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Traces and extensions of non-abelian gauge covariant Sobolev spaces

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TRACES AND EXTENSIONS OF NON-ABELIAN GAUGE COVARIANT SOBOLEV SPACES

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Introduction

The laws of Physics, whether subatomic or astral, are all connected through mathematics. Electromagnetic fields distort the subatomic world and planets bend space-time itself. Two different worlds, but only one mathematical concept : curvature. The motion of a fluid or the conduction of heat, two seemingly different phenomena yet one mathematical theory : partial differential equations. What happens when curvature meets partial differential equations ? We will look for some answers in geometric analysis, a branch of mathematics in which one can find *gauge covariant Sobolev spaces*.

Sobolev-type spaces have a first row seat in mathematical analysis, particularly in partial differential equations and calculus of variations. Finding solutions to obstacles such as initial value problems emerging from theoretical physics is one of the goals of these two branches. However, finding analytic solutions is not a simple task. That is why mathematicians came up with the notions of weak derivatives and weak solutions.

The usual Sobolev space $W^{1,p}(\Omega)$ on an open domain $\Omega \subset \mathbb{R}^n$ contains $L^p(\Omega)$ functions that are weakly differentiable and whose weak derivative is itself L^p. As elements of a Lebesgue space, Sobolev mappings are only defined up to a set of measure zero, and since the boundary of a domain is of measure zero, it is not clear how to translate a boundary value problem, whose solution is a smooth function, into a weak boundary value problem, whose solution is a weakly differentiable function. While it is impossible to give a meaning to L^p functions on the boundary, it is possible to do so for a Sobolev function, and this is called the *trace* of the function. The *trace space* of a Sobolev space is the space of all the traces of said Sobolev space.

Gauge covariant Sobolev spaces are Sobolev-type spaces which behave well under *gauge transformations*, changes of coordinates. Such spaces appear naturally in gauge theories such as electromagnetism or Yang-Mills theories which involve *gauge fields* A. The aim of this thesis is to give a rigorous definition and characterisation of the trace spaces of gauge covariant Sobolev spaces. In Chapter 3, we show that the trace U of a gauge covariant Sobolev mapping $u \in W_A^{1,p}(\Omega)$ on a domain $\Omega \subset \mathbb{R}^n$ with gauge field A satisfies

$$\iint\limits_{\substack{(x,y)\in\partial\Omega\times\partial\Omega \\ \operatorname{dist}_{\partial\Omega}(y,x)<\operatorname{inj}_{\partial\Omega}}} \frac{\left|R(x,y)U(y)-U(x)\right|^p}{|y-x|^{n+p-1}}\,\mathrm{d}y\,\mathrm{d}x < +\infty,$$

R depending on A. This integral is called the *gauge covariant Gagliardo seminorm* and is invariant under changes of gauge. Conversely, we prove that any function $U \in L^p(\partial\Omega)$ with finite gauge covariant Gagliardo seminorm is the trace of a gauge covariant Sobolev function on Ω. Therefore, we can say that the trace space of $W_A^{1,p}(\Omega)$ is the set of all $U \in L^p(\partial\Omega)$ with finite seminorm.

Although this result is new, it is not surprising per se. In fact, it has been know since 1957 in the case of a trivial gauge field $A = 0$. This result, due to Gagliardo [17] is known today as the *trace theorem* and is rather well documented in textbooks regarding the subject. The involved integral, known as the *Gagliardo seminorm* is

$$\iint_{(x,y)\in\partial\Omega\times\partial\Omega} \frac{\left|U(y) - U(x)\right|^p}{|y - x|^{n+p-1}}\, \mathrm{d}y\, \mathrm{d}x$$

and one can observe that the main difference is that R, and thus the dependence in a field A, is gone. Furthermore, a generalisation of the trace theorem was established by Nguyen and Van Schaftingen [34], stating that the gauge covariant Sobolev space valued in $\mathbb{R}^2$, known as the *magnetic Sobolev space*, satisfies the result we announced. The particularity of their result is that the gauge transformations for the mangetic Sobolev space are abelian. This simple yet important property allowed Nguyen and Van Schaftingen to obtain some estimates with methods which do not work in the non-abelian case. Our result is essentially the generalisation of Nguyen and Van Schaftingen's result to the non-abelian case, that is, to gauge covariant Sobolev spaces valued in any finite dimensional vector space.

This thesis is divided into three chapters. Chapter 1 is dedicated to an introduction to *vector bundles*, geometric objects. Vector bundles are a generalisation of the Cartesian product and formalise the notion of curvature, magnetic fields and gauge theories in general. They are very well documented in literature. In order to have a good understanding of gauge covariant Sobolev spaces, it is fundamental to have a good understanding of the geometry behind, that is, of vector bundles.

We then begin in Chapter 2 our study of bundle valued functional analysis, introducing bundle valued Lebesgue spaces, bundle valued Sobolev spaces and, last but not least, gauge covariant Sobolev spaces. The particular case of magnetic Sobolev spaces is the object of recent research, but it is not easy to find a textbook are a paper giving the basics. Some textbooks such as Lieb and Loss's *Analysis* [28] mention the magnetic Sobolev spaces but do not go into much detail. Chapter 2 is therefore a relatively unique text, as we construct gauge covariant Sobolev spaces piece by piece and give some basic results.

Finally, in Chapter 3, we attack the theory that gives its name to this thesis : trace theory of gauge covariant Sobolev spaces. Although covering the non-abelian case and therefore extending [34] which is restricted to the abelian case, our proof is very similar and clearly inspired by Nguyen and Van Schaftingen's proof. The key estimates that had to be modified are obtained with a technique presented by Chanillo and Van Schaftingen in their article [8] which aimed to do the first step in the proof in the non-abelian case. A big part of the proof was understanding [34] in detail, identifying the parts that needed to be adapted, and adapting them, usually with a technique similar to the one exposed in [8].

Chapter 1

Vector bundle theory

Gauge covariant Sobolev spaces are build with a strong geometric background, namely vector bundles, an important object of differential geometry. Vector bundles emerge naturally as soon as *geometry* is mixed with *differential*. Indeed, the very notion of a differential for a map valued in a manifold $\mathcal{N}$ is tightly related to the *tangent bundle* $T\mathcal{N}$ of $\mathcal{N}$, a first example of a vector bundle.

Vector bundles are nowadays omnipresent in mathematics. A few examples are algebraic topology [5, 35], algebraic geometry [15] and K-theory [19, 29, 35]. They formalise the notion of *covariant derivative* which appears in many branches of physics, such as quantum physics [2, 18, 20, 36, 38, 44, 45], electromagnetism [16, 18, 30, 38, 44, 45] and general relativity [16, 30].

For this work in particular, we shall be using covariant derivatives and therefore need to have some basic notions of vector bundle theory. Gauge covariant Sobolev spaces are spaces of *sections* of a bundle. The main differences between gauge covariant and the usual Sobolev spaces are of geometric nature rather than analytic.

We therefore must begin with some of the geometric background necessary to define and understand the objects which we will be working with. Other geometric concepts associated to gauge covariant Sobolev spaces will be introduced as needed.

Section 1.1 contains a discussion explaining how vector bundles arise naturally. It is inspired by [19]. In Section 1.2, we define what a vector bundle is and give a few examples to help build an intuition. Afterwards, in Section 1.3, we study some related objects like bundle morphisms. Finally, in Section 1.4, we study a particular class of functions known as *sections*. These will be the main functions with whom we will be working. Together with these functions, we introduce the notions of *connection* and *covariant derivatives*. One can roughly say that a gauge covariant Sobolev space is a Sobolev space, but instead of a weak derivative, one has a weak covariant derivative. It is thus a fundamental object for our work.

We assume that the reader is familiar with basic notions of differential geometry, such as abstract manifolds, tangent spaces, differential forms and so on. A familiarity with

Riemannian geometry is recommended. Riemannian geometry already includes many concepts related to vector bundles, such as curvature, covariant derivatives, etc. Our main references on the subject are [21,24,25,38,39]. However, all mentioned references contain the necessary information, and the interested reader can chose the references of his liking.

1.1 Products and twisted products

Roughly speaking, a vector bundle is a *twisted product* of a smooth manifold and a vector space, a generalisation of the usual Cartesian product. In order to understand why such a notion is relevant and motivate the definition, let us look into two examples inspired by the discussion made in [19, Ch. 1].

Notation 1.1.1. We denote by $\|\cdot\|$ the usual Euclidean norm on $\mathbb{R}^{n+1}$ and by $\mathbb{S}^n$ the Euclidean unit n-sphere, that is,

$$\mathbb{S}^n = \left\{ x \in \mathbb{R}^{n+1} \,\middle|\, \|x\| = 1 \right\}. \qquad \triangle$$

First, let us consider the two-dimensional sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. To each point $x \in \mathbb{S}^2$, we can associate the plane $T_x\mathbb{S}^2 \subset \mathbb{R}^3$ tangent to the sphere at this point. This plane can be viewed as the vector space

$$T_x = \left\{ v \in \mathbb{R}^3 \,\middle|\, (x \mid v) = 0 \right\}$$

orthogonal to x which was translated by x, $(\cdot \mid \cdot)$ denoting the usual scalar product in $\mathbb{R}^3$. In other words,

$$T_x\mathbb{S}^2 \cong \left\{ v_x \in \mathbb{S}^2 \times \mathbb{R}^3 \,\middle|\, v_x = (x, v_0),\ v_0 \in T_x \right\}$$

where the first component is understood to be the new starting point of v_0. The plane $T_x\mathbb{S}^2$, called the tangent space of the sphere at x, is thus endowed with a natural two-dimensional vector space structure given by

$$\begin{aligned}(x, v_0) + (x, w_0) &= (x, v_0 + w_0),\\ \alpha \cdot (x, v_0) &= (x, \alpha v_0),\end{aligned}$$

for all $v_0, w_0 \in T_x$ and $\alpha \in \mathbb{R}$. Figure 1.1 represents the sphere, a tangent space, as well as a tangent vector v_x and its analogue $v_0 \in T_x$ (i.e. $v_x = (x, v_0)$).

