

Louvain School of Management

Credit Value Adjustment

Comparison between constant and dynamic
market-implied recovery

Auteur : Ange Asseau
Promoteur(s) : Frédéric Vrins
Année académique 2019-2020
Master 120 crédits en Ingénieur de Gestion, à finalité spécialisée

Abstract

The 2008 financial crisis highlighted the flaws in the system regarding counterparty risk. The assessment of credit quality mainly relied on credit ratings provided by specialized agencies but with restricted accuracy. In this submission we develop the credit value adjustment model proposed by the Basel Committee to consider counterparty credit risk in estimating the value of a derivative. Different approaches existing, we decide to address the following problematic: *what is the impact of a market-implied recovery on the CVA compared to a constant recovery rate?*

Our first practical approach considers the recovery rate, one of the components of the model, to be constant. While this limitation facilitates the CVA calculations, it underestimates the accuracy of the actual amount to be added to the value of the security. To counter this problem, we compare our results with a model whose recovery is implicit to the market and is now dynamic. Considering two financial counterparties, namely Morgan Stanley and the Deutsche Bank, we observe that for a very short maturity, the first model overvalues the CVA for two good credit-quality institutions. However, for a longer term of one or two years, our second model provides an adjustment that is almost twice that obtained for a constant recovery rate.

DEDICATION AND ACKNOWLEDGEMENTS

I, Ange Asseau, would like to acknowledge everyone who played a role in my academic accomplishments. First of all, my parents, who supported me with love, understanding and coffee. Without you, I wouldn't be the person I am today on the gates to a first success.

Secondly, I would like to thank my thesis supervisor, Mr. Frédéric Vrans, for all his advice and guidance during this research process. My knowledge in the field of finance would not be what it is today without you as my professor and supervisor.

AUTHOR'S DECLARATION

I, Ange Asseau, do hereby declare that the dissertation entitled *Credit Value Adjustment comparison between constant and dynamic market-implied recovery rate* has been undertaken by me for the award of Master of Business Engineering. I have completed this study under the guidance of Mr. Frédéric Vrins, Professor of Financial Engineering. .

TABLE OF CONTENTS

	Page
List of Tables	xi
List of Figures	xiii
Introduction	1
I Theoretical perspective	3
1 Context	5
1.1 Introduction	5
1.2 Derivatives market	6
1.2.1 Derivative pricing	7
1.2.2 Credit derivatives	7
1.3 The failure of 2008	10
1.4 Counterparty credit risk	13
1.5 Mitigating counterparty credit risk	14
1.5.1 Default-remote entities	14
1.5.2 Termination events	14
1.5.3 Payment netting and close-out netting	15
1.5.4 Collateralisation	15
2 Call option pricing	17
2.1 Introduction	17
2.2 Earliest model of stock price and Brownian Motion	17
2.2.1 Random Walk	19
2.2.2 Brownian Motion with Drift	20
2.3 Geometric Brownian Motion with Itô calculation	22
2.4 Call option pricing	22
2.4.1 Risk-neutral derivation of the Black Scholes formula	23

TABLE OF CONTENTS

2.4.2	Assumptions discussion and benefits of Black-Scholes model	23
3	Credit Value Adjustment	25
3.1	Introduction	25
3.2	Description of counterparty risk components	26
3.2.1	Credit exposure	26
3.2.2	Probability of default	29
3.2.3	Recovery Rate and Loss-Given Default	33
3.3	Credit Value Adjustment	33
3.3.1	Probabilistic model	34
3.3.2	Unilateral CVA formula	34
II	Practical perspective	37
4	No Wrong-way Risk	39
4.1	Introduction	39
4.2	No-Wrong way Risk formula for call option	39
4.3	Black-Scholes inputs	40
4.3.1	Apple's stock price	41
4.3.2	Volatility parameter	41
4.3.3	Risk-free interest rate	44
4.4	European call option value	45
4.5	Counterparty risk components results	46
4.5.1	Constant recovery rate	46
4.5.2	Implied default probability from CDS inputs	47
4.6	UCVA results	49
4.7	Discussion	50
5	CVA with market-implied recovery	51
5.1	Introduction	51
5.2	Cox-Ross-Rubinstein	52
5.3	Model calibration	54
5.4	Calibrated term structures	55
5.5	UCVA calculation	56
5.6	Discussion	58
6	Conclusion	61
A	Appendices	63
A.1	Brownian motion with drift	63

A.1.1	Mean	63
A.1.2	Variance	63
A.2	Geometric Brownian motion solution	64
A.3	Geometric Brownian motion lognormal distribution parameters	66
A.3.1	Mean	66
A.3.2	Variance	66
A.4	Black-Scholes formula derivation	66
A.5	Global Corporate Average Cumulative Default Rates	68
A.6	Apple's stock price implied volatilities for different call options	69
A.7	Hazard rate curves for Morgan Stanley and the Deutsche Bank for a 40% recovery rate	69
A.8	Hazard rate values for a 20% and 60% constant recovery rate	70
A.9	Three-period algorithm for computing p_i^k	71
A.10	Binomial tree of the stock price for for the first twenty periods	72

LIST OF TABLES

TABLE	Page
1.1 Illustration of the impact of close-out netting when contracting with two different counterparties.	15
2.1 Properties of Brownian motion.	21
4.1 Historical volatility results for Apple's stock.	43
4.2 Implied volatilities data for different exercise dates and strike prices.	44
4.3 Daily Treasury Yield Curve Rates. Source: U.S. Department of the Treasury - https://www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/TextView.aspx?data=yield	45
4.4 Inputs summary table for a strike price $K = 380$	45
4.5 Inputs summary table for a strike price $K = 390$	45
4.6 Black-Scholes results with respect to $K = 380$	46
4.7 Black-Scholes results with respect to $K = 390$	46
4.8 Counterparties' CDS spreads.	47
4.9 Survival probabilities with respect to the hazard rates for a 40% recovery rate. . . .	48
4.10 UCVA results for a strike price $K = 380$	49
4.11 UCVA results for a strike price $K = 390$	49
5.1 Calibrated parameters.	55
5.2 UCVA results with market-implied recovery (MIR) for a strike price $K = 380$	58
5.3 UCVA results with market-implied recovery (MIR) for a strike price $K = 390$	58
A.1 Survival probabilities with respect to the hazard rates for a 20% recovery rate. . . .	70
A.2 Survival probabilities with respect to the hazard rates for a 60% recovery rate. . . .	71

LIST OF FIGURES

FIGURE	Page
1.1 Illustration of a CDS contract on a single reference entity.	8
1.2 General structure of a Credit-Linked Note.	9
1.3 General structure of a Total Return Swap.	9
1.4 Credit Default Swaps - Gross Notional Principal Outstanding Over Time.	10
1.5 Repackaging of Subprime Bonds into CDOs	12
2.1 Sample paths of a stochastic process, corresponding to two outcomes ω_1 and ω_2 . . .	19
2.2 Visual interpretation of the scaled random walk.	20
3.1 Illustration of the potential future exposure of a contract where each point on the vertical line of the shaded area represents a PFE related to a certain probability. . . .	27
3.2 Annual historical default probabilities. For example, for AA-rated firms, the point at 4 years is the default probability in the interval 3 years to 4 years, which is $0.24\% - 0.13\% = 0.09\%$	30
4.1 Historical performance of Apple's stock price.	41
4.2 Historical daily log returns of Apple's stock price.	42
4.3 Survival and Default curves for our two counterparties.	48
5.1 Cox-Ross-Rubinstein binomial tree.	52
5.2 Market-implied recovery term structures.	55
5.3 Default intensity term structures.	56
5.4 Ten Monte Carlo simulated paths under GBM.	57
5.5 Call option discounted values.	57
5.6 Call option discounted value term structure for our second maturity on 17th June 2022.	57
A.1 Global Corporate Average Cumulative Default Rates (1981-2019) (%). Source: Standard & Poor's (Default, Transition, and Recovery: 2019 Annual Global Corporate Default And Rating Transition Study).	68
A.2 Effect of Time on Implied Volatility. Source: Refinitiv Eikon.	69

LIST OF FIGURES

A.3 Hazard rate curve for Morgan Stanley using CDS spreads and 40% recovery rate. .	69
A.4 Hazard rate curve for the Deutsche Bank using CDS spreads and 40% recovery rate.	70
A.5 Morgan Stanley's stock price tree for the first 20 periods.	72
A.6 Deutsche Bank's stock price tree for the first 20 periods.	72

INTRODUCTION

Thanks to the events of recent years, today's financial world is no longer the same. Since the financial crisis of 2008 which highlighted the innumerable flaws in the financial system, regulations have multiplied within the large institutions to better counter a concept that has become essential to consider: the risk. After the collapse of the financial crisis, numerous criticisms have been raised about the regulations in effect, particularly those related to counterparty credit risk and the Basel II model on bank capital adequacy.

To improve the sensitivity of the system with regard to the accuracy of credit risk measurement, Basel Committee has introduced a new capital charge, the CVA (Credit Valuation Adjustment). By definition, it is the amount to be added to the value of an asset to take counterparty risk into account in the process. We find different models in the literature to assess this amount. Due to the stochastic dynamics of its components, several models consider one of them being constant: the recovery rate. This simplification has a direct impact on the credit value adjustment and the evidence shows that it is underestimated under this assumption. In order to determine the veracity of these statements, we compare in our work two practical parts: one considering a constant and independent recovery rate, the other considering a dynamic time-dependent market-implied recovery that depends on the counterparty's survival.

We begin the thesis with an introductory chapter: *Chapter 1* sets the context and describes the development of counterparty credit risk through 2008 financial crisis and the ways to mitigate it. *Chapter 2* introduces and defines the derivative we focus on to compute its related credit value adjustment: the European call option. We describe and demonstrate in this chapter the derivation of the Black-Scholes formula for pricing our derivative and discusses its benefits. *Chapter 3* explains the three components of the CVA formula: the recovery rate, the expected exposure and the default intensity. Then, we derive the unilateral credit value adjustment formula. *Chapter 4* is dedicated to the first practical part considering independent constant recovery rate. Finally, *Chapter 5* continues our analysis by comparing the previous results with those for a market-implied recovery.

Part I

Theoretical perspective

1.1 Introduction

Institutions have undergone a radical revolution over the last three decades, especially in the area of Financial Risk Management. FRM was adopted by the Basel Committee in 2001 as a four-step process: (i) the identification of events in broader classes of operational, market, liquidity, credit and other risks and in more specific sub-categories; (ii) the assessment of these risks using data and appropriate risk models; (iii) the ongoing monitoring and reporting of risk assessments carried out in a timely and effective manner; (iv) the supervision of these risks by senior management ([Alexander, 2003](#)).

In order to define the various risks mentioned above and to set out their scope of application, we relied on the definitions provided in [Cornford \(2005\)](#)'s work. He describes them as follows:

1. **Market risk** is that of loss resulting from changes in the market value of a position before it can be offset or liquidated.
2. **Operational risk** can refer to any of several hazards to which a bank is exposed simply as a result of being in business. For the purpose of Basel II, it is defined as the risk of loss due to inadequate or failed internal processes, people and systems or from external events. As such it includes legal but excludes strategic and reputational risk.
3. **Liquidity risk** can be defined as having two dimensions - cash liquidity or funding risk and market liquidity risk. The first is due to periodic needs for liquid funds which cannot be accurately predicted in advance. The second is the risk that a bank's own transactions will have an adverse effect on the prices at which it buys or sells assets.

4. **Credit risk**, the most pervasive of banking risks, relies on the possibility that a counterparty will be unable to meet its obligations.

Although financial risk management can be subdivided into several areas, we will focus our interest only on the latter, the credit risk, with particular attention to the counterparty's concept. In this risk category, the greatest complexity will reside in the involvement of financial instruments, the derivatives.

1.2 Derivatives market

By definition, a derivative is "*a financial instrument whose value depends on (or derives from) the values of other, more basic, underlying variables. Very often the variables underlying derivatives are the prices of traded assets*" (Hull, 2017). The most common underlying assets we can meet on the market are bonds, commodities, stocks, currencies, market indexes or interest rates, and they can be included among the four main forms of derivatives: swaps, futures, forwards and options. A stock option, for instance, is a derivative whose value depends on the underlying stock price.

These products can be traded within 2 markets : the *over-the counter (OTC) market* and the *exchange market*. The potential number of transactions in the *OTC market* far exceeds the number of trades in the exchange market. As a matter of fact, according to the statistical data released by the Bank for International Settlements, OTC derivatives reached an estimated outstanding notional amount of over \$559 trillion at end-December 2019 (BIS, 2020). Unlike the exchange-traded market, where investors trade standardized contracts that have been defined by the exchange and that are highly regulated, the terms of a contract don't have to be pointed out in OTC market and it was still sorely lacking in regulations until 2007 (discussed more in details in Section 1.3).

The main issue in the derivatives market in recent decades is that these OTC derivatives, which constitute the bulk of the derivatives that can be traded, comprise a greater counterparty risk, i.e. a greater risk that one of the two parties involved in the contract will default at some point during the term of the contract. Furthermore, such products enable global financial market participants to better deal with risk, including the risk of variations in commodity and stock prices, exchange rates, interest rates and market liquidity. Over the past 50 years, the range of futures and options traded has grown steadily. New hedging opportunities have emerged, allowing participants to reduce their surrounding economic uncertainty at a market-determined price, as well as new possibilities for those who are more willing and financially able to assume certain risks (Heinemann, 2011).

1.2.1 Derivative pricing

To discuss the pricing of our derivative product and to begin the mathematical writing that we will keep throughout this work, we have been inspired by the book of [Ekstrand \(2011\)](#) that provides us an introduction to the modeling of financial derivatives.

As mentioned above, derivative theory depends on a set $\{S^i\} = \{S^1, S^2, S^3, \dots\}$ of financial traded assets. The price of an asset S is expressed by a real number and in order to describe its change over time, we will use the notation S_t or $S(t)$ to represent its market value at time t . We also assume that their today's prices, $\{S_{t=0}^i\}$, are available information. Since derivative represents a contract between parties, we have a vested interest in how we will price it. We note V_t or $V(t)$ the financial product for which the price at time t is known in accordance with the price S_t of the underlying.

The particularity of such financial products lies in the uncertainty of the future value of this contract. Depending on the underlying asset, the future value of the contract at a possible default time will be the net value of all remaining future cashflows to be carried out under the terms. With respect to the party's status, the contract may have a positive or negative value. This point means that the counterparty risk is bilateral. It seems coherent to state that each party to the contract takes a risk with respect to the other party in a derivative transaction. The main complexity hidden in this bilateral nature was in theory resolved by the fact that the large players in derivatives transactions were nearly free of default. However, this assumption played a key role in the financial turmoil experienced in 2008, as explained in the [Section 1.3](#).

1.2.2 Credit derivatives

In the context of this work, it is essential to deepen the understanding of credit derivatives since they play a crucial part in trading, hedging and counterparty risk diversification. The key role of these financial instruments is to allow players to trade in credit as an asset to transfer the credit risk. By definition, « *a credit derivative is an agreement designed explicitly to shift credit risk between the parties; its value is derived from the credit performance of one or more corporations, sovereign entities, or debt obligations* » ([Mengle, 2007](#)).

The majority of credit derivative instruments on the market exists in the form of credit default swaps (CDS), a financial insurance derivative that allows a security owner to oust the risk behind a counterparty's failure to make payments. In return of a periodic premium (called *CDS spread*) generally expressed in basis points (bps) or as a percentage of the notional amount ($100bps = 1\%$), the protection/CDS seller commits to reimburse the buyer's loss in the occurrence of a credit events. Indeed, if the reference entity encounters a credit event that prevents it from fulfilling its obligations towards its counterparty, the protection seller will

have to compensate the protection buyer for the loss incurred. In addition, the CDS buyer must usually pay an accrued premium at this stage as compensation for the fact that premiums are paid in arrears ([Gregory, 2010](#)).

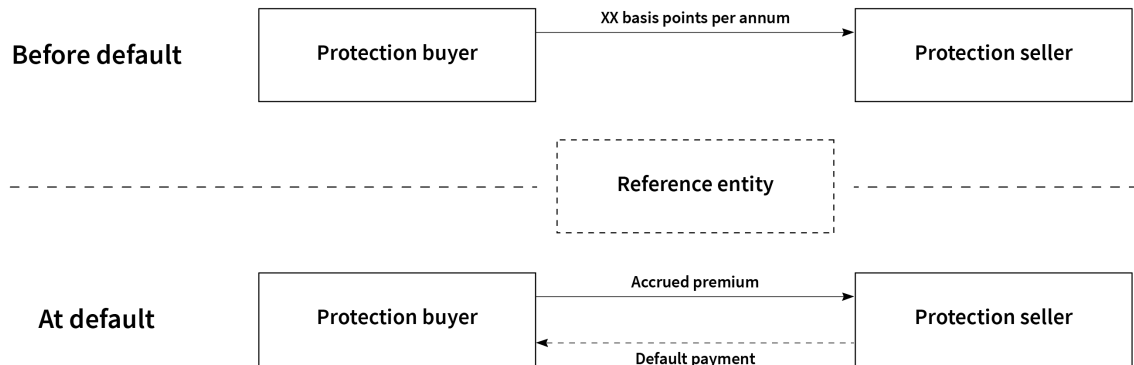


Figure 1.1: Illustration of a CDS contract on a single reference entity.

The concept to be understood here is that the reference entity is not part of the agreement and that CDS contract can be settled without its consent. In this type of product, the seller plays on the non-occurrence of that accident. Indeed, as unexpected incidents are rare, the seller is on average winner, whereas he does not commit any capital. It is therefore an a priori profitable activity for financial institutions that provide such credit default cover. The seller must still have great financial capacity to guarantee its obligation towards the CDS counterparty in the event of a suddenly debt default or credit event. The benefit of this arrangement for the buyer is that it transforms a risky security into a risk-free one ([Mathieu and Sterdyniak, 2009](#)).

CDS is an example of unfunded derivative credits since the credit protection seller commits to make a payment to cover a CDS buyer's losses only when a credit event occurs, without involving any upfront payment and without setting money aside in advance. Yet, an equivalent funded instrument exists, the credit-linked notes (CLN). It can be defined as a combination of a standard bond and a credit default swap. In this credit product, the holder of the CLN (the protection seller) will pay the notional principal to the CLN issuer at the inception of the contract and will receive in return a series of coupons until the maturity of the contract or until the default of the reference entity if it occurs. Credit events may include bankruptcy, any default of payment, restructuring or repudiation in respect of the issuer of the reference asset. CLN has a higher return than a conventional bond - which makes it an attractive product to improve the performance of asset managers. Our investor will have in fact assumed the credit risk of the loan between the CLN seller and the reference entity, as well as the risk that the CLN issuer itself may default to carry out its periodic payments ([Hui and Lo, 2002](#)). Typically, the investor receives a floating rate (e.g. LIBOR) or fixed rate plus a spread as compensation for default risk.

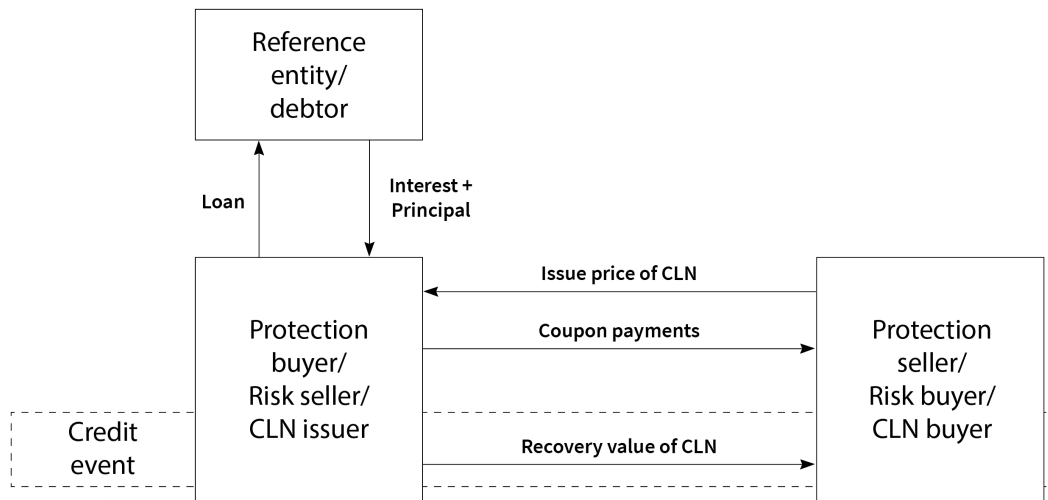


Figure 1.2: General structure of a Credit-Linked Note.

The last type of credit derivative we introduce is the total return swap (TRS). This time, the product not only takes into account the possibility of a counterparty defaulting on its obligations, but also the return on this asset. This bilateral swap agreement is established between a party that will make payments based on a commonly floating rate (the total return receiver) and a party that will make payments based on the total return of a reference security (the total return payer). As shown in the following figure, in the case of a bond representing the reference asset for example, our party *A* holds a bond issued by party *C* that is subject to default. If Company *C* does not default until the maturity of the bond, it will pay the principal amount of the bond plus the corresponding interest. Otherwise, to cover the risk of default of *C*, companies *A* and *B* enter into a total return swap contract.

