, $\ll\supset\gg$

p	q	$p \wedge q$	$p \vee q$	$p \supset q$
1	1	1	1	1
1	0	0	1	0
0	1	0	1	1
0	0	0	0	1

Note that if we follow **CL**, connectives can be redefined in terms of one another. In particular, the implicational connective, called material implication, is equivalent to a particular disjunction, $\ll\neg p \vee q\gg$:

p	q	$\neg p$	$\neg p \vee q$
1	1	0	1
1	0	0	0
0	1	1	1
0	0	1	1

³⁹ Our numbering of extra-textual elements is unique across all categories (tables, figures, examples and so on). It indicates the number of the chapter first (here we used the letter I for « introduction ») and then its position in the chapter's extra-textual elements (here is the first extra-textual element of the introduction).

Indeed, the truth table for material implication can be read as « an implication is true if the proposition in the consequent ('q') is true or the proposition in the antecedent ('p') is false, and false otherwise ».

I.4 The Classical Account of validity and the Deduction Equivalence

There is an important link between our formal language and the concept of validity that is guaranteed by two important philosophical ideas, from which the truth table of material implication follows.

The first idea is called *the Classical Account (of Validity)*, which states the condition for an argument to be valid. It holds that the impossibility of true premises and false conclusion constitutes the necessary and sufficient condition for the validity of an argument. The philosophical motivation behind this choice is retraced by Read :

A first thought is this: the point of a valid argument is to establish the truth of the conclusion by adducing premises from which it follows. That is, it is necessary to the validity of an argument that if the premises are true, so is the conclusion. Hence, if an argument has true premises and a false conclusion, or even⁴⁰ if it is logically possible for its premises to be true and its conclusion false, then the truth of the premises is clearly inadequate to guarantee the truth of the conclusion, and so the argument cannot be valid. That is, a necessary condition of the validity of an argument is that it be impossible for the premises to be true and the conclusion false. But more: suppose conversely that it is impossible for the premises to be true and the conclusion false. Then the truth of the premises would appear to rule out the falsity of the conclusion, and so, it would seem, we may validly infer the truth of the conclusion from that of the premises, and so the argument is valid. Hence, we seem to have shown that the impossibility of true premises and false conclusion is not only necessary, but also sufficient, for validity⁴¹.

The second idea creates the bridge between this account of validity and material implication. It is referred to as the *Deduction Theorem* or the *Deduction Equivalence*⁴². Its formulation is the following :

« B follows from $A_1, \dots, A_n, A$
if and only if
'if A then B' follows from $A_1, \dots, A_n$ »⁴³.

⁴⁰ Note that this « or even » particle introduces a rather different criterion, clearly characterizing validity as entailment-related rather than implication-related. See our final note on section I.2.3.1.

⁴¹ READ, S., *PEI*, p. 19.

⁴² We use « Deduction Equivalence » to refer to the philosophical idea of creating a bridge between deducibility and implication ; and « Deduction Theorem » to the particular result for a formal system.

⁴³ *Ibidem*, p. 21. We omitted Read's italic police on letters and developed the « iff » expression.

The second expression can be further developed for **CL**, which is tantamount to rewriting it as :

B follows from $A_1, \dots, A_n, A$
if and only if
' $A \supset B$ ' follows from $A_1, \dots, A_n$.

Note that the comma « , » appearing here is used to group the premises, it is equivalent to an « and » expression.

We can show that the truth table of implication follows from the Classical Account and the Deduction Equivalence combined in the following way⁴⁴. We represent the expression « follows from » with the turnstile symbol, and use the version of the Deduction Equivalence that makes explicit the use of material implication. The other logical particles are left unsymbolized, to show that the reasoning is not actually part of a logical system (it is a « meta » reasoning).

1. $\text{not-}P$ and $P \vdash Q$ (Classical Account⁴⁵)
2. $\text{not-}P \vdash P \supset Q$ ((1), Deduction Equivalence)

The truth of $\text{not-}P$ is sufficient to establish the truth of $P \supset Q$. If we accept that $\text{not-}P$ is true if and only if P is false, then the falsity of P is sufficient to establish the truth of $P \supset Q$.

1. Q and $P \vdash Q$ (Classical Account⁴⁶)
2. $Q \vdash P \supset Q$ ((1), Deduction Equivalence)

The truth of Q is sufficient to establish the truth of $P \supset Q$.

The last case to examine is the truth value of $P \supset Q$ in which P is true and Q is false.

$P \supset Q$ is equivalent to $P \vdash Q$ by the Deduction Equivalence. But by hypothesis, the premise P is true and the conclusion Q is false. Since by the Classical Account the impossibility of true premises and false conclusion is a necessary condition for the validity of an argument, the argument is invalid : $P \not\vdash Q$. So, the material implication $P \supset Q$ cannot be true, (because if so, we would have $P \vdash Q$ by the Deduction Equivalence) and so it is false when P is true and Q is false. Combining those three conditions gives us the truth table for material implication, see the end of previous section.

⁴⁴ See READ, S., *PEI*, p. 22.

⁴⁵ « $\text{not-}P$ and P » is always false and so anything follows from it. The necessary falsity of « $\text{not-}P$ and P » ensures us that we don't have true premises, and *a fortiori*, that we don't have true premises and a false conclusion.

⁴⁶ « Q and P » cannot be true if Q is false. So, if « Q and P » is true, *a fortiori*, Q is necessarily true. And so, our premises cannot be true with our conclusion false.

I.5 CarL's problems

Till now, we have seen that CarL's background logic is classical logic, that it has a truth-functional semantics associated to it, and that it is founded on the acceptance of the Classical Account and of the Deduction Equivalence. With those considerations in mind, we can finally grasp what bothers us in CarL's set of possible assumptions. We provide here a review of the more acknowledged ones⁴⁷.

1. Arguments with necessarily true conclusions : remember CarL's arguments, that from « some humans like chocolate » we can validly infer that « there is no integer n greater than or equal to 3 such that for any non-zero integers x, y, z , $x^n = y^n + z^n$ ». If we consider mathematical truths as necessarily so, it is impossible for them to be false. So, by the Classical Account, if we take it as the conclusion of an argument, any premise will validly enable us to infer them. For it is *a fortiori* impossible for the conclusion to be false *and* the premise true. If the premise is true, the argument is also *sound*. Another example of this unintuitive form of inference can take the logical form $P \vdash Q \vee \neg Q$. Since $Q \vee \neg Q$ is a tautology, it is impossible for it to be false. And so, by the Classical Account, any « P » allows us to validly infer it. As a consequence, CarL is allowed to say such things as « Macron is the president of France, therefore either water freezes at 0°C or water doesn't freeze at 0°C ».
2. Arguments with necessarily false (inconsistent) premises : this oddity, as the previous one, was used in our previous section (see our notes 45 and 46). The Classical Account ensures us that from such premises anything can be inferred. So, CarL is licensed to perform weird inferences of the kind « the sun is shining and the sun is not shining, therefore I am a stone ». To understand what, on a syntactic level, is responsible for this peculiarity, known as *ex falso [sequitur] quodlibet* (EFQ) or *the principle of explosion*, we will later present what we'll refer to as the « Lewis Argument »⁴⁸.
3. Thanks to the Deduction Equivalence, the arguments expressed above can be formalized in a syntactic system with the help of material implication, taking the form of tautologies. This would give us, respectively :

$$(M1) \quad \vdash B \supset (A \vee \neg A)$$

$$(EFQ) \quad \vdash (A \wedge \neg A) \supset B$$

⁴⁷ More to come later : see section 1.7.3. For the present examples, see MARES, E.D., *RLPI*, pp. 3-10.

⁴⁸ See section 3.3.

But other unintuitive tautologies using material implication can be proved in **CL**. They are sometimes referred to as the *paradoxes of material implication*. Here are the most common ones, followed by an example of corresponding natural language proposition, that is susceptible to be expressed by CarL :

- (M2) $\vdash A \supset (B \supset A)$
 « If it's raining, then if I'm a cat, it's raining »
- (M3) $\vdash \neg A \supset (A \supset B)$
 « If it's not raining, then if it's raining, I'm a cat »
- (M4) $\vdash (A \supset B) \vee (A \supset B)$
 « Either if it's raining, then I'm a cat, or if I'm a cat, then it's raining »
- (M5) $\vdash (A \supset B) \vee (B \supset C)$
 « Either if it's raining, then I'm a cat, or if I'm a cat, Santa Claus is bald »

Remember that those are tautologies : the truth values of those wffs are 'true' regardless of the truth values of their components. We may note that there are some similarities of form between some of these expressions. In (M1) and (M2) for example, any « B » implies an expression because that expression is considered true, in (M1) because it's a tautology ; and in (M2) because we affirmed its truth in the antecedent of the main connective (the « $\supset$ » outside the parentheses). This should be expected given the Classical Account.

This kind of observation can help find solutions to CarL's problems. In all the cases, it seems that we have been authorized to create an implicational or inferential connection between formulae that are irrelevant to one another. There seems to be no link whatsoever between the taste of chocolate and Fermat's last theorem ; the political status of Macron and the freezing point of water ; or between the weather and my place in the animal reign ! The inferences we make and the conditionals⁴⁹ we express require some kind of link between, respectively, the premises and the conclusion ; and the antecedent and the consequent. So, if our logical system is supposed to formalize our reasoning, as is expected from a pure logic, then it is too permissive, and the relation of implication too weak.

And so, it seems that at least one of the two philosophical ideas of section I.4 should be abandoned. This led some philosophers to develop new logical systems with a stronger notion of consequence and a new definition of validity. The system that has been the most studied and has constituted the most serious contender to **CL** is the system **R** of relevant implication. The central aim of the present work is to

⁴⁹ The term « conditional » is used to refer to our (natural language) « if...then... » uses.

present **R** and discuss the philosophical issues that surround it. Only marginally will we discuss close logical systems that are usually considered to share a « relevant » aspect with it.

I.6 Related literature⁵⁰

The most important works in the development and justification of **R**, as well as the study of other related logical systems, are due to Anderson and Belnap⁵¹ (with the contribution of Dunn Michael), with Ackermann's system as the starting point⁵²; and Routley and Meyer⁵³ (in particular for the development of its semantics). Important contributions have been made by Restall⁵⁴, Mares⁵⁵, Fine⁵⁶ and Read⁵⁷. Our reader may also find extra information on other systems of substructural logics as well as on **R** in Restall's introduction⁵⁸. For the literature on informal logics, we redirect our reader to the authors and books cited in subsection I.2.3.2. More references can be found in *LFA*⁵⁹.

I.7 Plan of the work

Our presentation of **R** and of the philosophical issues that surround it will take the following structure. Our Part I will be concerned with a first approach to **R**. We resituate the motivation for its construction in chapter 1, then develop and present its Hilbert-style system as well as its natural deduction system in chapters 2 and 3. Part II will be focused on providing different philosophical perspectives for **R**, in particular, we will try to find resources to deal with the problems met in **R**'s development of chapter 3. Chapter 4 presents the structural aspects of the system, while chapter 5 discusses the role of a semantics and introduces the possible worlds semantics. Chapter 6 will be dedicated to

⁵⁰ This section regroups references of authors concerned with our subject or subjects related to ours. The references are indicative, this does not constitute an exhaustive list of important publications or authors, but more of an indication.

⁵¹ See ANDERSON, A.R., and BELNAP, N., *Entailment, the Logic of Relevance and Necessity*, vol. I ; (now cited ANDERSON, A.R., and BELNAP, N., *ELRN I*) and ANDERSON, A.R., BELNAP, N., and DUNN, M., *Entailment, the Logic of Relevance and Necessity*, vol. II.

⁵² See ACKERMANN, W., « Begründung einer strengen Implikation » in *The Journal of Symbolic Logic*, 21 (1956).

⁵³ See ROUTLEY, R., MEYER, R., PLUMWOOD, V., and ROSS, B., *Relevant Logics and their Rivals*, vol. I and ROUTLEY, R., MEYER, R., « Semantics of entailment » in *Truth, Syntax, and Modality* (Leblanc, H. ed.), 68 (1973).

⁵⁴ See RESTALL, G., « Information Flow and Relevant Logics » in *Logic, Language and Computation* (Seligman, J., Westerståhl, D. eds.), vol. I.

⁵⁵ See MARES, E.D., *RLPI* and MARES, E.D., « Relevant logic and the theory of information » in *Synthese*, 109 (1996). More references for him can be found in *RLPI*, p. 220.

⁵⁶ See FINE, K., « Semantics for quantified relevance logics » in *Journal of Philosophical Logic*, 17 (1988) and FINE, K., « Models for entailment » in *Journal of Philosophical Logic*, 3 (1974).

⁵⁷ See READ, S., *PEI*, and READ, S., « What is wrong with disjunctive syllogism ? » in *Analysis*, 41 (1981).

⁵⁸ See RESTALL, G., *An introduction to substructural logics*.

⁵⁹ See BOUQUIAUX, L., and LECLERCQ, B., *LFA*, pp. 120-159.

Mares' presentation of the Routley-Meyer semantics. Part III regroups the critical aspects, with chapter 7 dedicated to a discussion of the semantics as well as what will be introduced as « Geach's challenge », and chapter 8 dedicated to several philosophical issues, from the question of the metalanguage to the question of pluralism.

Part I

First approach to **R**

- 1 – Motivating **R**
- 2 – The implicational fragments **R_h**[→] and **NDR**[→]
- 3 – The systems **R_h** and **NDR** of relevant implication

Chapter 1 : Motivating **R**

Before we can jump into our presentation of **R**'s syntactic systems, it is important to go back to its motivations. We showed in our introduction the existence, in **CL**, of very unintuitive results. Some are concerned with theorems using the connective « $\supset$ », and have been labelled « paradoxes of material implication » ; and some are concerned with the corresponding inferences, which constitute « non-sequiturs ». Such unintuitive results, we said, motivated the construction of **R**. But it may be the case that other strategies are able to explain away such oddities as well. The present chapter's goal is to discuss such strategies and situate **R**'s development among them.

1.1 Finding solutions

Let's start by emphasizing the problematic character of these results. To our knowledge, no philosopher has tried to deny the unintuitive character of the theorems or inferences at hand. So, any logician interested in giving an account of reasoning must give us a story to explain them away (that is, a plausible story dissolving our feeling of discomfort towards the said results). Different types of strategies can be pursued to accomplish this task : (1) one can amend **CL**'s syntactic system and/or semantics appropriately, and correct at least one of the two philosophical ideas that led to the truth table of material implication, or (2) one can justify its retaining. Those options are not exclusive to one another, nor exhaustive, for it is possible for one to (3) embrace some kind of logical pluralism, which is to say that several logical systems can or must be used to provide an account of reasoning. The development of **R** falls within the type of strategy (1) (and potentially (3), of course), but one might think of other ways to pursue such strategy type.

1.2 Changing the truth table of material implication

Indeed, one may be tempted to embrace what might appear at first glance as a more conservative approach : instead of developing a new formal system like **R**, requiring a new semantics as well, one might try to change the truth table for the material implication only. If we are in a Hilbert-style system, we can always take an axiomatic system expressed without the implicational connective and inter-define it in another way⁶⁰.

But on second thought, this appears way more clunky than it may seem. For the behaviour of material implication in the natural deduction system and in the structural

⁶⁰ See Appendix A, sections A.1 and A.2. One could even take Lukasiewicz' axioms presented in our example A.2, translate it into the disjunctive language using the equivalence between $A \supset B$ and $\neg A \vee B$, and use the resulting system, since it contains only negation and disjunction.

one would be dramatically modified. And its truth table has not been picked at random : remember that it is a consequence of endorsing the Classical Account and the Deduction Equivalence. So, one would need to reformulate and motivate the change on one of those meta-results, as well as defending the plausibility of the new truth table for « $\supset$ ».

This last task seems in turn compromised. Indeed, consider the potential candidates for the replacement of the material implication's truth table. There are $2^4 = 16$ possible truth tables for a binary connective. If we want the truth table to match our conditionals' uses, we want it to have « true » as an output when both the consequent and the antecedent are true. We also want it, in this perspective, to have « false » as an output when the antecedent is false and the consequent true. This leaves us with $2^2 = 4$ possibilities, since we fixed two lines of the truth table. One of these is the actual, 'old' truth table for implication, so it won't do anything for us. Another of these is just the truth table for the conjunction connective « $\wedge$ ». If we don't want to conflate our uses of « and » with our uses of « if... then... », this won't do. So, we are left with 2 candidates. One of them has its truth table equivalent to that of the consequent. It gives the output « true » whenever the consequent's input is « true », and it gives the output « false » whenever the consequent's input is « false ». So, if affirming an implication is different from affirming its consequent, this possibility too must be ruled out. Our last contender has « true » as an output when both the antecedent and the consequent's inputs are « true », and when both their inputs are « false ». Its output is « false » otherwise. But this truth table is the same as the one that is usually retained for logical equivalence. So it is judged more appropriate to codify our « ...if and only if... » uses, and we should expect this requirement too strong : it would hold for a too little number of instances compared to that of our 'old' truth table.

1.3 Extensionality and intensionality

The conclusion to draw from our previous subsection is that it is very unlikely that one can solve the paradoxes of material implication by « simply » changing the truth table for « $\supset$ ». That is, if one is committed to a revisionist strategy of type (1), no truth-functional analysis of the implicational connective will do.

All the connectives of **CL** have a common property, they are all extensional connectives. We call the *extension* of a term the (set of) object(s) it denotes – the « morning star » and the « evening star » have consequently the same extension, since they both refer to the planet Venus –, and the *extension* of a sentence its truth value – all the false sentences have the same extension, and all the true sentences also have the same

extension⁶¹. An *extensional connective* is a connective such that it maps extensions to extensions⁶². That is, its input is an extension, and its output is an extension. Since all the connectives of **CL** have as inputs the truth values of propositions, that is, the extensions of sentences ; and as outputs truth values, that is, the extensions of (compound) sentences, all the connectives of **CL** are extensional. The same holds for **CL_P**, the predicative conservative extension⁶³ of **CL**. But as soon as one follows a strategy of type (1), our previous section showed that such property would be lost, because at least the implicational connective would need an analysis different from that enabled by truth tables, and so at least the implicational connective would lose its extensional character.

Where the extension of a term refers to the (set of) objects it denotes, its intension refers to its meaning and definitional traits. Connectives that are not extensional are sometimes also labelled *intensional*. There is some equivocation here. Sometimes, it seems that the term « intensional » is taken as a synonym for « non-extensional »⁶⁴, and sometimes the term is used more precisely in connection with normal (classical) modal logics⁶⁵. We would like to say a few words on the latter characterization of a connective. The preceding definitions of extensions stem from the more general idea, taken from Carnap, that « the *extension* of a term is what the term shares with all terms that are equivalent to it »⁶⁶. A similar idea was pursued for the definition of the *intension* of a term, which is « what it shares with all the terms that are *logically* equivalent to it »⁶⁷. In this perspective, it is the notion of *possible world* that has been investigated. Two terms have the same *intension* when they are both true in the same set of possible worlds ; the intension of a term being precisely the set of possible worlds in which it is true. Intuitively, a possible world is a world conceivably different from ours, which respect the same necessary constraints as ours⁶⁸.

⁶¹ For predicate logic, we consider the *extension* of a n-ary predicate as the n-tuples that satisfy it. These definitions come from PEREGRIN, J., « Extensional vs. Intensional Logic » in *Handbook of the Philosophy of Science*, 5 (2007), p. 918.

⁶² This definition comes from NEALE, S., « On a Milestone of Empiricism » in *Knowledge, Language and Logic: Questions for Quine* (Orenstein, A., and Kotatko, P. eds.), §7, p. 246.

⁶³ See Appendix A, section A.3 for the definition of a conservative extension.

⁶⁴ See the uses of Read in READ, S., *PEI*, in particular p. 3 : « the resulting formal system contains several non-truth-functional (i.e. intensional) operators. When formal systems with such operators (modal or relevant) were first proposed, they were challenged to produce a semantics which would show them to be coherent and viable logics ».

⁶⁵ See PEREGRIN, J., « Extensional vs. Intensional Logic » in *Handbook of the Philosophy of Science*, 5 (2007), pp. 918-928. It should be noted that while the « intension » with an « s » is linked to the meaning of a word, the behaviour of a connective and the like, the « intention » with a « t » is linked to purposes, conscious states and the like.

⁶⁶ *Ibidem*, p. 918. Italic in the text.

⁶⁷ *Ibidem*. Italic in the text.

⁶⁸ For all this, see *Ibidem*, pp. 918-928.

For example, the « morning star » and the « evening star » share the same extension, but do not share the same intension, for they do not *necessarily* correspond to Venus. Nothing logical force those expressions to correspond to the star Venus, and so there is a possible world where Venus is, say, the morning star, but not the evening star ; or the other way around. And there is a possible world where Venus is neither. We can define an *intensional connective* as a connective that maps intensions to intensions. Following Berto, we can say that where extensional connectives allow for « substitution *salva veritate* of sentences with the same extension »⁶⁹, intensional connectives allow for « substitution *salva veritate* of sentences with the same intension »⁷⁰. Note that in this perspective, it is possible to study (and indeed, this is Berto's goal) operators that fail to be extensional or intensional, and are rather « hyperintensional », for they do not allow substitutions of sentences which are nonetheless true in the same possible worlds⁷¹.

1.4 Adding necessity

The notion of possible worlds is closely tied to a certain family of formal systems. Different possible worlds semantics can indeed be shown to be sound and complete with respect to different systems of *normal (classical) modal logics*⁷². Modal logics contain two unary (intensional) connectives, « $\Box$ » and « $\Diamond$ », sometimes interpreted as, respectively, necessity and possibility. The different axioms encoding the behaviour of the connectives correspond to different constraints on the accessibility relation between the worlds in the semantics. We keep the formal presentation for later⁷³ : we only need, for now, to examine whether those logics can solve the oddities of **CL** with which we are concerned in this chapter.

C. I. Lewis, who expressed his dissatisfaction with such results in 1917⁷⁴, tried to adopt a strategy of type (1) while remaining conservative in another way than keeping the truth tables. Instead, he tried to preserve the implicational connective itself, and to explain away the oddities of **CL** by holding that implication « is really *strict* implication »⁷⁵, in a system that is a conservative extension of **CL**. The claim is that our

⁶⁹ From the lecture of Franz Berto, BERTO, F., « Some Problems of Possible Worlds Semantics » (lecture 1) in *The Topics of Thought* (mini-course), slides 5-6.

⁷⁰ *Ibidem*.

⁷¹ See *ibidem*. Consider for example the belief « I have 30% chance of success » in comparison with the belief « I have 70% chance of failing ». Although those expression are logically equivalent (they have the same intensions), the corresponding beliefs can be different, as they can lead people to take different decisions (the first has a more positive connotation).

⁷² See section 5.2 for the characterization of a normal modal logic.

⁷³ See chapter 5.

⁷⁴ LEWIS, C.I., « The Issues Concerning Material Implication » in *The Journal of Philosophy, Psychology and Scientific Methods*, 14 (1917), pp. 350-356.

⁷⁵ MARES, E.D., *RLPI*, p. 8. Italic added by us.

inferential practices are to be captured, not by « $A \supset B$ », but by « $\Box(A \supset B)$ », where « $\Box$ » is to be interpreted as a connective encoding necessity.

But such maneuver seems insufficient : the intensions of necessary truths are identical, because the corresponding sentences are true in every possible world. The same holds for necessary falsities, which are false in every possible world. So, the paradoxes of material implication with necessary false antecedents or necessary true consequents remain, and we have new versions of the paradoxes we are trying to run away from. Indeed, even with strict implication « all valid formulae follow from any proposition »⁷⁶ and « any proposition [follows from] any logical falsehood »⁷⁷, which is contrary to our pre-theoretical intuitions. Where « $A \rightarrow_{\text{ML}} B$ » stands for « $\Box(A \supset B)$ », we have in those systems⁷⁸ :

- | | |
|--|----------------|
| (S1) $\vdash B \rightarrow_{\text{ML}} (A \vee \neg A)$ | matching (M1) |
| (EFQ ₂) $\vdash (A \wedge \neg A) \rightarrow_{\text{ML}} B$ | matching (EFQ) |

1.5 The classicist's answer

Far from convincing one that we should be more radical in our revisionist approach, our present situation may be taken as the clue that we've got things wrong since the beginning. One may think that the oddities of **CL** will vanish once we've properly understood the matter at hand. And so, one may pursue a conservative strategy of type (2) instead (as your friend suggested, in order to solve CarL's problems). In what follows, we refer to the philosophers and logicians that advocated this type of strategy as « the classicists ». Their defence generally takes as a basis Grice's analysis in *Studies in the Way of Words*, chapters 2 and 3.

1.5.1 Grice's theory of conversational maxims⁷⁹

The general idea of Grice is to show that the gap between our natural language uses and formal language devices can be filled by certain general features of our conversations⁸⁰. In these pages, Grice distinguishes, in the total signification of our communications, between what is said and what is implicated. The term *implicature* is used to denote the latter. We can illustrate what an implicature is using one of Grice's example : « A is standing by an obviously immobilized car and is approached by B ; the following exchange takes place :

⁷⁶ MARES, E.D., *RLPI*, p. 8.

⁷⁷ *Ibidem*, p. 9.

⁷⁸ Adapted from *ibidem*, p. 8.

⁷⁹ From GRICE, P., *Studies in the Way of Words*, chapters 2-3, pp. 22-57.

⁸⁰ See *ibidem*, p. 24.

A : *I am out of petrol.*

B : *There is a garage round the corner.* »⁸¹.

Here the implicature would be that the garage is open, or may be open. Even though this information is not explicitly said, it is implicated by the nature of the exchange.

The implicatures can be either conventional or nonconventional. The former are related to general features of the words used whereas the latter concern features that are more context-dependent⁸². Among nonconventional implicatures, there is a further distinction made between those that are conversational and those that are nonconversational. Among nonconversational implicatures one may find those that are related to aesthetic, social or moral norms that are « normally respected by participants in talk exchange »⁸³. Linked to those implicatures are certain maxims, that is, requirements expected to be respected by talk exchange participants, such as « be polite ».

But the attention of Grice is primarily focused on conversational implicatures and the maxims that are correlatively respected in most talk exchanges. These conversational maxims « and the conversational implicatures connected with them are specially connected [...] with the particular purposes that talk (and so, talk exchange) is adapted to serve and is primarily employed to serve »⁸⁴. The participants of talk exchange generally observe, notes Grice, a certain principle, the « Cooperative Principle » which roughly requires them to « make [their] conversational contribution such as is required, at the stage at which it occurs, by the accepted purpose or direction of the talk exchange in which [they] are engaged »⁸⁵. And in order to satisfy this general principle, the participants must respect a certain number of maxims (and submaxims). Conversational maxims fall into four categories, reproduced here with the associated maxims⁸⁶ :

1. Quantity : « Make your contribution as informative as is required (for the current purposes of the exchange). »
« Do not make your contribution more informative than is required. »
2. Quality : « Do not say what you believe to be false. »
« Do not say that for which you lack adequate evidence. »
3. Relation : « Be relevant. »

⁸¹ GRICE, P., *Studies in the Way of Words*, p. 32. *Italic in the text.*

⁸² Grice doubts that there is a decisive test to distinguish nonconventional and conventional implicatures, even though he provides five features that the conversational implicatures of the nonconventional type usually possess. See *ibidem*, pp. 39 ; 42-43.

⁸³ *Ibidem*, p. 28.

⁸⁴ *Ibidem*.

⁸⁵ *Ibidem*, p. 26.

⁸⁶ See *ibidem*, pp. 26-27.

4. Manner : « Avoid obscurity of expressions. »
 « Avoid ambiguity. »
 « Be brief (avoid unnecessary prolixity). »
 « Be orderly. »

1.5.2 *The assertibility conditions' theory*

Grice's analysis may allow one to retain **CL** as the correct analysis of consequence and inference while providing a story about the unintuitive results with which we are concerned. Let's follow Read and call « Equivalence Thesis » the thesis which states that the truth conditions of conditionals are given by the truth table of material implication⁸⁷. The classicist may defend this thesis by holding that the unintuitive examples with which we are concerned do not fulfil the criterion for an acceptable participation in a talk exchange. All truths are consequently not assertible : if we want our participation in talk exchanges to follow the maxim of quality, we want our contributions to be informative. And so only informative conditionals will be assertible. Read explains :

One popular view is that the assertibility conditions of conditionals, 'if A then B', are given by the conditional probability, $\Pr(B/A)$. So 'if A then B' will be highly assertible when the probability of B relative to that of A is high, that is, when $\Pr(B \ \& \ A)/\Pr(B)$ is high. Suppose A is false. On the equivalence thesis, 'if A then B' will be true. But it may well not be (highly) assertible, if in particular, $\Pr(B \ \& \ A)$ is very much less than $\Pr(B)$ ⁸⁸.

The same holds for cases of clearly irrelevant premises or antecedent : our participations in talk exchanges must follow the maxim of relation too. If we come back to our introductory robot, CarL, the classicist's move to solve his problems seems to keep intact his original code for reasoning and add a new « layer » of code that specifies what he can express, and under what circumstances. This appeal to pragmatics finds its motivation in the virtues of extensionality. Indeed, remember our previous conclusion that modifying the behaviour of our implicational connective would lead us to lose its extensional character.

1.5.3 *The virtues of extensionality*

Interestingly enough, one of the greatest defenders of extensional logic was Quine, the same philosopher and logician who developed the holist epistemology referred to in subsection I.2.3.1. The first virtue of an extensional logic like **CL** or its first-order predicate extension lies in the definitions of logical truth that can be recovered from it, if

⁸⁷ See READ, S., *PEI*, p. 26.

⁸⁸ *Ibidem*.

we decide to not take it as a primitive⁸⁹. Indeed, three equivalent definitions can be provided⁹⁰, which, if we put apart the potential philosophical advantages of one over another, is a remarkable sign of the stability of the notion studied. One of these definitions is in terms of substitution, where « a logical truth, then, is definable as a sentence from which we get only truths when we substitute sentences for its simple sentences »⁹¹; the second mobilizes set-theoretic notions, where the logical truths are the sentences « true in all the interpretations on all the non-empty universes »⁹²; and the third is defined in terms of provability in a formal system.

The second virtue of extensional logic is that it is very well designed to deal with inferences in certain contexts, that can be labelled ‘extensional’ for the occasion. We can think of the discipline of mathematics, that commonly uses classical results like Excluded Middle (either ‘A’ is true or its negation, ‘¬A’ is true : if I can prove that ¬A is false, then I proved A, because one of the two must be true) and the principle of noncontradiction (‘A’ and ‘¬A’ cannot be true together : in so called « proofs by contradiction », showing from B that A and ¬A would follow is enough to prove ¬B). We can also think of a number of everyday reasonings that seem to happen in such contexts, and that can be appropriately formalized using **CL** (or its first-order predicate extension) : from « either Prof. Plum or Col. Mustard committed the murder » we would infer « if Prof. Plum did not commit the murder, then Col. Mustard did »⁹³ (and we have « $A \vee B \vdash_{\text{CL}} \neg A \supset B$ »)⁹⁴,...

Extra virtues can be advanced apart from those two important ones : the elegance of the system studied, its simplicity, and its alleged capacity to constitute the basis for scientific inquiry. The last virtue in particular has been, as Rivenc notes⁹⁵, held by Quine, if we understand the scientific inquiry as the study of « systems of the world »⁹⁶. To his view we now turn our attention, in particular to understand the importance of conditionals in a program that is allegedly and exclusively concerned with science.

⁸⁹ Which is the road initially taken by Quine in QUINE, W.V.O., *Mathematical Logic*, §5, p. 28. See RIVENC, F., *ILP*, pp. 8-11.

⁹⁰ See *ibidem*, pp. 8-17.

⁹¹ QUINE, W.V.O., *Philosophy of Logic*, p. 50.

⁹² RIVENC, F., *ILP*, p. 13.

⁹³ Example from MARES, E.D., *RLPI*, pp. 137-138 ; who uses it as a version of an argument by Grice.

⁹⁴ When judged appropriate, we will now on indicate the system in which a result or a proof is presented in the indice of some metalinguistic sign, following common practice. If our reader finds our game-related example artificial, she can imagine what happens in her mind when, having lost her keys, her (very meticulous) significant other yells : « the keys are in the kitchen or on your nightstand ». If she can’t find it in the kitchen, then she’ll infer that the keys are on her nightstand (where they obviously are, because significant others are never wrong...).

⁹⁵ See RIVENC, F., *ILP*, p. 16.

⁹⁶ The expression « systems of the world » comes from QUINE, W.V.O., *Word and Object*, §47.

1.5.4 Quine, grammar, and logic

The notion of logical truth is, according to him, « grammar-relative »⁹⁷. The three definitions given above only concern a language built on truth functions and quantification. Even when we try to generalize the notion for any language, which would lead to a definition (given appropriate technical tweaks) of the kind : « a logical truth is a sentence that cannot be turned false by substituting for lexicon, even under supplementation of lexical resources »⁹⁸, we still wouldn't have a « transcendent » notion, because it would still depend « on the notion of a grammatical construction, or, in the complementary phrasing, the notion of lexicon »⁹⁹. So the « notion of logical truth is dependent on the way we built the grammar of our language, underpinned by linguistic data »¹⁰⁰, which leads to Quine's famous quote : « logic is, in the jargon of mechanics, the resultant of two components : grammar and truth »¹⁰¹.

We can now discern a twofold movement that puts Quine in an odd position. On the one hand, he motivates the acceptance of classical predicate logic mainly as a tool for « systems of the world », as an « organon bent to the purposes and norms of scientific thinking (at least as [he] considers it) »¹⁰². And this leads him to neglect any intensional logic, any applied logics designed to study propositional attitudes or uses of natural language as a work of second importance. On the other hand, « given the relativity of logic to the grammatical structures privileged [in the language] »¹⁰³, Quine feels compelled to show that his analysis of truth-functional implication is acceptable as a « translation of numerous uses of conditional »¹⁰⁴.

1.6 Choosing a logical theory

At this point it seems that we've made a full circle in our argumentation. Before we go any further, it may be useful to clarify our position. In our introduction, we have noted that some inferences and some implicational theorems of **CL** were problematic, because they did not seem to make sense when compared to our inferential practices and our uses of conditionals. We have seen that trying to solve those problems while adopting a revisionist strategy of type (1) confronted us with the abandon of complete extensionality of the system. Then we have explored the classicist's answer, (which constitutes a

⁹⁷ See RIVENC, F., *ILP*, p. 15.

⁹⁸ QUINE, W.V.O., *Philosophy of Logic*, p. 59.

⁹⁹ *Ibidem*, p. 60.

¹⁰⁰ RIVENC, F., *ILP*, p. 15.

¹⁰¹ QUINE, W.V.O., *Philosophy of Logic*, p. 60.

¹⁰² RIVENC, F., *ILP*, p. 16. Quine's view of science is qualified by Rivenc of « dreamt, physicalist and completed » (« rêvée, physicaliste et achevée », op. cit.).

¹⁰³ *Ibidem*, p. 17.

¹⁰⁴ *Ibidem*.

conservative strategy of type (2)), with the assertibility conditions' theory used as a basis to explain away apparent non-sequiturs and the allowance of the paradoxes of material implication in our conditionals. We just saw that this move was motivated by the virtues of extensional logic : the nice theory of logical truth that underpins **CL** (and **CL_P**), the conditionals in certain contexts being captured by material implication, the elegance and simplicity of the system, and its usefulness for scientific inquiry. Then we explained Quine's position on this matter, and ended up claiming that, even if his main interest was in the « systems of the world », his grammar choices for **CL_P** had to confront our conditional uses.

Are we contradicting ourselves in claiming first that **CL** is not adapted to capture our conditionals by appeal to the paradoxes of material implication and then that it really is ? And similarly in holding first that **CL** vindicates non-sequiturs, and then that it really encodes our inferential practices ? We need to be careful here, because evaluating any theory is a difficult and intricate matter. Evidence here goes in both ways. There are some of our conditionals that are well formalized by **CL**, and similarly there are some of our arguments that are well formalized by **CL**. But the theory is not perfect, because it seems too permissive in allowing the paradoxes of material implication in our conditionals and in validating non-sequiturs. Should we admit a conversationalist layer to save our theory, or build a new logical system to replace it, or admit that multiple logical systems encode different conditional uses and inferential practices ? From the rather pragmatic attitude we sketched in section I.2.3 it may seem clear that when choosing a theory over another in any discipline that is concerned with explanation, we believe there is no mechanical or systemic way to choose 'the correct' one.

All elements must come in the balance, as Kuhn¹⁰⁵ has noted : scope, fruitfulness, accuracy, even elements like simplicity and elegance can play an important role in accepting a theory. We should also be aware that the *ad hoc* character of new explanatory elements of a theory weight against it in the balance. But « there are no crucial experiments »¹⁰⁶ nor knock-down arguments that should make us oversee the qualities of a theory. Now, we do believe that there are rational, or at least reasonable arguments to accept a theory over another, that this is not only a matter of taste or social force. But the task is difficult and, as we said, does not admit any mechanical procedure of decision for the theorist.

With our philosophical stance made more precise, we will now display new evidence to put in the balance : we will present in the next section a criticism of Quine's « systems

¹⁰⁵ See KUHN, T.S., *The Structure of Scientific Revolution*, postscript, sections 5 and 6.

¹⁰⁶ READ, S., *PEI*, p. 24.

of the world », Read's criticism of the assertibility conditions' theory and we'll end the present chapter with some extra examples of problematic conditionals.

1.7 Criticizing CL_P

1.7.1 Science and CL_P

The idea that scientific theories use a language that can be formalized (and made clearer) thanks to classical logic is not particularly original. Quine, we said it, believed so. But it is at least a doubtful claim. Some recent developments in metaphysics have put to light the role that a notion like possibility could play in the scientific theories themselves¹⁰⁷. A notion, we said earlier, that cannot be analysed using the extensional resources of a system like CL_P. Dialetheists have also defended the idea that contradictions exist in our world, and that we consequently need a logical system that does not validate the principle of explosion, or (EFQ)¹⁰⁸ if we want to explain it.

Furthermore, if we want to be attentive to the *practice*¹⁰⁹, and not only to the *theories* of scientists, it is likely that CL (or CL_P) won't give us the resources we need for our investigation, because it should be expected that such practices are filled with (hyper)intensional traits, just like natural languages and propositional attitudes.

1.7.2 Read's criticism of the assertibility conditions' theory¹¹⁰

Putting in perspective the idea that only CL_P matters because it is the one true framework to formalize « systems of the world » is a way to reach the classicist by showing that one of the virtues of her logical system needs extra grounds to be held, or should be relativized. Another element may be showed wanting, namely, the assertibility conditions' theory.

To understand Read's criticism, we have to quickly reproduce the example he presents. We suppose that « a certain Roy Dyckhoff has claimed that John Slaney was in Edinburgh on a certain day and that Crispin Wright has denied it »¹¹¹. Then we construct

¹⁰⁷ See for example the ontology developed by Kastner in KASTNER, R., *The Transactional Interpretation of Quantum Mechanics*. An idea going in that ontological direction for biological individuals has been presented by professor Alexandre Guay in a conference entitled *Dazed and Confused : reflections about biological individuality* in Louvain-La-Neuve.

¹⁰⁸ A logical system that does not validate (EFQ) is said to be *paraconsistent*. For the arguments and examples of dialetheists, see PRIEST, G., *In Contradiction*. See also HESPEL, B., « Physique et non-dualité » in *La non-dualité : perspectives philosophiques, scientifiques, spirituelles* (Counet, J-M. ed.) for an interesting perspective on dialetheism linked to the mind-body problem.

¹⁰⁹ Philosophers who took part in the « practice turn » in philosophy of science have indeed enlarged their perspective to be more attentive to scientific practice. See for example SOLER, L., ZWART, S., LYNCH, M., ISRAEL-JOST, V., (eds.), *Science after the Practice Turn in the Philosophy, History, and Social Studies of Science*.

¹¹⁰ For this subsection, see READ, S., *PEI*, pp. 23-28.

¹¹¹ *Ibidem*, p. 23.

the following propositions relating to this situation (with their logical form and truth value expressed) :

(1) If John was in Edinburgh, Roy was right. $(1) P \supset Q$

Which is true given Roy's claims.

(2) If Crispin was right, so was Roy. $(2) R \supset Q$

That is equally obviously false, given the logic of denial.

(3) If John was in Edinburgh, Crispin was right. $(3) P \supset R$

That too is false, for Crispin denied it

And from those propositions, we construct the following argument (with their logical form expressed) :

If John was in Edinburgh, then Roy was right. $P \supset Q$ (T)

It's not the case that if Crispin was right, so was Roy. $\neg(R \supset Q)$ (T)

Hence, if John was in Edinburgh, Crispin was right. $P \supset R$ (F)

The premises of this argument, (1) and the denial of (2), look true, and its conclusion, (3), false. So given the Classical Account, this argument should not be valid.

But the sequent « $P \supset Q, \neg(R \supset Q) \vdash_{\text{CL}} P \supset R$ » is classically valid¹¹². The classicist's first answer is that one of (2) or (3) must be true, because « $\vdash_{\text{CL}} (R \supset Q) \vee (P \supset R)$ » is classically valid. Now, the classical logician can resort to the assertibility conditions' theory to explain this oddity :

If John was indeed in Edinburgh, then, according to the Equivalence Thesis, (2) is true, while if John was not there, it says that (3) is true. But their conditional probabilities are each low, differing from zero only to the extent that the probability that Roy or Crispin did not make the assertion I have attributed to them itself differs from zero¹¹³.

So, we may conclude that none of (2) or (3) is highly assertible. But Read goes on, and wonders :

What does the claim that the truth-condition is extensional, or material, amount to? It is very arguable whether the conditional probability really does provide the correct account of when a conditional is assertible ; but prescinding from that, what work does the other claim, the one about truth-conditions (the Equivalence Thesis, itself), do? Its only plausible role would seem to be to account for the behaviour of embedded conditionals. For assertibility only applies to whole

¹¹² The « , » here acts as a conjunctive connective. The corresponding embedded implication, using the Deduction Equivalence, can be verified as a classical tautology. The notion of sequent is presented in chapter 4.

¹¹³ READ, S., *PEI*, p. 26.

utterances, to sentences. The semantic contribution of an embedded clause to a sentence of which it is only a part cannot be anything to do with assertibility, but only with truth¹¹⁴.

This classicalist claim is, for Read, mistaken, for if we construct the disjunction of (2) and (3) :

Either if Crispin was right, so was Roy,

or if John was in Edinburgh, Crispin was right. $(R \supset Q) \vee (P \supset R)$

We obtain a true disjunction, with at least one of the two disjuncts true.

But this disjunction is not highly assertible. Why ? « The plausible answer is that it is because neither disjunct is highly assertible »¹¹⁵. But the answer cannot be so simple, for there are « cases of highly assertible compound sentences whose parts are not highly assertible. For example,

If the earth is flat then it has an edge »¹¹⁶.

As a result, we expect the classicist to explain

why a disjunction of true disjuncts is not highly assertible which does not depend simply on the fact that the disjuncts themselves are not highly assertible. But then it will clearly miss the point, because it is precisely because the disjuncts are not highly assertible that the whole is not so. In fact, the mistake is not to recognise that the disjuncts are not highly assertible because they are false, and it is its falsity which makes the disjunction not highly assertible¹¹⁷.

We leave to our reader the conclusion she feels adequate from this critical perspective. As we said, there are no knock-down arguments, but knowing that the assertibility conditions' theory might need another *ad hoc* explanation for explaining the assertibility of disjunctions should be another element to put in the balance when evaluating the conservation of **CL_P** (or **CL**).