Let us define the tangent bundle of the sphere $T\mathbb{S}^2$ as the set of all tangent vectors v_x as x ranges the whole sphere, that is,

$$T\mathbb{S}^2 = \left\{ (x, v) \in \mathbb{S}^2 \times \mathbb{R}^3 \,\middle|\, (x \mid v) = 0 \right\}.$$

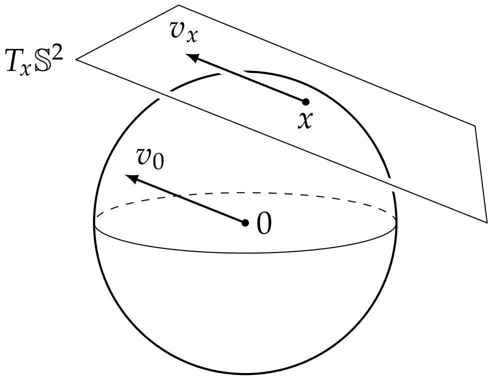


Figure 1.1: The sphere $\mathbb{S}^2$ and a tangent plane $T_x\mathbb{S}^2$; inspired by [19, Ch. 1].

In particular, $T\mathbb{S}^2$ is the disjoint union of the tangent spaces $T_x\mathbb{S}^2$ and, additionally, a topological space as subset of $\mathbb{S}^2 \times \mathbb{R}^3$.

What exactly have we constructed ? To each point $x \in \mathbb{S}^2$, we have associated a two-dimensional vector space. In other words, $T\mathbb{S}^2$ is a bunch of two-dimensional vectors spaces that were *glued together* following the structure of the sphere in some sense. That is exactly what the Cartesian product $\mathbb{S}^2 \times \mathbb{R}^2$ is. Could it be that the set $T\mathbb{S}^2$ is simply $\mathbb{S}^2 \times \mathbb{R}^2$ in disguise? More precisely, does there exist a homeomorphism $h\colon T\mathbb{S}^2 \to \mathbb{S}^2 \times \mathbb{R}^2$ mapping $T_x\mathbb{S}^2$ to $\{x\} \times \mathbb{R}^2$ via a linear isomorphism? If such a homeomorphism h were to exist, the vectors

$$v_x = h^{-1}(x, v), \quad x \in \mathbb{S}^2$$

for some $v \in \mathbb{R}^2_* = \mathbb{R}^2 \setminus \{0\}$ would define a continuous family of non-zero tangent vectors on $\mathbb{S}^2$. The following classical *algebraic topology* theorem, whose proof can be found in [9, Thm. 34.1], refutes this possibility.

Theorem 1.1.2 (Hairy ball theorem). *Let $v\colon \mathbb{S}^2 \to T\mathbb{S}^2\colon x \mapsto v_x$ be a continuous map such that $v_x \in T_x\mathbb{S}^2$ for all $x \in \mathbb{S}^2$. Then there exists $x \in \mathbb{S}^2$ such that*

$$v_x = (x, 0).$$

In other words, a continuous family of tangent vectors on the sphere $\mathbb{S}^2$ must vanish. This conveys that the set $T\mathbb{S}^2$ is indeed more complicated than a usual Cartesian product, it is a twisted product.

Lowering the dimension by one, let us examine the case of the circle $\mathbb{S}^1$. A similar construction to the above can be done, defining the tangent bundle of the circle $T\mathbb{S}^1$. However, in this case, we can define a non-vanishing family of tangent vectors. Given $x \in \mathbb{S}^1$, we consider its rotation by $\pi/2$, denoted by $x^\perp$, and define $v_x \in T_x\mathbb{S}^1$ by

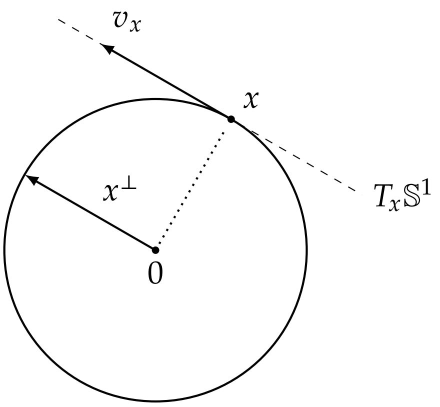


Figure 1.2: The circle $\mathbb{S}^1$ and a tangent space $T_x\mathbb{S}^1$.

translating $x^\perp$ to have its origin at x. In other words, we define $v_x = (x, x^\perp) \in T_x\mathbb{S}^1$. To help visualise this construction we refer the reader to Figure 1.2.

With this family of tangent vectors, we define the homeomorphism between $\mathbb{S}^1 \times \mathbb{R}$ and $T\mathbb{S}^1$

$$h\colon \mathbb{S}^1 \times \mathbb{R} \to T\mathbb{S}^1 \colon (x, t) \mapsto tv_x.$$

In particular, h maps $\{x\} \times \mathbb{R}$ to $T_x\mathbb{S}^1$ by the means of a linear isomorphism. This expresses that $T\mathbb{S}^1$ is, in contrast to the previously examined $T\mathbb{S}^2$, equivalent to the usual Cartesian product $\mathbb{S}^1 \times \mathbb{R}$.

Naturally, one might wonder whether this phenomenon of twisted product arises when the above construction is applied to spheres of higher dimensions. It is rather surprising that only $\mathbb{S}^0$, $\mathbb{S}^1$, $\mathbb{S}^3$ and $\mathbb{S}^7$ have a trivial tangent bundle. Precisely, a homeomorphism between $T\mathbb{S}^n$ and $\mathbb{S}^n \times \mathbb{R}^n$ taking $T_x\mathbb{S}^n$ to $\{x\} \times \mathbb{R}^n$ by a linear isomorphism exists only for $n = 0, 1, 3, 7$ (see [19, Thm. 2.16]).

The general notion of twisted products between two arbitrary manifolds is described by what is known as *fiber bundles*. The interested reader shall find a large amount of information on fiber bundles in the corresponding chapter(s) of the books [21, 24, 25, 38, 39] dedicated to the study of *differential geometry*. In the scope of this thesis, we shall only study a particular case of fiber bundles referred to as *vector bundles*. These bundles have an additional linear structure enabling a study from the point of view of *functional analysis* conducted in the subsequent chapters.

1.2 Vector bundles : definition and examples

In order to ensure more concise notations, we will always denote by $\mathcal{N}$ a manifold of dimension n. Manifolds and diffeomorphisms will be assumed smooth. Furthermore, F will denote a real vector space of dimension m and $GL(F)$ will denote the Lie group of linear automorphisms of F, that is, linear isomorphisms from F to itself. Finally, $G \subset GL(F)$ will be a Lie subgroup of $GL(F)$. If we deviate from any of the above notations, it will be explicitly stated.

Our definition of vector bundle is the combination of [24, Def. 6.12 + 6.21], [24, Def. 6.12 + 6.21] or [42, Def. 2.8 + 2.22 + 2.78]. This definition avoids the necessity of introducing additional concepts, mainly *principal bundles*. Vector bundles can in fact be defined as quotient spaces of principal bundles, see [21, Sect. I.5 +Ch. III] and [38, Def. 3.3]. We choose to work with the former since it is the more direct definition of vector bundle and is thus more suited for our work.

Definition 1.2.1 (Vector bundle)**.** *A manifold P together with a smooth surjective map $\pi_P \colon P \to \mathcal{N}$ is called a* vector bundle *over $\mathcal{N}$ with* model fiber *F and* structure group *G if*

(i) *For all $x \in \mathcal{N}$, the* fiber above x*, defined by $F_x = \pi_P^{-1}(\{x\})$, is a closed submanifold of P and has a real vector space structure isomorphic to F,*

(ii) *The manifold P is* locally trivial*, that is, there exists an open covering $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of $\mathcal{N}$ such that, for all $\alpha \in \mathcal{A}$, there exists a diffeomorphism*

$$\varphi_\alpha \colon \pi_P^{-1}(U_\alpha) \to U_\alpha \times F$$

with the property that, for all $x \in U_\alpha$, φ_α maps F_x to $\{x\} \times F$ by the means of a linear isomorphism. We call the pair $(U_\alpha, \varphi_\alpha)$ (or simply U_α) a local trivialisation *or* local gauge *of P,*

(iii) *Local trivialisations are G-*compatible*, that is, if $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are two intersecting local trivialisations of P, there exists a smooth function $g_{\alpha\beta} \colon (U_\alpha \cap U_\beta) \to G$ such that for all $x \in U_\alpha \cap U_\beta$ and $v \in F$*

$$\varphi_{\alpha\beta}(x, v) = \big(\varphi_\alpha \circ \varphi_\beta^{-1}\big)(x, v) = \big(x, g_{\alpha\beta}(x)[v]\big).$$

We call $g_{\alpha\beta}$ the local gauge transformation from φ_β to φ_α.

Similarly to a manifold $\mathcal{N}$ which is locally the Euclidean space $\mathbb{R}^n$, a vector bundle P is locally the Cartesian product $\mathcal{N} \times F$. For manifolds, this is translated into mathematics by local charts whereas for vector bundles, it is translated by local trivialisations. Going from one local chart to another on a manifold is done via change of charts and, in the case of vector bundles, moving from one trivialisation to another one is done by the gauge transformations. In other words, local trivialisations are in some sense the *charts* of a vector bundle.

Remark 1.2.2. Definition 1.2.1 (i) carries an implicit choice of a (potentially different) vector space structure for every fiber F_x. △

Since we will not be working with general fiber bundles but only vector bundles, we will often omit the term *vector* and refer to vector bundles simply as *bundles*. Furthermore, P will always denote a bundle as in Definition 1.2.1 and the map $\pi_P \colon P \longrightarrow \mathcal{N}$ will often be implicit.

Before presenting a few examples to build an intuition, we state this fundamental result concerning the dimension of a vector bundle.

Proposition 1.2.3. *If P is a vector bundle over $\mathcal{N}$ with model fiber F, then*

$$\dim P = \dim \mathcal{N} + \dim F.$$

Proof. Our goal is to construct an atlas for P such that P is locally $\mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$ where $n = \dim \mathcal{N}$ and $m = \dim F$. Let $p \in P$ and consider a local trivialisation (U, φ) such that $p \in \pi_P^{-1}(U)$. Let (V, ϕ) be a chart of $\mathcal{N}$ centred at $\pi_P(p)$ such that $V \subset U$ and let $L \colon F \longrightarrow \mathbb{R}^m$ be a linear isomorphism. Then

$$\begin{aligned} \Phi = (\phi, L) \circ \varphi \colon \pi_P^{-1}(V) &\longrightarrow \mathbb{R}^n \times \mathbb{R}^m, \\ q &\longmapsto \big(\phi(x), L(v)\big), \quad \varphi(q) = (x, v) \end{aligned}$$

defines a chart $\big(\pi_P^{-1}(V), \Phi\big)$ of P centred at p. □

Example 1.2.4 (The Cartesian product)**.** Let $\mathcal{N}$ be a manifold and let F be a vector space. The Cartesian product $P = \mathcal{N} \times F$ together with the projection on the first component $\pi_P(x, v) = pr_1(x, v) = x$ is a vector bundle. The pair $(\mathcal{N}, \mathrm{id}_P)$ defines a global trivialisation of P, that is, it satisfies Definition 1.2.1 (ii) by itself. The associated gauge transformation is the identity map on F. Therefore, the structure group G can be taken to be trivial, $G = \{\mathrm{id}_F\}$. The Cartesian product is thus called a *trivial* bundle. △

Note that the sentence *the structure group can be taken to be trivial* implies that we could take another Lie subgroup of $GL(F)$ as structure group for a Cartesian product. In fact, the meticulous reader might have noticed that the structure group in Definition 1.2.1 is far from unique, and, although one would lose information, one could always take $G = GL(F)$.

