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A Maxwell-Chern-Simons Theory in a Disk Coupled to a Scalar Field on its Boundary

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Contents

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Introduction

Motivations : from BMS_3 to a Maxwell-Chern-Simons theory

In 1962, Bondi, Van Den Burg and Metzner on the one hand and Sachs on the other hand ("BMS"), studied the question of gravitational waves emitted by a bounded, axially and reflexively symmetric source in an asymptotically flat space [2][9]. The first three began addressing the following four questions : "Does a source lose mass in the process of emitting gravitational waves? and if so, through which mechanism?", "What do the different algebraic types of Riemann tensors mean physically?", "Can one identify those contributions of the total gravitational fields that are due to the excitation of the radiation modes of the field?" and finally "What is the nature of the asymptotic symmetry group which leaves invariant the form of the boundary conditions for asymptotic flat spaces?". The last interrogation will be the one interesting us.
One would expect the answer to be the Poincaré group symmetry. However, the authors found out that one needs to use a more general group, containing the Poincaré one, in the hope of identifying all the diffeomorphisms leaving the asymptotic flatness unchanged. Such a group, called the BMS group, is the semi-direct product of the homogeneous Lorentz group with the group constituted by the supertranslations, an infinite abelian group containing the Poincaré translations.

This matter gained renewed interest with the emergence of the AdS/CFT correspondence around 1990. As explained in [4], AdS/CFT consists in a 1:1 correspondence between local fields in the gravity theory and operators in the quantum field theory, in particular gauge invariant composite operators. In this correspondence, the mass of a field in the gravity theory is related to the scale dimension of the corresponding operator, and asymptotic conditions on the fields of the gravity theory provide the bridge between the two sides of the correspondence.
This is how research on quantum field theories in bounded spatial domains or with non trivial asymptotical conditions at infinity came back under the spotlights. In particular, Stephen Hawking showed the crucial role that the BMS symmetry could play in the context of black holes dynamics and their quantum states, especially regarding the question of the origins of the entropy of the black holes [5] [6].

Recently as well, Battle, Campello and Gomis ("BCG") worked on the extended BMS group, this time being the semi-product of the the superrotations with the supertranslations, the superrotations being the equivalent of the supertranslations group but now for for the Lorentz transformations [1].
They aimed to build a $\mathfrak{bms}_3$ algebra associated to a free Klein-Gordon field in $2+1$ dimensions in the massive and the non-massive case.

On this basis, in his Master thesis [3] Gilles Buldgen tackled the question of whether one could apply the work of Battle, Campello and Gomis to another physical theory than the Klein-Gordon one, in particular he wanted to verify whether it was possible for a topologically massive gauge field, the Maxwell-Chern-Simons ("MCS") theory.
Managing to do so, he considered the following action of the MCS theory in $(2+1)$ dimensions in the flat Minkowski spacetime ($\mathcal{M}^{2+1}$) geometry:

$$S[A^\mu,\varphi] = \int_{\mathcal{M}^{2+1}} d^3x\Big(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+\frac{\kappa}{2}\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho}\Big), \tag{1}$$

which is the sum of the Maxwell ($-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$) and the Chern-Simons ($\frac{\kappa}{2}\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho}$) actions, with $F^{\mu\nu}$ being the electromagnetic tensor, A_μ the gauge potential and κ the Chern-Simons coupling constant, which induces a mass $m = 2|\kappa|$ for the photon.

In his Master thesis [7], Baptiste Lemaire then undertook to study this action in a finite space, in particular a disk of radius R, but coupled to a scalar field φ living exclusively on its boundary with its own dynamics and being coupled to the gauge potential, but with the Chern-Simons constant κ being set to zero. In polar coordinates, one then has,

$$\begin{aligned} S[A^\mu,\varphi] &= \int dt \int_{\mathcal{D}} d^2x\Big(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\Big) \\ &\quad + \int dt \int_0^{2\pi} d\theta\Big(\mu\bar{A}_\mu\epsilon^{\mu\nu}\partial_\nu\varphi + \frac{1}{2}(\partial_0\varphi)^2 - \frac{1}{2}\frac{1}{R^2}\partial_\theta^2 - \frac{1}{2}m^2\varphi^2\Big), \end{aligned} \tag{2}$$

where μ is the coupling constant between A_μ and φ and the $\bar{\ }$ indicates that we only consider the value on the boundary of the quantity underneath.

Finally, in this work I tried to follow his steps, but this time with a non-zero κ, in the hope of achieving satisfying results, especially now that we were working with a Chern-Simons theory, thought to be of particular interest in the study of the AdS/CFT correspondence we just introduced [8]. Unfortunately, as we will see later, the very presence of the non-zero κ didn't meet our expectations.

Master thesis' outline

This Master thesis will be divided in three main parts.

The aim of the first one will be to sum up as efficiently as possible the mathematical and physics tools, mainly the concepts of symmetries, in particular the Noether theorem and the concept of Noether charges that will prove to be very important for what follows, and what the hamiltonian formalism brings fourth that the lagrangian formalism lacks, as well as the physics research context in which this study takes place, namely the initial work about the BMS group, its extended version, and the introduction of the Maxwell-Chern-Simons theory. All of this is introduced in order for the reader to be provided everything he could need to understand the numerous steps that lead to this precise study and the hopes it carries.

The next chapter is the main core of this Master thesis. It introduces the full action field theory on which everything that follows in this work is based, starting with its Maxwell (M) and Chern-Simons (CS) part and then with the massive scalar theory living on the boundary of the space we consider, namely a $(2+1)$ dimensional space-time with a Minkoswki of metric signature $\eta_{\mu\nu} = +,-,-$ whose space-like section is an R-radius disk. From this full action we will extract the inherent symmetries through the equations of motion, as well as study the $U(1)$ gauge invariance carried by the gauge potential A^μ. Then, once all the constraints inherent to the system will be clear, we will construct analytic solutions for the various fields, $B, E^r, E^\theta, \varphi$ and gauge potentials A^t, A^r, A^θ to the point where we will find ourselves stuck, which will be the subject of the third chapter.

Indeed, at some point, it inevitably appeared that there was an inconsistency between the invariance properties of the scalar field φ and the boundary conditions, whenever the Chern-Simons constant $\kappa \neq 0$. We tried to revise our gauge choice before arriving to the conclusion that, in its current form, our theory could only be studied further in a context where each component of the electromagnetic field is shut down, or where every gauge value would be equal to zero or for a constant scalar field φ, each of these possibilities presenting very poor interest.

After these three main steps, we will try to present some alternatives to our theory and what could be hoped for.

Chapter 1

Symmetries, hamiltonian formalism and the BMS group

1.1 Symmetries in classical field theories

1.1.1 The action and the principle of least action

Let us consider n scalar fields $\phi^i(x^\mu)$, $i = 1, 2, ..., n$, living in a D-dimensional Minkowski spacetime of coordinate system x^μ, $\mu = 0, 1, 2, ..., D-1$.

In classical field theory, the action is the mathematical tool one uses to globally describe the state of a system and its evolution. It is written as the integral over a time interval of the Lagrangian, or the integral over time and space of the lagrangian density, functions of the fields and their derivatives

$$S[\phi^i] = \int_{t_1}^{t_2} dt\ L(\phi^i, \partial_\mu \phi^i) = \int_{t_1}^{t_2} dt \int_\Omega d^{D-1}\vec{x}\mathcal{L}(\phi^i, \partial_\mu \phi^i), \tag{1.1}$$

where Ω represents the space over which we integrate.
Through the principle of least action,

"The path taken by the field between times t_1 and t_2 and configurations q_1 and q_2 is the one for which the action is stationary to first order."

one can extract the equations of motion, and by solving them, should be able to describe the dynamics of the system. Explicitly, the action is"stationary [with respect to ϕ^i] to first order" if

$$\frac{\delta}{\delta\phi^i} S[\phi^i] = S[\phi^i + \delta\phi^i] - S[\phi^i] = 0, \tag{1.2}$$

with $\frac{\delta}{\delta\phi^i}$ being called the variational derivative. More precisely,

$$\begin{aligned}
\frac{\delta}{\delta\phi^i} S[\phi^i] &= \int_\Omega d^D x [\delta\phi^i \frac{\partial\mathcal{L}}{\partial\phi^i} + \delta(\partial_\mu\phi^i)\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^i)}] \\
&= \int_\Omega d^D x [\delta\phi^i \frac{\partial\mathcal{L}}{\partial\phi^i} - \delta\phi^i \partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^i)})] - \int_{\partial\Omega} d^D x\ \ \partial_\mu(\delta\phi^i \frac{\partial\mathcal{L}}{\partial\phi^i}) = 0,
\end{aligned} \tag{1.3}$$

where the last term is a surface term, $\partial\Omega$ denoting the boundary of the space Ω. In this case, it is assumed to cancel as we consider variations $\delta\phi^i$ vanishing on the spatial boundaries. We then obtain the previously mentioned equations of motion thanks to the Euler-Lagrange equation, *i.e.*

$$\frac{\partial\mathcal{L}}{\partial\phi^i} - \partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^i)}) = 0. \tag{1.4}$$

For example, let us determine that our fields are living in a flat spacetime, namely the D-dimensional Minkowski spacetime with the following metric tensor

$$\eta^{\mu\nu} = diag(-,+,+,...,+), \tag{1.5}$$

in this space, we use cartesian coordinates with the index 0 being the one we assign to the time t.
Let us consider for example the following action,

$$S[\phi^i] = \int d^Dx(-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^i - \frac{1}{2}m^2\phi^i\phi^i), \tag{1.6}$$

The Euler-Lagrange equation associated to this Lagrangian gives

$$(\eta^{\mu\nu}\partial_\mu\partial_\nu - m^2)\phi^i(x^\mu) = (\Box - m^2)\phi^i(x^\mu) = 0, \tag{1.7}$$

which are the famous Klein-Gordon equations.

1.1.2 Symmetries of the system

A general symmetry of the system is defined as a general transformation

$$\phi^i(x) \longrightarrow \phi^{i\prime}(x'), \tag{1.8}$$

such that if $\phi^i(x^\mu)$ is a solution of the equations of motion associated to an action, then $\phi^{i\prime}(x^{\mu\prime})$ is also a solution of these equations. Most of the time, we settle for transformations leaving the action invariant, but there will be theories for which the action is invariant up to a total time derivative.

Noether currents and charges

Let our ϕ^i be fields such that they allow us to consider the transformations leaving the action invariant as symmetries, and let us evaluate the action while undergoing a transformation, before having determined whether it is a symmetry or not. We will first define a generic infinitesimal symmetry variation of the fields as

$$\delta\phi^i(x^\mu) \equiv \epsilon^A\Theta_A\phi^i(x^\mu), \tag{1.9}$$

with ϵ^A being constant parameters and Θ_A representing a linear matrix or differential operator,
$A = 1, 2, ..., DimG$ for G being an arbitrary connected Lie group of the internal symmetries for this system. Here I use some of the notations and the train of thoughts presented in [4]. If this transformation happens to be a symmetry, then the action of the theory associated to the fields will be left invariant and the variation of the lagrangian density is a total derivative. Explicitly, this means

$$\delta\mathcal{L} \equiv \epsilon^A(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^i}\partial_\mu\Theta_A\phi^i + \frac{\delta\mathcal{L}}{\delta\phi^i}\Theta_A\phi^i) = \epsilon^A\partial_\mu K^\mu_A, \tag{1.10}$$

where K^μ_A hasn't been defined yet, but recalling the form of the Euler-Lagrange equation, multiplying it by ϵ^A on the left and by $\Theta_A\phi^i$ on the right, we can try to recover the term in parenthesis in (1.10),

$$\begin{aligned} 0 &= \epsilon^A(-\partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^i} + \frac{\delta\mathcal{L}}{\delta\phi^i})\Theta_A\phi^i \\ &= \epsilon^A(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^i}\partial_\mu\Theta_A\phi^i - \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^i}\partial_\mu\Theta_A\phi^i - \partial_\mu(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^i})\Theta_A\phi^i + \frac{\delta\mathcal{L}}{\delta\phi^i}\Theta_A\phi^i) \\ &= \epsilon^A\Big(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^i}\partial_\mu\Theta_A\phi^i + \frac{\delta\mathcal{L}}{\delta\phi^i}\Theta_A\phi^i - \partial_\mu(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^i}\Theta_A\phi^i)\Big), \end{aligned} \tag{1.11}$$

we can find a conserved current, $\mathcal{J}^\mu_A$ such that $\partial_\mu\mathcal{J}^\mu_A = 0$. We define $\mathcal{J}^\mu_A$ from K^μ_A as

$$\mathcal{J}^\mu_A = -\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^i}\Theta_A\phi^i + K^\mu_A, \tag{1.12}$$

such that it is compatible with (1.10). We can now give meaning to the Noether theorem,

"*Given a simply connected Lie group G, its infinitesimal transformations can be written as*

$$\delta x^\mu = \epsilon^A t^\mu_A; \quad \delta\phi^i(x^\mu) = \epsilon^A \phi^i_A(\phi^i(x^\mu), x^\mu); \quad \mathcal{J}^\mu = \epsilon^A \mathcal{J}^\mu_A, \tag{1.13}$$

with constant ϵ^A, $A = 1, 2, ..., DimG$. If the action of a field theory deriving from an action principle is invariant under such a symmetry group, there are DimG independent linear combinations of the equations of motion which are total derivatives

$$(\phi^i_A - t^\nu_A \partial_\nu \phi^i)\Big(\frac{\delta\mathcal{L}}{\delta\phi^i} - \partial_\mu \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi^i)}\Big) = \partial_\mu j^\mu_A, \quad A = 1, 2, ..., DimG \tag{1.14}$$

with $j^\mu_A = \mathcal{J}^\mu_A - t^\mu_A\mathcal{L} - (\phi^i_A - t^\nu_A\partial_\nu\phi^i)\frac{\partial\mathcal{L}}{\partial(\partial_\mu\mathcal{L})}$."

If one is to talk about conserved currents, then one also has to talk about conserved charges. Indeed, the Noether theorem tells us that for each of these j^μ_A, one has a conserved charge $\mathcal{Q}_A$. Explicitly,

$$\partial_\mu j^\mu_A(x^\mu) = -\partial_t j^0_A(x^\mu) + \vec{\nabla}\cdot\vec{j}_A(x^\mu) = 0, \tag{1.15}$$

we have,

$$\partial_t j^0_A(x^\mu) = \text{div } \vec{j}_A(x^\mu), \tag{1.16}$$

which lets us think that j^0_A is the charge density and $\vec{j}_A$ is the current density of a total conserved charge $\mathcal{Q}_A$, that we can relate as follows.
Let us suppose there exists a subspace of Ω called Σ such that the conserved current j^μ_A circulates in it without ever crossing its boundaries, then we have

$$\mathcal{Q}_A = \int_\Sigma d^{D-1}\vec{x} j^0_A(x^\mu). \tag{1.17}$$

The canonical formalism

Furthermore, let us keep the previous discussion in mind and consider the symmetries in the canonical formalism. Still using ϕ^i, the canonical coordinates are, at time $t = 0$, $\phi^i(0, \vec{x})$ for each point $\vec{x}$ of space, and the associated canonical momenta π_i are defined as $\pi_i(0, \vec{x}) = \frac{\delta\mathcal{L}}{\delta\partial_t\phi^i(0,\vec{x})}$.
The introduction of these two quantities allows us to write the Poisson brackets, which, for two observables $A(\phi^i, \pi_i)$ and $B(\phi^i, \pi_i)$ in the phase space, is defined as

$$\{A, B\} \equiv \int d^{D-1}\vec{x}\Big(\frac{\delta A}{\delta\phi^i(\vec{x})}\frac{\delta B}{\delta\pi_i(\vec{x})} - \frac{\delta A}{\delta\pi_i(\vec{x})}\frac{\delta B}{\delta\phi^i(\vec{x})}\Big). \tag{1.18}$$

Going further in our canonical analysis, we reconsider the particular case of the last subsection. Provided K^μ_A has a vanishing time component, equation (1.17) becomes

$$\mathcal{Q}_A = -\int d^{D-1}\vec{x}\frac{\delta\mathcal{L}}{\delta\partial_t\phi^i}\Theta_A\phi^i = -\int d^{D-1}\vec{x} \;\; \pi_i\Theta_A\phi^i. \tag{1.19}$$

One can see that the Noether charge $\mathcal{Q}_A$ generates the infinitesimal transformations $\Theta_A\phi^i$ thanks to the Poisson brackets,

$$\{\mathcal{Q}_A, \phi^i(x^\mu)\} = -\int d^{D-1}\vec{x'}\{\pi_j(\vec{x'})\Theta_A\phi^j(\vec{x'}), \phi^i(\vec{x})\} = \Theta_A\phi^i(x^\mu). \tag{1.20}$$

In the case of continuous global symmetries related to a Lie algebra, one can demonstrate that the algebra of the Poisson brackets of the Noether charges takes the following form,

$$\{\mathcal{Q}_A, \mathcal{Q}_B\} = f^C_{AB}\mathcal{Q}_C, \tag{1.21}$$

where f^C_{AB} are called structure constants of the Lie algebra. With all these tools in hand, we can take a new look to the Noether theorem.
Each symmetry of a field theory is associated, provided the said theory is defined by a variational principle, to a conserved charge, and the Poisson brackets of two such conserved charges generate the algebra of the symmetry group of the system.