1.8 New problems with conditionals

We would also like to add in the balance new examples where material implication translates our conditionals poorly (remember that we already showed some 'paradoxes of material implication' in section I.5). The conditionals considered are indicative, by opposition to « counterfactuals », where the antecedent is considered contrary to reality¹¹⁸ :

'If Oswald hadn't shot Kennedy, someone else would have.'

¹¹⁴ READ, S., *PEI*, p. 27.

¹¹⁵ *Ibidem*.

¹¹⁶ *Ibidem*.

¹¹⁷ *Ibidem*.

¹¹⁸ Example from MARES, E.D., *RLPI*, p. 12.

is a counterfactual, « someone uttering it would assume for the sake of the conversation at least that the antecedent is false »¹¹⁹ ; whereas

‘If Oswald did not shoot Kennedy, then someone else did’

is an indicative conditional (which is true), someone uttering it does not « assume that the antecedent is true or assume that it is false »¹²⁰.

Our examples are from Rivenc¹²¹.

1) Inferences from the negation of a conditional,

CL validates $\neg(A \supset B) \vdash_{\text{CL}} A$ but the following inference is judged problematic :

It is not true that if God exists, the righteous are rewarded.

So, God exists.

2) Strengthening of the antecedent,

CL validates $A \supset C \vdash_{\text{CL}} (A \wedge B) \supset C$ but not all indicative conditionals allow this move :

If the weather is good, I’ll go out,

So, if the weather is good and I am victim of a mortal accident, I’ll go out.

3) Transitivity,

CL validates $A \supset B, B \supset C \vdash_{\text{CL}} A \supset C$ but the following inference seems to not hold:

If the other candidates withdraw, Jean will get the job,

If Jean gets the post, the other candidates will be furious,

So, if the other candidates withdraw, they will be furious.

4) Contraposition,

CL validates all the instances of contraposition, including $A \supset B \vdash_{\text{CL}} \neg B \supset \neg A$ but our uses do not seem to completely validate it :

If it’s past 3 o’clock, it’s not much past 3 o’clock,

So, if it’s much past 3 o’clock, it’s not past 3 o’clock.

5) Conditions of negation of an indicative conditional,

CL considers as equivalent $\neg A \supset B$ and $A \vee B$ but the equivalence between :

(a) « If Jean didn’t do it, then Paul did » and (b) « either Jean did it, or Paul did it » doesn’t seem to hold, because someone who knows for sure that Paul didn’t do it could deny the truth of (a) while agreeing on (b) in some manner, « yes one of the two did it, but it’s not Paul ». If the conditionals were equivalent, they should « call for the same answer »¹²². If

¹¹⁹ MARES, E.D., *RLPI*, p. 12.

¹²⁰ *Ibidem*.

¹²¹ See RIVENC, F., *ILP*, pp. 22-23.

¹²² *Ibidem*, p. 23.

those elements were not put in the balance completely against **CL**, it is because **R** validates some of them (2-4) and so they can be put as elements weighing against it as well.

1.9 Looking for something new, R

The elements presented in section 1.7, as well as the non-sequiturs and paradoxes of material implication presented in the introduction, constitute important motivations for **R**, as well as the desire to avoid an *ad hoc* theory of assertibility conditions. All these things being said, we only motivated our desire for a new syntactic system. If we want to properly evaluate if we prefer to keep **CL** or embrace **R**, or instead embrace pluralism, we need to know what exactly **R** consists in. Only when its virtues and possible problems become clear will we be in a position to complete our balance of choice between the competing theories (this balance including the possibility of pluralism). So, it is now time to present the system **R**, and we'll start with its Hilbert-style and natural deduction systems.

Chapter 2 : The implicational fragments $\mathbf{R}_h \rightarrow$ and $\mathbf{NDR} \rightarrow$

The structure of the present and next chapters follow Rivenc's presentation¹²³, where the discussions around the different « relevant » aspects of the system is central. We denote the implicational connective with the symbol « $\rightarrow$ » to differentiate it from the truth-functional material implication of $\mathbf{CL}$, « $\supset$ ». $\mathbf{R}$ actually has presentations, based on the kind of tools one wants to use in her logic. $\mathbf{R}_h$ will now denote the Hilbert-style presentation of $\mathbf{R}$, $\mathbf{NDR}$ will now denote its natural deduction presentation, and $\mathbf{TR}$ will now denote its structural presentation¹²⁴. « $\mathbf{R}$ » refers indistinctly to any of the presentations (or syntactic logical systems) or, if one prefers, to the set of syntactic logical systems $\{\mathbf{R}_h, \mathbf{NDR}, \mathbf{TR}\}$ which are all sound and complete with respect to one another. Our uses of « the system $\mathbf{R}$ » and $\mathbf{R}$ have hitherto been linked to the motivation of it, and so the particularities of the different presentations could be ignored.

2.1 Hilbert-style implicational fragment for $\mathbf{R}, \mathbf{R}_h \rightarrow$

The language contains an infinity of atomic formulae, or parameters. Italic lowercase letters like p, q, r , are used to represent any of them, and one (non-extensional) binary connective « $\rightarrow$ » is contained within the language. We use uppercase letters like A, B, C to represent metavariables over wffs¹²⁵. We define wffs as follows :

- the atomic formulae are wffs ;
- if A and B are wffs, then $(A \rightarrow B)$ is a wff (recursive definition) ;
- only the expressions obtained through application of the preceding rules are wffs.

The implicational fragment of $\mathbf{R}_h, \mathbf{R}_h \rightarrow$, can be characterized as follows¹²⁶. Its only primitive rule of inference is Modus Ponens, MP : from $A \rightarrow B$ and A ¹²⁷, infer B . Its axioms, presented hereafter, are Identity and one of each pair : {Prefixing, Sufficing}, {Permutation, Assertion}, {Contraction, Self-distribution}¹²⁸.

¹²³ The content of the present chapter stems from RIVENC, F., *ILP*, pp. 37-128.

¹²⁴ Our uses of « $\mathbf{CL}$ » relate in the same way to its different Hilbert-style ($\mathbf{CL}_h$), natural deduction ($\mathbf{NDCL}$) and structural presentations ($\mathbf{TCL}$) than « $\mathbf{R}$ » to its own.

¹²⁵ We will not systematically make explicit the 'grammar rules' of the systems presented, which provide the recursive definitions to obtain well-formed formulae from the atomic formulae and the connective(s). But those are generally fairly easy to establish, see for $\mathbf{CL}$ and the present definitions, BOUQUIAUX, L., and LECLERCQ, B., *LFA*, p. 68. Furthermore, we follow common practice and use the term « formula » under the assumption that it is well formed.

¹²⁶ If our reader desires a systematic definition of the notions of axiom, rule of inference and theorem, she can consult Appendix A, section A.2, §2.

¹²⁷ Note that for MP and the other rules of inference in axiomatic systems, the order of apparition of the conditions in the proof in which we want to use the said rule doesn't matter.

¹²⁸ See RIVENC, F., *ILP*, p. 41.

Axioms 2.1 : Axioms of $R_h \rightarrow$

$(R_h \rightarrow 1) \vdash A \rightarrow A$	Identity
$(R_h \rightarrow 2) \vdash (A \rightarrow B) \rightarrow ((C \rightarrow A) \rightarrow (C \rightarrow B))$	Transitivity (Prefixing)
$(R_h \rightarrow 3) \vdash (A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$	Permutation
$(R_h \rightarrow 4) \vdash (A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$	Contraction
$(R_h \rightarrow 2') \vdash (A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$	Transitivity (Suffixing)
$(R_h \rightarrow 3') \vdash A \rightarrow ((A \rightarrow B) \rightarrow B)$	Assertion
$(R_h \rightarrow 4') \vdash (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$	Self-distribution

Church¹²⁹, in 1951, presented the system with $(R_h \rightarrow 1)$, $(R_h \rightarrow 2)$, $(R_h \rightarrow 3)$ and $(R_h \rightarrow 4)$ while Anderson and Belnap's presentation¹³⁰ use the axioms $(R_h \rightarrow 1)$, $(R_h \rightarrow 2')$, $(R_h \rightarrow 3)$, $(R_h \rightarrow 4')$ in order to easily get the Relevant Deduction Theorem¹³¹. Dunn showed that the different choices of axioms were independent and so that one could pick in the different pairs as she wishes¹³². We prove hereafter the interdeducibility of the axioms in the presence of some others, that is, from certain sets of axioms (and MP), we can obtain as a theorem a wff that matches another possible choice of axiom. We follow common practice and indicate the justification of each step (or « line »), on its right (or below when impossible to write it on the right), and attribute a numeral to each step, to indicate its position in the list of theorems and axioms. We start by proving that one can get Assertion from Permutation and vice-versa (in the presence of transitivity).

Proof 2.2¹³³: Assertion $(R_h \rightarrow 3')$ from Permutation $(R_h \rightarrow 3)$

1. $\vdash (A \rightarrow B) \rightarrow (A \rightarrow B)$ $(R_h \rightarrow 1)^{134}$
2. $\vdash ((A \rightarrow B) \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow ((A \rightarrow B) \rightarrow B))$ $(R_h \rightarrow 3)$

¹²⁹ The references of this paragraph stem from RIVENC, F., *ILP*, pp. 40-41. For Church's, see CHURCH, A., « The weak theory of implication » in *Kontrolliertes Denken* (Menne, A., Wilhelmy, A., Angsil, H. eds.), pp. 22-37.

¹³⁰ See ANDERSON, A.R., and BELNAP, N., *ELRN I*.

¹³¹ See section 2.3.

¹³² See DUNN, M., « Relevance logic and entailment » in *Handbook of philosophical logic*, vol. III, *Alternatives to classical logic* (Gabbay, D., and Guenther, F. eds.), pp. 117-224.

¹³³ From RIVENC, F., *ILP*, p. 40.

¹³⁴ For readability, we generally indicate instances of the axioms simply by indicating the corresponding axiom. Remember that wffs in the axioms can be replaced uniformly by any wff. Here we must replace the occurrences of « A » in the axiom $(R_h \rightarrow 1)$ by « $(A \rightarrow B)$ » uniformly in order to obtain the step of the proof. We can indicate this by writing « $A/(A \rightarrow B)$ ».

3. $\vdash A \rightarrow ((A \rightarrow B) \rightarrow B)$

1,2 by MP¹³⁵

For the other direction, we first need a peripheral result if we want to follow Rivenc's proof. We need something like a rule of transitivity.

First rule of transitivity (T_1) : from $\vdash D \rightarrow B$ and $\vdash A \rightarrow (B \rightarrow C)$, infer $\vdash A \rightarrow (D \rightarrow C)$.

Proof 2.3 : First rule of transitivity (T_1)

1. $\vdash D \rightarrow B$

Postulated condition

2. $\vdash (D \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (D \rightarrow C))$

($R_h \rightarrow 2'$)

3. $\vdash (B \rightarrow C) \rightarrow (D \rightarrow C)$

1,2 by MP

4. $\vdash A \rightarrow (B \rightarrow C)$

Postulated condition

5. $\vdash (A \rightarrow (B \rightarrow C)) \rightarrow (((B \rightarrow C) \rightarrow (D \rightarrow C)) \rightarrow (A \rightarrow (D \rightarrow C)))$

($R_h \rightarrow 2'$) with A/A , $B/(B \rightarrow C)$, $C/(D \rightarrow C)$

6. $\vdash ((B \rightarrow C) \rightarrow (D \rightarrow C)) \rightarrow (A \rightarrow (D \rightarrow C))$

4,5 by MP

7. $\vdash A \rightarrow (D \rightarrow C)$

3,6 by MP

Proof 2.4¹³⁶ : Permutation ($R_h \rightarrow 3$) from Suffixing ($R_h \rightarrow 2'$), Assertion ($R_h \rightarrow 3'$) and (T_1)

1. $\vdash B \rightarrow ((B \rightarrow C) \rightarrow C)$

($R_h \rightarrow 3'$)

2. $\vdash [B \rightarrow ((B \rightarrow C) \rightarrow C)] \rightarrow [(((B \rightarrow C) \rightarrow C) \rightarrow (A \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))]$

($R_h \rightarrow 2'$)¹³⁷

3. $\vdash (((B \rightarrow C) \rightarrow C) \rightarrow (A \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$

1,2 by MP

4. $\vdash (A \rightarrow (B \rightarrow C)) \rightarrow (((B \rightarrow C) \rightarrow C) \rightarrow (A \rightarrow C))$

($R_h \rightarrow 2'$)

5. $\vdash (A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$

1,4 by (T_1)¹³⁸

We are now concerned with the interdeducibility of the two transitivity axioms, Prefixing and Suffixing, which is very easy to show in presence of Permutation. As one

¹³⁵ We indicate the use of a rule by citing the steps that enable us to use it, then by citing its abbreviation.

¹³⁶ From RIVENC, F., *ILP*, p. 41.

¹³⁷ With substitutions A/B , $B/((B \rightarrow C) \rightarrow C)$, $C/(A \rightarrow C)$.

¹³⁸ Where 1 is an instance of $\vdash D \rightarrow B$ and 4 is an instance of $\vdash A \rightarrow (B \rightarrow C)$ with appropriate substitutions : $A/(A \rightarrow (B \rightarrow C))$, $B/((B \rightarrow C) \rightarrow C)$, $C/(A \rightarrow C)$, D/B . We only needed ($R_h \rightarrow 2'$) and MP to derive the rule so there is no problem in using it here. Note that Rivenc simply uses the justification « Transitivity » for the step, as he does for instances of Suffixing, see *ibidem*.

may expect, the two proofs follow the exact same structure (only the substitutions change)¹³⁹.

Proof 2.5 : Suffixing ($R_h \rightarrow 2'$) from Prefixing ($R_h \rightarrow 2$) and Permutation ($R_h \rightarrow 3$)

1. $\vdash (A \rightarrow B) \rightarrow ((C \rightarrow A) \rightarrow (C \rightarrow B))$ ($R_h \rightarrow 2$)
2. $\vdash ((A \rightarrow B) \rightarrow ((C \rightarrow A) \rightarrow (C \rightarrow B))) \rightarrow ((C \rightarrow A) \rightarrow ((A \rightarrow B) \rightarrow (C \rightarrow B)))$
($R_h \rightarrow 3$) with $A/(A \rightarrow B)$, $B/(C \rightarrow A)$, $C/(C \rightarrow B)$
3. $\vdash (C \rightarrow A) \rightarrow ((A \rightarrow B) \rightarrow (C \rightarrow B))$ 1,2 by MP
4. $\vdash (A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$ 3 with A/B , B/C , C/A ¹⁴⁰

Proof 2.6 : Prefixing ($R_h \rightarrow 2$) from Suffixing ($R_h \rightarrow 2'$) and Permutation ($R_h \rightarrow 3$)

1. $\vdash (A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$ ($R_h \rightarrow 2'$)
2. $\vdash ((A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))) \rightarrow ((B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)))$
($R_h \rightarrow 3$) with $A/(A \rightarrow B)$, $B/(B \rightarrow C)$, $C/(A \rightarrow C)$
3. $\vdash (B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$ 1,2 by MP
4. $\vdash (A \rightarrow B) \rightarrow ((C \rightarrow A) \rightarrow (C \rightarrow B))$ 3 with A/C , B/A , C/B

We continue this series of proofs by showing that Contraction and Self-distribution are interdeducible. We first need a peripheral result alike (T_1), but slightly different. The proof is very similar (once correct substitutions made), except that we also need a use of Prefixing.

Second rule of transitivity (T_2): from $\vdash C \rightarrow D$ and $\vdash A \rightarrow (B \rightarrow C)$, infer $\vdash A \rightarrow (B \rightarrow D)$.

Proof 2.7 : Second rule of transitivity (T_2)

1. $\vdash C \rightarrow D$ Postulated condition
2. $\vdash (C \rightarrow D) \rightarrow ((B \rightarrow C) \rightarrow (B \rightarrow D))$ ($R_h \rightarrow 2$)
3. $\vdash (B \rightarrow C) \rightarrow (B \rightarrow D)$ 1,2 by MP
4. $\vdash A \rightarrow (B \rightarrow C)$ Postulated condition
5. $\vdash (A \rightarrow (B \rightarrow C)) \rightarrow (((B \rightarrow C) \rightarrow (B \rightarrow D)) \rightarrow (A \rightarrow (B \rightarrow D)))$

¹³⁹ The two proofs follow the indications of Rivenc, see RIVENC, F., *ILP*, p. 40.

¹⁴⁰ This last step just so the result fits exactly ($R_h \rightarrow 2$), it just changes the names of the wffs.

- ($R_h \rightarrow 2'$)
6. $\vdash ((B \rightarrow C) \rightarrow (B \rightarrow D)) \rightarrow (A \rightarrow (B \rightarrow D))$ 4,5 by MP
7. $\vdash A \rightarrow (B \rightarrow D)$ 3,6 by MP

Proof 2.8¹⁴¹ : Self-distribution ($R_h \rightarrow 4'$) from ($R_h \rightarrow 2$), ($R_h \rightarrow 2'$), ($R_h \rightarrow 3$), (T_2) and ($R_h \rightarrow 4$)

1. $\vdash (A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$ ($R_h \rightarrow 3$)
2. $\vdash (A \rightarrow (A \rightarrow C)) \rightarrow (A \rightarrow C)$ ($R_h \rightarrow 4$)
3. $\vdash (B \rightarrow (A \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow (A \rightarrow C)))$ ($R_h \rightarrow 2$)
4. $\vdash (B \rightarrow (A \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$ 2,3 by (T_2)¹⁴²
5. $\vdash [(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))] \rightarrow \{[(B \rightarrow (A \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))] \rightarrow [(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))]\}$ ($R_h \rightarrow 2'$)¹⁴³
6. $\vdash [(B \rightarrow (A \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))] \rightarrow [(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))]$ 1,5 by MP
7. $\vdash (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$ 4,6 by MP

Proof 2.9¹⁴⁴ : Contraction ($R_h \rightarrow 4$) from ($R_h \rightarrow 3$) and ($R_h \rightarrow 4'$)

1. $\vdash (A \rightarrow (A \rightarrow B)) \rightarrow ((A \rightarrow A) \rightarrow (A \rightarrow B))$ ($R_h \rightarrow 4'$)
2. $\vdash [(A \rightarrow (A \rightarrow B)) \rightarrow ((A \rightarrow A) \rightarrow (A \rightarrow B))] \rightarrow [(A \rightarrow A) \rightarrow ((A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B))]$ ($R_h \rightarrow 3$)¹⁴⁵
3. $\vdash (A \rightarrow A) \rightarrow ((A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B))$ 1,2 by MP
4. $\vdash A \rightarrow A$ ($R_h \rightarrow 1$)
5. $\vdash (A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$ 3,4 by MP

¹⁴¹ With modifications, from RIVENC, F., *ILP*, p. 41.

¹⁴² Where 2 is an instance of « $\vdash C \rightarrow D$ » and 3 is an instance of « $\vdash A \rightarrow (B \rightarrow C)$ » (with appropriate substitutions).

¹⁴³ So it is an instance of « $\vdash (A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$ » with substitutions $A/(A \rightarrow (B \rightarrow C))$, $B/(B \rightarrow (A \rightarrow C))$, $C/((A \rightarrow B) \rightarrow (A \rightarrow C))$.

¹⁴⁴ With modifications, from *ibidem*, p. 40.

¹⁴⁵ With substitutions $A/(A \rightarrow (A \rightarrow B))$, $B/(A \rightarrow A)$, $C/(A \rightarrow B)$. Steps 2 and 3 display a pattern corresponding to a permutation rule, that can be inferred from the corresponding axiom : from $\vdash A \rightarrow (B \rightarrow C)$ infer $\vdash B \rightarrow (A \rightarrow C)$ (the proof would use as postulated condition $\vdash A \rightarrow (B \rightarrow C)$ and the axiom Permutation $\vdash (A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$ with an application of MP).

We end our series of proof with that of « Specialized Assertion », the theorem $\vdash ((A \rightarrow A) \rightarrow B) \rightarrow B$ discussed later. It is rather straightforward.

Proof 2.10 : Specialized Assertion

- | | |
|---|------------------------|
| 1. $\vdash (A \rightarrow A) \rightarrow (((A \rightarrow A) \rightarrow B) \rightarrow B)$ | $(R_h \rightarrow 3')$ |
| 2. $\vdash A \rightarrow A$ | $(R_h \rightarrow 1)$ |
| 3. $\vdash ((A \rightarrow A) \rightarrow B) \rightarrow B$ | 1,2 by MP |

From this series of proofs, we can see that the axiomatic formulation of Church¹⁴⁶ is indeed equivalent to the axiomatic formulation of Anderson and Belnap¹⁴⁷ (we now denote the expression « Anderson and Belnap » by « A&B » for convenience). Where the arrows display proof-dependence and are to be read « is used in the proof of », we can construct the following diagrams.

Diagram 2.11 : Proof-dependence of A&B's axioms from Church's

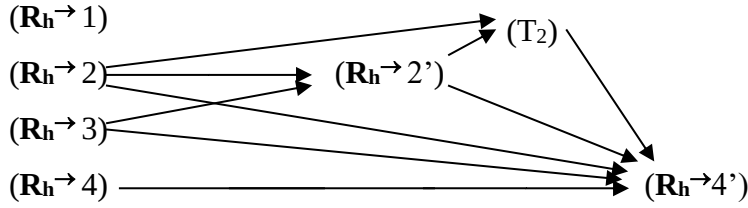
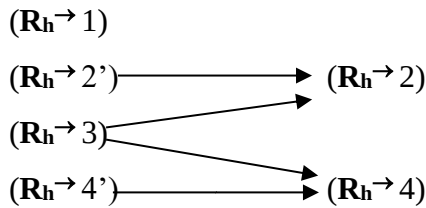


Diagram 2.12 : Proof-dependence of Church's axioms from Anderson and Belnap's



Now for our diagrams to be really useful, we need that each proven result is proven without any further assumption than the ones it is linked to with an arrow, together with the assumption that our primitive rule MP holds. Our reader can go back in the proofs and check that this is indeed the case. So we have that Church's system is indeed sufficient to obtain A&B's missing axioms as theorems, and that A&B's system is

¹⁴⁶ Which uses $(R_h \rightarrow 1)$, $(R_h \rightarrow 2)$, $(R_h \rightarrow 3)$, and $(R_h \rightarrow 4)$ as axioms.

¹⁴⁷ Which uses $(R_h \rightarrow 1)$, $(R_h \rightarrow 2')$, $(R_h \rightarrow 3)$, and $(R_h \rightarrow 4')$ as axioms.

sufficient to obtain Church's missing axioms as theorems : so Church and A&B's systems are equivalent.

2.2 The classical Deduction Theorem

Let's take a moment to reflect on the notion of proof-dependence we just discussed. Hilbert proofs are such that we can display any axiom (or theorem) desired and « do nothing » with it to obtain the conclusion, and this does not affect the validity of the proposed proof. Take our proof 2.5 of $(\mathbf{R}_h \rightarrow 2')$, Suffixing. In our proof, and as is represented in the above diagram, we only used, apart from MP, $(\mathbf{R}_h \rightarrow 2)$ and $(\mathbf{R}_h \rightarrow 3)$. But we could have added an instance of $(\mathbf{R}_h \rightarrow 4)$ and $(\mathbf{R}_h \rightarrow 1)$ at the beginning of our proof, say as steps 1 and 2, with the rest of the proof identical but moved two steps ahead, and it would still be a correct proof if we refer to our definition of a Hilbert-proof in section A.2. But those new axioms would not be « doing any work » in our proof. The idea of a relevant logic like $\mathbf{R}$ is to force every resource displayed in the proof to be of some use in establishing the conclusion.

In our diagrams, this relevant connexion is showed by the arrows. Note that the search for relevance is not the search for minimality : the derived rule (T_2) was used to establish $(\mathbf{R}_h \rightarrow 4')$, but (T_2) only depends on $(\mathbf{R}_h \rightarrow 2)$ and $(\mathbf{R}_h \rightarrow 2')$ (and also of MP). It would seem odd to claim that (T_2) was useless in our proof, when it clearly provided us with a step that none of $(\mathbf{R}_h \rightarrow 2)$ nor $(\mathbf{R}_h \rightarrow 2')$, as axioms, could provide¹⁴⁸.

The general idea here is to change what counts as a « good » deduction in our formal system. The Deduction Theorem presented in section I.4 does not hold for $\mathbf{R}$. The Deduction Equivalence indeed needs to be instantiated and proven if we want to show that the system accomplishes what we expect from it. Classically, the concept of deduction can be expressed along the following lines :

Relatively to a formal system $\mathbf{S}$, a deduction of C from a list of hypotheses $A_1, \dots, A_n$, is a sequence of formulae $B_1, \dots, B_m$, such that $B_m = C$ (the last formula is called the « conclusion »), and each B_j is either (1) an axiom of $\mathbf{S}$, either (2) one of the formulae A_i , either (3) obtained from previous

¹⁴⁸ Looking at Rivenc's proofs could give the opposite impression. He does not make the detour through the proofs of (T_1) or (T_2) in his presentation, but instead gives the justification « Transitivity » in the step. He is not to blame, of course, because he warns her reader that he does not provide the proofs *in extenso* (see RIVENC, F., *ILP*, p. 40), as experienced logicians usually jump steps that seem obvious to them. But it is clear that the valid formulae of the proofs are not instances of a transitivity axiom at all, but rather follows from what is a « transitivity move » of the form expressed by our derived rules. But we should note that the role played by our derivative rules is mostly heuristic, we can always replace their use by the appropriate instances of axioms and uses of MP that can be found in their (own) proofs.

formulae by application of a rule [...] a proof or a demonstration of C is a deduction of C from the empty set of hypotheses¹⁴⁹.

This notion of deduction can fit our official definition of a Hilbert-style proof. In our system it is still the case that this concept of deduction does not force any use of the formulae put in it, that we can « repeat » any result in the proof, import any other theorem of the system in it (so any theorem T can be a proof of T if it's the only step of the proof¹⁵⁰), that the conclusion can consequently be any step of the proof if we repeat it last. In such deduction, the notions of « premise » and « conclusion » are very liberal, and do not correspond to what we'd want to accept as such. The only

link between the [classical] official definition of a deduction and a deduction that *proves* its conclusion, is at best, the following : the official definition leaves open the possibility that a list of formulae fitting the official concept of deduction be in effect a *true deduction* of a consequence from premises¹⁵¹.

But this does not mean that this *must* be accepted as the notion of deduction for our system. If a deduction exists¹⁵² of C from $A_1, \dots, A_n$, we write : $A_1, \dots, A_n \vdash C$.

Which means that C is deducible from $A_1, \dots, A_n$. The Deduction Theorem¹⁵³ in **CL** can therefore be represented as :

$$A_1, \dots, A_n, A \vdash_{\text{CL}} C$$

if and only if

$$A_1, \dots, A_n \vdash_{\text{CL}} A \supset C$$

It is possible to prove that this theorem holds for any system stronger, or containing, the implicational system **H**→ (for Heyting), so for any system that contains MP, and the axioms¹⁵⁴ :

$\vdash A \rightarrow A$	Identity
$\vdash A \rightarrow (B \rightarrow A)$	Weakening
$\vdash (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$	Self-distribution

Where « $\rightarrow$ » is the implicational connective of the system. The Hilbert-style implicational fragment of **CL**, **CL_h[⊃]** can be axiomatized with¹⁵⁵ :

¹⁴⁹ RIVENC, F., *ILP*, pp. 47-48. Numbers between parentheses added by us.

¹⁵⁰ Of course, our only way to *know* that it is a theorem is through a different proof (except if it's also an axiom).

¹⁵¹ *Ibidem*, p. 49. Italic in the text.

¹⁵² For this paragraph and the next, see *ibidem*, p. 48, except for this version of the Theorem, which is stronger than Rivenc's. For it, see READ, S., *PEI*, p. 21.

¹⁵³ This is a reformulation of our introductory presentation.

¹⁵⁴ See RIVENC, F., *ILP*, p. 48.

¹⁵⁵ See *ibidem* p. 44.

$\vdash A \supset (B \supset A)$	Weakening
$\vdash (A \supset B) \supset ((B \supset C) \supset (A \supset C))$	Suffixing
$\vdash (A \supset B) \supset A \supset A$	Peirce's law

Where « $\supset$ » represents, as usual, the classical implicational connective. We do not present the proof of the Deduction Theorem for the Heyting's system here, and our reader can easily check that identity and weakening can be obtained in **CL** using truth tables. The important lesson is that this theorem can't be obtained for **R** : if it was, then Weakening would be provable in it, a result that **R** was precisely designed to avoid¹⁵⁶. So, we would like to change our notion of deduction so that a revised version of the Deduction Theorem can hold for it, since the link between implicational connective and deducibility seems so natural. This can be done : **R** can prove another theorem, relating to another, stronger, notion of deduction, where we require that the premises really do something in the deduction in which they appear. We prove it here for **R_h** $\rightarrow$ specifically.

2.3 The relevant Deduction Theorem

Definition 2.13 : Relevant deduction – « A deduction of C from $A_1, \dots, A_n$ is *relevant with respect to* A_i if and only if A_i has been used to derive C »¹⁵⁷.

The use of a hypothesis A in a deduction can be tracked : A is marked with a *, as well as any derived formula obtained from an application of MP where at least one of the conditions is a marked formula. It is then required, for the deduction of C to be *relevant with respect to* A, that C be also marked with a * (this is called *the star formulation*)¹⁵⁸.

Theorem 2.14 : Relevant Deduction Theorem¹⁵⁹ – There is a deduction in **R_h** $\rightarrow$ of C from the hypothesis $A_1, \dots, A_n, A$, which is relevant with respect to A, if and only if there is a deduction of $A \rightarrow C$ from $A_1, \dots, A_n$:

$$A_1, \dots, A_n, A^* \vdash_{\mathbf{R}_h \rightarrow} C^*$$

if and only if

$$A_1, \dots, A_n \vdash_{\mathbf{R}_h \rightarrow} A \rightarrow C$$

Proof 2.15 : Relevant Deduction Theorem¹⁶⁰

¹⁵⁶ We show that Weakening is not provable in **R**, see section 2.7.

¹⁵⁷ RIVENC, F., *ILP*, p. 49.

¹⁵⁸ From *ibidem*, pp. 49-50.

¹⁵⁹ From *ibidem*, p. 50.

¹⁶⁰ From *ibidem*, pp. 50-51 and ANDERSON, A.R., and BELNAP, N., *ELRN I*, §3.

From bottom to top : if $A \rightarrow C$ is deducible from the hypothesis $A_1, \dots, A_n$, then adding A to the number of hypothesis allows for the derivation, using MP, of C . And A has been used to derive C in this application of MP, so if it's starred, C is starred as well.

From top to bottom : Let $B_1, \dots, B_k$ be the initial deduction where A^* is among the hypotheses and where all the formulae obtained using A^* are starred. We show that for any B_i , if it's starred, then there is a deduction of $A \rightarrow B_i$ from $A_1, \dots, A_n$. We will have shown that there is a deduction of $A \rightarrow C$ from those hypotheses, since C is also starred. There are three possible cases.

1. $B_i = A^*$: in this case, since $B_i \rightarrow B_i$ is an axiom, Identity, it is deducible from $A_1, \dots, A_n$ (basis case of the induction on the complexity of formulae).

2. B_i is an axiom in which case A^* is not used to deduce it (but it is deducible from $A_1, \dots, A_n$) ; same if B_i is one of the hypothesis.

3. B_i is obtained by MP from the formulae $B_j, (B_j \rightarrow B_i)$. If none is starred, then B_i does not depend on A , and is instead deducible from $A_1, \dots, A_n$. Three new cases are to be distinguished now. This is the « inductive step » : we show the cases in which B_i is not A , but directly connected to it.

3.1 The minor premise, B_j , is starred, but not the major one, B_i . In the axiomatic proof we have $A \rightarrow B_j$ and $B_j \rightarrow B_i$. Suffixing and MP ensure us that from these two we can infer $A \rightarrow B_i$. And so that we have $A \rightarrow B_i$ from $A_1, \dots, A_n$.

3.2 The major premise, B_i , is starred, but not the minor one, B_j . In the axiomatic proof we have $A \rightarrow (B_j \rightarrow B_i)$ and B_j . Permutation and MP ensure us that from these we can infer $A \rightarrow B_i$. And so that we have $A \rightarrow B_i$ from $A_1, \dots, A_n$.

3.3 Both premises are starred. In the axiomatic proof we have $A \rightarrow (B_j \rightarrow B_i)$ and $A \rightarrow B_j$. Self-distribution and MP ensure us that from these we can infer $A \rightarrow B_i$. And so that we have $A \rightarrow B_i$ from $A_1, \dots, A_n$.

As we can see, the choice of axioms in A&B's system was made exactly to provide this result, and so the latter justifies the former. The first notion of relevance displayed by $\mathbf{R}_h$ (because we want it to be a conservative extension of $\mathbf{R}_h \rightarrow$) is linked to the idea of « being used » in a proof, and displayed in the Relevant Deduction Theorem. How can we build a natural deduction system that is faithful to this idea of relevance ?

2.4 The natural deduction implicational fragment for $\mathbf{R}, \mathbf{NDR} \rightarrow$

The equivalent of $\mathbf{R}_h \rightarrow$ in natural deduction form makes the preceding requirement of relevance way more explicit. Natural deduction systems, unlike Hilbert-style systems,

do not heavily rely on axioms. Rather, the proof-theoretic work is made through the application of a number of rules of inference. A proof in $\mathbf{NDR} \rightarrow$ is a list of steps, each being the application of a rule of inference of the system. The wffs and connectives are the same than $\mathbf{R}_h \rightarrow$, except that the formulae now include underwritten indices, the use of which is dictated by the rules of inference. Those rules are the following :

Rules 2.16 : Rules of inference of $\mathbf{NDR} \rightarrow$ ¹⁶¹

1. **Hypothesis introduction** : a new hypothesis can be introduced in the course of a proof. The hypothesis introduced constitutes the first step of a subproof, and is written at the top of the subproof it initiates. A subproof is written with a shift on the right of the previous proof, which is either the main proof or another subproof. It should be conceived as an « item » of the main proof. The hypothesis is introduced with an underwritten indice, $\{k\}$, where k is a single numeral ($\{k\}$ is a singleton), new for each hypothesis.

2. **Repetition** : A_a can be repeated inside a (sub)proof, keeping its indice « a »¹⁶².

3. **Reiteration** : A_a can be reiterated, that is, « imported » from a (sub)proof in one of its subproofs.

4. **$\rightarrow$ Elimination** : from A_a and $(A \rightarrow B)_b$, infer $B_{a \cup b}$ where « $\cup$ » means that the inferred formula has for indices the union of indices of the preceding expressions.

5. **$\rightarrow$ Introduction** : from a proof of B_a under hypothesis $A_{\{k\}}$, infer $(A \rightarrow B)_{a - \{k\}}$ where the new indice « $a - \{k\}$ » is the indice a from which we removed the indice $\{k\}$ which otherwise figures in it.

The justification in each step of the proof is still written on the right of the step. We abbreviate them following common practice, in order : « hyp. », « rep. », « reit. », « $\rightarrow E$ », « $\rightarrow I$ ». We can visualise the last two rules as follows :

$\rightarrow E$	$(A \rightarrow B)_b$ A_a $B_{a \cup b}$	$\rightarrow I$	$A_{\{k\}}$ B_a $(A \rightarrow B)_{a - \{k\}}$ (on the condition that $k \in a$)
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¹⁶¹ From RIVENC, F., *ILP*, p. 43.

¹⁶² Indices a , b can take any set of integers as a value. Here we do not always specify the set-theoretic particles.

Note that the rule « $\rightarrow I$ » discharges the hypothesis A , and so gets us back into the previous (sub)proof, and that « $\rightarrow E$ » is our version of MP. The condition on the indices for « $\rightarrow I$ » is crucial, only it ensures us that the deduction is relevant, and that the inferred implication is so. Cleaning our system of any reference to indices allows for the deduction of the forbidden theorem *par excellence*, « $A \rightarrow (B \rightarrow A)$ » and, in fact, it leads to $\mathbf{NDCL}^\supset$, the implicational fragment of $\mathbf{NDCL}$. In $\mathbf{NDCL}^\supset$ we indeed have the following proof, where our symbol « $\rightarrow$ » is replaced, as usual, with the symbol « $\supset$ ». We show on the right of the proof how $\mathbf{NDR}^\rightarrow$ can block it.

Proof 2.17 : Weakening in $\mathbf{NDCL}^\supset$

1.	A	hyp.	1.	A_1	hyp.
2.	B	hyp.	2.	B_2	hyp
3.	A	reit.	3.	A_1	reit.
4.	$(B \supset A)$	2,3 $\supset I$	4.	$(B \rightarrow A)_{1-2}$	$\rightarrow I ? X$
5.	$A \supset (B \supset A)$	3,4 $\supset I$			

In $\mathbf{NDR}^\rightarrow$, since the indice of A on line 3 does not contain that of B on line 2, we can't proceed to apply $\rightarrow I$.

It can be shown (1) that the axioms of $\mathbf{R}_h^\rightarrow$ are provable in $\mathbf{NDR}^\rightarrow$ and (2) that any proof in $\mathbf{NDR}^\rightarrow$ can be turned into a proof in $\mathbf{R}_h^\rightarrow$ ¹⁶³. So the two fragments are shown to be sound and complete with respect to one another ; and so it is also shown that the Relevant Deduction Theorem holds for $\mathbf{NDR}^\rightarrow$.

2.5 Hilbert-style and natural deduction systems for $E^\rightarrow$ ¹⁶⁴

A&B developed a system close to $\mathbf{R}$, but which was designed to capture necessity as well as relevance. The system is abbreviated « $\mathbf{E}$ » for « entailment ». $\mathbf{E}$ was supposed to avoid « fallacies » of modality. In our discussion of necessity in subsection I.2.3.1, we considered the example of the number 7, and of mathematics in general. For the sake of our comparison, we assume that numbers really are necessary entities. If we consider a sentence of the kind « $7+3=10$ », then it is clear that if this sentence is true, then it is necessarily so. Its truth depends on necessary entities and the way they relate to one another. So, it seems like its truth value should *not* depend on the way things are in our world that do not depend on necessities. That « the sky is sometimes blue » and « Macron

¹⁶³ See RIVENC, F., *ILP*, pp. 52-53.

¹⁶⁴ Section from *ibidem*, pp. 53-57.

is the president of France » cannot have an implicative connection with « $7+3=10$ » because contingent statements are just not the kind of entity that can determine the truth value of « $7+3=10$ ». As our blackboard example tried to show, the entities and truths that are necessary have a particular status that make them detached from our empirical world. There may be a sense in holding that our empirical world is just irrelevant to establish the truths of necessities. A defense of the rejection of those fallacies of modality can be found in the « Platonic Principle » which states that « necessary knowledge cannot derive from experience »¹⁶⁵. As Mares notes, versions of this argument have been widely defended in the history of philosophy, from Saint Augustine to Hume¹⁶⁶. On the other hand, more recent authors, like Putnam, have refused the principle. For instance, Putnam argued that even though we discovered that water was H_2O empirically, it is a necessary truth¹⁶⁷.

It may not be the best strategy to appeal to the Platonic Principle in the first place. If we admit that necessary truths indeed have a special modal status, we can still hold that empirical statements should not entail them. Most of us have learnt about fractions using pies or the representations of pies. But it is doubtful that, from the fact that we have learnt about fractions using pies, we want to hold that the way we cut pies *entails* propositions relating to fractions¹⁶⁸. Maybe we need to distinguish between the context of discovery or learning and the context of justification.

Mares introduces the concept of « necessitive » where if $\Box A$ is a necessitive (« $\Box$ » is a necessity connective), so is $\Box A \wedge A$, but not $\Box A \vee A$. It seems that on A&B's account, $\Box A \rightarrow A$ is also a necessitive¹⁶⁹. We can define a fallacy of modality : « a statement commits a fallacy of modality if and only if it says that a necessitive is entailed by some non-necessitive »¹⁷⁰. A&B tried to conceive a system where we do not have fallacies of modality, that is, where we do not have $\vdash A \rightarrow \Box A$. In **CL**, if we read « $\supset$ » as a relation of entailment (necessary implication), we have $\vdash_{CL} (A \supset A)$ as a necessitive (a theorem, so necessary true), and $\vdash_{CL} A \supset ((A \supset A) \supset A)$ as an instance of the Weakening axiom. So if A holds and is contingent it entails $((A \supset A) \supset A)$, a necessitive. This constitutes a fallacy of modality.

This result also holds for **R**. Even if Weakening is not a theorem of it, an instance of Assertion is $\vdash_R A \rightarrow ((A \rightarrow A) \rightarrow A)$. Under our previous assumptions, the implicational connective of **R** can't represent a relevant *and* necessary relation of implication. This is

¹⁶⁵ See ANDERSON, A.R., and BELNAP, N., *ELRN* I, §22.1.2.

¹⁶⁶ See MARES, E.D., *RLPI*, p. 107.

¹⁶⁷ See *Ibidem*.

¹⁶⁸ For instance, it is doubtful that « if we cut a pie in two halves and then cut again each half in two, then we have four pieces of pie » entails « $(1/2)/2 = 1/4$ ».

¹⁶⁹ *Ibidem*.

¹⁷⁰ *Ibidem*.

where **E** enters the scene. Its language can be defined in the same way as **R**'s. The Hilbert-style presentation of its implicational fragment, $\mathbf{E}_h \rightarrow$, is characterized by the MP and the same choice of axioms than $\mathbf{R}_h \rightarrow$, except that the axiom of Assertion is replaced by the axiom of Restricted Assertion ; or equivalently, except that the axiom of Permutation is replaced by the axiom of Restricted Permutation :

Axioms 2.18 : Axioms of $\mathbf{E}_h \rightarrow$

The same exact list than $\mathbf{R}_h \rightarrow$'s, except that Permutation ($\mathbf{R}_h \rightarrow 3$) and Assertion ($\mathbf{R}_h \rightarrow 3'$) are replaced, respectively, by :

- ($\mathbf{E}_h \rightarrow 3$) $\vdash_E (A \rightarrow ((B \rightarrow C) \rightarrow D)) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow D))$ Restricted Permutation
 ($\mathbf{E}_h \rightarrow 3'$) $\vdash_E (A \rightarrow B) \rightarrow (((A \rightarrow B) \rightarrow C) \rightarrow C)$ Restricted Assertion

Here, a particular case of Restricted Assertion is $\vdash_E (A \rightarrow A) \rightarrow (((A \rightarrow A) \rightarrow A) \rightarrow A)$, which, with Identity and MP 'only' gives

$$(1) \vdash_E ((A \rightarrow A) \rightarrow A) \rightarrow A,$$

committing no « modal fallacy ». It only says that if A is the consequent of a necessary proposition, then A is true.

In its natural deduction formulation, $\mathbf{NDE} \rightarrow$ has the same rules than $\mathbf{NDR} \rightarrow$, except that the rule (3) Reiteration is replaced by the rule of Restricted Reiteration, where the import of formulae only works for the implicational ones :

Restricted Reiteration : A formula of the form $(A \rightarrow B)_a$ can be reiterated, that is, « imported » from a (sub)proof in one of its subproofs.

$\mathbf{NDE} \rightarrow$ can be proven equivalent to $\mathbf{E}_h \rightarrow$ ¹⁷¹. From the definition of a necessitive, the entailment of a formula by a necessitive must itself be a necessitive. It seems that we can define a necessary formula in $\mathbf{E} \rightarrow$ as : $NA = (A \rightarrow A) \rightarrow A$. We may rewrite (1) as

$$(1) NA \rightarrow A,$$

saying that if A is necessary, then A is true. Which is the axiom characteristic of **T** in modal logics. With the same writing convention, the theorem obtained from $\mathbf{R} \rightarrow$ above, $\vdash_R A \rightarrow ((A \rightarrow A) \rightarrow A)$ can be rewritten as

$$(2) A \rightarrow NA.$$

But our proof 2.10 shows that Specialized Assertion is a theorem of **R**, one of the instances of which is precisely

¹⁷¹ See ANDERSON, A.R., and BELNAP, N., *ELRN I*, §4.1.