Definition 1.2.5 (Trivial bundle)**.** *A bundle P over a manifold $\mathcal{N}$ is called a* trivial bundle *if there exists a* global *trivialisation, i.e. a trivialisation of the form $(\mathcal{N}, \varphi)$.*

We have for example encountered in Section 1.1 the vector bundle $T\mathbb{S}^1$ which is a trivial bundle. If P is a trivial bundle, one may always assume that its structure group is the trivial group $G = \{\mathrm{id}_F\}$. However, the global trivialisation is not unique since we may compose it with the map $(\mathrm{id}_{\mathcal{N}}, L) \colon \mathcal{N} \times F \longrightarrow \mathcal{N} \times F$ for any linear isomorphism $L \colon F \longrightarrow F$. Similarly to working in charts for a manifold, one should always verify that objects defined in a trivialisation are independent of the particular choice of trivialisation.

The first and probably most important example of bundle that is in general not trivial is the *tangent bundle* [25, Example 10.4]. We have previously mentioned the tangent bundles of the circle $\mathbb{S}^1$ and the sphere $\mathbb{S}^2$, the latter being non-trivial. Tangent bundles are typically first encountered by students in undergraduate-level classes serving as introduction to differential geometry. We recall that the *tangent space* $T_x\mathcal{N}$ at a point $x \in \mathcal{N}$ is defined by the vector space containing all linear and continuous maps $v\colon C^\infty(\mathcal{N}, \mathbb{R}) \to \mathbb{R}$ such that [24, Sect. 2.1.3]

$$v(fg) = v(f)g(x) + f(x)v(g).$$

Furthermore, if (U, ϕ) is a chart of $\mathcal{N}$ centred at $x \in \mathcal{N}$, then

$$\begin{aligned} \mathrm{D}\big(\phi^{-1}\big)\big(\phi(x)\big)\colon \mathbb{R}^n &\to T_x\mathcal{N}, \\ w &\mapsto v = \mathrm{D}\big(\phi^{-1}\big)\big(\phi(x)\big)[w] \end{aligned}$$

where, for all $f \in C^\infty(\mathcal{N}, \mathbb{R})$,

$$v(f) = \mathrm{D}\big(\phi^{-1}\big)\big(\phi(x)\big)[w](f) = \mathrm{D}\big(f \circ \phi^{-1}\big)\big(\phi(x)\big)[w]$$

defines a linear isomorphism $\mathbb{R}^n \cong T_x\mathcal{N}$ with inverse $\mathrm{D}\phi(x)\colon T_x\mathcal{N} \to \mathbb{R}^n$, see for example [24, Thm. 2.10].

Example 1.2.6 (Tangent bundle)**.** The tangent bundle $T\mathcal{N}$ of an n-dimensional manifold $\mathcal{N}$ is defined as the disjoint union

$$T\mathcal{N} = \bigsqcup_{x \in \mathcal{N}} T_x\mathcal{N} = \bigcup_{x \in \mathcal{N}} \big(\{x\} \times T_x\mathcal{N}\big)$$

of all the tangent spaces $T_x\mathcal{N}$ of $\mathcal{N}$. By definition, there exists a natural surjective projection

$$\begin{aligned} \pi_{T\mathcal{N}}\colon T\mathcal{N} &\to \mathcal{N}, \\ (x, v) &\mapsto x. \end{aligned}$$

Let us show that $T\mathcal{N}$ is a manifold. Let (U, ϕ) be a chart of $\mathcal{N}$. Then, the map

$$\begin{aligned} \Phi\colon \pi_{T\mathcal{N}}^{-1}(U) \subset T\mathcal{N} &\to \phi(U) \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n, \\ (x, v) &\mapsto \big(\phi(x), \mathrm{D}\phi(x)[v]\big) \end{aligned}$$

defines a chart $\big(\pi_{T\mathcal{N}}^{-1}(U), \Phi\big)$ for $T\mathcal{N}$ and the collection of all so-constructed charts defines an atlas for $T\mathcal{N}$. Indeed, let (V, ψ) be another chart of $\mathcal{N}$ and denote by $\big(\pi_{T\mathcal{N}}^{-1}(V), \Psi\big)$ the chart of $T\mathcal{N}$ constructed as above. Assume that $\pi_{T\mathcal{N}}^{-1}(U) \cap \pi_{T\mathcal{N}}^{-1}(V)$ is not empty. Then, whenever defined for $(x, v) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$\big(\Phi \circ \Psi^{-1}\big)(x, v) = \Big(\phi \circ \psi^{-1}(x), \mathrm{D}\big(\phi \circ \psi^{-1}\big)(x)[v]\Big)$$

defines a smooth map since the change of charts $\phi \circ \psi^{-1} \colon \mathbb{R}^n \to \mathbb{R}^n$ is smooth. The tangent bundle $T\mathcal{N}$ is thus a smooth manifold of dimension $2n$ and the natural projection $\pi_{T\mathcal{N}}$ is smooth since, when expressed in charts, it is the projection on the first component.

Let us now prove that $T\mathcal{N}$ together with the projection $\pi_{T\mathcal{N}}$ defines a vector bundle. The fibers are by definition the tangent spaces

$$T_x\mathcal{N} = \pi_{T\mathcal{N}}^{-1}\big(\{x\}\big).$$

Let (U, ϕ) be a chart of $\mathcal{N}$. Then, the map

$$\begin{aligned} \varphi \colon \pi_{T\mathcal{N}}^{-1}(U) &\to U \times \mathbb{R}^n, \\ (x, v) &\mapsto \big(x, \mathrm{D}\phi(x)[v]\big) \end{aligned}$$

defines a local trivialisation of $T\mathcal{N}$. Repeating the construction for an atlas of $\mathcal{N}$ gives the local triviality of the tangent bundle. Let (V, ψ) be another chart of $\mathcal{N}$ such that $U \cap V$ is not empty and denote by $\tilde{\varphi}$ the trivialisation of $T\mathcal{N}$ constructed as above. Then, for all $(x, v) \in \tilde{\varphi}(U \cap V)$, we have

$$\big(\varphi \circ \tilde{\varphi}^{-1}\big)(x, v) = \big(x, \mathrm{D}\big(\phi \circ \psi^{-1}\big)(x)[v]\big).$$

The map $\mathrm{D}\big(\phi \circ \psi^{-1}\big) \colon U \cap V \to GL(\mathbb{R}^n)$ is smooth and defines the gauge transformation from $(V, \tilde{\varphi})$ to (U, φ). We conclude that $T\mathcal{N}$ with $\pi_{T\mathcal{N}}$ is a vector bundle over $\mathcal{N}$ with model fiber $\mathbb{R}^n$ and structure group $GL(\mathbb{R}^n)$. △

The structure group $G \subset GL(\mathbb{R}^n)$ of the tangent bundle is, with this construction, dependent on the chosen charts of $\mathcal{N}$. We will see later that under some additional structure on $\mathcal{N}$, one can give a better description of G.

The *cotangent bundle* [25, Ch. 11; 38, Sect. 3.4] is often introduced together with the tangent bundle. Its fibers are the dual spaces to the fibers of the tangent bundle. The cotangent bundle is therefore said to be *dual* to the tangent bundle. We recall two standard notions associated to duality.

Definition 1.2.7 (Dual space)**.** *The* dual space *of a real vector space V is the vector space V^* of all linear and continuous maps from V to $\mathbb{R}$, that is,*

$$V^* = \{\, L \colon V \to \mathbb{R} \mid L \text{ is linear and continuous} \,\}.$$

We say that V^* is dual *to V.*

If $m = \dim V$ and $(v_1, \dots, v_m)$ is a basis of V, one can construct the *dual basis* $(v_1^*, \dots, v_m^*)$ of V^* by setting

$$v_i^*(v_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

In particular, $V \cong V^*$ when V is finite dimensional.

Definition 1.2.8 (Adjoint map)**.** *Let $L\colon V \to W$ be a linear map between two vector spaces. The* adjoint *of L is the linear map*

$$L^*\colon W^* \to V^*$$

defined by

$$\big(L^*(\zeta)\big)[v] = \zeta\big[L(v)\big], \quad \zeta \in W^*, v \in V.$$

Example 1.2.9 (The cotangent bundle)**.** The cotangent bundle $T^*\mathcal{N}$ of an n-dimensional manifold $\mathcal{N}$ is defined as the disjoint union

$$T^*\mathcal{N} = \bigsqcup_{x \in \mathcal{N}} T_x^*\mathcal{N}$$

where $T_x^*\mathcal{N} = (T_x\mathcal{N})^*$ is the dual space of the tangent space at x, called the *cotangent space* at x. The elements of $T_x^*\mathcal{N}$ are called *covectors* at x. As for the tangent bundle, there is a natural projection $\pi_{T^*\mathcal{N}}\colon T^*\mathcal{N} \to \mathcal{N}$ mapping an element from $T_x^*\mathcal{N}$ to $x \in \mathcal{N}$.

One proves that $T^*\mathcal{N}$ is a manifold in a similar manner to the tangent bundle, see Example 1.2.6. Essentially, it suffices to replace $\mathrm{D}\phi(x)$ by its adjoint $\big(\mathrm{D}\phi(x)\big)^*$ whenever (U, ϕ) is a chart of $\mathcal{N}$. The projection $\pi_{T^*\mathcal{N}}$ is also smooth. For details, see [25, Ch. 11].