1.1.3 The hamiltonian formalism

We can also construct the action of a field theory through another formalism, the hamiltonian one. Though the lagrangian formalism is largely used when one wishes to describe the symmetries of a certain field theory, the informations provided by the hamiltonian on the symmetries will be far richer, in particular when it comes to quantum physics.
The hamiltonian density itself is defined as the Legendre transformation of the lagrangian density.

$$\mathcal{H}(\phi^i(x^\mu), \partial_\mu \phi^i(x^\mu)) = \dot{\phi}^i \pi_i - \mathcal{L}(\phi^i(x^\mu), \partial_\mu \phi^i(x^\mu)), \tag{1.22}$$

where we can see that, now, it also depends on the previously introduced momenta π_i. We recall that the sum over the i's is implicit.
The Hamiltonian itself is the integration over space of the hamiltonian density. It describes the time evolution of the system as

$$\frac{\delta H}{\delta \phi^i} = -\dot{\pi}_i, \qquad \frac{\delta H}{\delta \pi_i} = \dot{\phi}^i. \tag{1.23}$$

which can also be expressed with the Poisson Brackets we introduced earlier. Indeed,

$$\{\phi^i, H\} = \dot{\pi}_i, \qquad \{\pi_i, H\} = \dot{\phi}^i. \tag{1.24}$$

Through a inverse Legendre transformation, we can now define a lagrangian density depending this time not only on the fields ϕ^i and its first derivatives, but also on the momenta π_i

$$\mathcal{L}(\phi^i, \partial_\mu \phi^i, \pi_i) = \dot{\phi}^i \pi_i - \mathcal{H}(\phi^i, \partial_\mu \phi^i, \pi_i). \tag{1.25}$$

The action associated to this Lagrangian is now a functional of the phase space degrees of freedom,

$$S_H[\phi^i, \pi_i] = \int d^D x^\mu \mathcal{L}_H(\phi^i(x^\mu), \partial_\mu \phi^i(x^\mu), \pi_i(x^\mu)). \tag{1.26}$$

1.1.4 The Energy-Momentum Tensor

Thanks to all our previous work, we are now going to briefly introduce the Energy-Momentum tensor,"EMT", T^μ_ν. With the help of a Noether current associated to a spacetime translations invariant field theory, we could hope that we would find an EMT such that it describes the Energy-Momentum conservation of a classical field theory. We will also quickly see, through the work of Battle, Campello and Gomis, and of Gilles Buldgen after them, that one cannot settle for such a naive EMT if one is to keep being that ambitious.

Let us write the infinitesimal form of the spacetime translations, *i.e.* the Poincaré spacetime translations, we just talked about

$$\begin{aligned} x^\mu &\longrightarrow x^\mu + \xi^\mu, \\ \delta x^\mu &= \xi^\nu \delta^\mu_\nu, \end{aligned} \tag{1.27}$$

which leaves the action associated to any scalar, vector or spinor field theory invariant. The conserved Noether current one finds with these transformations is the Energy-Momentum tensor T^μ_ν such that

$$T^\mu_\nu = -\delta^\mu_\nu \mathcal{L} + \partial_\nu \phi^i \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^i}. \tag{1.28}$$

It is an interesting mathematical object as

$$T^{00} = -\mathcal{L} + \dot{\phi}^i \frac{\delta \mathcal{L}}{\delta \dot{\phi}^i} = \dot{\phi}^i \pi_i - \mathcal{L} = \mathcal{H}, \tag{1.29}$$

where $T^{\mu\nu}$ as well as $T_{\mu\nu}$ are simply found using the Minkowski metric, and the total conserved charges associated to the current densities $T^{0\nu}$ are none other than the 4-momentum pseudo-vector

$$P^\nu = \int_\Sigma d^{D-1}\vec{x} \;\; T^{0\nu}. \tag{1.30}$$

Through the examples of the pure abelian gauge theory of Maxwell

$$S_M[A^\mu] = -\frac{1}{4}\int_\Omega d^D x^\mu \;\; F_{\mu\nu}F^{\mu\nu}, \tag{1.31}$$

and the $U(1)$ abelian Chern-Simons theory (in a $2+1$ dimensional spacetime)

$$S_{CS}[A^\mu] = \int_\Omega d^3\vec{x}\frac{\kappa}{2}\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho}, \tag{1.32}$$

Gilles Buldgen noted that, in both cases, the Energy-Momentum tensor was neither symmetric nor gauge invariant, though we would wish it to be. He then studied an alternative to the canonical EMT, the Hilbert one, that was used in the context of Einstein's equations of General relativity, and saw, through various examples once again, that this Hilbert EMT didn't meet the same issues as its canonical counterpart.

Finally he used the work of Belinfante and Rosenfeld to "reconcile" the two approaches. This initiative consisted in noting that when one whishes to study relativistic theories, one has to consider Lorentz invariance, as well as invariance under the Poincaré group as a whole. This way, he recovered the Belinfante-Rosenfeld EMT, sum of the canonical one and a mathematical quantity called the Belinfante-Rosenfeld superpotential, verifying the same conservation property as the canonical EMT but this time also meeting our expectations regarding Poincaré symmetry and gauge invariance.

1.2 From BMS_3 to an MCS theory in a cylindric spacetime

The aim of this section, with the help of the mathematical and physical tools we introduced in the last section, is to go through the work of Battle, Campello and Gomis about the construction of a canonical realization of the $\mathfrak{bms}_3$ algebra associated to a free Klein-Gordon field theory in $(2+1)$ dimensions, in both the massless and the massive cases. Then we will summarize the study of Gilles Buldgen about the application of what precedes in an other context than gravitational fields, in particular in a topologically massive gauge field, namely a Maxwell (M) and Chern-Simons (CS) theory in a flat Minkowski spacetime. Finally, we will briefly overview the conclusions of Baptiste Lemaire that followed about an MCS theory in a finite space that lead us to my own study of an MCS theory in a $2+1$ bounded spacetime.

1.2.1 A canonical realization of the $\mathfrak{bms}_3$ algebra in a Klein-Gordon field theory

Let us recall that for a massive Klein-Gordon theory (of a scalar field ϕ) in a $2+1$ dimensional Minkowski spacetime $\mathcal{M}^{2+1}$, one has this action,

$$S[\phi] = -\frac{1}{2}\int_{\mathcal{M}^{2+1}} d^3x^\mu(\partial_\mu\phi\partial^\mu\phi + m^2\phi^2), \tag{1.33}$$

with $\mu = 0,1,2$ and the Minkowski metric $\eta^{\mu\nu} = \{-,+,+\}$, and the subsequent Euler-Lagrange equations

$$(\Box - m^2)\phi = 0. \tag{1.34}$$

The Fourrier modes expansion provides us the following classical solution to this equation,

$$\phi(t,\vec{x}) = \int \frac{d^2\vec{k}}{\Omega(\vec{k})}(a(\vec{k})a^{ik_\mu x^\mu} + \bar{a}(\vec{k})a^{-ik_\mu x^\mu}), \tag{1.35}$$

where $a(\vec{k})$ are the said Fourrier modes, the "bar" here denoting its complex counterpart, the 3-momentum k^μ is such that $k_\mu k^\mu = -{k^0}^2 + k_1^2 + k_2^2 = -m^2$ and $\Omega(\vec{k}) = (2\pi)^2 2k^0(\vec{k})$ is the integration measure for the hyperbolic plane H_2. From now on, we will note the product $k_\mu x^\mu$ in its contracted form kx.

BCG then evaluate the charges associated to Poincaré translations, P^μ, and to Lorentz rotations and boosts,

M^{ij}, M^{0j} $(i,j=1,2)$, in terms of the Fourrier modes, before adapting them in order to obtain convenient generators for the algebra and calculating their Poisson brackets.

$$P^\mu = \int \frac{d^2\vec{k}}{\Omega(\vec{k})}\bar{a}(\vec{k})k^\mu a(\vec{k}), \qquad M^{ij} = -i\int \frac{d^2\vec{k}}{\Omega(\vec{k})}\bar{a}(\vec{k})(k^i\frac{\partial}{\partial k^j} - k^j\frac{\partial}{\partial k^i})a(\vec{k}),$$

$$M^{0j} = tP^j - i\int \frac{d^2\vec{k}}{\Omega(\vec{k})}\bar{a}(\vec{k})(k^0\frac{\partial}{\partial k^j})a(\vec{k}), \tag{1.36}$$

$$\{P^\mu, P^\nu\} = 0, \qquad \{M^{\mu\nu}, P^\rho\} = P^\mu\eta^{\nu\rho} - P^\nu\eta^{\mu\rho},$$
$$\{M^{\mu\nu}, M^{\rho\sigma}\} = M^{\mu\sigma}\eta^{\nu\rho} + M^{\nu\rho}\eta^{\mu\sigma} - M^{\mu\rho}\eta^{\nu\sigma} - M^{\nu\sigma}\eta^{\mu\rho}.$$

The next step is to define the generators J, K_1, K_2, verifying the $SO(1,2)$ algebra in terms of the differential operators ∂_{k^i},

$$\begin{aligned} -\sqrt{\vec{k}^2+m^2}\partial_{k^1} &\equiv iK_1, \\ -\sqrt{\vec{k}^2+m^2}\partial_{k^2} &\equiv iK_2, \\ k^1\partial_{k^2} - k^2\partial_{k^1} &\equiv iJ. \end{aligned} \tag{1.37}$$

After that, the main goals are to generalize the 3-momenta k^μ, the realization of the charge of the translations on-shell and the Lorentz transformations, respectively into an infinite set of supertranslations, the associated generators P_l and superrotations, in both the massive and the massless cases.
In summary, we have.

Supertranslations, massive case For the parametrization of the $k_0 > 0$ sheet of the hyperbolic plane H_2, $-k_0^2 + k_1^2 + k_2^2 = -m^2$, we take

$$\begin{aligned} k_0 &= mz, \\ k_1 &= m\sqrt{z^2-1}\cos\phi, \\ k_2 &= m\sqrt{z^2-1}\sin\phi, \end{aligned} \tag{1.38}$$

in which $z \in [1, +\infty)$ is the infinite central axis of the hyperboloid and $\phi \in [0, 2\pi)$. We then have these expressions for the Lorentz generators

$$\begin{aligned} K_1 &= -i\frac{z}{\sqrt{z^2-1}}\sin\phi\frac{\partial}{\partial\phi} + i\sqrt{z^2-1}\cos\phi\frac{\partial}{\partial z}, \\ K_2 &= i\frac{z}{\sqrt{z^2-1}}\cos\phi\frac{\partial}{\partial\phi} + i\sqrt{z^2-1}\sin\phi\frac{\partial}{\partial z}, \end{aligned} \tag{1.39}$$

$$J = -i\frac{\partial}{\partial\phi}, \tag{1.40}$$

and this one for the supertranslations generators

$$P_l = \int \frac{d^2\vec{k}}{\Omega(\vec{k})} w_l(\vec{k})\bar{a}(\vec{k})a(\vec{k}), \tag{1.41}$$

with l the modes associated to the infinite set of supermomenta w_l. Later defined as the general solutions of the following equation,

$$\Delta_{LB} w_l = -\frac{2}{m^2}w_l, \tag{1.42}$$

where the factor 2 comes from the dimension of the hyperboloid, and Δ_{LB} is the Laplace-Beltrami operator, written as follows in the current parametrization,

$$\Delta_{LB} = \frac{1}{m^2}\Big((z^2-1)\frac{\partial^2}{\partial z^2} + 2z\frac{\partial}{\partial z} + \frac{1}{z^2-1}\frac{\partial^2}{\partial\phi^2}\Big). \tag{1.43}$$

These supermomenta are of crucial importance for the rest of the realization of the $\mathfrak{bms}_3$ algebra.
We now have the following Poisson brackets, with $K_\pm = K_1 \pm iK_2$

$$\begin{aligned}
\{P_l, P_{l'}\} &= 0, \\
\{J, P_l\} &= -ilP_l, \\
\{K_1, P_l\} &= \frac{1}{2}(1-l)P_{l+1} + \frac{1}{2}(1+l)P_{l-1}, \\
\{K_2, P_l\} &= -\frac{i}{2}(1-l)P_{l+1} + \frac{i}{2}(1+l)P_{l-1}, \\
\{K_\pm, P_l\} &= (1 \mp l)P_{l\pm1}.
\end{aligned} \tag{1.44}$$

Supertranslations, massless case With m being now equal to zero, we are here working on a conical space $(-k_0^2 + k_1^2 + k_2^2 = 0)$ with the following parametrization,

$$\begin{aligned}
k_0 &= r, \\
k_1 &= r\cos\phi, \\
k_2 &= r\sin\phi,
\end{aligned} \tag{1.45}$$

with the same definition for ϕ than previously and $r > 0$ for $k_0 > 0$.
As for the massive case, we give the expressions for the $SO(1,2)$ and the supertranslations generators,

$$\begin{aligned}
J &= -i\frac{\partial}{\partial\phi}, \\
K_1 &= ir\cos\phi\frac{\partial}{\partial r} - i\sin\phi\frac{\partial}{\partial\phi}, \\
K_2 &= ir\sin\phi\frac{\partial}{\partial r} + i\cos\phi\frac{\partial}{\partial\phi}, \\
P_l &= \int \frac{d^2\vec{k}}{\Omega(\vec{k})} w_l(\vec{k})\bar{a}(\vec{k})a(\vec{k}).
\end{aligned} \tag{1.46}$$

This first part allowed BCG to note that whether we considered a massive or massless scenario, the action of the supertranslations on the $a(\vec{k})$ is the same.

Superrotations, massless case Using the Witt algebra, BCG were able to construct differential operators, L_n, later allowing them to define on-shell generators for superrotations. These differential operators are defined as

$$L_n = (k_1^2 + k_2^2)^{-\frac{n}{2}}(k_1 + ik_2)\Big((ink_1 + k_2)\frac{\partial}{\partial k_1} + (ink_2 - k_1)\frac{\partial}{\partial k_2}\Big), \tag{1.47}$$

and as for the superrotations generators, the supertranslations ones and their algebra, the sought $\mathfrak{bms}_3$ one,

$$\mathcal{R}_n = \int \frac{d^2\vec{k}}{\Omega(\vec{k})} \bar{a}(\vec{k})L_n a(\vec{k}), \tag{1.48}$$

$$\begin{aligned}
\{\mathcal{R}_m, \mathcal{R}_n\} &= (m-n)\mathcal{R}_{m+n}, \\
\{\mathcal{R}_m, P_n\} &= (m-n)P_{m+n}, \\
\{P_m, P_n\} &= 0.
\end{aligned} \tag{1.49}$$

Superrotations, massive case They finally ingeniously found a way to define infinite-dimensional sets of superrotations, first by working out the generators T_n such that,

$$T_n = e^{in\phi}\Big(\frac{z-1}{z+1}\Big)^{\frac{|n|}{2}}\Big(-\frac{|n|(|n|+z)+z^2-1}{z^2-10}\frac{\partial}{\partial\phi} + in(|n|+z)\frac{\partial}{\partial z}\Big), \tag{1.50}$$

where the parameter z has the same definition as before and $n \in \mathbb{Z}$, and then realizing that these wouldn't form an algebra, so they had the idea to "divide" them into two infinite sets, $\mathcal{L}_n$ and $\mathcal{Q}_n$ such that

$$\begin{aligned} \mathcal{L}_n &= T_n, \qquad n \geq -1, \\ \mathcal{Q}_n &= T_n, \qquad n \leq 1, \end{aligned} \tag{1.51}$$

obeying

$$\begin{aligned} \left[\mathcal{L}_n, \mathcal{L}_m\right] &= i(n-m)\mathcal{L}_{n+m}, \quad n, m \geq -1, \\ \left[\mathcal{Q}_n, \mathcal{Q}_m\right] &= i(n-m)\mathcal{Q}_{n+m}, \quad n, m \leq -1, \end{aligned} \tag{1.52}$$

$$\mathcal{Q}_0 = \mathcal{L}_0 = -iJ, \qquad \mathcal{Q}_1 = \mathcal{L}_1 = K_+, \qquad \mathcal{Q}_{-1} = \mathcal{L}_{-1} = -K_-. \tag{1.53}$$

Finally, BCG manage to prove that supertranslations and superrotations leave the Klein-Gordon equations invariant. In parallel, Gilles Buldgen shows that on top of everything that preceded, the supertranslations constitute a continuous off-shell symmetry of the massive Klein-Gordon hamiltonian action.

1.2.2 Maxwell-Chern-Simons theory

Consecutively, he studied the combination of Maxwell and Chern-Simons theories, still in infinite Minkowski flat spacetime, in the attempt to show that the work done with the BMS symmetries could be applied in an other context than gravity theories.
Through constraint analysis and the splitting of the vector field degrees of freedom into the gauge invariant and the gauge dependent ones, he was able to show that MCS dynamics were equivalent to topologically massive gauge invariant Klein-Gordon dynamics.

On this basis, Baptiste Lemaire, in his Master thesis, considered the pure Maxwell theory in a $2+1$ dimensional bounded spacetime, namely an infinite cylinder of radius R whose height is parametrized by the time $t \in [0, +\infty)$. However, in this case it is coupled to a scalar field living on the boundary of the disk. This scalar field having its own dynamics following Klein-Gordon equations. After a prior analysis similar to the one we will have achieved at the end of chapter 2, he was able to notice that the quantum states of this system all share the same spectrum of massive states, whether he considered the boundary or the volume of the disk, even though the initial mass of the photon is equal to zero and that the scalar field possesses its own mass.
Both the electromagnetic and the scalar field are coupled, sort of resonate with each other, and thus have to share the same vibration modes, which echoes the AdS/CFT correspondence, minus the fact that there is no proper gravitational interaction in the volume of the disk.

This paved the way for us to finally consider a Maxwell-Chern-Simons theory in a disc coupled to a scalar field on its boundary, this time with a non-zero Chern-Simons coupling constant, thus adding the previously missing proper gravitational interaction. Hopefully we will have similar results than the ones of Baptiste Lemaire, up to a redefinition of the quantum states.

1.3 Summary

In this chapter, we could note how important the action was as one wishes to describe a classical field theory's dynamics, through the principle of least action allowing us to extract the Euler-Lagrange equations and then find the equations of motion, and the system's symmetries, mostly thanks to the Noether theorem and the concepts of Noether currents and charges it introduces.

Futhermore, the hamiltonian formalism proved to be crucial as well in the process of describing a system's symmetries with the help of the Poisson brackets which, applied to Noether conserved charges, allow to generate a symmetry group's algebra. The Poisson brackets of the canonical coordinates and the associated momenta with the Hamiltonian provide a description of a system's dynamics as well.

We then introduced the Energy-Momentum tensor, supposed to play the role of a Noether charge as we expect it to describe the Energy-Momentum conservation of a classical field theory, but it quickly appeared not to fulfill its role in its canonical form as it wasn't neither gauge invariant nor symmetric in its indices though it should be. Fortunately we could see that an alternative was found with the Hilbert form of the EMT, reconciled with the canonical one thanks to the Belinfante-Rosenfeld superpotential.