$\vdash_{\mathbf{R}} ((A \rightarrow A) \rightarrow A) \rightarrow A,$

i.e. : (1) $NA \rightarrow A$.

So, we have that in $\mathbf{R}^{\rightarrow}$, modal distinctions are erased, since A and NA are interdeducible, and the formula $((A \rightarrow A) \rightarrow A) \rightarrow A$ is sometimes called *demodalisation* for this reason. In $\mathbf{E}^{\rightarrow}$, (1) is a theorem, but not (2). In $\mathbf{DNE}$, the restriction on Reiteration blocks the proof of (2). Therefore, $\mathbf{R}^{\rightarrow}$ strictly contains $\mathbf{E}^{\rightarrow}$ since $\mathbf{R}^{\rightarrow}$ proves all the axioms of $\mathbf{E}^{\rightarrow}$ and more, and has no extra rule of inference. And so $\mathbf{E}^{\rightarrow}$ has the same relevant properties than $\mathbf{R}^{\rightarrow}$: it also satisfies the relevant deduction theorem and the variable-sharing properties (see next section). One can check that the operator « N » in $\mathbf{E}^{\rightarrow}$ has all the properties expected in a **S4** modal system¹⁷²:

$\vdash_{\mathbf{E}} (A \rightarrow B) \rightarrow (NA \rightarrow NB)$

$\vdash_{\mathbf{E}} (A \rightarrow B) \rightarrow N(A \rightarrow B)$

$\vdash_{\mathbf{E}} NA \rightarrow NNA$

Unfortunately for it, $\mathbf{E}$ was conjectured to be equivalent to $\mathbf{R}\Box$, the latter being, in a Hilbert-style system, $\mathbf{R}_h$ extended with a necessity operator, appropriate axioms and « A entails B » taken to be translated by « $\Box(A \rightarrow B)$ ». Maksimova proved that a certain formula, « very plausible for the necessary consequence »¹⁷³ and provable in $\mathbf{R}\Box$ given appropriate translation, is not a theorem of $\mathbf{E}$ ¹⁷⁴. The formula is : « $((A \rightarrow (B \rightarrow C)) \wedge (B \rightarrow (A \vee C))) \rightarrow (B \rightarrow C)$ ». This result justifies the subsequent focus, on the part of relevant logicians, on $\mathbf{R}\Box$ rather than on $\mathbf{E}$ to formalize necessity and relevance.

2.6 Proofs using matrices¹⁷⁵

Before coming back to our main subject and appreciate the second relevance property displayed by $\mathbf{R}^{\rightarrow}$, we must first introduce our reader to a particular proof technique that will be used to show the said property. The principle underlying such proof method is expressed by $A\&B$:

Suppose that: (1) all the axioms of a logistic theory $\mathbf{S}$ have a certain property P ; (2) all the rules of $\mathbf{S}$ are such that if their premises each have P , then so do their conclusions; (3) a certain formula A does not have the property P .

¹⁷² For further discussion and the demonstrations, see RIVENC, F., *ILP*, pp. 56-57. **S4** is discussed in section 5.2.

¹⁷³ *Ibidem*, p. 53.

¹⁷⁴ See MAKSIMOVA, L., « A semantics for the calculus E of entailment » in *Bulletin of the section of logic*, 2 (1973).

¹⁷⁵ For this section, see ANDERSON, A.R., and BELNAP, N., *ELRN I*, §8.4.

Then we may conclude that A is not a theorem of **S**, i.e., "the postulates of **S** do not permit a proof of A", since everything deducible in **S** has P, by (1) and (2) - but A does not have P, by (3)¹⁷⁶.

The property P considered by A&B can be anything, but a lot of the proofs so conceived actually use as a property that of satisfying a matrix, where « we say that a formula satisfies a matrix if every assignment of values to the variables in it yields, on computation, a designated value »¹⁷⁷. The designated value is indicated with an asterisk in the matrix, and the computation takes place with analogy to a multiplication table. Let's take a simple example, still from A&B.

If we want our system **E**→ to avoid « fallacies of modality », we want it to not have as theorems any formula where the antecedent is an atomic formula (a propositional variable if we allow ourselves some interpretation of the language) and the consequent an entailment (so an implicational formula). We consider the following matrix :

→	0	1	2
0	2	2	2
1	0	2	0
*2	0	0	2

For any implicational formula, the antecedent's value is to be read in the first column and the consequent's value is to be read in the first line. Then we look, for each value assignment, at the intersection of the line corresponding to the antecedent and the column corresponding to the consequent, and we have the value of the implication. For instance, if A has value 0 and B has value 1 then $(A \rightarrow B)$ has value 2 :

		<u>2nd input (consequent), B</u>			
		→	0	1	2
1 st input (antecedent), A	0	2	2	2	
	1	0	2	0	
	*2	0	0	2	

Consider now one of the possibilities for the second axiom of **E**_h→, Suffixing. We want to show that for any assignment of values to A, B, and C, the resulting value output is the designated value, 2. One of the computations required would be with the assignment of values A=0, B=1, C=2. We get for this computation, following the order of operations imposed by the parentheses :

¹⁷⁶ ANDERSON, A.R., and BELNAP, N., *ELRN* I, §8.4, with slight visual adjustment.

¹⁷⁷ *Ibidem*.

$$\begin{aligned}
&\vdash (A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C)) \\
&\quad (0 \rightarrow 1) \rightarrow ((1 \rightarrow 2) \rightarrow (0 \rightarrow 2)) \\
&\quad 2 \rightarrow (0 \rightarrow 2) \\
&\quad 2 \rightarrow 2 \\
&\quad 2
\end{aligned}$$

2 being the designated value, we are good for this assignment. Now, we must, for $\mathbf{E}_h \rightarrow$, (1) show that we get 2 as the resulting value of *all* possible assignments of values for A (and B, C...) for *all* the axioms (that is, we must show that all the axioms satisfy the matrix). Then we must (2) show that MP preserves this property. So, we must show that when B has an undesigned value, then $(A \rightarrow B)$ has an undesigned value or A has an undesigned value. And ultimately, we must (3) show that if p is an atomic formula (a variable), then $p \rightarrow (B \rightarrow C)$ does *not* satisfy the matrix. Then we have shown that there is no theorem of the form $p \rightarrow (B \rightarrow C)$ in $\mathbf{E}_h \rightarrow$.

Step (1) takes some time to be done manually, for the preceding matrix and 3 variables, A, B and C, there are $3.3.3=27$ possible assignments of values to examine only for the axiom Suffixing. So it is usually skipped in the proofs and left to be discovered for the interested reader. Step (2) is fairly easy to see for little matrices. If we check the columns with B having the value 0 or 1, we see that for $(A \rightarrow B)$ to get the value 2, we need A with value 0 or 1. And if A takes the value 2, in those same columns, then $(A \rightarrow B)$ has value 0.

Step (2) is also generally left to be discovered by the reader in the matrix during the proofs. For step (3), the problematic value assignment is usually provided by the person doing the proof. Here step (3) would be : since p is a variable, it can assume the value 1. Furthermore, since $(B \rightarrow C)$ has the implicational form, it can only take as value one of the outputs of the matrix, 0 or 2. Looking at the line of p with value 1, we see that for $p \rightarrow (B \rightarrow C)$ to have as an output a designated value, $(B \rightarrow C)$ could only have the value 1, which it can't. So $p \rightarrow (B \rightarrow C)$ does not satisfy the matrix, and no theorem of $\mathbf{E} \rightarrow$ can have this logical form.

Note that the property studied here is sometimes called « Ackermann's property » and implies that $A \rightarrow (A \rightarrow A)$ is not a theorem of $\mathbf{E} \rightarrow$. Even though this instance also happens to not be a theorem of $\mathbf{R} \rightarrow$, the more general property can't hold for it, because $p \rightarrow ((p \rightarrow p) \rightarrow p)$ is a theorem of $\mathbf{R} \rightarrow$, as an instance of the axiom Assertion, and has only one variable in the antecedent¹⁷⁸.

¹⁷⁸ For this paragraph, see RIVENC, F., *ILP*, pp. 62-63.

2.7 Variable-sharing property and beyond¹⁷⁹

Theorem 2.19 : Variable-sharing property – If $A \rightarrow B$ is a theorem of $\mathbf{R}\rightarrow$, A and B share at least one variable.

Note that this excludes from the theorems of $\mathbf{R}\rightarrow$ formulae of the form $A \rightarrow (B \rightarrow B)$.

Proof 2.20 : Variable-sharing property

We consider the following matrix, with the distinguished values indicated with * :

$\rightarrow$	-2	-1	+1	+2
-2	+2	+2	+2	+2
-1	-2	+1	+1	+2
*+1	-2	-1	+1	+2
*+2	-2	-2	-2	+2

All the axioms of $\mathbf{R}_h\rightarrow$ satisfy the matrix, and MP preserves the property. But if A and B share no variable, we can assign the value +2 to all the variables of A (so $A=+2$) and +1 to all the variables of B (so $B=+1$), so $(A \rightarrow B)$ takes the undistinguished value -2. So, if A and B do not share a variable, $(A \rightarrow B)$ is not provable.

A consequence of this result is that there is no mutual implication of all the theorems of $\mathbf{R}\rightarrow$, for $(A \rightarrow A) \rightarrow (B \rightarrow B)$ ¹⁸⁰ do not share a variable ; but both the antecedent and the consequent are instances of Identity. The variable-sharing property is a necessary but not a sufficient condition to belong to the set of theorems. Our most dreaded logical result, $A \rightarrow (B \rightarrow A)$ satisfies the variable-sharing property, but is not provable in $\mathbf{R}\rightarrow$. We can show it with a stronger result than what has just been proven.

Theorem 2.21 : ‘Stronger’ variable-sharing property – If A is a theorem of $\mathbf{R}\rightarrow$, any variable occurring in A appears at least once as *antecedent part* and at least once as *consequent part* of A .

Definitions :

¹⁷⁹ For all this section, all results and proofs included, see RIVENC, F., *ILP*, pp. 58-61 and ANDERSON, A.R., and BELNAP, N., *ELRN I*, §5.1.2.

¹⁸⁰ This is unlike **CL**, where all the tautologies imply each other (recall the first problem of CarL, section 1.5).

1. A is a consequent part of A ;
2. if $B \rightarrow C$ is a consequent part of A, then B is an antecedent part of A, and C is a consequent part of A ;
3. if $B \rightarrow C$ is an antecedent part of A, then C is an antecedent part of A, and B is a consequent part of A.

Proof 2.22 : ‘Stronger’ variable-sharing property

Consider the matrix of proof 2.20 again, which is satisfied by all the theorems of $\mathbf{R}\rightarrow$ and where MP preserves this property. It can be shown that if a variable p appears only as a consequent or an antecedent part of a formula A, then A is not provable. We consider the following value assignment. If a variable appears only as an antecedent part of A, we give it the value +2. If a variable appears only as a consequent part of A, we give it the value -2. We assign to all other variables the value +1. By induction on the length of proofs, it can be shown that for any B, subformula of A¹⁸¹ :

1. if B does not contain p , the value of B is +1 ;
2. if B contains p and is an antecedent part of A, the value of B is +2 ;
3. if B contains p and is a consequent part of A, the value of B is -2.

But we know that A is a consequent part of A, so the value of A is -2. So A doesn’t satisfy the matrix and is not a theorem.

As an example, we can consider the formula $p \rightarrow (q \rightarrow p)$, an instance of weakening. Its consequent parts are $p \rightarrow (q \rightarrow p)$, $(q \rightarrow p)$, and the second p and its antecedent parts are the first p and q . So, q appears only as an antecedent part. If we give it the value +2 and if we give the value +1 to p , we have :

$$+1 \rightarrow (+2 \rightarrow +1),$$

$$+1 \rightarrow -2$$

$$-2.$$

A corollary of this result is that no variable can appear only once in a theorem of $\mathbf{R}\rightarrow$ (so the formula of Peirce’s law, provable in $\mathbf{CL}$, is not provable here : $((A \rightarrow B) \rightarrow A) \rightarrow A$ is not a theorem of $\mathbf{R}\rightarrow$).

¹⁸¹ For more information on this proof by induction, see ANDERSON, A.R., and BELNAP, N., *ELRN* I, §12, p. 120.

2.8 Two criteria of relevance¹⁸²

The notion of relevance is present in $\mathbf{R} \rightarrow$ in two rather different ways. First, as we saw through the « Relevant Deduction Theorem », with the notion of « being used in a proof » central : a formula A is relevant with respect to another formula C if A has been used in a derivation of C (A is starred). In the previous section, we saw a different notion of relevance, where a formula A was relevant with respect to a formula C if A and C shared a variable. Some have argued from this ambiguity that the notion of relevance at stake was irretrievably confused, and that trying to « characterize valid inferences on the basis of a criterion of relevance was to put the cart before the horse »¹⁸³ and that it is the « validity, otherwise recognized, of an inference that shows that there is a relation of relevance between the premises and the conclusion »¹⁸⁴. So, the proof of EFQ showing that from a contradiction, anything follows, would appear as the proof that a contradiction is relevant to any proposition. So, it must be noted that the coexistence of the two criteria for relevance that we have explored in the present chapter is not necessarily a good thing for $\mathbf{R}$. To obtain our full system $\mathbf{R}$, we need to extend its implicational fragment appropriately. To this task we now turn our attention.

¹⁸² For this section, see RIVENC, F., *ILP*, p. 62.

¹⁸³ *Ibidem*.

¹⁸⁴ *Ibidem*.

Chapter 3 : The systems $\mathbf{R_h}$ and $\mathbf{NDR}$ of relevant implication

The present chapter completes the Hilbert-style and natural deduction presentations of $\mathbf{R}$. We start by adding negation to the implicational fragments before completing the systems by adding conjunction and disjunction. Some interesting results are presented along the way, with some elements of criticism discussed. In particular, we will discuss disjunctive syllogism, notions of deduction, intensional connectives and the relationship between $\mathbf{R}$ and $\mathbf{CL}$.

3.1 Adding negation : the fragments $\mathbf{R_h}^{\rightarrow \neg}$ and $\mathbf{NDR}^{\rightarrow \neg}$ ¹⁸⁵

For both $\mathbf{R_h}^{\rightarrow \neg}$ and $\mathbf{NDR}^{\rightarrow \neg}$, we add to the language the unary connective « $\neg$ », meant to represent negation, with appropriate grammar rule (of the kind, « if ‘A’ is a formula, then ‘ $\neg A$ ’ is a formula »). For $\mathbf{R_h}^{\rightarrow \neg}$, we add to a choice of four implicational axioms of section 2.1 the following¹⁸⁶ ones :

$(\mathbf{R_h}^{\neg 5}) \vdash (A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)$ Intuitionist Contraposition

$(\mathbf{R_h}^{\neg 6}) \vdash (\neg \neg A \rightarrow A)$ Double Negation

To extend $\mathbf{E_h}^{\rightarrow}$ with negation we also need the axiom of Reduction, « $\vdash (A \rightarrow \neg A) \rightarrow \neg A$ » which is obtainable as a theorem in $\mathbf{R_h}^{\rightarrow \neg}$:

Proof 3.1 : Reduction in $\mathbf{R_h}^{\rightarrow \neg}$ ¹⁸⁷

1. $\vdash A \rightarrow ((A \rightarrow \neg A) \rightarrow \neg A)$ $(\mathbf{R_h}^{\rightarrow 3'})$
2. $\vdash ((A \rightarrow \neg A) \rightarrow \neg A) \rightarrow (A \rightarrow \neg(A \rightarrow \neg A))$ $(\mathbf{R_h}^{\neg 5})$ ¹⁸⁸
3. $\vdash [A \rightarrow ((A \rightarrow \neg A) \rightarrow \neg A)] \rightarrow \{[(A \rightarrow \neg A) \rightarrow \neg A] \rightarrow (A \rightarrow \neg(A \rightarrow \neg A))\} \rightarrow [A \rightarrow (A \rightarrow \neg(A \rightarrow \neg A))]$ $(\mathbf{R_h}^{\rightarrow 2'})$ ¹⁸⁹
4. $\vdash [((A \rightarrow \neg A) \rightarrow \neg A) \rightarrow (A \rightarrow \neg(A \rightarrow \neg A))] \rightarrow [A \rightarrow (A \rightarrow \neg(A \rightarrow \neg A))]$ 1,3 MP
5. $\vdash A \rightarrow (A \rightarrow \neg(A \rightarrow \neg A))$ 2,4 MP
6. $\vdash (A \rightarrow (A \rightarrow \neg(A \rightarrow \neg A))) \rightarrow (A \rightarrow \neg(A \rightarrow \neg A))$ $(\mathbf{R_h}^{\rightarrow 4})$
7. $\vdash A \rightarrow \neg(A \rightarrow \neg A)$ 5,6 MP
8. $\vdash (A \rightarrow \neg(A \rightarrow \neg A)) \rightarrow ((A \rightarrow \neg A) \rightarrow \neg A)$ $(\mathbf{R_h}^{\neg 5})$
9. $\vdash (A \rightarrow \neg A) \rightarrow \neg A$ 7,8 MP

¹⁸⁵ For all this section, see RIVENC, F., *ILP*, pp. 69-80.

¹⁸⁶ See *ibidem*, p. 70.

¹⁸⁷ Adapted from *ibidem*.

¹⁸⁸ With $A/(A \rightarrow \neg A)$, B/A .

¹⁸⁹ With A/A , $B/((A \rightarrow \neg A) \rightarrow \neg A)$, $C/(A \rightarrow \neg(A \rightarrow \neg A))$.

The converse of axiom (**R**¬6) is also easily provable in **R_h→¬**:

Proof 3.2 : $(A \rightarrow \neg\neg A)$ in **R_h→¬**

1. $\vdash (\neg A \rightarrow \neg A) \rightarrow (A \rightarrow \neg\neg A)$ (**R_h¬5**)
2. $\vdash (\neg A \rightarrow \neg A)$ (**R_h→ 1**)
3. $\vdash (A \rightarrow \neg\neg A)$ 1,2 MP

To obtain **NDR→¬**, we need to add the following rules of inference to those of **NDR→** presented in section 2 :

6. **Contraposition (ctr.)** : Given a proof of $\neg B_b$ from the hypothesis $A_{\{k\}}$, under the assumption B_a , where k appears in **b**, infer $\neg A_{(a \cup b) - \{k\}}$:

ctr.	B_a $A_{\{k\}}$ $\neg B_b$ $\neg A_{(a \cup b) - \{k\}}$
	(on the condition that $k \in b$)

7. **Double Negation Elimination (¬¬E)** : from $\neg\neg A_a$, infer A_a .

NDE→¬ requires another rule, alike its axiom of Reduction, which states that from a proof of $\neg A_a$ under hypothesis $A_{\{k\}}$, one may infer $\neg A_{a-\{k\}}$. It can be seen that the natural deduction rules are alike their axiomatic counterparts, and indeed it can be proven that **NDR→¬** and **R_h→¬** are, as one would obviously require, equivalent¹⁹⁰.

Furthermore, the systems obtained are conservative extensions of the corresponding purely implicational fragments¹⁹¹. One can also prove a nice « intersubstitutability » result for interdeducible formulae : the Replacement Theorem holds for **R→¬**.

Theorem 3.3 : The Replacement Theorem¹⁹² – If $(A \rightarrow B)$ and $(B \rightarrow A)$ are theorems (also said : if A *co-implies* B , or if A and B are interdeducible), and if $F(B)$ is the result

¹⁹⁰ See RIVENC, F., *ILP*, p. 72.

¹⁹¹ See Appendix A, section A.3 for the definition of a conservative extension. For the proof of this result, see ANDERSON, A.R., and BELNAP, N., *ELRN I*, §14.4 (and §13.3).

¹⁹² RIVENC, F., *ILP*, pp. 73-75. Rivenc also provides a proof and examples.

of replacing in $F(A)$ one or several occurrences of A by B , then $(F(A) \rightarrow F(B))$ and $(F(B) \rightarrow F(A))$ are theorems.

We also obtain for $\mathbf{R}^{\rightarrow\neg}$ extended versions of our theorems 2.19 and 2.21, that is, of the ('stronger') variable-sharing properties. For the second theorem, we need the following additions to the definitions¹⁹³ :

4. if $\neg B$ is a consequent part of A , B is an antecedent part of A ;
5. if $\neg B$ is an antecedent part of A , B is a consequent part of A .

The presented negation is rather classical. With the replacement theorem, ($\mathbf{R}^{\neg 6}$), and the result proven in our proof 3.2, we have that A and $\neg\neg A$ are intersubstitutable. We can also prove all the forms of Contraposition, including the classical one, $(\neg A \rightarrow B) \rightarrow (\neg B \rightarrow A)$ ¹⁹⁴ (here proven in $\mathbf{NDR}^{\rightarrow\neg}$) :

Proof 3.4 : Classical Contraposition¹⁹⁵

- | | | |
|-----|---|----------------------|
| 1. | $(\neg A \rightarrow B)_1$ | hyp. |
| 2. | $\neg B_2$ | hyp. |
| 3. | $\neg A_3$ | hyp. |
| 4. | $(\neg A \rightarrow B)_1$ | reit. |
| 5. | $B_{3,1}$ | 3,4 $\rightarrow E$ |
| 6. | $\neg B_4$ | hyp. |
| 7. | $\neg B_4$ | rep. |
| 8. | $\neg\neg B_{3,1}$ | 5,6,7 ctr. |
| 9. | $\neg\neg A_{2,1}$ | 2,3,8 ctr. |
| 10. | $A_{2,1}$ | $\neg\neg E$ |
| 11. | $(\neg B \rightarrow A)_1$ | 2,10 $\rightarrow I$ |
| 12. | $(\neg A \rightarrow B) \rightarrow (\neg B \rightarrow A)$ | 1,11 $\rightarrow I$ |

Note that the steps 5-8 show that from a formula, we can introduce its double negation at the same indice. So, we get a derived Double Negation Introduction rule in our system. Likewise, we can always derive $(\neg B \rightarrow A)$ from $(\neg A \rightarrow B)$, as steps 1 and 11 show.

¹⁹³ See RIVENC, F., *ILP*, pp. 77-78.

¹⁹⁴ See *ibidem*, pp. 72-73.

¹⁹⁵ *Ibidem*, p. 73.

It is possible to introduce negation using a certain propositional constant, « f », of falsity, like in minimal logic¹⁹⁶. Then we can define $\neg A$ as $A \rightarrow f$, and add the only axiom for negation to the 4 of the implicational fragment : $\vdash ((A \rightarrow f) \rightarrow f) \rightarrow A$. The axiom of Intuitionist Contraposition being, in this context, an instance of Permutation : $\vdash (A \rightarrow (B \rightarrow f)) \rightarrow (B \rightarrow (A \rightarrow f))$; Reduction (for $E_h \rightarrow$), an instance of Contraction : $\vdash (A \rightarrow (A \rightarrow f)) \rightarrow (A \rightarrow f)$; and Double Negation, the new axiom itself. But defining negation in this way can have peculiar consequences. Urquhart¹⁹⁷, unsatisfied with a « classical » negation for $R^{\rightarrow \neg}$ and defining negation this way, showed that if we buy the idea that a rule for $\vee$ Elimination should have its conclusion proven by both disjuncts :

$\vee E$	1.	$(A \vee B)_a$	
	2.	A_a	B_a
		—	—
		—	—
	n	$C_{a \vee b}$	$C_{a \vee b}$
	$n+1$	$C_{a \vee b}$	

then we would not be in a good position to prove excluded middle (which is provable in **R**) without committing a fallacy of relevance. We would need some proof of the kind :

1.	A	f	
2.		$A \rightarrow f$	$\rightarrow I$
3.	$A \vee \neg A$	$A \vee \neg A$	$\vee I$ and abbreviation
4.	$A \vee \neg A$		$\vee E$

But step 2 commits a fallacy of relevance. Urquhart uses this as an argument against a negation of « classical style » for $R^{\rightarrow \neg}$, but this could simply be seen as an argument against a definition of negation as provided. Urquhart didn't prove that « another way of characterizing negation that could justify Excluded Middle was in itself incompatible with the idea of relevant implication »¹⁹⁸.

3.2 Validity of first degree entailment¹⁹⁹

In order to present one of the more criticized results of **R**, we have to make a detour through some sideways considerations. We start by adding to our language the usual extensional connectives, « $\vee$ » and « $\wedge$ » with appropriate grammar rules. We define a

¹⁹⁶ For all this paragraph, see RIVENC, F., *ILP*, pp. 76-80.

¹⁹⁷ URQUHART, A., « What is Relevant Implication ? » in *Directions in Relevant Logic* (Norman, J., and Richard, S. eds.), chapter 11.

¹⁹⁸ RIVENC, F., *ILP*, p. 80.

¹⁹⁹ For all this section, see *ibidem*, pp. 81-91.

zero degree formula (zdf) as a formula which only contains the connectives $\wedge$, $\vee$ and $\neg$; and a *first degree entailment* (fde) as a formula of the form $(A \rightarrow B)$, where A and B are zdfs. The system $\mathbf{R}_h^{\text{fde}}$ (actually identical to $\mathbf{E}_h^{\text{fde}}$) is the fragment of $\mathbf{R}_h$ that captures the notion of entailment concerning fdes. In the present section, we will only explore the conditions of validity in the system through the notion of tautological entailment, we do not provide the axiomatic presentation of $\mathbf{R}_h^{\text{fde}}$ ²⁰⁰.

It is clear that our notion of validity cannot be identical to that of classical logic : recall that in $\mathbf{CL}$, « $(A \wedge \neg A)$ » implies any B , while in $\mathbf{R}$ a contradiction cannot be relevant for any arbitrary proposition. We start by defining a primitive entailment and the corresponding notion of validity :

Definition 3.5 : Primitive entailment – A *primitive entailment* is a fde where the antecedent is a *primitive conjunction* (that is, a conjunction of atoms) and where the consequent is a *primitive disjunction* (that is, a disjunction of atoms). It is a formula of the form :

$$A_1 \wedge \dots \wedge A_n \rightarrow B_1 \vee \dots \vee B_m,$$

where every A_i and B_j is an atom (n or m or both can be equal to 1).

Definition 3.6 : Valid primitive entailment – A primitive entailment is *valid*, or *explicitly tautological*, if and only if there is an atom in its antecedent that is identical to an atom in its consequent : that is, if and only if one of the A_i is identical to one of the B_j . Examples : the formulae $(p \rightarrow p)$, $(p \wedge q) \rightarrow q$, $p \rightarrow (p \vee q)$ are explicitly tautological.

But the following formulae are not explicitly tautological, and so are considered invalid :

$$(p \wedge \neg p) \rightarrow q, \quad p \rightarrow (q \vee \neg q), \quad (p \wedge \neg p) \rightarrow (q \vee \neg q).$$

All fdes do not have the form of a primitive entailment. Consider the following formula, where the consequent is a conjunction : $p \rightarrow (p \wedge (r \vee \neg r))$. In what circumstances should it be accepted ? It seems that we ought to answer : if both primitive entailments $(p \rightarrow p)$ and $p \rightarrow (r \vee \neg r)$ were explicitly tautological, which is not the case. Why such answer ? Because if we were to accept as valid formulae of the form $A \rightarrow (A \wedge (B \vee \neg B))$, since the conjunction $A \wedge (B \vee \neg B)$ implies each of its conjuncts, we'd have in particular $(A \wedge (B \vee \neg B)) \rightarrow (B \vee \neg B)$ and by transitivity $A \rightarrow (B \vee \neg B)$, which

²⁰⁰ Our reader can consult the axiomatic presentation of $\mathbf{R}_h^{\text{fde}}$ in RIVENC, F., *ILP*, p. 92 ; and the proof that validity in $\mathbf{R}_h^{\text{fde}}$ is captured by tautological entailment in *ibidem*, pp. 92-95.

is one of the theorems of **CL** we are trying to avoid, (M1)²⁰¹. It is possible to block the inference with the rejection of the transitivity of implication, but A&B and the proponents of **R** do not find this solution acceptable²⁰². With this in mind, we are led to establish the following definitions.

Definition 3.7 : Normal form of fdes – A fde is in normal form if and only if the antecedent is a disjunction of primitive conjunctions, and the consequent is a conjunction of primitive disjunctions. It is a formula of the form :

$$A_1 \vee \dots \vee A_n \rightarrow B_1 \wedge \dots \wedge B_m,$$

where every A_i is a primitive conjunction and every B_j is a primitive disjunction.

Examples : $p \rightarrow (p \wedge (q \vee \neg q))$, $(p \vee q) \rightarrow (p \wedge q)$ are in normal form. So is the following formula : $(\neg p \vee (p \wedge q)) \rightarrow ((\neg p \vee p) \wedge (\neg p \vee q))$.

Definition 3.8 : Valid entailment in normal form – A fde in normal form :

$A_1 \vee \dots \vee A_n \rightarrow B_1 \wedge \dots \wedge B_m$ is *valid*, or *explicitly tautological* if and only if for any A_i and any B_i , the primitive entailment $A_i \rightarrow B_i$ is explicitly tautological.

Examples : from the examples of the preceding definition, the two first formulae aren't explicitly tautological while the third is.

Now, those definitions do not cover all the possible cases, for all fdes are not in normal form (for instance : $(p \wedge (\neg p \vee q)) \rightarrow q$. When they are not, we must use all the usual replacement rules to transform the expressions at hand so they get the required form : associativity, commutativity, double negation elimination, distributivity and De Morgan's law for entering negation in parentheses²⁰³. We are at last able to formulate the following definition, a syntactic notion of validity for **R_h^{fde}**:

²⁰¹ See section 1.5.

²⁰² See notably ANDERSON, A.R., and BELNAP, N., *ELRN I*, §15.

²⁰³ Our reader can find those rules among those presented in BOUQUIAUX, L., and LECLERCQ, B., *LFA*, pp. 52-53. Why are we allowed to use those rules ? It seems that we are constructing a notion of validity that just accepts those as validity-preserving. Rivenc just says that « we can admit that those rules of transformation preserve validity » (RIVENC, F., *ILP*, p. 85) while A&B seem to use a detour through a set-theoretic framework to show that they are (ANDERSON, A.R., and BELNAP, N., *ELRN I*, §20.3.2, p. 229). But one may say that this latter strategy already implicitly accepts that such replacement rules should hold. It seems that this acceptance is reasonable because relevance logic is all about changing the notion of consequence. The connectives « $\wedge$ », « $\vee$ » and « $\neg$ » are not the primary targets of the system, so they are, at least initially, not to be interpreted differently than in **CL** and are expected to behave classically. But see sections 3.4, 3.6, 6.7 and 8.2.

Definition 3.9 : Valid first degree entailment – $A \text{ fde } A \rightarrow B$ is a *tautological entailment* or *valid entailment* if and only if it has a normal form $A_1 \vee \dots \vee A_n \rightarrow B_1 \wedge \dots \wedge B_m$ that is explicitly tautological.

3.3 The rejection of disjunctive syllogism

It follows from the notion of validity for fdes just established that the famous Disjunctive Syllogism (DS) is invalid for $\mathbf{R}_h^{\text{fde}}$, and so for $\mathbf{R}_h$, which is a conservative extension of the former. Consider indeed the formula : $(\neg A \wedge (A \vee B)) \rightarrow B$. One of its instances is $(\neg p \wedge (p \vee q)) \rightarrow q$, which, in normal form, becomes $[(\neg p \wedge p) \vee (\neg p \wedge q)] \rightarrow q$. To be valid, we would need $(\neg p \wedge p) \rightarrow q$ and $(\neg p \wedge q) \rightarrow q$ to be valid primitive entailments, so in each expression, the antecedent and the consequent should share an atomic formula. But in the former, $(\neg p \wedge p) \rightarrow q$, the antecedent and the consequent do not share an atom. Therefore, the formula $(\neg A \wedge (A \vee B)) \rightarrow B$ is not valid. And the associated rule of inference : ‘from $\neg A$ and $(A \vee B)$, infer B ’, is rejected.

This appears as a big problem, because DS is generally considered as a valid rule of inference, that is fairly common in our inferential practices. To see what might be wrong about this mode of inference, we can consider what is known as Lewis’ argument, showing a proof of EFQ using different rules of inference²⁰⁴. The rules are :

- | | |
|---|----------------|
| 1. from ‘A and B’, we can infer ‘A’, and we can infer ‘B’ ; | Simplification |
| 2. from ‘A’, we can infer ‘A or B’ ; | Addition |
| 3. from ‘A or B’, and from ‘not-A’, we can infer ‘B’. | DS |

Proof 3.10 : Lewis’ argument

- | | |
|----------------|----------------|
| 1. A and not-A | Premise |
| 2. A | Simplification |
| 3. Not-A | Simplification |
| 4. A or B | Addition on 2 |
| 5. B | 3,4 DS |

If we accept the three principles of inference and the transitivity of consequence, then $(A \wedge \neg A) \rightarrow B$ must be accepted as a valid consequence, since B can be deduced from a contradiction. But as we have seen in our introduction, the formula so obtained, EFQ, commits a typical fallacy of relevance. So proponents of $\mathbf{R}$, unsatisfied with the idea of

²⁰⁴ A similar proof was already acknowledged in the Middle Ages. See RIVENC, F., *ILP*, p. 88 and READ, S., *PEI*, p. 31.

an intransitive relation of consequence, have advocated for the rejection of the rule DS. What is wrong, then, one may ask, with DS ?

Read's answer is that the problem rests²⁰⁵, in substance, on an equivocation of the particle « or ». Our uses of « or » can have (at least²⁰⁶) two different meanings, and so call for two different translations in formal systems. One of these uses is indeed purely truth-functional, and is to be translated with our « $\vee$ » in the formal system, and the other is intensional, to be translated with another particle, sometimes called fission, and symbolized in the present work with « + ».

In the first sense, « the truth of 'P or Q' follows immediately from that of P or of Q, and it is false if both are false »²⁰⁷. For this sense, addition is naturally acceptable, but not disjunctive syllogism : « if the ground for asserting 'P or Q' is P, say, then to learn that P is false (i.e. 'not-P' is true), far from enabling one to proceed to Q, undercuts the warrant for asserting 'P or Q' in the first place »²⁰⁸. The position of the person affirming a truth-functional disjunction is weakened : « ' $P \vee Q$ ' is weaker than P, and does not, even with the addition of 'not-Q', entail P, even though P clearly does »²⁰⁹. In the second sense, « 'P or Q' is equivalent to 'if not-P then Q', taking from 'if' its non-truth-functional character »²¹⁰. For this sense, DS is valid, but not addition.

If we come back to Lewis' argument, the problem stems from the use of rules that cannot be used together. If one proceeds to establish line 4 by addition, then it is the first, truth-functional, sense of « or » that is being used. So, the use of DS in line 5 is prohibited. Those considerations help answer the worry that we actually use DS in our reasonings (and so that DS should be valid), and softens the criticism claiming that we are touching to an indispensable ingredient of our inferential practice. But they do not explain why « the DS, the 'good old' DS with its extensional disjunction (and its negation supposed

²⁰⁵ See READ, S., *PEI*, pp. 31-34.

²⁰⁶ Classical logicians sometimes use another connective to translate some of our « or » uses. The connective is sometimes symbolized by « **W** », as in BOUQUIAUX, L., and LECLERCQ, B., *LFA*. This is an « exclusive » disjunction, which is true if exactly one of the disjunct is true (so it is also false if both disjuncts are true). Think of a canteen situation, when asked « cereal or fruit juice ? », where if one takes cereal, then she can't have fruit juice. The inference : « P or Q, P $\vdash$ not-Q » is correct for this sense of 'or'. Here too, we don't want our uses of « or » conflated, for if addition was to hold together with the previous rule of inference (let's call it (FJE) for 'fruit juice elimination') we could prove the « incompatibility of any pair of propositions whatever » :

given	P	
then	P or Q	Addition
whence	not-Q	(FJE).

Citation, example and explanation from READ, S., *PEI*, p. 34.

²⁰⁷ *Ibidem*, p. 32.

²⁰⁸ *Ibidem*, p. 33.

²⁰⁹ *Ibidem*.

²¹⁰ *Ibidem*.

so) is not valid »²¹¹. Indeed, « if $A \vee B$ is true, then one of the disjuncts is so ; and since A is false, B must be true, *point à la ligne* »²¹². This is what has been called the « Geach's Challenge »²¹³ by Read, and will only be discussed later in the present work²¹⁴.

An important note may be appropriate here. The acceptability of DS is to be distinguished from a certain property of theories (including logical theories), that is referred to as the γ -rule. The rule « holds for a theory T if whenever ' $\neg P \vee Q$ ' and ' P ' belong to T , ' Q ' belongs to T »²¹⁵. The γ -rule notoriously holds for **R** and **E**, among other systems. But the question is different from « whether one can legitimately infer Q from P and ' $\neg P \vee Q$ ' »²¹⁶. Indeed, a rule of inference can « be invalid, but applicable when its premises are theorems, in which case the conclusion is too ».²¹⁷

3.4 The system R_h

Our parenthesis on the rejection of DS closed, we can continue our presentation, and this time provide the whole axiomatic system for **R**, **R_h** . We present only one possible choice of the implicational fragment, but as noted in section 2.1, other equivalent possibilities are open. Our language is the same as **R_h^{fde}** , except that we add intensional connectives, « **o** » and « **+** » (respectively, intensional conjunction and intensional disjunction) ; and sometimes propositional constants (which will not be discussed here, we added them for the sake of completeness).

Axioms 3.11 : Axiomatic presentation of R_h ²¹⁸

Implication (one choice among others) :

(R_h 1) $\vdash A \rightarrow A$	Identity
(R_h 2) $\vdash (A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$	Transitivity (Suffixing)
(R_h 3) $\vdash A \rightarrow ((A \rightarrow B) \rightarrow B)$	Assertion
(R_h 4) $\vdash (A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$	Contraction

²¹¹ RIVENC, F., *ILP*, p. 91.

²¹² *Ibidem*.

²¹³ See READ, S., *PEI*, chapter 7. The name refers to a review of A&B : GEACH, P., « Review of *Entailment* » in *Philosophy*, 52 (1977), pp. 493-495.

²¹⁴ See section 7.2.

²¹⁵ READ, S., *PEI*, p. 30.

²¹⁶ *Ibidem*.

²¹⁷ RIVENC, F., *ILP*, p. 91, footnote 2. Another example given by Rivenc of such rule that is not a valid mode of inference but is applicable to theorems is the rule of necessitation in modal logics ('from $\vdash A$, infer $\vdash \Box A$ ').

²¹⁸ From *ibidem*, pp. 110-111.

Negation :

$$(R_h 5) \vdash (A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)$$

Intuitionist Contraposition

$$(R_h 6) \vdash (\neg\neg A \rightarrow A)$$

Double Negation

Conjunction :

$$(R_h 7) \vdash (A \wedge B) \rightarrow A$$

$$(R_h 8) \vdash (A \wedge B) \rightarrow B$$

$$(R_h 9) \vdash ((A \rightarrow B) \wedge (A \rightarrow C)) \rightarrow (A \rightarrow (B \wedge C))$$

Disjunction :

$$(R_h 10) \vdash A \rightarrow (A \vee B)$$

$$(R_h 11) \vdash B \rightarrow (A \vee B)$$

$$(R_h 12) \vdash ((A \rightarrow C) \wedge (B \rightarrow C)) \rightarrow ((A \vee B) \rightarrow C)$$

Distributivity of conjunction on disjunction :

$$(R_h 13) \vdash (A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee C)$$

Rules :

MP : from $A \rightarrow B$ and A infer B

Modus Ponens

$\wedge I$: from A and B , infer $(A \wedge B)$

Adjunction (or $\wedge$ Introduction)²¹⁹

We can add to the system the following definitions :

$$D1 \ A \bullet B \stackrel{\text{df}}{=} \neg(A \rightarrow \neg B)$$

Fusion = intensional conjunction

$$D2 \ A + B \stackrel{\text{df}}{=} \neg A \rightarrow B$$

Fission = intensional disjunction

And optionally axioms for propositional constants :

$$\vdash A \rightarrow (t \rightarrow A)$$

t Introduction

$$\vdash (t \rightarrow A) \rightarrow A$$

t Elimination

$$\vdash f \leftrightarrow \neg t$$

$$\vdash A \rightarrow T$$

$$\vdash F \rightarrow A$$

²¹⁹ With some restriction, see next subsection.

The system is, as one should expect, a conservative extension of $\mathbf{R}_h \rightarrow$, $\mathbf{R}_h \rightarrow \neg$ and $\mathbf{R}_h^{\text{fde}}$ ²²⁰.

3.4.1 The Adjunction rule²²¹

A few words must be said on the presence of the extra rule of inference. Even though it doesn't seem suspect, one may wonder why we did not simply add an axiom instead, of the form : $\vdash A \rightarrow (B \rightarrow (A \wedge B))$ which would allow to derive, from A and B and two applications of MP, $(A \wedge B)$. The answer is that such axiom would allow to derive the forbidden theorem : $\vdash A \rightarrow (B \rightarrow A)$ in the following way :

1. $\vdash (A \wedge B) \rightarrow A$ ($\mathbf{R}_h$ 7)
2. $\vdash [(A \wedge B) \rightarrow A] \rightarrow [(B \rightarrow (A \wedge B)) \rightarrow (B \rightarrow A)]$ Prefixing²²²
3. $\vdash (B \rightarrow (A \wedge B)) \rightarrow (B \rightarrow A)$ 1,2 MP
4. $\vdash A \rightarrow (B \rightarrow (A \wedge B))$ Axiom of Adjunction ?
5. $\vdash \{A \rightarrow (B \rightarrow (A \wedge B))\} \rightarrow \{[(B \rightarrow (A \wedge B)) \rightarrow (B \rightarrow A)] \rightarrow [A \rightarrow (B \rightarrow A)]\}$ ($\mathbf{R}_h$ 2)
6. $\vdash [(B \rightarrow (A \wedge B)) \rightarrow (B \rightarrow A)] \rightarrow [A \rightarrow (B \rightarrow A)]$ 4,5 MP
7. $\vdash A \rightarrow (B \rightarrow A)$ 3,6 MP

And since this formula is not a theorem of $\mathbf{R}_h \rightarrow$, our system would no longer be a conservative extension of it. So the addition of such axiom is unacceptable. But things get dirtier. From our rule of inference, from the two premises A, B, the conjunction $(A \wedge B)$ is deducible, and it seems that both premises have been used in the process. But according to the relevant Deduction Theorem, we then should have :

$A \vdash B \rightarrow (A \wedge B)$ and by a second application of it, $\vdash A \rightarrow (B \rightarrow (A \wedge B))$, which we just ruled out as problematic.

How are we to solve this issue ? There seems to be two possible solutions, « either refuse that from two hypothesis A, B, their conjunction is deducible, by tweaking the notion of ‘used premise’, which is tantamount to placing a restriction on the Adjunction rule »²²³, or « resign to lose the substance of the relevant Deduction Theorem, where it would no longer be true that every time we have ‘A, B $\vdash$ C’ we can conclude ‘A $\vdash$ B $\rightarrow$ C’ »²²⁴.

²²⁰ See RIVENC, F., *ILP*, p. 121.

²²¹ The subsection stems from *ibidem*, pp. 111-116.

²²² Recall that the non-chosen axioms for the implicational fragment are deducible from the chosen ones.

²²³ *Ibidem*, p. 113.

²²⁴ *Ibidem*, with slight adjustments.