Concerning the bundle structure, the fibers are by definition

$$T_x^*\mathcal{N} = \pi_{T^*\mathcal{N}}^{-1}\big(\{x\}\big).$$

Given a chart (U, ϕ) of $\mathcal{N}$, the map

$$\begin{aligned} \varphi\colon \pi_{T^*\mathcal{N}}^{-1}(U) &\to U \times (\mathbb{R}^n)^*, \\ (x, v) &\mapsto \Big(x, \big(\mathrm{D}\phi(x)\big)^*[v]\Big) \end{aligned}$$

defines a local trivialisation (U, φ) of $T^*\mathcal{N}$. As for the tangent bundle, repeating this construction for an atlas of $\mathcal{N}$ yields the local triviality of $T^*\mathcal{N}$. Furthermore, we may observe that the model fiber of $T^*\mathcal{N}$ is $(\mathbb{R}^n)^*$, the dual of $\mathbb{R}^n$. Let (V, ψ) be another chart of $\mathcal{N}$ such that $U \cap V$ is not empty and denote by $\tilde{\varphi}$ the trivialisation of $T^*\mathcal{N}$ as constructed above. Then, for all $(x, v) \in \tilde{\varphi}(U \cap V)$, a computation shows that

$$\big(\varphi \circ \tilde{\varphi}^{-1}\big)(x, v) = \Big(x, \big(\mathrm{D}(\phi \circ \psi^{-1})(x)\big)^*[v]\Big),$$

i.e. the gauge transformation from (U,ϕ) to $(V,\tilde{\varphi})$ is $\mathrm{D}\big(\phi\circ\psi^{-1}\big)^*\colon U\cap V\to GL\big((\mathbb{R}^n)^*\big)$. Note that this is the adjoint map of the gauge transformation for the tangent bundle $T\mathcal{N}$, see Example 1.2.6. We conclude that the cotangent bundle $T^*\mathcal{N}$ with $\pi_{T^*\mathcal{N}}$ defines a vector bundle over $\mathcal{N}$ with model fiber $(\mathbb{R}^n)^*$ and structure group $GL\big((\mathbb{R}^n)^*\big)$. △

As for the tangent bundle, the structure group of the cotangent bundle depends on the chosen charts of $\mathcal{N}$.

The example of the cotangent bundle generalises to exterior powers of the tangent space [25, Ch. 14].

Definition 1.2.10 (Exterior power and k-forms)**.** *Let V be a finite dimensional vector space and let $k\geq 1$. An* exterior k-form *ω on V is a (k-)multilinear map*

$$\omega\colon \underbrace{V\times\cdots\times V}_{k\ \text{times}}\to\mathbb{R}$$

such that

$$\omega(v_1,\dots,v_k)=0$$

whenever there exists $i,j\in\{1,\dots,k\}$ such that $i\neq j$ and $v_i=v_j$. The set of all exterior k-forms on V is called the kth exterior power *of V and denoted by $\bigwedge^k V^*$.*

When $V=T_x\mathcal{N}$ is a tangent space, one writes $\bigwedge^k T_x^*\mathcal{N}=\bigwedge^k(T_x\mathcal{N})^*$. Then, for all $k\geq 1$,

$$\bigwedge^k T^*\mathcal{N}=\bigsqcup_{x\in\mathcal{N}}\bigwedge^k T_x^*\mathcal{N}$$

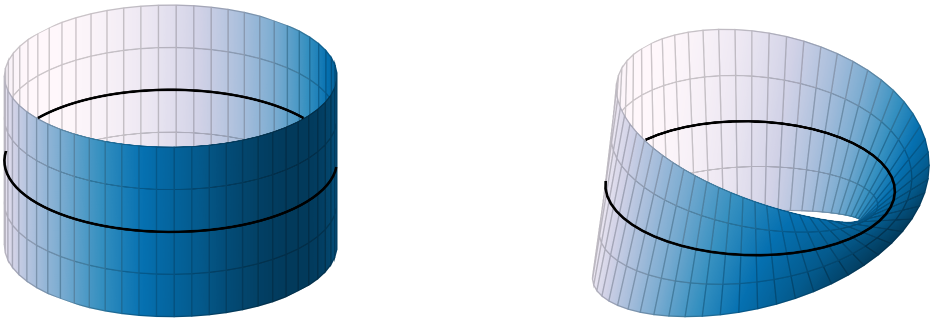
defines a vector bundle over $\mathcal{N}$ with model fiber $\bigwedge^k(\mathbb{R}^n)^*$ [39, Sect. 21]. This bundle is important as it is closely related to differential k-forms, see Example 1.4.4. When $k=1$, $\bigwedge^1 T^*\mathcal{N}=T^*\mathcal{N}$ is the cotangent bundle.

Definition 1.2.11 (Terminology)**.** *If P is a vector bundle, then we call $\mathcal{N}$ the* base space*, P the* total space *and π_P the* projection*. A family $\big\{(U_\alpha,\varphi_\alpha)\big\}_{\alpha\in\mathcal{A}}$ of G-compatible local trivialisations is called a* (G-)bundle atlas *for P if $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ covers $\mathcal{N}$. For an open subset $U\subset\mathcal{N}$ of the base space, we define the* bundle above U *as the open set*

$$P_{|U}=\pi_P^{-1}(U)\subset P.$$

A local trivialisation (U,φ) is said to be centred *at a point $p\in P$ if $p\in P_{|U}$.*

Intuitively, the total space P can be seen as a multitude of different copies of the model fiber F that were somehow *glued* to the base space $\mathcal{N}$, which is how one might visualise a Cartesian product or the construction of tangent bundles we did in Section 1.1, as



(a) The Cylinder : trivial. (b) The Möbius strip : not trivial.

Figure 1.3: Two bundles over $\mathbb{S}^1$ with model fiber $\mathbb{R}$.

well as the Examples 1.2.6 and 1.2.9. The gluing method need not be *straight* as for the Cartesian product, which is mathematically translated by a topology on P that is not trivial, i.e. not the product topology.

To help the reader have a clear mental image of specific examples of vector bundles rather than abstract definitions, let us consider the two equivalence relations $\sim$ and $\approx$ on $\mathbb{R}^2$ defined by $(x, y) \sim (x+n, y)$ and $(x, y) \approx (x+n, (-1)^n y)$ for all $n \in \mathbb{Z}$. Then the quotient spaces

$$\mathcal{N}_1 = {}^{\mathbb{R}^2}\!/\!_{\sim} \quad \text{and} \quad \mathcal{N}_2 = {}^{\mathbb{R}^2}\!/\!_{\approx}$$

induced by these relations define different bundles over the circle $\mathbb{S}^1$ with model fiber $\mathbb{R}$. The first bundle $\mathcal{N}_1$ is the ordinary cylinder (of infinite height) which we have already encountered as $T\mathbb{S}^1$ [21, Example V.4.4 (1)]. The second bundle $\mathcal{N}_2$ is the Möbius strip (of infinite width), also called twisted cylinder [21, Example V.4.4 (3)], [25, Example 10.3], [39, Example 7.3]. Both manifolds are depicted in Figure 1.3. Observe how on each point x of the circle, depicted in black, there is a line, a vector space, depicted in a shade blue, passing through that point x perpendicularly to the circle.

1.3 Cocycles, bundle maps and subbundles

Now that we have built an intuition on what a vector bundle is, let us look at some basic properties and related objects. For starters, let us establish a fundamental property of the gauge transformations of a bundle P. They form what is called a *cocycle* [24, Def. 6.10.], [39, Sect. 27.1] and encode a great deal of information on the bundle. Conversely, cocycles can be used to construct vector bundles, see the *construction theorem* [24, Thm. 6.30].

Definition 1.3.1 (Cocycle)**.** *Let G be any Lie group and $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ an open covering of a manifold $\mathcal{N}$. A G-*cocycle *on $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ is an assignment of a smooth map*

$$g_{\alpha\beta}\colon (U_\alpha \cap U_\beta) \to G$$

to each intersecting ordered pair (U_α, U_β), $\alpha, \beta \in \mathcal{A}$, such that

$$g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma. \tag{1.1}$$

*The family $\{g_{\alpha\beta}\}$ forms a G-*cocycle *(or simply* cocycle*) for the covering $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ and one refers to the relation (1.1) as the* cocycle condition.

Let us note that if $\{g_{\alpha\beta}\}$ is a cocycle for a covering $\{U_\alpha\}_{\alpha\in\mathcal{A}}$, there are always *two* functions assigned to a non-empty intersection $U_\alpha \cap U_\beta$ ($\alpha \neq \beta$), namely $g_{\alpha\beta}$, associated to the pair (U_α, U_β), and $g_{\beta\alpha}$, associated to the pair (U_β, U_α). Moreover, the pair (U_α, U_α) must also be considered, yielding the map $g_{\alpha\alpha}$ for all $\alpha \in \mathcal{A}$.

The following lemma [39, Prop. 27.8], which some authors include in the definition of cocyle (e.g. [24, Def. 6.10] or [45, Sect. 16.1]), states that in a G-cocyle $\{g_{\alpha\beta}\}$, we always have the relation $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$, where the inverse is taken in the group G.

Lemma 1.3.2. *Let G be any Lie group and let us denote its neutral element by $e \in G$. If $\{g_{\alpha\beta}\}$ is a G-cocycle for a covering $\{U_\alpha\}_{\alpha\in\mathcal{A}}$, then*

$$g_{\alpha\alpha} = e \quad \text{on } U_\alpha$$

for all $\alpha \in \mathcal{A}$. Furthermore, if $U_\alpha \cap U_\beta$ is non-empty, then

$$g_{\alpha\beta}g_{\beta\alpha} = e \quad \text{on } U_\alpha \cap U_\beta.$$

The neutral element e in the above equalities is understood the be the constant map $x \mapsto e \in G$ on the corresponding domain.

Proof. Let $\alpha \in \mathcal{A}$ and $x \in U_\alpha$. The cocycle condition for $\alpha = \beta = \gamma$ reads

$$g_{\alpha\alpha}(x)g_{\alpha\alpha}(x) = g_{\alpha\alpha}(x).$$

Multiplying both sides by the inverse of $g_{\alpha\alpha}(x) \in G$ yields $g_{\alpha\alpha}(x) = e$.

If $U_\alpha \cap U_\beta$ is non-empty, the cocycle condition applied to $\alpha = \gamma$ and β gives

$$g_{\alpha\beta}g_{\beta\alpha} = g_{\alpha\alpha} = e \quad \text{on } U_\alpha \cap U_\beta. \qquad \square$$

Proposition 1.3.3. *Let P be a bundle with a G-bundle atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\mathcal{A}}$. Then the associated gauge transformations $\{g_{\alpha\beta}\}$ form a G-cocyle for the covering $\{U_\alpha\}_{\alpha\in\mathcal{A}}$, that is,*

$$g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma.$$

By the previous lemma, we also have

$$g_{\alpha\alpha} = \mathrm{id}_F \quad \text{and} \quad g_{\alpha\beta}g_{\beta\alpha} = \mathrm{id}_F$$

on U_α and $U_\alpha \cap U_\beta$ respectively.