Consecutively, we analyzed the Klein-Gordon theory and thanks to all the previously introduced tools, we checked the algebra of the Poincaré translations and Lorentz transformations, in order to further do the same with the infinite sets of supertranslations and superrotations and see that they left the Klein-Gordon equations invariant.

We were finally able to briefly introduce the study of the pure Maxwell theory on a disc coupled to a scalar sector on its boundary, and showed the conclusion that were made after the quantum states of such a theory were established, leading to a first step towards a potential AdS/CFT correspondence.

Chapter 2

Study of the EM fields, scalar field and the gauge sector

2.1 Action of the system

Our action comprises four distinct terms. We first have the Maxwell field theory,

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{2.1}$$

with $F^{\mu\nu}$ the electromagnetic tensor or Maxwell tensor, here in (2+1) dimensions as we recall, we then have the $U(1)$ abelian Chern-Simons part of the action,

$$\frac{1}{2}\kappa\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho}, \tag{2.2}$$

where κ is the Chern-Simons coupling constant, A_μ the electromagnetic gauge potential and $\epsilon^{012} = 1$. These two first terms will be added to define what we called the Maxwell-Chern-Simons theory. Next we have the Klein-Gordon action for a topologically massive scalar field,

$$\frac{1}{2}(\partial_\mu\varphi\partial^\mu\varphi - m^2\varphi^2). \tag{2.3}$$

Finally, we have the term representing the interaction between the gauge potential and the scalar field,

$$\mu\bar{A}_\mu\epsilon^{\mu\nu}\partial_\nu\varphi. \tag{2.4}$$

We will regroup the last two terms as the ones living exclusively on the boundary of the considered space, presently the R-radius disc, and the bar symbol above A_μ means that we are only considering the boundary value of the gauge potential (or of any electromagnetic field component as will occur).
In summary, we have,

$$S[A_\mu,\varphi] = \int dt\int_{\mathcal{D}} d^2\vec{x}\; r\mathcal{L}_{MCS} + \int dt\int_0^{2\pi} d\theta R\mathcal{L}_{\mathcal{C}} = \int dt(L_{MCS} + L_{\mathcal{C}}), \tag{2.5}$$

where $\mathcal{C}$ stands for "circle" to designate the interactions exclusively living on it. All there is to recall about the variable changes from cartesian coordinates to polar ones is written in the Appendices of this master thesis.

Let us first study the Maxwell part of the action by recalling the definition of the electromagnetic tensor,

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{2.6}$$

We can start by noting that for F^{0i}, $i = 1, 2$ (0 standing for the time variable t) one has

$$F^{0i} = \partial^t A^i - \nabla^i A^t = \partial_t A^i + \nabla_i A^t, \tag{2.7}$$

which is the definition of the i^th component of the electric field up to its sign, as

$$F^{ij} = \partial^i A^j - \partial^j A^i \approx \vec{\nabla} \times \vec{A}, \tag{2.8}$$

which this time would be the definition of the magnetic field B in 3 dimensions, but as our space is in 2 dimensions, B cannot be a vector, *a minori* be orthogonal to the disk as it would be in 3 dimensions, so B will be a (pseudo-)scalar field.
We further observe that Maxwell equations, in the absence of Chern-Simons coupling, correspond to

$$\partial_\mu F^{\mu\nu} = 0. \tag{2.9}$$

This equation, used in parallel with the decomposition we realized just above, tells us

$$\vec{\nabla} \cdot \vec{E} = 0 \quad \text{and} \quad \vec{\nabla} \times B - \partial_t \vec{E} = 0, \tag{2.10}$$

which are respectively the third and fourth Maxwell equations (again, the × symbol is not to strictly consider as the curl operator in 3 dimensions).
One can find the two other Maxwell equations by noticing that the electromagnetic tensor verifies the Bianchi identity,

$$\partial_\mu F_{\nu\rho} + \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} = 0,$$

or, put differently,

$$\epsilon^{\mu\nu\rho} \partial_\mu F_{\nu\rho} = 0. \tag{2.11}$$

Indeed, provided the previously established definitions of $\vec{E}$ and B are used, one can see

$$\vec{\nabla} \cdot B = 0, \quad \text{and} \quad \vec{\nabla} \times B + \partial_t \vec{E} = 0. \tag{2.12}$$

With everything that preceded in mind, one can write the Maxwell Lagrangian in terms of E^r, E^θ and B,

$$\mathcal{L}_M = \frac{1}{2}({E^r}^2 + {E^\theta}^2 - B^2), \tag{2.13}$$

and the same can also be done for the Chern-Simons part of the Maxwell-Chern-Simons Lagrangian,

$$\mathcal{L}_{CS} = \kappa(-A^t B + A^r E^\theta - A^\theta E^r). \tag{2.14}$$

We may now focus on the part of the action living on the boundary of the disk . Immediately, we find, rewriting the Klein-Gordon equation in the polar coordinates,

$$\mathcal{L}_\mathcal{C} = \mu(\bar{A}^t \frac{1}{R} \partial_\theta \varphi - \bar{A}^\theta \partial_t \varphi) + \frac{1}{2}((\partial_t \varphi)^2 - (\frac{1}{R} \partial_\theta \varphi)^2 - m^2 \phi^2). \tag{2.15}$$

U(1) Gauge invariance

A priori, the φ scalar field is gauge invariant, whereas the A^μ gauge field transforms as follows, for any $U(1)$ gauge transformation parametrized by a $\chi(x^\mu)$ spacetime function,

$$\tilde{A^\mu}(x^\mu) = A^\mu(x^\mu) + \partial^\mu \chi(x^\mu). \tag{2.16}$$

This implies the following,

$$\tilde{A^t} = A^t + \partial_t \chi, \qquad \tilde{A^r} = A^r - \partial_r \chi, \qquad \tilde{A^\theta} = A^\theta - \frac{1}{r} \partial_\theta \chi. \tag{2.17}$$

Consequently, we have $\tilde{F}_{\mu\nu}(x^\mu) = F_{\mu\nu}(x^\mu)$, but the Chern-Simons contribution to the action isn't unmistakably gauge invariant as it transforms non trivially. However at first we naively took $A^t = 0$, as was chosen in Baptiste Lemaire's master thesis. We will later see that it was a poor gauge choice when $\kappa \neq 0$ and come back to this discussion.

2.2 Hamiltonian formalism

2.2.1 Momenta, Hamiltonian and hamiltonian Lagrangian

Let us now use the hamiltonian formalism we introduced earlier. The first step is to write the field momenta, according to this formula $\pi_\phi = \frac{\partial(r\mathcal{L})}{\partial(\partial_t\phi)}$. The aim of doing so is to construct new degrees of freedom on which the hamiltonian density and the hamiltonian form of the lagrangian density will depend, thus providing more information on the systems dynamics than when only relying on the canonical coordinates.

$$\begin{aligned}
\pi_t &= \frac{\partial(r\mathcal{L}_{MCS})}{\partial(\partial_t A^t)} = 0, \\
\pi_r &= \frac{\partial(r\mathcal{L}_{MCS})}{\partial(\partial_t A^r)} = \frac{r}{2}2(\partial_t A^r + \partial_r A^t) + r\kappa A^\theta, \\
\pi_\theta &= \frac{\partial(r\mathcal{L}_{MCS})}{\partial(\partial_t A^\theta)} = \frac{r}{2}2(\partial_t A^\theta + \frac{1}{r}\partial_\theta A^t) - r\kappa A^r, \\
\pi_\varphi &= \frac{\partial(R\mathcal{L}_{\mathcal{C}})}{\partial(\partial_t\varphi)} = -\mu R\bar{A}^\theta + R\partial_t\varphi,
\end{aligned} \tag{2.18}$$

that allow us to notice their relations to the electric field components,

$$\begin{aligned}
E^r &= -\frac{1}{r}(\pi_r - r\kappa A^\theta), \\
E^\theta &= -\frac{1}{r}(\pi_\theta + r\kappa A^r).
\end{aligned} \tag{2.19}$$

We can now find the Hamiltonian by applying the Legendre transformation on the Lagrangians,

$$\begin{aligned}
\mathcal{H}_{MCS} &= \partial_t(A^\mu)\pi_\mu - r\mathcal{L}_{MCS} \\
&= \partial_t A^r\pi_r + \partial_t A^\theta\pi_\theta - \frac{r}{2}(({E^r}^2 + {E^\theta}^2 - B^2) + \kappa(-A^tB + A^rE^\theta - A^\theta E^r)) \\
&= -\partial_t A^r(rE^r) - \partial_t A^\theta(rE^\theta) + \frac{r}{2}((-{E^r}^2 - E^\theta + B^2) - \kappa(A^tB + A^rE^\theta - A^\theta E^r) \\
&= -r(\partial_t A^r + \partial_r A^t)E^r - r(\partial_t A^\theta + \frac{1}{r}\partial_\theta A^r)rE^\theta + r(\frac{1}{r}\partial_r(rA^\theta) - \frac{1}{r}\partial_\theta A^r)B \\
&\quad + \frac{r}{2}(-{E^r}^2 - {E^\theta}^2 - B^2) - \frac{\kappa}{2r}(A^tB + A^rE^\theta - A^\theta E^r), \\
&= r\partial_r A^tE^r + \frac{1}{r}\partial_\theta A^t rE^\theta - (\frac{1}{r}\partial_r(rA^\theta) - \frac{1}{r}\partial_\theta A^r)rB \\
&\quad + \frac{r}{2}({E^r}^2 + {E^\theta}^2 - B^2) - \frac{r\kappa}{2}(A^tB + A^rE^\theta - A^\theta E^r),
\end{aligned} \tag{2.20}$$

$$\begin{aligned}
\mathcal{H}_\varphi &= \partial_t(\varphi)\pi_\varphi - \mathcal{L}_\varphi \\
&= \frac{1}{2R}(\pi_\varphi - \mu R\bar{A}^\theta)^2 - \mu\bar{A}^t\partial_\theta\varphi + \frac{R}{2}(\frac{1}{R}\partial_\theta\varphi)^2 + \frac{R}{2}m^2\varphi^2,
\end{aligned} \tag{2.21}$$

where the particular choice for the way we wrote the MCS Hamiltonian is justified by conveniency for the form of the hamiltonian form of the Lagrangian, that we will write immediately,

$$\begin{aligned}
\mathcal{L}^{\mathcal{H}}_{MCS} &= \partial_t(A^\mu)\pi_\mu - \mathcal{H}_{MCS} \\
&= -r(\partial_t A^r + \partial_r A^t)E^r - r(\partial_t A^\theta + \frac{1}{r}\partial_\theta A^t)E^\theta - r(\frac{1}{r}\partial_r(rA^\theta) - \frac{1}{r}\partial_\theta A^r)B \\
&\quad - \frac{r}{2}({E^r}^2 + {E^\theta}^2 - B^2) + \frac{r\kappa}{2}(A^tB + A^rE^\theta - A^\theta E^r),
\end{aligned} \tag{2.22}$$

an alternative form in terms of the momenta $\pi_{(r,\theta)}$ is the following,

$$\begin{aligned}\mathcal{L}^{\mathcal{H}}_{MCS} &= \pi_r\partial_r A^t + \pi_\theta\frac{1}{r}\partial_\theta A^t - \frac{1}{2r}((\pi_r - r\kappa A^\theta)^2 + (\pi_\theta - r\kappa A^r)^2) \\ &-r(\frac{1}{r}\partial_r(rA^\theta) - \frac{1}{r}\partial_\theta A^r)(\kappa A^t + \frac{1}{2}(\frac{1}{r}(rA^\theta) - \frac{1}{r}\partial_\theta A^r)) \\ &-A^t(\partial_r\pi_r + \frac{1}{r}\partial_\theta\pi_\theta) + \partial_r(A^t\pi_r) + \frac{1}{r}\partial_\theta(A^t\pi_\theta), \end{aligned} \tag{2.23}$$

and finally the hamiltonian form of the scalar field Lagrangian

$$\begin{aligned}\mathcal{L}^{\mathcal{H}}_{\varphi} &= \pi_\varphi(\frac{\pi_\varphi}{R} - \mu\bar{A}^\theta) - \frac{1}{2R}(\pi_\varphi - \mu R\bar{A}^\theta)^2 + \mu\bar{A}^t\partial_\theta\varphi \\ &-\frac{R}{2}(\frac{1}{R}\partial_\theta\varphi)^2 - \frac{R}{2}m^2\varphi^2. \end{aligned} \tag{2.24}$$

2.2.2 Equations of motion

Now, we can find the equations of motion by varying the action according to the various fields, momenta and gauge potentials,

$$\begin{aligned}
A^r &: \partial_t\pi_r = -\kappa((\pi_\theta + r\kappa A^r) + \partial_\theta A^t) - \partial_\theta B, \\
A^\theta &: \partial_t\pi_\theta = \kappa((\pi_r - r\kappa A^\theta) + r\partial_r A^t) + r\partial_r B, \\
A^t &: \partial_r\pi_r + \frac{1}{r}\partial_\theta\pi_\theta = -r\kappa B, \\
\pi_r &: \partial_t A^r + \partial_r A^t = \frac{1}{r}(\pi_r - r\kappa A^\theta) = -E^r, \\
\pi_\theta &: \partial_t A^\theta + \frac{1}{r}\partial_\theta A^t = \frac{1}{r}(\pi_\theta + r\kappa A^r) = -E^\theta, \\
\varphi &: \partial_t\pi_\varphi = -R(\mu\frac{1}{R}\partial_\theta\bar{A}^t + m^2\varphi^2 - \frac{1}{R^2}\partial_\theta^2\varphi), \\
\bar{A}^\theta &: \mu\partial_t\varphi = \bar{B}, \\
\bar{A}^t &: \frac{\mu}{R}\varphi = \bar{E}^r.
\end{aligned} \tag{2.25}$$

The third relation, for $\kappa = 0$, is the Gauss' law while the two last give us boundary conditions.
Strong of all the equations we now have in hand, we can dig in the definition of the B field,

$$\begin{aligned}
B &= -\frac{1}{r\kappa}(\partial_r\pi_r + \frac{1}{r}\partial_\theta\pi_\theta) \\
&= -\frac{1}{r\kappa}(\partial_r(\pi_r - r\kappa A^\theta) + \frac{1}{r}\partial_\theta(\pi_\theta + r\kappa A^r)) \\
&\quad -\frac{1}{r\kappa}(\partial_r(r\kappa A^\theta) + \frac{1}{r}\partial_\theta(-r\kappa A^r)) \\
&= -\frac{1}{r\kappa}(\partial_r(\pi_r - r\kappa A^\theta) + \frac{1}{r}\partial_\theta(\pi_\theta + r\kappa A^r)) \\
&\quad -B
\end{aligned} \tag{2.26}$$

$$B = -\frac{1}{2r\kappa}(\partial_r(\pi_r - r\kappa A^\theta) + \frac{1}{r}\partial_\theta(\pi_\theta + r\kappa A^r)). \tag{2.27}$$

This new expression for B will immediately allow us to establish interesting relations for the first and second time derivatives of the magnetic field,

$$\begin{aligned}
\partial_t B &= \frac{1}{r}\partial_r(r\partial_t A^\theta) - \frac{1}{r}\partial_\theta(\partial_t A^r) \\
&= \frac{1}{r}(\partial_r(\pi_\theta + r\kappa A^r) - \frac{1}{r}\partial_\theta(\pi_r - r\kappa A^\theta)),
\end{aligned} \tag{2.28}$$

$$\begin{aligned}
\partial_t^2 B &= \frac{1}{r}(\partial_r(\partial_t\pi_\theta + r\kappa\partial_t A^r) - \frac{1}{r}(\partial_t - r\kappa\partial_t A^\theta)), \\
&= \frac{1}{r}(\partial_r(r\partial_r B + \kappa(\pi_r - r\kappa A^\theta + r\partial_r A^t) + r\kappa\partial_t A^r) \\
&\quad -\frac{1}{r}\partial_\theta(-\partial_\theta B - \kappa(\pi_\theta + r\kappa A^r) - r\kappa(\frac{1}{r}\partial_\theta A^t + \partial_t A^\theta))) \\
&= \frac{1}{r}(\partial_r(r\partial_r B + \kappa(\pi_r - r\kappa A^\theta + r\kappa(\partial_r A^t + \partial_t A^r))) \\
&\quad -\frac{1}{r}\partial_\theta(-\partial_\theta B - \kappa(\pi_\theta + r\kappa A^r) - r\kappa(\frac{1}{r}\partial_\theta A^t + \partial_t A^\theta))) \\
&= \frac{1}{r}\partial_r(r\partial_r B) + \frac{1}{r^2}\partial_\theta^2 B + \frac{2\kappa}{r}(\partial_r(\pi_r - r\kappa A^\theta) + \frac{1}{r}\partial_\theta(\pi_\theta + r\kappa A^r)).
\end{aligned} \tag{2.29}$$

We keep 2.28 in mind for later, when we will try to find the analytic expressions of the momenta. Right now, let us notice the terms multiplied by $\frac{2\kappa}{r}$ in the last line of 2.29, reminding us of 2.27, thus allowing us to finally find the following equation governing the magnetic field,

$$(\partial_t^2 - \frac{1}{r}\partial_r(r\partial_r) - \frac{1}{r^2}\partial_\theta^2 + 4\kappa^2)B(t,r,\theta) = 0, \tag{2.30}$$

which is precisely the Klein-Gordon equation for a field of mass $m = 2|\kappa|$.