A&B choose clearly the first option for their natural deduction presentation²²⁵ and seem to choose the second for their axiomatic version. They note²²⁶ that the ‘law’ of Exportation, $\vdash ((A \wedge B) \rightarrow C) \rightarrow (A \rightarrow (B \rightarrow C))$, should not be accepted. Even if « it is true that from the three premises A , B , and $(A \wedge B) \rightarrow C$, C is deducible since one application of Adjunction gives $A \wedge B$ from A and B , from which we obtain C by Modus Ponens »²²⁷, the law is to be rejected because it goes against the spirit of the relevant Deduction Theorem, « B is not, in fact, a separate hypothesis, which could be discharged *alone and in isolation*, and this is how we understand a collection of axioms at the top of a theory »²²⁸ that is, « as acting as a conjunction, without any implicational link suggested between them »²²⁹.

But instead of concluding that from a collection of premises, say « A, B » one is not in general allowed to derive $A \wedge B$, because the comma does not always have the value of a conjunction, A&B prefer to admit, in the axiomatic presentation, a new concept of deduction. With this new concept, we say that there’s a proof that the premises $A_1, \dots, A_n$ infer a conclusion C if and only if there is a relevant deduction of C from the premises where « *all* the premises are marked with $*$ and, for an application of Adjunction to be allowed, both formulae must be marked with $*$ »²³⁰. So according to this new concept of deduction, which we can symbolize with « $\vdash^*$ », we have $A, B \vdash^* A \wedge B$, but it is in fact tantamount to $A \wedge B \vdash A \wedge B$ (with the « $\vdash$ » referring to the previous, ‘official’ notion of deduction). This new concept provides a

form of the Deduction Theorem called ‘Entailment Theorem’: which states that if there’s a proof that the premises $A_1, \dots, A_n$ establish (in the sense of $\vdash^*$) a conclusion C , then $(A_1 \wedge \dots \wedge A_n) \rightarrow C$ is a theorem of **R**. [So] we do get $A, B \vdash^* A$ and it is true that $(A \wedge B) \rightarrow A$ is a theorem of **R**. But according to the ‘official’ concept of deduction, if we admit on the other hand that $A, B \vdash A \wedge B$ the idea of the [first, relevant] Deduction Theorem must be relinquished. Any attempt to try to prove it would fail, since [from $A, B \vdash^* A$] we cannot infer $B \rightarrow A$ from A to start the construction of a proof of $B \rightarrow (A \wedge B)$ under the sole hypothesis A , since we do not have access to Weakening²³¹.

Another option is possible. We adapt the natural deduction’s choice to the axiomatic system, where we require as before that A and B are « *both conjointly* »²³² used in a derivation of $A \wedge B$. But the different premises now get indices instead of

²²⁵ See section 3.5.

²²⁶ See ANDERSON, A.R., and BELNAP, N., *ELRN* I, §22.2.2.

²²⁷ RIVENC, F., *ILP*, p. 114.

²²⁸ *Ibidem*. Italic in the text.

²²⁹ *Ibidem*.

²³⁰ *Ibidem*, p. 115. Italic in the text.

²³¹ *Ibidem*.

²³² *Ibidem*.

being marked by a *, as in natural deduction. We get a « notion of relevant deduction adapted to assume the presence of Adjunction, where another theorem of Deduction for $[R_h]$ is provable »²³³ namely that

$$\text{if } A_1, \dots, A_n \vdash C \text{ then } \vdash B_1 \rightarrow (B_2 \dots (B_p \rightarrow C) \dots),$$

where « the B_j , $1 \leq j \leq p$ are, among the A_i , $1 \leq i \leq n$, the hypothesis *used* in the initial deduction »²³⁴.

3.4.2 The use of premises and intensional conjunction

Part of the difficulty here may be due to the following fact : « while the relation of entailment generally holds between a collection of premises and a conclusion, the way such collection of premises should be understood is not at the forefront of [A&B's] book »²³⁵. Is the comma « , » linking the premises to be understood as a conjunction, so we have : $A, B \vdash A \wedge B$ and so $A, B \vdash A$? « But then the idea of the relevant Deduction Theorem, linking implication and deducibility, is lost, or limited to the simultaneous discharge of hypothesis in their conjunctive form »²³⁶, or should we take the link between premises as something stronger, « which precisely ensures us that when we have : $A, B \vdash C$, we also have $A \vdash B \rightarrow C$ »²³⁷ ?

As Rivenc notes²³⁸, it may have been wiser for A&B to acknowledge that there was more than one way of combining premises, and so to clearly differentiate between one use of comma that is to be translated by the extensional « and », ‘ $\wedge$ ’ and one use to be translated by the intensional « and », ‘ $\bullet$ ’. In fact, it is traditional, as we’ll see in the next chapter, to avoid ambiguity by grouping premises intensionally with another punctuation mark, the semi-colon « ; ». This is reminiscent of Read’s move of section 3.3, except that here we acknowledge, not the ambiguity of « or », but the ambiguity of « and ».

We introduced intensional connectives in the system through definitions, but intensional conjunction can be introduced using axioms instead ; and the two formulations can be proven equivalent²³⁹. The axioms are :

$$\bullet 1 \vdash A \rightarrow (B \rightarrow (A \bullet B))$$

$$\bullet 2 \vdash (A \rightarrow (B \rightarrow C)) \rightarrow ((A \bullet B) \rightarrow C)$$

²³³ RIVENC, F., *ILP*, pp. 115-116.

²³⁴ *Ibidem*, p. 116.

²³⁵ *Ibidem*.

²³⁶ *Ibidem*, pp. 116-117.

²³⁷ *Ibidem*, p. 117.

²³⁸ See *Ibidem*.

²³⁹ See *Ibidem*, pp. 125-126.

In fact, the converse of **o2** holds, and so a stronger result holds as a theorem of **R**, which « manifests the very narrow link between **o** and $\rightarrow$ »²⁴⁰ :

$$\vdash ((A \mathbf{o} B) \rightarrow C) \leftrightarrow ((A \rightarrow B) \rightarrow C)$$

where « $\leftrightarrow$ » symbolizes an implicational connective that can be read in both directions. We have, put otherwise :

$A \mathbf{o} B \vdash C$ if and only if $A \vdash B \rightarrow C$.

The link encoded by the « **o** » connective is stronger than the one encoded by the « $\wedge$ » connective : « it can be paraphrased intuitively this way ‘*A combined with B implies C if and only if A implies that B implies C*’ »²⁴¹. This result, when read from left to right, is exactly the ‘law’ of exportation rejected previously, with an intensional rather than an extensional conjunction. We note that, on the other hand, it is not true of this relation that it implies both conjuncts : $A \mathbf{o} B$ doesn’t imply A nor B . If this was the case, we could deduce the forbidden theorem : $\vdash (A \rightarrow B) \rightarrow A$.

On a last note concerning intensional conjunction here, it is suggested that the relationship between new data and scientific theories is of this kind when inferring predictions : « when altitude increases, air cools down *combined* with the news that clouds are approaching the mountains, allow to infer that it’s going to rain »²⁴². So the former theory implies that the new data will lead to the prediction. By contrast, as we’ve said earlier, the axioms at the top of a theory are to be taken extensionally, without any implicational link suggested.

3.4.3 Excluded Middle and the relation to classical tautologies²⁴³

A rather important result of **R** is that it allows to derive the famous Excluded Middle, that is, $\vdash (A \vee \neg A)$ ²⁴⁴. From this point on, using the validity of Excluded Middle in **R**, an ingenious construction enables to prove that the system **R** actually contains the tautologies of **CL** ; and more : that the extensional fragment of **R** is exactly propositional classical logic. This calls for clarification. Didn’t we say that **R** couldn’t prove some theorems of **CL**, namely, the paradoxes of material implication ? In fact, the trick is that we must take all the theorems of **CL** and transform them into a form that does *not* contain the implicational connective. The proof actually requires to use classical equivalences of form²⁴⁵ to transform *all* expressions of **CL** in big formulae containing only disjunction

²⁴⁰ RIVENC, F., *ILP*, p. 126.

²⁴¹ *Ibidem*. This formulation avoids the ambiguity of our « and » uses.

²⁴² *Ibidem*, p. 127. *Italic in the text.*

²⁴³ For this section, see *ibidem* pp. 117 ; 121-124.

²⁴⁴ For a proof, see *ibidem*, p. 117.

²⁴⁵ One can easily see from the truth tables, for instance, that $A \supset B$ is equivalent to $\neg A \vee B$ (see section I.3) and that $A \wedge B$ is equivalent to $\neg(\neg A \vee \neg B)$.

and negation as connectives. So the real difference between **R** and **CL** is the way they define deducibility and consequence : in **R** ‘what follows from what’ is different than in **CL**, and that is a very big difference, since it is usually recognized that the main concern of (pure) logic is validity and consequence²⁴⁶.

3.5. The system **NDR**²⁴⁷

We end the present chapter by providing the promised presentation of **NDR** and briefly discuss distributivity. For the sake of completeness, we rewrote down the rules of inference of **NDR**’s implicational-negation fragment. Note that, as one desires, **NDR** is a conservative extension of **NDR** $\rightarrow\neg$ and can be proven sound and complete with respect to **R_h**. Its language is identical to **R_h**’s. The intensional connectives can be introduced using the same definitions as for **R_h**.

Rules 3.12 : Rules of inference of **NDR**

General rules

1. **Hypothesis introduction (hyp.)** : a new hypothesis can be introduced in the course of a proof. The hypothesis introduced constitutes the first step of a subproof, and is written at the top of the subproof it initiates. A subproof is written with a shift on the right of the previous proof, which is either the main proof or another subproof. It should be conceived as an « item » of the main proof. The hypothesis is introduced with an underwritten indice, $\{k\}$, where k is a single numeral ($\{k\}$ is a singleton), new for each hypothesis.
2. **Repetition (rep.)** : A_a can be repeated inside a (sub)proof, keeping its indice « **a** »²⁴⁸.
3. **Reiteration (reit.)** : A_a can be reiterated, that is, « imported » from a (sub)proof in one of its subproofs.

Implication

4. **$\rightarrow$ Elimination ($\rightarrow$ E)** : from A_a and $(A \rightarrow B)_b$, infer $B_{a \cup b}$ where « $\cup$ » means that the inferred formula has for indices the union of indices of the preceding expressions.
5. **$\rightarrow$ Introduction ($\rightarrow$ I)** : from a proof of B_a under hypothesis $A_{\{k\}}$, infer $(A \rightarrow B)_{a - \{k\}}$ where the new indice « $a - \{k\}$ » is the indice **a** from which we removed the indice $\{k\}$ which otherwise figures in it.

²⁴⁶ See the first paragraph of section I.2.

²⁴⁷ This section stems from RIVENC, F., *ILP*, pp. 118-121.

²⁴⁸ Indices **a**, **b** can take any set of integers as a value. In examples we do not specify the set-theoretic particles.

Negation

6. **Contraposition (ctr.)** : Given a proof of $\neg B_b$ from the hypothesis $A_{\{k\}}$, under the assumption B_a , where k appears in b , infer $\neg A_{(a \cup b) - \{k\}}$.

7. **Double Negation Elimination ($\neg\neg E$)** : from $\neg\neg A_a$, infer A_a .

Conjunction

8. **$\wedge$ Elimination ($\wedge E$)**:

$(A \wedge B)_a$	$(A \wedge B)_a$
_____	_____
A_a	B_a

9. **$\wedge$ Introduction ($\wedge I$)**, with restriction on indices²⁴⁹ :

A_a
B_a

$(A \wedge B)_a$

Disjunction

Since disjunction is interpreted classically, $A \vee B$ can be defined as $\neg(\neg A \wedge \neg B)$ and the rules for disjunction obtained as derived rules.

10. **$\vee$ Introduction ($\vee I$)** : From A_a , infer $(A \vee B)_a$; from B_a , infer $(A \vee B)_a$.

Proof 3.6 : $\vee$ Introduction in NDR²⁵⁰

1.	A_1	hyp.
2.	$(\neg A \wedge \neg B)_2$	hyp.
3.	$\neg A_2$	2 $\wedge E$
4.	$\neg(\neg A \wedge \neg B)_1$	1,2,3 ctr.

²⁴⁹ The restriction is one the options considered earlier, see section 3.4.1. « The conjunction must depend on one and same premise, which has been used to derive both conjuncts. Which blocks, not only the deduction of ' $A \wedge B$ ' from ' A, B ', but also the deduction of $A \wedge (B \vee \neg B)$ from A , theorems having no indices. » RIVENC, F., *ILP*, p. 118.

²⁵⁰ Based on *ibidem*. For the other one, just change the A of lines 1 and 3 with B (the other $\wedge E$ rule is used).

11. **$\vee$ Elimination ($\vee E$)** : From $(A \vee B)_a$, $(A \rightarrow C)_b$, $(B \rightarrow C)_b$, infer $C_{a \cup b}$.

Proof 3.13 : $\vee$ Elimination in $\mathbf{NDR}^{251}$

1.	$(\neg(\neg A \wedge \neg B))_1$	hyp.
2.	$[(A \rightarrow C) \wedge (B \rightarrow C)]_2$	hyp.
3.	$(A \rightarrow C)_2$	2 $\wedge E$
4.	$(B \rightarrow C)_2$	2 $\wedge E$
5.	$\neg C_3$	hyp.
6.	$(B \rightarrow C)_2$	4 reit.
7.	$(\neg C \rightarrow \neg B)_2$	6 ctr2
8.	$\neg B_{2,3}$	5,7 $\rightarrow E$
9.	$\neg C_3$	5 rep.
10.	$(A \rightarrow C)_2$	3 reit.
11.	$(\neg C \rightarrow \neg A)_2$	10 ctr2
12.	$\neg A_{2,3}$	9,11 $\rightarrow E$
13.	$(\neg A \wedge \neg B)_{2,3}$	8,12 $\wedge I$
14.	$\neg C \rightarrow (\neg A \wedge \neg B)_2$	5,13 $\rightarrow I$
15.	$[(\neg(\neg A \wedge \neg B)) \rightarrow \neg \neg C]_2$	14 ctr2
16.	$\neg(\neg A \wedge \neg B)_1$	1 reit.
17.	$(\neg \neg C)_{1,2}$	15,16 $\rightarrow E$
18.	$C_{1,2}$	$\neg \neg E$

Intermediary result, classical contraposition (2)

1.	$(A \rightarrow B)_1$	hyp.
2.	$\neg B_2$	hyp.
3.	A_3	hyp.
4.	$(A \rightarrow B)_1$	reit.
5.	$B_{3,1}$	3,4 $\rightarrow E$
6.	$\neg \neg B_{3,1}$	$\neg \neg I$
7.	$\neg A_{2,1}$	2,3,6 ctr.
8.	$(\neg B \rightarrow \neg A)_1$	2,7 $\rightarrow I$

We use as a derived rule (ctr2): from $(A \rightarrow B)_a$, infer $(\neg B \rightarrow \neg A)_a$.

3.6 The problem of distributivity

The restriction on the indices of the rule $\wedge I$ prevents one to properly demonstrate the rule of distributivity, $(\mathbf{R}_h 13) \vdash A \wedge (B \vee C) \rightarrow (A \wedge B) \vee C$; and the effect of lifting this restriction leads to the dropping out of the relevant Deduction Theorem²⁵², which would also prevent the proof to go through²⁵³. Since the formula « satisfies all the acknowledged criteria of relevance »²⁵⁴, A&B have decided to add it as a primitive rule in $\mathbf{NDR}$, so we

²⁵¹ Based on ANDERSON, A.R., and BELNAP, N., *ELRN* I, §23.2.

²⁵² See section 3.4.1.

²⁵³ See RIVENC, F., *ILP*, p. 119.

²⁵⁴ *Ibidem*, p. 120.

must add to the rules just presented : from $(A \wedge (B \vee C))_a$, infer $((A \wedge B) \vee C)_a$. Rivenc notes :

This can be judged unsatisfying. On the one hand, it breaks the typical symmetry of Introduction and Elimination rules of natural deduction. On the other, we can blame the rules of the system, since they do not allow to derive a formula recognized as valid²⁵⁵.

This can lead to a version of **R** free of distributivity, so without (**R_h** 13) in the Hilbert-system and without the primitive rule in natural deduction. We note that **R** without distributivity is decidable, as well as its implicational and implicational-negation fragments ; but **R** is not²⁵⁶. We say that a logical system is *decidable* if there's a decision procedure that determines whether arbitrary formulae are logically valid in the system²⁵⁷. **CL**, for a reminder, is decidable, and so are 'standard normal modal logics'.

Interestingly enough, Mares²⁵⁸ presents the $\vee E$ rule of Brady, which can replace the $\vee E$ rule just presented, and from which, together with another rule introduced, the (**E**) rule, one can derive distributivity. The rules are :

(Brady $\vee E$) From $A \vee B_a$, infer A_a, B_a .

(**E**) From A_a, A_a infer A_a .

Note that « the two subscripted formulae separated by a comma constitute a single step in a proof. They introduce two lines of proof. Within a line of proof, we can use all of the standard rules »²⁵⁹.

Proof 3.14 Distributivity using Brady's rules

1. $(A \wedge (B \vee C))_1$	hyp.
2. A_1	1 $\wedge E$
3. $(B \vee C)_1$	1 $\wedge E$
4. B_1, C_1	3 Brady $\vee E$
5. $(A \wedge B)_1, C_1$	2,4 $\wedge I$
6. $(A \wedge B)_1, (A \wedge C)_1$	2,5 $\wedge I$
7. $((A \wedge B) \vee (A \wedge C))_1, (A \wedge C)_1$	6 $\vee I$
8. $((A \wedge B) \vee (A \wedge C))_1, ((A \wedge B) \vee (A \wedge C))_1$	7 $\vee I$

²⁵⁵ See RIVENC, F., *ILP*, pp. 85 ; 120.

²⁵⁶ See *ibidem*, p. 120. The proof that **R** is not decidable can be found in URQUHART, A., « The Undecidability of Entailment and Relevant Implication » in *Journal of Symbolic Logic*, 49 (1984).

²⁵⁷ So the idea is that for decidable systems, we can build up a method to check systematically in a step-by-step procedure whether a formula is one of its theorems.

²⁵⁸ See MARES, E.D., *RLPI*, p. 34.

²⁵⁹ *Ibidem*.

9. $((A \wedge B) \vee (A \wedge C))_1$

8 ,E

3.7 Summary of part I

Our part I has been focused on the motivations of **R** and on its syntactic presentations in a natural deduction system and in a Hilbert-style system. In chapter 1, we have seen different strategies that could be used to explain away the oddities of **CL**. In particular, we have shown that if one was dissatisfied with a conservative strategy, through the defense of the assertibility conditions' theory, then one had the possibility to embrace a revisionist strategy and to develop a new syntactic system that avoided the pitfalls of **CL** diagnosed earlier (i.e., the paradoxes of material implication and non-sequiturs).

We have decided to present the alternative theory of reasoning, presentation which appears to us as a necessary condition for the evaluation of the best strategy to adopt. We started by presenting in chapter 2 the implicational fragment of **R** in natural deduction and Hilbert-style systems, and have shown how the fragment was displaying nice properties of relevance (like the relevant Deduction Theorem and the variable-sharing property).

In extending the system $\mathbf{R}^{\rightarrow}$ with negation, conjunction and disjunction however, trouble has found us. In chapter 3, we have seen that our system rejected disjunctive syllogism (DS), a well-recognized mode of inference. We have shown that we could soften this apparently disastrous result by noting that our uses of « or » could be ambiguous (between an intensional and extensional meaning), and that DS could be valid for the intensional form. We also introduced Geach's challenge, forcing us to show how our « good old » extensional DS could be invalid, a challenge still to be faced. Then we have seen that the introduction of the adjunction rule in our system was not without dire consequences : we had to either limit its use or lose the substance of the relevant Deduction Theorem. Furthermore, introducing distributivity in our system felt rather unnatural.

Part of our difficulties in extending the fragment $\mathbf{R}_h^{\rightarrow\neg}$ may be due to a misunderstanding, or at least to a lack of clarification, of how we can group premises. And the rejection of DS may appear less counterintuitive once a semantics for **R** presented. To clarify the way(s) premises can be grouped, we now present the Gentzen, or structural, system for **R**. This presentation will be followed by semantical considerations, first with the possible worlds semantics and secondly with the Routley-Meyer semantics.

Part II

Structural and semantical approach to **R**

- 4 – A structural presentation of **R**
- 5 – Semantics and possible worlds
- 6 – The Routley-Meyer semantics for **R**

Chapter 4 : A structural presentation of **R**

4.1 Thoughts on extensional connectives²⁶⁰

The situation for A&B seems, at this point, rather precarious. We saw in the previous chapter that they developed a rather well-justified implicational-negation fragment for **R**, showing that it satisfied a nice relevant Deduction Theorem as well as a variable-sharing property. Apart from a criticism on ‘ambiguity’ of relevance, the system seemed to keep its promises intact. But when we added the traditional extensional connectives to it, the system quickly led to disaster²⁶¹. Not only did we lose DS, a rejection still in need of explanation through an answer to Geach’s challenge, but more importantly, we don’t seem to be able to get an acceptable notion of deducibility.

This point is made by Arnon Avron in an important article, « Whither Relevance Logic ? »²⁶². The author considers the different notions of deducibility developed by A&B and shows that none of them is satisfying when extensional connectives are present. He notes that « in order for $\rightarrow$ to serve as the ‘heart of logic’ it should be possible to reduce the general case »²⁶³ of the inference of a conclusion C from a collection of premises $A_1, \dots, A_n$, to the case reflected by the inference of a conclusion C from a formula. How can we operate this reduction ?

First, we can use the ‘reduction principle’. Only one half is actually explicitly accepted by A&B: « if there’s a proof that the premises $A_1, \dots, A_n$ establish a conclusion C, then $(A_1 \wedge \dots \wedge A_n) \rightarrow C$ is a theorem of **R** »²⁶⁴. But this leads to discrepancy, for we have « easy examples of cases in which $C \rightarrow D$ is not a theorem of **R** or **E** although the validity of D follows from that of C in those systems, with a constructive proof that meets any plausible relevant criterion »²⁶⁵. In particular, « every proof of C can easily be

²⁶⁰ This section stems from AVRON, A., « Whither Relevance Logic ? » in *Journal of Philosophical Logic* and RIVENC, F., *ILP*, pp. 191-194.

²⁶¹ Taking an expression from DUNN, M., « Relevance logic and entailment » in *Handbook of philosophical logic*, vol. III, *Alternatives to classical logic* (Gabbay, D., and Guenther, F. eds.), p. 137. Cited from AVRON, A., « Whither Relevance Logic ? » in *Journal of Philosophical Logic*, p. 255.

²⁶² See *ibidem*, pp. 243-262.

²⁶³ *Ibidem*, p. 247.

²⁶⁴ The other half of which being the conditional the other way around. This half was not endorsed because it led to validating a fallacy of relevance. Because we have $(A \wedge B) \rightarrow (A \wedge B)$ and so by this half we would have a proof of $(A \wedge B)$ from A, B. But then we’d have a relevant proof of $A \rightarrow (B \rightarrow (A \wedge B))$:

$$\frac{\frac{\frac{A_1, B_2}{(A \wedge B)_{1,2}}}{(B \rightarrow (A \wedge B))_1}}{A \rightarrow (B \rightarrow (A \wedge B))}$$

leading, as subsection 3.4.1 shows, to $\vdash A \rightarrow (B \rightarrow A)$. From RIVENC, F., *ILP*, p. 193.

²⁶⁵ AVRON, A., « Whither Relevance Logic ? » in *Journal of Philosophical Logic*, p. 250.

extended to proofs of $C \wedge (B \rightarrow B)$ and of $(C \wedge B) \vee (C \wedge \neg B)$, but the corresponding implications fail in **R** and **E** »²⁶⁶.

Second, we can use the « close link of deducibility between (say) A_n and B in the context of the other premises $A_1, \dots, A_{n-1}$ »²⁶⁷ with a notion of proof « warranting that A_n has indeed been used in the deduction of B »²⁶⁸ to conclude « the entailment $A_n \rightarrow B$ from the other premises »²⁶⁹. The general formulation of this idea, recall, led to a relevant Deduction Theorem of the form : $A_1, \dots, A_n \vdash C$ if and only if $\vdash B_1 \rightarrow (B_2 \dots (B_p \rightarrow C) \dots)$. But, as Avron notes, « there is just no generally accepted, official definition of what a proof from assumptions is in the bulk of the relevantists work ! »²⁷⁰ and the different versions presented also lead to severe issues²⁷¹. The reason, as we understand it, is that all of them try to circumvent the extensional connectives from proving their associated relevant fallacies. And so they must incorporate restrictions in the proof systems that (1) are *ad hoc* ; (2) violate the extensional nature of expressions that are extensional ; (3) lead to unacceptable results. To illustrate (3), we can consider the case in which :

Applications of adjunction are [...] limited. They are allowed in a proof of B from $A, \dots, A_n$ in two cases :

1. Each of the two premises was proved using some of the assumptions $A, \dots, A_n$ (not necessarily all, and not necessarily the same assumptions in both. The intersection may even be empty).
2. Both premises were proved without using any assumptions (i.e., both are logical theorems)²⁷².

But this version causes a paradox : « we have that $A, B \rightarrow B \vdash A \wedge (B \rightarrow B)$ but $A \not\vdash A \wedge (B \rightarrow B)$ although $(B \rightarrow B)$ is a theorem. In order to derive $A \wedge (B \rightarrow B)$ we should pretend that we do not know that $B \rightarrow B$ is true and ‘assume’ it »²⁷³ which, according to Avron, is not only ridiculous, but also « violates one of the two basic conditions about every consequence relation : the transitivity condition (or ‘cut’) »²⁷⁴. Moreover, we cannot infer, with this notion of proof, « $A \wedge C \leftrightarrow B \wedge C$ from $A \leftrightarrow B$. The principle of

²⁶⁶ AVRON, A., « Whither Relevance Logic ? » in *Journal of Philosophical Logic*, p. 250.

²⁶⁷ RIVENC, F., *ILP*, p. 192.

²⁶⁸ *Ibidem*.

²⁶⁹ *Ibidem*.

²⁷⁰ AVRON, A., « Whither Relevance Logic ? » in *Journal of Philosophical Logic*, p. 255. The term « relevantist » denotes a person who believes that the correct analysis of validity is that of the relevant logic **R**. The relevantist is to **R** what a classicist is to **CL**.

²⁷¹ See *ibidem*, pp. 256-261.

²⁷² *Ibidem*, p. 259.

²⁷³ *Ibidem*.

²⁷⁴ *Ibidem*, pp. 259-260. See section 4.3 for an explanation of what the ‘cut’ rule is.

substitution of equivalents fails therefore for **R** and **E** (at least as a derivable rule, although not as an admissible rule) »²⁷⁵. Finally, with this notion of proof, we have that we

cannot infer C from A when we know (or assume) $A \wedge B \rightarrow C$ and $A \wedge \neg B \rightarrow C$ (The possibility of doing this for every C is equivalent to deriving $A \wedge (\neg B \vee B)$ from A !). This is really a strong limitation, which an *intuitionist* will be glad to accept. **R** and **E**, however, accept both the excluded middle and the double negation laws, so they are completely non-intuitionistic in spirit²⁷⁶.

In addition to this criticism, the above interpretation of the grouping of premises is problematic : « [A&B themselves] recognize that this is not how we ordinarily understand a collection of premises »²⁷⁷, we can recall notably their understanding of a collection of axioms at the top of a theory : « when it is said that the axioms of group theory imply that the identity element is unique we understand that their conjunction implies this »²⁷⁸.

The conclusion Avron draws from those elements is that matters of extensionality are simply incompatible with matters of relevance. One can be interested in one or the other, but not integrate them both together in a formal system. Anyways, as we've highlighted earlier, the confusion in A&B's work may rest on a lack of elucidation on the grouping of premises. To this task we now turn our attention, through the structural presentation of **R**.

4.2 Preliminaries²⁷⁹

We start by presenting the basic notions of Gentzen systems, which will be the starting point of our framework. Instead of considering the relation of deducibility between a set of sentences (or formulae) and a sentence (or formula) : $\Gamma \vdash C$, we want to consider the premises as a finite *sequence*. Sets enable to group elements in a way that is insensitive to order and repetition (so the set $\{b, a, a, c\}$ is the same as the set $\{b, a, c\}$ or $\{a, b, c\}$); while sequences are sensitive to both these ideas. So a more appropriate notation would be : $A_1, \dots, A_n \vdash C$, which means that the conclusion C is deducible from the finite sequence of premises $A_1, \dots, A_n$. The comma here has the meaning, in Gentzen's presentation, of $a(n)$ (extensional) conjunction. It is also possible to accept several formulae as conclusions, in which case the comma is interpreted as a disjunction in the conclusion.

²⁷⁵ AVRON, A., « Whither Relevance Logic ? » in *Journal of Philosophical Logic*, p. 260.

²⁷⁶ *Ibidem*. Italic in the text.

²⁷⁷ RIVENC, F., *ILP*, p. 194.

²⁷⁸ ANDERSON, A.R., and BELNAP, N., *Entailment, the Logic of Relevance and Necessity*, vol. I, §22.2.2.

²⁷⁹ This section stems from RIVENC, F., *ILP*, pp. 195-199 and RESTALL, G., *An introduction to substructural logics*, pp. 19-30.

The fact that we consider sequences rather than sets enables us to put to light certain rules of transformation of the premises, which are called *structural rules*. As Rivenc notes, « the more complex are the initial structured entities, the more numerous are the rules that we can make explicit and that are susceptible to reduce this complexity »²⁸⁰ and « once those rules are explained, it is possible to refuse them for exploration or providing a particular motive »²⁸¹.

A logic that does not accept the totality of the structural rules for the presentation of **CL** is called for this reason a *substructural logic*. But if one is sensitive to the move of « first exhibiting complex *structured* sequences and *then* exhibiting structural rules to enable the simplification of this complexity, then those systems would deserve the name of *structural logics* as well »²⁸². A special mention must be made of the structural rule of Weakening, which enables one to add extra premises in the antecedent of sequents. The rule is « linked to a plausible property of the relation of consequence, monotony : if $\Gamma \vdash C$, then $\Gamma; A \vdash C$ »²⁸³ but « its admission affects the properties of the structured premises »²⁸⁴.

We follow Rivenc and start by giving a structural presentation of **NDR**, which allows a smoother transition to the new formalism. We then provide the proper rules for **TR**. Our propositional language defines formulae as usual, recursively from atomic formulae, using as connectives $\rightarrow$, **o**, $\wedge$ and $\vee$. The connective of arity zero, **t** also counts as a formula. We introduce in our language punctuation marks, having, like connectives, a certain arity. Two binary punctuation marks « , » and « ; » are introduced, as well as « 0 », a punctuation mark of arity zero. We can now recursively define the notion of *structure*, and a related notion of *substructure* that parallelizes that of subformula²⁸⁵:

Definition 4.1 : Structure

- (1) A formula is a *structure* ;
- (2) if P is a punctuation mark, of arity 0 or two, the result of applying P to zero or two structures (according to the number of arguments) is a *structure* (for instance « 0 », « X;Y », « Z,(X;Y) » etc. are structures. The convention is to put between parentheses binary structures).

²⁸⁰ RIVENC, F., *ILP*, p. 196.

²⁸¹ *Ibidem*.

²⁸² *Ibidem*. We hope that choosing the shorter name for our chapter's title did not confuse or upset or reader.

²⁸³ *Ibidem*.

²⁸⁴ *Ibidem*, pp. 196-197.

²⁸⁵ For the definitions 4.1-4.6, see *ibidem*, p. 198.

Definition 4.2 : Substructure

- (1) Any structure is a *substructure* of itself ;
- (2) if a structure Z has the form PXY , the substructures of X and of Y are *substructures* of Z . We note $X(Y)$ the structure X which has Y as a substructure.

We now have the building blocks of our language, but we need to organize those entities in the appropriate way. So we add the following definitions :

Definition 4.3 : Substitution

A substructure can be *substituted* for another. Let $X(Y)$ be the structure X with Y as a substructure. $X(Z)$ is the structure obtained by substituting the structure Z for the structure Y in $X(Y)$.

Definition 4.4 : (One conclusion) sequent (or one-conclusion consecution)

A (one conclusion) *sequent* (sometimes called a consecution) is a figure of the form $X \vdash A$, where X is a structure, the antecedent, and A is a formula, the consequent.

Definitions 4.5 : Inference and related notions

An *inference* is a pair. Its first member is a set of zero, one, two or three sequents, the premises ; and its second member is a sequent, the conclusion. If there is no premise, the inference is an *axiom*. The *proof* of a sequent is a *tree*, where the *root* is the sequent to be proven, the *tops* are axioms, and where each step, or *node*, is the application of a rule of inference. A *proof with assumption* is a proof where the tops can be axioms or *assumptions*.

4.3 Natural deduction systems in structural form

We have replaced the notion of « sequence of formulae » in the antecedent of sequents by binary structures, because they allow for a better presentation and understanding of the structural rules²⁸⁶. We distinguish here between structural rules, which affect, as we said, the ways in which antecedents can be rearranged ; and operational rules which concern the ways connectives can be introduced or eliminated in the course of a proof.

²⁸⁶ See RIVENC, F., *ILP*, p. 198. The innovation was due to Belnap, see BELNAP, N., « Display Logic » in *Journal of Philosophical Logic*, 11 (1982).

The first category of rules can be integrated in the systems as one prefers, they are in « self-service », while the second category, as well as the unique axiom scheme, are common to all the systems built on the basis of the language described.

Rules and axiom 4.6²⁸⁷

(1) Axiom scheme (zero premise rule), where A is any formula

$$A \vdash A \quad (\text{Id})$$

(2) Operational rules

$$\begin{array}{c}
 \frac{X;A \vdash B}{X \vdash A \rightarrow B} \rightarrow I \qquad \frac{X \vdash A \rightarrow B \quad Y \vdash A}{X;Y \vdash B} \rightarrow E \\
 \\
 \frac{X \vdash A \quad Y \vdash B}{X;Y \vdash A \circ B} \circ I \qquad \frac{X \vdash A \circ B \quad Y(A;B) \vdash C}{Y(X) \vdash C} \circ E \\
 \\
 \frac{X \vdash A \quad X \vdash B}{X \vdash A \wedge B} \wedge I \qquad \frac{X \vdash A \wedge B}{X \vdash A} \wedge E \quad \frac{X \vdash A \wedge B}{X \vdash B} \wedge E \\
 \\
 \frac{X \vdash A}{X \vdash A \vee B} \vee I \quad \frac{X \vdash B}{X \vdash A \vee B} \vee I \qquad \frac{X \vdash A \vee B \quad Y(A) \vdash C \quad Y(B) \vdash C}{Y(X) \vdash C} \vee E \\
 \\
 0 \vdash t \quad tI \qquad \frac{X \vdash t \quad Y(0) \vdash C}{Y(X) \vdash C} tE
 \end{array}$$

(3) Structural rules

Since we do not have « empty structures », we replace sequents of the form « $\vdash A$ » in Gentzen presentations, which correspond to theorems, by sequents of the form « $0 \vdash A$ ». We start the presentation of structural rules by those who concern the introduction and elimination of that particular punctuation mark, then present those concerning semicolon.

Rules for « 0 »

²⁸⁷ From RIVENC, F., pp. 199-201.

$$\frac{X \vdash C}{0;X \vdash C} \quad +0$$

$$\frac{0;X \vdash C}{X \vdash C} \quad -0$$

$$\frac{X \vdash C}{X;0 \vdash C} \quad 0+$$

$$\frac{X;0 \vdash C}{X \vdash C} \quad 0-$$

Associativity (**B**)²⁸⁸

$$\frac{V(X;(Y;Z)) \vdash C}{V((X;Y);Z) \vdash C}$$

Converse associativity (**Bc**)

$$\frac{V((X;Y);Z) \vdash C}{V(X;(Y;Z)) \vdash C}$$

Associativity with permutation (**B'**)

$$\frac{V(X;(Y;Z)) \vdash C}{V((Y;X);Z) \vdash C}$$

Strong commutativity (**C**)

$$\frac{V((X;Y);Z) \vdash C}{V((X;Z);Y) \vdash C}$$

Weak commutativity

$$\frac{V(X;Y) \vdash C}{V(Y;X) \vdash C}$$

Strong contraction (**W**)

$$\frac{V((X;Y);Y) \vdash C}{V(X;Y) \vdash C}$$

Weak contraction

$$\frac{V(X;X) \vdash C}{V(X) \vdash C}$$

Weakening (**K**)

$$\frac{V(X) \vdash C}{V(X;Y) \vdash C}$$

Weakening on the left

$$\frac{V(X) \vdash C}{V(Y;X) \vdash C}$$

With those rules explicated, it is possible to denote a certain formal system by the connectives it uses and the structural rules it admits. So the frameworks enable relatively welcome comparisons. Since we did not introduce negation yet, we are able to represent the positive fragment of **E**, **E**⁺: +0-00-**BB'****W**. Note that this is the version without distributivity, and that the absence of mention of the connectors indicates that they are all present. The positive fragment of **R**, again without distributivity, **R**⁺ is represented with

²⁸⁸ The abbreviations refer to operators, also called combinators, in the combinatory logic of Curry. In that logic « (**B**) is the combinator representing the composition of two functions, (**C**) is the transformation of a function to its converse, (**W**) is the function of one argument obtained from a two arguments function by identification of the two arguments, and (**K**) is the expression of a constant as function of a variable » RIVENC, F., *ILP*, p. 201, footnote. Our reader can find more information on combinators in MARES, E.D. and MEYER, R.K., « Relevant Logics » in *The Blackwell Guide to Philosophical Logic* (Goble, L. ed.), pp. 293-300 ; or in the work of Curry, CURRY, H.B., and FEYES, R., *Combinatory Logic*, vol. I.

+0-0**BCW**²⁸⁹ ; and intuitionist logic²⁹⁰ is represented by the system **BCWK**, with all the main structural rules, and the « false constant » added to its language via a connective « $\perp$ » with the additional rule :

$$\frac{X \vdash \perp}{Y(X) \vdash A} \quad \perp E$$

where the negation of A is defined as $A \rightarrow \perp$. The system with the absence of this last connective matches minimal logic²⁹¹.

We can get a taste of how proofs are constructed in this framework with a few examples. Note that the proofs can take the form of a tree, but since it's not always easy to use the notation, they can also take a linear form. In this case, we have a numbered list of steps as a proof and the justification of the steps mentions the appropriate previous lines. We provide the two forms for our first example.

Proof 4.7 : $A \circ (B \circ C) \vdash (A \circ B) \circ C$ in the presence of (Bc) ²⁹²

(Tree form)

$$\frac{\frac{\frac{A \vdash A \quad B \vdash B}{A;B \vdash A \circ B} \quad C \vdash C}{(A;B);C \vdash (A \circ B) \circ C} \quad B \circ C \vdash B \circ C}{A \circ (B \circ C) \vdash A \circ (B \circ C) \quad A;(B \circ C) \vdash (A \circ B) \circ C} \quad \text{oI} \quad \text{oI} \quad \text{(Bc)} \quad \text{oE} \quad \text{oE}$$

(Linear form)

1. $A \vdash A$ (Id)
2. $B \vdash B$ (Id)
3. $A;B \vdash A \circ B$ 1,2 oI

²⁸⁹ In presence of commutativity, one introduction and elimination rule for $\circ$ are enough. The two others can do as well. See RIVENC, F., *ILP*, p. 206.

²⁹⁰ Intuitionist logic is a type of logical system that has been constructed as a basis for mathematical theory as an alternative to classical logic. The idea of its founders was to avoid certain doubtful mathematical proofs, notably those using Excluded Middle. As a result, intuitionist logic interprets negation in a way that is allegedly different than **CL**, where double negation elimination is not accepted, just like Excluded Middle.

²⁹¹ See for this paragraph, *ibidem*, pp. 205-206. More systems are presented in RESTALL, G., *An introduction to substructural logics*, pp. 39-41.

²⁹² From RIVENC, F., *ILP*, p. 204.

4. $C \vdash C$ (Id)
5. $(A;B);C \vdash (A \circ B) \circ C$ 3,4 $\circ I$
6. $A;(B;C) \vdash (A \circ B) \circ C$ 5, (Bc)
7. $B \circ C \vdash B \circ C$ (Id)
8. $A;(B \circ C) \vdash (A \circ B) \circ C$ 6,7 $\circ E$
9. $A \circ (B \circ C) \vdash A \circ (B \circ C)$ (Id)
10. $A \circ (B \circ C) \vdash (A \circ B) \circ C$ 8,9 $\circ E$

Proof 4.8 : Assertion $0 \vdash A \rightarrow ((A \rightarrow B) \rightarrow B)$ in the presence of weak commutativity²⁹³

$$\begin{array}{c}
 \frac{A \rightarrow B \vdash A \rightarrow B \quad A \vdash A}{\quad} \rightarrow E \\
 \frac{\quad}{A \rightarrow B; A \vdash B} \text{Weak (C)} \\
 \frac{\quad}{A; A \rightarrow B \vdash B} \rightarrow I \\
 \frac{\quad}{A \vdash (A \rightarrow B) \rightarrow B} +0 \\
 \frac{\quad}{0; A \vdash (A \rightarrow B) \rightarrow B} \rightarrow I \\
 \frac{\quad}{0 \vdash A \rightarrow ((A \rightarrow B) \rightarrow B)} \rightarrow I
 \end{array}$$

Proof 4.9 : Classical results using weakening on the left²⁹⁴

$$\begin{array}{c}
 \frac{A \vdash A}{A; B \vdash A} \text{Left (K)} \quad \frac{B \vdash B}{A; B \vdash B} \\
 \frac{\quad}{A \vdash B \rightarrow A} \rightarrow I \quad \frac{\quad}{A \vdash B \rightarrow B} \rightarrow I \\
 \frac{\quad}{0; A \vdash B \rightarrow A} +0 \quad \frac{\quad}{0; A \vdash B \rightarrow B} +0 \\
 \frac{\quad}{0 \vdash A \rightarrow (B \rightarrow A)} \rightarrow I \quad \frac{\quad}{0 \vdash A \rightarrow (B \rightarrow B)} \rightarrow I
 \end{array}$$

In the way characterized above, $\mathbf{R}^+$ does not admit distributivity : $A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)$. Now, one can decide, as we saw, to introduce it into the system by a separate axiom. Here distributivity cannot be proven without Weakening and Weak contraction. But adding Weakening to the structural rules of $\mathbf{R}$ leads to fallacies of irrelevance... So instead of accepting it for « ; », we add a new punctuation mark in our language. This second punctuation mark is the comma « , ». For it, we accept all the ‘strong’ forms of

²⁹³ From RIVENC, F., *ILP*, p. 204.

²⁹⁴ From *ibidem*, p. 205. We put the justifications in-between the two proofs since they are identical for the lines which face each other.

structural rules, so it enables us to obtain distributivity. Most operational rules, such as $\rightarrow I$, $\bullet I$, etc. require in the antecedent-parts of some of their sequents the presence of structures linked by a semicolon. So our introduction of « , » for which, in particular, Weakening and Contraction are permitted, is expected to not affect anything that is not linked to disjunction and conjunction ; the connectives whose rules do not involve the semicolon directly. The following structural rules are accepted, as we said, in order to construct a proof of distributivity²⁹⁵ :

Extensional associativity

$$\frac{V(X,(Y,Z)) \vdash C}{V((X,Y),Z) \vdash C}$$

Extensional Contraction

$$\frac{V(X,X) \vdash C}{V(X) \vdash C}$$

Extensional commutativity

$$\frac{V(X,Y) \vdash C}{V(Y,X) \vdash C}$$

Extensional weakening

$$\frac{V(X) \vdash C}{V(X,Y) \vdash C}$$

The term « extensional » indicates that the rules apply to the punctuation mark « , » which has the same properties as extensional conjunction. The comma mimics the behaviour of « $\wedge$ » just like the semicolon mimics the behaviour of « $\bullet$ ». We note that, as one would require, it can be shown that « adding the comma with these structural rules does not alter the properties of any connectives other than $\wedge$ and $\vee$ »²⁹⁶, which « is not obvious »²⁹⁷ :

There just might be a consecution involving just the intensional fragment (say the semicolon, fusion and implication), which can be proved by taking a detour through the extensional language (using comma, $\wedge$ and $\vee$) but which cannot be proved without taking such a detour²⁹⁸.