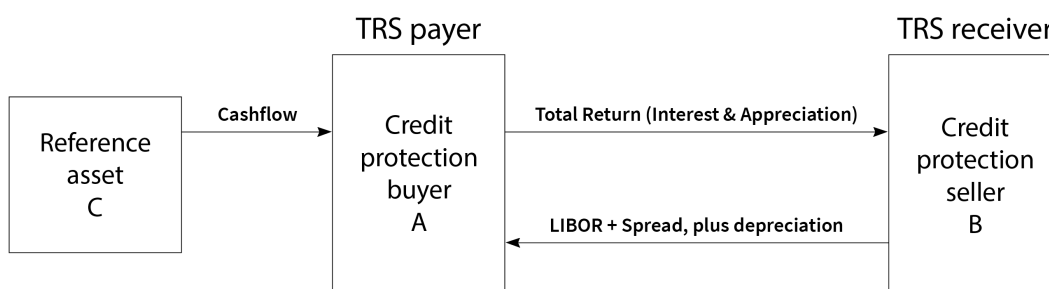


Figure 1.3: General structure of a Total Return Swap.

To protect itself against default and decrease in value of the reference asset, the TRS payer will pay the credit protection party *B* all the cashflows involved plus any market value appreciation throughout the contract lifetime. Furthermore, the TRS receiver will pay the credit protection buyer a LIBOR-based payment plus a negotiated spread, as well as any depreciation in the value of the asset. If the obligor of the reference asset defaults, the investor has the choice

of either making a net payment based on the difference between the price of the reference asset at the start of the deal and the price of the reference asset at the time of default, or agreeing to receive the defaulted reference asset and pay its initial price to the TRS payer. After one of these two events has occurred, neither the payer nor the receiver has any further obligation to the other party, and the TRS is terminated (Wang and Ye, 2014).

1.3 The failure of 2008

The financial crisis of 2008 sounds for most people as a crash in subprime mortgages and the real estate market. Commercial banks definitely lost a lot of money, resulting in a higher value of loans granted than the value of houses acquired following payment defaults by individuals. This important disaster was in fact only one component of the veracity of the meltdown suffered. This financial crisis was truly rooted in pricing errors in the credit default swaps (CDS), a derivative widely used at that time by financial institutions (Murphy, 2008). The following figure shows the growth of CDS from 2001 to 2008 according to the data provided by the International Swaps and Derivatives Association (ISDA). Its volume shot up from less than \$1 trillion in 2001 to \$54.6 trillion in 2008. In other words, a factor of 50 has been applied in only 7 years (Terzi and Uluçay, 2009).

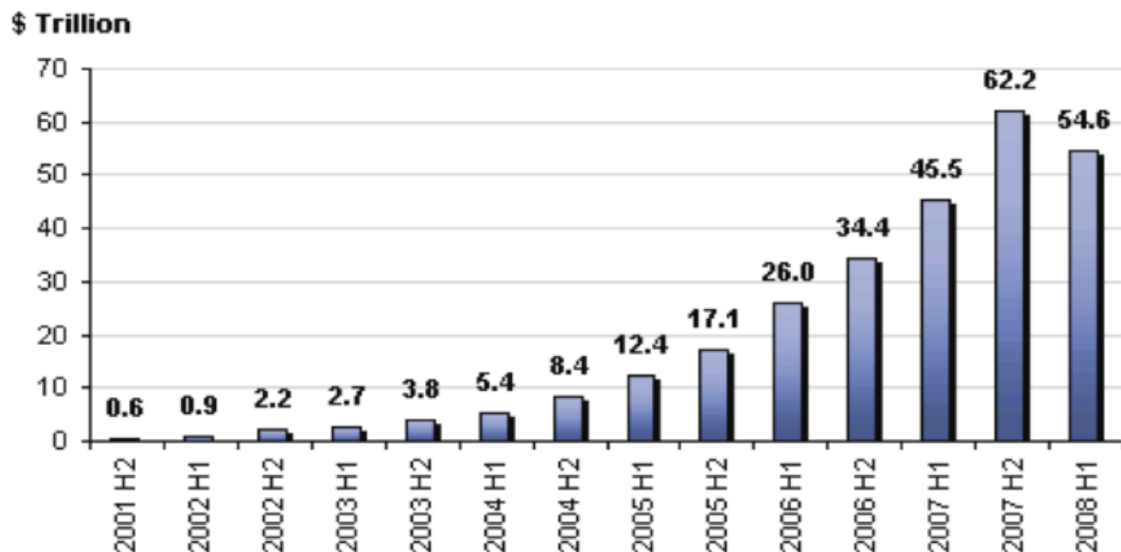


Figure 1.4: Credit Default Swaps - Gross Notional Principal Outstanding Over Time.

Such growth resides in the ability of CDSs to reduce banks' capital requirements since they provide a guarantee against the risk of default. As a result, CDSs are a risk hedging tool widely used by financial agents. Thanks to their presence on the OTC market and the distinctive feature of these debt insurance instruments, which make it possible to avoid the regulations normally imposed in insurance contracts, this product has become increasingly popular on the

derivatives market (Bouveret, 2009).

The financial crisis was fostered by the idea that some corporations, particularly large financial institutions, were so powerful and so entrenched in the global financial and economic system that their failure would negatively impact the entire economy. In the case of liquidity troubles, the government could bail out these businesses to prevent that impact. 'Too big to fail' is the term used to describe such entities but it has been contradicted by the failure of large corporations, including the multinational investment bank Lehman Brothers and one of the largest insurance companies at the time, American International Group (AIG).

For such companies, it was unthinkable that they would fail since they had the resources to recruit the best talent, as well as the best risk management practices. In addition to its major global insurance function, AIG was also active in the finance domain through its Financial Services and Asset Management departments. As part of their Financial Services unit, they were involved in capital market transactions, including CDS transactions with, in the early stages of the economic meltdown, a \$526 billion portfolio of credit default swaps (Sjostrom, 2009).

The subsidiary of the AIG group in charge of selling such protection against third party credit default focused mainly on the super senior risk tranches, i.e. the tranche considered risk-free since it was the first to be paid and therefore not to undergo any loss (Sjostrom, 2009). These tranches consisted mainly of diversified loans and debt securities and were often rated AAA by credit rating agencies. It was common practice to pool such loans consisting mainly of commercial, automobile and student loans, mortgages and other similar assets, which generated cash flows and were often considered illiquid, in investment security, i.e. asset-backed securities (ABS), through securitization process.

Securitization is the transformation of certain receivables into financial securities traded on the markets. In order to "securitize" and then trade them on the markets, a bank sets up a company specially dedicated to the purchase of these receivables, called a Special Purpose Vehicle (SPV). This SPV will then be responsible for issuing these receivables on the markets, after first aggregating them into portfolios, also known as CDOs (Collateralized Debt Obligations). CDOs can be defined as « *a type of asset-backed security whose underlying pool of assets consists of tranches of other asset-backed securities and other debt obligations* » (Lucas et al., 2007). In order to be diversified, each portfolio is made up of several categories of obligations, called "tranches": some are very safe (senior), others are more risky (equity) and are subject to a significant risk of default.

The main benefit of securitization is that it spreads the risk of default over a wide range of players. The risk thus becomes "collateral": by issuing CDOs on the market, a bank first protects itself from excessive credit risk. By pooling debt securities within a CDO, a financial institution seeks to limit risks and stabilise financial products. It also allows to write off all the loans in its securitized portfolios, thereby improving its profit and loss account. The other side of the coin is almost mechanical: by spreading the risk to a multitude of international players, a large-scale credit default is likely to destabilize a large number of financial institutions (banks, pension funds, insurance companies) in many countries around the world.

These CDOs have shot up around 2003, as illustrated below, thanks to the development of subprimes in the United States. Subprime system allowed households to repay only a small portion of the loan during the first two or three years of their loan. If house prices continued to rise, borrowers could refinance their loans before monthly payments increased. However, almost all of these subprime mortgages were pooled into CDOs ([Barnett-Hart, 2009](#)).

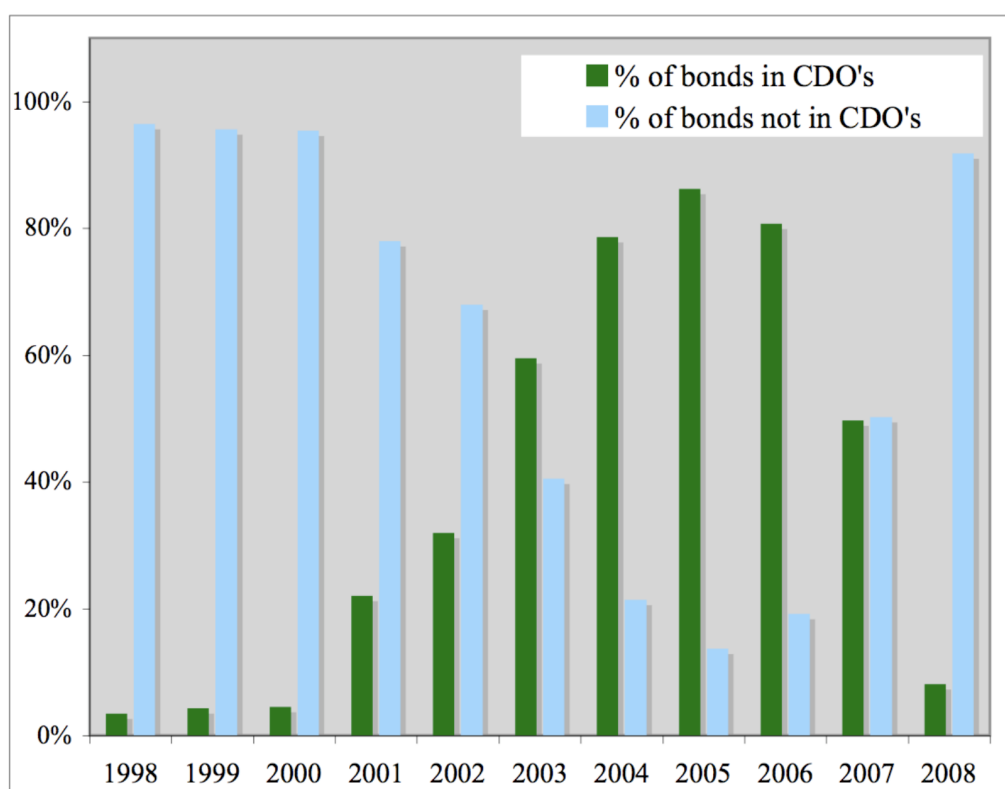


Figure 1.5: Repackaging of Subprime Bonds into CDOs

The American International Group was the largest seller of CDSs for CDOs. However, the banks granted loans to individuals who were financially modest and who failed to repay their loans 2-3 years later. Since a sufficiently large number of credits composing the CDO had not been repaid, it was sufficient for the holder of the CDS to activate its insurance policy and

receive the compensatory damages. AIG viewed these CDSs as easy money but the reality was completely different and the company lost billions of dollars. Indeed, their risk models assumed that the financial assets that made up these CDOs would never default and that the company would therefore cash the CDS spreads without committing any capital out of their own pocket (Lucas et al., 2007). This strategy was a disaster because of their confidence in the ratings provided by the major rating agencies that were Moody's, S&P, and Fitch.

The rating agencies did not take the necessary vigilance. They agreed to give AAA ratings to securities based on subprime credit portfolios without measuring the risk linked to the rise in interest rates and the end of the real estate bubble. Some banks mixed subprime loans with other types of loans to obtain these more diversified portfolios and theoretically less risky, which added to the opacity. This has resulted in impairment of securitized assets and institutions were forced both to report these losses and to reintegrate credits into their balance sheets. Their credit-to-capital ratio was sharply deteriorated, affecting their ability to lend (Mathieu and Sterdyniak, 2009).

A company such as the insurer AIG was considered too big to fail and received close to \$182 billion in government bailout (Source: U.S. Department of the Treasury), which was not the case for all the institutions that made the same mistake. This financial turmoil has highlighted the need for investors to take credit events and counterparty's default into account.

1.4 Counterparty credit risk

Despite the huge impact of the financial crisis discussed above, this experience enabled the financial sector to correct previous concerns and to focus more on the assessment and management of a specific form of risk: the *Counterparty Credit Risk*. CCR corresponds to the risk that the counterparty in a derivatives transaction will default its obligations prior to maturity of a trade and so won't be able to respect the current and future contract payments (Ballotta and Fusai, 2015). The behavioral model of financial entities has undergone radical changes aimed at reducing the risk of bank collapses and mainly at improving their financial stability not to repeat the crash of 2008. This idea that some participants would never fail, or at least would never be allowed to fail, had for many years reduced the emphasis on counterparty risk in the market to the lowest possible extent.

The pre-crisis regulatory framework has been heavily criticised and the Basel II model on bank capital adequacy was seen as one of the factors contributing to the subprime financial crisis. At that point in time, standards on banking laws and regulations were provided by the Basel II Accord (issued by the Basel Committee on Banking Supervision). Even though

this Accord is more risk-sensitive than Basel I and deals with 3 major pillars (the minimum capital requirements, the supervisory review and the market discipline), it turns out that one of the accusations made is that the Basel II framework contributed to the undercapitalization of several banks in terms of their risk exposure. In addition, under Basel II, the assessment of credit risk was attributed to non-bank institutions, such as rating agencies, which were likely to be subject to potential conflicts of interest ([Cannata and Quagliariello, 2009](#)). All the errors made in the system in place before the collapse highlighted the need to improve the sensitivity of the system with regard to the accuracy of credit risk measurement. As a result, The Basel Committee has introduced a new capital charge, the CVA (Credit Valuation Adjustment), as part of the Basel III regulations. This additional charge was created in order to improve banks' resistance to a deterioration in the market value of their assets in times of major credit crisis. The spirit remains the same as in previous regulations, i.e. to force banks to hold sufficient capital to cope with periods of stress ([Ballotta and Fusai, 2015](#)).

1.5 Mitigating counterparty credit risk

In this section, based on the methods provided in the book of [Gregory \(2010\)](#), we discuss different ways to mitigate the counterparty risk that companies may face when contracting a financial product. These different methods focus on reducing current credit exposure and potential future exposure (two concepts later described) but rarely change the probability of experiencing a loss.

1.5.1 Default-remote entities

Although this method has been heavily criticised and questioned in recent years as explained in Section 1.3, the most basic method used to mitigate this counterparty risk is to trade with so-called "too big to fail" institutions that have a near-zero probability of default. But this concept was dismantled and qualified as fundamentally wrong and should now be taken with caution. To this end, there are other methods to avoid the trap experienced in 2008.

1.5.2 Termination events

A termination event is a circumstance that allows a contracting party to terminate a deal on a certain date between the inception of the agreement and the maturity date. Its benefit arises from the ability of an entity to terminate a transaction contracted with a counterparty whose credit quality deteriorates during the term of the agreement. These break clauses have the advantage of being bilateral, providing the possibility for both (or more) parties to a contract to terminate the transaction. Such a counterparty risk mitigation tool is particularly useful when trading with a party of good credit quality on a long-term transaction. For such a long contract,

there is ample time for the mark-to-market of the underlying product to become positive and for the counterparty's credit quality to deteriorate to the point of bankruptcy.

1.5.3 Payment netting and close-out netting

For netting to take place and to be able to benefit from it to mitigate counterparty risk, one condition is necessary: an institution must simultaneously buy from and sell to another company. Today, there are two types of netting commonly used in the derivatives market:

- *Payment netting*: this is the situation when several counterparties have a number of obligations to each other, they agree to offset and net these obligations to transform a series of cashflows at a given day into a single net payment paid by the party that owes such an amount. This procedure helps to reduce settlement risk, i.e. the possibility that a party may fail to deliver within the agreed time the terms of its contract, and, in particular, to improve operational efficiency.
- *Close-out netting*: this second procedure is more relevant to counterparty risk since it reduces pre-settlement risk, i.e. the possibility that a party to a contract may fail to fulfil its obligations under that contract, resulting in a default prior to the settlement date. In such an agreement, the parties agree that if one of them defaults, the obligations they hold with respect to each other are immediately terminated and their positions are netted to arrive at a single due value.

When a credit event prevents an institution from continuing its obligations, in the majority of cases, there is a solvent and an insolvent party. Close-out procedure allows derivatives contracting parties to net their exposures. As we will see in Section 3.2.1, a credit exposure occurs only if the MtM of a financial asset is positive relative to the time of contraction. Considering the case of two trades related to a counterparty with and without netting, a netting benefit only occurs when at least one mark-to-market is negative. This method thus makes it possible to reduce the total exposure at the time of termination.

	MtM		Total exposure		
	Trade 1	Trade 2	No netting	Netting	Netting benefit
Counterparty 1	20	25	45	45	0
Counterparty 2	30	-15	30	15	15

Table 1.1: Illustration of the impact of close-out netting when contracting with two different counterparties.

1.5.4 Collateralisation

Widely used in the 1990s, collateralisation is a transferable asset or security, or even a promise of guarantee, used to cover credit risk in financial transactions in the event that the beneficiary

cannot meet its payment obligations. In the first place, these collaterals were mainly cash or government securities whereas today this instrument covers various forms such as mortgage-backed securities, corporate bonds, letters of credit and guarantees, equity and cover almost half of OTC derivatives exposures. In his work, [Gregory \(2010\)](#) explains that using the netting, which first reduces risk exposures, the remaining net exposure is further mitigated through the use of collateral, reducing total market exposure by about 93%.

While being a handy tool in reducing credit exposure, collateralisation can give rise to other risks such as foreign exchange (FX) risk, market risk, operational risk and liquidity risk, and is therefore a process to be managed carefully. For example, in the case of a home loan, the mortgagee may have pledged his house as collateral to cover his default risk. Nevertheless, this coverage could bring the risks outlined above to the fore:

- Between the contracting parties, they may not have the same local currency. In this case, one currency may be depreciated in relation to the other, particularly in the case of cash collateral. As a result, the party receiving the collateral in the event of a counterparty default may be less compensated than originally planned. This scenario is referred as *foreign exchange risk*.
- Since the property is pledged as collateral, there is a risk that its value will decrease during the term of the loan and that the bank will acquire a house that is worth less than it was (as experienced during the financial crisis). This is *market risk*.
- It is also possible that if the borrower defaults on payment, the mortgage giver may not be able to obtain the property directly and it may face costs (e.g. legal costs) to evict the ex-owners and to sell the good. It corresponds to *operational risk*.
- When the lender receives the property as a result of a default in payment, the house may still not sell directly due to a lack of buyers and may drop in value during this period. Or, in order to sell the property, the bank may decide to sell the property at a discount to its fair value. This risk corresponds to *liquidity risk*.

CALL OPTION PRICING

2.1 Introduction

In this chapter, we will mathematically develop the derivative on which will be applied our valuation method taking into account counterparty default, namely CVA. We focus on the European call option which, unlike forward contract, gives the buyer the right but not the obligation to buy the asset at maturity at a predetermined price. Although the reference asset may take the form of commodities, bonds, interest rates, or other, we deal with stock options in this paper.

The dynamics of the stock price were first described by Louis Bachelier in his doctoral dissertation, highlighting the Brownian motion to describe its random structure. First used in the field of science, this concept has revolutionized finance, making it possible today to accurately estimate the value of financial instruments and to build certain predictive patterns. We will analyze in this chapter the geometric Brownian motion concept through Ito's lemma to derive the mathematical model for pricing an option contract: the Black-Scholes formula. We will close this part by discussing the benefits associated with the assumptions of the model.

2.2 Earliest model of stock price and Brownian Motion

A modern approach to financial engineering with respect to the valuation of a stock was proposed by Louis Bachelier in the early 1900s. As stressed, the price of a stock option depends on the price of the underlying stock, which fluctuates unpredictably over time. Although first used to describe the random motion of a particle immersed in a fluid subject to no interaction other than shocks with the molecules of the surrounding fluid, this French mathematician used

the Brownian motion phenomenon to model the price movement of a stock. To understand the description of a random motion, we must first understand what a stochastic process is.

Definition 2.2.1. *A stochastic process is a time-indexed collection of random variables whose outcome ω is unknown before observing the result of the random experience:*

$$X : \mathbb{T} \times \Omega \rightarrow \mathbb{R}, (t, \omega) \mapsto X_t(\omega)$$

Usual notation: $(X_t)_{t \in \mathbb{T}}$

The interpretation of the parameter t as a time index introduces a dynamic aspect: to model that the uncertainty of the events of ω becomes less and less uncertain as time elapses, i.e. we have more and more information, we introduce the notion of filtration ([Pham, 2011](#)).

Definition 2.2.2. *The collection $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{T}}$ is a filtration on $\mathbb{T} \times (\Omega, \mathcal{F})$ if $\mathcal{F}(t) \subseteq \mathcal{F}$ are σ -fields for all $t \in \mathbb{T}$ and $\mathcal{F}(s) \subseteq \mathcal{F}(t) \forall s \leq t$.*

$\mathcal{F}(t)$ is interpreted as the information known at date t and increases over time whereas filtration $\mathbb{F}$ determines how information is revealed through time.