2.3 Analytic solutions for the electromagnetic field

2.3.1 The magnetic field

Given the system's invariance under time translations and under rotations of the disk, the normal modes of vibration for this wave equation ((pseudo-)scalar) can be decomposed in an eigenstates basis for these two symmetries, namely the ones characterized by well defined frequency and angular momentum. Separating these variables, the linear differential equation above becomes a second order ordinary differential equation with respect to the radial dependency, let us illustrate this by first writing the supposed angular and time dependencies of B

$$B(t,r,\theta) = \sum_{l=-\infty}^{\infty} B_l(t,r)e^{il\theta}, \qquad B_l^*(t,r) = B_{-l}(t,r), \tag{2.31}$$

$$B_l(t,r) = e^{-i\omega_l t}B_l(\omega_l, r), \tag{2.32}$$

where the $*$ designates the complex conjugate of the quantity.
The Klein-Gordon equation accordingly changes into the following form

$$\begin{aligned}
(-\omega_l^2 - \frac{1}{r}\partial_r(r\partial_r) + \frac{l^2}{r^2} + 4\kappa^2)B_l(\omega_l, r) &= 0, \\
(\partial_r^2 + \frac{1}{r}\partial_r + \Big(\omega_l^2 - 4\kappa^2) - \frac{l^2}{r^2}\Big)B_l(\omega_l, r) &= 0.
\end{aligned} \tag{2.33}$$

Let us now introduce a new notation, for each possible value of the angular momentum $l \in \mathbb{Z}$, corresponding to the wavenumber $k_l = \sqrt{\omega_l^2 - 4\kappa^2} \geq 0$, $2|\kappa| \leq \omega_l \leq \infty$. If we make the following change of variable on 2.33, $u = k_l r$ and divide the whole by k_l^2, one then obtains

$$(\partial_u^2 + \frac{1}{u}\partial_u + (1 - \frac{l^2}{u^2}))B_l(u) = 0, \tag{2.34}$$

which is basically the equation defining Bessel functions with a length scale factor equal to the wavenumber one, but since we wish the B field to remain regular everywhere and finite over the extent of the disc's volume, only one of the two linearly independent solutions for this radial, finite for $r = 0$, equation is possible, either the one in terms of the first species Bessel function with a positive integer index for a non-zero wavenumber, or a pure dependency as a power of r for a wavenumber equal to zero.
The solution for the B field can then be expressed under the following form

$$\begin{aligned} B(t,r,\theta) &= \sum_{l=-\infty}^{+\infty} \sum\!\!\!\!\!\!\int_{2|\kappa|}^{\infty} d\omega_l (a_l(\omega_l) e^{-i\omega_l t + il\theta} + a_l^\dagger(\omega_l) e^{i\omega_l t - il\theta}) J_{|l|}(k_l r) \\ &+ \sum_{l=-\infty}^{+\infty} (\frac{r}{R})^{|l|} (\beta_l e^{-i2|\kappa|t + il\theta} + \beta_l^\dagger e^{i2|\kappa|t - il\theta}). \end{aligned} \tag{2.35}$$

In this expression, $a_l(\omega_l)$ and β_l are complex integration constants ($(a_l(\omega_l))^\dagger$ and $\beta_l^\dagger$ being their complex conjugates, for they become operator and their adjoints, satisfying Fock bosonic algebrae, up to normalisation factors, while working with quantised fields) associated to each possible value for the angular momentum l and for the angular frequency, $\omega_l \geq 2|\kappa|$. The $\sum\!\!\!\!\!\!\int$ symbol indicates that the frequencies spectrum has yet to be determined, whether it be discrete or continuous, and that one has yet to check whether the minimal value $\omega_l = 2|\kappa|$ is included or not (if it is, for a continuous spectrum it is an equal to zero measure accumulation point in the integral over the frequency spectrum).

2.3.2 Electric field components

Having constructed the solution for $B(t,r,\theta)$, let us consider the previously established relations for this field,

$$\partial_t B = \frac{1}{r}(\partial_r(\pi_\theta + r\kappa A^r) - \frac{1}{r}\partial_\theta(\pi_r - r\kappa A^\theta)), \qquad \partial_r \pi_r + \frac{1}{r}\partial_\theta \pi_\theta = -r\kappa B. \tag{2.36}$$

However, in this form, this did not provide us anything of use. We then decided to switch from the momenta $\pi_{(r,\theta)}$ to the electric field components $E^{(r,\theta)}$. Let us remind that

$$E^r = -\frac{1}{r}(\pi_r - r\kappa A^\theta), \qquad E^\theta = -\frac{1}{r}(\pi_\theta + r\kappa A^r), \tag{2.37}$$

and recalling as well that so far we were still working with the initial, naive intuition of a gauge such that $A^t = 0$ and instead of the equations 2.36, we decided to focus on the equations provided by the variations of the action relative to A^r and A^θ, respectively

$$\begin{aligned} \partial_t \pi_r &= -\kappa((\pi_\theta + r\kappa A^r) + \partial_\theta A^t) - \partial_\theta B, \\ \partial_t \pi_\theta &= \kappa((\pi_r - r\kappa A^\theta) + r\partial_r A^t) + r\partial_r B, \end{aligned} \tag{2.38}$$

that become, using $\partial_t A^r = \frac{1}{r}(-rE^r)$ and $\partial_t A^\theta = \frac{1}{r}(-rE^\theta)$,

$$\begin{aligned} \partial_t(-rE^r) &= -r\kappa\frac{1}{r}(-rE^\theta) - \partial_\theta B, \\ \partial_t E^r &= -2\kappa E^\theta + \frac{1}{r}\partial_\theta B, \end{aligned} \tag{2.39}$$

$$\begin{aligned} \partial_t(-rE^\theta) &= r\kappa\frac{1}{r}(-rE^r) + r\partial_r B, \\ \partial_t E^\theta &= 2\kappa E^r - \partial_r B. \end{aligned} \tag{2.40}$$

Finally, deriving the $E^{(r,\theta)}$ a second time with respect to their time dependence, one can obtain the two following equations,

$$(\partial_t^2 + 4\kappa^2)E^r = (2\kappa\partial_r + \frac{1}{r}\partial_\theta\partial_t)B(t,r,\theta), \tag{2.41}$$

$$(\partial_t^2 + 4\kappa^2)E^\theta = (2\kappa\frac{1}{r}\partial_\theta - \partial_r\partial_t)B(t,r,\theta). \tag{2.42}$$

From this basis, I tried to find the expressions of the electric field components, by first replacing the right hand terms of 2.41 and 2.42 by a generic function $f(t,r,\theta)$ and then separating the equations into homogeneous and inhomogeneous parts, before adapting my generic solutions to the relevant right hand terms,

$$
\begin{aligned}
E^r(t,r,\theta) &= \sum_{l=-\infty}^{\infty} \sum\!\!\!\!\!\!\int_{2|\kappa|}^{\infty} (a_l(\omega_l)e^{-i\omega_l t+il\theta} + a_l^\dagger(\omega_l)e^{i\omega_l t-il\theta}) \\
&\quad \Big(-\frac{2\kappa}{k_l}J'_{|l|}(k_l r) - \frac{l\omega_l}{rk_l^2}J_{|l|}(k_l r)\Big)d\omega_l \\
&\quad \begin{cases} - \sum_{l=-\infty}^{\infty} \frac{2|\kappa l|}{k_l^2}(\beta_l e^{-i2|\kappa|t+il\theta} + \beta_l^\dagger e^{i\omega_l t-il\theta})\frac{1}{R}(\frac{r}{R})^{|l|-1} \\ + \sum_{l=-\infty}^{\infty}(e^{-i2|\kappa|t+il\theta}g_l^r(r) \quad + \quad e^{i2|\kappa|t-il\theta}g_l^{r\dagger}(r)), \end{cases}
\end{aligned} \tag{2.43}
$$

$$
\begin{aligned}
E^\theta(t,r,\theta) &= \sum_{l=-\infty}^{\infty} \sum\!\!\!\!\!\!\int_{2|\kappa|}^{\infty} (a_l(\omega_l)e^{-i\omega_l t+il\theta} - a_l^\dagger(\omega_l)e^{i\omega_l t-il\theta}) \\
&\quad \Big(\frac{-2il\kappa}{rk_l^2}J_{|l|}(k_l r) - \frac{i\omega_l}{k_l}J'_{|l|}(k_l r)\Big)d\omega_l \\
&\quad \begin{cases} - \sum_{l=-\infty}^{\infty} \frac{2|\kappa}{k_l^2}l|(\beta_l e^{-i2|\kappa|t+il\theta} - \beta_l^\dagger e^{i\omega_l t-il\theta})\frac{1}{R}(\frac{r}{R})^{|l|-1} \\ + \sum_{l=-\infty}^{\infty}(e^{-i2|\kappa|t+il\theta}g_l^\theta(r) - e^{i2|\kappa|t-il\theta}g_l^\theta(r)), \end{cases}
\end{aligned} \tag{2.44}
$$

where the $g_l^{(r,\theta)}(r)$ are arbitrary complex integration constants from the previous resolution of 2.41 and 2.42 with a generic $f(t,r,\theta)$. We chose to "confront" the $g_l^{(r,\theta)}(r)$ terms with the β_l ones, inherited from the expression of $B(t,r,\theta)$. In order to do that, we will use the equations 2.36 we had left behind until now, which one can translate as such,

$$
\partial_t B = \frac{1}{r}\partial_\theta E^r - \frac{1}{r}\partial_r(rE^\theta), \tag{2.45}
$$

$$
2r\kappa B = \frac{1}{r}\partial_r(rE^r) + \frac{1}{r}\partial_\theta E^\theta. \tag{2.46}
$$

By recalling the relation inherent to the Bessel functions $J_{|l|}(k_l r)$, $(\partial_u^2 + \frac{1}{u}\partial_u + (1 - \frac{l^2}{u^2}))J_{|l|}(u) = 0$, one could easily see that the $a_l(\omega_l)$ terms were obeying both 2.45 and 2.46, while the "confrontation" between the β_l and the $g_l^{(r,\theta)}(r)$ terms was a little bit more tricky but eventually were solved as well, as is demonstrated in the appendix B.

The next step was to realize the same work for the scalar field φ as we did for the $E^{r,\theta}$ components, using the boundary conditions. Unfortunately, we could very quickly notice that the said boundary conditions were incompatible as they were, which lead us to a reconsideration of our gauge choice.

2.4 Reconsidering the $U(1)$ invariance gauge

As we said earlier, while under local gauge transformations, noted $\tilde{\ }$, we have $\tilde{F}_{\mu\nu}(x^\mu) = F_{\mu\nu}(x^\mu)$. The Chern-Simons contribution to the action is not exactly gauge invariant by itself as it transforms like a surface term, namely,

$$
\int d^3x^\mu \frac{1}{2}\kappa\epsilon^{\mu\nu\rho}\partial_\mu\chi F_{\nu\rho} = \int d^3x^\mu \partial_\mu(\frac{1}{2}\kappa\chi\epsilon^{\mu\nu\rho}F_{\nu\rho}), \tag{2.47}
$$

given the Bianchi identity ($\epsilon^{\mu\nu\rho}\partial_\mu F_{\nu\rho} = 0$).
In the case of a symmetry transformation leaving the equations of motion invariant, a time surface term (a total time derivative) is acceptable as an action variation. However it does not apply to space-like surface terms. Nonetheless, in the present case, using the divergence theorem on a disc, such a surface term outlives the radial derivative transformation on the boundary of the disk . Consequently, the above mentioned $U(1)$ gauge transformations still define a local or gauge symmetry for the system provided that the functions $\chi(t,r,\theta)$ are such that

$$
\chi(t,R,\theta) = 0. \tag{2.48}
$$

This is the boundary condition that gauge transformations have to obey in order for the Chern-Simons contribution to the action to be gauge invariant given the finite domain of the disk.
This implies the following, first remembering that theses gauge transformations can be written

$$\tilde{A}^t = A^t + \partial_t\chi, \qquad \tilde{A}^r = A^r - \partial_r\chi, \qquad \tilde{A}^\theta = A^\theta - \frac{1}{r}\partial_\theta\chi. \tag{2.49}$$

Thus on the disk's boundary we have

$$\tilde{\bar{A}}^t = \bar{A}^t + \partial_t\chi(t,R,\theta) = \bar{A}^t, \quad \tilde{\bar{A}}^r = \bar{A}^r + \partial_r\bar{\chi}(t,r,\theta), \quad \tilde{\bar{A}}^\theta = \bar{A}^\theta - \frac{1}{R}\partial_\theta\chi(t,R,\theta) = \bar{A}^\theta. \tag{2.50}$$

Subsequently, $\bar{A}^t(t,r,\theta) = A^t(t,R,\theta)$ and $\bar{A}^\theta(t,r,\theta) = A^\theta(t,R,\theta)$ are physical gauge invariant degrees of freedom as such, when $\kappa \neq 0$ (in contradistinction to the $\kappa = 0$ case).

Furthermore, and for the very same reason, when $\kappa \neq 0$, it isn't possible to justify the choice of a gauge such that $A^t(t,r,\theta) = 0$ (which is to say that, given an arbitrary configuration for $A^t(t,r,\theta)$, we can't find a gauge transformation such that $\tilde{A}^t(t,r,\theta) = 0$ for a function $\chi(t,r,\theta)$ obeying the boundary conditions, namely, $\chi(t,R,\theta) = 0$), if only for the fact that $A^t(t,R,\theta)$ is gauge invariant under the class of gauge transformations obeying the boundary conditions. Nevertheless, considering the aforementioned gauge transformations, it appears that, given an arbitrary configuration for $A^r(t,r,\theta)$ one can always find a gauge transformation $\chi(t,r,\theta)$ such that $\chi(t,R,\theta) = 0$ and $\tilde{A}^r(t,r,\theta) = 0$. Indeed we only need to fix

$$\chi(t,r,\theta) = -\int_r^R dr' A^r(t,r',\theta), \qquad \tilde{A}^r(t,r,\theta) = 0. \tag{2.51}$$

Later, we will be able to work in the $A^r = 0$ gauge and afterwards to re-insert the gauge degrees of freedom in the system, in order to express the solutions to the equations of motion for A^t, A^r and A^θ using an arbitrary gauge transformation parametrized with an arbitrary function $\chi(t,r,\theta)$ such that $\chi(t,R,\theta) = 0$.

As we will see later, these considerations, specific to the Chern-Simons coupling on the boundary, also affect the boundary conditions that couple the scalar field to the gauge field.

2.5 Reconstructing the electric field components by a development in eigenmodes and variable separations

Fortunately, this change of gauge didn't make us restart from scratch, as the important results remained untouched until 2.41 and 2.42 included.
The equations to be solved, included the one for $B(t,r,\theta)$ are, on the one hand, linear in the $B, E^{(r,\theta)}$ fields but also on the other hand invariant under time translations and under rotations inside the disk around its center.
Subsequently, these equations can be considered for $B, E^{r,\theta}$ taking complex values (in order to then extract the real parts), whereas the time and angular dependencies can be developped in "plane waves", with each of these waves corresponding to an eigenmode factorised according to a term of the form $e^{-i\omega t+il\theta}$ ($l \in \mathbb{Z}$ and $\omega \geq |\kappa|$).
We can then convert the partial differential equations into simple algebraic equations with ordinary derivatives with respect to r, and linear in the components $B, E^{(r,\theta)}$, whose solutions may be expressed in terms of complex integration constants. The complete final solution is then obtained by simple linear combinations from which we eventually extract the real parts. Thus, if a mode's dependency is of the form

$$e^{-i\omega t+il\theta} f(r), \tag{2.52}$$

we have the following associations,

$$\partial_t \to -i\omega, \qquad \partial_\theta \to il, \qquad \partial_r \to \frac{d}{dr} \equiv d_r. \tag{2.53}$$

Accordingly, the considered equations become

$$\begin{aligned}
\Big(-\omega^2 - \frac{1}{r}d_r r d_r + \frac{l^2}{r^2} + 4\kappa^2\Big)B &= 0, \qquad\qquad k = \sqrt{\omega^2 - 4\kappa^2}, &(2.54)\\
\Big(\frac{1}{r}d_r r d_r + (\omega^2 - 4\kappa^2) - \frac{l^2}{r^2}\Big)B &= 0, &(2.55)
\end{aligned}$$

with

$$\begin{aligned}
-i\omega E^r + 2\kappa E^\theta - \frac{il}{r}B &= 0,\\
-i\omega E^\theta - 2\kappa E^r + d_r B &= 0, \qquad (2.56)
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{r}d_r(rE^r) + \frac{il}{r}E^\theta &= 2\kappa B,\\
-\frac{1}{r}d_r(rE^\theta) + \frac{il}{r}E^r &= -i\omega B. \qquad (2.57)
\end{aligned}$$

In the expressions above, the quantities $B, E^{(r,\theta)}$ now represent complex quantities, functions of r, and (implicitly) dependent of ω and $l \in \mathbb{Z}$ (one should then write, *e.g.*, $B_l(\omega_l, r) \in \mathbb{C}$, etc.).
In this form, the resolution of the equations immediately becomes more transparent.

Furthermore, at first we are simply going to solve them, before even considering the regularity condition when $r \to 0$ of the yet to be constructed fields, the said condition will only be applied last. This will also make things clearer when one wishes to consider the integration constants that must cancel in the expressions for the fields, according to the sign of the product κl.

Regarding whether the frequency spectrum is discrete or continuous, which is to say whether the case $\omega_l = 2|\kappa|$ is included in the $\not\!\!\int$ or not, as one wishes to isolate this particular case for this particular value (and not to "lose" it during the analysis in the case of a continuous spectrum), the $\omega_l = 2|\kappa|$ contributions are the ones specifically corresponding to the β_l contributions (namely in the case of a discrete spectrum with a finite interval between the value $2|\kappa|$ and the next value in the frequency spectrum, this one contribution isn't being explicitly included in the notation $\not\!\!\int$).

Thus it is this precise expression for the solution for $B(t, r, \theta)$ that serves as a starting point for the resolution of the equations in the complete gauge sector. It is possible (as it is for the β_l) that the integration constants $a_l(\omega_l)$ and β_l must obey some restrictions for one to obtain the general and complete solution to these equations. In particular the sign of the non-zero values of l will be correlated to the sign of κ, and one will have to distinguish between the values of l such that κl is positive or negative. We will use the following notations for these respective situations,

$$\begin{aligned}
\sum_{\kappa l>0} &: \qquad \text{sum over every value of } l \neq 0 \text{ such that sgn}(l) = \text{sgn}(\kappa);\\
\sum_{\kappa l\leq 0} &: \qquad \text{sum over every value of } l \neq 0 \text{ such that sgn}(l) = \text{-sgn}(\kappa); \qquad (2.58)
\end{aligned}$$

This way, we will have, for any infinite series over l of coefficients f_l,

$$\sum_{l=-\infty}^{+\infty} f_l = f_0 + \sum_{\kappa l>0} f_l + \sum_{\kappa l\leq 0} f_l. \qquad (2.59)$$

Because of the care one should use analysing the k-zero modes, we will consider the various cases of whether $\omega > 2|\kappa|$ or $\omega = 2|\kappa|$, or put differently, whether $k > 0$ or $k = 0$, each in turn.

k-non-zero modes

Let us first consider the $\omega > 2|\kappa|$ case. Then the solution to the equation for B, namely,

$$(\frac{1}{r}d_r r d_r + (\omega^2 - 4\kappa^2) - \frac{l^2}{r^2})B = 0, \qquad k = \sqrt{\omega^2 - 4\kappa^2}, \tag{2.60}$$

are clearly given by a linear combination of the function $J_{|l|(kr)}$ and $N_{|l|}(kr)$, with still arbitrary complex coefficients (integration constants) - at the moment we aren't worrying about the regularity condition of the solutions around $r = 0$, for which as a matter of fact $N_{|l|}(kr)$ diverges logarithmically.