Another important result can be proven, this time for any system built using the above operational rules²⁹⁹. The result concerns the « Cut » rule (or « Transitivity » rule) which states that an intermediary formula appearing both in the antecedent and consequent of the two sequent-premises can be suppressed and so disappears in the course of a proof. The rule can be represented as :

The Cut rule

$$\frac{X \vdash A \quad Y(A) \vdash B}{Y(X) \vdash B}$$

²⁹⁵ See RESTALL, G., *An introduction to substructural logics*, p. 36. So R^+ with distributivity requires the addition of the comma in its language with the presented extensional structural rules. For the proof of distributivity using those rules, see op. cit.

²⁹⁶ *Ibidem*, p. 37.

²⁹⁷ *Ibidem*.

²⁹⁸ *Ibidem*.

²⁹⁹ For this paragraph and the proof, see *ibidem*, pp. 37-39 and RIVENC, F., *ILP*, pp. 207-209.

and is admissible for the natural deduction systems characterized here : adding it to the systems doesn't increase the stock of their theorems. So, whatever can be proven with it can be proven without it.

4.4 Sequent calculus for R^{300}

We keep the same language, structural rules and introduction rules as before. The big change is that the symmetry between introductory and elimination rules is replaced by a symmetry between introduction to the left and introduction to the right. The previous introductory rules constitute our rules of introduction to the right of the turnstile. The rules for introduction to the left are :

$$\frac{X \vdash A \quad Y(B) \vdash C}{Y(A \rightarrow B; X) \vdash C} \rightarrow\vdash$$

$$\frac{X(A; B) \vdash C}{X(A \circ B) \vdash C} \circ\vdash$$

$$\frac{X(A) \vdash C}{X(A \wedge B) \vdash C} \quad \frac{X(B) \vdash C}{X(A \wedge B) \vdash C} \quad \wedge\vdash$$

$$\frac{X(A) \vdash C \quad X(B) \vdash C}{X(A \vee B) \vdash C} \vee\vdash$$

$$\frac{X(0) \vdash C}{X(t) \vdash C} t\vdash$$

Given a natural deduction system in structural form **S**, the corresponding system in sequent calculus **TS** is the system determined by the presence of the same connectors and the same structural rules³⁰¹. We note that the system **TI** of intuitionist sequent calculus can be built using the connectives $\rightarrow$, $\wedge$, $\vee$ with their rules, the false constant $\perp$, the punctuation mark « , » for which all the structural rules are admitted, the structure 0 and the constant **t**. Remember the false constant has the associated rule $\perp E$ and that we have **tI** and **tE** for **t**. Now, it is common to actually formulate negation in another way, since $\perp E$ enables one to make a formula, the false constant, disappear ; and it is common to

³⁰⁰ This section stems from RIVENC, F., *ILP*, pp. 210-225.

³⁰¹ For the proof concerning **TR**⁺, see *ibidem*, pp. 211-214.

require that no formula disappears in the course of a proof. Instead, the negation is defined with the following rules, with the admission of sequents with empty consequents (consequents of sequents are considered as sets containing at most one formula as an element)³⁰² :

$$\frac{X \vdash A}{X, \neg A \vdash} \quad \frac{X, A \vdash}{X \vdash \neg A} \quad \frac{X \vdash}{X \vdash B}$$

The systems characterized so far do not allow structures to figure in the consequents of the sequents. To allow such presence is to render the systems « completely symmetric »³⁰³, and so accommodates it to a presentation with multiple conclusions. Our sequents can now have structures in their consequent. The connective creating the bridge for a multiple-conclusion presentation is the negation, which has for rules :

$$\frac{X \vdash A; Y}{X; \neg A \vdash Y} \quad \neg \vdash \quad \frac{X; A \vdash Y}{X \vdash \neg A; Y} \quad \vdash \neg$$

This negation enables one to derive the sequent $A \vdash \neg \neg A$, as well as $\neg \neg A \vdash A$; and the forms of contraposition in the presence of Associativity and Permutation³⁰⁴. Now that we allow for structures in the consequents, our rules gain in generality. And the idea of understanding the connective of negation as a translation in the object language of a structural operation is generalised to other connectives. Consider the rules for $\rightarrow$. They now can be represented as :

$$\frac{X \vdash Y(A) \quad X'(B) \vdash Y'}{X'(A \rightarrow B; X) \vdash Y(Y')} \rightarrow \vdash \quad \frac{X; A \vdash B; Y}{X \vdash A \rightarrow B; Y} \vdash \rightarrow$$

The rule of introduction on the left expresses MP. Consider :

$$\frac{X \vdash A \quad B \vdash C}{(A \rightarrow B; X) \vdash C} \rightarrow \vdash$$

Rivenc notes that « if X allows to deduce A, then X with $A \rightarrow B$ gives B ; and if on the other hand B implies C, then by transitivity X with $A \rightarrow B$ allows to deduce C »³⁰⁵. In fact,

the rules ensure that the deducibility relation holds between A and the formula $B \rightarrow C$ exactly when B added to A deduces C : the formula $B \rightarrow C$ « says » that given the context A, C is deducible from A. So it is no surprise that the different forms of implication in the different systems inherit their

³⁰² For all this paragraph, see RIVENC, F., *ILP*, pp. 214-215.

³⁰³ *Ibidem*, p. 217.

³⁰⁴ See *ibidem*, p. 218.

³⁰⁵ *Ibidem*, p. 219.

characteristics from the various properties of the relation of deducibility. Those properties being in turn dependent on the assumptions concerning the punctuation marks, that is, on the accepted or refused structural rules³⁰⁶.

The fusion connective, « **o** » clearly reproduces the behaviour of the semicolon. We have :

$$\frac{X(A;B) \vdash C}{X(A \text{ o } B) \vdash C} \quad \text{o}\vdash$$

and if we reread the rule of **oE** as a rule of « cut » admissible ranging on ‘A **o** B’ and ‘A;B’, we have :

$$\frac{X \vdash A \text{ o } B \quad Y(A;B) \vdash C}{Y(X) \vdash C}$$

which is a transitivity move tantamount, together with **o**⊢, to the identification of A **o** B with the structure A;B.

What about the extensional conjunction ? It has the properties of the punctuation mark « , » if it’s present in the language. If Contraction and Weakening are authorized for the semicolon in addition to Associativity and Permutation, the structures A,B and A;B end up behaving identically and the two conjunctive connectives can be identified with one another³⁰⁷. That’s what happens for **CL**, which erases intensional distinctions in the logic. Furthermore, since we have the implication connective introduced thanks to the semicolon and the identification of this latter structure with A **o** B, we have the expected relationships³⁰⁸ :

$$\frac{A \text{ o } B \vdash C}{A \vdash B \rightarrow C} \quad \frac{A \vdash B \rightarrow C}{A \text{ o } B \vdash C}$$

It is possible to add an intensional disjunction « + » to the language. It also collapses to extensional disjunction in the presence of all the structural rules for semicolon. Its introduction rule on the right is motivated by the idea of Gentzen that conclusions are grouped disjunctively, while premises are grouped conjunctively. All those considerations lead to the system **TR**, deductively equivalent to the system **Rh**³⁰⁹. We summarize its rules hereafter.

System 4.10 : TR

³⁰⁶ RIVENC, F., *ILP*, p. 219.

³⁰⁷ See *ibidem*, pp. 220-221.

³⁰⁸ See *ibidem*, p. 221.

³⁰⁹ See for the proof, *ibidem*, pp. 225-228.

(1) Axiom scheme (zero premise rule), where A is any formula

$$A \vdash A \quad (\text{Id})$$

(2) Operational rules

$$\frac{X \vdash Y(A) \quad X'(B) \vdash Y'}{X'(A \rightarrow B; X) \vdash Y(Y')} \rightarrow\vdash \qquad \frac{X; A \vdash B; Y}{X \vdash A \rightarrow B; Y} \vdash \rightarrow$$

$$\frac{X(A; B) \vdash Y}{X(A \circ B) \vdash Y} \circ\vdash \qquad \frac{X \vdash Y(A) \quad X' \vdash Y'(B)}{X; X' \vdash (Y; A \circ B); Y'} \vdash \circ$$

$$\frac{X(A) \vdash Y}{X(A \wedge B) \vdash Y} \quad \frac{X(B) \vdash Y}{X(A \wedge B) \vdash Y} \wedge\vdash \qquad \frac{X \vdash Y(A) \quad X \vdash Y(B)}{X \vdash Y(A \wedge B)} \vdash \wedge$$

$$\frac{X; A \vdash Y \quad B; X' \vdash Y'}{(X; A + B); X' \vdash Y; Y'} +\vdash \qquad \frac{X \vdash Y(A; B)}{X \vdash Y(A + B)} \vdash +$$

$$\frac{X(A) \vdash Y \quad X(B) \vdash Y}{X(A \vee B) \vdash Y} \vee\vdash \qquad \frac{X \vdash Y(A)}{X \vdash Y(A \vee B)} \quad \frac{X \vdash Y(B)}{X \vdash Y(A \vee B)} \vdash \vee$$

$$\frac{X \vdash A; Y}{X; \neg A \vdash Y} \neg\vdash \qquad \frac{X; A \vdash Y}{X \vdash \neg A; Y} \vdash \neg$$

$$\frac{X(0) \vdash Y}{X(\mathbf{t}) \vdash Y} \mathbf{t}\vdash \qquad 0\vdash \mathbf{t} \qquad \vdash \mathbf{t}$$

(3) Structural rules

Rules for « 0 »

$$\frac{X \vdash Y}{0; X \vdash Y} +0 \qquad \frac{0; X \vdash Y}{X \vdash Y} -0$$

Associativity (**B**)³¹⁰

$$\frac{V(X;(Y;Z))}{V((X;Y);Z)}$$

Strong commutativity (**C**)

$$\frac{V((X;Y);Z)}{V((X;Z);Y)}$$

Strong contraction (**W**)

$$\frac{V((X;Y);Y)}{V(X;Y)}$$

Extensional associativity

$$\frac{V(X,(Y,Z)) \vdash C}{V((X,Y),Z) \vdash C}$$

Extensional commutativity

$$\frac{V(X,Y) \vdash C}{V(Y,X) \vdash C}$$

Extensional Contraction

$$\frac{V(X,X) \vdash C}{V(X) \vdash C}$$

Extensional weakening

$$\frac{V(X) \vdash C}{V(X,Y) \vdash C}$$

This presentation completes the present chapter. We have seen that the structural presentation has the advantage of making explicit different rules for combining premises, while keeping identical, for a number of systems, their operational rules. In the structural framework, we can appreciate the difference between **R** and **CL** solely in terms of structural rules. The notion of structure enabled us to show that there may be two ways to combine premises (and, similarly, conclusions), one extensional using the comma, and one intensional using the semicolon. This distinction, we noted, is erased in a purely extensional logic like **CL**. The structure of semicolon « ; » in **R** and its close link both with the implicational connective and with the intensional conjunction suggests that **R** may be better seen if we understand it as a system where premises are being intensionally grouped. This idea, pursued by Read, will be elaborated in our sections 7.2.6 and 8.5. For now, we'd like to present the Routley-Meyer semantics for **R**. Since the semantics is very heavily based on possible worlds semantics, we start by exposing the latter.

³¹⁰ For all the structural rules, V represents an antecedent or a consequent structure.

Chapter 5 : Semantics and possible worlds

5.1 The role of a semantics

In section 1.2, we held that logicians usually like to provide a semantics for their syntactic logical systems. We distinguished between pure semantics and applied semantics, where the latter also have the entities of the semantics fitting notion(s) we are acquainted with. Multiple reasons can motivate the logician's desire for a semantics : (1) syntactic systems are sometimes considered as mere formal languages with rules and axioms in need of an interpretation, which is provided by the semantics (so only at a semantical level can we even say that **R** has a propositional language) ; (2) it reinforces the idea that the notion of consequence is stable and interesting ; (3) it can help settle particular philosophical issues³¹¹ ; and (4) it can help prove theorems otherwise difficult to obtain.

The first of these motivations is sometimes grounded in the idea that our logical systems need semantical analysis because they are basically just constituted of empty strings of symbols. This is reminiscent of a formalist approach to logic that is not necessarily justified, and it has been challenged by those who find explanatory power in the syntactic systems themselves. Logical systems are generally built with some goal in mind : in our approach, some phenomenon, linguistic or not, is being explained by the analysis. The different elements used in the formal language have some natural language counterparts they are supposed to represent. In building **R**, we were trying to provide an analysis of validity and of (some) uses of « if... then... » that avoided the pitfalls met in **CL**. So we already have some idea, true, maybe confused, but some idea at least of what **R** consists in. To say that any logical system without a semantics is useless or meaningless appears as an exaggeration at odds with the pragmatic attitude we have described in section I.2.3³¹². Rivenc, Read and Copeland have on their own accounts defended a position along this line. Rivenc has held that the meaning of connectives could be obtained through the operational rules of a sequent calculus, and he has used it as an

³¹¹ See for example section 7.2.

³¹² Our discussion is reminiscent of an episode in the development of quantum physics. One of the ways initially pursued to understand it was with the aid of matrices, mathematical devices that were not picturable in the physical reality. This has led to a lot of controversies, because some physicists didn't believe that this was a legit way to understand a physical phenomenon ; at least not without some further physical interpretation of the numbers appearing in the matrices. Alike, syntactic systems are sometimes thought to not be a legit way of understanding or analysing without a semantics. We do think that those syntactic systems give some intelligibility to the phenomenon they try to analyse without the further requirement of a semantics *needed*. But certainly, semantics provide extra virtues to a logical theory. The events discussed here are well presented in DE REGT, H.W., *Understanding Scientific Understanding*, chapter 7.

argument to show that relevant negation was the same as classical negation³¹³, since those rules are the same in the structural framework. Read has held the same view, preferring to distinguish not between pure and applied semantics, but between formal semantics and theory of meaning, both enlightening in their own rights. The role of formal semantics is, according to him, to provide technical results « such as soundness, decidability, independence, compactness, variable-sharing and so on »³¹⁴. And it is clear that, for him, a theory of meaning can only be achieved through syntactic means, in particular through structural systems :

The use of an expression includes both correct and incorrect uses. The task of logic is to establish a theory demarcating the correct from the incorrect uses of logical expressions, that is, the valid from the invalid arguments. Unless we mark the correct uses from the mistakes, no content will be given to the expression. The meaning of the logical expressions is constituted by those correct uses. We must examine those arguments which are accounted valid, those invalid, and it is they which determine the sense of the connectives. Gentzen's insight, taken up by Prawitz, was that the sense of a connective is given by its introduction rule. A theory of meaning for logic should be formulated in terms of rules of proof. Such a theory requires a harmony between rules for asserting a formula (introduction rules) and rules for drawing conclusions from it (elimination rules)³¹⁵.

Nonetheless, when we consider his argument against a theory of meaning obtained through a pure semantics accommodated with an appropriate interpretation, we may find his position wanting. He holds indeed that the translation from a language to another is insufficient to obtain the meaning of the first, « unless the meaning of the second is known »³¹⁶. So how can we, one may wonder, understand the propositions of the possible worlds semantics (that we will be presenting in a moment), which map from worlds to truth values ? Read answers :

If by a further map, the problem is merely postponed. If by some other means, then the theory is, as Blackburn has so well put it, dog-legged. Whatever (we know not what, yet) can now be done for the proposition, can be done for the formula directly. So at some point in this chain, meaning must be given differently than by such a translation³¹⁷.

But precisely, not all languages are created equal. There is some truth in holding that the meaning of a language appears in a much more intelligible way in a language than in

³¹³ See RIVENC, F., *ILP*, chapter 8. For more on negation, see our sections 6.7, 7.2 and 8.2. Rivenc also acknowledges the semantical differences between classical and relevant negation, and ends his book on the open question asking « which of those two points of view, one syntactic, the other semantical, gives the best representation of negation in relevant logics ? ». op cit., p. 246.

³¹⁴ READ, S., *PEI*, p. 160.

³¹⁵ *Ibidem*, p. 179.

³¹⁶ *Ibidem*, p. 161.

³¹⁷ *Ibidem*.

another. Just like there is some truth in holding that a language comes more handily to perform proofs than another. One of the main virtues of possible worlds semantics is that it makes talks about modality easier to represent and to work with. And maybe the best proof of this « intelligibility » gap lies in the work of the logician itself. If she bothers translating natural languages and theories into a formal language, it is precisely because she thinks the latter makes a theory of validity for the former easier, and that it sheds new light on the meaning of natural language expressions. Certainly Read agrees with this point of view, and this is precisely why he thinks that we do not need applied semantics, and that meaning can be found in syntactic systems. But it might well be that during her translation, the logician finds herself precisely in the situation described for the possible world semanticist by Read. She may find useful and enlightening to translate her abstract talk into something she finds much more intelligible, a language, that is, with different virtues than that of her syntactic system : interpreted thanks to a notion we are acquainted with, where the meanings of the expressions appear more intelligibly. And we see no legitimate reason to deny this second move. Our position on this issue is in fact well exposed by Copeland :

There are, indeed, many ways by which one may be brought to an understanding of a set of connectives besides mastering an applied semantics for them. [...] [O]ne may come to grasp the meanings of a group of connectives through being presented with natural deduction rules for them, or, in the most basic case, by simply participating in a community whose inferential practices are an embodiment of those rules. Not only does the construction of an applied semantics presuppose an understanding of some set of connectives: it will often be the case that the target connectives of the semantics are themselves already perfectly well understood. [...] It may be formulated in order to make explicit, and unify, an understanding gained through exposure to inferential practice, which may be community wide or confined to a small group or a specialized area [...] [It can also] relate one's understanding of a set of connectives in a systematic way to an account of the deducibility relation—the possible worlds account, say³¹⁸.

The acquainted notion expressed in applied semantics can be spelled out in terms of possession of a property. Copeland writes :

The conversion of a pure semantics to an applied semantics consists, then, in the interpretation of the mathematical recursion as a statement of necessary and sufficient conditions for the possession, by sentences of the forms involved, of an antecedently understood property Φ . Φ must be related to linguistic activity in such a way that a grasp of the conditions under which sentences of the forms

³¹⁸ COPELAND, B.J., « What is a semantics for classical negation ? » in *Mind*, 95 (1986), pp. 482-483.

involved possess this property is sufficient for an understanding of the targeted elements of the sentences³¹⁹.

For instance, the semantics for **CL** exposed in our introduction holds that when a proposition A has the property of being true, then the proposition $B \supset A$ has the property of being true as well. Copeland has claimed that two different requirements were to be expected from such property³²⁰. First, the property should be so that its (non-)possession by a compound sentence can be traced back to its (non-)possession by its elements : it must be *iterative*. Second, the property should be so that it gives to the connectives their intended meaning, that is, the properties yield a semantics that is *faithful* to these connectives.

We will be discussing the interpretation of the Routley-Meyer (R-M) semantics in the light of those constraints in part III. For now, we will present possible worlds³²¹ semantics, which is a famous semantics for normal modal logics. We chose to present possible worlds semantics because it possesses interesting links with the R-M semantics, and because we have already talked about modal logics several times in the present work. Furthermore, it is a very popular semantics and so our reader may find interest in it for its own right. Concerning the R-M semantics, also called the ‘relational semantics’ we are presenting it because it is the most discussed semantics in relevant literature. But our reader must be aware that alternatives exist : truthmaker semantics, Urquhart’s semilattice semantics, Kit Fine’s operational semantics and Dunn’s algebraic semantics³²².

5.2 Normal modal logics

We start by exposing the syntactic counterparts of the semantics. We use a Hilbert-style presentation here, and we only turn our attention to propositional modal logic, leaving matters related to quantification aside. We add to the language of **CL** the unary connectives « $\Box$ » and « $\Diamond$ » with appropriate rules of wff formation. Their readings depend on the interpretation one wants to give to the logic. Furthermore, the modal logics

³¹⁹ COPELAND, B.J., « What is a semantics for classical negation ? » in *Mind*, 95 (1986), p. 480.

³²⁰ *Ibidem*.

³²¹ Sometimes referred to as « possible world semantics » without an ‘s’ on world. There seems to be no universally used spelling.

³²² See (in order) : JAGO, M., « Truthmaker Semantics for Relevant Logic » in *Journal of Philosophical Logic*, 49 (2020) ; URQUHART, A., « Semantics for relevant logics » in *Journal of Symbolic Logic*, 37 (1972) ; FINE, K., « Models for entailment » in *Journal of Philosophical Logic*, 3 (1974), ANDERSON, A.R., and BELNAP, N., *ELRN* I, §18 (written by DUNN, J., M.). The three last references stem from MARES, E.D., *RLPI*, pp. 28-29.

characterized here are all *normal modal logics*. A few definitions are welcome. The first uses the equivalence symbol that needs to be added to the language³²³:

Definition 5.1 : Classical modal logic

A *classical modal logic* is a logic that contains the scheme (Def $\Diamond$) : $\vdash \Diamond A \equiv \neg \Box \neg A$ and that is closed under the rule :

where $A \equiv B$ is a theorem of the logic, so is $\Box A \equiv \Box B$.

Definition 5.2 : Regular modal logic

A *regular modal logic* is a logic that includes (Def $\Diamond$) and is closed under the rule :

where $A \supset B$ is a theorem of the logic, so is $\Box A \supset \Box B$

Definition 5.3 : Normal modal logic

A *normal modal logic* is a regular modal logic that is closed under the rule of necessitation :

where A is a theorem of the logic, so is $\Box A$.

We note that every regular and normal modal logic is also classical. To get a working logical system, we also need axioms. The axiom schemes encoding the behaviour of the new connectives are generally found among the following list³²⁴ :

List of axioms 5.4 : Possible axioms for normal modal logics

T :	$\Box A \supset A$		$A \supset \Diamond A$
D :	$\Box A \supset \Diamond A$		
4 :	$\Box A \supset \Box \Box A$		$\Diamond \Diamond A \supset \Diamond A$
5 (E) :	$\Diamond A \supset \Box \Diamond A$		$\Diamond \Box A \supset \Box A$
B :	$A \supset \Box \Diamond A$		$\Diamond \Box A \supset A$
D1 :	$\neg \Box (\Box A \supset B) \supset \Box (\Box B \supset A)$		
G1 :	$\Diamond \Box A \supset \Box \Diamond A$		
T_c :	$A \supset \Box A$		$\Diamond A \supset A$

³²³ The definitions come from MARES, E.D., *RLPI*, p. 22.

³²⁴ From LECLERCQ, B., *Épistémologie des modalités, modalités épistémiques*.

Some of these axioms have two logical expressions associated (separated by a « | ») : this is due to the fact that in normal modal logic, one element of the pair taken as an axiom allows the derivation of the other. One is free to pick one formulation over the other, but one will always get the other half anyways. Note that a certain formula, usually labelled **K**, can be obtained as a theorem in all normal modal logics. The formula **K** is : $\Box(A \supset B) \supset (\Box A \supset \Box B)$. So whatever the interpretation of « $\Box$ » given, this formula will be valid in the semantics. This being said, the choice of a set of axioms is dependent on the philosophical goal pursued ; goal which also determines the reading of the connectives. For instance, in some deontic interpretation, one could read « $\Box A$ » as « A is morally/legally mandatory » and « $\Diamond A$ » as « A is morally/legally allowed ». And in this interpretation, one would probably not want **T** as an axiom, because it is very plausible to hold that there are states of affairs holding in which people do not respect their moral or legal obligations ; and there are states of affairs holding in which people do immoral or illegal actions. On the other hand, **D** seems acceptable, because we may desire that what is mandatory be also allowed.

Another important feature to note is that some axiomatic systems are stronger than others. For instance, it can be proven that from the axiom **T**, **D** is derivable ; and from **T** and **5** (sometimes referred to as the axiom « **E** »), **B** is derivable³²⁵. So one should, as usual, be careful about the potential axioms she'd like in her axiomatic system. Two sets of axioms are commonly discussed : the system **S4** includes the axioms **K**, **T**, and **4** ; and the system **S5** includes the axioms **K**, **T**, and **5**. The axiom **4** can be proven in **S5**, so the system **S4** is contained in **S5** (and since **S5** also has **K** and **5**, it allows for a derivation of **D** and **B**). A lot more could be said about the choice of axioms and their interconnexions, but this would lead us far away from our primary goal, which is to present the semantics associated with it. To introduce it, we need the following items.

5.3 Possible worlds semantics

Recall our classical interpretation $\mathcal{I}$ that assigned a truth value to each propositional variable and complex formula of the language where no modal connective was involved for **CL**. An interpretation $\mathcal{I}$ is sometimes called a model $\mathcal{M}$. A model consisted in a pair $\langle I, v \rangle$, where 'I' was the set of individuals and 'v' was the assignation of a truth value. To have an adequate framework for our new connectives, however, we need a different notion of model. So we define instead a model $\mathcal{M}$ as a triple $\langle W, R, v \rangle$ where W is the set of indices called possible worlds and denoted $w_0, w_1, w_2, \dots$, R is the binary

³²⁵ See LECLERCQ, B., *Épistémologie des modalités, modalités épistémiques*, section B.IV.

accessibility relation between worlds, and v is the assignment function which associates to each propositional variable a truth value in each individual world. When the possible world w_1 is accessible from the possible world w_0 we note : Rw_0w_1 . The idea here is to have a notion of truth that « is relativised to worlds. So, instead of talking about the truth or falsity of a formula in a model, we talk about the truth or falsity of a formula in a particular world in a model »³²⁶. Intuitively, a possible world is a world just like ours, where the same logical rules hold (this is why they are ‘possible’), but which differs from ours in some respect : certain propositions have their truth value changed, and so we can picture these worlds as having states of affairs that hold there differently than from our world. These worlds can also be seen as maximal consistent classes of sentences : maximal because they specify « in as much detail as possible how things are. That’s what entitles [them] to be called ‘world[s]’ »³²⁷. Consistent because they do not accept contradictions : of any proposition, if p is true, then $\neg p$ is false ; and if $\neg p$ is true, then p is false.

The notion of truth in a world in a model permitted by the semantics enables us to provide the following truth conditions for the modal operators³²⁸ :

$\Box A$ is true in world w in a model $\mathcal{M}$ if and only if for every world w' such that w' is accessible from w , A is true at w' ;

and

$\Diamond A$ is true in world w in a model $\mathcal{M}$ if and only if there is some world w' such that w' is accessible from w and A is true at w' .

For the sake of completeness, we write the different stipulations for the valuations of formulae (those concerning the connectives of **CL** are just relativized to worlds in the models, but are basically the truth-functional ones from truth tables)³²⁹ :

1. For all atomic wffs p and all $a \in W$, let $v(p, w) \in \{T, F\}$.

So our valuation is a total function on the atomic wffs and always returns either the value true (‘T’) or the value false (‘F’). ‘ $v(A, w)$ ’ is to be read : ‘the value of the formula A at the index (or indice, or world) w ’. We need to recursively define the outputs of the valuation function for complex formulae. Since the output will always be ‘T’ or ‘F’, our logic is said to be bivalent. In the syntactic language, we define ‘ $\Box(A \supset B)$ ’ as ‘ $A \rightarrow_{ML} B$ ’.

2. (i) $v(\neg A, w) = T$ iff $v(A, w) = F$;

(ii) $v(A \wedge B, w) = T$ iff $v(A, w) = T$ and $v(B, w) = T$;

³²⁶ MARES, E.D., *RLPI*, p. 23.

³²⁷ READ, S., *PEI*, pp. 80-81.

³²⁸ From MARES, E.D., *RLPI*, p. 23.

³²⁹ With minor changes, from READ, S., *PEI*, pp. 79-80.

- (iii) $v(A \vee B, w) = T$ iff $v(A, w) = T$ or $v(B, w) = T$;
- (iv) $v(A \supset B, w) = T$ iff $v(A, w) = F$ or $v(B, w) = T$;
- (v) $v(\Box A, w) = T$ iff for all w' such that Rww' , $v(A, w') = T$ ³³⁰ ;
- (vi) $v(A \rightarrow_{ML} B, w) = T$ iff for all w' such that Rww' , $v(A, w') = F$ or $v(B, w') = T$.

3. Otherwise $v(A) = F$.

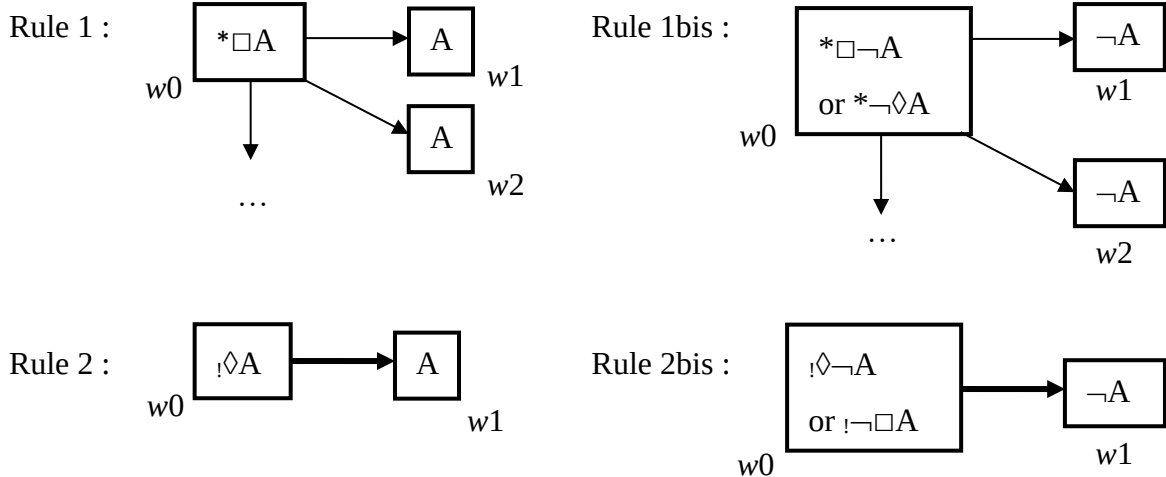
We can use the notion of structure from chapter 4 and note « $X \models A$ » to indicate that the formula A follows from the structure X in the semantics. We simply specify that $X \models A$ if and only if for all worlds w , in all models $\mathcal{M}$, $v(A, w) = T$ whenever $v(B, w) = T$ for all $B \in X$. Since we are dealing with an extension of **CL**, the structure X is a set of wffs (that is, formulae linked with an extensional ‘,’). Moreover, we have « $\models A$ » if and only if for all worlds w in all models $\mathcal{M}$, $v(A, w) = T$.

It is convenient to represent the models in diagrams, with a few appropriate rules³³¹. When a formula of the form $\Box A$ appears in a world w , we must add the formula A in any world w' accessible from it. When a formula of the form $\Diamond A$ appears in a world w , we must construct a world accessible from it and where the formula A appears. The connective $\Box$ does not force us to construct any world, we note $*\Box A$ to indicate this fact. The connective $\Diamond$, on the contrary, forces us to construct a new accessible world from our original one, we note $!\Diamond A$ to acknowledge this fact. Since $\Diamond A \equiv \neg\Box\neg A$, we have $\neg\Diamond A \equiv \neg\neg\Box\neg A$ so by double negation elimination $\neg\Diamond A \equiv \Box\neg A$. $\Box\neg A$ forces us to indicate the formula $\neg A$ in any world accessible, since it is a particular case of $\Box A$, so we have $*\Box\neg A$ and $*\neg\Diamond A$. On the other case involving negation, we have $\Diamond\neg A \equiv \neg\Box\neg\neg A$ and so $\Diamond\neg A \equiv \neg\Box A$. $\Diamond\neg A$ is a particular case of $\Diamond A$, so we have $!\Diamond\neg A$ and $!\neg\Box A$. The equivalences can be easily understood when we think about $\Diamond$ and $\Box$ as representing possibility and necessity : something is possible when it is false that its negation is necessary ; and when it is false that a negation is necessary, then it means that the negated formula is possible. Alike, when a negation is necessary, it means that the negated formula is not possible, etc. We end up with the following rules :

Rules 5.5 : Schematic rules encoding the behaviour of modal connectives in possible worlds semantics for normal modal logics

³³⁰ Recall that our modal logic is normal, so the valuation of ‘ $\Diamond A$ ’ is that of ‘ $\neg\Box\neg A$ ’ thanks to (Def $\Diamond$). We can alternatively add directly : $v(\Diamond A, w) = T$ iff there is a w' such that Rww' and $v(A, w') = T$.

³³¹ For the remaining of this section, we follow Leclercq’s presentation in LECLERCQ, B., *Épistémologie des modalités, modalités épistémiques*, section C.



Inside each world, formulae can be decomposed using the traditional techniques of semantic tables. For instance, if we have in a world $\neg(A \supset B)$, we write under the formula $A, \neg B$, which represents indeed the model in which it is false that A implies B in **CL**. The sense of the « , » is that of a conjunction. The general method followed to establish the validity of a formula in the model is through a proof by contradiction. We'd like to know if A is always valid in our semantics (valid in all the models). We consider the models of $\neg A$. If all the models lead to a contradiction, we have shown that $\neg A$ is never valid in our models, and so A must be valid in all the models. Thus, we have shown that A is valid in our semantics. Consider the formula **K**. We show that it must always be valid in our semantics. We start by taking its negation.

Proof 5.6 : K is valid in possible world semantics for normal modal logics

$\neg[\Box(A \supset B) \supset (\Box A \supset \Box B)] \quad (\neg K)$ $\Box(A \supset B), \neg(\Box A \supset \Box B)$ (we simplify the negation of the implication in the model) $*\Box(A \supset B), *\Box A, !\neg\Box B$ (again, and we add our notation to know what rules to apply)			w0
$\neg B, A, \quad (A \supset B)$ / \ $\neg B, A, \neg A \quad \neg B, A, B$ X X	$(!\neg\Box B \text{ forces us to construct a model where } \neg B \text{ holds ; we carry over the other expressions according to rule 1})$ $(\text{we have two ways to construct a model where } (A \supset B) \text{ holds ; one where } A \text{ is false and one where } B \text{ is true. We consider both and see that each leads to a contradiction})$		
		w1	

So no model satisfies $\neg K$ and so K is valid in all the models ; so K is valid in our semantics. Now if we consider the other, optional axioms. We end up in a situation where, as such, our semantics cannot « disprove » their negations. Let's consider, for example, the axiom T ; or rather, its negation. We have :

$\neg(\Box A \supset A)$	$(\neg T)$
$*\Box A, \neg A$	(simplification)

w_0

No contradiction can be obtained from this world alone. And even if there was some accessible world to it, $\neg A$ would not carry over. So how could we prove the contradiction ? The answer is : by imposing that the world w_0 be accessible to itself. Because then we would have A carried over in world w_0 by $*\Box A$, and a plain contradiction with $\neg A$:

$\neg(\Box A \supset A)$	$(\neg T)$
$*\Box A, \neg A$	(simplification)
$A, \neg A$	(by rule 1)

w_0

This is called the *reflexivity* condition on the binary accessibility relation. It forces any world to be accessible from itself. It matches the formal condition in our models (using quantification)³³² : $\forall x_w (R x_w x_w)$, where x_w is any possible world of the model. We show the different conditions one can impose on the binary accessibility relation and the related axioms they allow to derive. We illustrate the imposed condition with red coloured items in the diagrams ; note that $x_w, y_w, \dots$ are variables ranging over worlds.

List 5.7 : Possible conditions on the binary accessibility relation and related axioms

Axioms (one formulation)	Corresponding condition	Diagram representation
T $\Box A \supset A$	Reflexivity $\forall x_w (R x_w x_w)$	
D $\Box A \supset \Diamond A$	Seriality $\forall x_w \exists y_w (R x_w y_w)$	

³³² We omit the italic police on the indices attached to variables ranging over worlds for readability.

4 $\Box A \supset \Box \Box A$	Transitivity $\forall x_w \forall y_w \forall z_w [(R_{x_w y_w}) \wedge (R_{y_w z_w}) \supset R_{x_w z_w}]$	
5 (E) $\Diamond A \supset \Box \Diamond A$	Euclidean $\forall x_w \forall y_w \forall z_w [(R_{x_w y_w}) \wedge (R_{x_w z_w}) \supset R_{y_w z_w}]$	
B $A \supset \Box \Diamond A$	Symmetry $\forall x_w \forall y_w (R_{x_w y_w} \supset R_{y_w x_w})$	
D1 $\neg \Box (\Box A \supset B) \supset \Box (\Box B \supset A)$	Connectedness $\forall x_w \forall y_w \forall z_w [(R_{x_w y_w} \wedge R_{x_w z_w}) \supset (R_{y_w z_w} \vee R_{z_w y_w})]$	
G1 $\Diamond \Box A \supset \Box \Diamond A$	Convergence $\forall x_w \forall y_w \forall z_w [(R_{x_w y_w} \wedge R_{x_w z_w}) \supset \exists t_w (R_{y_w t_w} \wedge R_{z_w t_w})]$	
T_c $A \supset \Box A$	Isolation $\forall x_w [\exists y_w (R_{x_w y_w} \supset (x_w = y_w))]$	

As one expects, we are presenting this semantics because it can be proven sound and complete with the corresponding axiomatic systems. So adding a condition on the accessibility relation gives us a semantics that is sound and complete with the axiom matching the condition³³³. So for any A , « $\models A$ » in our semantics if and only if « $\vdash A$ » in the corresponding axiomatic system. Note that with no condition on the accessibility relation, the semantics is sound and complete with respect to all the normal modal logics that do not include any axiom of the list 5.4.

5.4 Philosophical perspectives

As Mares indicates³³⁴, the possible worlds semantics possesses several virtues that one may find philosophically attractive. First, the semantics is compositional, which guarantees that the property it encodes, once interpreted, is iterative : the « truth

³³³ A presentation of the proof can be found in LECLERCQ, B., *Épistémologie des modalités, modalités épistémiques*, section D.

³³⁴ See MARES, E.D., *RLPI*, pp. 23-25.

conditions of complex sentences are treated as being functionally dependent on the truth conditions of the simple sentences that make them up »³³⁵. Second, the semantics has the important virtue of couching modal talk into the language of set theory. Mares notes :

We talk about sets of points (i.e. possible worlds) and relations between these points. Relations, moreover, can themselves be treated as sets of a sort ; in the case of binary relations, as sets of ordered pairs of points. Thus, for example, if the accessibility relation holds between w and w' , then the ordered pair (w, w') is among the members of the set that makes up the accessibility relation. [And ordered pairs are themselves sets. The pair (w, w') can be defined as $\{w, \{w, \{w'\}\}\}$]³³⁶.

This language is extensional, and set theory is a tool that « has been around for a long time and a lot of its mathematical properties are very well understood »³³⁷. It is also easier to construct proofs in this language than directly in the language of modal logic, which incorporates, as we have seen, intensional connectives, notoriously hard to treat mathematically. The third virtue concerns the range of application of possible worlds semantics. We already said it could support several interpretations, and so be used for different purposes (through temporal, deontic, alethic notions ; to provide analysis of believes, knowledge). It has also been used as a basis for an analysis of counterfactuals ! Fourth and last virtue pinpointed by Mares, the possible worlds semantics allows for natural and intuitive interpretations of modality : it provides intelligibility and « can make modal talk easier to understand »³³⁸. Mares illustrates this point with the example of nomic necessity (necessity linked to the laws of nature)³³⁹. We can think of it in our semantics in the following way. A world w' is nomically accessible from a world w if and only the laws of nature obeyed at w are also obeyed at w' . If we denote the nomic necessity operator ' $\Box_N$ ', we have that the formula $\Box_N A$ is true at w « if and only if for every world w' that is nomically accessible from w , A is true at w' »³⁴⁰. Another virtue, not put to light by Mares, includes the rather simplicity of use and accessibility of the semantics through the use of diagrams. This should not be overlooked : the intelligibility of a theory is a key to its acceptance³⁴¹.

The semantics also come with its fair share of criticism. We have already seen in section 1.4 that necessary material implication validates theorems committing fallacies of relevance, i.e. a version of (M1) and (EFQ). Other criticisms have been formulated,

³³⁵ MARES, E.D., *RLPI*, p. 24.

³³⁶ *Ibidem*, p. 23. The text between « [] » is from his footnote.

³³⁷ *Ibidem*.

³³⁸ *Ibidem*, p. 24.

³³⁹ See *ibidem*.

³⁴⁰ *Ibidem*. This example is aimed at showing how talk about possible worlds can make talk about modality intuitive. There are philosophical disputes on the notion of « law of nature » in which Mares (and us) wish not to enter here. See op. cit., footnote.

³⁴¹ As is explained and illustrated in DE REGT, H.W., *Understanding Scientific Understanding*.

this time around directly to the intensional component involved in modality. Quine has indeed held that « intensions were not sufficiently clearcut to be included into a subject matter of a science so rigorous as logic »³⁴². Now, our analysis seems to rely on well-known notions, with clearcut boundaries and no fuzziness : functions and set theory. But one may hold that the « fuzziness Quine diagnosed did find its way even into our notion of intension, hidden into the Trojan horse of the concept of possible world. Is there a possible world where Berlin is east from Paris, while Paris is *not* west of Berlin ? »³⁴³ And if we come to answer the question by considering worlds as maximal consistent classes of sentences, where the meanings of expressions are already specified and the relation of compatibility between sentences so determined, then it becomes unclear why we built our semantics. Because such achievement, it seems, is required for languages in need of analysis, not for those who are already fully analysed. But as Peregrin explains, we could answer to Quine's criticism along the following line :

The fact that something is fuzzy does not automatically imply that it could not be useful to explicate (or model) it in a non-fuzzy way. After all, everything we encounter within the empirical world is (more or less) fuzzy, and we often profit from making idealized models. The whole point is to be aware of the level of idealization employed and not to employ the model where a higher resolution is required³⁴⁴.

A last issue of philosophical interest that we would like to raise here concerns the status of possible worlds. How should we understand them ? For any theory, one can either be a realist about it, holding that the entities of the theory do in fact exist, or an anti-realist about it, holding that the theory is merely a fiction or a tool used to produce explanatory work of some sort. There are at least two ways one can be a realist :

Lewis distinguishes between vertebrate realist theories and ersatzist theories about possible worlds. Vertebrate realist theories hold that other possible worlds are places like our own universe. They contain individuals standing in relation in space-time to one another. Ersatzist theories, on the other hand, claim that other possible worlds are either abstract objects or abstract constructions, such as sets of sentences or sets of SOA [states of affairs]³⁴⁵.

It should be clear both from the pragmatic stance put forward in the present work and from the previous qualifications of possible worlds as sets of indices and maximal

³⁴² PEREGRIN, J., « Extensional vs. Intensional Logic » in *Handbook of the Philosophy of Science*, 5 (2007), p. 936.

³⁴³ *Ibidem*, p. 937. Italic in the text. The citation goes on : « If we imagine possible worlds in some intuitive (in the Kantian sense) way, then it seems not, but could we not have possible worlds with some bizarre non-Euclidean geometries in which something like this would be possible? (And should we insist on intuitivity of possible worlds at all?) And again, who is to decide? »

³⁴⁴ PEREGRIN, J., « Extensional vs. Intensional Logic » in *Handbook of the Philosophy of Science*, 5 (2007), p. 938.