Here, $\mathrm{id}_F\colon F \longrightarrow F$ is the identity map on F and the neutral element of the structure group $G \subset GL(F)$. In the above proposition, it is understood to be the constant map $x \mapsto \mathrm{id}_F$ on U_α or $U_\alpha \cap U_\beta$.

Proof. Let $\alpha, \beta, \gamma \in \mathcal{A}$ such that $U = U_\alpha \cap U_\beta \cap U_\gamma$ is not empty. By definition of the gauge transformations, for all $x \in U$ and $v \in F$,

$$\begin{aligned}\left(x, g_{\alpha\gamma}(x)v\right) = \left(\varphi_\alpha \circ \varphi_\gamma^{-1}\right)(x, v) &= \left[\left(\varphi_\alpha \circ \varphi_\beta^{-1}\right) \circ \left(\varphi_\beta \circ \varphi_\gamma^{-1}\right)\right](x, v)\\ &= \left(\varphi_\alpha \circ \varphi_\beta^{-1}\right)\left(x, g_{\beta\gamma}(x)v\right)\\ &= \left(x, g_{\alpha\beta}(x)g_{\beta\gamma}(x)v\right)\end{aligned}$$

which yields $g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$. □

Comparably to many mathematical structures, bundles have a special kind of map which respect their structure. Such a map is called a *vector bundle morphism* or a *vector bundle map* [25, Ch. 10].

Definition 1.3.4 (Vector bundle map). *Let P, Q be two vector bundles with projections π_P, π_Q, base spaces $\mathcal{N}, \mathcal{M}$ and model fibers F, V respectively. A map $B\colon P \longrightarrow Q$ is called a* vector bundle map *if there exists a map $b\colon \mathcal{N} \longrightarrow \mathcal{M}$ such that the diagram*

$$\begin{tikzcd} P \arrow[r, "B"] \arrow[d, "\pi_P"'] & Q \arrow[d, "\pi_Q"] \\ \mathcal{N} \arrow[r, "b"'] & \mathcal{M} \end{tikzcd}$$

commutes, i.e. $\pi_Q \circ B = b \circ \pi_P$, and, for all $x \in \mathcal{N}$, the restriction to the fiber F_x

$$B_{|F_x}\colon F_x \longrightarrow V_{b(x)}\colon v \mapsto B(v)$$

is linear. One says that the map B covers b*. Moreover, if $\mathcal{N} = \mathcal{M}$, we say that the vector bundle map $B\colon P \longrightarrow Q$ is a* bundle map over $\mathcal{N}$*, provided that the diagram*

$$\begin{tikzcd}
P \arrow[rr, "B"] \arrow[dr, "\pi_P"'] & & Q \arrow[dl, "\pi_Q"] \\
& \mathcal{N} &
\end{tikzcd}$$

commutes, i.e. B covers the identity map $\mathrm{id}_{\mathcal{N}}$ *of $\mathcal{N}$.*

Here again, since we will only work with vector bundles and thus vector bundle maps, we shall omit the term *vector* and call these morphisms *bundle maps*. Note how, in this particular text, we do not assume that bundle maps are smooth. This is due to the fact that this work is included in the branch of *geometric analysis*, with emphasis on *analysis*. In the following chapters, we will be working with functions that are not continuous, such as weakly differentiable functions. To make the distinction between a smooth and non-smooth bundle map, we will call a smooth bundle map a *bundle morphism*.

Example 1.3.5. Given a smooth map $f\colon \mathcal{N} \longrightarrow \mathcal{M}$ between two smooth manifolds, the differential of f at $x \in \mathcal{N}$ is a linear map

$$\mathrm{D}f(x)\colon T_x\mathcal{N} \to T_{f(x)}\mathcal{M}.$$

The differential of f between the tangent bundles

$$\begin{gathered}
\mathrm{D}f\colon T\mathcal{N} \to T\mathcal{M}, \\
v \in T_x\mathcal{N} \mapsto \mathrm{D}f(x)[v] \in T_{f(x)}\mathcal{M}
\end{gathered}$$

is a bundle morphism covering f. The differential $\mathrm{D}f$ is sometimes called the *tangent map* of f and denoted Tf. This notation has the advantage that we can consider T as a functor between the categories of smooth manifolds (where arrows are smooth maps) and smooth vector bundles (where arrows are vector bundle morphisms) since

$$\begin{tikzcd}
T\mathcal{N} \arrow[r, "Tf=\mathrm{D}f"] \arrow[rr, bend left, "T(f\circ g)=\mathrm{D}(f\circ g)"] \arrow[d, "\pi_{T\mathcal{N}}"] & T\mathcal{M} \arrow[r, "Tg=\mathrm{D}g"] \arrow[d, "\pi_{T\mathcal{M}}"'] & T\mathcal{L} \arrow[d, "\pi_{T\mathcal{L}}"] \\
\mathcal{N} \arrow[r, "f"'] \arrow[rr, bend right, "f\circ g"'] & \mathcal{M} \arrow[r, "g"'] & \mathcal{L}
\end{tikzcd}$$

commutes, i.e. $T(f \circ g) = Tf \circ Tg$, and $T(\mathrm{id}_{\mathcal{N}}) = \mathrm{id}_{T\mathcal{N}}$. △

Example 1.3.6. For $\lambda \in \mathbb{R}$, we define the map $m_\lambda \colon P \longrightarrow P$ by fiberwise multiplication

$$m_{\lambda|F_x} \colon F_x \longrightarrow F_x \colon v \longmapsto \lambda v,$$

where the multiplication is taken with respect to the linear structure of the fiber F_x. Locally, m_λ is by definition the multiplication by λ on the fiber component. Indeed, if (U, φ) is a trivialisation of P, then

$$\left(\varphi \circ m_\lambda \circ \varphi^{-1}\right)(x, v) = (x, \lambda v).$$

The map m_λ is thus in particular smooth and a diffeomorphism when $\lambda \neq 0$. By definition,

$$\begin{tikzcd} P \arrow[rr, "m_\lambda"] \arrow[dr, "\pi_P"'] & & P \arrow[dl, "\pi_P"] \\ & \mathcal{N} & \end{tikzcd}$$

commutes, i.e. $m_\lambda \colon P \longrightarrow P$ is a bundle map over $\mathcal{N}$. In fact, m_λ is an example of a vector bundle *isomorphism*, that is, an invertible bundle morphism, as long as $\lambda \neq 0$. Its inverse is the bundle morphism $m_{1/\lambda} \colon P \longrightarrow P$. △

Much like subsets of manifolds can be manifolds themselves and therefore called submanifolds, subsets of vector bundles can be vector bundles themselves, in which case they are called *subbundles* [39, Def. 20.1].

Definition 1.3.7 (Subbundle)**.** *Let P be a vector bundle over $\mathcal{N}$ and let Q be a vector bundle over a submanifold $\mathcal{M} \subset \mathcal{N}$. Then Q is called a* subbundle *of P if $Q \subset P$ is a submanifold of P and the inclusion map $i \colon Q \longrightarrow P$ defines a bundle morphism covering the injection $j \colon \mathcal{M} \longrightarrow \mathcal{N}$, that is,*

$$\begin{tikzcd} Q \arrow[r, hook, "i"] \arrow[d, "\pi_Q"'] & P \arrow[d, "\pi_P"] \\ \mathcal{M} \arrow[r, hook, "j"'] & \mathcal{N} \end{tikzcd}$$

commutes. When $\mathcal{M} = \mathcal{N}$, we call Q a full subbundle *of P.*

Different yet equivalent definitions can be found for example in [24, 25]. We chose to work with the definition of [39] as we find it to be the more elegant one. The term *full subbundle* can probably not be found in literature as it is our own designation.

We shall come across two important examples in Section 1.4. For now, we only consider this simple example justifying the terminology introduced in Definition 1.2.11 concerning the bundle $P_{|U}$ above an open set $U \subset \mathcal{N}$.

Example 1.3.8. Let us show that, given an open subset $U \subset \mathcal{N}$, the set $P_{|U} = \pi_P^{-1}(U)$ defines a subbundle of P. Since π_P is smooth and U open, $P_{|U}$ is a submanifold of P. Since P is a vector bundle, one verifies that $P_{|U}$ with $\pi_{P_{|U}} = (\pi_P)_{|U}$ defines a vector bundle by restricting the trivialisations of P to $P_{|U}$. Since the fiber of $P_{|U}$ at $x \in U$ is the same as the fiber of P at x, the injection map $i\colon P_{|U} \to P$ is a bundle morphism. The bundle $P_{|U}$ above U is thus indeed a subbundle of P. △

1.4 Sections, connections and covariant derivatives

Let us consider a trivial bundle $P = \mathbb{R}^n \times \mathbb{R}^m$ and a function $s\colon \mathbb{R}^n \to P$ such that $\pi_P \circ s = \mathrm{id}_{\mathbb{R}^n}$. In other words, we assume that s has the form

$$x \in \mathbb{R}^n \mapsto s(x) = \big(x, f(x)\big)$$

for some function $f\colon \mathbb{R}^n \to \mathbb{R}^m$. We may then observe that s is nothing but the graph of the function f. If we replace $\mathbb{R}^n$ with an arbitrary manifold $\mathcal{N}$, this statement is still true. In particular, if $P = T\mathbb{S}^1 = \mathbb{S}^1 \times \mathbb{R}$ is the tangent bundle to the circle, the map $s\colon \mathbb{S}^1 \to T\mathbb{S}^1$ is a tangent vector field on the circle, that is, for a given point of the circle, it associates one and only one vector tangent to that point.

Moving away from the trivial bundle, the functions $s\colon \mathcal{N} \to P$ with $\pi_P \circ s = \mathrm{id}_{\mathcal{N}}$ are still of great interest. To each point $x \in \mathcal{N}$ of the base space, such a function associates one and only one element of the fiber F_x. Those special functions are called *sections* and formalise the notion of vector fields on a manifold (Example 1.4.2), differential forms (Example 1.4.3) and many more [16, 21, 25, 30, 38, 39].