Thus, the function $B(r)$ is supposed to be specified this way, and it is as such that it contributes as a source term for the other equations yet to be considered.
The next one on our plate is the system,

$$\begin{aligned} -i\omega E^r + 2\kappa E^\theta &= \frac{il}{r}B, \\ -2\kappa E^r - i\omega E^\theta &= -d_r B, \end{aligned} \tag{2.61}$$

whose solutions are

$$\begin{aligned} E^r &= \frac{-1}{\omega^2 - 4\kappa^2}(\frac{\omega l}{r} + 2\kappa d_r)B, \\ E^\theta &= \frac{-i}{\omega^2 - 4\kappa^2}(\frac{2\kappa l}{r} + \omega d_r)B. \end{aligned} \tag{2.62}$$

We still must check whether these two last equations,

$$\begin{aligned} \frac{1}{r}d_r(rE^r) + \frac{il}{r}E^\theta &= 2\kappa B, \\ -\frac{1}{r}d_r(rE^\theta) + \frac{il}{r}E^r &= -i\omega B, \end{aligned} \tag{2.63}$$

are also satisfied. With a simple and direct substitution of the expressions for the $E^{(r,\theta)}$ in these two equations, and by using the (Bessel) equation that $B(r)$ obeys, it is immediate to verify that, indeed, no further restriction arises for these solutions.

When $\omega > 2|\kappa|$, the solution are then given by the above expressions for $B(r)$ (as a complex linear combination of $J_{|l|}(kr)$ and $N_{|l|}(kr)$) and subsequently also for $E^r(r)$ and $E^\theta(r)$ expressed in terms of $B(r)$, with $k > 0$.

k-zero modes

Let us now consider the situation when $k = 0$ (or $\omega = 2|\kappa|$). We are thus interested in the following equations. For the field $B(r)$,

$$(\frac{1}{r}d_r r d_r - \frac{l^2}{r^2})B = 0, \tag{2.64}$$

for the electric field components,

$$\begin{aligned} -i|\kappa|E^r + 2\kappa E^\theta &= \frac{il}{r}B, \\ -i|\kappa|E^\theta - 2E^r &= -d_r B, \end{aligned} \tag{2.65}$$

and finally, for the consistency relations,

$$\begin{aligned} -\frac{1}{r}d_r(rE^\theta) + \frac{il}{r}E^r &= -i2|\kappa|B, \\ \frac{1}{r}d_r(rE^r) + \frac{il}{r}E^\theta &= 2\kappa B. \end{aligned} \tag{2.66}$$

Consequently the solution for $B(r)$ is given, a priori, by a complex linear combination of $(\frac{r}{R})^l$ and $(\frac{r}{R})^{-l}$, or consequently of $(\frac{r}{R})^{|l|}$ and $(\frac{r}{R})^{-|l|}$.

Furthermore, the two coupled equations for the two components of the electric field are also written, with $s = sgn(\kappa) = \pm 1$,

$$\begin{aligned} E^r + isE^\theta &= -\frac{l}{|\kappa|}\frac{1}{r}B, \\ -isE^r + E^\theta &= -\frac{i}{|\kappa|}d_r B, \end{aligned} \tag{2.67}$$

but as the last one can also be written

$$E^r + isE^\theta = \frac{s}{|\kappa|}d_r B, \tag{2.68}$$

the internal consistency of these two equations imposes the following linear condition,

$$\frac{s}{|\kappa|}d_r B = -\frac{l}{|\kappa|}\frac{1}{r}B, \qquad d_r B = -sl\frac{1}{r}B, \tag{2.69}$$

whose solution is necessarily proportional to $(\frac{r}{R})^{-sl}$, thus leaving possible only one of the two solutions to the second order equation in d_r for the $B(r)$ field. For the present needs, let us momentarily represent this solution in the form

$$B(r) = (\frac{r}{R})^{-sl}B_0. \tag{2.70}$$

B_0 being a complex integration constant. This being noted, the single remaining equation for the two electric field components is

$$E^r(r) + isE^\theta(r) = -\frac{l}{|\kappa|}\frac{1}{r}B(r). \tag{2.71}$$

Obviously, the general solution to this equation can be expressed under the following form

$$E^r(r) = \gamma(r) - \frac{l}{4|\kappa|}\frac{1}{r}B(r), \qquad E^\theta(r) = is\gamma(r) + is\frac{l}{4|\kappa|}\frac{1}{r}B(r), \tag{2.72}$$

where $\gamma(r)$ is an arbitrary complex function, but this function still has to be such that these last two following conditions are fulfilled,

$$\begin{aligned} -\frac{1}{r}d_r(rE^\theta) + \frac{il}{r}E^r &= -i2|\kappa|B, \\ \frac{1}{r}d_r(rE^r) + \frac{il}{r}E^\theta &= 2\kappa B. \end{aligned} \tag{2.73}$$

By a direct substitution into these two equations of the expressions above for $E^r(r)$ and $E^\theta(r)$, in each case it happens that they impose the following restriction for the $\gamma(r)$ function,

$$\frac{1}{r}d_r(r\gamma(r)) - \frac{sl}{r}\gamma(r) = 2\kappa B(r), \tag{2.74}$$

which consists in an inhomogeneous linear first-order differential equation in r. It is immediate to convince oneself that the general solution to the homogeneous equation is given by a complex integration constant that multiplies the function

$$(\frac{r}{R})^{sl-1}. \tag{2.75}$$

As for a particular solution to the inhomogeneous differential equation, supposing that it is given by a function homogeneous with respect to r as is $B(r)$ above, it is easy to establish that it is given by

$$\frac{\kappa R}{(1-sl)}(\frac{r}{R})^{-sl+1}B_0. \tag{2.76}$$

As such, $\gamma(r)$is necessarily given by

$$\gamma(r) = (\frac{r}{R})^{sl-1}\Gamma_0 + \frac{\kappa R}{(1-sl)}(\frac{r}{R})^{-sl+1}B_0, \tag{2.77}$$

where Γ_0 is a complex integration constant. In conclusion, we finally have, for the k-zero modes,

$$\begin{aligned} B(r) &= (\frac{r}{R})^{-sl}B_0, \\ E^r(r) &= (\frac{r}{R})^{sl-1}\Gamma_0 + \frac{\kappa R}{(1-sl)}(\frac{r}{R})^{-sl+1}B_0 - \frac{l}{4|\kappa|}\frac{1}{r}(\frac{r}{R})^{-sl}B_0, \\ E^\theta(r) &= is\Big((\frac{r}{R})^{sl-1}\Gamma_0 + \frac{\kappa R}{(1-sl)}(\frac{r}{R})^{-sl+1}B_0 + \frac{l}{4|\kappa|}\frac{1}{r}(\frac{r}{R})^{-sl}B_0\Big). \end{aligned} \tag{2.78}$$

Regularity with respect to r for $r = 0$

Now, one still has to consider the restrictions that are imposing themselves on these solutions' integration constants as one requires regularity at $r = 0$ for the $B(r), E^{(r,\theta)}(r)$ fields. For the k-non-zero modes, we know one has to exclude the $N_{|l|}(kr)$ branch for the $B(r)$ field's solution, only leaving an $aJ_{|l|}(kr)$ term, a being a complex integration constant,

$$B(r) = aJ_{|l|}(kr). \tag{2.79}$$

Subsequently, the electric field components are given by

$$E^r(r) = \frac{-1}{\omega^2 - 4\kappa^2}a(\frac{\omega l}{r}J_{|l|}(kr) + 2\kappa kJ'_{|l|}(kr)), \qquad E^\theta(r) = \frac{-i}{\omega^2 - 4\kappa^2}a(\frac{2\kappa l}{r}J_{|l|}(kr) + \omega kJ'_{|l|}(kr)), \tag{2.80}$$

where $J'_{|l|}(kr)$ represents the derivative with respect to its argument of the Bessel function $J_{|l|}(kr)$.

As for the k-zero modes for which the above solutions are given, we have a slightly more subtle situation, according to the sign of the product sl. In the case where $l = 0$, it clearly appears that we have to put $\Gamma_0 = 0$, which is to say,

$$l = 0: \quad B(r) = B_0, \quad E^r(r) = r\kappa B_0, \quad E^\theta(r) = r\kappa B_0. \tag{2.81}$$

However, when $sl > 0$ (or, put differently, when $sl = |l|$), B_0 must be equal to 0,

$$sl > 0: \quad B(r) = 0, \quad E^r(r) = (\frac{r}{R})^{sl-1}\Gamma_0, \quad E^\theta(r) = is(\frac{r}{R})^{sl-1}\Gamma_0. \tag{2.82}$$

On the other hand, when $sl < 0$, we need to have $\Gamma_0 = 0$, this time leaving

$$\begin{aligned} sl < 0: \quad B(r) &= (\frac{r}{R})^{-sl}B_0, \\ E^r(r) &= \Big(\frac{\kappa R}{(1-sl)}(\frac{r}{R})^{-sl+1} - \frac{l}{4|\kappa|R}(\frac{r}{R})^{-sl-1}\Big)B_0, \\ E^\theta(r) &= is\Big(\frac{\kappa R}{(1-sl)}(\frac{r}{R})^{-sl+1} + \frac{l}{4|\kappa|R}(\frac{r}{R})^{-sl-1}\Big)B_0. \end{aligned} \tag{2.83}$$

Let us note that, putting $l = 0$ in these last expressions, they reproduce the solution for this very value of l as it should.

All this being said, it is now possible to regroup every case for each field in order to have the final and complete solutions to the Maxwell equations for our system. First for the magnetic field we have,

$$\begin{aligned} B(t,r,\theta) &= \sum_{l=-\infty}^{+\infty} \int_{2|\kappa|}^{\infty} d\omega_l (a_l(\omega_l)e^{-i\omega_l t + il\theta} + a_l^\dagger(\omega_l)e^{i\omega_l t - il\theta})J_{|l|}(k_l r) \\ &+ \sum_{\kappa l \le 0} (\beta_l e^{-i2|\kappa|t + il\theta} + \beta_l^\dagger(\omega_l)e^{i\omega_l t - il\theta})(\frac{r}{R})^{|l|}. \end{aligned} \tag{2.84}$$

where we recall that $a_l(\omega_l)$ and β_l are the complex integration constants associated to this field, as the sum $\sum_{\kappa l\leq 0}$ represents the infinite series over all the values of l such that $\kappa l \leq 0$. Now for the two electric field components, we have, for the radial one,

$$\begin{aligned} E^r(t,r,\theta) &= \sum_{l=-\infty}^{+\infty} \fint_{2|\kappa|}^{\infty} d\omega_l \frac{-1}{\omega_l^2 - 4\kappa^2} (a_l(\omega_l)e^{-i\omega_l t + il\theta} + a_l^\dagger(\omega_l)e^{i\omega_l t - il\theta})(\frac{\omega_l l}{r} J_{|l|}(k_l r) + 2\kappa k_l J'_{|l|}(k_l r)) \\ &+ \sum_{\kappa l>0} (\gamma_l e^{-i2|\kappa|t+il\theta} + \gamma_l^\dagger(\omega_l) e^{i\omega_l t - il\theta})(\frac{r}{R})^{|l|-1} \\ &+ \sum_{\kappa l\leq 0} (\beta_l e^{-i2|\kappa|t+il\theta} + \beta_l^\dagger(\omega_l)e^{i\omega_l t-il\theta})\Big(\frac{\kappa R}{|l|+1}(\frac{r}{R})^{|l|+1} + \frac{|l|}{4\kappa R}(\frac{r}{R})^{|l|-1}\Big). \end{aligned} \tag{2.85}$$

where the γ_l again are complex integration constants, while for the angular component of the electric field we have

$$\begin{aligned} E^\theta(t,r,\theta) &= \sum_{l=-\infty}^{+\infty} \fint_{2|\kappa|}^{\infty} d\omega_l \frac{-i}{\omega_l^2 - 4\kappa^2} (a_l(\omega_l)e^{-i\omega_l t + il\theta} - a_l^\dagger(\omega_l)e^{i\omega_l t - il\theta})(\frac{2\kappa l l}{r} J_{|l|}(k_l r) + \omega_l k_l J'_{|l|}(k_l r)) \\ &+ \sum_{\kappa l>0} is(\gamma_l e^{-i2|\kappa|t+il\theta} - \gamma_l^\dagger(\omega_l) e^{i\omega_l t - il\theta})(\frac{r}{R})^{|l|-1} \\ &+ \sum_{\kappa l\leq 0} is(\beta_l e^{-i2|\kappa|t+il\theta} - \beta_l^\dagger(\omega_l)e^{i\omega_l t-il\theta})\Big(\frac{\kappa R}{|l|+1}(\frac{r}{R})^{|l|+1} - \frac{|l|}{4\kappa R}(\frac{r}{R})^{|l|-1}\Big). \end{aligned} \tag{2.86}$$

One note must be made here, about the k-zero modes contributions. Indeed only the $\kappa l \leq 0$ chirality modes contribute to the magnetic field, as the $\kappa l > 0$ chirality ones are absent for $\kappa \neq 0$. This only chirality in $B(t,r,\theta)$ is the one opposed to the sign of κ, a parameter that violates the invariance under parity in the disk (as well as time reversal symmetry). Furthermore, these k-zero chiral modes that contribute to the magnetic field aslo determine completely the k-zero modes of the same chirality contributing to the electric field. However the electric field also has k-zero modes of opposite chirality, such that $\kappa l > 0$, with a configuration that is parametrized by the complex constants γ_l, which are consequently completely independent from the magnetic field ones.

2.6 Construction of the gauge potentials

Having identified the magnetic and electric fields, it becomes possible to determine a set of gauge potentials, A^t, A^r and A^θ associated to them such that

$$B = \frac{1}{r}\partial_r(rA^\theta) - \frac{1}{r}\partial_\theta A^r, \quad E^r = -(\partial_t A^r + \partial_r A^t), \quad E^\theta = -(\partial_t A^\theta + \frac{1}{r}\partial_\theta A^t). \tag{2.87}$$

Because of the system's $U(1)$ gauge symmetry, we know that these potentials are defined up to the following gauge transformations, with $\chi(t,r,\theta)$ an arbitrary function such that $\chi(t,R,\theta) = 0$,

$$\tilde{A}^t = A^t + \partial_t\chi, \qquad \tilde{A}^r = A^r - \partial_r\chi, \qquad \tilde{A}^\theta = A^\theta - \frac{1}{r}\partial_\theta\chi. \tag{2.88}$$

As explained earlier, this symmetry would allow us to fix the radial gauge $A^r(t,r,\theta) = 0$, and as such define the potentials $A^t(t,r,\theta)$ and $A^\theta(t,r,\theta)$ from the representation obtained for the $B(t,r,\theta), E^{(r,\theta)}(t,r,\theta)$ fields. Even though one can use this approach, it would lead us to express the gauge potential as specific integrals of the Bessel functions and their derivatives, which are not fit for the still to be solved equations regarding the scalar field $\varphi(t,\theta)$. Again, it is possible to consider the problem under a form taking advantage of an $e^{-i\omega t+il\theta}$ kind of mode decomposition for the electromagnetic fields and the associated differential equations, as the core relation of this more direct approach that doesn't address the question of a radial gauge but allows one to identify a collection of gauge potentials fitting the aforementioned equations. We are left with the Bessel equations obeyed by $B(r)$, namely

$$\frac{1}{r}d_r(rkJ'_{|l|}(kr)) = -k^2(1 - \frac{l^2}{(kr)^2})J_{|l|}(kr). \tag{2.89}$$

If, once again, $B(r), E^{(r,\theta)}(r)$ represent the complex modes associated to specific values of $\omega \geq 2|\kappa|$ and $l \in \mathbb{Z}$, the equations we consider are written the following way

$$\frac{1}{r}d_r(rA^\theta) - \frac{il}{r}A^r = B, \qquad i\omega A^r - d_r A^t = E^r, \qquad i\omega A^\theta - \frac{il}{r}A^t = E^\theta, \tag{2.90}$$

with $B(r), E^{(r,\theta)}(r)$ given by the previously established expressions. If we manage to determine a collection of gauge potentials that are solutions to these equations, the general solution is obtained by applying a general gauge transformation parametrized by an arbitrary function $\chi(t,r,\theta)$ such that $\chi(t,R,\theta) = 0$. We now have to distinguish the two situations where $k = 0$ or $k > 0$.

k-non-zero modes

When $\omega > 2|\kappa|$, the equations we have to consider are

$$\begin{aligned}
\frac{1}{r}d_r(rA^\theta) - \frac{il}{r}A^r &= J_{|l|}(kr), \\
i\omega A^r - d_r A^t &= \frac{-1}{\omega^2 - 4\kappa^2}\left(\frac{\omega l}{r}J_{|l|}(kr) + 2\kappa k J'_{|l|}(kr)\right), \qquad (2.91)\\
i\omega A^\theta - \frac{il}{r}A^t &= \frac{-i}{\omega^2 - 4\kappa^2}\left(\frac{2\kappa l}{r}J_{|l|}(kr) + \omega k J'_{|l|}(kr)\right). \qquad (2.92)
\end{aligned}$$

Because of the equation that $rJ_{|l|}(kr)$ obeys and on the basis of the first of this set of equations, it is possible to take, as a starting point for the construction of a solution to these three equations the fact that $A^\theta(r)$ may be proportional to $J'_{|l|}(kr)$ and $A^r(r)$ to $J_{|l|}(kr)$ multiplied by a certain function of r. Indeed, such an *ansatz* substituted in the first of the three equations above allows one to determine $A^\theta(r)$ and $A^r(r)$ with an amplitude that we are still leaving free for $A^\theta(r)$. By substituting these two configurations for these two potentials in the two other equations, it eventually becomes possible to also determine the $A^t(r)$ potential as well as the normalisation factors, hence leading to a solution for the equations above.