Remarks:

- Since X is a set of random variables, it is defined on $\mathbb{T} \times (\Omega, \mathcal{F})$.
- For each $t \in \mathbb{T}$, X_t is a random variable on $(\Omega, \mathcal{F})$.
- For each $(t, \omega) \in \mathbb{T} \times \Omega$, $X_t(\omega)$ is a real number.
- A stochastic process is discrete with respect to time if $\mathbb{T}$ is countable and is continuous otherwise.
- In discrete-time applications, we usually take $\mathbb{T} = \{0, 1, \dots\}$ or $\mathbb{T} = \{0, t_1, \dots\}$ whilst $\mathbb{T} = \mathbb{R}^+$ is commonly chosen in continuous-time applications.
- If $\{X_t \mid t \in \mathbb{T}\}$ is a stochastic process, then the random variable $X_t - X_s$, where $s < t$, is called an increment of the process.

For an investor, the dream being to predict this future set of information and thereby to forecast the stock path over time, it has become of great interest to develop theories and models that can explain its market price behaviour. One of the models that has emerged is the *random walk*, an economic theory stating that stock prices trail randomly up and down, unrelated to previous price movements and making it impossible to accurately predict the path the market would take over time ([Chitenderu et al., 2014](#)).

Because of this inability to predict with exactitude the unpredictable, they simulate the asset price by generating a multitude of price paths the stock could follow in the future.

Definition 2.2.3. The sample path of a stochastic process $(X_t)_{t \in \mathbb{T}}$ is the function:

$$\Omega \rightarrow \mathbb{R}, \omega \mapsto X_t(\omega)$$

that assigns to any possible outcome ω the trajectory $(X_t(\omega))_{t \in \mathbb{T}}$.

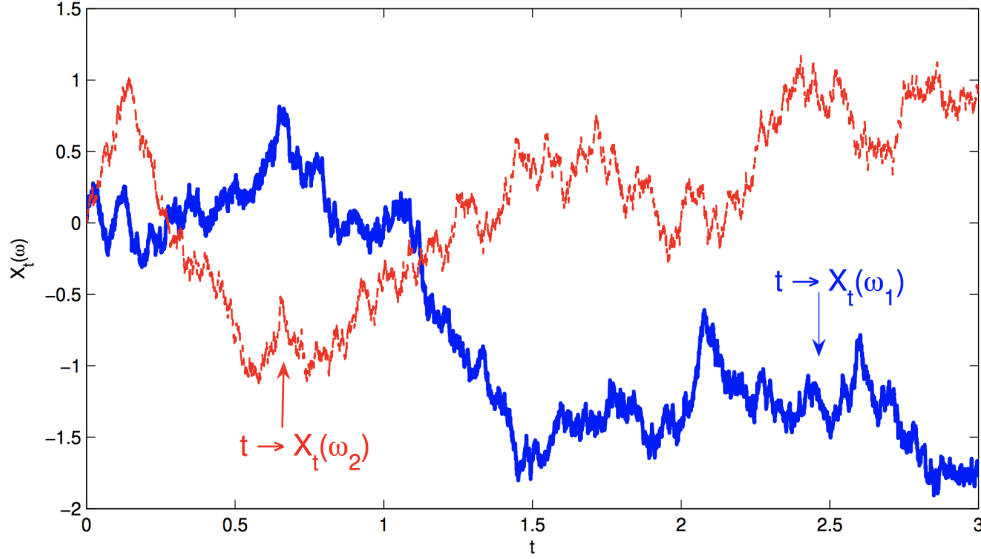


Figure 2.1: Sample paths of a stochastic process, corresponding to two outcomes ω_1 and ω_2 .

2.2.1 Random Walk

Random walk hypothesis is the first step in understanding Brownian motion. It is a financial theory claiming that stock market prices are subject to a random process and thus cannot be foreseen. By definition, the process $M = (M_i)_{i \in \mathbb{T}}$ is a random walk if it takes the form

$$M_i = \sum_{k=1}^i Z_k \quad \text{with } M_0 = 0$$

where $Z_1, Z_2, \dots$ are independent and identically distributed random variables on $(\Omega, \mathcal{F})$ satisfying

$$Z_k = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } 1-p \end{cases}$$

Remark: if $p = 1/2$, the process is a *symmetric random walk*.

Let us now consider the scenario whereby we **scale down the values** taken between each step and **speed up time** of our random walk M_t . The *scaled random walk* is defined as

$$W_t^{(n)} = \frac{1}{\sqrt{n}} M_{nt}$$

Remark: M_j is defined for $j \in \{0, 1, 2, \dots\}$, but $W_t^{(n)}$ is defined for $j \in \{0, 1/n, 2/n, \dots\}$. Hence, we can summarize the scaled random walk by the following three features:

1. The initial value of the process is: $W_0^{(n)} = 0$.
2. The layer spacing between two successive jumps is equal to $1/n$.
3. The "Up" and "Down" jumps are equal and of size $1/\sqrt{n}$ with equal probability.

For visual interpretation, here is the sample paths of a random walk M_t , of the accelerated random walk M_{2t} and of the scaled random walk $W_t^{(2)} = M_{2t}/\sqrt{2}$ (for $n = 2$).

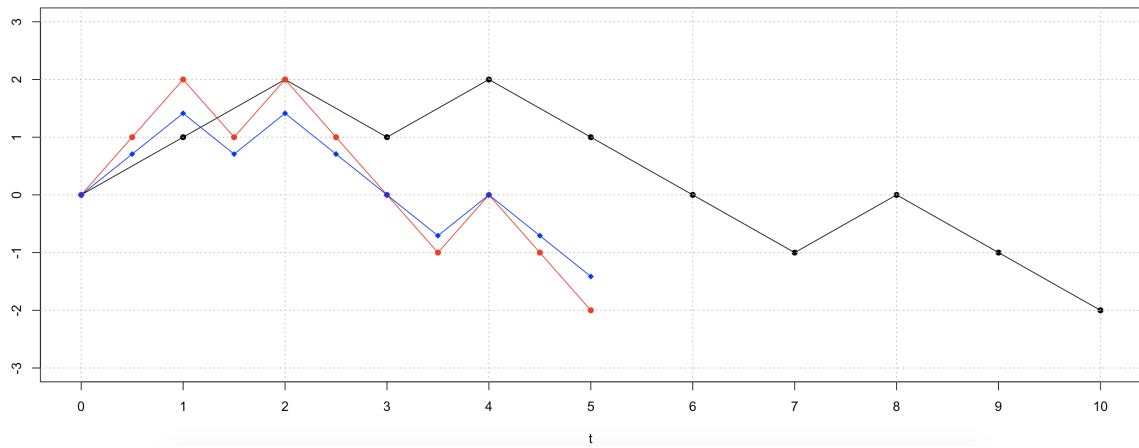


Figure 2.2: Visual interpretation of the scaled random walk.

The limit of the random walk lies in its nature to characterize a discrete dynamics between successive random steps. When n becomes large, the shape of the scaled random walk curve will increasingly look like the shape of the price of a stock. In the limit case where $n \rightarrow \infty$, the continuous symmetric scaled random walk $W^{(n)}$ is called a Brownian motion.

2.2.2 Brownian Motion with Drift

The Brownian motion is the central process to describe the continuous motion of a stock price and is defined as the scaling limit of a random walk in one dimension as the number of steps tends to become large, meaning that the time steps become close to 0.

In his work dedicated to the modelling of stock prices, Bachelier describes the future values of stock price by considering a drift term μt , taking the form of a constant multiple of time t , to

the standard Brownian motion and a scale parameter to obtain a Brownian motion with drift X_t . This process is also called "*generalized random walk*" and is denoted by

$$(2.1) \quad X_t = \mu t + \sigma W_t$$

where $\{W_t \mid t \geq 0\}$ is the standard Brownian motion, $\mu \in \mathbb{R}$ is the drift parameter and $\sigma \in \mathbb{R}_+$ is the volatility.

Definition 2.2.4. A continuous stochastic process $W = \{W_t \mid t \geq 0\}$ is a standard Brownian motion process if

- it is grounded: $W_0 = 0$
- it has independent increments: for any $q \leq r \leq s \leq t$,
 $(W_r - W_q) \perp (W_t - W_s)$
- the increments are stationary (Normal): $W_t - W_s \sim W_{t-s} \sim \mathcal{N}(0, t - s)$

	BM	BM increment
Expectation	$E[W_t] = 0$	$E[W_t - W_s] = 0$
Variance	$Var[W_t] = t$	$Var[W_t - W_s] = t - s$

Table 2.1: Properties of Brownian motion.

Over any time t , the distribution of a Brownian motion with drift satisfies a normal distribution whose parameters μt and $\sigma^2 t$ represent its mean and variance respectively (proof in Appendix A.1). However, it turns out that the model proposed by Louis Bachelier has been reconsidered for two main reasons:

1. Even if the initial value of the stock is positive $S_0 > 0$, there is a weak (but existing) probability that our stock would reach negative value under certain paths, which is obviously unrealistic to price stocks.
2. For a same time interval, a cheaper stock being sold at small price will tend to have small increments in price while other more expensive stocks being sold at higher price will tend to have larger increments in price. However, Brownian motion with drift has a variance that only depends on time without any other parameter consideration specific to the company's stock. Consequently, this assumption is also unrealistic for stock prices.

To overcome these two problems, it is no longer the price of the stock that we simulate, but rather its rate of return, which can be positive or negative over time and that can be defined in the following manner:

$$(2.2) \quad \frac{dS_t}{S_t} = dX_t = \mu dt + \sigma dW_t$$

where S_t is the stock price at time t and $\frac{dS_t}{S_t}$ is the corresponding stock rate of return. The 2.2 expression refers to a stochastic differential equation (SDE) and is described as an extension to ordinary differential equations except that it includes at least one stochastic process. While equivalent, this stochastic differential equation is more often found in the literature in the form:

$$(2.3) \quad dS_t = \mu S_t dt + \sigma S_t dW_t$$

and whose solution is called *geometric Brownian motion* (GBM).

2.3 Geometric Brownian Motion with Itô calculation

By definition, a geometric Brownian motion is a continuous-time stochastic process that satisfies equation 2.3. The rules being different for differentiation and integration in a stochastic environment, we introduce an identity to find the differential of a time-dependent function of a stochastic process: the Itô's lemma.

Itô's lemma. Let X_t be a stochastic process and $f(t, X_t)$ a smooth function. Then,

$$(2.4) \quad df(t, X_t) = f_t(t, X_t)dt + f_x(t, X_t)dX_t + \frac{1}{2}f_{xx}(t, X_t)(dX_t)^2$$

The solution to the stochastic differential equation 2.3 using Itô's lemma is

$$(2.5) \quad S_t = S_0 \cdot \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right)$$

and whose demonstration details are provided in Appendix A.2.

At any point in time, the geometric Brownian motion follows a lognormal distribution with parameters $\left(\ln(S_0) + \mu t - \frac{1}{2}\sigma^2 t\right)$ and $\sigma\sqrt{t}$ (Dunbar, 2016). With $S_0 > 0$, it justifies that S_t is positive for all t , resolving the problem of a possible negative stock value.

2.4 Call option pricing

A European call option on an underlying stock with exercise date T is said to be in the money (ITM) when $S_T > K$, at the money (ATM) when $S_T = K$, or out of the money (OTM) if $S_T < K$, where K represents the price fixed at inception at which the stock could be purchasable at maturity. This specific price is called strike price K or exercise price.

Since we wouldn't buy a stock at a higher price than market value, the payoff at maturity of a European call option is

$$(2.6) \quad C_T = \max\{S_T - K, 0\} = \begin{cases} S_T - K & \text{if } S_T > K \\ 0 & \text{if } S_T \leq K \end{cases}$$

2.4.1 Risk-neutral derivation of the Black Scholes formula

European call options can be priced using the risk-neutral measure, meaning that we can value options in terms of their expected payoffs discounted from expiration T to the present. In a risk-neutral environment, we can additionally consider that the mean rate of return μ is equal to the risk-free rate that we assume constant. At the present time $t = 0$ the valuation of European call option takes the form

$$(2.7) \quad V_0 = e^{-rT} E[C_T]$$

From this equation and the geometric Brownian motion of the stock price, the mathematical model for estimating the option contract is derived: the *Black-Scholes formula*

$$(2.8) \quad V_0 = S_0 \mathcal{N}(d_1) - K e^{-rT} \mathcal{N}(d_2)$$

where $\mathcal{N}(\cdot)$ is the cumulative distribution function of the standard normal distribution with

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T} = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

and whose detailed derivation can be found in Appendix A.4 (Shreve, 2004). However, a number of assumptions must be met to use this model:

- The risk-free rate is constant during the life of the option
- The stock doesn't pay any dividend
- The option must be European, i.e. it can't be exercised before its maturity
- Transaction costs of trading the option are not taken into account
- Stock prices follow a lognormal distribution
- The variance of the returns is constant.

2.4.2 Assumptions discussion and benefits of Black-Scholes model

The Black-Scholes model is an attempt to simplify financial asset and derivative markets into a set of relevant mathematical rules. Although it provides only a theoretical objective price for an option contract, the model is an important basis for market analysis. The main advantage of the model is that it is based entirely on objective numbers rather than human judgment. Another advantage is that the formula is relatively simple in mathematical terms, not requiring

sophisticated computer software to perform the calculations.

As we see in the Black-Scholes formula, the ultimate result is a pricing formula that takes into account only five parameters:

- S_0 , the current price of the asset on which the option is based.
- K , the price at which the option holder has the right to buy the asset.
- T , the amount of time left until the contract expires.
- σ , the volatility of the asset price, which determines how predictable the market price will be at the exercise date.
- r , the current rate on offer for risk-free investments.

CREDIT VALUE ADJUSTMENT

3.1 Introduction

The historical context now explained allows us to highlight in this chapter the valuation method for derivatives that takes into account the risk of counterparty default: the *credit value adjustment* (hereafter CVA). As a matter of fact, the past events have proven that it is no longer possible today for investors to value products without taking this risk into account or neither to consider the largest institutions, these 'too big to fail', as not being subject to a potential bankruptcy.

Counterparty risk has been brought to the forefront by the mistakes made to such an extent that the third Basel Accord presented by the Basel Committee on Banking Supervision (BCBS) introduced a capital charge associated with the creditworthiness of counterparty in OTC derivative transactions. The purpose of this charge is to force financial institutions to set aside capital to mitigate the risk of MtM losses on contracts. According to the Bank of International Settlements, approximately two-thirds of the losses incurred during the 2008 financial crisis were related to MtM losses and only one-third to actual defaults ([Ahlberg, 2013](#)). The pricing feature resulting from this risk is the CVA that we can define as *the value that needs to be added to the price of a product to account for the risk of the counterparty to default* ([Ekstrand, 2011](#)).

In this chapter, we will outline the key components of counterparty risk and how these components are mathematically quantified in order to derive the general CVA formula that we will use in the different practical applications.

3.2 Description of counterparty risk components

Counterparty risk consists of three elements: the risk exposure, the default probability of the counterparty and the recovery rate at default. To illustrate each of these concepts, we have been guided by the explanations provided in [Gregory \(2010\)](#)'s book, detailing their implications in the calculation of the CVA.

3.2.1 Credit exposure

The credit exposure is described as the loss incurred if opposing party involved in the contract defaults. This exposure is inherent to the analysis of counterparty risk because, for many financial products and particularly for the derivatives, the creditor is not exposed to the total principal amount of the transaction, but rather to the mark-to market (MtM), a method valuing an asset to its actual and current market level. In counterparty risk, the MtM is more representative than the replacement costs which take into account additional costs related to the amount that would be required to replace the asset with a similar one at the current market price. This second method is more appropriate for dealing with liquidity risk and doesn't accurately represent what we could actually lose.

Depending on whether or not a trade is in favour of an institution during the evolution of the underlying asset, MtM will be positive or negative and two situations may therefore exist at the time of a counterparty default:

- *Positive MtM*: by defaulting and consequently failing to make future commitments, the loss an institution faces is equal to the mark-to-market amount less any possible recovery.
- *Negative MtM*: in this scenario, it is the institution that owes the contracting party something although it defaults. The entity is still legally constraints to pursue its obligation but is not exposed to any loss.

The mathematical writing that describes the exposure of an entity in general is therefore as follows:

$$Exposure = \max(MtM, 0) = MtM^+$$

or more often expressed as

$$E(t) = \max\{V(t, T), 0\}$$

where $V(t, T)$ is the present value of the contract at time t .

Exposure characterization essentially consists in answering the following two questions:

- (i) What is the current exposure, i.e. the maximum loss if the contracting party defaults today?

(ii) What is the exposure in the future, i.e. what could be the loss if the contracting party defaults at some point in the future?

The second item listed above is more complex to deal with than the first, except in some basic circumstances. To be relevant, this exposure must be conditional on the default of the counterparty and must be quantified by asking the following questions:

- (i) What will be the exposure in 1 year assuming the contracting party will default in 1 year?
- (ii) What will be the exposure in 2 years assuming the contracting party will default in 2 years?
- (iii) And so on.

For simplification purposes, this exposure is often considered independent of any default event and of the credit quality of the counterparty. This assumption is described by the term "no-wrong way risk".

This need for characterizing what exposure would be in the future is described by the concept of potential future exposure (PFE). This PFE represents the possible exposure by considering a multitude of paths for a certain contract value defined today. The difficulty with this concept arises from the uncertainty regarding the future exposure that the institution would face. As we have just seen, exposure corresponds only to the case where the MtM is positive and whose an example of its distribution can be represented by the grey area of Figure 3.1.

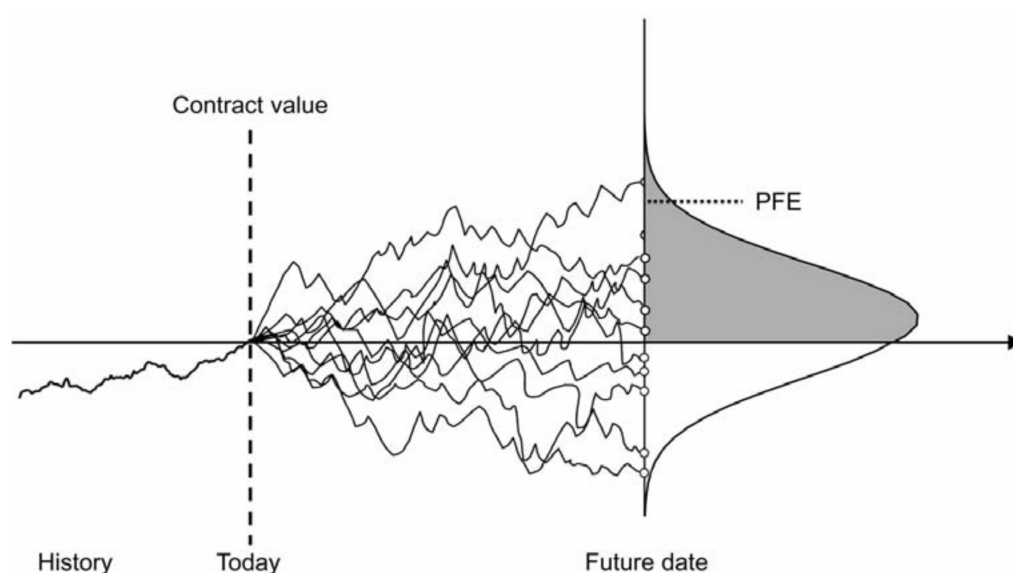


Figure 3.1: Illustration of the potential future exposure of a contract where each point on the vertical line of the shaded area represents a PFE related to a certain probability.

3.2.1.1 Exposure with netting agreements

Without a netting agreement, it is clear that the total exposure is simply the sum of the different exposures and that a positive value of one contract cannot be offset by the negative value of another as seen in Section 1.5.3. In this scenario, the exposure for N trades between the same two counterparties can be expressed as

$$E(t) = \sum_{i=1}^N \max\{V_i(t, T), 0\}.$$

Alternatively, if the two counterparties enter into an agreement allowing netting, the total exposure will be represented by

$$E^{Net}(t) = \max\left\{\sum_{i=1}^N V_i(t, T), 0\right\}$$

and whose amount will always be less than or equal to the amount without netting.

In their book, [Brigo et al. \(2013\)](#) interpret the efficiency of netting by comparing the correlation between different contracts included in the netting set. Their results indicate that if contracts are significantly negatively correlated, the impact of netting would be greater because the exposure of a contract of positive value would be offset by other contracts of negative value. In the opposite case, where the contracts would be positively correlated, netting agreement would have only a small beneficial effect on total exposure.

In our credit value adjustment, the presence of netting could have a significant impact on that value to be added since it has the potential to reduce the exposure of a derivatives portfolio to a single counterparty and thus reduce the counterparty's credit risk. Nevertheless, we will consider in our practical parts the presence of only one contract in order to ease the calculations and because it is not a big simplification.