Definition 1.4.1 (Section). *A* local section (or cross section) *of a bundle P on $U \subset \mathcal{N}$ is a function $s\colon U \to P$ such that the diagram*

$$\begin{tikzcd} U \arrow[rr, "s"] \arrow[dr, hook] & & P \arrow[dl, "\pi_P"] \\ & \mathcal{N} & \end{tikzcd}$$

commutes, i.e. $\pi \circ s(x) = x$ for all $x \in U$.

As mentioned previously, it is worth noting that in contrast to classical works in differential geometry, such as [21, 24, 25, 30, 38, 39], we do not ask for smoothness or continuity. Since our goal is to define gauge covariant Sobolev spaces, see for example [13; 18, Sect. 23.3; 28, Sect. 7.19; 34], restricting ourselves to smooth functions is a rather bad idea. In fact, Sobolev-type spaces often contain functions that are not smooth nor continuous.

Here are some classical examples of sections which emerge naturally and almost immediately in differential geometry [21,24,25,30,38,39]. An undergraduate level differential geometry class or textbook should always mention and study Examples 1.4.2 to 1.4.4 to some extent, even if omitting the notion of vector bundle.

Example 1.4.2 (Tangent vector field)**.** A tangent vector field on $U \subset \mathcal{N}$ is a section $v\colon U \longrightarrow T\mathcal{N}$ of the tangent bundle $T\mathcal{N}$. At each point $x \in U$, $v_x = v(x) \in T_x\mathcal{N}$ is a vector tangent to $\mathcal{N}$ at x. When $\mathcal{N} \subset \mathbb{R}^3$, one can easily visualise such a vector field. △

Example 1.4.3 (Differential form)**.** A differential form on a subset $U \subset \mathcal{N}$ is a section $\omega\colon U \longrightarrow T^*\mathcal{N}$ of the cotangent bundle $T^*\mathcal{N}$. At each point $x \in U$, $\omega_x = \omega(x) \in T^*_x\mathcal{N}$ defines a covector at x, i.e. $\omega_x\colon T_x\mathcal{N} \longrightarrow \mathbb{R}$ is a linear form on the tangent space $T_x\mathcal{N}$. △

Example 1.4.4 (Differential k-form)**.** A differential k-form (or simply k-form), $k \geq 1$, on $U \subset \mathcal{N}$ is a section $\omega\colon U \longrightarrow \bigwedge^k T^*\mathcal{N}$ of the bundle $\bigwedge^k T^*\mathcal{N}$. At each point $x \in U$, $\omega_x = \omega(x) \in \bigwedge^k T^*_x\mathcal{N}$ defines an exterior k-form

$$\omega_x\colon \underbrace{T^*_x\mathcal{N} \times \cdots \times T^*_x\mathcal{N}}_{k \text{ times}} \longrightarrow \mathbb{R}. \qquad \triangle$$

The example of differential k-forms generalises to *vector valued k-forms* [39, Sect. 21].

Definition 1.4.5 (Vector valued exterior form)**.** *Let V, W be finite dimensional vector spaces and let $k \geq 1$. A W* valued exterior k-form *on V is a (k-)multilinear map*

$$\omega\colon \underbrace{V \times \cdots \times V}_{k \textit{ times}} \longrightarrow W$$

such that

$$\omega(v_1, \ldots, v_k) = 0 \in W$$

whenever there exists $i, j \in \{1, \ldots, k\}$ such that $i \neq j$ and $v_i = v_j$. The set of all W valued exterior k-forms on V is denoted by $\bigwedge^k(V^*, W)$.

If $m = \dim W$, a W valued k-form ω can be expressed in any basis $(w_i)_{i=1}^m$ of W as

$$\omega = \sum_{i=1}^m \omega_i w_i\colon \underbrace{V \times \cdots \times V}_{k \text{ times}} \longrightarrow W$$

where, for all $1 \leq i \leq m$, $\omega_i \in \bigwedge^k V^*$ is a $\mathbb{R}$ valued exterior k-form. Similarly to the bundle $\bigwedge^k T^*\mathcal{N}$ of exterior k-forms, one can define the bundle [39, Sect. 21]

$$\bigwedge^k (T^*\mathcal{N}, V) = \bigsqcup_{x \in \mathcal{N}} \bigwedge^k (T_x^*\mathcal{N}, V)$$

for any vector space V.

Example 1.4.6 (Vector valued k-form)**.** A V valued (differential) k-form on $U \subset \mathcal{N}$, where V is a vector space and $k \geq 1$, is a section $\omega \colon U \to \bigwedge^k (T^*\mathcal{N}, V)$ of the bundle $\bigwedge^k (T^*\mathcal{N}, V)$. At each point $x \in U$, $\omega_x = \omega(x) \in \bigwedge^k (T_x^*\mathcal{N}, V)$ defines a V valued exterior k-form

$$\omega_x \colon \underbrace{T_x^*\mathcal{N} \times \cdots \times T_x^*\mathcal{N}}_{k \text{ times}} \to V.$$

If $m = \dim V$ and $(v_i)_{i=1}^m$ is a basis of V, then

$$\omega = \sum_{i=1}^{n} \omega_i v_i \in \bigwedge^k (T^*\mathcal{N}, V)$$

where, for all $1 \leq i \leq m$, $\omega_i \in \bigwedge^k T^*\mathcal{N}$ is a differential k-form (valued in $\mathbb{R}$). △

One can for example see a matrix valued differential k-form as a matrix whose entries are all differential k-forms.

Remark 1.4.7. The reader should be aware that non-degenerate continuous sections do not always exist globally on a manifold. As mentioned in the introduction of this chapter, the famous *Hairy Ball Theorem* (Theorem 1.1.2) states that it is impossible to have a global continuous non-vanishing vector field on the sphere $\mathbb{S}^2$. △

Before moving on with our study of sections, we present, without going into details, a last object that can be viewed as a section. This object is called a *bundle metric* on a vector bundle P [39, Sect. 10.4] and generalises the notion of *Riemannian metric* on a manifold [23], [39, Ch. 1].

Definition 1.4.8 (Bundle metric)**.** *A* bundle metric *on a vector bundle P is a map Ξ that assigns to each $x \in \mathcal{N}$ a scalar product*

$$\Xi_x = \Xi(x) = \langle \cdot \mid \cdot \rangle_x \colon F_x \times F_x \to \mathbb{R}$$

on the fiber F_x. It is required to be smooth in the sense that, if s_1, $s_2\colon U \subset \mathcal{N} \to P$ are two smooth sections, then

$$\begin{aligned}\Xi(s_1, s_2)\colon U &\to \mathbb{R},\\ x &\mapsto \Xi_x\big(s_1(x), s_2(x)\big) = \big\langle s_1(x) \mid s_2(x)\big\rangle_x\end{aligned}$$

is smooth.

Example 1.4.9 (Riemannian metric)**.** A Riemannian metric on a manifold $\mathcal{N}$ is a map Ξ that assigns to each $x \in \mathcal{N}$ a scalar product Ξ_x on the tangent space $T_x\mathcal{N}$. In other words, a Riemannian metric is a bundle metric on the tangent bundle $T\mathcal{N}$. △

With this class of functions, the sections, comes an inconvenience. If we consider a smooth global section $s\colon \mathbb{R}^n \to P = \mathbb{R}^n \times \mathbb{R}^m$ of a trivial bundle, that is, the graph of a smooth function $f\colon \mathbb{R}^n \to \mathbb{R}^m$, its differential at $x \in \mathbb{R}^n$ is a linear map

$$\mathrm{D}s(x)\colon T_x\mathbb{R}^n \to T_{s(x)}P.$$

We may explicitly compute that

$$\begin{aligned}T\mathbb{R}^n &= \mathbb{R}^n \times \mathbb{R}^n,\\ TP &= (\mathbb{R}^n \times \mathbb{R}^m) \times (\mathbb{R}^n \times \mathbb{R}^m)\end{aligned}$$

and, for $(x, v) \in T_x\mathbb{R}^n = \{x\} \times \mathbb{R}^n$,

$$\mathrm{D}s(x)\big[(x, v)\big] = \big(x, f(x), v, \mathrm{D}f(x)[v]\big) \in T_{s(x)}P = \{s(x)\} \times (\mathbb{R}^n \times \mathbb{R}^m)\,.$$

The first three components on the right-hand side are redundant and lead to an unwanted explosion of the dimension. In fact, since s is the graph of f, the important information encoded in the differential $\mathrm{D}s$ is already encoded in the derivative of f, which is the last component of the right-hand side.

In some sense, we would like to consider the section $s\colon \mathbb{R}^n \to P$ as a function valued in $\mathbb{R}^m$ only, that is, to identify it with $f\colon \mathbb{R}^n \to \mathbb{R}^m$. This would enable the use of the usual derivative in the Euclidean space $\mathrm{D}f(x)\colon \mathbb{R}^n \to \mathbb{R}^m$ from real analysis. For a given $v \in \mathbb{R}^n$, we can then look at the graph of $x \mapsto \mathrm{D}f(x)[v]$ which is the map

$$\begin{aligned}\mathbb{R}^n &\to P = \mathbb{R}^n \times \mathbb{R}^m,\\ x &\mapsto \big(x, \mathrm{D}f(x)[v]\big),\end{aligned}$$

which we can observe is a section of P. The map

$$\begin{aligned}\mathrm{D}^P s\colon T\mathbb{R}^n &\to P,\\ (x, v) &\mapsto \big(x, \mathrm{D}f(x)[v]\big)\end{aligned}$$

is then a new kind of derivative for the section s. It is a P *valued* 1-*form* in the sense that, for all $x \in \mathbb{R}^n$,

$$\mathrm{D}^P s(x) \in \mathrm{Lin}(T_x\mathbb{R}^n, F_x)$$

where $F_x = \{x\} \times \mathbb{R}^m$ is the fiber of $P = \mathbb{R}^n \times \mathbb{R}^m$ above x. One writes $\mathrm{D}^K s \in \bigwedge^1(T^*\mathbb{R}^n, P)$.

The operator D^P is a differential operator acting on differentiable sections $C^1_{\pi_P}(\mathcal{N}, P)$ of P which is better suited than the usual differential of functions between manifolds. In some sense, it respects the bundle structure of P. This kind of operator is called a *covariant derivative* on P. The archetype of covariant derivatives is the *Levi-Civita* covariant derivative which can be associated to any Riemannian manifold in a natural manner, see for example [23, Sect. 4.3] or [39, Def. 6.4].