Such an analysis leads to the following expressions for any $l \in \mathbb{Z}$, with $k^2 = \omega^2 - 4\kappa^2$,

$$\begin{aligned}
A^t(r) &= \frac{2\kappa}{4\kappa^2 - \omega^2}J_{|l|}(kr) = \frac{\kappa}{k^2}J_{|l|}(kr), \\
A^r(r) &= \frac{il}{\omega^2 - 4\kappa^2}J_{|l|}(kr) = \frac{il}{r}\frac{1}{k^2}J_{|l|}(kr), \\
A^\theta(r) &= \frac{-k}{\omega^2 - 4\kappa^2}J'_{|l|}(kr) = -\frac{1}{k}J'_{|l|}(kr).
\end{aligned} \tag{2.93}$$

k-zero modes

In the k-zero mode sector, it is necessary to distinguish the two separate solutions according to whether $\kappa l > 0$ or $\kappa l \leq 0$. In the first case, we must have

$$\begin{aligned}
\frac{1}{r}d_r(rA^\theta) - \frac{il}{r}A^r &= \left(\frac{r}{R}\right)^{|l|}\beta_l, \\
i2|\kappa|A^r - d_r A^t &= \Big(\frac{\kappa R}{|l|+1}\left(\frac{r}{R}\right)^{|l|+1} + \frac{|l|}{4\kappa R}\left(\frac{r}{R}\right)^{|l|-1}\Big)\beta_l, \\
i2|\kappa|A^r - \frac{il}{r}A^t &= is\Big(\frac{\kappa R}{|l|+1}\left(\frac{r}{R}\right)^{|l|+1} - \frac{|l|}{4\kappa R}\left(\frac{r}{R}\right)^{|l|-1}\Big)\beta_l.
\end{aligned} \tag{2.94}$$

Again, examining the form of these equations starting from the first of the three equations of each situation, it is immediately established that the following solution is given by the following expressions, for $\kappa l > 0$,

$$A^t(r) = -\frac{R}{|l|}\left(\frac{r}{R}\right)^{|l|}\gamma_l, \qquad A^r(r) = 0, \qquad A^\theta(r) = 0, \tag{2.95}$$

and for $\kappa l \leq 0$,

$$\begin{aligned}
A^t(r) &= -\frac{1}{4\kappa}(\frac{r}{R})^{|l|}\beta_l, \\
A^r(r) &= -is\frac{R}{2(|l|+1)}(\frac{r}{R})^{|l|+1}\beta_l, \\
A^\theta(r) &= \frac{R}{2(|l|+1)}(\frac{r}{R})^{|l|+1}\beta_l.
\end{aligned} \tag{2.96}$$

The complete gauge potentials

On the basis of the set of results established earlier for the $A^{(t,r,\theta)}(r)$ modes and applying another general gauge transformation, it follows that the general configurations of the gauge potentials associated to the solutions constructed earlier for the electromagnetic field are represented by the following expressions,

$$\begin{aligned}
A^t(t,r,\theta) &= \sum_{l=-\infty}^{+\infty} \oint\!\!\!\!\!\!\int_{2|\kappa|} d\omega_l \frac{-2\kappa}{k_l^2}(a_l(\omega_l)e^{-i\omega_l t+il\theta} + a_l^\dagger(\omega_l)e^{i\omega_l t-il\theta})J_{|l|}(k_l r) \\
&+ \sum_{\kappa m>0}(\gamma_l e^{-2|\kappa|t+il\theta} + \gamma_l^\dagger e^{i2|\kappa|t-il\theta})(\frac{-R}{|l|})(\frac{r}{R})^{|l|} \\
&+ \sum_{\kappa l\leq 0}(\beta_l e^{-i2|\kappa|t+il\theta} + \beta_l^\dagger e^{i2|\kappa|t-il\theta})(\frac{-1}{4\kappa})(\frac{r}{R})^{|l|} \\
&+\partial_t\chi(t,r,\theta), \\
A^r(t,r,\theta) &= \sum_{l=-\infty}^{+\infty} \oint\!\!\!\!\!\!\int_{2|\kappa|} d\omega_l \frac{il}{k_l^2}\frac{1}{r}(a_l(\omega_l)e^{-i\omega_l t+il\theta} - a_l^\dagger(\omega_l)e^{i\omega_l t-il\theta})J_{|l|}(k_l r) \\
&+ \sum_{\kappa l\leq 0}(-is)(\beta_l e^{-i2|\kappa|t+il\theta} - \beta_l^\dagger e^{i2|\kappa|t-il\theta})\frac{R}{2(|l|+1)}(\frac{r}{R})^{|l|+1} \\
&-\partial_r\chi(t,r,\theta), \\
A^\theta(t,r,\theta) &= \sum_{l=-\infty}^{+\infty} \oint\!\!\!\!\!\!\int_{2|\kappa|} d\omega_l \frac{-1}{k_l}(a_l(\omega_l)e^{-i\omega_l t+il\theta} + a_l^\dagger(\omega_l)e^{i\omega_l t-il\theta})J'_{|l|}(k_l r) \\
&+ \sum_{\kappa l\leq 0}(\beta_l e^{-i2|\kappa|t+il\theta} + \beta_l^\dagger e^{i2|\kappa|t-il\theta})\frac{R}{2(|l|+1)}(\frac{r}{R})^{|l|+1} \\
&-\frac{1}{r}\partial_\theta\chi(t,r,\theta),
\end{aligned} \tag{2.97}$$

where $\chi(t,r,\theta)$ is an arbitrary function such that $\chi(t,R,\theta) = 0$.

2.7 Analytic solution for the scalar field

In the scalar sector on the boundary of the disk, we have the following boundary conditions,

$$\mu\partial_t\varphi = \bar{B} + \kappa\bar{A}^t, \qquad \mu\partial_\theta\varphi = R\bar{E}^r - \kappa R\bar{A}^\theta, \tag{2.98}$$

as well as the following equations of motion

$$(\partial_t^2 - \frac{1}{R^2}\partial_\theta^2 + m^2)\varphi = \mu\bar{E}^\theta, \qquad \pi_\varphi = R(\partial_t\varphi + \mu\bar{A}^\theta) \tag{2.99}$$

The goal here is to harness the previously established expressions for B and $E^{(r,\theta)}$ to finally construct the analytic solution for the scalar field, last in our list.

By first considering the relation that involves the first time derivative of φ, we find

$$\begin{aligned}
\mu\partial_t\varphi(t,\theta) \quad = \quad & \sum_{l=-\infty}^{\infty}\fint_{2|\kappa|}^{\infty} d\omega_l (a_l(\omega_l)e^{-i\omega_l t+il\theta}+a_l^\dagger(\omega_l)e^{i\omega_l t-il\theta})(1+\frac{2\kappa^2}{k_l^2})J_{|l|}(k_l R) \\
& -\sum_{\kappa l>0}\frac{\kappa R}{|l|}(\gamma_l e^{-i2|\kappa|t+il\theta}+\gamma_l^\dagger e^{i2|\kappa|t-il\theta}) \\
& -\sum_{\kappa l\leq 0}(1-\frac{\kappa}{4|\kappa|})(\beta_l e^{-i2|\kappa|t+il\theta}+\gamma_l^\dagger e^{i2|\kappa|t-il\theta}),
\end{aligned} \tag{2.100}$$

lead to a first expression for φ,

$$\begin{aligned}
\varphi(t,\theta) \quad = \quad & \sum_{l=-\infty}^{\infty}\fint_{2|\kappa|}^{\infty} d\omega_l \frac{1}{-i\mu\omega_l}(1+\frac{2\kappa^2}{k_l^2})(a_l(\omega_l)e^{-i\omega_l t+il\theta}-a_l^\dagger(\omega_l)e^{i\omega_l t-il\theta})J_{|l|}(k_l R) \\
& -\sum_{\kappa l>0}\frac{\kappa R}{\mu|l|}(\gamma_l e^{-i2|\kappa|t+il\theta}-\gamma_l^\dagger e^{i2|\kappa|t-il\theta}) \\
& -\sum_{\kappa l\leq 0}(1-\frac{s}{4})(\beta_l e^{-i2|\kappa|t+il\theta}-\gamma_l^\dagger e^{i2|\kappa|t-il\theta})+f(\theta),
\end{aligned} \tag{2.101}$$

where s is the same as the one we used to express $E^{(r,\theta)}$ and $f(\theta)$ is the integration constant from the point of view of the time dependence.
Let us now do the same starting from the second equation in 2.97,

$$\begin{aligned}
\mu\partial_\theta\varphi(t,\theta) \quad = \quad & \sum_{l=-\infty}^{\infty}\fint_{2|\kappa|}^{\infty} d\omega_l (a_l(\omega_l)e^{-i\omega_l t+il\theta}+a_l^\dagger(\omega_l)e^{i\omega_l t-il\theta})(-\frac{l\omega_l}{k_l^2}J_{|l|}(k_l R)-\frac{\kappa R}{k_l}J'_{|l|}(k_l R)) \\
& +\sum_{\kappa l>0}R(\gamma_l e^{-i2|\kappa|t+il\theta}+\gamma_l^\dagger e^{i2|\kappa|t-il\theta}) \\
& -\sum_{\kappa l\leq 0}(\frac{\kappa R^2}{2(1+|l|)}-\frac{l}{4|\kappa|})(\beta_l e^{-i2|\kappa|t+il\theta}+\gamma_l^\dagger e^{i2|\kappa|t-il\theta}),
\end{aligned} \tag{2.102}$$

from which we find an alternative expression for φ,

$$\begin{aligned}
\varphi(t,\theta) \quad = \quad & \sum_{l=-\infty}^{\infty}\fint_{2|\kappa|}^{\infty} d\omega_l \frac{1}{-i\mu\omega_l}(a_l(\omega_l)e^{-i\omega_l t+il\theta}-a_l^\dagger(\omega_l)e^{i\omega_l t-il\theta})(\frac{i\omega_l}{\mu k_l^2}J_{|l|}(k_l R)-\frac{\kappa R}{ilk_l}J'_{|l|}(k_l R)) \\
& +\sum_{\kappa l>0}\frac{R}{\mu il}(\gamma_l e^{-i2|\kappa|t+il\theta}-\gamma_l^\dagger e^{i2|\kappa|t-il\theta}) \\
& -\sum_{\kappa l\leq 0}(\frac{\kappa R^2}{2il(1+|l|)}-\frac{1}{4i|\kappa|})(\beta_l e^{-i2|\kappa|t+il\theta}-\gamma_l^\dagger e^{i2|\kappa|t-il\theta})+g(t).
\end{aligned} \tag{2.103}$$

Unfortunately, it appears these two equations are completely incompatible, starting with a quick dimensional comparison of the two results, unmistakingly because of the boundary conditions or our gauge choice when $\kappa\neq 0$. The next chapter will be dedicated in finding why and what could perhaps be done.

2.8 Summary

In this second chapter, we used everything that has been previously set up in order to study our full version of a topologically massive gauge field theory in the hope that we would be able to describe its symmetries and the dynamics of each of the fields and gauge potentials involved.

We naturally began introducing the various parts of our action, composed of Maxwell and Chern-Simons field theories, as well as a massive scalar field exclusively interacting with the gauge potential on the boundary of our space-time, namely a (2+1)-dimensional cylinder. After briefly recalling the Maxwell equations inherent to the Maxwell theory, we had a very brief discussion about $U(1)$ gauge invariance.

In the following section, we proceeded further with the hamiltonian formalism, expressing the various momenta each obtained through the variation relative to each of the relevant field or gauge potential, which allowed us to express the hamiltonian density and the lagrangian density in its hamiltonian form. We used the latter to extract the equations of motion and the constraints that came along. We finished by writing the Klein-Gordon equation governing the magnetic field B.

The next step was to find an analytic expression for the very same magnetic field as well as for the electric field components and the gauge potential. We did so by first noticing that the solution to the Klein-Gordon equation obeyed by $B(t, r, \theta)$ was a Bessel function, provided each wavenumber associated to an angular momentum l was such that $k_l = \sqrt{\omega_l^2 - 4\kappa^2} \geq 0$, which led us to introduce the symbol $\sum\!\!\!\!\!\!\int$ as long as we weren't sure whether the frequency spectrum was discrete or continuous. We kept the discussion going with our first attempt to construct the analytic expression of the electric components, "abandoning" the momenta π_r, π_θ, but we got stuck, realizing we couldn't go further with the gauge choice $A^t = 0$.

We thus took a deeper look at the $U(1)$ gauge invariance of the system and concluded the proper gauge choice had to be $A(t, R, \theta) = 0$, noticing that when $\kappa \neq 0$, A^t and A^θ on the boundary of the disk are physical gauge invariant degrees of freedom.

We were then able to pursue the construction of the electric field components analytic expressions, by first transforming the equations relating them to the magnetic field and to each other into linear ordinary derivatives with respect to the radial degree of freedom. Once it was done, we decided to separate the analysis according to whether we were in a $\omega > 2|\kappa|$ or a $\omega = 2|\kappa|$ case for each of these cases separates the contributions according to the sign of κl. The final step of this analysis is to require regularity of the fields at $r = 0$.

Eventually, the exact same process was followed for the gauge potentials, but when finally proceeding to find an analytic solution for the scalar field φ, it was quite immediate to see that there was a problem, coming from the boundary conditions and/our the gauge choice.

Chapter 3

Reconsidering the question of the gauge invariance

3.1 Discussion over the consistency between the boundary conditions and φ's gauge invariance property

It appeared that, in the presence of the Chern-Simons term of our action, we cannot restore this action's invariance for all gauge transformations, *i.e.* inclusive to those that do not meet the boundary condition $\chi(t,R,\theta)=0$. For a moment, my supervisor and I hoped that it would be possible to restore the situation by adding a boundary term proportional to $\bar{A}^t\bar{A}^\theta$, but this suggestion eventually proved ill-fated.

For a gauge transformation, the Chern-Simons term, $\int d^3x^\mu \frac{1}{2}\kappa\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho}$ transforms as follows in its contribution to the action (ignoring a total time derivative term),

$$\Delta S_{CS}=\int dt\int_0^{2\pi}d\theta(-\kappa)(R\bar{\chi}\bar{E}^\theta). \tag{3.1}$$

With the form of action we consider, it is not possible to find any other term whose gauge variation may counterbalance this contribution. It is not excluded that, by modifying the φ field coupling to the electromagnetic sector but with a scalar field that is then not gauge invariant anymore, we may find a way to bypass this situation, but presently we won't address the question of how to possibly modify our choice of action, give the time that was left to complete the Master thesis.
Searching a bit in the literature, it is also the conclusion that the other authors found (when one makes the choice not to add other degrees of freedom to the system). Under such conditions, in order to ensure the gauge invariance of the action we are interested in, either we restrict ourselves to gauge transformations such that $\bar{\chi}(t,R,\theta)=0$ as discussed previously or we restrict ourselves to electric field configurations such that $\bar{E}^\theta=0$, such that the space-like surface term above cancels but without any other particular restriction to be imposed for $\chi(t,r,\theta)$.

Whether it be one or another of the two possible choices (that happen not to be exclusives), it is then also possible to explicitly check that the action for the scalar field $\varphi(t,\theta)$ on the boundary of the disk is gauge invariant by itself, without any restriction on $\bar{\chi}(t,R,\theta)$ (as it is the case when $\kappa=0$).
Nonetheless we also still have these two following boundary conditions,

$$\mu\partial_t\varphi=\bar{B}+\kappa\bar{A}^t, \qquad \mu\partial_\theta\varphi=R\bar{E}^r-\kappa R\bar{A}^\theta, \tag{3.2}$$

while the gauge transformations of the electromagnetic potentials are

$$\tilde{A}^t=A^t+\partial_t\chi, \qquad \tilde{A}^r=A^r-\partial_r\chi, \qquad \tilde{A}^\theta=A^\theta-\frac{1}{r}\partial_\theta\chi, \tag{3.3}$$

and the scalar field φ is gauge invariant by itself. The consistency between this property and the transformation properties under gauge transformations of the two boundary conditions above requires that the gauge parameters are such that $\bar{\chi}(t,R,\theta) = 0$ anyway.

Hence indeed, because of gauge symmetry, we have to restrict to gauge transformations such that $\bar{\chi}(t,R,\theta) = 0$, whether the condition $\bar{E}^\theta(t,R,\theta) = 0$ is imposed or not.

However, let us analyze further the two previous boundary conditions, that are correct as they are (and determined as being the consequence of the equations of motion associated to the action choice we made). Taking the angular derivative of the first one and the time derivative of the second one, as either way the left hand member gives $\mu\partial_t\partial_\theta$, one needs the right hand expressions to be equal. Which is to say,

$$\partial_\theta\bar{B} + \kappa\partial_\theta\bar{A}^t = \partial_t\bar{E}^r - \kappa R\bar{A}^\theta, \tag{3.4}$$

or equivalently

$$\partial_t\bar{E}^r - \frac{1}{R}\partial_\theta\bar{B} = \kappa(\partial_t\bar{A}^\theta + \partial_\theta\bar{A}^t), \tag{3.5}$$

or, eventually, when $\kappa \neq 0$, the consistency of the entirety of these equations and boundary conditions impose that we have to set, apart from the condition $\bar{\chi}(t,R,\theta) = 0$, the following restriction in the electromagnetic sector anyway,

$$\bar{E}^\theta(t,R,\theta) = 0. \tag{3.6}$$

3.2 The final form of the solutions to each field and gauge potential

The expressions for the solutions for $B(t,r,\theta), E^{(r,\theta)}(t,r,\theta), A^{(t,r,\theta)}(t,r,\theta)$ are known, and for the electromagnetic potentials it is even the case for an arbitrary gauge. There remains to solve the equations for $\varphi(t,\theta)$ and $\pi_\varphi(t,\theta)$, but this time imposing that $\bar{E}^\theta(t,R,\theta) = 0$.