³⁴⁵ MARES, E.D., *RLPI*, p. 68.

consistent classes of sentences that we do not think that anyone subscribing to it is in a strong position to be a vertebrate realist about it. The reason is that the vertebrate realist confuses the entities that are being talked about with the way they are represented in the semantics. This may not seem totally obvious, because people who are realists about a certain theory are not always confronted with this problem. For instance, when physicists construct a theory to explain the way objects move in space-time and interact with each other, say, the realist takes the entities of the theory and tries to understand how they can relate to the entities of our world. But the theory relates to the empirical world in a certain way that seems to be ‘immediate’, and not through a third-party device designed to make the theory more intelligible.

To see the vertebrate realist’s problem, we can use another example. Consider the theory of Evolution. When one is interested in the evolutionary history of a particular species, it is common practice for this person to represent the way that particular species relate to other species in the course of Evolution through the construction of a « phylogenetic tree »³⁴⁶. Certainly, a realist about the theory of evolution would hold that species, in one way or another, really do exist (even as structures, or ‘individual species’, or whatever), and that a given species can ‘evolve’ or ‘give birth’ (or whatever) to another species in the course of Evolution. But we would be very surprised to see her hold that the tree exists. What tree, may our reader wonder ? The phylogenetic tree ! The tree in which we represented the relations between the species in the course of Evolution. Never have we walked in a forest and accidentally stumbled upon a phylogenetic tree, with at the end of each branch, a species... We understand that the tree is a way to represent how species can relate to one another, and that we should not take too seriously the expression « tree », for our theory is *not* about a certain « phylogenetic tree » at all. In more recent years, many phylogenetic trees do not even take the form of a tree, if a further argument felt desired to establish our point.

The same goes for our vertebrate realist. The possible worlds semantics is a theory about, say, (necessary) propositions, formulae, sets of sentences, states of affairs or some other entity of the kind. So if one is to be a realist about it, those should be the target entities for the ontology. Not possible worlds. The vertebrate realist commits the same mistake as our phylogenetic trees’ supporter, but in a less obvious way. It is more difficult for us to hold that such a bizarre entity as a phylogenetic tree exists, because we ‘know’ that there are no such things in our empirical world as a phylogenetic tree. It is less

³⁴⁶ Our reader can find examples of phylogenetic trees on the site of the Université de Lyon : THOMAS, P. « Comment et pourquoi représenter l'arbre phylogénétique du vivant ? La réponse du Musée des Confluences de Lyon » published as a *Image de la semaine*, by Dequincey, O., on the 06th/04/2015, available at the address : <https://planet-terre.ens-lyon.fr/ressource/Img492-2015-04-06.xml>

difficult to hold that possible worlds exist, precisely because, by definition, they are not in « our world ». But in substance, the phylogenetic tree supporter and the vertebrate realist commit the same mistake of taking too seriously a terminological term used to refer to an apparatus built for better intelligibility of a theory.

Should the person endorse a form of Ersatzism instead ? We believe this option is open, much like we left open the modal status of logic discussed in subsection I.2.3.1, but one may alternatively satisfy herself with the idea that the semantics is just a tool to understand modal talk more conveniently. The general conditions for commitment to a particular ontology constitute a subject matter beyond the scope of the present work. Our methodology has been to treat logical system as a way to provide a theory of validity, and as a tool to clarify natural languages as well as a range of concepts like those concerned by applied logics. So we have deliberately left aside metaphysical issues. To some extent, we do not feel compelled to embrace anti-realism, and we believe that some form of scientific realism is perfectly compatible with the view of logic herein presented. This is in continuity with our previous talk on the status of pure logics.

This being said, we underlined in subsection 1.7.1 the fact that possibility could play an important part in scientific explanation. We do not think that possibility cannot play any part in one's ontology (it may actually be a very important part of it), but we surely think that one cannot embrace the existence of possible worlds merely on grounds provided by possible worlds semantics.

A further note seems appropriate, concerning the *interpretations* of the normal modal logic and of possible worlds semantics. The matter is totally different when we analyse, say, belief, or, say, nomic necessity using a normal modal logic. There we may want to hold that beliefs do exist, or that laws of nature do in a sense exist as well. But the arguments for their existence is independent from the possible worlds semantics³⁴⁷, and we do not infer from such analysis the existence of possible worlds themselves. We merely use the modal theory and the semantics as a way of understanding how those entities can relate to one another. And even then, we need *some amount* of pre-understanding of the notion considered to choose the appropriate axioms and conditions on the accessibility relation.

In the present chapter, we have defended the idea that a semantics, even if it was not indispensable to give a meaning to the connectives of a syntactic logical system, or to

³⁴⁷ It is clear that if we display a well-constructed theory for some entity, it might be used as an argument to show that the entity may very well exist and behave according to the theory. So through such indirect considerations, one may « use » the possible worlds semantics to show the plausibility of the existence of some entity. But the use here is very different than the possible worlds vertebrate realist's, and is indeed usually backed up by extra evidence and/or arguments.

give them explanatory power, could be very useful for different reasons. It can, indeed, help the logician settle philosophical issues, prove theorems, get a grasp of the meaning of the expressions of her logical system and render her system or certain features of it more intelligible. The semantics also reinforces the idea that the notion of consequence of the syntactic system is stable and interesting. We have presented Copeland's distinction between pure and applied semantics, and have endorsed his claim that a more philosophically satisfying semantics should be of the second type ; where the property used to interpret the semantics is iterative and faithful.

The role of a semantics made clearer, we have moved on to present the syntactic systems of normal modal logics, including their potential axioms. All those systems, which are conservative extensions of **CL**, can be, as we've seen, semantically analysed in the framework of possible worlds semantics. We introduced, in this context, the binary accessibility relation with its possible constraints. We ended our present chapter by exposing the different virtues of this particular semantics, raising the question of the fuzziness of the concept of 'possible worlds', and discussing the ontological status of the entities in the semantics. On this last point, we claimed that the vertebrate realist about possible worlds misinterpreted the ontology of the theory and that our view was compatible with an ersatzist type of realism. We now use our acquaintance with the possible worlds semantics to present the more abstract and less straightforward Routley-Meyer semantics for **R**.

Chapter 6 : The Routley-Meyer semantics for **R**

In this chapter, we present the Routley-Meyer semantics for **R**. We chose to follow the presentation of Edwin Mares³⁴⁸, because of the careful attention he gives to the interpretational task associated to it. Our reader can find a different presentation in Read's book³⁴⁹, which has the advantage of showing how different logics sharing aspects of relevance can be semantically associated to the relational semantics with different conditions on the accessibility relation, but is not concerned with the interpretational task. Indeed, Read endorses, as we discussed, a proof-theoretic reading for the meaning of the connectives of his logic, and doesn't think necessary to have a theory of meaning linked to a semantics. It is important to note that where Read treats intensional conjunction as a primitive, Mares does not, and rather treats the implicational connective as a primitive.

6.1 From possible worlds to situations

We start by giving a few cursory characteristics of the R-M semantics. Its authors had in mind, while developing it, the possible worlds semantics for modal logic. They changed the latter to fit **R**, and in the process replaced the binary accessibility relation with a ternary accessibility relation. We follow Mares and « instead of talking about possible worlds, we will now switch to discussing 'situations' »³⁵⁰. The reason being that the indices in our semantics are different than in possible worlds semantics.

The R-M semantics differs indeed from possible worlds semantics on several aspects. In order to avoid the paradoxes of material implication, our logical truths, like « $A \vee \neg A$ » and « $A \rightarrow A$ », cannot hold at all the situations. So « whereas worlds are complete, situations can be incomplete »³⁵¹, unlike worlds, situations « do not decide every issue »³⁵². At some situations, the information concerning p can be lacking, and this property « is sometimes expressed by saying that at some situations 'the principle of bivalence fails' »³⁵³. This is required to avoid that $(A \vee \neg A)$ holds at all the situations, and so it is required to prevent the validity of a formula translating a paradox of material implication with our implicational connective : $\vdash B \rightarrow (A \vee \neg A)$. Unlike worlds, situations can also be inconsistent, and so some situations can make contradictions true. This is required to avoid a version of (EFQ), i.e. $\vdash (A \wedge \neg A) \rightarrow B$ ³⁵⁴.

³⁴⁸ See MARES, E.D., *RLPI*.

³⁴⁹ See READ, S., *PEI*, pp. 59-60 ; 78-113.

³⁵⁰ MARES, E.D., *RLPI*, p. 27.

³⁵¹ *Ibidem*.

³⁵² *Ibidem*.

³⁵³ *Ibidem*.

³⁵⁴ See RIVENC, F., *ILP*, p. 130.

It is harder to picture the ternary accessibility relation of the R-M semantics than it is to picture the binary accessibility relation of possible worlds. For now, we can say that if we represent the ternary relation with « R », then if $Rabc$ holds, we may say that the situations « a and b are compatible from the point of view of the situation c »³⁵⁵ or in informational content, that « the information in situation a combined to the information in situation b is available at situation c »³⁵⁶. The truth condition for relevant implication is

« ' $A \rightarrow B$ ' is true at a situation s if and only if for all situations x and y , if $Rsxy$ and ' A ' is true at x , then ' B ' is true at y »³⁵⁷.

Giving a reasonable interpretation of this truth condition is important for Mares, since he intends to give the R-M semantics « the same sort of status as a theory of meaning and a theory of truth as the possible world semantics »³⁵⁸. The emphasis in his approach is put on valid inferences rather than on theorems, and so on proofs rather than on valid formulae. But as he notes, « we cannot avoid the topic of valid formulae completely, at least not in relevant logic »³⁵⁹.

Since there are situations where logical truths fail to hold, how can we define a notion of validity ? That of possible worlds won't do. The trick is to define validity in some particular situation(s), precisely, situation(s) where logical truths hold. Sometimes this is accomplished by an indice, a situation representing the « real world », and sometimes by indices representing « logical situations ». Mares takes the second option.

He then defines validity in a model : « a formula is *valid* [or verified] *in a model* if and only if it holds at all the logical situations »³⁶⁰. He also gives the following definition : « a one-premise deduction from A to B is said to be valid on $\mathcal{M}$ if and only if for every situation s in $\mathcal{M}$, if A is true at s according to $\mathcal{M}$, then B is also true at s according to $\mathcal{M}$ »³⁶¹.

6.2 Hereditariness relation and n-ary accessibility relation

A formal apparatus comes handily to prove the semantic entailment theorem, which is a semantical equivalent to the deduction theorem.

³⁵⁵ RIVENC, F., *ILP*, p. 120.

³⁵⁶ *Ibidem*, 129.

³⁵⁷ MARES, E.D., *RLPI*, p. 28.

³⁵⁸ *Ibidem*.

³⁵⁹ *Ibidem*, p. 29.

³⁶⁰ *Ibidem*, p. 30. Italic in the text. The text added in « [] » stems from his footnote.

³⁶¹ *Ibidem*.

Theorem 6.1 : The semantic entailment theorem³⁶²

If the deduction from A to B is valid in $\mathcal{M}$, then 'A $\rightarrow$ B' is a valid formula in $\mathcal{M}$.

The formal apparatus consists in a binary relation on situations, denoted with « $\leq$ ». It is sometimes called the *persistence* or *hereditariness* relation. We read it : « for any situations s and t , if $s \leq t$, we say that t extends s »³⁶³. The relation is reflexive, transitive and anti-symmetrical :

Relation 6.2 : The hereditariness relation

(Reflexivity) for any situation s , $s \leq s$;

(Transitivity) for any situations s , t , and u , if $s \leq t$ and $t \leq u$, then $s \leq u$;

(Anti-symmetry) for any situations s , t , if $s \leq t$ and $t \leq s$, then $s = t$.

It is consequently called a *partial order*³⁶⁴.

Furthermore, the relation of hereditariness can be defined as follows, for any situations t and u ³⁶⁵ :

$t \leq u$ if and only if there is some logical situation s such that $Rstu$.

In order to prove the semantic entailment theorem, we need the persistence lemma, stating that « for any formula 'A', value assignment v , and situations s and t , if 'A' is true at s according to v and $s \leq t$, then 'A' is true at t according to v »³⁶⁶. Since we read ' $s \leq t$ ', as ' t extends s ', it seems natural to require that what is true in s according to a value assignment v should also be true in t given the same value assignment. To obtain this persistence lemma, we constrain the accessibility relation in the following way³⁶⁷ :

if $s' \leq s$ and $Rstu$, then $Rs'tu$;

and the value assignment in the following way :

for any propositional variable p , if p is true according to assignment v at s and $s \leq t$, then p is also true at t according to v . With these different elements, it can be shown that the

³⁶² From MARES, E.D., *RLPI*, p. 30.

³⁶³ *Ibidem*.

³⁶⁴ Intuitively, a partial order is a way to 'order' elements with one another while endorsing that some of these elements cannot be so 'ordered', this is why the order is partial. Formally, a partial order is just a reflexive, transitive, and anti-symmetric relation.

³⁶⁵ See *ibidem*, p. 31.

³⁶⁶ *Ibidem*.

³⁶⁷ For the two constraints, see *ibidem*.

persistence lemma follows, and that the semantic entailment theorem can be obtained as well³⁶⁸.

If we want to treat inferences with more than one premise, we need more than what is permitted by our hereditariness relation. We need to construct, using our ternary accessibility relation, « relations of ‘arity’ greater than three »³⁶⁹. So we need to be able to construct relations that allow for more than three arguments. This can be technically achieved in the following way. « We introduce the notation R^n for $n \geq 0$ and give it meaning as follows »³⁷⁰ :

R^0st if and only if $s \leq t$;

R^1stu if and only if $Rstu$;

$R^{n+1}s_1 \dots s_{n+2}t$ if and only if $\exists x(R^n s_1 \dots s_{n+1}x \text{ and } Rx s_{n+2}t)$, for $n \geq 1$.

For instance :

R^2stuv if and only if $\exists x(Rstx \text{ and } Rxuv)$,

and

R^3stuvw if and only if $\exists x(R^2stux \text{ and } Rxvw)$, where we can simplify further down the second expression to $\exists x \exists y (Rsty \text{ and } Ryux \text{ and } Rxvw)$ ³⁷¹.

Mares subsequently defines the validity of an inference in a R-M model as follows :
« an inference from the premises $A_1, \dots, A_n$ to the conclusion B is *relevantly valid* in a model $\mathcal{M}$ if and only if for all situations $s_1, \dots, s_n$, if for each i such that $1 \leq i \leq n$, A_i is true at s_i in $\mathcal{M}$, then for any situation t such that $R^n s_1 \dots s_n t$, B is true at t in $\mathcal{M}$ »³⁷².

If we consider the natural deduction system, we can « think of the subscripts as referring to situations »³⁷³. So the expression ‘ A_1 ’ in the formal system is interpreted in our semantics as « there is a situation (call it ‘ s_1 ’) in which the formula ‘ A ’ is true »³⁷⁴. And for the expression ‘ $A_{1,2}$ ’, the subscript ‘1,2’ is « taken to refer to some arbitrary situation t such that $R s_1 s_2 t$ »³⁷⁵ ; the subscript ‘1,2,3’ would refer similarly to some arbitrary situation t such that $R^2 s_1 s_2 s_3 t$, etc.

Two commentaries can be made. First, the semantic entailment theorem gives us a (semantic) correlation between implication and inference. So « it makes sense to connect the accessibility relation – the element of the semantics that treats implication – with

³⁶⁸ See MARES, E.D., *RLPI*, p. 31.

³⁶⁹ *Ibidem*, p. 32.

³⁷⁰ *Ibidem*.

³⁷¹ Characterization and examples from *ibidem*, p. 32.

³⁷² *Ibidem*.

³⁷³ *Ibidem*.

³⁷⁴ *Ibidem*.

³⁷⁵ *Ibidem*.

inference more generally »³⁷⁶, and this is why the relation has been generalised to relations of arbitrary arity and « inference treated in terms of this class of relations »³⁷⁷. Secondly, the superscripts on the relation can be omitted altogether since « it is the ‘same relation’ construed as holding between different numbers of situations. It has the same function in each of these cases – to provide a semantics for deductive inference »³⁷⁸. We follow Mares in omitting them.

6.3 Further constraints on the accessibility relation³⁷⁹

Since we interpret the subscript ‘1,2’ in **NDR** as representing an arbitrary situation t such that Rs_1s_2t (where 1 refers to s_1 , 2 refers to s_2 , ...) and since in **NDR** it does not matter in which order we state a hypothesis or in which order it is used in a proof (the « order in which information is presented in a deduction is not crucial »³⁸⁰), we want ‘1,2’ and ‘2,1’ to coincide. So to represent « the arbitrariness of the order of premises in our semantics »³⁸¹ we constrain the relation so that if Rs_1s_2t then we also have Rs_2s_1t . This way of « interchanging » the situations in the relation can be generalised. Only the last situation of the relation must stay fixed. So we have that for any situations $s_1, \dots, s_i, s_j, \dots, s_n$ if $Rs_1 \dots s_i s_j \dots s_n t$ then $Rs_1 \dots s_j s_i \dots s_n t$.

For the second constraint we can consider the following deduction in **NDR** :

- | | | |
|----|----------------------------------|---------------------|
| 1. | $((A \rightarrow B) \wedge A)_1$ | hyp. |
| 2. | $(A \rightarrow B)_1$ | 1, $\wedge E$ |
| 3. | A_1 | 1, $\wedge E$ |
| 4. | B_1 | 2,3 $\rightarrow E$ |

The subscript ‘1’ refers to a situation s . So the deduction tells us that if $(A \rightarrow B) \wedge A$ is true in a situation s , then B is also true in that situation s . To make the argument valid, we postulate that for any situation s , we have $Rsss$. Generalising, we obtain that for any number of s ’s greater than 2, $Rs \dots s$.

The last constraint on the accessibility relation ensures us that « we can use the information in a situation as many times as we want in a deduction »³⁸² ; so it ensures that in the natural deduction system, we can « use any single hypothesis as many times as we want in a given deduction »³⁸³. The constraint states that if $R \dots s \dots t$ holds, (with at least

³⁷⁶ MARES, E.D., *RLPI*, p. 32.

³⁷⁷ *Ibidem*.

³⁷⁸ *Ibidem*, pp. 32-33.

³⁷⁹ *Ibidem*, pp. 35-36.

³⁸⁰ *Ibidem*, p. 35.

³⁸¹ *Ibidem*.

³⁸² *Ibidem*.

³⁸³ *Ibidem*.

three situations linked by R), then $R...ss...t$ also holds. So « if we have $Rstu$ in a model, then we also have $Rsstu$ and $Rsttu$ (but we do not necessarily have $Rstuu$) »³⁸⁴.

6.4 The R-M semantics for the positive fragment of R, R^+

We can summarize the different elements that have been hitherto presented in the following way. Note that the extensional disjunction and conjunction do not present any particularities in the framework. Their truth is relativized to indices, (so here, to situations) just in the same way their truth was relativized to worlds in normal modal logics.

Presentation 6.3 : The R-M Semantics for R^+

We start with a *positive relational frame*, which is a triple $\langle sit, logical, R \rangle$ such that *sit* is a non-empty set (of situations), *logical* (the set of logical situations) is a non-empty subset of *sit*, and R satisfies the following definitions and postulates³⁸⁵ :

- (1) $t \leq u =_{df} \exists x (x \in logical \text{ and } Rxst)$, to be read « $t \leq u$ if and only if there is some logical situation s such that $Rstu$ » ;
- (2) $R^1stu =_{df} Rstu$
- (3) $R^{n+1}s_1...s_{n+2}t$ iff $\exists x (R^n s_1...s_{n+1}x \text{ and } Rxs_{n+2}t)$, for $n \geq 1$;
- (4) if $R...st...u$ then $R...ts...u$ (interchange) ;
- (5) if $R...s...t$ then $R...ss...t$ (repetition) ;
- (6) $Rsss$ (complete reflexivity) ;
- (7) if $Rstu$ and $s' \leq s$, then $Rs'tu$.

A *positive relational model* is a quadruple $\langle sit, logical, R, v \rangle$ such that $\langle sit, logical, R \rangle$ is a positive relational frame and v is a value assignment, that is, a function from propositional variables to sets of situations.

We write ' $s \models_v A$ ' to indicate that A is true in a situation s according to valuation v . So ' $s \models_v A$ ' is another way to say $v(A, s) = T$. For any value assignment v , any propositional variable p , and any situations s and t , if $s \models_v A$ and $s \leq t$, then $t \models_v A$. The truth clauses for formulae can now be stated. For any propositional variable p and any formulae A and B , we have:

- (i) $s \models_v p$ iff $v(p, s) = T$;
- (ii) $s \models_v A \wedge B$, iff $s \models_v A$ and $s \models_v B$;

³⁸⁴ MARES, E.D., *RLPI*, pp. 35-36.

³⁸⁵ All this presentation is from *ibidem*, pp. 36 ; 210.

(iii) $s \models_v A \vee B$, iff $s \models_v A$ or $s \models_v B$;

(iv) $s \models_v A \rightarrow B$ iff $\forall x \forall y ((R_s x y \text{ and } x \models_v A) \supset y \models_v B)$.

Where in clause 4, « $\supset$ » is indeed material implication : Mares endorses a completely classical metalanguage for his presentation³⁸⁶. A formula is *valid* in a R-M model iff for all $s \in \text{logical}$ (and all valuations), $s \models_v A$.

6.5 Situating implication³⁸⁷

Before we can add negation to our semantics, we need to consider the interpretation of the clause for the implicational connective offered by Mares. This is the most important connective of the logic, and this is also the one that requires the ternary accessibility relation in its valuation. So if one desires to turn the R-M semantics into an applied semantics, this interpretation is crucial.

Back to the characterization of situations, Mares takes them to be a way to model our communication of partial information, which is a pretty common thing³⁸⁸ :

We do not know what is going on in the whole world, but rather tell people what is going on in our own surroundings. These surroundings are situations. But a situation, as I use the term, need not be an agent's physical location. We can use situations, for example, to model an agent's state of beliefs. Situations are structures of information, rather than physical locations, and can include information from various sources³⁸⁹.

The situations considered can be either abstract or concrete. A concrete situation is just a part of the world, while an abstract situation is an abstract representation of a part of the world. Mares thinks of his situations as abstract situations (and indeed, the only realist reading he offers of his theory is through an ersatzist point of view). Mares notes that even if our inferences are made from « the limited perspective of some situation »³⁹⁰, they can contain information, and indeed consist in inferences about, distant parts of the world. Mares starts by the interpretation of an inference using a natural deduction system. For each inference, the world is held fixed. So in Mares' semantics,

1. A_1 hyp.
2. B_2 hyp.
- ...

³⁸⁶ See section 8.4.

³⁸⁷ This section stems from MARES, E.D., *RLPI*, pp. 39-56.

³⁸⁸ The semantics for situations developed by Mares originates from Barwise and Perry's work, see BARWISE, J. and PERRY, J., *Situations and attitudes*.

³⁸⁹ MARES, E.D., *RLPI*, p. 40.

³⁹⁰ *Ibidem*, p. 41.

is to be read « as saying that we hypothesise that there is a situation in a world in which ‘A’ holds and that there is a situation in the same world in which ‘B’ holds »³⁹¹. When several indices are indicated, the indices track back the origin of the information, so if we obtain ‘B_{1,2}’ from the hypotheses ‘(A → B)₁’ and ‘B₂’, then we have assumed a situation 1 containing the information that A implies B ; and we have assumed « a situation 2 in the same world that supports the proposition A »³⁹² so we can infer « that there is a situation in that world in which B obtains »³⁹³. Looking back at the rule for →I in **NDR** : from a proof of B_a under hypothesis A_{k}, infer (A → B)_{a - {k}} (with {k} ∈ a).

From Mares’ interpretation, this is to be read : « if we can infer, on the basis of the hypothesis that there is a situation in the world in which A holds, that there is a situation 2 in which B holds, then we can infer that there is a situation in which A → B holds »³⁹⁴. This gives the following reading of the truth condition for implication :

« A → B is true at s if and only if there is information in s such that on the hypothesis that there is some situation t in the same world as s in which A holds, we can legitimately derive that there is some situation u in the same world in which B holds »³⁹⁵.

The role played by information in Mares’ view calls for further characterization. Mares calls « any fact that licenses a situated inference an ‘informational link’ »³⁹⁶ and he holds that « a link is information that tells us that if there is a situation in this world in which a proposition A holds, then there is also a situation in the same world in which B holds (for some propositions A and B) »³⁹⁷. So informational links consist just in information contained in situations that enable one, given extra information on certain (other) situations, to infer that other situation(s) should hold. As a typical example, we could see some scientific theories as informational links such that given certain observations (which give rise to proposition A in situation s), we can infer certain predictions (which give rise to proposition B in a situation t). The « informational links are themselves contained as information in situations, and vary from situations to situations »³⁹⁸. Furthermore, they must be perfectly reliable, and since our « inferences about other situations are in fact defeasible »³⁹⁹, implication is taken as an idealisation.

³⁹¹ MARES, E.D., *RLPI*, p. 42.

³⁹² *Ibidem*.

³⁹³ *Ibidem*.

³⁹⁴ *Ibidem*, pp. 42-43.

³⁹⁵ *Ibidem*, p. 43.

³⁹⁶ *Ibidem*.

³⁹⁷ *Ibidem*, p. 44.

³⁹⁸ *Ibidem*. We took the example of a scientific theory, but a convention would constitute another example of informational link, this time which clearly varies from situations to situations.

³⁹⁹ *Ibidem*, p. 45.

We lack a way to understand the ternary accessibility relation and the binary relation on situations « $\leq$ ». The hereditariness relation is a part-whole relation on situations, « if $s \leq t$ we can say that s is a part of t »⁴⁰⁰. To understand the ternary accessibility relation, Mares starts by introducing a relation ‘I’, « that relates a pair of situations to propositions »⁴⁰¹, where propositions are some sets of situations (see end of section 6.6). We note ‘|A|’ the proposition expressed by the formula A (the set of situations in which A is true). We have $Ist|A|$ if and only if we can « legitimately infer from the hypothesis that s and t exist in the same world to the conclusion that there is some situation in the same world in which $|A|$ obtains »⁴⁰². We obtain the truth condition for implication⁴⁰³ :

Truth condition 6.4 : Implication

‘ $A \rightarrow B$ ’ is true in s iff $\forall x (‘A’ \text{ is true in } x \supset Isx|B|)$

which can be read « an implication is true at a situation if and only if hypothesising the existence in the same world of a situation which makes the antecedent true allows us to infer the existence of a situation in that world in which the consequent holds »⁴⁰⁴. We say that if $Ist|A|$, then $|A|$ is *in the neighbourhood* of s and t .

For his interpretative task, Mares seems to explain the different elements of the R-M semantics while making use of intuitive notions one can find in neighbourhood semantics. This is how we understand his sections 3.5-3.8 So he needs to show that we can recover the rules of **NDR** from this neighbourhood semantics, and this is what he seems to do by translating propositions of complex formulae into set-theoretic constructions of propositions of simple formulae.

For conjunction, we have $|A \vee B| = |A| \cap |B|$, so the « proposition expressed by a conjunction is identical to the set-theoretic intersection of the propositions expressed by the conjuncts »⁴⁰⁵. In order to validate in the neighbourhood semantics the inferential move : from ‘ $A \rightarrow B$ ’ and ‘ $A \rightarrow C$ ’ at s , infer ‘ $A \rightarrow (B \wedge C)$ ’ at s , Mares adds the following condition on I :

$(Ist|A| \text{ and } Ist|B|) \supset Isx|A| \cap |B|$

which says that if « from the hypotheses that s and t hold at the same world, we can infer that there is some situation in $|A|$ that holds in that world and some situation in $|B|$ that

⁴⁰⁰ MARES, E.D., *RLPI*, p. 52.

⁴⁰¹ *Ibidem*, 46.

⁴⁰² *Ibidem*. With slight notational adjustments.

⁴⁰³ *Ibidem*.

⁴⁰⁴ *Ibidem*.

⁴⁰⁵ *Ibidem*, p. 47.

holds in this world, then we can infer that there is some situation in both $|A|$ and $|B|$ that holds in that world »⁴⁰⁶. Mares notes that the semantics does not warrant the natural deduction rule :

from A_a and B_b , infer $A \wedge B_{a \cup b}$

which would be unacceptable, since, as we've shown in subsection 3.4.1 this would lead to the introduction of the paradoxes of material implication in our system. What this inference would say in the semantics is that

given a situation in which A holds and a situation in which B holds, we can infer that there is a situation in which both A and B hold. Whereas it may be true of any world that for any two situations s and t in it there is a situation u in it in which all the propositions of each of s and t also hold in u , the inference to the existence of this sort of situation is hardly justified by the information in every situation. It is too much to ask of all our situations that they contain this much information about the nature of worlds⁴⁰⁷.

For disjunction, we have $|A \wedge B| = |A \cup B|$, so the proposition expressed by a disjunction is identical to the set theoretic intersection of the propositions expressed by the disjuncts, which was to be expected.

We are close to reaching our goal of giving an interpretation of the ternary accessibility relation. We need one more notion, that of intersection of all the neighbourhoods of two situations. We can picture this in the following way : assume we are in a situation s , and we hypothesise that there is a situation t containing an object i that is uniformly green and occupies a certain region of space r . We denote the proposition « i is uniformly green » with ' g ' and the proposition « i occupies a region of space r » with ' o '. From our situation s , the proposition ' $g \wedge o$ ' « implies every proposition of the form r' is occupied by a uniformly green surface, where r' is a subregion of r »⁴⁰⁸. From what has just been said about conjunction, it follows that the « conjunction[s] of pairs of these propositions are also implied (and conjunctions of these conjunctions, and so on) »⁴⁰⁹.

Now we may want to say that $g \wedge o$ implies more than each of these propositions and their finite conjunctions. We may want to say that it implies the « conglomerate of these propositions »⁴¹⁰. Indeed, when we think of $g \wedge o$, « it is intuitive to say that it implies that there is a situation in the present world in which every subregion of r is uniformly

⁴⁰⁶ MARES, E.D., *RLPI*, p. 47. With slight notational adjustments.

⁴⁰⁷ *Ibidem*, p. 49.

⁴⁰⁸ *Ibidem*, p. 50.

⁴⁰⁹ *Ibidem*.

⁴¹⁰ *Ibidem*.

green »⁴¹¹. This intuition can be captured in the semantics by « postulating that, for all situations s and t , the intersection of all the neighbourhoods of s and t is also a neighbourhood of s and t »⁴¹². We denote this intersection by $P(s, t)$.

What does $P(s, t)$ tells us ? If we take two propositions, say $|A|$ and $|B|$, and if $|A|$ as well as $|B|$ are in the neighbourhood of s and t , then $|A| \cap |B|$ would also be in the neighbourhood of s and t :

This tells us that if s and t obtain in the same world, then so does some situation from $|A| \cap |B|$. We need no extra information to know that some situation obtains in that world in which $|A|$ holds and some situation obtains in which $|B|$ holds. Thus, $Ist|A|$ provides superfluous information as does $Ist|B|$ ⁴¹³.

We can generalise this reasoning to all intersections. So it appears that our $P(s, t)$ gives us more information than does the information « that any other proposition is a neighbourhood of s and t . Therefore, we really need to state only that this generalised intersection $P(s, t)$ is such that $IstP(s, t)$ »⁴¹⁴. We can finally define a ternary relation on situations, which is just the Routley-Meyer accessibility relation⁴¹⁵ :

$Rstu$ if and only if u is in $P(s, t)$, that is, if and only if u is in the intersection of the neighbourhoods of s and t .

And $Rstu$ can be read :

Given the hypotheses that s and t obtain in the same world, u is among the candidates that we can infer also to obtain in that world. The situation u will contain all the information that can be inferred from the information in s and t . Given the information in s and t , we cannot deductively infer which of the situations u such that $Rstu$ obtain. But we can know that on the hypothesis that s and t obtain at the same world, at least one such u also obtains at that world⁴¹⁶.

With this definition, we can get the following theorem in the neighbourhood semantics⁴¹⁷ :

‘ $A \rightarrow B$ ’ is true at s iff $\forall x \forall y ((Rsxy \text{ and ‘} A \text{’ is true at } x) \supset \text{‘} B \text{’ is true at } y)$

which can be read « ‘ $A \rightarrow B$ ’ is true in a situation s if and only if, when we postulate the existence of any situation in the same world in which ‘ A ’ holds, we can infer that there is a situation in that world in which ‘ B ’ holds »⁴¹⁸, which is just our truth condition 6.4.

⁴¹¹ MARES, E.D., *RLPI*, p. 50.

⁴¹² *Ibidem*.

⁴¹³ *Ibidem*, p. 51. With slight notational adjustments.

⁴¹⁴ *Ibidem*.

⁴¹⁵ From *ibidem*.

⁴¹⁶ *Ibidem*.

⁴¹⁷ See *ibidem*.

⁴¹⁸ *Ibidem*.

6.6 The virtues of the R-M semantics⁴¹⁹

Mares notes that the R-M semantics has some of the virtues of the possible world semantics. It is indeed compositional : checking the truth conditions for formulae in the models should convince us of this fact. The semantics also has a wide range of applicability : Mares provides a theory of conditionals and counterfactuals based on it, and argues that it is a good candidate for a theory of relevant identity as well as for grounding a deontic logic⁴²⁰.

He also intends to show that the semantics can be couched in purely extensional terms. His strategy is to build the semantical theory using only the following primitive entities : possible worlds, individuals and sets⁴²¹. The general reduction move takes properties and relations to be functions from worlds to sets⁴²² ; states of affairs (SOA) to be sequences (set-theoretic constructs) including relations of a certain type and entities of the corresponding type⁴²³ ; and situations to be ordered pairs. A situation $s = (SOA(s), R(s))$, where « $SOA(s)$ is a set of states of affairs; these are the states of affairs that obtain at s . $R(s)$ is a set of ordered pairs. (t, u) is in $R(s)$ if and only if $Rstu$ holds in our frame. Thus, the set of situations itself determines which situations are related to which in the frame »⁴²⁴. The relation $\trianglelefteq$, the informational part-whole relation on situations, can now be defined⁴²⁵ : we say that $s \trianglelefteq t$ if and only if $SOA(s) \subseteq SOA(t)$ and $R(t) \subseteq R(s)$. The second part of the definition says that the « fewer pairs of situations related to a given situation u , the more implicational propositions hold at u . Thus, if $R(t)$ is a subset of $R(s)$ then at least as many implications hold at t as hold at s »⁴²⁶.

To finish his reduction, Mares constructs possible situations. He first defines :

Definition 6.5 : Minimally obtains

A situation s minimally obtains at a world w if and only if (a) for all σ in $SOA(s)$, σ holds at w and (b) for all situations t , if $SOA(s) = SOA(t)$, then $s \trianglelefteq t$.

⁴¹⁹ We present the virtues of the semantics before introducing negation in the language. The reason being that part of Mares' presentation of negation concerns worldly situations, and worldly situations require, to be defined, notions that Mares introduces in the reduction of the language of the semantics. Nothing that is said about negation contravenes what is said here.

⁴²⁰ See MARES, E.D., *RLPI*, pp. 116-122 ; 125-160.

⁴²¹ *Ibidem*, p. 58. Mares adds : « if there is a way of analysing possible worlds in terms of more primitive (and more satisfactory) entities, then my theory can accept this analysis. We need possible worlds, wherever they come from », op. cit., p. 59.

⁴²² *Ibidem*, p. 60.

⁴²³ *Ibidem*, p. 61.

⁴²⁴ *Ibidem*, p. 65.

⁴²⁵ See *ibidem*.

⁴²⁶ *Ibidem*, pp. 65-66.

which says that « a situation minimally obtains at a world if it represents that world in terms of its SOA and, if from the set of situations that have the same SOA as it does, it makes the fewest implications true »⁴²⁷. We need this definition because in some cases, « all the SOA in a situation hold at a world, but the situation itself is not possible »⁴²⁸. We obtain the definition of a possible situation⁴²⁹ :

Definition 6.6 : Possible situation

A situation s is possible if and only if there is some situation t such that there is a world at which t minimally obtains and $s \preceq t$.

which simply says that « a possible situation is just a situation that obtains at some possible worlds, or perhaps in more than one possible world »⁴³⁰. Mares also add that there is a strong relationship between logical situations and possible worlds, which is stated⁴³¹ :

for each world w there is a logical situation s such that there is some situation t for which both $s \preceq t$ and t minimally obtains at w .

He comments :

I am not exactly sure how to justify this. Some support comes from its consequence that all valid formulae are also necessarily true. Perhaps we could rely on our theory of situated inference for help in justifying our claim. It seems reasonable to suggest that in each possible world there are the salient facts about the intended model as a whole. In our world there are the truths about the set-theoretic universe (if there is such a thing) and these facts are in every possible world. Moreover, in every possible world there are facts about the set of situations and the structures that we are constructing here. Thus, in each world there is information about what propositions semantically entail which propositions in the model. In each possible world, therefore, the links are available to make true ‘ $A \rightarrow B$ ’ where ‘ A ’ semantically entails ‘ B ’ in the intended model⁴³².

A last reduction is performed, that of propositions, which are « piece(s) of information of a certain sort »⁴³³. A proposition is just « a set of situations closed upward under $\preceq$. This

⁴²⁷ MARES, E.D., *RLPI*, p. 66.

⁴²⁸ *Ibidem*.

⁴²⁹ See *ibidem*.

⁴³⁰ *Ibidem*.

⁴³¹ *Ibidem*, pp. 66-67.

⁴³² *Ibidem*.

⁴³³ *Ibidem*.

means that if ϕ is a proposition and s in ϕ , then for any t such that $s \leq t$, t is in ϕ as well »⁴³⁴.

We leave to our reader the care to evaluate to what extent Mares' interpretation provides intelligibility on the notions of inference and implication it is trying to clarify. It is clear that the semantics is more fine-grained than the possible worlds semantics, and so, to some extent, that we should expect the former to be more difficult to present and to master than the latter. It may not be so obvious that it makes talk about inference easier, so we leave open the question of whether Mares' semantics also possesses the fourth virtue he attributed to possible worlds semantics.

6.7 Adding negation

If we want a semantics for **R**, we also need a way to treat negation in it. Mares introduces negation through a binary relation on situations. This can appear *prima facie* as a curious move. Why wouldn't we be able to introduce it similarly to the way negation was defined in normal modal logic, through a specification of the sort :

' $\neg A$ ' is true in s if and only if ' A ' fails to be true in s .

With a classical meta-language, this is indeed tantamount to the specification expressed in section 5.3. But recall that in order to avoid the paradoxes of material implication, we need some inconsistent situations, and some incomplete situations. This specification would force any situation to be consistent and bivalent, so we can't accept it⁴³⁵.

The binary relation introduced by Mares is the relation of compatibility between situations. « Csx » is to be read as « s is compatible with x ». It is added as a primitive relation, with the truth condition for negation being :

$s \models_v \neg A$ if and only if $\forall x(Csx \supset x \not\models_v A)$.

Where $\supset$ is material implication. Note that Mares' notation differs from our previous notation of section 5.3. He writes ' $s \models_v A$ ' to indicate that A is true in a situation s according to valuation v . And ' $s \not\models_v A$ ' indicates that A is not true in a situation s according to valuation v .

We may understand different features of this negation through examples. This specification allows for partial situations. Consider the situation of Mares writing his book. The situation contains the information available to Mares : what is going on in his office and what he can see through the windows. He writes :

⁴³⁴ MARES, E.D., *RLPI*, p. 70.

⁴³⁵ As Priest argued, we can interpret the term « fails » in a non-classical way (that is, using a non-classical metalanguage) and thus refuse bivalence and in particular the consistency of all situations while endorsing the specification. But Mares endorses « a completely classical metalanguage », see *ibidem*, pp. 73-74.

Nothing happening here makes it true that it is currently raining in Toronto (which is on the other side of the globe). But situations in which it is raining in Toronto are compatible with my current situation. So neither ‘It is raining in Toronto’ nor ‘It is not raining in Toronto’ is true in my current situation. Thus, bivalence fails for this situation and the situation is partial⁴³⁶.

The treatment for negation also allows for inconsistency. This requires the acceptance that a situation may be incompatible with itself :

Consider a situation s in which my dog Ramsey is both black and white all over at the same time [...] The situation s is not compatible with itself ! Put like this it sounds bizarre, but consider someone who says incompatible things about something, say, about the colour of Ramsey’s coat. This person’s representation of the world is incompatible with itself⁴³⁷.

We allow for A and $\neg A$ to both be true at some situation s : in the case in which A is true in s and s is not compatible with itself, with A failing to be true at every situation compatible with s , and so $\neg A$ is also true in s .

What constraints should we put on the relation of compatibility ? We just said that the relation should not be reflexive, so we do not want to force, for any s , Css . But it should be symmetrical, so for all situations s and t , we have :

if Cst , then Cts .

The symmetry « makes valid the inference at a situation from any proposition A to its double negation $\neg\neg A$ »⁴³⁸. So we have double negation introduction. But we need something different to make valid the converse rule of double negation elimination. Mares suggests to add the following postulate :

(star postulate) $\exists x(Csx \text{ and } \forall y(Csy \supset y \sqsubseteq x))$.

From here we can define the so-called ‘Routley star operator’ : « the star of a situation s , s^* , is the maximal situation compatible with s . The star postulate ensures that there is an s^* for every situation s »⁴³⁹. The star postulate and operator enable to derive the following equivalence in the semantics :

‘ $\neg A$ ’ is true in s iff ‘ A ’ fails to be true in s^* .

The following postulate enables us to make valid double negation elimination :

$s^{**} = s$.

The intuitive interpretation of the postulate is that it « takes a situation to the maximal situation that is compatible with it »⁴⁴⁰.

⁴³⁶ MARES, E.D., *RLPI*, p. 75.

⁴³⁷ *Ibidem*.

⁴³⁸ *Ibidem*, p. 78.

⁴³⁹ *Ibidem*, p. 79.

⁴⁴⁰ *Ibidem*.

In his formal presentation, in appendix B, Mares also adds the following postulates that do not seem to be discussed in the course of the book :

(1) if $s \leq t$ then $t^* \leq s^*$. This seems reasonable, the less information a situation s contains, the more can be contained in its star mate (the maximal situation compatible with it), since its mate can « say more » without turning incompatible with it. Furthermore, since t extends s , t^* should not contain information that is not in s^* . So $t^* \leq s^*$ seems to appear reasonable.

(2) if $Rstu$, then Rsu^*t^* .

We can complete our presentation of section 6.4 in the following way.

Presentation 6.7 : The R-M semantics for R

An *R-frame* is a quadruple $\langle sit, logical, R, C \rangle$ such that $\langle sit, logical, R \rangle$ is a positive relational frame and C is a binary relation on *sit*. We add to the definitions and postulates of section 6.4 the following postulates :

(8) for each situation s there is a unique $\leq$ -maximal situation s^* such that Css^* ;

(9) if $Rstu$, then Rsu^*t^* ;

(10) if $s \leq t$ then $t^* \leq s^*$;

(11) $s^{**} = s$.

An *R-model* is a quintuple $\langle sit, logical, R, C, v \rangle$ such that $\langle sit, logical, R, C \rangle$ is an *R-frame* and v is a value assignment, that is, a function from propositional variables to sets of situations. The valuation satisfies all the clauses expressed in section 6.4, and the new clause for negation :

(v) $s \models_v \neg A$ iff $\forall x (Csx \supset x \not\models_v A)$.

We note that the proviso in clause (8) that the $\leq$ -maximal situation s^* be unique is not discussed as such in the core text of the book and is in fact very weird⁴⁴¹. It means that there is *only* one situation that is a $\leq$ -maximal situation compatible with s . But arguably, if my situation s doesn't contain any information on the weather in Toronto, then there is a $\leq$ -maximal situation compatible with it in which it is raining in Toronto, and a $\leq$ -maximal situation compatible with it in which it is not raining in Toronto. But the requirement that the $\leq$ -maximal situation s^* be unique seems, to our understanding, linked to the clause (11), according to which $s^{**} = s$. And this forces s^* to make both « it is raining in Toronto » *and* « it is not raining in Toronto » true. Mares writes :

⁴⁴¹ Thanks to Pr. Verdée for pointing this out !