3.2.1.2 Exposure with collateralisation

Collateralisation is a tool for reducing credit risk by guaranteeing the acquisition of assets in the form of cash, securities or physical assets in the event of a counterparty default. With the presence of collaterals, future exposure must take into account this compensation related to a potential loss. If we denote $Coll(t)$ as the value of the collateral that would be owed to the investor at time t , the exposure of a contract with a collateral agreement is

$$E^{Coll}(t) = \max\{V_i(t, T) - Coll(t), 0\}.$$

The formulas that define the amount of collateral required by each party of a contract are governed in an credit support annex (CSA). This document notably includes ([Hull and White, 2012](#))

- *the threshold*, i.e. the present value limit above which collateral must be posted.
- *the minimum transfer amount*, i.e. the minimum difference between the present value of the contract and the value of the collateral to benefit from it.
- *the eligible assets*, i.e. which assets are allowed as collateral.
- *the frequency* of marking-to-market and rebalancing of collateral.

Although this agreement can significantly reduce credit risk, we decide not to include it in our calculation of exposure. As described in 1.5.4, other risks may emerge and we do not include them within the scope of this work.

3.2.2 Probability of default

The probability of default of a counterparty is a probabilistic representation of its credit quality and therefore its solvency. In this work, $P_D(t)$ will refer to the default cumulative distribution function (CDF). For a contracting party that has not yet defaulted on its obligations, $P_D(t)$ has the properties of being a monotonically increasing function of time with $P_D(0) = 0$ and $\lim_{t \rightarrow +\infty} P_D(t) = 1$. In other terms, the counterparty's probability of default at the time of contracting is zero and the probability subsequently increases to the value of one when time tends to infinity (Ahlberg, 2013).

As Gregory (2010) describes in his book, $P_D(t)$ forms a term structure of the default probability (how it changes over time) whose the trend can generally be described, with respect to the creditworthiness, as:

- A counterparty with a poor credit quality will tend to have a term structure that is steeply sloping at the beginning and then tend to decrease. It is more likely for such a company to default soon after the contract begins. This tendency results in a higher probability that failure will occur, if it is to occur, earlier than at the maturity of the contract. A low future default probability is not the result of perfect creditworthiness, but rather that the company will not survive until that moment and will be more likely to fail in the short term.
- A counterparty with a high credit quality that is likely to deteriorate over time will have a flat curve at the beginning, followed by a sharp rise thereafter.
- A counterparty with a high credit quality and stable prospects will have a slowly rising curve.

Although there are several ways to construct a term structure of default, we focus on two estimation methods, namely the historical estimation and the market-implied default probabilities, and discuss their respective benefits.

3.2.2.1 Historical estimation

This is perhaps the most mathematically straightforward method of assessing a counterparty's probability of default. It involves examining historical default data and using past experience to predict the probability of future default. By taking, for example, the average cumulative default rates between 1981 and 2019 shown in Appendix A.5, we can easily draw the historical default curve for S&P's AA-, BB-, B- and CCC/C-rated firms.

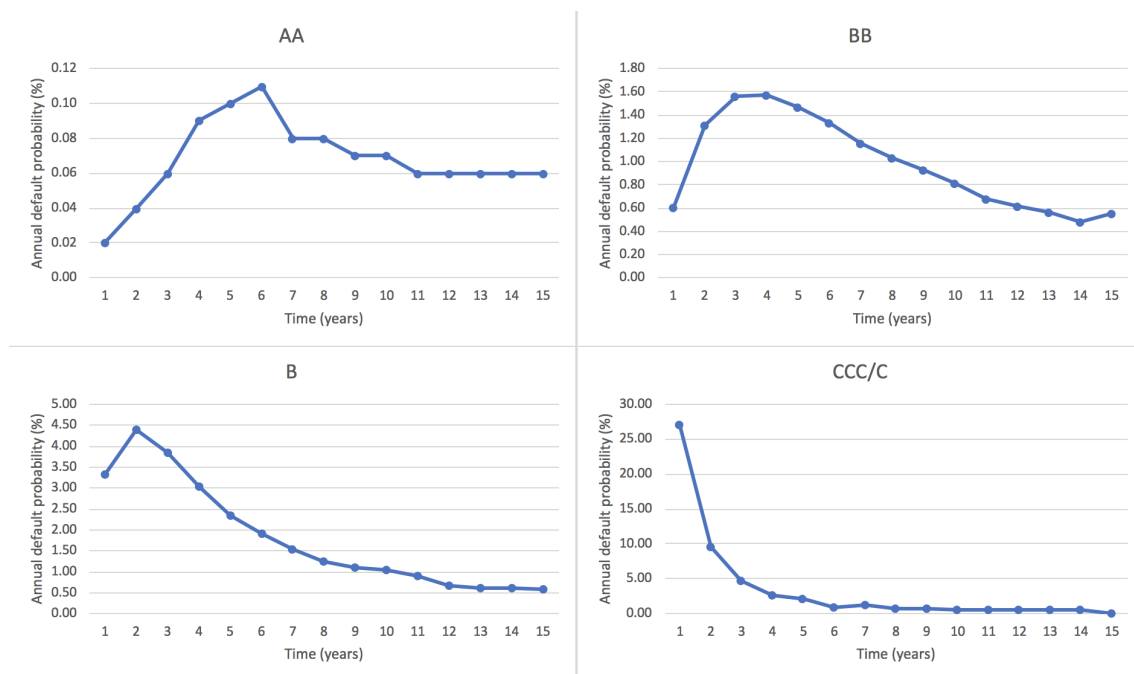


Figure 3.2: Annual historical default probabilities. For example, for AA-rated firms, the point at 4 years is the default probability in the interval 3 years to 4 years, which is $0.24\% - 0.13\% = 0.09\%$.

As described, we observe that historical values confirm the hypothesis that firms with poor credit quality have default probabilities concentrated in the short-term. This is not justified by their expected improvement of credit quality, but rather by the fact that default in the long run could potentially occur only by surviving at the beginning.

3.2.2.2 Market-implied default probabilities

The default probability curve of a firm may also be implicit in the market prices of credit default swap (CDS) contracts related to the counterparty in question, in which case the curve

will be market implied and hence risk-neutral (Ahlberg, 2013). Since CDS contracts are only traded for certain reference institutions, this method does not apply to small and medium-sized businesses, for which other methods will have to be used, such as estimation by bond spread analysis, by the available analysis of defaults of similar companies or by the analysis of proxy variables that can be weighted together to form a probability of default curve. However, we will not go into the details of the latter.

In default curve modeling, time to the occurrence of a failure (if it occurs) can be thought of as an independent continuous random variable. If $F(t)$ represents the probability that our security will default prior to time t , and τ is the time of default, we can represent the distribution of time until default as

$$F(t) = P(\tau \leq t) \quad t \geq 0$$

The relationship between the survival function and the default function is described as

$$S(t) = 1 - F(t) = P(\tau > t) \quad t \geq 0$$

where $S(t)$ represents the probability that the security will survive until time t . Using actuarial notation to describe the mathematical pattern, we can formulate a discrete-time probability statement on a security that has survived x years as

$$(3.1) \quad q_x = P(\tau \leq x + 1 | \tau > x)$$

The symbol q_x is named the marginal probability of default and represents the probability that a security will default within the next year knowing it has survived x years.

Without a modeling process for the default event itself, the above equations for time to default are fairly useless. To overcome this, reduced-form models assume that default events occur in accordance with a Poisson counting process which has the following properties:

1. The numbers of changes in non-overlapping intervals are independent for all intervals.
2. The probability of two or more changes in a sufficiently small interval is 0.

These two relationships are meaningful when related to business failures, since it is realistic to assume that the probability of failure in any given period is independent of all other periods and that the probability of two corporate defaults in any period is equal to 0. In modelling a firm's default, we will only consider the first Poisson event or the first arrival of the default since a firm can only default once (Nunweiler and Watti, 2006).

Considering the intensity of the process as a continuous process, we can rewrite our conditional probability formula in equation 3.1 so that it is based on a small variation in time:

$$(3.2) \quad P(\tau \leq t + dt | \tau > t) = \lambda(t)dt$$

The expression can be interpreted as the probability that a default occurs in the time interval $[t, t + dt]$ conditional on surviving up to time t . We observe that the probability is proportional to a time function $\lambda(t)$, referred to as the hazard rate or the intensity of default, and the duration of the time interval dt . We make the simplifying assumption that the hazard rate process is deterministic and that is a step-wise constant function.

Poisson process is generally suitable for modeling the default. This model is used to model rare or discretely countable events, which is perfectly appropriate for our case study. Here, the default time is described as the first jump time of a Poisson process with a stochastic or deterministic arrival intensity (hazard rate). In a Poisson process, the probability of achieving exactly k successes (i.e. the existence of default) is described by the binomial distribution

$$P(k|n) = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}.$$

Since our sample size n is large because we want to determine when the default would occur among a large number of time steps, the formula for the Poisson process will approximate a Poisson distribution probability function,

$$p(k) = P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

where k represents the number of events occurring in the time interval. This formula can be extended to give the probability that $\tau > t$, i.e. the probability that τ (time to default) does not occur in the time interval $[0, t]$. In this case, no credit event would occur and we can replace k in the formula above by 0 to give us the survival probability $S(t)$ for a deterministic step-wise hazard rate:

$$\begin{aligned} S(t) = P(\tau > t) &= \frac{(\lambda t)^0}{0!} e^{-\lambda t} \\ &= e^{-\lambda t} \end{aligned}$$

However, since we are more interested in modeling the default rather than the survival, we deduce the cumulative default probability $F(t)$ as being:

$$\begin{aligned} F(t) = P(\tau < t) &= 1 - S(t) \\ &= 1 - P(\tau > t) \\ &= 1 - e^{-\lambda t} \end{aligned}$$

for a constant hazard rate, or

$$= 1 - e^{-\int_0^t \lambda(x) dx}$$

for a step-wise constant intensity of default.

For some reasons of simplification and because a single-name CDS market is mainly based around 5-year instruments since other maturities will be rather illiquid, we may only take the CDS spread for a 5-year maturity to represent the overall default intensity under the relation

$$(3.3) \quad \lambda \approx \frac{X_{CDS}}{1-R}$$

proposed by [Gregory \(2010\)](#) for the approximation of the hazard rate based on the CDS premium to be paid to cover counterparty credit risk and where X_{CDS} is the CDS premium (as a percentage) and R is the assumed recovery rate (also a percentage).

3.2.3 Recovery Rate and Loss-Given Default

In the event of a breach of its obligations, the derivative holder who contracted with the defaulting party generally recovers a certain ratio of the gross exposure. This percentage is called the recovery rate, denoted by R , and can be considered deterministic for simplicity at the time of contracting, meaning that the recovery rate is a constant percentage of the exposure over time, or stochastic and correlated with the evolution of the default probability. This rate plays a crucial role in the quantification of credit derivatives and is an important figure in the computation of credit value adjustment as it potentially has a significant effect on the amount lost. This concept may also be described as the "loss given default", which is related to the recovery rate component by the relation: *loss given default* = 1 - *recovery rate*.

3.3 Credit Value Adjustment

As described, Credit Value adjustment is the price of the counterparty credit risk and is in itself a financial instrument that must be estimated using market data. In this work, we decide to define the price of this instrument as the risk-neutral price under the assumption that the market is complete.

In their collective work, [Brigo et al. \(2013\)](#) define the CVA as *the market price of counterparty credit risk on that contract given by the risk-neutral expectation of the loss that is due to counterparty default between inception and maturity weighted with the risk-neutral probability of the counterparty defaulting*. Although all the components have been developed, before deriving the CVA formula, it is important to specify that we will focus on the unilateral credit value adjustment, i.e. the model that considers the counterparty to be risky but the investor to be free of default.

3.3.1 Probabilistic model

In order to derive the CVA formula, the probabilistic framework must first be specified as well as its specific notations. We first describe the probabilistic market model by defining $(\Omega, \mathcal{G}, \mathcal{G}_t, \mathbb{Q})$ as the probability space, where Ω is the sample space that collects all possible outcomes, $\mathcal{G}$ the σ -field with $(\mathcal{G}_t)_{t \geq 0}$ being the complete market filtration up to time t , i.e. $\mathcal{G}_t$ provides information about paths of the asset up to t , and finally $\mathbb{Q}$ being the risk-neutral measure under which all primary assets discounted at the risk-free rate (S^i/β) are $(\mathbb{Q}, \mathbb{F})$ -martingales.

Our sample space Ω can be thought as a *product space* that would include all possible scenarios within the world of market and credit risk. Mathematically speaking, $\Omega = \mathcal{M} \times \mathcal{C}$ where

- $\mathcal{M}$ is the collection of all possible market risk factor outcomes,
- $\mathcal{C}$ is the set of all possible credit risk factor outcomes.

In addition, the filtration is an enlarged filtration that we describe as $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, where

- $\mathcal{F}_t$ is the credit risk-free market filtration,
- $\mathcal{H}_t$ is the the default filtration generated by the default event: $\mathcal{H}_t = \sigma(\{\tau \leq u\} : u \leq t)$, where τ is the counterparty default time.

The price of derivatives free of default risk is determined in relation to the credit risk-free market filtration $\mathcal{F}_t$ under the risk-neutral measure $\mathbb{Q}_{\mathcal{M}}$ on the space of market risk $\mathcal{M}$ while derivatives that are exposed to the risk of default are priced in regard to the complete market filtration $\mathcal{G}_t$ under the risk-neutral measure $\mathbb{Q}$. Consequently, the expectations $E^{\mathbb{Q}_{\mathcal{M}}}[\cdot | \mathcal{F}_t]$ and $E^{\mathbb{Q}}[\cdot | \mathcal{G}_t]$ used in the derivation of the CVA formula represent respectively the expectations conditional on the credit risk-free market filtration $\mathcal{F}_t$ and on the complete market filtration $\mathcal{G}_t$ under the corresponding risk-neutral measures ([Ahlberg, 2013](#)).

3.3.2 Unilateral CVA formula

Default risk being by definition the possibility that the counterparty will default before the maturity of the contract, two situations exist:

1. The first case is where the counterparty survives until the maturity of the contract ($\tau > T$), in which point all payments under the terms of the contract will normally occur. In that case, the present value at time zero for the investor is

$$(3.4) \quad PV_{\tau > T} = V(0, T)$$

2. The second case is where the counterparty defaults at some point prior to the maturity of the contract, failing to meet its future obligations and triggering a risk to the investor of credit exposure. If the present value is positive for the investor, this value will be definitively lost, unless a recovery policy has been decided at the time of the contract. And if the present value is negative, the investor will always have to compensate the creditors of the defaulting counterparty. For that reason, in the case of defaulting before maturity ($\tau < T$), the present value at time zero for the investor is

$$(3.5) \quad PV_{\tau \leq T} = V(0, \tau) + (R \max\{V(\tau, T), 0\} + \min\{V(\tau, T), 0\})D(0, \tau)$$

where $D(s, t)$ is the discount factor between the times s and t .

Therefore, the resulting present value of the contract with unilateral credit value adjustment is given by the risk-neutral expectation of the two present values 3.4 + 3.5. Furthermore, let $\mathbb{1}_{\{A\}}$ denote the indicator function, i.e. the function that takes the value 1 if the event A occurs and 0 otherwise. We thus get

$$\begin{aligned} PV &= E^Q \left[\mathbb{1}_{\{\tau > T\}} V(0, T) + \mathbb{1}_{\{\tau \leq T\}} V(0, \tau) + \mathbb{1}_{\{\tau \leq T\}} (R \max\{V(\tau, T), 0\} + \min\{V(\tau, T), 0\}) D(0, \tau) \middle| \mathcal{G}_0 \right] \\ &= E^Q \left[\mathbb{1}_{\{\tau > T\}} V(0, T) + \mathbb{1}_{\{\tau \leq T\}} V(0, \tau) + \mathbb{1}_{\{\tau \leq T\}} \left(-(1-R) \max\{V(\tau, T), 0\} \right. \right. \\ &\quad \left. \left. + \max\{V(\tau, T), 0\} + \min\{V(\tau, T), 0\} \right) D(0, \tau) \middle| \mathcal{G}_0 \right] \\ &= E^Q \left[\mathbb{1}_{\{\tau > T\}} V(0, T) + \mathbb{1}_{\{\tau \leq T\}} V(0, \tau) - \mathbb{1}_{\{\tau \leq T\}} (1-R) \max\{V(\tau, T), 0\} D(0, \tau) \right. \\ &\quad \left. + \mathbb{1}_{\{\tau \leq T\}} V(\tau, T) D(0, \tau) \middle| \mathcal{G}_0 \right] \\ &= E^Q \left[\left[\mathbb{1}_{\{\tau > T\}} + \mathbb{1}_{\{\tau \leq T\}} \right] V(0, T) - \mathbb{1}_{\{\tau \leq T\}} (1-R) \max\{V(\tau, T), 0\} D(0, \tau) \middle| \mathcal{G}_0 \right] \\ &= E^Q \left[V(0, T) \middle| \mathcal{F}_0 \right] - E^Q \left[\mathbb{1}_{\{\tau \leq T\}} (1-R) \max\{V(\tau, T), 0\} D(0, \tau) \middle| \mathcal{G}_0 \right] \end{aligned}$$

In our last equation, the terms are defined as the risk-free present value of the contract and the CVA. We are now able to observe mathematically that the CVA is an additive adjustment term as previously stated. Let us now focus on the development of the general formula of the UCVA:

$$\begin{aligned} CVA &= E^Q \left[\mathbb{1}_{\{\tau \leq T\}} (1-R) \max\{V(\tau, T), 0\} D(0, \tau) \middle| \mathcal{G}_0 \right] \\ &= E^Q \left[\int_0^T (1-R) \max\{V(s, T), 0\} D(0, s) \delta(s - \tau) ds \middle| \mathcal{G}_0 \right] \end{aligned}$$

where $\delta(\cdot)$ is the Dirac delta function. Using the Tower law on conditional expectations, $E[X|\mathcal{G}] = E[E[X|\mathcal{F}]|\mathcal{G}]$, we have

$$\begin{aligned}
 &= E^{\mathbb{Q}} \left[E^{\mathbb{Q}_{\mathcal{M}}} \left[\int_0^T (1-R) \max\{V(s, T), 0\} D(0, s) \delta(s - \tau) ds \middle| \mathcal{F}_t \vee \{\tau = t\} \right] \times \mathbb{1}_{\{\tau = t\}} \middle| t \leq T, \mathcal{G}_0 \right] \\
 &= E^{\mathbb{Q}} \left[E^{\mathbb{Q}_{\mathcal{M}}} \left[(1-R) \max\{V(t, T), 0\} D(0, t) \middle| \mathcal{F}_t \vee \{\tau = t\} \right] \times \mathbb{1}_{\{\tau = t\}} \middle| t \leq T, \mathcal{G}_0 \right] \\
 &= \int_0^T E^{\mathbb{Q}_{\mathcal{M}}} \left[(1-R) \max\{V(t, T), 0\} D(0, t) \middle| \mathcal{F}_t \vee \{\tau = t\} \right] dP_D(t)
 \end{aligned}$$

The general UCVA formula can therefore be written as follows

$$(3.6) \quad UCVA = \int_0^T E^{\mathbb{Q}_{\mathcal{M}}} \left[(1-R) \max\{V(t, T), 0\} D(0, t) \middle| \mathcal{F}_t \vee \{\tau = t\} \right] dP_D(t)$$

Part II

Practical perspective

NO WRONG-WAY RISK

4.1 Introduction

Although our recovery rate can be considered time-dependent and implied by the market, the first case analysis we consider in this part is the standard approach that assumes a constant recovery rate under the no-wrong way risk. We suppose that the recovery rate, the default of the counterparty and the credit exposure are not related even though evidence has shown a dependency between these patterns ([Altman et al., 2005](#)). We impose this simplification because the real interest of this practical approach consists in analyzing the UCVA results for two counterparties (one riskier than the other) issuing a call option and for different constant recovery rates.

To study this difference in results between a more secure counterparty and a less secure one, we chose the American multinational investment bank and financial services company Morgan Stanley and the largest German bank, the Deutsche Bank. For the underlying asset of our European call option, we opted for Apple's stock. As this company is rated AA⁺ according to our database, this information is a good indicator of its survival over time and could ensure that the company does not go bankrupt before the two issuers of the derivative.