Definition 1.4.10 (Covariant derivative)**.** *A* covariant derivative *on a bundle P is a linear map D^P on the set $C^1_{\pi_P}(\mathcal{N}, P)$ of differentiable sections of P such that, for all $x \in \mathcal{N}$ and $f \in C^1(\mathcal{N}, \mathbb{R})$,*

(i) $\mathrm{D}^P s(x)\colon T_x\mathcal{N} \to F_x\colon v \mapsto \mathrm{D}^P s(x)[v]$ *is linear,*

(ii) $\mathrm{D}^P(fs)(x) = \mathrm{D}f(x)s(x) + f(x)\mathrm{D}^P s(x)$

where $\mathrm{D}f$ *denotes the differential of* f*.*

Our notation for the covariant derivative and the here given definition are slightly different but equivalent to the commonly used ones, see for example [18, Ch. 22], [21, Ch. III] and [38, Sect. 10.5]. We choose to deviate from the conventions for multiple reasons that will appear throughout the text, but will not be explicitly stated.

Note that if $s\colon U \to P$ is a smooth section and if $X\colon U \to T\mathcal{N}$ is a tangent vector field, then

$$\begin{gathered} \mathrm{D}^P s(\cdot)[X]\colon U \to P, \\ x \mapsto \mathrm{D}^P s(x)[X_x] \in F_x \end{gathered}$$

is a section of P.

Example 1.4.11 (Levi-Civita)**.** Let $\mathcal{N}$ be a manifold endowed with a Riemannian metric $\Xi = \langle \cdot \mid \cdot \rangle$. The Levi-Civita covariant derivative (or connection) ∇^{L-C} is the unique covariant derivative on $T\mathcal{N}$ such that, for all smooth tangent vector fields X, $Y\colon \mathcal{N} \to T\mathcal{N}$,

$$\mathrm{D}\big(\Xi(X, Y)\big)(x) = \big\langle \nabla^{L-C} X(x) \mid Y_x \big\rangle_x + \big\langle X_x \mid \nabla^{L-C} Y(x) \big\rangle_x \tag{1.2}$$

and

$$[X; Y] = \nabla^{L-C} Y[X] - \nabla^{L-C} X[Y], \tag{1.3}$$

where $[\cdot\,;\,\cdot]$ is the Lie bracket of vector fields

$$[X;Y]_x = \mathrm{D}Y(x)[X_x] - \mathrm{D}X(x)[Y_x].$$

The Levi-Civita connection is fundamental in Riemannian geometry and completely determined by the formula [39, Thm. 6.6]

$$\begin{aligned}2\big\langle \nabla^{L-C} Y(\cdot)[X] \mid Z\big\rangle = \mathrm{D}\big(\Xi(Y,Z)\big)(\cdot)[X] + \mathrm{D}\big(\Xi(Z,X)\big)(\cdot)[Y] - \mathrm{D}\big(\Xi(X,Y)\big)(\cdot)[Z]\\ - \big\langle X \mid [Y;Z]\big\rangle + \big\langle Y \mid [Z;X]\big\rangle + \big\langle Z \mid [X;Y]\big\rangle\end{aligned}$$

where $X, Y, Z \colon \mathcal{N} \to T\mathcal{N}$ are smooth tangent vector fields. △

The Levi-Civita connection is said to be *compatible with the metric* of $\mathcal{N}$ (or the bundle metric of $T\mathcal{N}$) and *torsion-free* because it satisfies (1.2) and (1.3) respectively. Note that the second relation is a kind of commutator, as it interchanges the tangent vector fields X and Y. This only makes sense because both are sections of the bundle $T\mathcal{N}$. The torsion-free relation (1.3) is thus impossible to define for a covariant derivative on an arbitrary bundle P.

Our goal is now to construct covariant derivatives for arbitrary vector bundles. For this, we first observe that the above map $\mathrm{D}^P s$ can be written as $\mathrm{D}^P s = K \circ \mathrm{D}s$, where

$$K\colon TP = (\mathbb{R}^n \times \mathbb{R}^m) \times (\mathbb{R}^n \times \mathbb{R}^m) \to P = \mathbb{R}^n \times \mathbb{R}^m \tag{1.4}$$

is defined by

$$K(w,x,y,z) = (w,z)$$

and

$$\begin{tikzcd} TP \arrow[r, "K"] \arrow[d, "\pi_{TP}"'] & P \arrow[d, "\pi_P"] \\ P \arrow[r, "\pi_P"'] & \mathbb{R}^n \end{tikzcd}$$

commutes, i.e. K is a bundle map covering the projection π_P.

Let us recall that for all points x of the base space $\mathcal{N}$, the fiber $F_x \subset P$ is a closed submanifold of P. In particular, for any point $p \in F_x \subset P$, the tangent space of the fiber

$$\begin{aligned}T_pF_x = \big\{\, \gamma'(0) \in T_pP \;\big|\; \gamma\colon [-1,1] \to P,\ \gamma(t) \in F_x,\ \gamma(0) = p \,\big\}\\ = \big\{\, v \in T_pP \;\big|\; \mathrm{D}\pi_P(p)[v] = 0 \,\big\}\\ = \ker \mathrm{D}\pi_P(p) \subset T_pP\end{aligned}$$

is a canonical linear subspace of the tangent space T_pP of P. When $P = \mathcal{N} \times F$ is a trivial bundle, we can think of vectors of T_pF_x as being vertical to the base space $\mathcal{N}$. Since a bundle is locally trivial, we can (at least locally) still think of vectors tangent to a fiber as being vertical to $\mathcal{N}$ [21, Ch. II], [23, Ch. 4].

Definition 1.4.12 (Vertical space)**.** *The* vertical space *at $p \in F_x \subset P$ is defined as*

$$V_pP = T_pF_x = \ker \mathrm{D}\pi_P(p).$$

A vector $v \in V_pP$ is called a vertical vector *at $p \in P$.*

Since F_x is a vector space, it is canonically isomorphic to its tangent space, the vertical space, at any point.

Proposition 1.4.13. *For any $p \in P$ with $\pi_P(p) = x$, the linear map*

$$\mathrm{Vert}_p \colon F_x \longrightarrow V_pP \colon v \longmapsto \frac{\mathrm{d}}{\mathrm{d}t}(p + tv)_{|t=0} = v \in V_pP$$

defines an isomorphism and is called the vertical lift*.*

Proof. The map Vert_p is injective. Since $\dim F_x = \dim V_pP$, the conclusion follows. □

Furthermore, the disjoint union of the vertical spaces is also a canonical subbundle of TP.

Definition 1.4.14 (Vertical bundle)**.** *The* vertical bundle *of P is the full subbundle of TP whose fiber at $p \in P$ is the vertical space V_pP. In other words,*

$$VP = \bigsqcup_{p \in P} V_pP.$$

Since we already have a canonical subspace of T_pP of dimension $\dim F$ and $\dim P = \dim \mathcal{N} + \dim F$, we would like to construct another subspace $H_pP \subset T_pP$ of dimension $\dim \mathcal{N}$ with the property

$$T_pP = V_pP \oplus H_pP.$$

For instance, if $P = \mathcal{N} \times F$ is a Cartesian product, then, for all $p = (x, v) \in P$,

$$T_pP = T_x\mathcal{N} \oplus V_pP$$

and we may define

$$H_pP = T_x\mathcal{N} \times \{0\} \subset T_x\mathcal{N} \oplus V_pP = T_pP.$$

Note that we have $H_pP = \ker K_{|T_pP}$ with the map K defined similarly to (1.4).

We are naturally led to a central object in bundle theory : a *connection* [21,23,24,38,39,42]. Three different but equivalent [24, Ch. 12; 42, Sect. 3.3] definitions of a linear connection exist. We choose to work with [42, Def. 3.9] since it is nicely suited for our work. We remind the reader that we use the term *bundle* to refer to *vector bundles*.

Definition 1.4.15 (Linear Connection)**.** *A* linear connection *on a bundle P is a bundle morphism $K\colon TP \to P$ covering the projection $\pi_P\colon P \to \mathcal{N}$*

$$\begin{tikzcd}
TP \arrow[r, "K"] \arrow[d, "\pi_{TP}"'] & P \arrow[d, "\pi_P"] \\
P \arrow[r, "\pi_P"'] & \mathcal{N}
\end{tikzcd}$$

such that for all $p \in P$

$$K \circ \mathrm{Vert}_p = \mathrm{id}_{F_{\pi(p)}} .$$

As usually in this text, the term *linear* will be omitted and thus the term *connection* shall be used.

Definition 1.4.16 (Horizontal space)**.** *Let $K\colon TP \to P$ be a connection. The* horizontal space *H_pP at $p \in F_x$ is defined as the kernel of the restriction*

$$K_{|T_pP}\colon T_pP \to F_x$$

of the connection K to the fiber T_pP,

$$H_pP = \ker K_{|T_pP} \subset T_pP.$$

A vector $v \in H_pP$ is called a horizontal vector *at $p \in P$.*

Proposition 1.4.17. *Let $K\colon TP \to P$ be a connection. Then, for all $p \in P$, $\dim H_pP = \dim \mathcal{N}$ and*

$$T_pP = V_pP \oplus H_pP.$$

Proof. Label $x = \pi_P(p)$. Since $K \circ \mathrm{Vert}_p = \mathrm{id}_{F_x}$,

$$K_{|T_pP}\colon T_pP \to F_x$$

is surjective and

$$V_pP \cap H_pP = \{0\}. \tag{1.5}$$

By the rank-nullity theorem, we have

$$\dim T_pP = \dim F + \dim \mathcal{N} = \dim F_x + \dim H_pP$$

i.e. $\dim H_pP = \dim \mathcal{N}$. Since $\dim V_pP = \dim F$, the conclusion follows with (1.5). ☐

The name *horizontal space* is not only complementary to the vertical space, but also quite intuitive. Indeed, when $P = \mathcal{N} \times F$ is a trivial bundle, we have, for all $p \in F_x$,

$$T_pP = T_x\mathcal{N} \oplus V_pP$$

and a natural connection is given by $\ker K_{|T_pP} = T_x\mathcal{N} \times \{0\}$. In other words, the horizontal space consists of the vectors that are horizontal to $\mathcal{N}$.

Definition 1.4.18 (Horizontal bundle)**.** *Let $K\colon TP \to P$ be a connection. The* horizontal bundle *HP is defined as the full subbundle of TP whose fiber at $p \in P$ is the horizontal space H_pP. In other words,*

$$HP = \bigsqcup_{p\in P} H_pP.$$

Proposition 1.4.19. *Let $K\colon P \to TP$ be a connection. Then the linear map D_K defined for all differentiable sections $s \in C^1_{\pi_P}(\mathcal{N}, P)$ by*

$$\mathrm{D}_K s = K \circ \mathrm{D}s$$

is a covariant derivative on P.