Considering then the explicit expression for $E^\theta(t,r,\theta)$ and this boundary condition, it is possible to conclude that we necessarily need to have, for the k-zero modes,

$$\kappa l > 0: \quad \gamma_l = 0; \qquad \kappa l \leq 0: \quad \beta_l = 0, \tag{3.7}$$

as for the k-non-zero modes,

$$(\omega_l k_l J'_{|l|}(k_l R) + \frac{2\kappa l}{R} J_{|l|}(k_l R))a_l(\omega_l) = 0. \tag{3.8}$$

This equation is quite special. For non zero values of the $a_l(\omega_l)$ amplitudes, it imposes the following condition

$$\omega_l k_l J'_{|l|}(k_l R) + \frac{2\kappa l}{R} J_{|l|}(k_l R) = 0. \tag{3.9}$$

One might think that this condition so determines the frequency spectrum ω_l. However, as the values of $\omega_l, k_l, J_{|l|}(k_l R)$ and $J'_{|l|}(k_l R)$ do not depend on the sign of l, if a solution to this equation exists for a particular value of l, and as such determines the values for the corresponding k_l and ω_l with an non zero $a_l(\omega_l)$ amplitude, no solution can exist for the same value of l of opposite sign, and in this case, it must be the corresponding $a_{-l}(\omega_l)$ coefficient that has to cancel. Put another way, excepted for $l = 0$, in the best case scenario the infinite "half" of all the $a_l(\omega_l)$ must cancel, as for the other half the corresponding k_l and ω_l must obey the previously stated spectral condition. Let us note that for $l = 0$ the values of k_0 have to be part of the spectrum of the roots of the function $J'_0(x) = -J_1(x)$, *ie* $k_{0,n} = \frac{x_{0,n}}{R}$ where $x_n, (n = 1, 2, ...)$, are the positive roots, or the values such that $J_1(x_n) = 0$.

But once the above conclusion is reached, with the previous condition, it becomes possible to reconsider all the expressions for the $B(t,r,\theta), E^{(r,\theta)}(t,r,\theta), A^{(t,r,\theta)}(t,r,\theta)$ fields using the expression of $J'_{|l|}(k_l R)$ in terms of $J_{|l|}(k_l R)$ obtained from the next condition for the non-zero $a_l(\omega_l)$ amplitudes. By doing so, only the k-non-zero modes still contribute to all these fields, and only the functions $J_{|l|}(k_l R)$ still play a part in all of their expressions.

It then happens to be very easy to solve the two boundary conditions for the field φ and obtain their solution under the unique following form,

$$\varphi(t,\theta) = \sum_l \sum_{\omega_l} \frac{1}{\mu}(1+\frac{2\kappa^2}{k_l^2})(\frac{i}{\omega_l})(a_l(\omega_l)e^{-i\omega_l t+il\theta} - a_l^\dagger(\omega_l)e^{i\omega_l t-il\theta})J_{|l|}(k_l R) + \varphi_0, \tag{3.10}$$

where φ_0 is a last integration constant and the sum over the l is just applied to the contributions associated to the non-zero $a_l(\omega_l)$, and where the sum over the ω_l is about the spectrum of the values of ω_l determined by the previous spectral condition.

However, this field must still obey the Klein-Gordon equation,

$$(\partial_t^2 - \frac{1}{R^2}\partial_\theta^2 + m^2)\varphi = \mu\bar{E}^\theta = 0, \tag{3.11}$$

which implies, still for non zero $a_l(\omega_l)$,

$$\sum_l \sum_{\omega_l} \frac{1}{\mu}(1+\frac{2\kappa^2}{k_l^2})(\frac{i}{\omega_l})(-\omega_l^2 + \frac{l^2}{R^2} + m^2)(a_l(\omega_l)e^{-i\omega_l t+il\theta} - a_l^\dagger(\omega_l)e^{i\omega_l t-il\theta})J_{|l|}(k_l R) = 0, \tag{3.12}$$

and $\varphi_0 = 0$ if $m^2 \neq 0$. Hence one needs that, for every non zero value of $a_l(\omega_l)$, the two following conditions are fulfilled,

$$\omega_l k_l J'_{|l|}(k_l R) + \frac{2\kappa l}{R} J_{|l|}(k_l R) = 0, \qquad \omega_l^2 = m^2 + \frac{l^2}{R^2} \quad \text{or} \quad k_l^2 = m^2 - 4\kappa^2 + \frac{l^2}{R^2}. \tag{3.13}$$

As such, three mass scales "enter into competition", namely the inverse radius of the disc, $\frac{1}{R}$, as well as the two masses $2|\kappa|$ and m. In the $\kappa = 0$ case, the two scales $\frac{1}{R}$ and m combine themselves as a spectral relation determining the frequency spectrum of the system, included in the electromagnetic sector of mass equal to zero, but when $\kappa \neq 0$, it becomes essentially impossible to satisfy both equations at the same time for the values of l associated to non zero values of $a_l(\omega_l)$, in particular because, now, the electromagnetic sector isn't of mass equal to zero but equal to $2|\kappa|$.

For example, if the two scales m^2 and $4\kappa^2$ are degenerate, $4\kappa^2 = m^2$, we must have $k_l R = |l|$, but there is nothing that tells us that the values of $J_{|l|}(|l|)$ and $J'_{|l|}(|l|)$ are such that the first condition above is met as well, or reversely that, if values of $k_l > 0$ may be found such that they meet the first spectral condition, then the condition $k_l^2 R^2 = (m^2 - 4\kappa^2)R^2 + l^2$ is satisfied as well.

The inevitable conclusion for the generic situation is that it is excluded to satisfy all the equations of the system for arbitrary values of R, κ and m any other way than for the trivial solution with, addingly, $a_l(\omega_l) = 0$ for every value of l, which is to say having every electromagnetic field of the system equal to zero, or gauge values equivalent to being equal to zero for the electromagnetic potentials, and a constant value (and equal to 0 if $m^2 \neq 0$) for the scalar field φ (whose action is indeed invariant under the constant translations in its configurations, $\varphi(t,r,\theta) \longrightarrow \varphi(t,r,\theta) + \varphi_0$, in the case of a zero mass, $m^2 = 0$).

Because of the coupling of the electromagnetic sector both with the Chern-Simons term violating the invariance under parity as well as under time reversal of the disc's boundary coupling and the coupling of the type "electric dipolar" with the scalar field on the boundary of the disc, eventually the hereby defined dynamics is so constrained that it is impossible classically, except for a complete absence of any oscillation in these fields. The "vacuum" configuration is the only one still possible, as soon as $\kappa \neq 0$.

Chapter 4

Conclusions

Even though our study of the Maxwell-Chern-Simons theory in a disk coupled to a scalar field on its boundary proved to be so restrictive as to allow as only possible solution the trivial one, let us briefly sum up what has been done in this Master thesis and confront it to our initial expectations.

The first chapter was dedicated to the mathematical and physical concepts necessary as a common ground in order for any reader to understand the motivations and the approaches of the second chapter.
After that, in the latter we undertook to describe the action of the theory we have been studying, one contribution to it at a time, before writing the hamiltonian density and the hamiltonian form of the lagrangian density of this theory, in order to extract the equations of motions.
We then quickly presented the notion of gauge invariance and our first gauge choice before constructing solutions for the various fields and gauge potentials, starting with the magnetic field B. In order to do this, we separated its angular dependence from its radial and time ones, changed the latter to a dependence in the frequency, and noticed the equation obeyed by $B(\omega_l, r)$ defines the Bessel functions. This allowed us to write the final form of the analytic solution for $B(t, r, \theta)$, for which we separated the frequency development between the $\omega_l > 2|\kappa|$ case and the $\omega_l = 2|\kappa|$ case, as we weren't sure if the latter could be included in the sum/integral over the frequency spectrum, about which we weren't sure either whether it was discrete or not. By using the relations between the magnetic and electric fields components, we found analytic solutions for these as well. However, carrying on with the scalar field, we realized we had to change our gauge choice which, fortunately, had not been implemented yet at that stage in the construction of the solutions for all the gauge invariant fields in the electromagnetic sector of the system.
The discussion that followed made us realize that the only gauge potential on which we could impose a restriction was A^r, since the other two are physical gauge invariant degrees of freedom on the boundary of the disk when $\kappa \neq 0$.
We then started over the search of analytic solutions for the electric field components, rewriting the equations relating B to E^r and E^θ as ordinary derivatives with respect to r, and by distinguishing the following cases, $\omega \geq 2|\kappa|$, $\omega = 2|\kappa|$, $\kappa l > 0$ and $\kappa l \leq 0$ and taking into account regularity with respect to r for $r = 0$.
After that, we followed the same steps in order to write the analytic solutions for the gauge potentials, but once again, when we finally proceeded with the solution for the scalar field, it clearly appeared that the boundary conditions or the gauge choice needed to be reconsidered.

Eventually, in the third chapter we carefully reconsidered the consequences of a non-zero κ, namely for gauge invariance and the boundary conditions that couple both the electromagnetic and the scalar sectors on the disk's boundary. We concluded that either the allowed gauge transformations have to be trivial on the disk boundary, or else that the angular component of the electric field E^θ has to vanish at that boundary, noticing that these choices are not mutually exclusive. However, whether we chose one or the other, both choices ended in E^θ vanishing on the disk's boundary, among other restrictions that both options shared. Furthermore, given our choice of coupling term in the action between the boundary scalar field and the electromagnetic one, it turns out that only trivial oscillatory modes are allowed as solutions to the complete set of coupled equations of motion. Such a conclusion was totally unexpected and most disappointing indeed.

As presented in the introduction, we had strongly hoped that, in the scope of the AdS/CFT correspondence, we would find results similar to the ones obtained in the absence of the Chern-Simons coupling, up to a redefinition of the spectrum of the quantum states of the system. However, unmistakably, the presence of this coupling has been much more restrictive than expected and it proved impossible to even try to define the quantum states spectrum beyond the ground state or quantum vacuum in the case of the quantised theory.
The above is thus the main and unexpected conclusion of this work. However at the same time it raises a prospective question as well, that goes beyond the time limit of the present Master thesis.
Namely, how may one adopt another coupling of the Maxwell-Chern-Simons theory in the disk to some field, scalar or otherwise, living on its boundary, and then preserve all at once gauge invariance for all gauge transformations without restriction, as well as all dynamical modes of all fields albeit that their amplitude and energy spectra are thereby correlated and entangled, as is the case when $\kappa = 0$, namely for the pure Maxwell theory? Thereby providing another explicit example, still void of supersymmetry, of gauge field theory in the bulk coupled to a conformal field theory on the boundary, entangled with one another, very much in the spirit of the AdS/CFT correspondence, which after all remains the prime motivation for the study having been presented in this study.

Appendices

Appendix A

Vector analysis in cylindrical coordinates in 2 euclidian dimensions

Cartesian to polar coordinates change :

$$\begin{aligned} x &= r\cos(\theta), \qquad & y = r\sin(\theta), \\ r &= \sqrt{x^2+y^2}, \qquad & \theta = \arctan\frac{y}{x}, \end{aligned} \tag{A.1}$$

Cartesian and polar coordinates basis:

$$\begin{aligned} \hat{e}_r &= \hat{e}_x\cos\theta + \hat{e}_y\sin\theta; \qquad & \hat{e}_x = \hat{e}_r\cos\theta - \hat{e}_\theta\sin\theta; \\ \hat{e}_r &= -\hat{e}_x\sin\theta + \hat{e}_y\cos\theta; \qquad & \hat{e}_y = \hat{e}_r\sin\theta + \hat{e}_\theta\cos\theta; \end{aligned} \tag{A.2}$$

Vector field decomposition :

$$\vec{A} = \hat{e}_x A^x + \hat{e}_y A^y = \hat{e}_r A^r + \hat{e}_\theta A^\theta, \tag{A.3}$$

with

$$\begin{aligned} A^r &= A^x\cos\theta + A^y\sin\theta; \qquad & A^x = A^r\cos\theta - A^\theta\sin\theta \\ A^r &= -A^x\sin\theta + A^y\cos\theta; \qquad & A^y = A^r\sin\theta + A^\theta\cos\theta \end{aligned} \tag{A.4}$$

And

$$\vec{A}^2 = A^{x2} + A^{y2} = A^{r2} + A^{\theta 2}. \tag{A.5}$$

Vector analysis :

$$\begin{aligned} \frac{\partial}{\partial x} &= \cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}; \qquad & \frac{\partial}{\partial r} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}; \\ \frac{\partial}{\partial y} &= \sin\theta\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial}{\partial\theta}; \qquad & \frac{\partial}{\partial\theta} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}; \\ \vec{\nabla}\phi &= \hat{e}_x\frac{\partial\phi}{\partial x} + \hat{e}_y\frac{\partial\phi}{\partial y} = \hat{e}_r\frac{\partial\phi}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial\phi}{\partial\theta}; & \\ \vec{\nabla}\cdot\vec{A} &= \frac{\partial A^x}{\partial x} + \frac{\partial A^y}{\partial y} = \frac{\partial A^r}{\partial r} + \frac{1}{r}A^r + \frac{1}{r}\frac{\partial A^\theta}{\partial\theta} = \frac{1}{r}\partial_r(rA^r) + \frac{1}{r}\partial_\theta A^\theta; & \\ \vec{\nabla}^2\phi &= \vec{\nabla}\cdot(\vec{\nabla}\phi) = (\partial_x^2 + \partial_y^2)\phi = \frac{1}{r}\partial_r(r\partial_r)\phi + \frac{1}{r^2}\partial_\theta^2\phi. & \end{aligned} \tag{A.6}$$

Integration of a surface term over a disk D of radius R and its boundary C :

$$\int_{\mathcal{D}} d^2\vec{x}\vec{\nabla}\cdot\vec{A} = \int_0^R rdr\int_0^{2\pi} d\theta\frac{1}{r}\partial_r(rA^r) = \int_0^{2\pi} d\theta R A^r(R,\theta) \quad \text{if } A(r=0,\theta) \text{ is finite.} \tag{A.7}$$

Appendix B

Confronting the arbitrary complex integration constants β_l and $g_l^{(r,\theta)}(r)$

In this section of the appendices, we confront the complex integration constants β_l and $g_l^{(r,\theta)}(r)$. The motivation behind these calculations was to establish a unique expression for the $\omega_l = 2|\kappa|$ contribution to the analytic solutions of the electric field components.
Let us first remind the two following relations between the electromagnetic field components,

$$\begin{aligned} \partial_t B &= \frac{1}{r}\partial_\theta E^r - \frac{1}{r}\partial_r(rE^\theta), & \text{(B.1)} \\ 2r\kappa B &= \frac{1}{r}\partial_r(rE^r) + \frac{1}{r}\partial_\theta E^\theta. & \text{(B.2)} \end{aligned}$$

Now let us try to write each party of these equations, starting with the first time derivative of B,

$$\begin{aligned} \partial_t B = & \sum_{l=-\infty}^{\infty} \not\int_{2|\kappa|}^{\infty} (-i\omega_l)(a_l(\omega_l)e^{-i\omega_l t+il\theta} - a_l^\dagger(\omega_l)e^{i\omega_l t-il\theta})J_{|l|}(k_l r) \\ & + \sum_{l=-\infty}^{\infty} (-i2|\kappa|l)(\beta_l e^{-i2|\kappa|t+il\theta} - \beta_l^\dagger e^{i2|\kappa|t-il\theta})(\frac{r}{R})^{|l|}, \qquad \text{(B.3)} \end{aligned}$$

and then the spatial derivatives on the electric field components (where the prime over the $g_l^{(r,\theta)}(r)$ denotes the radial derivative),

$$\begin{aligned} \frac{1}{r}\partial_\theta E^r = & \frac{1}{r}(\sum_{l=-\infty}^{\infty} \not\int_{2|\kappa|}^{\infty} (a_l(\omega_l)e^{-i\omega t+il\theta} - a_l^\dagger(\omega_l)e^{i\omega_l t-il\theta})(\frac{-2il\kappa}{k_l}J'_{|l|}(k_l r) - \frac{il^2\omega_l}{rk_l^2}J_{|l|}(k_l r))) \\ & + \begin{cases} +\sum_{l=-\infty}^{\infty} \frac{(-2i|\kappa||l|l)}{k_l^2}(\beta_l e^{-i2|\kappa|t+il\theta} - \beta_l^\dagger e^{i2|\kappa|t-il\theta})\frac{1}{R^2}(\frac{r}{R})^{|l|-2} \\ \sum_{l=-\infty}^{\infty} \frac{il}{r}(e^{-i2|\kappa|t+il\theta}g_l^r(r) - e^{i2|\kappa|t-il\theta}g_l^{r\dagger}(r)), \end{cases} \qquad \text{(B.4)} \\ \frac{1}{r}\partial_r(rE^\theta) = & \frac{1}{r}(\sum_{l=-\infty}^{\infty} \not\int_{2|\kappa|}^{\infty} (a_l(\omega_l)e^{-i\omega_l t+il\theta} - a_l^\dagger(\omega_l)e^{i\omega_l t-il\theta})(\frac{-2il\kappa}{k_l}J'_{|l|}(k_l r) + \frac{(-i\omega_l)}{k_l}J'_{|l|}(k_l r) - (ir\omega_l)J_{|l|}''(k_l r)))d\omega_l \\ & \begin{cases} +\sum_{l=-\infty}^{\infty} \frac{(-2i|\kappa l|l)}{k_l^2}(\beta_l e^{-i2|\kappa|t+il\theta} - \beta_l^\dagger e^{i2|\kappa|t-il\theta})\frac{1}{R^2}(\frac{r}{R})^{|l|-2} \\ \\ \sum_{l=-\infty}^{\infty} \frac{1}{r}(re^{-i2|\kappa|t+il\theta}g_l^{\theta\prime}(r) - re^{i2|\kappa|t-il\theta}g_l^{\theta\prime\dagger}(r)) + (e^{-i2|\kappa|t+il\theta}g_l^\theta(r) - re^{i2|\kappa|t-il\theta}g_l^{\theta\dagger}(r)), \end{cases} \qquad \text{(B.5)} \end{aligned}$$

whose sum gives,

$$\frac{1}{r}\partial_\theta E^r - \frac{1}{r}\partial_r(rE^\theta) = -\frac{1}{r}(\sum_{l=-\infty}^{\infty}\int_{2|\kappa|}^{\infty}(a_l(\omega_l)e^{-i\omega_l t+il\theta} - a_l^\dagger(\omega_l)e^{i\omega_l t-il\theta})(\frac{il^2\omega_l}{rk_l^2}J_{|l|}(k_lr) - \frac{i\omega_l}{k_l}j'_{|l|}(k_lr) - ir\omega_l J_{|l|}"(k_lr))d\omega_l$$