4.2 No-Wrong way Risk formula for call option

The impact of our assumptions on the general formula for the unilateral CVA is straightforward. The recovery rate being considered constant we can, by the property of the expectation of a

constant, take this term out of our expectation and integral

$$UCVA = (1 - R) \int_0^T E^{\mathbb{Q}_M} \left[\max\{V(t, T), 0\} D(0, t) \middle| \mathcal{F}_t \vee \{\tau = t\} \right] dP_D(t)$$

The second term to be developed is the discounted credit exposure representing the risk that we will not realize our gain. In European call option, a special feature is that the contract value is always positive (see equation 2.6). We derive our formula as

$$(4.1) \quad UCVA = (1 - R) \int_0^T E^{\mathbb{Q}_M} \left[V(t, T) \cdot D(0, t) \middle| \mathcal{F}_t \vee \{\tau = t\} \right] dP_D(t)$$

Under risk-neutral valuation principle, the price $V(t, T)$ discounted at risk-free rate is a Martingale. Since we compute the credit value adjustment at time $t = 0$, it means that we only have information concerning the dynamics of the derivative at that time. The expectation expression can then be simplified in the following form under Martingale property:

$$E^{\mathbb{Q}_M} \left[V(t, T) \cdot D(0, t) \middle| \mathcal{F}_0 \right] = V(0, T) \cdot D(0, 0)$$

Back to previous (4.1) expression of UCVA formula, all we have left to do is to integrate.

$$\begin{aligned} UCVA &= (1 - R) \int_0^T V(0, T) \cdot D(0, 0) \cdot dP_D(t) \\ &= (1 - R) \cdot V(0, T) \int_0^T dP_D(t) \\ &= (1 - R) \cdot V(0, T) \cdot (P_D(T) - P_D(0)) \end{aligned}$$

If we assume that it is impossible for the counterparty to default at the time the contract is entered into, i.e. $P_D(0) = 0$, we get our final UCVA expression by incorporating our default probability function whose expression was deduced in the subsection 3.2.2.2:

$$(4.2) \quad UCVA = (1 - R) \cdot V(0, T) \cdot \left(1 - e^{-\int_0^T \lambda(x) dx} \right)$$

We notice that the CVA of a European call option is the product of deterministic values under the assumptions mentioned and whose model seems now simpler to code for our data.

4.3 Black-Scholes inputs

As introduced, we decided to take the Apple's stock price as the underlying asset in the context of the thesis. We chose this reference company because of the availability of historical data but mainly for the data already available on the Refinitiv Eikon financial analysis platform concerning the implied volatilities we will need in the Black-Scholes formula for different strike prices and maturities.

4.3.1 Apple's stock price

Before addressing the estimation of the call option value for a single stock, it is interesting to take a look at the historical performance of Apple.

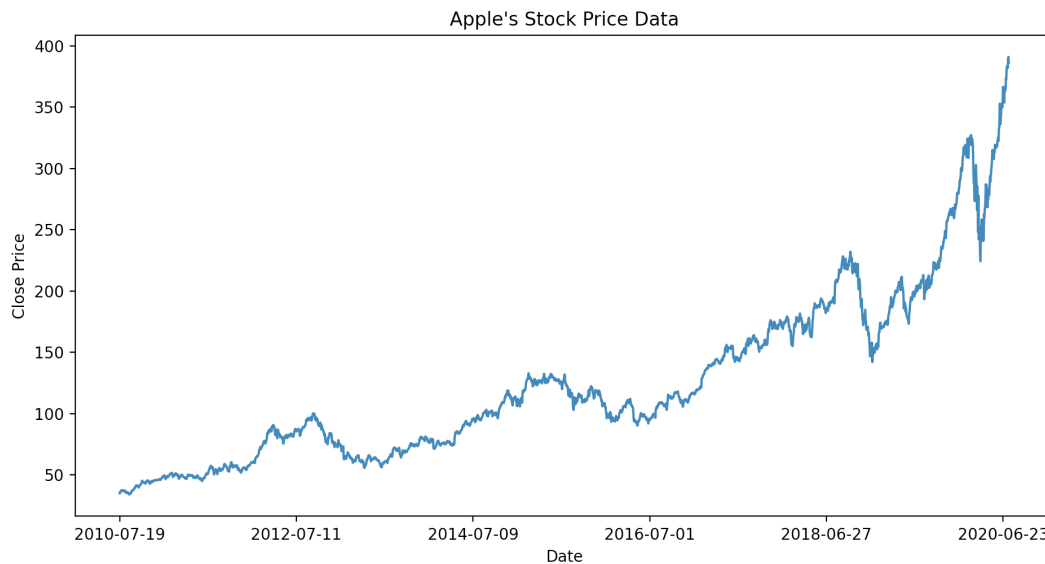


Figure 4.1: Historical performance of Apple's stock price.

Since one of the parameters of the Black-Scholes formula is the stock price at time $t = 0$, i.e. the time at which the value of the derivative is estimated, it was crucial to define that valuation time. We consider here the 16th of July 2020 stock price to be the initial value (S_0) for our calculation. According to the various sources available, the current stock price is estimated at \$386.09. The only simplifications made here are that we consider that no dividend is paid to shareholders between the inception of the contract and its maturity and that the volatility is constant.

4.3.2 Volatility parameter

The volatility parameter is the most complicated input to estimate and a wrong assessment can have a huge effect on the value of the option. To evaluate this parameter, measured as a percentage per year, there are two common ways:

1. By observing the historical volatility of the underlying asset. In this scenario, we assume that the stock will have a similar constant volatility in the future or we adjust this historical volatility according to our beliefs about the future (for example, we might estimate the volatility input higher than the calculated historical volatility because we expect an event that would significantly influence the movement of the stock price either positively or negatively).

2. By looking at the implied volatility of similar options, i.e. options with other strike prices or exercise dates for the same underlying or options on related securities, for instance options related to other stocks in the same business sector.

4.3.2.1 Historical volatility

When we refer to the historical volatility of a stock's price, we are really referring to the past volatility of returns. Even if it may seem insignificant, this interpretation is important to understand when calculating and interpreting historical volatility. We mathematically define our historical volatility as the annualized standard deviation of returns and, since it is easy for us to calculate past returns, this method does not require sophisticated calculations.

First, we need to calculate the continuously compounded return of each period. In our instance, we calculate the day-to-day returns using the formula

$$R_t = \ln\left(\frac{S_t}{S_{t-1}}\right)$$

We obtain quite easily the following graph showing all the daily returns of our stock.

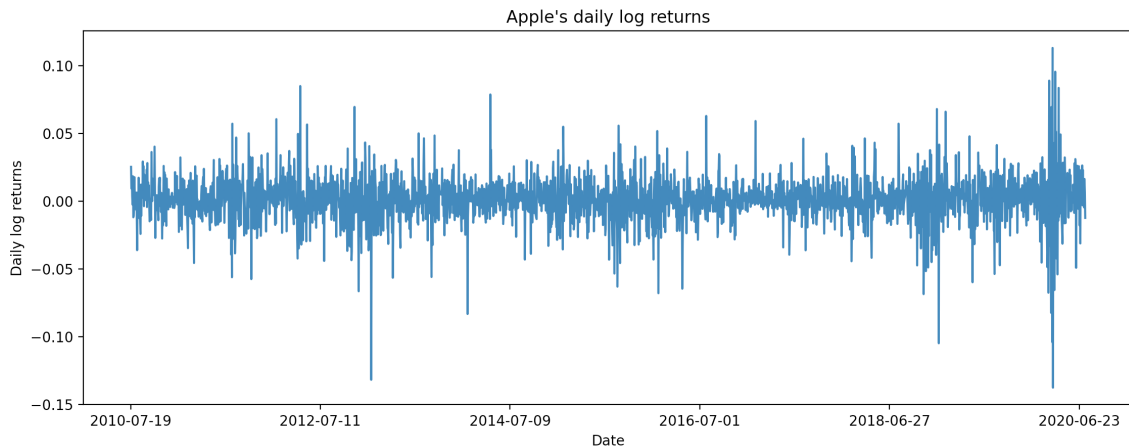


Figure 4.2: Historical daily log returns of Apple's stock price.

Thereafter, we need to calculate the standard deviation of the log returns we obtained in step 1. By definition, the standard deviation is the square root of the variance corresponding to the root mean square deviation from the mean. In order to calculate this volatility, the first step is to obtain the average of these log returns using

$$\begin{aligned} R_{avg} &= \frac{\sum_{i=1}^n R_i}{n} \\ &= 0.0009528636586951298 \end{aligned}$$

Then, we calculate the squared deviation from the average for each of the returns:

$$(R_i - R_{avg})^2$$

The sample variance is calculated from a sample and not from the total population. To find this value, we must apply the following formula:

$$\sigma_{day}^2 = \frac{\sum_{i=1}^n (R_i - R_{avg})^2}{n - 1}$$

Since standard deviation is the square root of variance, the whole formula therefore is:

$$\sigma_{day} = \sqrt{\frac{\sum_{i=1}^n (R_i - R_{avg})^2}{n - 1}}$$

where the number we got now (σ) is the 1-day historical volatility. The formula above is applicable for 1-period historical volatility. the parameter is commonly computed in its annualized form. To annualize 1-period of volatility, we simply multiply it by the square root of the number of periods per year. Since we estimated the daily standard deviation using returns from actual business days only, the average number of observations per year is 252, the median number of trading days per year in U.S. markets from 2010 to 2022. Hence,

$$\sigma_{annual} = \sqrt{252} \cdot \sigma_{day}$$

With respect to our stock price data, we get the following results:

	Start date	End date
	19/07/2020	16/07/2022
Average of the returns: R_{avg}	0.00095324	
Variance of the returns: σ_{day}^2	0.00029925	
1-day historical volatility: σ_{day}	0.01729896	
Annualized historical volatility: σ_{annual}	0.27461248	

Table 4.1: Historical volatility results for Apple's stock.

Historical volatility reflects the movement of the stock price on the basis of the past and would therefore predict the future by looking backward. Furthermore, the return in the years 2010 has the same weight, and therefore the same impact, than recent returns whereas the latter should be considered more representative for the calculation of future volatility. Hence, this technique is a good approach to get an idea of the past movement but investors pay more attention to implied volatility when calculating the value of an option.

4.3.2.2 Implied volatility

Implied volatility is a dynamic measure that evolves with activity in the options marketplace. When implied volatility increases, the price of options usually increases simultaneously assuming that everything else remains unchanged. Thus, when implied volatility increases after

a transaction has been entered into, it is good for the owner and bad for the seller of the option. The way implied volatility is assessed has been widely adopted and provides better information on future movements than historical volatility.

In our practical part, we directly found the implied volatilities of Apple's stock price for different maturities and strike prices in our financial analysis platform. We find below three maturities we will address in our analysis. However, data differs between short and longer maturities. It is obvious that for a call option to be exercised in the coming months, it is very unlikely to find contracts whose strike price is very different from the current stock price (as observed for an exercise date on 7th of August 2020). To study the impact of the maturity and the strike price chosen for the call option, we will therefore focus on the \$380 and \$390 strike prices available for the three maturities (see Appendix A.6 for original tables of collected data).

	Strike prices								
	360	370	377.5	380	385	390	392.5	400	410
7 August 2020			44.85	45.20	45.40	45.50	44.97		
17 June 2022	38.07	35.46		34.92		34.33		34.01	33.72
16 September 2022	34.94	35.36		35.06		35.20		34.39	33.71

Table 4.2: Implied volatilities data for different exercise dates and strike prices.

4.3.3 Risk-free interest rate

Like volatility, the risk-free interest rate is measured in percent per annum. For investors, it is the rate at which they could deposit or borrow cash during the term of the option. Naturally, the duration of the interest rate must correspond to the expiry date.

The risk-free interest rate theoretically represents the minimum return an investor would expect for any investment since he or she will not tolerate additional risk unless the potential rate of return is higher than the risk-free rate. Yet, in practice, an actual risk-free rate does not exist since even the safest investments involve a very low risk as past events have shown.

Risk-free interest rate commonly refers to the U.S. Treasury yield curve rates for US-based investors or to the yield on long-term government bonds. We use the Treasury yield curve rates provided by the U.S. Department of the Treasury to determine our risk-free rates for different time horizons. This curve establishing a relationship between the yield and the maturity of a security is based on closing market bid yields on Treasury securities actively traded in the over-the-counter market.

Date	1 Mo	3 Mo	6 Mo	1 Yr	2 Yr	3 Yr	5 Yr	7 Yr	10 Yr	20 Yr	30 Yr
07/13/20	0.11	0.14	0.15	0.16	0.16	0.19	0.30	0.48	0.64	1.11	1.33
07/14/20	0.11	0.15	0.14	0.17	0.14	0.19	0.28	0.47	0.63	1.09	1.30
07/15/20	0.12	0.16	0.15	0.15	0.16	0.18	0.28	0.47	0.64	1.11	1.33
07/16/20	0.12	0.11	0.13	0.14	0.16	0.17	0.28	0.46	0.62	1.09	1.31

Table 4.3: Daily Treasury Yield Curve Rates.

Source: U.S. Department of the Treasury -

<https://www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/TextView.aspx?data=yield>

4.4 European call option value

To clarify our analysis of the inputs that will allow us to compute the value of our derivative, here are two summary tables showing the parameters for a \$380 and \$390 exercise price respectively:

	Exercise dates		
	7 August 2020	17 June 2022	16 September 2022
Current stock price S_0	386.089996	386.089996	386.089996
Implied volatility σ	45.20%	34.92%	35.06%
Maturity T^*	0.06746	1.91667	2.16667
Risk-free rate r	0.12%	0.16%	0.16%

Table 4.4: Inputs summary table for a strike price $K = 380$.

	Exercise dates		
	7 August 2020	17 June 2022	16 September 2022
Current stock price S_0	386.089996	386.089996	386.089996
Implied volatility σ	45.50%	34.33%	35.20%
Maturity T^*	0.06746	1.91667	2.16667
Risk-free rate r	0.12%	0.16%	0.16%

Table 4.5: Inputs summary table for a strike price $K = 390$.

* Our maturities are obtained by calculating the number of business days between our valuation date and exercise date excluding weekends and public holidays in the United States and dividing it by the median number of trading days per year (252 days).

Now that we have all the elements enabling us to calculate the value of our derivative using the Black-Scholes formula and under the defined assumptions, the results obtained for $d1$, $d2$ and V_0 for our two exercise prices and our three maturities are shown in the following table:

	For $K = 380$		
	7 August 2020	17 June 2022	16 September 2022
$d1$	0.1948185	0.2809535	0.2955604
$d2$	0.07741994	-0.2024921	-0.2205088
V_0	21.15501	76.73428	81.61779

Table 4.6: Black-Scholes results with respect to $K = 380$.

	For $K = 390$		
	7 August 2020	17 June 2022	16 September 2022
$d1$	-0.02548965	0.2228903	0.2463084
$d2$	-0.1436674	-0.2523871	-0.2718216
V_0	16.40948	71.42754	77.91036

Table 4.7: Black-Scholes results with respect to $K = 390$.

The first observation we can draw here is that the contract value is higher for a lower exercise price. Since the implied volatilities are close for the two strike prices and for the respective maturities, we can quickly guess that the losses related to a counterparty default would be greater for a strike price of \$380 than for a strike price of \$390.

4.5 Counterparty risk components results

As a reminder, we derived the no wrong-way risk with constant recovery formula for the UCVA in the case of a call option in section 4.2.

$$(4.3) \quad UCVA = (1 - R) \cdot V(0, T) \cdot \left(1 - e^{-\int_0^T \lambda(x) dx}\right)$$

Although the current value of our derivative has been obtained using the Black-Scholes formula, we still need to determine the two missing parameters: the recovery rate and the implied probability of default.

4.5.1 Constant recovery rate

For the calculation of the UCVA under the no-wrong way risk assumption, the following question arises: what recovery rate should be applied? As mentioned, we decided to take a constant recovery rate which allowed us to simplify our UCVA formula. According to the literature, standard constant recovery rate of 40% (i.e. a loss given default of 60%) reflects a historical average for a long time period ([Elkamhi et al., 2010](#)).

In addition to its mathematical advantage, a constant recovery rate also makes it easy to apply the connection between the recovery rate, the probability of default through the hazard

rate and the CDS spreads of the counterparty issuing the derivative. The term "*credit triangle*" describes this relationship and refers to the simple formula that we've already met in the sub-section [3.2.2.2](#)

$$\lambda = \frac{X_{CDS}}{(1-R)}.$$

An interesting analysis would be to compare UCVA values for different constant recovery rates and to observe its influence on final results. We will see in the section [4.6](#) that we compare the credit value adjustment for three constant recovery rates, namely 20%, 40% and 60%.

4.5.2 Implied default probability from CDS inputs

To derive our implied probability of default using a Poisson process, the first step was to collect CDS spreads data regarding the counterparty issuing the European call option. Although explained in the subsection [1.2.2](#), let us recall that a CDS is a priori a good indicator of a company's default risk and such data are needed to construct our credit curve using the Credit Curve Bootstrapping Methodology using *Rstudio*.

This methodology allows us to obtain our survival curve by calculating hazard rates derived from the credit triangle rule. In order to see the impact of the counterparty's credit quality on our final results, we selected two institutions that could issue the derivative: the American bank Morgan Stanley (MS) and the largest German bank, the Deutsche Bank (DB). Thanks to our financial analysis platform, we were able to gather the different CDS premiums for these two institutions.

	6 Mo	1 Yr	2 Yr	3 Yr	4 Yr	5 Yr	7 Yr	10 Yr	20 Yr	30 Yr
MS	28.95	34.48	40.19	45.45	49.70	56.51	71.27	82.63	97.280	105.51
DB	50.10	50.31	58.77	67.17	76.50	86.03	98.07	104.57	111.94	116.21

Table 4.8: Counterparties' CDS spreads.

However, to find the hazard rate function that matches the market CDS spreads, we had to make the following assumptions when using the *CDS bootstrapping function* on *Rstudio*:

- Credit default swap premium payments are made quarterly following a business day calendar.
- Hazard rate is a piece-wise constant function of time.
- Recovery rate is constant.

Here are the results based on a constant recovery rate of 40% and the CDS premiums provided in the table [4.8](#):

Tenor	Morgan Stanley		Deutsche Bank	
	Survival Prob.	Hazard Rate	Survival Prob.	Hazard Rate
6 Mo	0.9975907	0.004824478	0.9958342	0.008349098
1 Yr	0.9942688	0.006670805	0.9916509	0.008419107
2 Yr	0.9866848	0.007657038	0.9805873	0.011219546
3 Yr	0.9774995	0.009352776	0.9669063	0.014050060
5 Yr	0.9538058	0.012268892	0.9303623	0.019263769
7 Yr	0.9193070	0.018419958	0.8907150	0.021774780
10 Yr	0.8691054	0.018718561	0.8380751	0.020305594
20 Yr	0.7154803	0.019451043	0.6840640	0.020305621
30 Yr	0.5729019	0.022223954	0.5501014	0.021794890

Table 4.9: Survival probabilities with respect to the hazard rates for a 40% recovery rate.

From these data and the implicit information about the credit quality of the entities for different time horizons, we are able to construct the survival and default probability functions. In the figure below, we find at the top of the graph the survival probabilities, and at the bottom the default probabilities under the relation $S_t = 1 - F_t$.

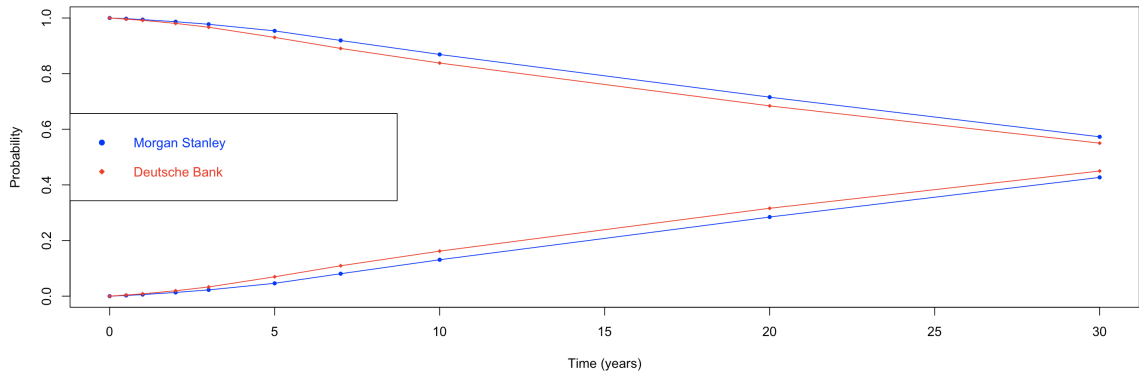


Figure 4.3: Survival and Default curves for our two counterparties.

Although the probabilities of survival may look quite similar on the graph, we must not forget that we are working in percentages and that it is by the hazard rate values obtained that we will be able to price the UCVA (for more clarity on hazard rate shape, see Appendix A.7). If we take a look at it, we can see that the hazard rates are different for a short horizon and that this difference will play a crucial role in our final results. This difference is mainly due to the deterioration of Deutsche Bank's credit quality in 2019 following a total annual loss of €5.3 billion. That year, the bank has cut staff and impaired its assets value and is considered as one of Europe's most distressed major lenders (Ewing, 2020).

4.6 UCVA results

In this results section, we provide the full development of UCVA for Morgan Stanley as counterparty, a constant recovery rate of 40%, a strike price of \$380 and an exercise date on 7th of August 2020. As the value of our derivative is reported under these conditions in the table 4.6, we can use our numerical values to price our UCVA:

$$\begin{aligned} UCVA &= (1 - 0.4) \cdot 21.15501 \cdot \left(1 - e^{-\int_0^{0.06746} \lambda(x) dx}\right) \\ &= (1 - 0.4) \cdot 21.15501 \cdot \left(1 - e^{-(\lambda(0.5) \cdot 0.06746)}\right) \end{aligned}$$

where $\lambda(0.5)$ is the constant hazard rate between 0 and 6 Mo.

$$= 0.004130384$$

This premium to be added to the value of our derivative to take into account the risk of default of our counterparty is very small and can be explained by a call exercise date very close to the valuation date (17 business days between the two dates) and by the very good credit quality of Morgan Stanley. However, all the interest is to analyse the UCVA for different counterparties, different maturities and for different constant recovery rates.