Should we believe that, for every situation s , there is some maximal situation s^* that is compatible with it ? I am not sure how to argue for either an affirmative or a negative answer to this question. Perhaps the intuitiveness of double negation elimination supports the acceptance of the star postulate. On the other hand, perhaps relevant logicians should take seriously the idea of abandoning double negation elimination and some of the DeMorgan laws and instead adopting some intuitionist-like negation⁴⁴².

Which one of the semantics or the classical negation of **R** is to blame for those unnatural results ? We leave the question open to our reader and we continue with our presentation.

Mares notes that we can define a worldly situation as « a logical situation that exactly covers a world. That is, a logical situation s is worldly if and only if there is some world w such that s minimally obtains at w and, for all situations t that obtain at w , $t \leq s$ »⁴⁴³. We can « postulate that the worldly situations are those situations that are *maximally consistent* [...] they are consistent, so they are compatible with themselves, but they are also complete »⁴⁴⁴. Worldly situations, indeed, are just the equivalent of possible worlds in our framework ! So if s is worldly and Cst , then t is a part of s ($t \leq s$). We can see that this should be the case with the following consideration :

Suppose that s is worldly and Cst . That is, t is compatible with s . Assume, for the sake of a reductio, that t is not part of s . Then t contains some information that is not in s . Thus, there is information that is not in s but does not conflict with the information in s . But this means that s is not complete. And so, if worldly situations are to be considered complete, then they are only compatible with situations that are their parts⁴⁴⁵.

For all worldly situations s , we have that $s = s^*$, so we recover a classical negation. Indeed, the principle of bivalence and of consistency both hold at worldly situations. Respectively, for all worldly situations s :

« either ‘ A ’ or ‘ $\neg A$ ’ is true in s » (with the scheme of excluded middle, ‘ $A \vee \neg A$ ’ true) and

« it is not the case that both ‘ A ’ and ‘ $\neg A$ ’ are true in s » (with the scheme of the law of non-contradiction, ‘ $\neg A \wedge A$ ’ true).

In fact, at worldly situations, we have all the theorems of **CL** expressed without the implicational connective true. This is to be expected since they are logical situations, and our logic contains all the theorems of **CL** written without the implicational connective.

⁴⁴² MARES, E.D., *RLPI*, p. 79.

⁴⁴³ *Ibidem*.

⁴⁴⁴ *Ibidem*, p. 80.

⁴⁴⁵ *Ibidem*.

We note that the semantics can be proven sound and complete with respect to **R**. It has been built specifically to obtain this result⁴⁴⁶.

6.8 Extending the semantics⁴⁴⁷

It is possible to arrange the semantics to include the treatment of modal operators, as well as the treatment of quantification. To treat modality, we first need to extend our base logic **R** to include in its language the unary propositional operator ' $\Box$ ', with the appropriate rule of formation for wffs. Mares then adds to **R_h**⁴⁴⁸:

a defined operator : (Def $\Diamond$) : $\Diamond A \equiv_{df} \neg \Box \neg A$;

the following axioms :

C $(\Box A \wedge \Box B) \rightarrow \Box(A \wedge B)$;

DV $\Box(A \vee B) \rightarrow (\Box A \vee \Diamond B)$;

and the following rule :

RM : from $\vdash A \rightarrow B$, infer $\vdash A \rightarrow \Box B$. This makes our logic a regular modal logic if we read the definition with relevant implication ' $\rightarrow$ ' instead of material implication ' $\supset$ '.

Then we adapt our frames to add a binary accessibility relation on situations. The modality is treated here alike in possible worlds semantics. An RC-frame is a quintuple $\langle sit, logical, R, C, M \rangle$ where $\langle sit, logical, R, C \rangle$ is an R-frame and M (for modality) is a binary accessibility relation such that :

(12) if Mst , then Ms^*t^* ;

(13) if $s \preceq s'$ and $Ms't'$, then there is a $t \preceq t'$ such that Mst .

An RC-model is a sextuple $\langle sit, logical, R, C, M, v \rangle$ such that $\langle sit, logical, R, C, M \rangle$ is an RC-frame and v is a value assignment, that is, a function from propositional variables to sets of situations. The valuation satisfies all the clauses expressed in section 6.4 and 6.7, and the new clause for the modal operator :

(vi) $s \models_v \Box A$ iff $\forall x(Msx \supset x \models_v A)$.

The postulate (12) enables to make valid the derivable clause :

(vii) $s \models_v \Diamond A$ iff $\exists x(Msx \text{ and } x \models_v A)$.

We can add conditions on the binary accessibility relation in order to capture the addition of desired axioms. We have :

⁴⁴⁶ Our reader can find the proof in RIVENC, F., *ILP*, pp. 133-142 or in READ, S., *PEI*, chapter 5.

⁴⁴⁷ From MARES, E.D., *RLPI*, pp. 211-212 ; 96-109.

⁴⁴⁸ From *ibidem*, p. 211. The presentation starts with a Hilbert-style presentation of **R**.

(N) from $\vdash A$, infer $\vdash \Box A$ (the rule of necessitation), made valid by the postulate : $\forall x \forall y ((x \in \text{logical} \text{ and } Mxy) \supset y \in \text{logical})$.

And identical conditions than those expressed in section 5.3 for the other axiom schemes. We simply use the letter ‘M’ rather than ‘R’ to represent the binary accessibility relation, link situations rather than worlds ; and have our axiom schemes use our relevant implicational connective rather than material implication. So for **T** ($\Box A \rightarrow A$), we have the condition $\forall s(Mss)$, etc.

It is also possible to extend the semantics as well as the syntactic systems with quantifiers (‘ $\forall$ ’ and ‘ $\exists$ ’). This task, however goes beyond the scope of the present work. We simply note that the extension, from a semantical perspective, is notoriously tricky. Mares adds 21 postulates issuing from Fine’s presentation⁴⁴⁹. One of the important insights of this semantics is that it « modifies the standard interpretation of the universal quantifier »⁴⁵⁰. If we take a relation Qst as holding only if the domain of s is a subset of the domain of t , then the truth condition for the universal quantifier is⁴⁵¹ :

‘ $\forall x A(x)$ ’ is true at s if and only if for all t such that Qst , every individual in the domain of t satisfies ‘A’ at t .

With the appropriate conditions on Q , Fine is able to show that all the situations related to s by Q are conservative extensions of s : they say the same things than s in s ’s vocabulary. Those considerations end the present chapter. We will discuss the significance of the present semantics in the last part of the present work.

6.9 Summary of part II

Our part II has been focused on providing new tools to get a better grasp on, and find solutions to, the problems of **R** encountered in our part I. Chapter 4 has clarified the notion of ‘group of premises’, and has in particular shown (1) the close link between intensional notions and implication in **R**, (2) the fact that we may distinguish, in **R**, two ways of grouping premises (and conclusions), (3) that differentiating between operational and structural rules enable one to appreciate some syntactic systems’ differences only in terms of the latter.

Chapter 5 introduced our view of semantics, that can be used for more philosophical and more technical applications by the logician. We followed Copeland in holding that

⁴⁴⁹ Kit Fine presented the extended semantics in FINE, K., « Semantics for quantified relevance logic » in *Journal of Philosophical Logic*, 17 (1988) and ANDERSON, A.R., BELNAP, N., and DUNN, M., *Entailment, the Logic of Relevance and Necessity*, vol. II, §38, §48.9 and §52-53 (those paragraphs are written by Kit Fine).

⁴⁵⁰ RIVENC, F., *ILP*, p. 249.

⁴⁵¹ From MARES, E.D., *RLPI*, p.115.

for most philosophical purposes, the semantics proposed should be interpreted in terms of an iterative and faithful property. We then presented possible worlds semantics, their virtues, the criticism of ‘fuzziness’ they had received, and the different views one could endorse on their ontological significance. On this last point, we argued that possible worlds were best seen as ersatz or tools for gaining intelligibility about modal talks.

The present chapter has been concerned with Edwin Mares’ interpretation of the Routley-Meyer semantics. We have seen that the semantics used situations, which, unlike possible worlds, were not always consistent (so not always possible), and not always complete (so not always worlds). We have provided the different interpretational elements of Mares, most notably, we noted that implication could be treated in inferential and informational terms. In particular, an implication of the form $A \rightarrow B$ in a situation is an informational link, that is, « information that tells us that if there is a situation in this world in which a proposition A holds, then there is also a situation in the same world in which B holds »⁴⁵². We then presented the introduction of negation in the semantics, which could not be done using the classical truth condition for negation. We noted that some of the clauses relative to the star operator required for the occasion were unintuitive.

Our part III will deepen some of the philosophical questions left open in the course of our presentation. In particular, our next chapter will answer the question that we still haven’t faced, on the nature of the present interpretation : is Mares’ interpretation a pure or an applied semantics ? We will also discuss Geach’s challenge, that still hasn’t been faced since its acknowledgement in section 3.3. We will see that a semantics for **R** can come handily in a response to it. Our chapter 8 will focus on several notions, whose importance will only start to get obvious in our next chapter : metalanguages, the issues concerning negation and the possibility of pluralism.

⁴⁵² MARES, E.D., *RLPI*, p. 395.

Part III

Perspectives on **R**

7 – On semantics and Geach's challenge

8 – Metalanguage and pluralism

Chapter 7 : On semantics and Geach's challenge

Our presentation of **R** completed, we can now focus on some philosophical aspects linked to it. Some of them have explicitly drawn our attention through this work, and some others have only been hinted at. The present chapter is dedicated to a discussion of the semantics just presented, and to the answers relevantists have formulated to Geach's challenge.

7.1 Mares' interpretation of the R-M semantics

Our first task is to examine whether Mares' interpretation of the R-M semantics lives up to its promises. That is, we want to answer the following question : does the interpretation provided in chapter 6 enable us to accept the R-M semantics as an applied semantics ? The two conditions required for this acceptance are, as we've seen, the *iterative* and the *faithful* character of the property used to interpret the semantics. But what property should be talked about exactly ? In the models, the formulae of the semantics are given truth values at indices, with some of these indices having a special status, and three relations on those indices were provided (if we consider the R-frames rather than the RC-frames). The indices are interpreted as being situations, the special situations being logical situations (or, provided slight modification, worldly situations) and the relations on those situations are interpreted as :

- a part-whole relation for « $\leq$ » ;
- a relation of inclusion in an intersection of neighbourhoods for R ;
- a compatibility relation for C (with the star of a situation interpreted as the maximal situation that is compatible with it).

It seems that our natural candidate for the iterative and faithful property should be the truth of formulae relativised to situations, whose relations Mares is trying to clarify. But there is a complication : recall that the notion of information has also been introduced. So situations have, in fact, a dual nature⁴⁵³. The context of Mares' remark on the dual nature of situations is in itself interesting, and it is linked to negation, with which we will be concerned soon, so we expose it briefly here.

Williamson and David Lewis have provided two closely related arguments intended to show that there was something contradictory in, respectively, denying bivalence or accepting situations which are inconsistent⁴⁵⁴. Both the arguments use a certain step, which Mares denotes (N)⁴⁵⁵ :

⁴⁵³ See MARES, E.D., *RLPI*, p. 86.

⁴⁵⁴ See WILLIAMSON, T., *Vagueness*, pp. 188-189 and LEWIS, D.K., *On the Plurality of Worlds*, p. 7.

⁴⁵⁵ See MARES, E.D., *RLPI*, p. 84.

(N) $s \models_v \neg A$ if and only if $\text{Not } s \models_v A$.

The arguments are then formalized in the following way by Mares :

Arguments 7.1 : Williamson and Lewis' arguments⁴⁵⁶

Williamson's argument :

- | | |
|---|----------------------------------|
| 1. $\text{Not } (s \models_v A \text{ or } s \models_v \neg A)$ | hyp. (denial of bivalence for s) |
| 2. $\text{Not } (s \models_v A \text{ or } \text{Not } s \models_v A)$ | 1, (N) |
| 3. $\text{Not } s \models_v A \text{ and } \text{Not } \text{Not } s \models_v A$ | 2, DeMorgan's law |
| 4. $\text{Not } s \models_v A \text{ and } s \models_v A$ | 3, Double negation elimination |

Lewis' argument

- | | |
|---|---|
| 1. $s \models_v A \wedge \neg A$ | hyp. (there is an inconsistent situation) |
| 2. $s \models_v A$ | 1, Truth Cond. for conjunction |
| 3. $s \models_v \neg A$ | 1, Truth Cond. for conjunction |
| 4. $\text{Not } s \models_v A$ | 3, (N) |
| 5. $s \models_v A \text{ and } \text{Not } s \models_v A$ | 2,4 Truth Cond. for conjunction |

Since Mares has « adopted truth value gaps (in some situations) and the existence of inconsistent situations »⁴⁵⁷, he needs a response to these arguments. Now, he cannot simply « claim that the 'not' used in these principles is classical. For if we were to replace it with relevant negation, the same arguments would hold »⁴⁵⁸. Rather, what Mares denies is (N), « the assumption that our object language negation ought to be interpreted in this straightforward way in terms of a metalinguistic negation »⁴⁵⁹. Why ? Clearly, the opposite would seem natural, because if we look for example at the clause for conjunction, « our object language conjunction is interpreted in terms of a metalinguistic conjunction »⁴⁶⁰ :

$s \models_v A \wedge B$ iff $s \models_v A$ and $s \models_v B$.

The answer lies, precisely, in the dual nature of situations. The situations are indeed « the primary indices at which sentences are true or false. But they also provide the basis

⁴⁵⁶ From MARES, E.D., *RLPI*, pp. 85-86.

⁴⁵⁷ *Ibidem*, p. 86.

⁴⁵⁸ *Ibidem*.

⁴⁵⁹ *Ibidem*.

⁴⁶⁰ *Ibidem*.

for a theory of information »⁴⁶¹ : a situation s makes a sentence ‘ A ’ true if and only if s contains the information that A . And so object language negation cannot be interpreted as a metalinguistic negation because it is not the case that « a situation contains the information that A if and only if it does not contain the information that not- A »⁴⁶². If we reconsider the situation, denoted s for the occasion, of Mares writing his book in his office in Wellington, we may ask the two following questions : « (1) is it raining now in Toronto ? (2) Does s contain the information that it is raining in Toronto ? »⁴⁶³ the answer to question (2) is simple : no, but (1) has him stumped, Mares has « no idea what the weather is in Toronto right now »⁴⁶⁴. So from the fact that a situation does not contain the information that A , we cannot infer that it contains the information that $\neg A$. This corresponds to the rejection of one half of (N), from right to left :

(Nrl) if not $s \models_v A$, $s \models_v \neg A$.

What about the other half ? From left to right, we have :

(Nlr) if $s \models_v \neg A$, not $s \models_v A$.

Its rejection requires (1) to « justify the acceptance in our models of inconsistent situations, i.e., situations that are incompatible with themselves »⁴⁶⁵, acceptance which was exposed in section 6.7 and (2) to « show that there is an intelligible semantics of negation that does not entail (Nlr) »⁴⁶⁶, which is precisely accomplished by the relation of compatibility C .

From those considerations, it seems like there are, in Mares’ framework, two ways to interpret $s \models_v A$. The first is in terms of truth, $s \models_v A$ means that A is true at situation s according to valuation v . The second is in terms of information, $s \models_v A$ means that the information A is contained, or accessible, at situation s . Which of « being true at » or « being an information contained in » is the property relativized to situations that we should expect to be iteratively transmitted in the semantics and get a faithful meaning to the connectives ? The question is not frontally answered by Mares, and this is probably a weakness of his presentation. Maybe he would argue that both properties can be attached to the formulae in a faithful and iterative way, and are anyways closely connected, so this should not matter much. But on the other hand, the justification for the rejection of (N) partly relies, it seems, on the reading of $\models_v$ as informational, so it may be the more

⁴⁶¹ MARES, E.D., *RLPI*, p. 86.

⁴⁶² *Ibidem*.

⁴⁶³ *Ibidem*, pp. 86-87.

⁴⁶⁴ *Ibidem*, p. 87.

⁴⁶⁵ *Ibidem*.

⁴⁶⁶ *Ibidem*. With slight notational adjustment.

primitive reading. On those remarks we end the present section, and leave to our reader the care to draw the conclusion she considers appropriate.

7.2 Geach's challenge⁴⁶⁷

We discussed in section 3.3 the rejection of disjunctive syllogism (DS) by the system **R**. Recall that DS was the inference of 'B' from ' $A \vee B$ ' and ' $\neg A$ '. We have pondered the problematic aspect of this rejection by noting that our uses of « or » were equivocal, so that we should not always be able to infer 'B' from 'A or B' and 'not A'. In particular, Read noted⁴⁶⁸ that the truth-functional reading of 'or' accepted addition, the inference of ' $A \vee B$ ' from 'A' or from 'B', but not DS ; and that the intensional reading of 'or' accepted DS, the inference of 'B' from 'A' and ' $A + B$ ', but not addition, the inference of ' $A + B$ ' from 'A' or from 'B'. This was indeed used to block C.I. Lewis' argument showing that from a contradiction, anything whatsoever followed.

Geach's challenge, in this context, is to provide an example where the « good old » disjunctive syllogism, with its extensional ' $\vee$ ', does not hold. To him, A&B did not provide such example and « it was clear that none could be forthcoming »⁴⁶⁹. Geach writes : « it would be difficult to describe even a possible set-up in which [...] ' $A \vee B$ ' should be true but both its disjuncts false »⁴⁷⁰. Indeed, if it is the case that A and B are both false, « then ' $A \vee B$ ' is false too. So contraposing, if ' $A \vee B$ ' is true, at least one of A, B is true »⁴⁷¹. And so if A is false, it must be B that is true, or the other way around. Period. Several answers to the challenge can be proposed. All of them must acknowledge the fact that the connective here is indeed the truth-functional connective ' $\vee$ ' which is at stake, « for the starting point is the relevant logician's denial that Disjunctive Syllogism, or detachment for ' $\supset$ ', is in the classical logician's sense of ' $\vee$ ' and ' $\supset$ ', valid »⁴⁷².

7.2.1 The dialetheic response

The first answer is the dialetheic response⁴⁷³. Recall that a dialetheist is a person who believes that there are true contradictions in our world. So she believes that some propositions are both true and false. Her counterexample to DS assumes a proposition A that is both true and false, and a proposition B that is simply false. Then the proposition

⁴⁶⁷ All this section stems from READ, S., *PEI*, pp. 31-34 ; 131-143. We slightly adapted the formalism in our citations when judged necessary (for example, we replaced his extensional « & » connective with our « $\wedge$ » in order to not confuse our reader).

⁴⁶⁸ See *ibidem*, pp. 31-34.

⁴⁶⁹ *Ibidem*, p. 132.

⁴⁷⁰ GEACH, P., « Review of *Entailment* » in *Philosophy*, 52 (1977), p. 495.

⁴⁷¹ READ, S., *PEI*, p. 132.

⁴⁷² *Ibidem*, p. 133.

⁴⁷³ See *ibidem*, pp. 133-134.

$A \vee B$ is both true and false, true because A is true and false because both A and B are false. $\neg A$ is also true because A is false (and it is also true because A is false). So we have that ' $A \vee B$ ' and ' $\neg A$ ' are true while B is false, which constitutes a counterexample to DS.

Read notes that « Geach did not conceive such a response »⁴⁷⁴ and that « there are at least two reasons why he should not »⁴⁷⁵. The first one is that he did not believe that such a proposition as the assumed A could be both true and false. The second one is that « it is no part of the relevantist challenge to classical logic, and so to the validity of DS, to claim that some propositions are both true and false »⁴⁷⁶. Now, the dialetheist needs extra evidence to support her claim that there are such contradictory propositions as A that are both true and false. She will find some, for example, in « the many and varied semantic paradoxes »⁴⁷⁷, such as the liar paradox, stating « this sentence is false » or a proposition construed in a similar way⁴⁷⁸. But the use of such move by the relevantist has laid her open « to irrelevant classical accusations and objections to positions to which, qua relevantist, [s]he is not committed »⁴⁷⁹. Indeed, at no point did the relevantist construct her system **R** to account for the existence of contradictions. She constructed it to avoid the paradoxes of material implication and non-sequiturs validated by **CL**. So, *as a relevantist*, she cannot simply give the answer of a *dialetheist*.

7.2.2 The Australian plan

Two semantical answers to the challenge have been provided by relevantists, answers that are not without connexion with the dialetheist's. Those two answers rely on two « recipes for particular way of doing semantics »⁴⁸⁰, and have been « affectionately dubbed the Australian and the American plans »⁴⁸¹, in reference, we believe, to the geographical location of their authors. The first of these answers, on the Australian plan, is provided by the insights of our previous chapter.

The counterexample can be constructed as follows. In a situation s , we let the value of ' $A \vee B$ ' be true. So the value of one of A , or B , is true, since $s \models_v A \vee B$, iff $s \models_v A$ or $s \models_v B$. We also want B false, so « it had best be A which is true »⁴⁸². But we want $\neg A$ to be true as well for our counterexample. This is permitted by our star operator and our

⁴⁷⁴ READ, S., *PEI*, p. 133.

⁴⁷⁵ *Ibidem*.

⁴⁷⁶ *Ibidem*.

⁴⁷⁷ *Ibidem*, p. 134.

⁴⁷⁸ For more information on the liar paradox and related paradoxes, our reader can consult KIRKHAM, R. L., *Theories of truth, A critical introduction*, chapter 9.

⁴⁷⁹ READ, S., *PEI*, p. 134.

⁴⁸⁰ *Ibidem*, p. 131.

⁴⁸¹ *Ibidem*.

⁴⁸² *Ibidem*, p. 134.

truth condition for $\neg A$. This is how the present plan keeps the metalanguage consistent : in the dialethic answer, the contradiction indeed permeated to the metalanguage, by accepting the classical truth condition for negation, $v(\neg A) = T$ iff $v(A) = F$, it was required that A be both true and not true.

In our R-M semantics, the value of $\neg A$ is given by some « less gross property of A »⁴⁸³ : $s \models_v \neg A$ iff $\forall x(Csx \supset x \not\models_v A)$. We need a situation s where A is true, and the requirement that all the compatible situations with s do not give to A the value true. Recall that the ‘star’ apparatus enabled us to derive, in our semantics, the truth condition :

‘ $\neg A$ ’ is true in s iff ‘ A ’ fails to be true in s^* .

So we do not directly require A to be « both true and not true »⁴⁸⁴, we require A to be true and $\neg A$ to be true at s . A enables us to derive $A \vee B$ at s and the fact that A fails to be true at s^* enables us to have $\neg A$ true at s as well. Hence, we have constructed our counterexample to DS.

7.2.3 About truth-functionality

Before we jump to the American plan, a little twist may be welcome. In the R-M semantics, negation may be seen as treated in a non-truth-functional way. If we define a truth-functional connective as a connective whose truth value depends on the truth value « of its immediate constituents, and [on] them alone »⁴⁸⁵, then our negation is not truth-functional. The value of $\neg A$ in a situation (at an index) does not entirely depend on the value of A in the same situation (at the same index).

Should we conclude that relevant negation is non truth-functional ? Read introduces an interesting perspective on the question. Consider the connective « $\wedge$ ». We may say that it is a truth-functional connective, if anything is. Read says :

Is there such a thing as a truth-functional connective? That may seem a surprising question. Of course there are truth-functional connectives: $\vee$, $\wedge$, $\supset$ are just three. One reason we say this is that if someone were to come along with a theory of ‘ $\wedge$ ’, say, which was not truth-functional, we would reply along roughly these lines: that may or may not be a very interesting theory about something – ‘and’, say – but, since it is not truth-functional, it is not a theory of ‘ $\wedge$ ’. ‘ $\wedge$ ’ is by definition truth-functional⁴⁸⁶.

Read has two worries with this line of thought. First it makes sense to ask if the connective ‘ $\wedge$ ’ really is truth-functional in the same way as « it makes sense to ask whether Aristotle – whom we pick out as the chap who wrote the *Metaphysics* – might

⁴⁸³ READ, S., *PEI*, p. 135.

⁴⁸⁴ *Ibidem*.

⁴⁸⁵ *Ibidem*.

⁴⁸⁶ *Ibidem*.

not actually have written that book »⁴⁸⁷. His second worry is this : « it is not that I wonder whether certain connectives do or do not have the appropriate properties to count as truth-functional. What I doubt is whether any single clear sense makes such a decision possible »⁴⁸⁸. The above characterization of a truth-functional connective, as having its truth value depend only on the truth value(s) of its constituent(s) « presupposes that the logic, and the connective, is presented semantically »⁴⁸⁹, but logics are not always constructed with a semantics already specified. So « although there may be a semantics for a system in which a connective is deterministically assigned such a truth-function, there may be another for the calculus in which it has a different treatment »⁴⁹⁰. Our ‘ $\neg$ ’ connective had, as we’ve seen in section 1.3, a purely truth-functional treatment in **CL**’s semantics. But we can build a semantics that does not give it such treatment, and that nevertheless captures a fragment of classical logic.

This can be done, for instance, for a logic **C**, with connectives « $\neg$ » and « $\supset$ », and where, in our semantics, our frame is a non-empty set W of indices. A valuation for our frame is a double $\langle W, v \rangle$ such that⁴⁹¹ :

$$v(\neg A, w) = \begin{cases} \text{T if for some } w' \in W, v(A, w') = \text{F} \\ \text{F otherwise.} \end{cases}$$

$$v(A \supset B, w) = \begin{cases} \text{T if either for some } w' \in W, v(A, w') = \text{F, or for all } w' \in W, v(B, w') = \text{T.} \\ \text{F otherwise.} \end{cases}$$

Indeed, it can be shown that « A is valid in **C** if and only if A is classically valid »⁴⁹². We can try, alternatively, to « find some syntactic analogue of truth-value »⁴⁹³, like the following inferences capturing the truth-functional behaviour of material implication :

$$Q \vdash P \supset Q$$

$$\neg P \vdash P \supset Q$$

$$P, \neg Q \vdash \neg(P \supset Q)$$

In words, « if Q is true or P is false, ‘ $P \supset Q$ ’ is true ; while if P is true and Q false, ‘ $P \supset Q$ ’ is false »⁴⁹⁴. But as Read notes, « this builds in implicitly that negation is truth-

⁴⁸⁷ READ, S., *PEI*, pp. 135-136.

⁴⁸⁸ *Ibidem*, p. 136.

⁴⁸⁹ *Ibidem*.

⁴⁹⁰ *Ibidem*.

⁴⁹¹ With adaptations, from *ibidem*. This is « Gödel’s translation of **C** into the semantics of **S5** », *ibidem*. See GÖDEL, K., « An Interpretation of the Intuitionistic Sentential Logic » in *The Philosophy of Mathematics* (Hintikka, J. ed.), pp. 128-129.

⁴⁹² READ, S., *PEI*, p. 136.

⁴⁹³ *Ibidem*.

⁴⁹⁴ *Ibidem*, p. 137.

functional. For we are assuming that ‘not-P’ is true if and only if P is false »⁴⁹⁵. Read suggests that we first characterize truth-functionality of a connective with respect to a certain semantics and then call a connective truth-functional *simpliciter* if it can be characterized truth-functionally in a semantics. Thus, relevant negation is truth-functional simpliciter because the American plan characterizes it truth-functionally. And this can be used as an argument against the classicist’s objection that relevantists haven’t reached their target in their counterexamples for DS because their negation is not truth-functional, unlike classical negation. Both can be seen as truth-functional simpliciter.

7.2.4 The American plan

According to the defenders of the American plan, « the fact that even with the star-semantics, a proposition and its negation may both be false (or both true) shows that the complication of star is unnecessary. For we can avoid the lack of functionality in simple-minded more easily »⁴⁹⁶. How can this be accomplished ? By admitting that propositions can take one of four truth values. Those values are : True, False, Both, and Neither. And they are obtained by combinations of the two usual truth values : true and false (in technical terms, they are « elements of the power set of the two usual values »⁴⁹⁷).

Let’s follow read and denote the two basic, usual truth values that can be combined, as ‘little true’ and ‘little false’. We will differentiate it with ‘big True’ and ‘big False’, which are distinguished, combined truth values. A formula A has value :

‘big True’ if and only if it is only ‘little true’ ;

‘big False’ if and only if it is only ‘little false’ ;

Both if and only if it is ‘little true’ and ‘little false’ ;

Neither if and only if it does not have the value ‘little true’ nor the value ‘little false’.

It is important to note that the « recursion on the connectives is governed by the normal rules for (little) truth and [little] falsity »⁴⁹⁸. So $\neg A$ is ‘big True’ (‘little true’ only) if and only if A is ‘big False’ (‘little false’ only), it is ‘big False’ if and only if A is ‘big True’, it is Neither if and only if A is Neither, and Both if and only if A is Both. We can summarize the truth dependence in a table, where ‘T’ stands for ‘big True’, ‘F’ for ‘big False’, ‘N’ for ‘Neither’, ‘B’ for ‘Both’, ‘t’ for ‘little true’ and ‘f’ for ‘little false’ ; and where one of the basic truth value is assigned to a formula when there is a ‘1’ in the row of its valuation ; and it is not assigned to a formula when there is a ‘0’ in the row of its valuation. The dependency of the truth value of $\neg A$ on the truth value of A is shown :

⁴⁹⁵ READ, S., *PEI*, p. 137.

⁴⁹⁶ *Ibidem*, p. 138.

⁴⁹⁷ *Ibidem*.

⁴⁹⁸ *Ibidem*.

Table 7.2 : Truth values in the American plan

A			$\neg A$		
t		f	t		f
1	T	0	0	F	1
1	B	1	1	B	1
0	N	0	0	N	0
0	F	1	1	T	0

We can build our counterexample to DS in the following way. Let B have the value ‘big False’ and A have the value ‘Both’. Then ‘ $A \vee B$ ’ takes the value ‘Both’ and ‘ $\neg A$ ’ takes the value ‘Both’. Since « truth-preservation, required for validity, is just that, (little) truth-preservation, and not (big) Truth-preservation »⁴⁹⁹, we have that « A follows from some hypotheses if and only if A is neither False nor Neither when all the hypotheses are either True or Both »⁵⁰⁰. And so DS is not truth-preserving : the hypotheses ‘ $A \vee B$ ’ and ‘ $\neg A$ ’ are Both, and the conclusion to draw ‘B’ is False.

7.2.5 Read’s dissatisfaction

Those two plans do not satisfy Read. He writes :

The dialethic account relies on the counterintuitive idea that some propositions are both true and false. The Australian plan interprets this as saying that a proposition and its negation might both be true ; and the American plan, as saying that Bivalence fails, and there are four possible semantic values. They thus give to relevant logic a semantic strangeness of which neither its original philosophical motivation nor its syntactic formulation (in particular, its natural deduction presentation) would give any hint [...]. This unfortunate connection between relevant ideals and dialetheism can be traced in each case to another feature which the three semantic plans mistakenly share : the Classical Account of Validity⁵⁰¹.

Now, it has also been argued that DS was acceptable in certain conditions. The counterexample provided by the Australian plan requires a proposition and its negation to be true at the same indice, and the counterexample provided by the American plan requires a proposition to have the value Both. So on the hypothesis of ‘local consistency’, that there are, in fact, no contradictions, one may use DS⁵⁰². But for Read this is still a

⁴⁹⁹ READ, S., *PEI*, p. 138.

⁵⁰⁰ *Ibidem*.

⁵⁰¹ *Ibidem*, p. 139.

⁵⁰² *Ibidem*, p. 140.

confused move from the relevantist, because there is a gap between her reasons for endorsing a relevant logic and the explanation she gives for his rejection of DS. He adds :

The reasons for giving a formal theory containing a non-truth-functional conditional, namely, that the Classical Account of Validity and the Classical Deduction Equivalence entail the identification of ‘if’ with material implication and so validate invalid arguments, give no reason to suppose there are such contradictory propositions [as those supposed by the Australian and the American plans]. This should make one suspect that some sleight of hand has gone on the above reasoning towards eclecticism⁵⁰³.

We may not be totally convinced by Read’s criticism. Lewis’ argument, showing that from $A \wedge \neg A$ one can derive any B whatsoever, is a typical fallacy of relevance. And we can justify the rejection of DS on the basis that it is a necessary loss to avoid this particular irrelevant deduction. Furthermore, we understand directly that the rejection of DS is linked to inconsistent contexts, since the irrelevant conclusion has been obtained, in this particular argument using DS, from the assumption of a contradiction. And as Mares shows⁵⁰⁴, we do in fact sometimes find ourselves in situations with inconsistent pieces of information, for example, when dealing with fictions. And in those situations, we do not ruin the fiction by holding that anything whatsoever is true in that fiction. So it is true that there are situations where we do not, in fact, validate DS, situations precisely such that the usual presupposition of consistence cannot be endorsed.

Now, one could use the criticism formulated in section 2.8 and hold that, precisely, what Lewis’ argument shows is that a contradiction is relevant to any proposition whatsoever. Recall that the argument used the duplicity of relevant criteria in **R** to establish that to « characterize valid inferences on the basis of a criterion of relevance was to put the cart before the horse »⁵⁰⁵. But this required the « validity, otherwise recognized »⁵⁰⁶ of the inferences at hand. And as the examples Mares provided show, it is far from obvious that the validity of Lewis’ argument is « otherwise recognized » : we do not use it when inconsistent information is available⁵⁰⁷.

7.2.6 The Scottish plan

Read’s approach, that we may call the Scottish plan, consists in giving up the classicality of the metalanguage, which he has diagnosed as the source of confusion in the relevantists’ answers to Geach’s challenge. Iseminger noted indeed that Geach’s challenge was in fact not free of criticism : « to insist that the showing of invalidity (or

⁵⁰³ READ, S., *PEI*, p. 141.

⁵⁰⁴ See MARES, E.D., *RLPI*, pp. 178-180 ; 194-195.

⁵⁰⁵ RIVENC, F., *ILP*, p. 62.

⁵⁰⁶ *Ibidem*.

⁵⁰⁷ Also, Read’s position might be weaker than it seems on this matter, see section 8.5.

‘formal incorrectness’) requires the production of actual (or possible) counter-examples is just to assume the classical account of validity (or equivalence of validity and ‘formal correctness’) which is in question »⁵⁰⁸. Indeed, if we interpret the challenge as demanding to show an example where ‘ $A \vee B$ ’ and ‘ $\neg A$ ’ are true, and ‘ B ’ false ; with ‘and’ interpreted classically, no counterexample can be forthcoming. We’d get ‘ $\neg A \supset B$ ’ and ‘ $\neg A$ ’, which rules out the possibility for ‘ B ’ to be false. Consequently, on the Classical Account, it follows that ‘ $\neg A$ ’ and ‘ $\neg A \supset B$ ’ entail ‘ B ’. But Read rejects the Classical Account : the impossibility for ‘ $\neg A$ ’ and ‘ $\neg A \supset B$ ’ to be true and ‘ B ’ false « does not show that $[\neg]A$ and ‘ $[\neg]A \supset B$ ’ entail B »⁵⁰⁹.

If in the light of those considerations, Geach « insists that he meant ‘and’ extensionally (that is, in such a way that the falsity of its conjuncts suffices for the falsity of the conjunction) then we can, nay must, agree that no counterexample is possible »⁵¹⁰. But « we must also deny that that shows that DS is valid. This move would be as irrelevant to the validity of DS as the earlier consideration of intensional disjunction for its major premise »⁵¹¹.

Read holds that the correct way to group premises in valid arguments is not with an extensional conjunction ‘ $\wedge$ ’ but with an intensional conjunction ‘ $\bullet$ ’. His relevant account of validity is the following⁵¹² :

Account 7.3 : Read’s account of validity

A entails B if and only if A cannot be true and (meaning fuse, ‘ $\bullet$ ’) B false.

Read has, in substance, retained from the presentation in sequent calculus of **R** the idea that the link between the implicational connective and the grouping of premises was to be obtained through an intensional way to group the said premises. This is how he can dissolve the various non-sequiturs of the Classical Account. His version of the Deduction Equivalence is⁵¹³ :

B follows from $C \bullet A$ if and only if ‘if A then B ’ follows from C .

Said otherwise : $C \bullet A \vdash B$ if and only if $C \vdash A \rightarrow B$. It is not too hard to see from our considerations of chapter 4 that this holds for **R**. This is the way Reads uniquely

⁵⁰⁸ ISEMINER, G., « Is Relevance Necessary for Validity ? » in *Mind*, 89 (1980), p. 198. From READ, S., *PEI*, p. 141.

⁵⁰⁹ *Ibidem*, p. 142.

⁵¹⁰ *Ibidem*.

⁵¹¹ *Ibidem*.

⁵¹² See *ibidem*, pp. 127-128.

⁵¹³ See *ibidem*, p. 39.

characterizes the notion of deducibility and escapes their multiplication (remember the different notions present in the work of A&B) and notable problems (remember Arnon Avron's criticism of section 4.1).

So the answer of the Scottish plan to Geach's challenge is to change what should be required by the example, by changing the metalanguage and embrace a relevant one. What is needed to be shown, then, is that it is possible for the premises to be true fuse the conclusion false in an argument following the form of DS. Then we will have shown that the validity of DS is not established. Read writes : « what we must show is that, for example, ' $A \wedge \neg A$ ' can be true fuse B false, not simply their extensional conjunction. Fusion is not truth-functional, in particular, the fusion of two propositions may be true even though both are false »⁵¹⁴.

We must now provide an « instance A, B such that, for example, ' $(A \vee B) \wedge \neg A$ ' fuse ' $\neg B$ ' is true, that is, for which B does not follow from ' $A \vee B$ ' and ' $\neg A$ ' »⁵¹⁵. We can show that there are suitable A, B by instancing them appropriately, just like « the advocate of the dialethic plan can only reply by instancing suitable A : for example, 'This sentence is false' or 'The Russell set belongs to itself' »⁵¹⁶.

Read suggests the following example : « let A be 'Socrates was a man' and B be 'Socrates was a stone' »⁵¹⁷. It rightly follows from A that 'Socrates was a man or a stone', so we have ' $A \vee B$ '. But « it does not follow that if Socrates was not a man he was a stone »⁵¹⁸, indeed, « the conditional is a stronger statement than the disjunction interpreted truth-functionally »⁵¹⁹. So to infer correctly « from 'Socrates was not a man' (' $\neg A$ ') that Socrates was a stone (B), the premise 'Socrates was a man or a stone' (' $A \vee B$ ') will not suffice, for we need the stronger (but false) premise that if Socrates was not a man, he was a stone »⁵²⁰.

In short, we cannot derive ' $\neg A \rightarrow B$ ' which is required to obtain B from ' $A \vee B$ ' and ' $\neg A$ ' in this set-up : « hence the reasoning is blocked, and B does not follow from ' $A \vee B$ ' and ' $\neg A$ ', nor from A and its negation »⁵²¹. So is provided the counterexample to Geach's challenge by the Scottish plan.

⁵¹⁴ READ, S., *PEI*, p. 142.

⁵¹⁵ *Ibidem*.

⁵¹⁶ *Ibidem*.

⁵¹⁷ *Ibidem*.

⁵¹⁸ *Ibidem*.

⁵¹⁹ *Ibidem*, pp. 142-143.

⁵²⁰ *Ibidem*, p. 143.

⁵²¹ *Ibidem*.

7.3 A note on Read's interpretation

We have said that Read took intensional conjunction as a primitive for his analysis of implication, where Mares preferred to take implication. This is permitted because, according to Read « we have a pretheoretic understanding of fusion [...]. This pretheoretic understanding is demonstrated by our ability to group together premises in inferences »⁵²². This is how Read could escape the accusation of putting the cart before the horse, accusation that was formulated against A&B in their desire to base their notion of validity on criteria of relevance and not the other way around.

But Rivenc doubts that Read's strategy is free from criticism. The former writes :

This is where doubts are allowed: I am not sure that a definition of the notion of logical consequence (of the true and unique notion : Read [...] is 'pertinentist') from the fusion connector sheds all the light that Read expects from it. This connector retains something artificial : is it still truly an homophonic explanation of meaning when one characterizes the connector **o** by the idea of intensional conjunction ?⁵²³

Indeed, for Rivenc, the introduction of '**o**' is more of a theoretic construct, partly motivated by « the desire to find a way to reject EFQ formulated with classical negation »⁵²⁴. But, he continues, what is our motivation to reject (EFQ) if it is not the fact that it constitutes an obvious fallacy of relevance ? It is « certainly not due to a prior knowledge of fusion, which would show us that this mode of inference is not admissible »⁵²⁵.

So Rivenc does not think that fusion can play the role that has been attributed to it by Read. It cannot be « the enlightening, primitive and fundamental notion from which one can define or redefine the notion of logical consequence : the idea of relevance, however indecisive, remains a heuristic rule for our evaluations »⁵²⁶.

In the present chapter, we have seen that Mares' interpretation was in a rather good position to be considered as an applied semantics, even if it wasn't quite clear whether information or truth was its important interpretative, iterative and faithful property. This being said, the interpretation, as we've seen in section 6.7, needs some tweaking or extra justification concerning the way it treats the star of a situation.

We have also seen how his interpretation, as part of the Australian plan, could answer Geach's challenge. We have defended the idea that it was a fairly interesting way of

⁵²² READ, S., *PEI*, p. 171.

⁵²³ RIVENC, F., *ILP*, p. 181. A connective is homophonic if it has the same meaning in the object language and in the metalanguage : « same sound, same meaning », READ, S., *PEI*, p. 151.

⁵²⁴ RIVENC, F., *ILP*, p. 181.

⁵²⁵ *Ibidem*.

⁵²⁶ *Ibidem*.

answering the challenge. We have also discussed the American plan's answer to the challenge, plan which, as we've seen, has the particularities to (1) ground the fact that relevant negation can be seen as a truth-functional connective simpliciter, and (2) use 4 truth values obtained as combinations of the two traditional ones, but renamed for the occasion, 'little' true and 'little' false. Those answers did not satisfy Read, who found them mistaken in endorsing the Classical Account. His solution, therefore, was to embrace a fully relevant metalanguage to answer the said challenge. We ended our chapter by taking our distances with Read's interpretation, and wondering if intensional conjunction could endorse the role he gave to it in his interpretation.

We will discuss in the next chapter the reason why metalanguages seem to become very important in dealing with the philosophical issues surrounding our system **R**. We will see that the appeal to the metalanguage can be used to provide a certain perspective on the question of the nature of **R**'s negation. Nonetheless, we will argue that the question of its nature is very difficult to answer, given that the semantics and structural presentations give it such different treatments with respect to **CL**'s. But we will argue that this might not be an issue as significant as some may think, once we have resituated the debates in their contexts. We will then complete our presentations of Mares' and Read' positions, with a note on Mares' metalanguage and a note on Read's general philosophical position.

Chapter 8 : Metalanguage and pluralism

8.1 A word on metalanguages

It should not be too surprising to see Read appeal to the level of the metalanguage to provide a solution to Geach's challenge. The logician, we said it, develops tools for different purposes. Sometimes to analyse a particular notion, sometimes to provide a theory of validity, and sometimes to clarify natural languages. It can happen that a single logical theory can serve as a basis for different philosophical purposes (like **R** which has been defended as a theory of validity, as a basis for a theory of conditionals, or for the analysis of certain modalities⁵²⁷, etc.), or that a certain framework (like possible worlds semantics, or sequent calculus) enables one to unify what may appear as very different logical systems.