$$\left\{\begin{array}{l} \sum_{l=-\infty}^{\infty}\frac{(-2i|\kappa|l^2)}{k_l^2}(\beta_l e^{-i2|\kappa|t+il\theta} - \beta_l^\dagger e^{i2|\kappa|t-il\theta})\frac{1}{R^2}(\frac{r}{R})^{|l|-2} \\ \sum_{l=-\infty}^{\infty}\frac{1}{r}(e^{-i2|\kappa|t+il\theta}(ilg_l^r(r) - rg_l^{\theta\prime}(r) - g_l^\theta(r)) \\ \qquad -e^{i2|\kappa|t-il\theta}((ilg_l^r(r)^\dagger - rg_l^{\theta\prime\dagger}(r) - g_l^\theta(r)^\dagger)) \end{array}\right. \tag{B.6}$$

as for B.2 we have

$$\frac{1}{r}\partial_r(rE^r) = \frac{1}{r}(\sum_{l=-\infty}^{\infty}\int_{2|\kappa|}^{\infty}(a_l(\omega_l)e^{-i\omega_l t+il\theta} + a_l^\dagger(\omega_l)e^{i\omega_l t-il\theta})(-\frac{2\kappa}{k_l}J'_{|l|}(k_lr) - 2\kappa J_{|l|}"(k_lr) - \frac{l\omega_l}{k_l}J'_{|l|}(k_lr)))$$

$$\left\{\begin{array}{l} \sum_{l=-\infty}^{\infty}\frac{(-2|\kappa|l^2)}{k_l^2}(\beta_l e^{-i2|\kappa|t+il\theta} - \beta_l^\dagger e^{i2|\kappa|t-il\theta})\frac{1}{R^2}(\frac{r}{R})^{|l|-2} \\ \sum_{l=-\infty}^{\infty}\frac{1}{r}(e^{-i2|\kappa|t+il\theta}g_l^r(r) - e^{i2|\kappa|t-il\theta}g_l^{r\dagger}(r) \\ \qquad +r(e^{-i2|\kappa|t+il\theta}g_l^{r\prime}(r) - e^{i2|\kappa|t-il\theta}g_l^{r\prime\dagger}(r)), \end{array}\right. \tag{B.7}$$

$$\frac{1}{r}\partial_\theta(E^\theta) = \frac{1}{r}(\sum_{l=-\infty}^{\infty}\int_{2|\kappa|}^{\infty}(a_l(\omega_l)e^{-i\omega_l t+il\theta} + a_l^\dagger(\omega_l)e^{i\omega_l t-il\theta})(-\frac{2\kappa l^2}{rk_l^2}J_{|l|}(k_lr) + \frac{l\omega_l}{k_l}J'_{|l|}(k_lr)))$$

$$\left\{\begin{array}{l} \sum_{l=-\infty}^{\infty}\frac{(2|\kappa||l|l)}{k_l^2}(\beta_l e^{-i2|\kappa|t+il\theta} + \beta_l^\dagger e^{i2|\kappa|t-il\theta})\frac{1}{R^2}(\frac{r}{R})^{|l|-2} \\ \sum_{l=-\infty}^{\infty}\frac{1}{r}(e^{-i2|\kappa|t+il\theta}g_l^\theta(r) - re^{i2|\kappa|t-il\theta}g_l^{\theta\dagger}(r)), \end{array}\right. \tag{B.8}$$

whose sum gives,

$$\frac{1}{r}\partial_r(rE^r) + \frac{1}{r}\partial_\theta(E^\theta) = \frac{1}{r}(\sum_{l=-\infty}^{\infty}\int_{2|\kappa|}^{\infty}(a_l(\omega_l)e^{-i\omega_l t+il\theta} + a_l^\dagger(\omega_l)e^{i\omega_l t-il\theta})(\frac{2l^2\kappa}{rk_l^2}J_{|l|} - \frac{-2\kappa}{k_l}J_{|l|'(k_lr)} - 2\kappa J_{|l|}"(k_lr))d\omega_l$$

$$\left\{\begin{array}{l} \sum_{l=-\infty}^{\infty}\frac{(-2|\kappa|l^2)}{k_l^2}(\beta_l e^{-i2|\kappa|t+il\theta} + \beta_l^\dagger e^{i2|\kappa|t-il\theta})\frac{1}{R^2}(\frac{r}{R})^{|l|-2} \\ \sum_{l=-\infty}^{\infty}\frac{1}{r}(e^{-i2|\kappa|t+il\theta}(ilg_l^\theta(r) + rg_l^{r\prime}(r) + g_l^r(r)) \\ \qquad +e^{i2|\kappa|t-il\theta}((ilg_l^\theta(r) + rg_l^{r\prime}(r) + g_l^r(r))^\dagger)). \end{array}\right. \tag{B.9}$$

As previously said, $J_{|l|}(u)$ obeys $(\frac{d^2}{du^2} + \frac{1}{u}\frac{d}{du} + (-\frac{l^2}{u^2}))J_{|l|}(u) = 0$, so the $J_{|l|}$ terms of the equations are equal. Indeed,

$$-i\omega_l(1 - \frac{l^2}{r^2k_l^2} + \frac{1}{rk_l}\partial_r + \partial_r^2)J_{|l|}(k_lr) = 0, \tag{B.10}$$

$$2\kappa(1 - \frac{l^2}{r^2k_l^2} + \frac{1}{rk_l}\partial_r + \partial_r^2)J_{|l|}(k_lr) = 0, \tag{B.11}$$

It remains to check that we can have

$$-i2|\kappa|)\beta_l\frac{r}{R}^{|l|} = \frac{1}{r}(e^{-i2|\kappa|t+il\theta}(ilg_l^r(r) - rg_l^{\theta\prime}(r) - g_l^\theta(r)) - c.c.), \tag{B.12}$$

$$2\kappa\beta_l(\frac{r}{R})^{|l|} = \frac{1}{r}(ilg_l^\theta(r) + g_l^r(r) + rg_l^{r\prime}(r)), \tag{B.13}$$

from which we isolate $g_l^r(r)$ and $g_l^\theta(r)$

$$\begin{aligned}
g_l^\theta(r) &= \frac{1}{il}(2r\kappa\beta_l\frac{r}{R}^{|l|} - g_l^r(r) - rg_l^r{}'(r)),\\
g_l^\theta{}'(r) &= \frac{1}{il}(2(|l|+1)\kappa\beta_l(\frac{r}{R})^{|l|} - 2g_l^r{}'(r) - rg_l^r{}"(r)),\\
g_l^r(r) &= \frac{1}{il}(-i2\kappa r\beta_l(\frac{r}{R})^{|l|} + rg_l^\theta{}'(r) + g_l^\theta(r)),\\
g_l^r{}'(r) &= \frac{1}{il}(-i2\kappa(|l|+1)\beta_l(\frac{r}{R})^{|l|} + 2g_l^\theta{}'(r) + rg_l^\theta{}"(r)).
\end{aligned} \tag{B.14}$$

Inserting one $g_l^{(r,\theta)}(r)$ into another we get

$$\begin{aligned}
-i2\kappa\beta_l(\frac{r}{R})^{|l|} &= \frac{1}{r}(ilg_l^r(r) - \frac{r}{il}(2(|l|+1)\kappa\beta_l(\frac{r}{R})^{|l|} - 2g_l^r{}'(r) - rg_l^r{}"(r)) - \frac{1}{il}(2r\kappa\beta_l(\frac{r}{R})^{|l|} - g_l^r(r) - rg_l^r{}'(r)))\\
&= \frac{1}{r}(il(1-\frac{1}{l^2})g_l^r(r) + \frac{3r}{il}g_l^r{}'(r) + \frac{r^2}{il}g_l^r{}"(r) - \frac{2\kappa\beta_l R}{il}((|l|+1)+1)(\frac{r}{R})^{|l|}),\\
2\kappa\beta_l(\frac{r}{R})(-i+\frac{|l|+2}{il}) &= \frac{il}{r}(1-\frac{1}{l^2})g_l^r(r) + \frac{3}{il}g_l^r{}'(r) + \frac{r}{il}g_l^r{}"(r),\\
2\kappa\beta_l(\frac{r}{R})^{|l|}(l+|l|+2) &= -\frac{l^2}{r}(1-\frac{1}{l^2}g_l^r(r) + 3g_l^r{}'(r) + rg_l^r{}"(r)),\\
2\kappa\beta_l(\frac{r}{R})^{|l|}(l+|l|+2) &= -\frac{l^2}{r}(1-\frac{1}{l^2})g_l^r(r) + 3g_l^r{}'(r) + rg_l^r{}"(r),\\
&= \frac{1}{r}(1-l^2)g_l^r + 3rg_l^r{}'(r) + rg_l^r{}"(r),\\
2R\kappa\beta_l(\frac{r}{R})^{|l|+1}(l+|l|+2) &= (1-l^2)g_l^r(r) + 3rg_l^r{}'(r) + r^2g_l^r{}"(r),
\end{aligned} \tag{B.15}$$

and

$$\begin{aligned}
2\kappa\beta_l(\frac{r}{R})^{|l|} &= \frac{1}{r}(ilg_l^\theta(r) + \frac{1}{il}(-i2\kappa r\beta_l(\frac{r}{R})^{|l|} + rg_l^\theta{}'(r) + g_l^\theta{}'(r))\\
&\quad + \frac{r}{il}(-i2\kappa(|l|+1)\beta_l(\frac{r}{R})^{|l|} + 2g_l^\theta{}' + rg_l^\theta(r))),\\
2iR\kappa\beta_l(\frac{r}{R})^{|l|+1}(l+|l|+2) &= -g_l^\theta(r)(1-l^2) + 3rg_l^\theta{}'(r) + r^2g^\theta{}"(r),
\end{aligned} \tag{B.16}$$

In order to solve the homogeneous equation for g^θ, let us put $g_l^\theta(r) \sim \delta_l^\theta r^\alpha$

$$\begin{aligned}
\delta_l^\theta(\alpha(\alpha-1))r^\alpha + 3\delta_l^\theta\alpha r^\alpha - \delta_l^\theta r^\alpha(1-l^2) &= 0,\\
\alpha(\alpha-1) + 3\alpha - (1-l^2) = 0, &
\end{aligned} \tag{B.17}$$

thanks to which we find that $\alpha = -1 \pm \sqrt{2-l^2}$. We exclude the "minus" solution as we would have a diverging solution around r=0, meaning that the solutions for l, l being a real integer, are $l = \{-1, 0, 1\}$, but we have to exclude the $l = \pm 1$ as well as $g_l^\theta(r)$ would be constant. We are left with

$$g_l^\theta(r) \sim \delta_l^\theta r^{\sqrt{2}-1}. \tag{B.18}$$

Now for the non-homogeneous equation let us put $g_l^\theta(r) \sim \gamma_l^\theta(\frac{r}{R})^{|l|+1}$

$$\begin{aligned}
\gamma_l^\theta|l|(|l|+1)(\frac{r}{R})^{|l|+1} + 3\gamma_l^\theta(|l|+1) + 3\gamma_l^\theta(|l|+1)(\frac{r}{R})^{|l|+1} - \gamma_l^\theta(1-l^2)(\frac{r}{R})^{|l|+1} &= 2iR\kappa\beta_l(\frac{r}{R})^{|l|+1}(l+|l|+2),\\
\gamma_l^\theta = \beta_l\frac{iR\kappa(l+|l|+2)}{l^2+2|l|+1}, &
\end{aligned} \tag{B.19}$$

compatible with an $l = 0$ solution. Let us now proceed the same way with $g_l^r(r)$.

$$\delta_l^r\alpha(\alpha-1)r^\alpha + 3\alpha\gamma_l^r r^\alpha + (1-l^2)\gamma_l^r r^\alpha = 0, \tag{B.20}$$

for which we have those solutions : $\alpha = \frac{-2\pm\sqrt{4-4(1-l^2)}}{2}$. We exclude the "minus" solution for regularity reason, thus we have

$$g_l^r(r) \sim \delta_l^r r^{|l|-1}, \tag{B.21}$$

for any l such that $|l| > 1$. Now the inhomogeneous part :

$$\begin{aligned} \gamma_l^r|l|(|l|+1)(\frac{r}{R})^{|l|+1} + 3\gamma_l^r(|l|+1)(\frac{r}{R})^{|l|+1} + \gamma_l^r(1-l^2)(\frac{r}{R})^{|l|+1} &= 2R\kappa\beta_l(\frac{r}{R})^{|l|+1}(l+|l|+2), \\ \gamma_l^r &= \beta_l\frac{R\kappa(l+|l|+2)}{2|l|+2}, \end{aligned} \tag{B.22}$$

which leaves us with this spectrum of solution : $g_l^r(r) \sim \beta_l\frac{R\kappa(l+|l|+2)}{2|l|+2}(\frac{r}{R})^{|l|+1}$. Now let us check back our previous equations,

$$\begin{aligned} \frac{1}{r}(ilg_l^r(r) - rg_l^{\theta\prime}(r) - g_l^\theta(r)) &= \frac{1}{r}(il(\delta_l^r r^{|l|-1} + \gamma_l^r(\frac{r}{R})^{|l|+1}) - r(\sqrt{2}-1)\delta_l^\theta r^{\sqrt{2}-1} - \delta_l^\theta r^{\sqrt{2}-1} \\ &\quad -\frac{r}{R}(|l|+1)\gamma_l^\theta(\frac{r}{R})^{|l|} - \gamma_l^\theta(\frac{r}{R})^{|l|+1}), \\ &= il(\delta_l^r r^{|l|-2} + \gamma_l^r\frac{1}{R}(\frac{r}{R})^{|l|}) - \sqrt{2}\delta_l^\theta r^{\sqrt{2}-2} - (\frac{|l|+1}{R}\gamma_l^\theta + \gamma_l^\theta)(\frac{r}{R})^{|l|}, \end{aligned} \tag{B.23}$$

from which we can exclude the non $(\frac{r}{R})^{|l|}$ terms, so $\delta_l^r = \delta_l^\theta = 0$.

$$\begin{aligned} \frac{1}{r}(il(\delta_l^\theta r^{\sqrt{2}-1} + \gamma_l^\theta(\frac{r}{R})^{|l|+1}) - r((|l|-1)\delta_l^r r^{|l|-2} &- \frac{1}{R}(|l|+1)\gamma_l^r(\frac{r}{R})^{|l|}) + \delta_l^r r^{|l|-1} + \gamma_l^r(\frac{r}{R})^{|l|+1}) \\ &= \\ il(\delta_l^\theta r^{\sqrt{2}-2} + \gamma_l^\theta\frac{1}{R}(\frac{r}{R})^{|l|}) &+ |l|\delta_l^r r^{|l|-2} + \frac{1}{R}((|l|+1)\gamma_l^r(\frac{r}{R})^{|l|} + \gamma_l^r). \end{aligned} \tag{B.24}$$

Similarly, we again need $\delta_l^r = \delta_l^\theta = 0$. We are then left with

$$\begin{aligned} -i2|\kappa|\beta_l(\frac{r}{R})^{|l|} &= \frac{il}{R}\gamma_l^r(\frac{r}{R})^{|l|} - \frac{|l|+2}{R}\gamma_l^\theta(\frac{r}{R})^{|l|}, \\ &= \begin{cases} \frac{il}{R}\frac{R\kappa}{|l|+1}\beta_l(\frac{r}{R})^{|l|} - \beta_l\frac{iR\kappa(l+|l|+2)}{l^2+2|l|+1}\frac{|l|+2}{R}(\frac{r}{R})^{|l|} & \text{if } l \le 0, \\ \frac{il}{R}\beta_l(\frac{r}{R})^{|l|} - \frac{|l|+2}{R}\frac{iR\kappa(2l+2)}{l^2+2l+1}(\frac{r}{R})^{|l|} & \text{if } l \ge 0 \end{cases} \\ &= \begin{cases} \beta_l(\frac{r}{R})^{|l|}(\frac{il\kappa}{1-l} - \frac{i\kappa(2l+4)}{l^2-2l+1}) & \text{if } l \le 0, \\ \beta_l(\frac{r}{R})^{|l|}(\frac{il\kappa}{1+l} - (l+2)\frac{i\kappa(2l+2)}{l^2+2l+1}) & \text{if } l \ge 0, \end{cases} \\ &= \begin{cases} -i\kappa\beta_l(\frac{r}{R})^{|l|}\frac{1+3l}{(l-1)^2} & \text{if } l \le 0, \\ -i\kappa\beta_l(\frac{r}{R})^{|l|}\frac{l^2+5l+4}{(1+l)^2} & \text{if } l \ge 0, \end{cases} \end{aligned} \tag{B.25}$$

meaning that, for some l, we need one of the therms between parenthesis to be equal to 2, which we get for $l = 2$.
Let us do the same for the other equation :

$$\begin{aligned} 2\kappa\beta_l(\frac{r}{R})^{|l|} &= \frac{il}{R}\gamma_l^\theta(\frac{r}{R})^{|l|} + \frac{2+|l|}{R}(\frac{r}{R})^{|l|}\gamma_l^r, \\ &= \begin{cases} \frac{2il}{R}(\frac{r}{R})^{|l|}iR\kappa\frac{l+1}{(l+1)^2}\beta_l + \frac{2-l}{R}(\frac{r}{R})^{|l|}\frac{R\kappa}{1-l} & \text{if } l \le 0 \\ \frac{il}{R}(\frac{r}{R})^{|l|}2iR\kappa\frac{l+1}{(l+1)^2}\beta_l + \frac{2+l}{R}(\frac{r}{R})^{|l|}\beta_l R\kappa & \text{if } l > 0, \end{cases} \\ &= \begin{cases} \kappa(\frac{r}{R})^{|l|}\beta_l(1 - \frac{l-1}{l+1}) & \text{if } l \le 0 \\ \kappa(\frac{r}{R})^{|l|}\beta_l\frac{1-l}{1+l} & \text{if } l > 0, \end{cases} \end{aligned} \tag{B.26}$$

for which the terms between parentheses are equal to 2 if $l = 0$ this time.

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