For our three maturities, selected recovery rates in 4.5.1 and the two different counterparties, we obtain the following UCVA results for two different strike prices (the hazard rates for a recovery rate of 20% and 60% are set out in Appendix A.8):

RR	Morgan Stanley			Deutsche Bank		
	7/08/20	17/06/22	16/09/22	7/08/20	17/06/22	16/09/22
20%	0.004130552	0.5848926	0.7285369	0.007147573	0.8533829	1.027947
40%	0.004130384	0.5840463	0.7273178	0.007147071	0.8515453	1.025452
60%	0.004130049	0.5823584	0.7248878	0.007146068	0.8478857	1.020485

Table 4.10: UCVA results for a strike price $K = 380$.

RR	Morgan Stanley			Deutsche Bank		
	7/08/20	17/06/22	16/09/22	7/08/20	17/06/22	16/09/22
20%	0.003203979	0.544443	0.6954436	0.005544216	0.7943652	0.9812537
40%	0.003203849	0.5436552	0.6942799	0.005543827	0.7926547	0.9788714
60%	0.003203589	0.542084	0.6919603	0.005543049	0.7892481	0.9741298

Table 4.11: UCVA results for a strike price $K = 390$.

4.7 Discussion

Our first discussion addresses the results obtained for two different exercise prices. We note that for the same exercise date, constant recovery rate and counterparty, the amount to be added to the price of our derivative to account for the risk of counterparty default is greater for an exercise price of \$380 than for an exercise price of \$390. This result is perfectly consistent since the value of our European call option is higher for a smaller strike price (and for relatively similar implied volatilities). That higher value, implying a greater potential unrealizable gain if a default occurs, explains such results.

A second interesting analysis is the comparison of our results between the two selected counterparties. As described in subsection 4.5.2, Deutsche Bank is considered riskier than Morgan Stanley. This higher default intensity resulting from a higher CDS premium to be paid quarterly to hedge has a huge impact on our results and it is consistent that the credit value adjustment is higher for the German bank.

Although our first two observations can be deduced logically, a third point seems stranger. Indeed, we observe that for different constant recovery rates, our credit value adjustment is almost identical. These results mean that we would add the same amount to the value of our derivative to account for counterparty default for a 20% recovery rate as we would for a 60% rate. This idea sounds inconsistent since we would expect to lose much more in the first case.

The explanation for this problem lies in a compensation principle in the credit triangle. Indeed, we notice in the formula 3.3 that for a given CDS premium, there are infinite possibilities for the hazard rate and the recovery rate as long as the product $\lambda \cdot (1 - R)$ is equal to our spread. In other words, for a constant loss given default $(1 - R)$ of 80% and 40% and the same CDS premium, our model will compensate these different possible losses by a lower default intensity for the first rate and a higher default intensity for the second (the results are apparent in Appendix A.8). We therefore conclude that independently of the recovery rate, this offsetting will always occur and makes the model irrelevant. Our second practical part aims precisely at obtaining a more relevant model that derives a market-implied recovery rate.

CVA WITH MARKET-IMPLIED RECOVERY

5.1 Introduction

In the previous model that considers recovery rates being constant, we noted that credit value adjustment was almost identical due to the offsetting effect from the credit triangle rule. In this second practical approach, we develop a more sophisticated model where recovery rate is still inferred from the term structure of CDS premiums of the counterparty issuing the derivative but also from the term structure of stock prices as a measure of the default intensity.

The development of this model is aimed at addressing the two main issues when considering recovery rate as a constant. First, it does not match the data and strong evidence has shown the existence of a recovery rate that is time-dependent but also dependent on the business sector (see for example [Altman and Kishore \(1996\)](#) and [Altman et al. \(2005\)](#)). Secondly, since the CVA is a mark-to-market correction impacting a bank's profit and loss account ([François and Jiang, 2017](#)), the oversimplification made in the first model could bias this estimate of the capital to be kept to cover the default.

We show in our results that UCVA is significantly underestimated for middle-term maturities when the recovery rate is considered constant. For this purpose, we use the model presented in [François and Jiang \(2017\)](#)'s article and where the adjustment to be made to the price of a derivative is calculated by a market-implied recovery through the Cox-Ross-Rubinstein binomial model.

5.2 Cox-Ross-Rubinstein

In this second model, the stochastic behaviour of the stock price is described by a Cox-Ross-Rubinstein binomial tree. For each of the nodes of our tree, the stock price can increase or decrease by a factor that depends on the volatility of the stock and on the length between two consecutive steps. We call these factors "up" and "down" and describe them by the letters "u" and "d" respectively. For a certain initial S_0 value of the stock, the binomial tree over the first three periods behaves like this:

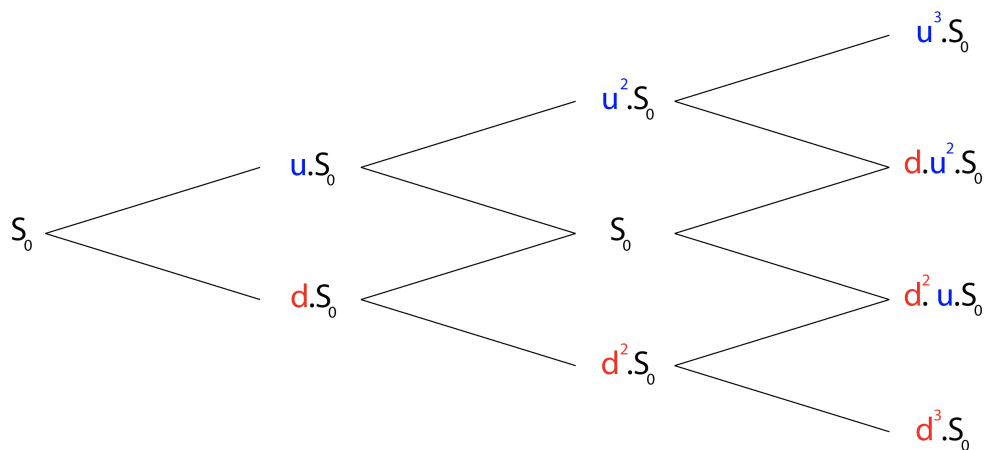


Figure 5.1: Cox-Ross-Rubinstein binomial tree.

In their paper, [François and Jiang \(2017\)](#) use the methodology proposed by [Das and Hanouna \(2009\)](#) and where an additional feature is assumed: the stock can jump into default. We denote each node of our tree by the index $[i, k]$, where k represents the indexes of time (from 0 to N) and the index i refers to the level of the node at time k . For the k^{th} period, i takes the value of 0 at the top and the value k at the bottom. We can describe the evolution of the stock price through our tree by

$$(5.1) \quad \begin{cases} S_i^{k+1} = S_i^k \exp(\sigma\sqrt{h}) & \text{with probability } q_i^k(1 - \varepsilon_i^k) \\ S_{i+1}^{k+1} = S_i^k \exp(-\sigma\sqrt{h}) & \text{with probability } (1 - q_i^k)(1 - \varepsilon_i^k) \\ 0 & \text{with probability } \varepsilon_i^k \end{cases}$$

where the stock price increases by a factor $u = \exp(\sigma\sqrt{h})$ or falls by the inverse factor $d = \exp(-\sigma\sqrt{h})$ conditional on no default. Regarding the volatility σ of the stock of our issuing counterparty, we decide to take the historical return volatility for the same time period. For Morgan Stanley, the annualized historical volatility is 35.979% and 41.896% for Deutsche Bank. The last parameter to define to build our tree is h , the length of each of our steps. To be consistent with our first practical part, our daily time step will be $h = 1/252$.

We notice that in the model of [Das and Hanouna \(2009\)](#), the nodes of a period $k + 1$ depend on two probabilities:

- ε_i^k , the one-period probability of jumping to default (also referred to as the default intensity). In the same way as seen in sub-section 3.2.2.2, the daily default intensity is related to the daily hazard rate λ_i^k as follows:

$$(5.2) \quad \varepsilon_i^k = 1 - \exp(-\lambda_i^k).$$

- q_i^k , the risk-neutral probability of an up move conditional on the survival of the counterparty. In the absence of arbitrage, that probability for each node is given by:

$$(5.3) \quad q_i^k = \frac{\frac{R^k}{1-\varepsilon_i^k} - \exp(-\sigma\sqrt{h})}{\exp(\sigma\sqrt{h}) - \exp(-\sigma\sqrt{h})}$$

where $R^k = e^{rh}$ is the one-period compound factor that we consider constant for each node (r is the constant risk-free rate).

The default intensity variable is said to be dynamic in our model due to its time- and state-dependence. We notice that it takes a different value for each of the nodes of our tree. Furthermore, for our financial instrument, we apply a dynamic recovery rate over $[i, k]$ that we denote ϕ_i^k . The hazard rate used to determine the default occurrence being inversely related to the stock price and the recovery rate being inversely correlated to the default intensity according to the various evidence, we define the two following functions for our dynamic default probability and recovery rate on node $[i, k]$:

$$(5.4) \quad \lambda_i^k = (S_i^k)^{-b}$$

$$(5.5) \quad \phi_i^k = \frac{1}{1 + \exp(a_0 + a_1 \varepsilon_i^k)}$$

This model includes three parameters: b , a_0 and a_1 . For b , representing the elasticity of the hazard with respect to the stock price, we make the restriction that $b \geq 0$ to ensure that the default risk of the model increases when the counterparty's stock price falls. For the recovery rate represented by a logit function to ensure that the value is comprised between 0 and 1, the parameter a_0 is the scale parameter of the function and a_1 captures the dependence between the default intensity and the recovery rate ([François and Jiang, 2017](#)).

To construct the term structure of risk-neutral default probabilities and recovery rates, the last step is to aggregate all the ε_i^k and ϕ_i^k for each state i . For this, we need to derive the probability of reaching node $[i, k]$ via all possible paths in our tree. This risk-neutral probability is denoted p_i^k and cannot be obtained analytically. As these probabilities depend on two non-constant probabilities, namely q_i^k and λ_i^k , we need to calculate them step by step. An example

of the calculation is provided for the first three periods in Appendix A.9. Once the algorithm has been created for a number of k periods, we are able to build our forward curve of default probabilities and recovery rates as

$$(5.6) \quad \varepsilon^k = \sum_{i=0}^k p_i^k \varepsilon_i^k$$

$$(5.7) \quad \phi^k = \sum_{i=0}^k p_i^k \phi_i^k$$

5.3 Model calibration

To find the value of our three parameters, we calibrate our optimization function using the term structures of the counterparty CDS premiums. If we define the fair spread of our CDS contract with maturity j as π_j , it is the one that makes the initial present value of the expected premiums (premium leg) equal to the present value of the expected losses (protection leg). Considering that protection premiums are paid on a quarterly basis, $u = 1, \dots, j \times 4$ refers in our model to the u^{th} quarter after inception and where T_u are the settlements dates.

Taking as the face value of the underlying security the unit notional, the expected present value of all premiums to be paid is

$$(5.8) \quad \sum_{u=1}^{j \cdot 4} \left(\pi_j \prod_{k=1}^{u \cdot 63} (1 - \varepsilon^k) D(0, T_u) \right)$$

where ε^k denotes the probability of jumping to default between periods k and $k - 1$. The upper bound of our product operator is designed to represent the number of business days existing in u quarter(s) (63 days per quarter).

The CDS protection leg representing the expected present value of loss payments includes the recovery rate variable and is expressed as follows:

$$(5.9) \quad \sum_{u=1}^{j \cdot 4} \left(\prod_{k=1}^{u \cdot 63 - 1} (1 - \varepsilon^k) \varepsilon^{u \cdot 63} (1 - \phi^{u \cdot 63}) D(0, T_u) \right).$$

By equalizing the two legs, the general no-arbitrage CDS pricing equation can be written

$$(5.10) \quad \pi_j = \frac{\sum_{u=1}^{j \cdot 4} \left(\prod_{k=1}^{u \cdot 63 - 1} (1 - \varepsilon^k) \varepsilon^{u \cdot 63} (1 - \phi^{u \cdot 63}) D(0, T_u) \right)}{\sum_{u=1}^{j \cdot 4} \left(\prod_{k=1}^{u \cdot 63} (1 - \varepsilon^k) D(0, T_u) \right)}$$

The theoretical premium depends only on the default intensity and recovery rate curves, the discount being deterministic, which in turn depend on our three parameters to be calibrated

with regard to the market values of CDS premiums. This set of parameters is calibrated by minimizing the total pricing error on eight selected maturities (6 months, 1 year to 5 years, 7 years and 10 years). Our model using market-implied recovery is therefore obtained by solving the optimization function

$$(5.11) \quad \min_{a_0, a_1, b} = \sum_{x=1}^8 \left(\pi_x(a_0, a_1, b) - \pi_x^{obs} \right)^2$$

where x is the index of our spread with corresponding maturity j (e.g. π_2 is the theoretical CDS spread with maturity 1 Yr) and π_x^{obs} the spreads observed in the financial analysis platform.

5.4 Calibrated term structures

Our model to be calibrated is based on CDS premiums we can observe on the market for our two counterparties. Data are already included in table 4.8 and are denominated in U.S. dollar to be consistent with the stock price data. To define the value of our three parameters b, a_0, a_1 that minimize (5.11), we used *rstudio's constrOptim* function that allowed us to optimize with constraints. Taking as initial values the averages of the inputs given in [François and Jiang \(2017\)](#) and [Das and Hanouna \(2009\)](#)'s work for corresponding credit rating, we inferred the forward default probabilities and recovery rates for both our counterparties and whose parameters are summarized here

	MS			DB		
	b	a_0	a_1	b	a_0	a_1
Initial values	0.93	-21	2600	0.86	-22	8600
Calibrated parameters	0.75	-21.019	2602.346	0.67	-21.764	8602.346
Pricing error	0.0023102			0.004630366		

Table 5.1: Calibrated parameters.

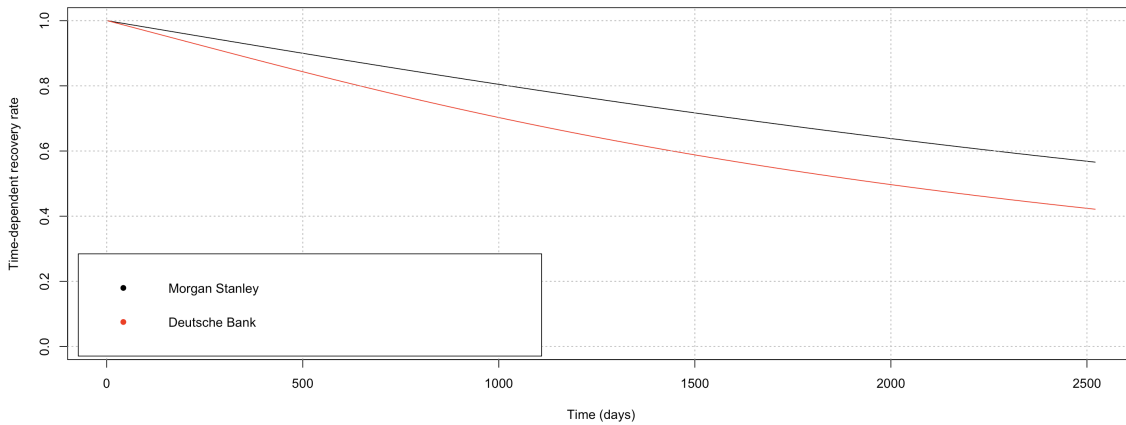


Figure 5.2: Market-implied recovery term structures.

The shape of these two curves seems consistent with the different analyses provided by [François and Jiang \(2017\)](#) and [Das and Hanouna \(2009\)](#). Since both of our counterparties belong to the banking environment, it is hard to imagine them going bankrupt quickly after contracting even if Deutsche Bank is riskier than Morgan Stanley with regard to their credit quality. This negative correlation not taken into account in our first practical approach between the recovery rate and the intensity of defects now appears in our two graphs (see Figures 5.2 and 5.3). Furthermore, the theory suggesting that surprise defaults of high credit quality businesses are likely to result in higher recovery rates ([Das and Hanouna, 2009](#)) is confirmed.

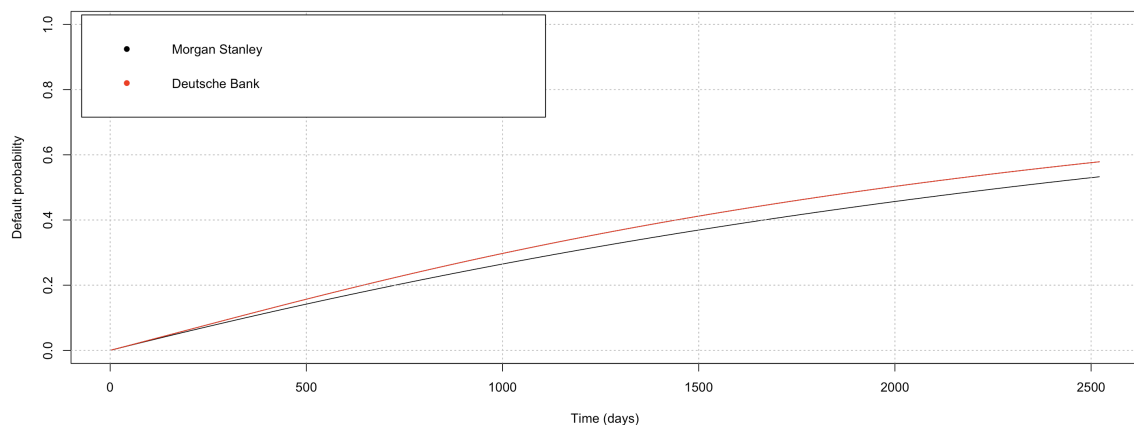


Figure 5.3: Default intensity term structures.

Now that we have calibrated our term structures, we can extend the [Das and Hanouna \(2009\)](#) model to the CVA calculation to compare *chapter 4* results.

5.5 UCVA calculation

For the calculation of the credit value adjustment in our second case study, we impose two limitations: the recovery rate is independent of the expected exposure, allowing us to separate our expectation into a product of two expectations, and the exposure is independent of the default time τ . we can rewrite our UCVA formula as

$$\begin{aligned}
 UCVA &= \int_0^T E^{\mathbb{Q}_M} \left[(1-R) \max\{V(t, T), 0\} D(0, t) \middle| \mathcal{F}_t \vee \{\tau = t\} \right] dP_D(t) \\
 &= \int_0^T E^{\mathbb{Q}_M} \left[(1-R) \middle| \mathcal{F}_t \vee \{\tau = t\} \right] E^{\mathbb{Q}_M} \left[V(t, T) D(0, t) \middle| \mathcal{F}_t \vee \{\tau = t\} \right] dP_D(t) \\
 &= \int_0^T \left(1 - E^{\mathbb{Q}_M} [R \middle| \mathcal{F}_t \vee \{\tau = t\}] \right) E^{\mathbb{Q}_M} [V(t, T) D(0, t)] dP_D(t)
 \end{aligned}$$

The dynamics of our recovery rate and default intensity having been obtained after calibration, we undertake Monte Carlo simulations to model the probability of the different outcomes in the stock price process. Among 10,000 geometric Brownian motion simulations of Apple's

stock price, we drew ten of the paths and for which we plotted as an example the call option discounted values relative to each of the points in Figure 5.5 for our second maturity on 17th June 2022.

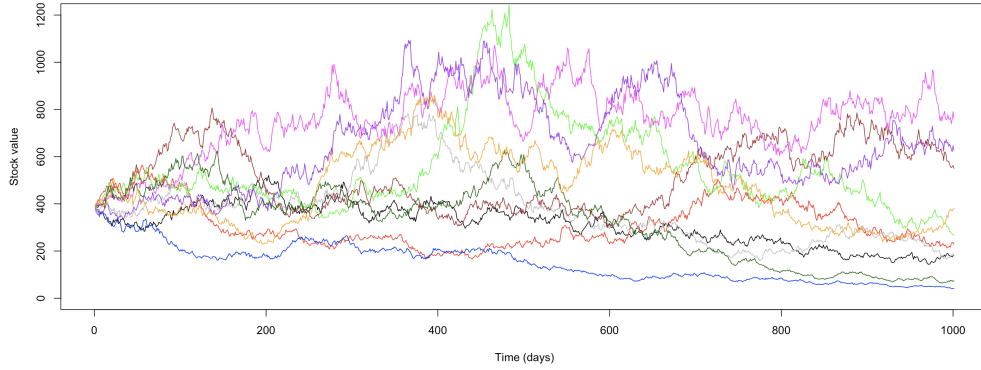


Figure 5.4: Ten Monte Carlo simulated paths under GBM.