Anyways, the logician who develops those tools uses a certain formal language ; and at no moment does she completely « get out » of linguistic practice in talking about that language or analysing it. Said otherwise, the logician can only study a language *from within a language*. The present work is written in English, and so our only way to communicate *about* classical logic, the paradoxes of material implication, the system **R** of relevant implication, etc. is through the resources offered by that language. But part of the logical systems we discuss try precisely to explain what is a good argument 'in English' or, alternatively, in any natural language. So when we discuss what should be a good argument, what should be expected from a logical theory, and so on, the question arises of the theory of validity that pertains to that discussion. And so the object of investigation finds a way to impact the discussion about the object that is investigated. This is why discussions about metalanguages can have considerable philosophical significance for the logician.

8.2 About negation

An interesting result can be obtained if one takes a relevant metalanguage and considers the question of negation. From the way negation was introduced in the semantics presented in our section 6.7, it is at least open to debate whether the negation of **R** is classical. We can call the negation introduced in the R-M semantics the « De Morgan negation ». Recall that we cannot introduce a negation that would have the classical valuation clause for it without introducing some of the paradoxes **R** was

⁵²⁷ See MARES, E.D., *RLPI*, pp. 116-122 ; 125-160.

designed to avoid⁵²⁸. Let's call this negation the « Boolean negation ». It can be shown that if we adopt a relevant metalanguage (so where the metalanguage negation is to be interpreted as the De Morgan negation) with four truth values, such as in the American plan, the two negations merge together⁵²⁹. So it is possible for the relevantist to pretend that she « doesn't even understand »⁵³⁰ the argument to the effect that the De Morgan's negation is not the Boolean negation. But the « distinction shown to be unassignable in the object language is presupposed in the metalanguage, as a characteristic of a relevant metalanguage. It is precisely that point which is questionable »⁵³¹.

But the question remains : does the negation of relevant logic (the De Morgan's negation) have the same meaning as the negation of classical logic (the Boolean negation) ? The present work has shown evidence going both for a positive and a negative answer to this question. The structural presentation of **R** shares with the structural presentation of **CL** its operational rules for the connectives, and in particular the rules for negation. So if the meaning of connectives is to be read through those rules, then we have an argument to say that relevant negation is the same as classical negation. On the other hand, we said it, from the point of view of the R-M semantics, the two negations are different. But this idea can also be at least partially pondered by the fact that, as we noted in section 7.2.3, both could be said to be truth-functional *simpliciter*, since relevant negation was truth-functional in the American plan.

Part of the importance of this question stems from the classicist's objection to the relevantist that when she denies that, say, DS or (EFQ) are not valid principles of inference, she is not denying the *classicist's* principles of inference, for those principles use a Boolean negation and that the relevantist's theory uses the De Morgan's negation. This move has been famously expressed by Quine : « here [...] is the deviant logician's predicament : when he tries to deny the doctrine he only changes the subject »⁵³².

Rivenc ended his book wondering which of the semantical or the proof-theoretic (i.e. structural) answer was to be provided to the question. It is a difficult question, but it may end up not being that important. The classicist can always just deny that any negation not validating DS or (EFQ) is the classical negation, but we may wonder, what are her grounds for holding that a negation should validate them ? Said otherwise, may she insist that classical negation must be such and such, we may ask what are her grounds for

⁵²⁸ See MARES, E.D., *RLPI*, pp. 73-74 ; 92-93. Mares argues that we can also get to this conclusion by other means : by the fact that we cannot introduce it without adopting a « radically non-classical view of Boolean negation and reject[ing] the real motivation behind it », op. cit., p. 93.

⁵²⁹ See RIVENC, F., *ILP*, pp. 187-190.

⁵³⁰ *Ibidem*, p. 187.

⁵³¹ *Ibidem*, p. 190.

⁵³² QUINE, W.V.O., *Philosophy of Logic*, p. 81.

accepting a negation that is such and such as her negation ? Her reasons would probably come from the various virtues of **CL** that have been investigated in the course of the present work (and maybe from other virtues that we have not foreseen). So it is still possible to lead the debate on different grounds, and one may already hear Mares from afar, holding that there are at least some circumstances in which it is just not true that most humans reason according to classical standards (using DS or (EFQ)), i.e. circumstances in which consistence cannot be assumed (say, for instance, confronted to certain fictions or given contradictory information). And we can come back with all the non-sequiturs and elements of criticism that have been presented through those pages on the table. We are back, it seems, to our evaluation of the different logical theories on the market, as we were in chapter 1. Such evaluations must be made holistically, and so the significance of the question of whether relevant negation really is classical negation must be accordingly relativised.

8.3 Pluralism

In chapter 1, we have exposed three strategies to take position on the problems encountered by **CL**. The first one was to look for a new logical system that did not encounter the said problems, this is the strategy adopted by the relevantists ; the second one was to justify the retaining of **CL**, which is at the origin of the classicist's answer ; and the third one was to embrace a form of pluralism. A lot of the discussions that have been exposed since then are the results of arguments between classicists and relevantists.

Since **R** has been developed by the relevantists as a reaction to **CL**, this is not surprising that, in order to understand precisely what have been the philosophical motivations behind it, we have to present how relevantists and classicists have argued against each other. Moreover, one could defend that it is more 'economical' to only endorse one logical system. Much like scientists like to unify and reduce their theories to a small amount of more « fundamental » ones, partly to avoid an unnecessary multiplication of the entities admitted in them, some logicians like to think that there is 'One True Logic' which can provide all one could require of a theory of validity.

Why should we endorse several logical theories when one can do the trick ? It is certainly common sense to prefer, all things being equal, fewer theories in order to explain a phenomenon. And consequently, having a wide explanatory range is, for a theory, an unquestionable virtue of it. Even for philosophers that are not especially responsive to reductionist programs, the methodological preference for unified theories is generally taken favourably. Indeed, some 'ontological' emergentists (philosophers that believe that not all the entities of the world are reducible to the entities of fundamental physics) like

to have a ‘neat’ ontology that is not too heavily populated ; and some ‘epistemic’ emergentists (philosophers that believe that not all the theories explaining the empirical world are reducible to the theories of fundamental physics) like to have a ‘neat’ set of theories that is not too heavily populated⁵³³. Back to our logical theories, the opposition between relevantists and classicists has forced both sides to develop new tools and to push their philosophical positions to their edges. The R-M semantics and its interpretation by Mares are very interesting theories, for example, that may not have been developed if no one was in a relevantist’s approach.

This being said, when all is said and done, a logician can still find some attractiveness in embracing logical pluralism, the idea that different logical systems are required to explain our reasonings, and to provide an appropriate theory of validity. Pluralism may be a way to account for the fact, say, that each theory of validity says something true about our reasonings, just in different inferential contexts. The present work has been focused on two logical systems, **R** and **CL**, but the pluralist (the person who endorses logical pluralism) can find some interest in intuitionist logic, or in a logic that is intransitive⁵³⁴, or in a logic that does not accept the scheme $A \rightarrow A$ ⁵³⁵, etc.

Some pluralists have grounded their view in some « meaning-invariant » framework. Rivenc, for example⁵³⁶, advocates a pluralism of this kind. Recall that in the structural presentation, several different logical systems have their operational rules identical. **R** and **CL** can be presented in this fashion with the same operational rules. So if the meaning of the connectives is to be found in those rules, then the pluralist can endorse several logical systems which attribute to their connectives the same meaning. And he can claim that the differences between those systems is just the way one is allowed to group premises, which may depend on the inferential context at hand. And this framework is « likely to provide the unified conceptual framework in which the idea of the unity of logic lives on »⁵³⁷.

We suggest, however, that a broader form of pluralism could be endorsed. There is a rather natural connexion between our pragmatic stance and pluralism. If we methodologically assume logic to be a tool for understanding human reasoning, clarifying

⁵³³ Our reader can find extra information on the concept(s) of emergence in SARTENAER, O., *Qu’est-ce que l’émergence ?*.

⁵³⁴ That is, a logic that does not validate the transitivity of implication. For example, in which the sequent $A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$ is not valid. Note that rejecting transitivity is another way to block Lewis’ argument.

⁵³⁵ Which is a way to interpret the idea that to repeat something is not to produce a reasoning. See MARES, E.D., *RLPI*, pp. 200-201.

⁵³⁶ See RIVENC, F., *ILP*, pp. XVI-XX ; 187-190. See also DICHER, B., « A Proof-Theoretic Defence of Meaning-Invariant Logical Pluralism » in *Mind*, 125 (2016), pp. 727-757.

⁵³⁷ RIVENC, F., *ILP*, p. XVIII.

some concepts and exploring scientific or philosophical possibilities (such as the possibility that there are contradictory entities in our world), then it seems that we should not be too picky on what constitutes an interesting logical theory. Obviously, if we want to retain, *in a second time*, the idea that there are necessary entities ; and/or if we want to start committing ourselves to some metaphysical theory, then we may want to start thinking about those logical theories that are not *only* tools. But our claim remains that, methodologically, our pragmatic stance seems to justify a fairly loose pluralism.

Our reader may develop a feeling of confusion when we suggest that we may want to commit ourselves to some metaphysical theory. The present work does not close the door to metaphysics, and our pragmatic stance is not incompatible, we believe, with a form of scientific realism, as we have argued in section 5.4. But we believe, to quote a friend of ours, that metaphysics should be to philosophy what a cherry is on the top of a cake. There is a feeling that philosophers who try to jump too quickly to metaphysical speculation tend to produce theories that are not of the finest kind. And we believe that philosophers should be attentive to the scientific fields and keep an open mind when doing metaphysics. So we think that an interesting way to stick to those requirements and avoid dogmatism is through a form of methodological pragmatism.

8.4 Mares' metalanguage

We would like to end the present chapter by coming back to Mares' metalanguage. We'd like to expose how he justifies his classical metalanguage, and what his vindication of classicality really amounts to. Before anything, we need to define the notion of strong consistency.

Definition 8.1 Strong consistency⁵³⁸

A theory is *strongly consistent* if and only if it is characterised by the set of its prime, consistent extensions. A theory Δ is *an extension* of a theory Γ if and only if Γ is a subset of Δ . A theory Γ is *characterised* by a set of theories if and only if Γ is just the intersection of that set, that is, Γ includes all and only those formulae that are in all of the theories in that set.

We have that a theory Γ is *prime* if and only if for every disjunction $A \vee B$ in Γ either A is in Γ or B is in Γ ; and a theory Γ is consistent if and only if for no formula does it contain both the formula and its negation.

⁵³⁸ From MARES, E.D., *RLPI*, p. 174.

Mares reconstructs the acceptability of DS as follows⁵³⁹. Suppose we have a theory, Th , that is strongly consistent. We have :

$$('A \vee B') \in Th$$

From this and the hypothesis that Th is strongly consistent, we can « *reasonably infer* »⁵⁴⁰, where ' $\Rightarrow$ ' symbolises the implicational symbol of a conditional :

$$(' \neg A') \in Th \Rightarrow ('B') \in Th$$

With the extra premise,

$$(' \neg A') \in Th$$

We can infer, using modus ponens for conditional :

$$('B') \in Th$$

This reasoning enables Mares to hold that for theories which are believed to be strongly consistent, DS can be justified as a *reasonable* mode of inference. This is why he uses his theory of conditionals (which is based on his theory of relevant implication) rather than the theory of relevant implication directly. Interestingly enough, we can see Mares' theory of conditionals as a way to answer our worries of section I.2.3.2. The framework establishing inference in terms of information and situations enables him to reach the contextuality of our conditionals and, by this way, of our non-deductive reasoning.

Validating such reasonable mode of inference is particularly important for certain people, notably those who are working in very extensional contexts, like mathematicians. In mathematics indeed, as we noted in section 1.5.3, this kind of reasoning is used all the time. Mares notes that DS

is not all that there is to classical reasoning. We need, for example, a set of axioms for classical logic. Mathematicians tend to help themselves to those theorems of classical logic that they know. For example, proofs by cases often appeal to the law of excluded middle. Thus, we can say that mathematical theories at least often contain a set of axioms for classical logic (even if they are not explicitly acknowledged). Now we have a vindication of classical reasoning about some theories⁵⁴¹.

One may see a problem, because we have already provided the example of a very bad proof in our introduction (which was adapted from Mares' book, and that he rejected as a non-sequitur). From « some humans like chocolate », to « there is no integer n greater than or equal to 3 such that for any non-zero integers x, y, z , $x^n = y^n + z^n$ ». This example

⁵³⁹ From MARES, E.D., *RLPI*, p. 184.

⁵⁴⁰ *Ibidem*.

⁵⁴¹ *Ibidem*, p. 187.

was permitted by **CL**, and Mares just vindicated the use of classical reasoning. Is Mares contradicting himself ?

He doesn't think so. This proof has the form : from A, infer B, where B « is a necessary truth according to the theory that we assume (in this case the theory of arithmetic) »⁵⁴². He notes that in terms of natural deduction system, the proof can be construed as follows⁵⁴³ :

1. ('B' $\in$ N)₁ hyp.
2. A₂ hyp.
3. ('B' $\in$ N)₁ 1, reit.

where N is the theory of arithmetic. He comments : « this is not a good relevant deduction, since the index for the second premise does not appear in the subscript for the conclusion »⁵⁴⁴. What is accepted instead « is the rather harmless inference form »⁵⁴⁵ :

1. ('A $\wedge$ B' $\in$ N)₁ hyp.
2. ('B' $\in$ N)₃ 1, (and a complicated version of $\wedge$ E)

Moreover, Mares notes that (EFQ) is not accepted. The second « one suggests that a contradiction might hold in a theory »⁵⁴⁶, the assumed strong consistency of that theory must be « held in suspension. Hypothesising that a contradiction holds in a theory forces us seriously to consider what would be the case if there is a contradiction in the theory »⁵⁴⁷. And so the « vindication of classical reasoning breaks down »⁵⁴⁸ preventing one from using (EFQ).

So the « *vindication of classical reasoning* does not really vindicate all of classical reasoning. It does not let us infer arbitrary truths (even arbitrary necessary truths) from any old proposition and it does not allow us to use *ex falso quodlibet* »⁵⁴⁹, in particular it does not vindicate a system like **NDCL** which allows « arbitrary hypotheses (such as inconsistent hypotheses) »⁵⁵⁰. What is vindicated is rather « the use of a Hilbert-style deduction system where theorems are deduced from a strongly consistent set of axioms and rules (and which is such that we have good reason to believe that it is strongly consistent) »⁵⁵¹.

⁵⁴² MARES, E.D., *RLPI*, p. 187.

⁵⁴³ From *ibidem*.

⁵⁴⁴ *Ibidem*.

⁵⁴⁵ *Ibidem*.

⁵⁴⁶ *Ibidem*.

⁵⁴⁷ *Ibidem*, p. 188.

⁵⁴⁸ *Ibidem*.

⁵⁴⁹ *Ibidem*.

⁵⁵⁰ *Ibidem*.

⁵⁵¹ *Ibidem*.

So the justification of Mares' classical metalanguage rests on the assumption that the metatheory used in his book is consistent. And he justifies his belief that the metatheory is so by noting that the R-M semantics « has been around for thirty years and has been well used by many logicians. No contradiction has been produced in it (although it has been used to model contradictions) »⁵⁵². The set-theory he uses in it also has a « proof of [its] relative consistency [...] with standard Zermelo-Frankel set theory, with the axiom of foundation (ZF). ZF in turn has been extremely widely used without mathematicians finding any contradictions »⁵⁵³. So Mares has « good inductive evidence for the belief that the theories [used] are consistent, even in the strong sense that we mean here »⁵⁵⁴.

8.5 Summary of part III

Our part III has been focused on several philosophical aspects of **R**. We started by showing how Mares interpretation of the R-M semantics could give it the status of an applied semantics, even if his presentation did not really enable us to say whether truth or information was to be taken as the important interpretative property. We have also shown that Mares could provide an answer to Geach's challenge by holding that, when we were in inconsistent situations (in dealing with fictions or receiving contradictory information for example), DS was no longer valid. This idea was made more precise in the present chapter, where we saw that his metalanguage vindicated, in certain contexts, classical reasoning. Moreover, we have noted that Mares' semantics offered interesting perspectives, most notably by providing a theory of conditionals⁵⁵⁵. Theory, as our reader may recall, that felt totally out of reach for **CL**. This being said, it must be recalled that Mares' treatment of negation suffered from its unintuitive constraints on s^* .

The question of the metalanguage showed its importance in our previous chapter, where Read used a relevant metalanguage to answer Geach's challenge through the Scottish plan. The origin of such move was to be found in Read's formulation of the Deduction Equivalence, and his rejection at the metalevel of the Classical Account. Read used the close relationship between '**o**' and ' $\rightarrow$ ' put to light in the structural presentation to justify '**o**' as the fundamental connective grouping premises in relevant deductions. We have seen that taking '**o**' as a fundamental notion could be problematic, it is far from obvious that we have an intuitive understanding of it comparable to that of ' $\wedge$ '.

We have also explored the issues concerning negation. We have said that relevant negation could be seen as a truth-functional connective simpliciter. But at the end of the

⁵⁵² MARES, E.D., *RLPI*, p. 188.

⁵⁵³ *Ibidem*.

⁵⁵⁴ *Ibidem*.

⁵⁵⁵ As we have said, the theory is built to circumvent the problematic results found in section 1.8

day, even if the semantical and the syntactic views do not give a unified view of what relevant negation consists in, we argued that this may not be such a big deal. For the acceptance of a logical system is holistic, the classicist and the relevantist must advocate for their views of deducibility on different grounds.

With all these elements put on the table, our reader is now in a much better position than in chapter 1 to choose between the two competing theories for explaining and modelling our reasonings that are **R** and **CL**. Obviously, she can also follow our suggestion that pluralism is methodologically preferable.

Conclusion

Our philosophical and logical presentation of the system **R** of relevant implication has come to an end. Our journey has been long, so it may be appropriate to summarize the different important parts that have constituted it. We started with the realization that our friend CarL may have something wrong going on in his software.

In particular, we've seen that he was coded to provide reasonings and express sentences that were judged valid by the well-known logical system **CL**. Our pragmatic stance made clear, and the potential limits of our approach acknowledged, we exposed the different characteristics of **CL**. The system, as we've seen, can be semantically analysed as purely truth-functional, with the truth value of each complex formula entirely determined by the truth value of its components. In particular, a formula of the form ' $A \supset B$ ' is true in the system if and only if its antecedent is false or its consequent is true. And it is false if and only if its antecedent is true and its consequent false. This truth condition finds its justification in the combination of two philosophical ideas, which, when accepted together, actually enable one to prove the said truth condition. The first of these ideas is the Classical Account of validity, which holds that it is a sufficient and necessary condition for the validity of an argument that it be impossible for its premises to be true and its conclusion false. The second of these philosophical ideas is the Deduction Equivalence, which states that a conclusion B follows from premises $A_1, \dots, A_n$, A if and only if the conclusion 'if A then B ' follows from the premises $A_1, \dots, A_n$.

Our next step was to display the different unintuitive results attached to the truth condition so obtained. We have exposed the paradoxes of material implication, with among them : $(M1) \vdash B \supset (A \vee \neg A)$, $(EFQ) \vdash (A \wedge \neg A) \supset B$, and $(M2) \vdash A \supset (B \supset A)$. The problem for each of these paradoxes rested in the fact that the antecedent was in no way *relevant* to the truth of the consequent. And so it appeared that **CL** validated arguments where the premises were irrelevant with respect to the conclusion they were supposed to establish.

Our chapter 1 has been dedicated to the different strategies one could follow in order to solve CarL's problems. We have seen that one could (1) be a revisionist and claim that another system, for example **R**, had to be develop in order to account for our reasonings ; (2) be a classicist and justify the retaining of classical logic as a theory of reasoning, strategy most notably advocated by the proponents of the assertibility conditions' theory ; or (3) be a pluralist and hold that more than one logical system could be acceptable in explaining our reasonings. We also mentioned that **CL** had several virtues, and so we held that the choice between **CL** and **R** was not obvious at all, as is, for that matters, the choice between two competing theories in general : everything must come in the

theorist's balance of evaluation. We concluded that it was necessary, in order to choose between our two systems, to present both of them properly. And so we started our presentation of the system **R** of relevant implication.

Our starting considerations of chapter 2 showed how the implicational fragment of **R** in Hilbert-style and natural deduction systems was nicely built : we saw that a relevant Deduction Theorem and the variable-sharing property held for it. Even if one could be dissatisfied with this duplication of our notion of relevance, things only really became difficult at our chapter 3.

Indeed, if the addition of negation did not cause any trouble to our system, the subsequent addition of the extensional conjunction and disjunction led to unexpected trouble. First, we ended up rejecting disjunctive syllogism (DS) : from $\neg A$ and $(A \vee B)$, infer B , a widely recognised mode of inference for which the fallacy of relevance was far from obvious. We pondered the severity of this rejection with three observations. First, that DS is used in Lewis' argument, showing that the validity of *ex falso quodlibet*, an obvious fallacy of relevance, is dependent on the acceptance of this apparently innocuous principle of inference. Second, we observed that our uses of « or » may very well be ambiguous, and there just might be an intensional sense of it that accepts DS but not addition (from 'A' or 'B', infer 'A or B'). Third, we were reminded that it is not because the principle of inference is not always valid that it never is. We nevertheless were confronted with Geach's challenge, asking us to show how it was possible to have both $\neg A$ and $(A \vee B)$ true and B false, in the classical sense of these connectives.

But our problems did not end so easily. In our final system, we could not have an axiom for disjunction introduction, on pain of validating fallacies of relevance, so we had to find a way to avoid such result. The solution was to add it a primitive a rule of inference, the adjunction rule, but with either a restriction on its application ; or with the loss of the relevant Deduction Theorem. Moreover, there seemed to be no natural way to prove distributivity in our system, so we had to add it as an axiom.

So the second step of our journey was to look for solutions and further insights on the nature of these problems. The framework of substructural logics, exposed in chapter 4, enabled us to consider **R** and its relation with **CL** in a new light. In particular, we've seen that it was possible to consider the differences between the two systems as entirely dependent on the way one is able to transform the groups of premises inside them. **CL** allows the rule of Weakening to be used for the structure « ; » where **R** rejects such use. It is that rule, then, that is responsible for the introduction of irrelevant premises in classical logic. It also became obvious that implication and intensional conjunction were closely related, a result which has inspired Read in his interpretation ; and that one might

consider two ways of combining premises, in particular if one desires distributivity : one way using the semicolon and the other using the comma.

Further insights on our problematic results were looked for into the realm of semantics. We started our chapter 5 by making clear our stance on the role of such theory : useful but not always essential to the work of the logician ; and by expressing the particular philosophical interest one could find in applied semantics. We then presented the possible worlds semantics, closely related to our object of interest, the R-M semantics. The novelty of the semantics was that truth was relativized to particular indices in the models, called possible worlds because they are maximal (and so, worlds) consistent (and so, possible) classes of sentences.

We finally got to present the R-M semantics itself in chapter 6, which replaces the binary accessibility relation of possible worlds with a ternary accessibility relation ; and replaces the possible worlds with situations (sometimes incomplete and sometimes inconsistent). The interpretation of Mares', as we've seen, explains the accessibility relation R as a relation of inclusion in an intersection of neighbourhoods ; the ' $\trianglelefteq$ ' as a part-whole relation ; and C as a compatibility relation. Associated to this last component was the star operator, which required suspicious constraints on the situations ; in particular, it required that the $\trianglelefteq$ -maximal situation s^* be unique for all s .

In our last part, we reflected on different elements that had not been brought to their conclusions. In chapter 7, we defended that Mares' interpretation could be seen as an applied semantics. We later presented his classical metalanguage, completing our review of his interpretation. In the meanwhile, we had presented the answers that had been proposed to the important Geach's challenge. Mares could answer it by showing that in some situations, where consistence of information was not ensured, DS was not valid.

Read, on the other hand, answered the challenge by defending a relevant metalanguage. His grounds for such defense were to be found in his claim that premises ought to be grouped intensionally in our inferences rather than extensionally. He provided a nice Deduction Theorem for his view, but as Rivenc has noted, the notion of intensional conjunction might just not be the primitive and enlightening notion Read wants it to be.

Our last chapter was the occasion for us to understand the significance of metalanguages ; and to discuss the nature of $\mathbf{R}$'s negation. This enabled us to refocus on the guiding idea of this work : the general evaluation of $\mathbf{R}$ as a theory of validity, supposed to provide an explanation of what a good argument is, in competition with the other theory that is $\mathbf{CL}$. We suggested that pluralism, although not especially put on the front scene of this work until now, may end up being the most reasonable position ; in particular given the pragmatic stance endorsed in the present work.

Throughout our presentation, we tried to give to our reader different elements to put in her balance of evaluation in order for her to choose between **R** and **CL**, while noting that the third option of pluralism was available. We tried to show how Mares' and Read's interpretations could overcome traditional criticism formulated against **R**, through the development of an interpreted semantics for the first ; and through the claim that intensional conjunction was the correct way to group premises for the second. We have also criticized their approaches, but those critics can be overcome : maybe there really is a satisfying story that Mares can give us to justify the weird clause for the star operator of his semantics ? Or maybe we should try to adopt an intuitionist negation in our system as he suggested ? And maybe Read can give us extra enlightenment on the notion of intensional conjunction that will make us see that we are more acquainted with it than we expected ?

Anyways, our reader must be aware that there are different tools than those presented herein on the philosophical market, and that at least some of them can be interesting to shed a new light on the system **R**. Like the truthmaker semantics for **R**, which figures among the works that we have barely pinpointed. The philosophical idea motivating a truthmaker semantics is that « truths are true *in virtue of* specific states [...] ». We have states of affairs and a method of combining them together, and sentences are true or false in virtue of those states »⁵⁵⁶. As Jago notes, the truthmaker semantics constitutes a natural framework for **R** :

First, invoking the 'true in virtue of' locution requires that truthmakers be relevant to what they make true. Not every state makes '*a is either red or not red*' true. *That a is red* does, but *that b is blue* does not. The notion of relevance here is left at the intuitive level, just as it is in relevance logic ; but still, there does seem to be a confluence of motivation⁵⁵⁷.

Second, the « truthmaker semantics pins down a suitable notion of 'putting two states together' which [...] gives rise to a suitable conditional. And third, [...] it provides an intuitive notion of validity »⁵⁵⁸. So if our reader believes in the virtues of semantical approaches but found Mares' position wanting, she might be interested in exploring the truthmaker semantics for **R**.

Our reader can alternatively look for different systems than **R**. Now, to our knowledge, not many other logical systems have been direct contenders of **CL** for the rank of 'One True Logic', but certainly a lot of systems can be used to model certain types of reasoning (our reader can find some of those in our section 8.3).

⁵⁵⁶ JAGO, M., « Truthmaker Semantics for Relevant Logic » in *Journal of Philosophical Logic*, 49 (2020), p. 681.

⁵⁵⁷ *Ibidem*, p. 684. Italic in the text.

⁵⁵⁸ *Ibidem*.

So we end the present work on a rather unfortunate note. It seems that we have not answered our initial challenge of helping our good friend CarL. Recall that this could have been accomplished by indicating the way to change in an appropriate manner his software. We did not give, that is, the required answer to the technical question : how can we make him sound more human ? The two initial options considered were to add a layer of code in order to make him better at having conversations while retaining his initial code, the logic **CL**, or to change completely his code and replace it with **R**. But what about our preferred solution ? What would we have to do to CarL to make him a « pluralist » ? CarL would need, it seems, different codes to coexist in his software. But how would he be able to ‘know’ which code to use at every moment ? Certainly, we’d need a particular layer of code that tells him which code to use given such and such inputs. This required ‘meta-code’ may be something, for our theory of reasoning, like the context of validity, or of the argumentation, which determines what logical system we should be using in a particular instance. Such theory may help provide some unity to the kind of pragmatism we have tried to endorse in the present work.

Speaking of unity, it may be important to note that the present work has not been focused on the different systems of relevant logics weaker than **R**, which constitute with it the unity of the « relevant logics »⁵⁵⁹ ; nor on the numerous approaches that share a ‘relevant aspect’ with **R**. For example, in the hyperintensional frameworks which require something like a relevant relation between the entities they study. The great insight of A&B’s work has been to demand more of a true implication or to a valid deduction than a mere connexion of truth value between, respectively, the antecedent and the consequent, and the premise(s) and the conclusion(s). And such insight has been, since then, greatly valuable to those who have been interested in clarifying philosophically important notions, let it be that of belief, reasoning, ethics, ... Who knows what else might be discovered in the distance from the high shoulders of relevance ?

⁵⁵⁹ Interestingly enough, Arnon Avron has noted that what can count as a « relevant logic » or « relevance logic » was not generally very clear. So he tried to produce appropriate definitions in an article, see AVRON, A., « What is relevance logic ? » in *Annals of Pure and Applied Logic*, 165 (2014), pp. 26-48.

Appendix A - Basic syntactic logical notions

The present appendix introduces the basic vocabulary and concepts that are used to construct syntactic logical systems. The focus is put on propositional classical logic, which is very quickly presented here because for the purposes of this work, the semantical presentation of the introduction is sufficient and comes more handily.

A.1 Formulae and connectives

To be able to represent reasoning in all generality, logical systems make use of formal languages⁵⁶⁰, where any piece of information whatsoever is represented with a *formula*, denoted by a capital letter, like A, B or C. To build up formulae, we start with *atomic formulae*, which can be seen as the basic, irreducible entities of our formal language. They are usually represented with italic lowercase letters, like *a*, *b* or *c*. Atomic formulae are our first formulae. Since our argumentations do not merely cite disconnected pieces of information, but instead establish link between them, or express special kind of informational content, we need more. The formal languages used must be able to reflect some structural features of natural languages, which are necessary for a comprehensible discourse. This task is accomplished by *connectives*. Connectives have an arity, which is the number of inputs they take⁵⁶¹. When atomic formulae are so grouped, we also obtain formulae. And when such built-up formulae are grouped with the help of other or the same type of connectives, we have, again, entities that count as formulae. Indeed, since formulae are very general, the group of two formulae linked by a connective counts as another formula. And each formula can count as one input of another formula. This yields an extra symbol for grouping formulae properly in expressions, the parentheses, « (» and «) »⁵⁶². It is common to also use a conventional order of applications in the reading of connectives, in order to limit the number of parentheses. For example, the unary connective « $\neg$ » is applied before binary connectives. So we can write « $\neg A \vee B$ » without parentheses as an equivalent to « $(\neg A) \vee B$ », not to be read as « $\neg (A \vee B)$ »⁵⁶³. With the presented method, we have a recursive way of constructing increasingly complex formulae from atomic formulae, by application of connectives⁵⁶⁴. Note that some special connectives have arity 0 and are meant to represent special

⁵⁶⁰ The general structure of the rest of section A.1 is partly based on WAGNER, P., « Qu'est-ce que la théorie des modèles ? » in *Enquête sur le concept de modèle*.

⁵⁶¹ See RESTALL, G., *An introduction to substructural logics*, p. 15.

⁵⁶² Some logicians also use dots with particular conventional rules in conjunction with parentheses, see for example ANDERSON, A.R., and BELNAP, N., *Entailment, the Logic of Relevance and Necessity*, vol. I.

⁵⁶³ See BOUQUIAUX, L., and LECLERCQ, B., *Logique formelle et argumentation*, p. 40 (now cited BOUQUIAUX, L., and LECLERCQ, B., *LFA*).

⁵⁶⁴ See *ibidem*, p. 68 for a more formal definition of the recursive way to build up formulae for **CL**.

formulae, like falsehood or truth. Table A.1 presents the connectives usually used in propositional classical logic and the typical natural language counterparts they are supposed to formalize⁵⁶⁵. Those counterparts are important when we think about the philosophical goal of our logical construction, but the formal language itself is sometimes considered as uninterpreted strings of symbols with rules of formation.

In some presentations, an extra symbol, the turnstile « $\vdash$ », represents the notion of deducibility. ‘ $A \vdash B$ ’ means that B is validly deducible from A , ‘ $\vdash B$ ’ means that B is a theorem of the system, that it is deducible from it without any premise. The symbol « $\vdash$ » doesn’t belong to the language of the system, though. It is a symbol used to talk *about* the formal system, so it belongs to the metalanguage. Sometimes an indice is used to indicate the logical system for which the validity is being expressed. The indice uses the abbreviation for the system considered. Those abbreviations are usually written in bold capital letters.

Table A.1 : the usual connectives of classical logic

Symbol of the connective	Arity	Name	Natural language counterpart
$\neg, \sim$	1	Negation	« not... », « it is not the case that... »
$\wedge$	2	Conjunction	« ... and ... »
$\vee$	2	Disjunction	« ... or... »
$\supset$	2	Material implication	« if... then... »

Note that other connectives can be introduced (like the equivalence one, « $\leftrightarrow$ »), being defined in terms of combinations of other connectives. Likewise, **CL** can be presented with less than those connectives.⁵⁶⁶ We chose to present those that will be regularly discussed in the present work.

A.2 Putting a logical system to work⁵⁶⁷

It is important to note that until now, we have only introduced the vocabulary of our logic, and gave an unformal glimpse of how we can properly form formulae, which are

⁵⁶⁵ It is adapted from RESTALL, G., *An introduction to substructural logics*, p. 16.

⁵⁶⁶ See BOUQUIAUX, L., and LECLERCQ, B., *LFA*, p. 66.

⁵⁶⁷ This section is partly based on *ibidem*, pp. 64-65.

in references to this proper form called « well-formed formulae », abbreviated « wffs », (« wff » for the expression in singular), but to get a fully working logical system we need more. Those extra components vary drastically with the way we want to build our logical system. Indeed, there exists several *proof systems*, which constitute different syntactic ways of presenting logical systems. In the present work, three such presentations are explored, in the form of Hilbert(-style) systems, natural deduction systems and (sub)structural systems. We briefly discuss Hilbert system's axioms and rules of inference here, because it will be useful to present important related notions in the next subsection, and we keep the two other presentations for chapters 3 and 4.

Hilbert systems contain special wffs that are accepted as valid within the system without proof, sometimes called *axioms* when the constituents of such wffs are atomic formulae, and called *axiom schemes* when their constituents are generic wffs. Sometimes the term *axiom* is used generically to refer to axiom schemes. The present work follows the latter practice. Such systems also contain *rules of inferences* that are expressed in a metalanguage. Those are special operations that can be applied to the wffs : if certain *conditions* are met, we are allowed to infer a certain (inferred) new wff⁵⁶⁸. A *primitive rule of inference*⁵⁶⁹ is a rule of inference that is accepted as holding for the system without being proven. A *theorem*, or a *valid formula*, is a wff that can be proven in the logical system, it is the last line of a Hilbert proof. A *Hilbert proof* is just a series of wffs, each being (1) an axiom or (an instance of) an axiom scheme, (2) a wff obtained from the application of a rule of inference, (3) a theorem. As can be seen in section 2.1, logicians usually like to use extra rules of inferences. We prove two extra rules (three if we count the permutation rule) in that section. So we feel like we need an adapted notion of proof for these. A *Hilbert rule proof* is a Hilbert proof where (1) one can introduce as a step in the proof a postulated wff if it matches a condition of the rule of inference we are trying to prove, (2) the last line of the proof matches the inferred new wff the rule we are trying to prove allows to infer. Example 2 provides an axiomatic, Hilbert-style, presentation for classical propositional logic. Note that a set of axioms and rules of inference do not always uniquely determine a logical system, even in a given proof system. Indeed, different sets of axioms and rules of inference can be proven to be equivalent⁵⁷⁰.

⁵⁶⁸ As an example, the famous Modus Ponens for **CL** in example A.2 has for conditions « p » and « $(p \supset q)$ » and for inferred wff « q ».

⁵⁶⁹ Our definitions of primitive rule of inference, theorem, and Hilbert proof are based on RESTALL, G., *An introduction to substructural logics*, p. 73.

⁵⁷⁰ For examples, see section 2.1.

Example A.2 : Lukasiewicz presentation of propositional classical logic in a Hilbert-style system⁵⁷¹

Axioms

A1 $(p \supset q) \supset [(q \supset m) \supset (p \supset m)]$

A2 $p \supset (\neg p \supset q)$

A3 $(\neg p \supset p) \supset p$

Rules of inference

R1 Rule of detachment : if p is an axiom or a theorem and that $(p \supset q)$ is also an axiom or a theorem, then q is a theorem.

R2 Rule of uniform substitution : any formula obtained by way of uniform substitution in an axiom or a theorem is a theorem.

Replacing the axioms with corresponding axiom schemes, using formulae instead of atomic formulae, enables one to dispense herself of rule R2. Axiom schemes are infinities of axioms of the same logical form, which is indeed equivalent to particular axioms with a rule of uniform substitution⁵⁷². Note that this axiomatic presentation only uses two connectives, but as said earlier, one may recover the other ones by ways of appropriate definitions.

One might wonder how we can know that presentations of different syntactic systems, such as Hilbert systems, natural deduction systems or structural systems, are those of the « same logic », since they can vary furiously in the way they work. The idea that a single logical system can be represented in several proof-systems stems from very important metatheorems that show *completeness* and *soundness* of a system with respect to another. Those results are called *metatheorems* because they are not expressed in any of the systems considered : they are, rather, theorems *about* the systems considered. To obtain them we have to think « outside » of those systems⁵⁷³. Then, we proceed to establish the proofs of completeness and soundness. These notions are also important for other comparisons in this work, so it may be useful to already clarify them.

Given the appropriate translation techniques, a *completeness* proof for a system **A** with respect to a system **B** is a proof that all the theorems provable in **B** are also provable in **A**. So the system **A** proves at least as much as the system **B**. Complementarily, a

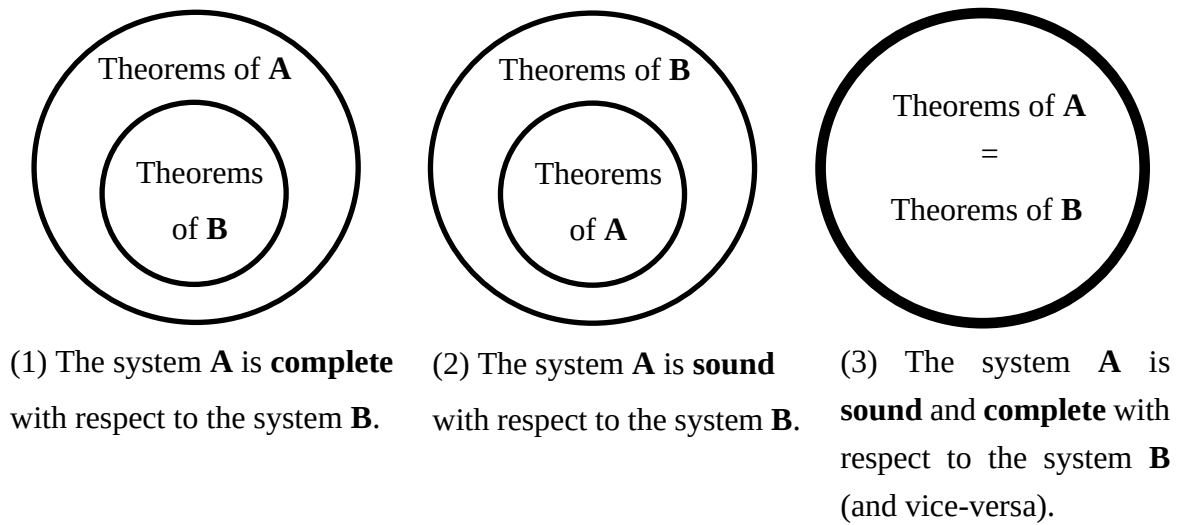
⁵⁷¹ From BOUQUIAUX, L., and LECLERCQ, B., *LFA*, p. 64. Our translation.

⁵⁷² Technically, an appropriate change of rule R1 is required too, by replacing the atomic formulae by formulae and the term « axiom » by « axiom scheme ».

⁵⁷³ See for example RESTALL, G., *An introduction to substructural logics*, pp. 81-88.

soundness proof for a system **A** with respect to a system **B** is a proof that all the theorems provable in **A** are also provable in **B**. So the system **A** is shown to not prove more than the system **B**. Those definitions can be applied to proof systems, but also to other mathematical systems like semantics (where we consider tautologies rather than theorems in the latter). When soundness and completeness are proven, the two systems are shown to perfectly fit one another theoremwise, for in this case the system **A** would prove all that is provable in, but not more than, **B**. Figure 3 represents the sets fitting those notions for two proof systems.

Figure A.3 : completeness and soundness of a proof system with respect to another



A.3 Extending a syntactic logical system

It is important to note at this point of our presentation that it is possible to study extensions of a given logic. A logical system **B** is an extension of a logical system **A** if and only if the set of theorems of **A** is a subset of the set of theorems of **B**. Instead of propositional classical logic, one can study *first-order (classical) logic*, also sometimes referred to as *first-order predicate calculus* where *quantifiers*, « $\forall$ » and « $\exists$ », respectively universal quantifier («for all...») and existential quantifier («there is at least one...») are added to the formal language as well as symbols to represent predicates and functions. This logic enables one to quantify for variables over individuals only, not over relations.

For example, we can express «there is at least one x such that x is blue» with the form $\exists x B(x)$ in first-order logic (where « B » represents the property of being blue). But we can't express something like «there is at least one colour such that the object x is of that colour»: $\exists C C(x)$. In order to express this, one has to change the rules to form wffs,

and this changes the properties of the logical system. So this would lead to the study of *second-order (classical) logic*. And to quantify over properties of properties (and so on), one needs a logic of a higher-order even. The term *higher-order (classical) logic* denotes systems starting from second-order and higher.

Each extension comes with new specifications in the logical system extended. New grammar rules, axioms and/or rules of inference are so integrated into Hilbert systems. Another important example, regarding the extension of propositional classical logic, is that of *normal (classical) modal logics*. Here, new symbols are introduced, « $\Box$ » and « $\Diamond$ », sometimes interpreted as, respectively, an operator of necessity and of possibility (they are unary connectives). In Hilbert presentations, different sets of axioms can be added to propositional classical logic to dictate the behaviour of those symbols and the theorems of the system considered can change dramatically as a consequence.

The choice of a set of axioms over another depends on the philosophical goal pursued. An important number of modal logics are applied logics that interpret « $\Box$ » and « $\Diamond$ »⁵⁷⁴ in different ways. Those logics try to formalize, for example, temporal notions, believes, knowledge, or moral duties. The set of axioms chosen consequently depends on the expected behaviour of the notion studied (which is sometimes philosophically debated)⁵⁷⁵.

Sometimes, as is the case for $\mathbf{R}_h$ in *Introduction à la logique pertinente*⁵⁷⁶, a logical system is presented by introduction of new connectives, starting with a fragment of the final system. Rivenc starts with the implicational fragment of $\mathbf{R}_h$, abbreviated for the occasion $\mathbf{R}_h \rightarrow$, and builds up the entire system by adding negation, conjunction, disjunction and so on, to the system $\mathbf{R}_h \rightarrow$ which has implication as its sole connective. For this purpose, as he notes⁵⁷⁷, we want the new, enriched system to be a *conservative extension* of the previous one, not a mere extension. A logical system $\mathbf{B}$ is a conservative extension of a logical system $\mathbf{A}$ if and only if (1) $\mathbf{B}$ is an extension of $\mathbf{A}$ and (2) any theorem of $\mathbf{B}$ using only the language of $\mathbf{A}$ is a theorem of $\mathbf{A}$. Said otherwise, clause (2) states that any wff of $\mathbf{B}$ not provable in $\mathbf{A}$ is not expressible in $\mathbf{A}$. When $\mathbf{B}$ is a conservative extension of $\mathbf{A}$, $\mathbf{A}$ is said to be a *fragment* of $\mathbf{B}$.

⁵⁷⁴ For the definition of applied logics, see section I.2.2.

⁵⁷⁵ See LECLERCQ, B., *Épistémologie des modalités, modalités épistémiques*, pp. 9-25 and 69-114.

⁵⁷⁶ See RIVENC, F., *ILP*. We also present $\mathbf{R}$ in this manner.

⁵⁷⁷ See *ibidem*, pp. 75-76.

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