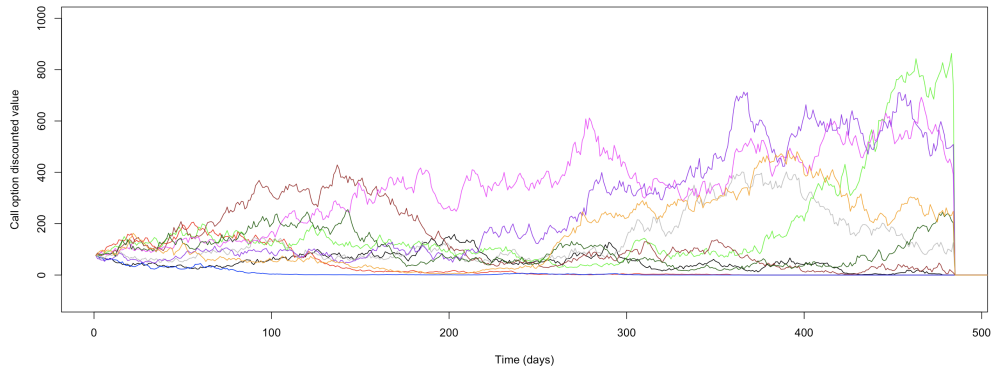


Figure 5.5: Call option discounted values.

Finally, we derive the general discounted value structure for our maturity representing the pattern of our exposure for a maturity $T_2 = 1.91667$ and a strike price $K = 380$.

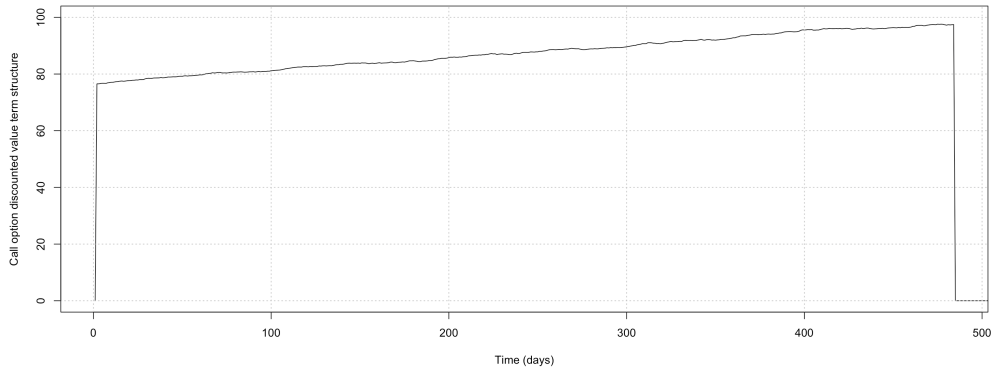


Figure 5.6: Call option discounted value term structure for our second maturity on 17th June 2022.

For the different exposure structures related to the different maturities and the two strike prices, we obtain our unilateral credit value adjustments with market-implied recovery (MIR) summarized in the following table to easily observe the value divergences with the constant model:

	Morgan Stanley			Deutsche Bank		
	7/08/20	17/06/22	16/09/22	7/08/20	17/06/22	16/09/22
MIR	0.000351013	0.9530269	1.271443	0.000566310	1.636664	2.186237
20%	0.004130552	0.5848926	0.7285369	0.007147573	0.8533829	1.027947
40%	0.004130384	0.5840463	0.7273178	0.007147071	0.8515453	1.025452
60%	0.004130049	0.5823584	0.7248878	0.007146068	0.8478857	1.020485

Table 5.2: UCVA results with market-implied recovery (MIR) for a strike price $K = 380$.

	Morgan Stanley			Deutsche Bank		
	7/08/20	17/06/22	16/09/22	7/08/20	17/06/22	16/09/22
MIR	0.000272273	0.8871181	1.213689	0.000439274	1.523476	2.086929
20%	0.003203979	0.544443	0.6954436	0.005544216	0.7943652	0.9812537
40%	0.003203849	0.5436552	0.6942799	0.005543827	0.7926547	0.9788714
60%	0.003203589	0.542084	0.6919603	0.005543049	0.7892481	0.9741298

Table 5.3: UCVA results with market-implied recovery (MIR) for a strike price $K = 390$.

5.6 Discussion

In this section, we discuss the validity of our results for our various counterparties, maturities and exercise prices and interpret the discrepancies with the adjustments obtained in our first practice chapter. Furthermore, we dedicate a paragraph to explain the limitations related to the coding of our problematic, notably concerning optimization.

Our first discussion is about the results obtained for the Morgan Stanley's counterparty. In our tables, we notice a significant difference in the impact of a market-implied recovery between a very short-term and a medium-term maturity. Focusing on our first maturity on 7th of August 2020, we obtained a UCVA of \$0.000351013 while this amount was around \$0.00413 for our three constant rates. This huge difference is mainly explained by the term structure of our recovery shown in Figure 5.4. We observe that our rate is close to 100% at the beginning, explaining our very low UCVA being inversely proportional to the recovery. For the two other maturities, we are close to a factor of 1.63 and 1.75 between the MIR UCVA and the UCVA considering a constant recovery for the maturities of 17/06/22 and 16/09/22 respectively.

For Deutsche Bank, the impact is more or less the same. Considered riskier as described in the section 4.5.2, it is normal that we get a recovery structure below that of Morgan Stanley.

However, this structure is still close to 100% for a very short maturity and we logically get a very small adjustment for our maturity of 7/08/20. For the other two exercise dates, we now obtain a factor of 1.917854 and 2.126799 with respect to the UCVA calculated in our first practical approach.

Our third observation deals with differences in results between our counterparties and maturities. In our case study, we note that time-dependent recovery has more influence the higher the maturity. This result appears to be consistent with the evidence in the literature, describing that the underestimation intensifies as the horizon of the position lengthens ([François and Jiang, 2017](#)). Moreover, the market-implied recovery model has a greater impact on the adjustment to be made when the counterparty is riskier. We see that the factors are higher for Deutsche Bank than for Morgan Stanley.

The consistency between our calibrated term structures and those found in the literature for financial institutions of similar risk indicates that our model seems fair. However, we must consider in our judgment that our optimization was done around a collection of initial values for our three parameters and that it provides the nearest local minimum. Having tried various optimization methods to find the global minimum of our function, it turned out that the code was running continuously, probably due to a lack of power of the computer used for the optimization. Therefore, the results obtained do not perfectly reflect those of an overall minimum, but reflect fairly well the general behavior.

CONCLUSION

The objective of our dissertation was to analyze the differences in adjustments considering constant recovery throughout the process and one implying a dynamic recovery conditional on the survival of the counterparty. To achieve this, we studied a first practical case considering the components of CVA independent and three constant recovery rates for various maturities. We then developed this first model by considering a dynamic recovery inferred from the CDS spread term structure but also from the term structure of the counterparty's stock price reflecting the intensity of default.

Through our results, we discovered that the market-implied model estimates the adjustment to be made to the value of our security much lower than the first model for a very close exercise date of the option. The explanation for this difference can be found in the assumptions made for each model. In the first, we assumed a piece-wise constant function for the hazard rates. The consequence of this limitation is that the default risk is the same for 17 days as for 6 months for example and partly discredits the accuracy of the adjustments. Moreover, the model has no dependency between variables and the UCVA formula becomes the simple product of deterministic values.

The reality is quite different. Through our second practical part, we derived the default and recovery curves on a daily basis, allowing us to better reflect the aggregate tendency. In addition, the negative correlation between default intensity and recovery rate is now taken into account. As a matter of fact, implied recovery model introduces probabilities that are conditional on the survival of the counterparty. This precisely constructs a dependence between recovery and default, and respect the empirical evidence found in the literature. As our default

intensities are very low at the beginning, the correlated recovery compensates for a possible sudden loss by a large rate close to 100% and implying a very low UCVA.

For our other two maturities, the factor to be applied to the UCVA obtained via a constant recovery rate to obtain the MIR UCVA is consistent with those found in the studies by [François and Jiang \(2017\)](#) and [Das and Hanouna \(2009\)](#). We get for the maturities of 17/06/22 and 16/09/22 a multiplier factor of 1.63 and 1.75 for Morgan Stanley respectively and 1.917854 and 2.12679 for Deutsche Bank. We complete our analysis of the results by concluding that the more sophisticated model can double this amount to be added to the value of a security and is therefore not to be taken lightly in order to best mitigate counterparty credit risk.

However, two limitations are important to note. We considered credit exposure independent of the creditworthiness of the counterparty. This relationship known as "wrong way risk" or "right way risk" depending on the movement of the exposure for a deterioration in credit quality was not the subject of this dissertation to not complicate the modeling and coding of our methods. In addition, we have always assumed independence between exposure and the default time τ . A study implicating a dependence between these two variables could improve this work to make the results even more accurate.



A.1 Brownian motion with drift

$$X(t) = \mu t + \sigma W(t)$$

A.1.1 Mean

$$\begin{aligned} E[X(t)] &= E[\mu t + \sigma W_t] \\ &= E[\mu t] + E[\sigma W_t] \\ &= \mu t + \sigma E[W_t] \end{aligned}$$

Since the mean of the standard Brownian motion is zero, i.e. $E[W_t] = 0$, we finally have:

$$= \mu t$$

A.1.2 Variance

$$\begin{aligned} Var[X(t)] &= Var[\mu t + \sigma W_t] \\ &= E[(\mu t + \sigma W_t)^2] - E[\mu t + \sigma W_t]^2 \\ &= E[(\mu t)^2] + E[(\sigma W_t)^2] + 2 \cdot E[\mu \sigma t W_t] - (\mu t)^2 \\ &= (\mu t)^2 + \sigma^2 E[(W_t)^2] + 2\mu \sigma t E[W_t] - (\mu t)^2 \\ &= \sigma^2 E[W_t^2] \end{aligned}$$

According to the definition of the variance we have:

$$E[W_t^2] = Var[W_t] + E[W_t]^2$$

$$= t + 0$$

$$= t$$

By substituting this value in the variance formula, we obtain:

$$\text{Var}[X(t)] = \sigma^2 t$$

A.2 Geometric Brownian motion solution

There are many different ways to solve the SDE and one of them is trying an ansatz of the form $S_t = A_t e^{\sigma W_t}$, where A_t is a deterministic function t . Using the Ito calculation and differential rule ($dt^2 = 0$, $dt dW_t = 0$ and $dW_t^2 = dt$) with $f(t, w) = A_t e^{\sigma w}$, so that $f_t = \dot{A} e^{\sigma w}$, $f_w = \sigma f$, and $f_{ww} = \sigma^2 f$, the the result is

$$\begin{aligned} d(A_t e^{\sigma W_t}) &= \dot{A}_t e^{\sigma W_t} dt + \sigma A_t e^{\sigma W_t} dW_t + \frac{1}{2} \sigma^2 A_t e^{\sigma W_t} dt \\ (A.1) \quad &= \left(\frac{\dot{A}_t}{A_t} + \frac{1}{2} \sigma^2 \right) S_t dt + \sigma S_t dW_t \end{aligned}$$

(A.1) satisfies (2.3) if

$$\frac{\dot{A}_t}{A_t} + \frac{1}{2} \sigma^2 = \mu,$$

implying that

$$\dot{A}_t = \left(\mu - \frac{1}{2} \sigma^2 \right) A_t.$$

The solution to this equation is a simple exponential

$$A_t = A_0 e^{(\mu - \frac{1}{2} \sigma^2) t}$$

If we set $t = 0$ and using that a standard Brownian motion is grounded (i.e. $W_0 = 0$), we get $A_0 = S_0$. Hence, the full SDE solution is

$$S_t = S_0 e^{\sigma W_t + (\mu - \frac{1}{2} \sigma^2) t}.$$

Another way to derive the geometric brownian motion would have been to consider the following nonlinear transformation

$$f(t, S_t) = \ln(S_t)$$

When applying Itô's lemma,

$$(A.2) \quad d\ln(S_t) = 0dt + \frac{1}{S_t}dS_t + \frac{1}{2} \cdot \frac{-1}{S_t^2}(dS_t)^2$$

where (see equation 2.3)

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

and

$$\begin{aligned} (dS_t)^2 &= (\mu S_t dt + \sigma S_t dW_t)^2 \\ &= \mu^2 S_t^2 dt^2 + \sigma^2 S_t^2 dW_t^2 + 2\mu\sigma S_t^2 dt dW_t \end{aligned}$$

Making use of the Itô rule

$$\begin{aligned} dt^2 &\rightarrow 0, \\ dt dW_t &\rightarrow 0, \\ (dW_t)^2 &= dt, \end{aligned}$$

we get the result

$$\begin{aligned} (dS_t)^2 &= 0 + \sigma^2 S_t^2 dt + 0 \\ &= \sigma^2 S_t^2 dt \end{aligned}$$

All the elements being in our possession, we can rewrite Itô's equation (A.2) as:

$$\begin{aligned} d\ln(S_t) &= 0dt + \frac{1}{S_t}dS_t + \frac{1}{2} \cdot \frac{-1}{S_t^2}(dS_t)^2 \\ &= \frac{1}{S_t}(\mu S_t dt + \sigma S_t dW_t) + \frac{1}{2} \cdot \frac{-1}{S_t^2}(\sigma^2 S_t^2 dt) \\ &= \mu dt + \sigma dW_t - \frac{1}{2}\sigma^2 dt \\ &= \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t \end{aligned}$$

We can then integrate the two sides and it follows that:

$$\begin{aligned} \int_0^t d\ln(S_s) &= \int_0^t \left(\mu - \frac{1}{2}\sigma^2\right)ds + \int_0^t \sigma dW_s \\ \left[\ln(S_s)\right]_0^t &= \left[\left(\mu - \frac{1}{2}\sigma^2\right)s\right]_0^t + \left[\sigma W_s\right]_0^t \\ \ln(S_t) - \ln(S_0) &= \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t \\ \ln(S_t) &= \ln(S_0) + \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t \\ S_t &= \exp\left(\ln(S_0) + \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right) \\ S_t &= S_0 \cdot \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right) \end{aligned}$$

A.3 Geometric Brownian motion lognormal distribution parameters

A.3.1 Mean

$$\begin{aligned}
 E[\ln(S_t)] &= E\left[\ln(S_0) + \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right] \\
 &= \ln(S_0) + \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma \underbrace{E[W_t]}_{=0} \\
 &= \ln(S_0) + \left(\mu - \frac{1}{2}\sigma^2\right)t
 \end{aligned}$$

A.3.2 Variance

$$\begin{aligned}
 Var[\ln(S_t)] &= E\left[\left(\ln(S_t) - E[\ln(S_t)]\right)^2\right] \\
 &= E\left[\left(\ln(S_0) + \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t - \ln(S_0) - \left(\mu - \frac{1}{2}\sigma^2\right)t\right)^2\right] \\
 &= E\left[\left(\sigma W_t\right)^2\right] \\
 &= \sigma^2 E[W_t^2] \\
 &= \sigma^2 t
 \end{aligned}$$

A.4 Black-Scholes formula derivation

We know from the theory that the payoff of a call option at maturity T is

$$C_T = \max\{S_T - K, 0\}.$$

Using the risk-neutral valuation, we value the option in terms of its expected payoffs discounted from expiration until the present day. If we assume that the payoffs grow on average at the risk-free rate, we get

$$\begin{aligned}
 V_0 &= e^{-rT} E[C_T] \\
 (A.3) \quad &= e^{-rT} E[\max\{S_T - K, 0\}]
 \end{aligned}$$

We see that the price of a call option depends on the stock value. Thus, one of the benefits of using the lognormal distribution of a stock is that it helps us calculate the probability that the call option expires in the money, which is analogous to calculating the probability that $S_T \geq K$.

$$P(S_T \geq K) = P\left(S_0 e^{\left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}Z} \geq K\right)$$

$$\begin{aligned}
 &= P\left(e^{\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}Z} \geq \frac{K}{S_0}\right) \\
 &= P\left(\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}Z \geq \ln\left(\frac{K}{S_0}\right)\right) \\
 &= P\left(Z \geq \frac{\ln\left(\frac{K}{S_0}\right) - \left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) \\
 &= P(Z \geq -d_2)
 \end{aligned}$$

Hence, we get

$$\max\{S_T - K, 0\} = \begin{cases} S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}Z} & \text{if } Z \geq -d_2 \\ 0 & \text{if } Z \leq -d_2 \end{cases}$$

Substituting this expression in equation A.3

$$\begin{aligned}
 (A.4) \quad V_0 &= e^{-rT} E\left[\max\{S_T - K, 0\}\right] \\
 &= e^{-rT} \int_{-d_2}^{\infty} \left[S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}x} - K\right] \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
 &= \underbrace{\frac{e^{-rT}}{\sqrt{2\pi}} \int_{-d_2}^{\infty} S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}x} \cdot e^{-\frac{x^2}{2}} dx}_{\textcircled{1}} - \underbrace{\frac{Ke^{-rT}}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{-\frac{x^2}{2}} dx}_{\textcircled{2}}
 \end{aligned}$$

Let's focus on part ① of the equation A.4,

$$\begin{aligned}
 (A.5) \quad &\frac{1}{\sqrt{2\pi}} S_0 e^{-rT} \int_{-d_2}^{\infty} e^{\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}x-\frac{x^2}{2}} dx \\
 \Rightarrow &\frac{1}{\sqrt{2\pi}} S_0 \int_{-d_2}^{\infty} e^{-\frac{1}{2}\sigma^2 T + \sigma\sqrt{T}x - \frac{x^2}{2}} dx \\
 \Rightarrow &\frac{1}{\sqrt{2\pi}} S_0 \int_{-d_2}^{\infty} e^{-\frac{1}{2}(x^2 - 2\sigma\sqrt{T}x + \sigma^2 T)} dx \\
 \Rightarrow &\frac{1}{\sqrt{2\pi}} S_0 \int_{-d_2}^{\infty} e^{-\frac{1}{2}(x - \sigma\sqrt{T})^2} dx
 \end{aligned}$$

Let $y = x - \sigma\sqrt{T}$ with $dy = dx$. For the new lower limit of the integral

$$\begin{aligned}
 y &= -d_2 - \sigma\sqrt{T} \\
 &= \left[\left(\frac{\ln\left(\frac{K}{S_0}\right) - \left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \right) - \sigma\sqrt{T} \right] \\
 &= \frac{\ln\left(\frac{K}{S_0}\right) - rT + \frac{1}{2}\sigma^2 T - \sigma^2 T}{\sigma\sqrt{T}} \\
 &= \frac{\ln\left(\frac{K}{S_0}\right) - \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}
 \end{aligned}$$

$$= -d_1$$

The new lower bound of the integral is $y = -d_1$. Then, back to equation A.5, we have

$$\begin{aligned} & \frac{1}{\sqrt{2\pi}} S_0 \int_{-d_1}^{\infty} e^{-\frac{1}{2}y^2} dy \\ \Rightarrow & S_0 [1 - \mathcal{N}(-d_1)] \\ \Rightarrow & S_0 \mathcal{N}(d_1) \end{aligned}$$

For the second part ② of our equation A.4, we solve

$$\begin{aligned} & K e^{-rT} \left[\frac{1}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{-\frac{x^2}{2}} dx \right] \\ \Rightarrow & K e^{-rT} [1 - \mathcal{N}(-d_2)] \\ \Rightarrow & K e^{-rT} \mathcal{N}(d_2) \end{aligned}$$

Now that the two parts have been derived, we obtain the formula for estimating the present value of a European call option

$$V_0 = S_0 \mathcal{N}(d_1) - K e^{-rT} \mathcal{N}(d_2)$$

A.5 Global Corporate Average Cumulative Default Rates

Rating	--Time horizon (years)--														
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
AAA	0.00	0.03	0.13	0.24	0.35	0.45	0.51	0.59	0.64	0.70	0.73	0.76	0.79	0.85	0.91
AA	0.02	0.06	0.12	0.21	0.31	0.42	0.50	0.58	0.65	0.72	0.78	0.84	0.90	0.96	1.02
A	0.05	0.14	0.23	0.35	0.47	0.62	0.79	0.93	1.08	1.24	1.37	1.50	1.63	1.75	1.89
BBB	0.16	0.45	0.78	1.17	1.58	1.98	2.33	2.67	3.00	3.32	3.65	3.92	4.16	4.42	4.69
BB	0.61	1.92	3.48	5.05	6.52	7.85	9.01	10.04	10.97	11.78	12.46	13.08	13.64	14.12	14.67
B	3.33	7.71	11.55	14.58	16.93	18.83	20.36	21.60	22.70	23.74	24.63	25.30	25.92	26.53	27.12
CCC/C	27.08	36.64	41.41	44.10	46.19	47.09	48.26	49.05	49.76	50.38	50.87	51.39	51.99	52.52	52.59

Figure A.1: Global Corporate Average Cumulative Default Rates (1981-2019) (%).
Source: Standard & Poor's (Default, Transition, and Recovery: 2019 Annual Global Corporate Default And Rating Transition Study).

A.6 Apple's stock price implied volatilities for different call options

Calls		Expiry Date	Strike
ImpVol Mid	Delta Mid		
44.85	0.60	07-Aug-2020	377.5
45.20	0.58	07-Aug-2020	380
45.85	0.56	07-Aug-2020	382.5
45.40	0.53	07-Aug-2020	385
45.61	0.51	07-Aug-2020	387.5
45.50	0.49	07-Aug-2020	390
44.97	0.46	07-Aug-2020	392.5

Calls		Expiry Date	Strike
ImpVol Mid	Delta Mid		
38.07	0.65	17-Jun-2022	360
35.46	0.62	17-Jun-2022	370
34.92	0.60	17-Jun-2022	380
34.33	0.58	17-Jun-2022	390
34.01	0.55	17-Jun-2022	400
33.72	0.53	17-Jun-2022	410
35.43	0.53	17-Jun-2022	415

Calls		Expiry Date	Strike
ImpVol Mid	Delta Mid		
34.94	0.64	16-Sep-2022	360
35.36	0.63	16-Sep-2022	370
35.06	0.60	16-Sep-2022	380
35.20	0.59	16-Sep-2022	390
34.39	0.56	16-Sep-2022	400
33.71	0.54	16-Sep-2022	410
33.82	0.52	16-Sep-2022	420

Figure A.2: Effect of Time on Implied Volatility.
Source: Refinitiv Eikon.

A.7 Hazard rate curves for Morgan Stanley and the Deutsche Bank for a 40% recovery rate

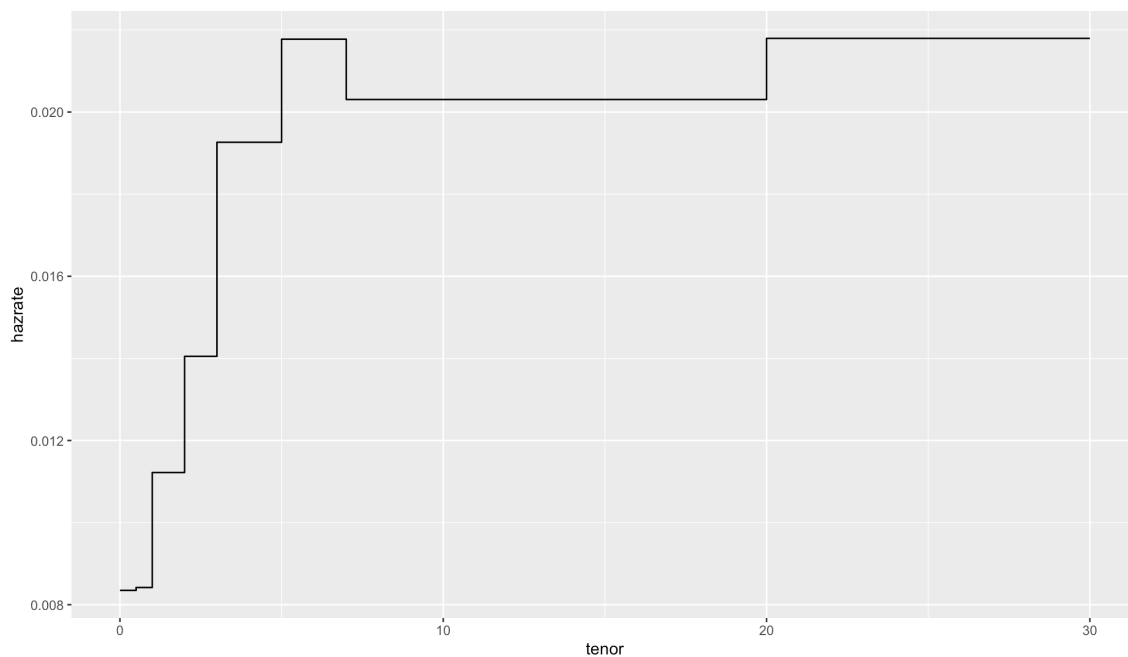


Figure A.3: Hazard rate curve for Morgan Stanley using CDS spreads and 40% recovery rate.

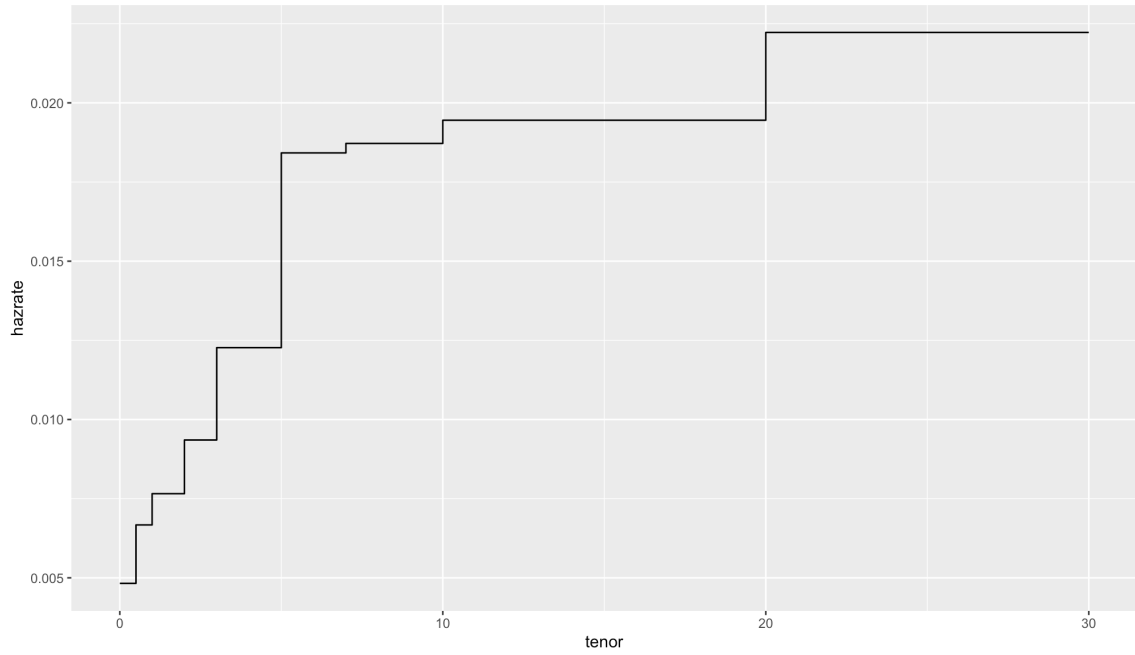


Figure A.4: Hazard rate curve for the Deutsche Bank using CDS spreads and 40% recovery rate.

A.8 Hazard rate values for a 20% and 60% constant recovery rate

Tenor	Morgan Stanley		Deutsche Bank	
	Survival Prob.	Hazard Rate	Survival Prob.	Hazard Rate
6 Mo	0.9981925	0.003618358	0.9968740	0.006261823
1 Yr	0.9956988	0.005002605	0.9937317	0.006314302
2 Yr	0.9899983	0.005741531	0.9854074	0.008412037
3 Yr	0.9830821	0.007010586	0.9750873	0.010528143
5 Yr	0.9651813	0.009188338	0.9473812	0.014412741
7 Yr	0.9389841	0.013758741	0.9170575	0.016265668
10 Yr	0.9004582	0.013964936	0.8762425	0.015175787
20 Yr	0.7790711	0.014480144	0.7529244	0.015167799
30 Yr	0.6612191	0.016401702	0.6403104	0.016201187

Table A.1: Survival probabilities with respect to the hazard rates for a 20% recovery rate.

Tenor	Morgan Stanley		Deutsche Bank	
	Survival Prob.	Hazard Rate	Survival Prob.	Hazard Rate
6 Mo	0.9963882	0.007236718	0.9937577	0.01252365
1 Yr	0.9914146	0.010008205	0.9875025	0.01262878
2 Yr	0.9800879	0.011490562	0.9710124	0.01683985
3 Yr	0.9664185	0.014045241	0.9507263	0.02111303
5 Yr	0.9313940	0.018457314	0.8970834	0.02903868
7 Yr	0.8809150	0.027860599	0.8399000	0.03293301
10 Yr	0.8090009	0.028387037	0.7660389	0.03068330
20 Yr	0.6014381	0.029647653	0.5633876	0.03072651
30 Yr	0.4255568	0.034592501	0.4035358	0.03337026

Table A.2: Survival probabilities with respect to the hazard rates for a 60% recovery rate.

A.9 Three-period algorithm for computing p_i^k

The algorithm is initiated with the starting point probability

$$p_0^0 = 1.$$

For the first step, we can write

$$\begin{aligned} p_0^1 &= p_0^0(1 - \lambda_0^0)q_0^0 \\ p_1^1 &= p_0^0(1 - \lambda_0^0)(1 - q_0^0) \end{aligned}$$

Then, since it is clear that the middle node of the second period can be reached by two different paths,

$$\begin{aligned} p_0^2 &= p_0^1(1 - \lambda_0^1)q_0^1 \\ p_1^2 &= p_0^1(1 - \lambda_0^1)(1 - q_0^1) + p_1^1(1 - \lambda_1^1)q_1^1 \\ p_2^2 &= p_1^1(1 - \lambda_1^1)(1 - q_1^1) \end{aligned}$$

Finally, for the third period

$$\begin{aligned} p_0^3 &= p_0^2(1 - \lambda_0^2)q_0^2 \\ p_1^3 &= p_0^2(1 - \lambda_0^2)(1 - q_0^2) + p_1^2(1 - \lambda_1^2)q_1^2 \\ p_2^3 &= p_1^2(1 - \lambda_1^2)(1 - q_1^2) + p_2^2(1 - \lambda_2^2)q_2^2 \\ p_3^3 &= p_2^2(1 - \lambda_2^2)(1 - q_2^2) \end{aligned}$$

A.10 Binomial tree of the stock price for for the first twenty periods

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]	[,7]	[,8]	[,9]	[,10]	[,11]	[,12]	[,13]	[,14]	[,15]	[,16]	[,17]	[,18]	[,19]	[,20]
[1,]	52.64	53.84670	55.08106	56.34371	57.63531	58.95652	60.30801	61.69049	63.10465	64.55124	66.03098	67.54465	69.09301	70.67687	72.29704	73.95434	75.64964	77.38380	79.15772	80.97229
[2,]	0.00	0.00	51.46034	52.64000	53.84670	55.08106	56.34371	57.63531	58.95652	60.30801	61.69049	63.10465	64.55124	66.03098	67.54465	69.09301	70.67687	72.29704	73.95434	75.64964
[3,]	0.00	0.00000	0.00000	50.30713	51.46034	52.64000	53.84670	55.08106	56.34371	57.63531	58.95652	60.30801	61.69049	63.10465	64.55124	66.03098	67.54465	69.09301	70.67687	72.29704
[4,]	0.00	0.00000	0.00000	0.00000	49.17975	50.30713	51.46034	52.64000	53.84670	55.08106	56.34371	57.63531	58.95652	60.30801	61.69049	63.10465	64.55124	66.03098	67.54465	69.09301
[5,]	0.00	0.00000	0.00000	0.00000	0.00000	48.07764	49.17975	50.30713	51.46034	52.64000	53.84670	55.08106	56.34371	57.63531	58.95652	60.30801	61.69049	63.10465	64.55124	66.03098
[6,]	0.00	0.00000	0.00000	0.00000	0.00000	0.00000	47.00023	48.07764	49.17975	50.30713	51.46034	52.64000	53.84670	55.08106	56.34371	57.63531	58.95652	60.30801	61.69049	63.10465
[7,]	0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	45.94696	47.00023	48.07764	49.17975	50.30713	51.46034	52.64000	53.84670	55.08106	56.34371	57.63531	58.95652	60.30801
[8,]	0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	44.91729	45.94696	47.00023	48.07764	49.17975	50.30713	51.46034	52.64000	53.84670	55.08106	56.34371	57.63531
[9,]	0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	43.91070	44.91729	45.94696	47.00023	48.07764	49.17975	50.30713	51.46034	52.64000	53.84670	55.08106
[10,]	0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	42.92667	43.91070	44.91729	45.94696	47.00023	48.07764	49.17975	50.30713	51.46034	52.64000
[11,]	0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	41.96469	42.92667	43.91070	44.91729	45.94696	47.00023	48.07764	49.17975	50.30713
[12,]	0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	41.02427	41.96469	42.92667	43.91070	44.91729	45.94696	47.00023	48.07764
[13,]	0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	40.10492	41.02427	41.96469	42.92667	43.91070	44.91729	45.94696
[14,]	0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	39.20617	40.10492	41.02427	41.96469	42.92667	43.91070
[15,]	0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	38.32757	39.20617	40.10492	41.02427	41.96469
[16,]	0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	37.46855	38.32757	39.20617	40.10492
[17,]	0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	36.62898	37.46855	38.32757
[18,]	0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	35.80813	36.62898	37.46855
[19,]	0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	35.00568	35.80813
[20,]	0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	34.22121

Figure A.5: Morgan Stanley's stock price tree for the first 20 periods.

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]	[,7]	[,8]	[,9]	[,10]	[,11]	[,12]	[,13]	[,14]	[,15]	[,16]	[,17]	[,18]	[,19]	[,20]
[1,]	10.04	10.308503	10.584186	10.867242	11.157868	11.456267	11.762645	12.077217	12.400282	12.731824	13.072315	13.421912	13.780859	14.149404	14.527806	14.916328	15.315240	15.724820	16.145354	16.577134
[2,]	0.00	9.778491	10.040000	10.308503	10.584186	10.867242	11.157868	11.456267	11.762645	12.077217	12.400282	12.731824	13.072315	13.421912	13.780859	14.149404	14.527806	14.916328	15.315240	15.724820
[3,]	0.00	0.000000	9.523793	9.778491	10.040000	10.308503	10.584186	10.867242	11.157868	11.456267	11.762645	12.077217	12.400282	12.731824	13.072315	13.780859	14.149404	14.527806	14.916328	15.315240
[4,]	0.00	0.000000	0.000000	9.275729	9.523793	9.778491	10.040000	10.308503	10.584186	10.867242	11.157868	11.456267	11.762645	12.077217	12.400282	12.731824	13.072315	13.780859	14.149404	14.527806
[5,]	0.00	0.000000	0.000000	0.000000	9.034127	9.275729	9.523793	9.778491	10.040000	10.308503	10.584186	10.867242	11.157868	11.456267	11.762645	12.077217	12.400282	12.731824	13.072315	13.421912
[6,]	0.00	0.000000	0.000000	0.000000	0.000000	8.798818	9.034127	9.275729	9.523793	9.778491	10.040000	10.308503	10.584186	10.867242	11.157868	11.456267	11.762645	12.077217	12.400282	12.731824
[7,]	0.00	0.000000	0.000000	0.000000	0.000000	0.000000	8.569637	8.798818	9.034127	9.275729	9.523793	9.778491	10.040000	10.308503	10.584186	10.867242	11.157868	11.456267	11.762645	12.077217
[8,]	0.00	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	8.346426	8.569637	8.798818	9.034127	9.275729	9.523793	9.778491	10.040000	10.308503	10.584186	10.867242	11.157868	11.456267
[9,]	0.00	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	8.129029	8.346426	8.569637	8.798818	9.034127	9.275729	9.523793	9.778491	10.040000	10.308503	10.584186	10.867242
[10,]	0.00	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	7.917294	8.129029	8.346426	8.569637	8.798818	9.034127	9.275729	9.523793	9.778491	10.040000	10.308503
[11,]	0.00	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	7.711075	7.917294	8.129029	8.346426	8.569637	8.798818	9.034127	9.275729	9.523793	9.778491
[12,]	0.00	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	7.510226	7.711075	7.917294	8.129029	8.346426	8.569637	8.798818	9.034127	9.275729
[13,]	0.00	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	7.314610	7.510226	7.711075	7.917294	8.129029	8.346426	8.569637	8.798818
[14,]	0.00	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	7.124088	7.314610	7.510226	7.711075	7.917294	8.129029	8.346426
[15,]	0.00	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	6.938529	7.124088	7.314610	7.510226	7.711075	7.917294
[16,]	0.00	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	6.757803	6.938529	7.124088	7.314610	7.510226
[17,]	0.00	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	6.581784	6.757803	6.938529	7.124088
[18,]	0.00	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	6.410350	6.581784	6.757803
[19,]	0.00	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	6.243381	6.410350
[20,]	0.00	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	6.080762

Figure A.6: Deutsche Bank's stock price tree for the first 20 periods.

BIBLIOGRAPHY

- Ahlberg, J. (2013). *Credit Value Adjustment*. PhD thesis, University of Lund.
- Alexander, C. (2003). The present and future financial risk management. *ISMA Centre Discussion Papers in Finance* 2003-12.
- Altman, E. I., Brady, B., Resti, A., and Sironi, A. (2005). The link between default and recovery rates: Theory, empirical evidence, and implications. *The Journal of Business*, 78(6):2203–2228.
- Altman, E. I. and Kishore, V. M. (1996). Almost everything you wanted to know about recoveries on defaulted bonds. *Financial Analysts Journal*, 52(6):57–64.
- Ballotta, L. and Fusai, G. (2015). Counterparty credit risk in a multivariate structural model with jumps. *Finance*, 1(36):39–74.
- Barnett-Hart, A. K. (2009). The story of the cdo market meltdown: An empirical analysis. *Mossavar-Rahmani Center for Business Government*, 4.
- BIS (2020). Statistical release: Otc derivatives statistics at end- december 2019. *Bank for international settlements*.
- Bouveret, A. (2009). Le marché des credit default swap (cds). *Économie & prévision*, 3(189):133–140.
- Brigo, D., Morini, M., and Pallavicini, A. (2013). *Counterparty Credit Risk, Collateral and Funding: With Pricing Cases For All Asset Classes*. Wiley Finance Series.
- Cannata, F. and Quagliariello, M. (2009). The role of basel ii in the subprime financial crisis: Guilty or not guilty? *CAREFIN Research Paper*, 3(9).
- Chitenderu, T. T., Maredza, A., and Sibanda, K. (2014). The random walk theory and stock prices: Evidence from johannesburg stock exchange. *International Business Economics Research Journal*, 13(6):1241–1250.
- Cornford, A. (2005). Everything you wanted to know about basel ii but were afraid to ask. *Finance Bien Commun*, 1(23):41–56.

- Das, S. R. and Hanouna, P. (2009). Implied recovery. *Journal of Economic Dynamics and Control*, 33(11):1837–1857.
- Dunbar, S. R. (2016). Stochastic processes and advanced mathematical finance: Properties of geometric brownian motion. *Department of Mathematics*, pages 1837–1857.
- Ekstrand, C. (2011). *Financial Derivatives Modeling*. Springer.
- Elkamhi, R., Jacobs, K., and Pan, X. (2010). The cross-section of recovery rates and default probabilities implied by credit default swap spreads. *Cambridge University Press*.
- Ewing, J. (2020). Deutsche bank reports huge loss for 2019. *The New York Times*.
- François, P. and Jiang, W. (2017). Credit value adjustment with market-implied recovery. *Journal of Financial Services Research*, 56:145–166.
- Gregory, J. (2010). *Counterparty Credit Risk*. The new challenge for global financial markets. John Wiley & Sons.
- Heinemann, S. (2011). Financial derivatives and responsibility – how to deal ethically with financial risk. *Finance & Bien Commun*, 1(39):45–56.
- Hui, C.-H. and Lo, C. C.-F. (2002). Effect of asset value correlation on credit-linked note values. *International Journal of Theoretical and Applied Finance*, 6(5):455–478.
- Hull, J. (2017). *Options, Futures, and Other Derivatives*. Pearson Education Limited.
- Hull, J. and White, A. (2012). Collateral and credit issues in derivatives pricing. *Rotman School of Management Working Paper No. 2212953*.
- Lucas, D. J., Goodman, L. S., and Fabozzi, F. J. (2007). Collateralized debt obligations and credit risk transfer. *Yale ICF Working Paper*, 7(6).
- Mathieu, C. and Sterdyniak, H. (2009). La globalisation financière en crise. *Revue de l’OFCE*, 3(110):13–73.
- Mengle, D. (2007). Credit derivatives: An overview. *Federal Reserve Bank of Atlanta*.
- Murphy, A. (2008). An analysis of the financial crisis of 2008: Causes and solutions. *Oakland University - School of Business Administration*.
- Nunweiler, C. and Watti, P. (2006). *Estimating Implied Corporate Default Probabilities from Bond Prices*. PhD thesis, University of Simon Fraser.
- Pham, H. (2011). *Optimisation et contrôle stochastique appliqués à la finance*. Springer.
- Shreve, S. E. (2004). *Stochastic Calculus for Finance II: Continuous-Time models*. Springer.

- Sjostrom, W. K. (2009). The aig bailout. *Washington and Lee Law Review*, 66:943–991.
- Terzi, N. and Uluçay, K. (2009). The role of credit default swaps on financial market stability. *Procedia - Social and Behavioral Sciences*, 24:983–990.
- Wang, A. and Ye, Z. (2014). Total return swap valuation with counterparty risk and interest rate risk. *Abstract and Applied Analysis*, pages 1–12.