Proof. We have to verify that D_K satisfies Definition 1.4.10. Let $x \in \mathcal{N}$, $c \in \mathbb{R}$ and v, $w \in T_x\mathcal{N}$. Then

$$\begin{aligned}\mathrm{D}_K s(x)[cv + w] = K\big(\mathrm{D}s(x)[cv + w]\big) &= K\big(c\mathrm{D}s(x)[v] + \mathrm{D}s(x)[w]\big)\\ &= cK\big(\mathrm{D}s(x)[v]\big) + K\big(\mathrm{D}s(x)[w]\big) = c\mathrm{D}_K s(x)[v] + \mathrm{D}_K s(x)[w],\end{aligned}$$

that is, $\mathrm{D}_K s(x)\colon T_x\mathcal{N} \to F_x$ is linear.

Let $f \in C^1(\mathcal{N}, \mathbb{R})$. Then, by the Leibniz rule for the differential of fs, we have, for all $v \in T_x\mathcal{N}$

$$\begin{aligned}\mathrm{D}_K(fs)(x)[v] = K\big(\mathrm{D}(fs)(x)[v]\big) &= K\big(\mathrm{D}f(x)[v]s(x) + f(x)\mathrm{D}s(x)[v]\big)\\ &= K\big(\mathrm{D}f(x)[v]s(x)\big) + K\big(f(x)\mathrm{D}s(x)[v]\big)\end{aligned}$$

and since $f(x)$ and $\mathrm{D}f(x)$ are scalars, the linearity of K yields

$$\mathrm{D}_K(fs)(x)[v] = \mathrm{D}f(x)[v]K\big(s(x)\big) + f(x)K\big(\mathrm{D}s(x)[v]\big) = \mathrm{D}f(x)[v]s(x) + f(x)\mathrm{D}_K s(x)[v]$$

where the last equality follows from the definition of D_K and $K \circ \mathrm{Vert}_p = \mathrm{id}_{F_x}$. ☐

In the following paragraph, we use the *Einstein summation* convention, that is, we sum over the variables that appear as sub- *and* superscript. For example, if $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, then

$$(x \mid y) = \sum_{i=1}^{n} x_i y_i = x^i y_i.$$

Let $\mathcal{N}$ be a Riemannian manifold endowed with the Levi-Civita covariant derivative ∇^{L-C}. Let (U, ϕ) be a chart of $\mathcal{N}$ and denote by

$$\partial_i(x) = \mathrm{D}(\phi^{-1})(x)[e_i], \quad 1 \le i \le n$$

the standard basis of $T_x\mathcal{N}$ associated to the chart (U, ϕ), $(e_i)_{1\le i\le n}$ being the standard basis of $\mathbb{R}^n$. One defines the *Christoffel symbols* $(\Gamma^k_{ij})_{1\le i,j,k\le n}$ by [24, Sect. 4.3], [39, Sect. 13.3]

$$\nabla^{L-C}\partial_j(x)[\partial_i(x)] = \Gamma^k_{ij}(x)\partial_k(x).$$

The Christoffel symbols are such that, if $s = s^j\partial_j \colon U \to T\mathcal{N}$ is a smooth tangent vector field on $\mathcal{N}$ and $v = v^i\partial_i(x) \in T_x\mathcal{N}$ is a tangent vector, then

$$\nabla^{L-C}s(x)[v] = \nabla^{L-C}(s^j\partial_j)(x)\big[v^i\partial_i(x)\big] = v^i\big(\mathrm{D}s^j(x)\big[\partial_i(x)\big]\partial_j(x) + s^j(x)\Gamma^k_{ij}\partial_k(x)\big) \quad (1.6)$$

and therefore completely determine the Levi-Civita covariant derivative above the chart (U, ϕ).

To establish the general local expression of a connection, we consider the following definition [39, Sect. 7.6] generalising the idea of the above vector fields

$$\partial_i \colon U \to T\mathcal{N}, \quad 1 \le i \le n$$

which, at each point $x \in U$, form a basis of $T_x\mathcal{N}$, the fiber of the tangent bundle $T\mathcal{N}$ above x.

Definition 1.4.20 (Local frame)**.** *A* (local) frame *of P on an open subset $U \subset \mathcal{N}$ is an ordered collection $(e_1, \dots, e_m)$ of smooth sections $e_i \colon U \to P$ such that, for all $x \in U$, the collection $(e_1(x), \dots, e_m(x))$ forms a basis of the fiber F_x.*

Lemma 1.4.21. *An open subset $U \subset \mathcal{N}$ is a local trivialisation (U, φ) of P if and only if it admits a local frame.*

Proof. Assume (U, φ) is a local trivialisation of P. Let $(f_1, \dots, f_m)$ be a basis of F and define the smooth sections

$$e_i \colon U \longrightarrow P \colon x \mapsto \varphi(x, f_i), \quad 1 \leq i \leq m.$$

Since $\varphi(x, \cdot) \colon F \longrightarrow F_x$ is a linear isomorphism for all $x \in U$,

$$(e_1(x), \dots, e_m(x))$$

defines a basis of F_x and the ordered collection $(e_i)_{1 \leq i \leq m}$ is a frame on U.

Conversely, let $(e_i)_{1 \leq i \leq m}$ be a local frame on U and $(f_i)_{1 \leq i \leq m}$ a basis of F. Let us define

$$\varphi \colon U \times F \longrightarrow \pi_P^{-1}(U)$$

which maps

$$(x, v) = \left(x, \sum_{i=1}^{m} v_i f_i \right) \in U \times F$$

to the element

$$\varphi(x, v) = \sum_{i=1}^{m} v_i e_i(x) \in F_x.$$

By definition, φ restricts to a linear isomorphism from $\{x\} \times F$ to F_x. Its inverse is given by

$$\varphi^{-1}(p) = \left(\pi_P(p), \sum_{i=1}^{m} p_i f_i \right)$$

where $p = \sum_{i=1}^{m} p_i e_i(x) \in F_x$. Since the sections e_i and the projection π_P are smooth, φ is a diffeomorphism and $\left(U, \varphi^{-1}\right)$ a local trivialisation of P. $\square$

We are now able to express any covariant derivative locally [21, Ch. III, Sect. 7], [24, Prop. 12.6]. In physical models involving covariant derivatives, such as *quantum physics* with *electromagnetism*, *Yang-Mills theories* [2, 18, 20, 36, 44, 45] or *general relativity* [16, 30] and related subjects [3, 4, 13, 16, 23, 33, 37, 38], one usually uses the local expression of the covariant derivative as it enables one to do more advanced computations.

Proposition 1.4.22. *Let (U, φ) be a local trivialisation of a vector bundle P endowed with a connection $K \colon TP \longrightarrow P$. Then there exists a* $\mathrm{Lin}(F, F)$ *valued 1-form A,*

$$A(x) \in \bigwedge^1 \left(T_x^* \mathcal{N}, \mathrm{Lin}(F, F)\right),$$

on U such that if $s \colon U \longrightarrow P$ is a differentiable section with $\varphi \circ s(x) = \left(x, \tilde{s}(x)\right)$, we have, for *all* $v \in T_x\mathcal{N}$*,*

$$\varphi \circ \mathrm{D}_K s(x)[v] = \big(x, \mathrm{D}\tilde{s}(x)[v] + A(x)[v]s(x)\big). \tag{1.7}$$

Proof. Let $(f_j)_{1\le j\le m}$ be a basis of F and consider the frame $(e_j)_{1\le j\le m}$ associated to (U,φ) and $(f_j)_{1\le j\le m}$ through Lemma 1.4.21, that is, for $x \in U$,

$$e_j(x) = \mathrm{D}\big(\varphi^{-1}\big)(x)[f_j], \quad 1 \le j \le m.$$

Let $s = \sum_{j=1}^m s_j e_j \colon U \longrightarrow P$ be a smooth section and let $v \in T_x\mathcal{N}$. Then, linearity and the Leibniz rule for covariant derivatives, see Definition 1.4.10, yield

$$\mathrm{D}_K s(x)[v] = \sum_{j=1}^m \mathrm{D}_K(s_j e_j)(x)\big[v\big] = \sum_{j=1}^m \Big(\mathrm{D}s_j(x)\big[v\big]e_j(x) + s_j(x)\mathrm{D}_K e_j(x)[v]\Big). \tag{1.8}$$

We define the for $1 \le j,k \le m$ the 1-forms A_{jk} on U by

$$\mathrm{D}_K e_j(x)[v] = \sum_{k=1}^m A_{jk}(x)[v]e_k(x)$$

which are essentially the analogue of the Christoffel symbols. By (1.8), we have

$$\begin{aligned}&(pr_2 \circ \varphi)\big(\mathrm{D}_K s(x)[v]\big)\\ &\quad= \sum_{j=1}^m \Big(\mathrm{D}s_j(x)[v]f_j + s_j(x)\sum_{k=1}^m A_{jk}(x)[v]f_k\Big) = \mathrm{D}\tilde{s}(x)[v] + \sum_{j=1}^m s_j(x)A_j(x)[v]\end{aligned} \tag{1.9}$$

where $pr_2 \colon U \times F \longrightarrow U$ is the projection on the second component and

$$A_j = \sum_{k=1}^m A_{jk} f_k \in \bigwedge^1(T^*\mathcal{N}, F)$$

is a F valued 1-form on U. Finally, we define the $\mathrm{Lin}(F,F)$ valued 1-form A on U by setting

$$\begin{gathered}A(x)[v] \in \mathrm{Lin}(F,F),\\ w \mapsto A(x)[v]w = \sum_{j=1}^m A_j(x)[v]w_j(x)\end{gathered}$$

for all $x \in U$, $v \in T_x\mathcal{N}$ and $w = \sum_{j=1}^m w_j f_k \in F$. By (1.9), we have

$$(pr_2 \circ \varphi)\big(\mathrm{D}_K s(x)[v]\big) = \mathrm{D}\tilde{s}(x)[v] + A(x)[v]\tilde{s}(x)$$

which is equivalent to (1.7) and therefore yields the conclusion. □

In other words, we have shown that a covariant derivative D_K is completely determined by $\mathrm{D}_K = \mathrm{D} + A$ in the trivialisation (U, φ). This A is a $\mathrm{Lin}(F, F)$ valued 1-form,

$$A\colon U \to \bigwedge^1 \big(T^*\mathcal{N}, \mathrm{Lin}(F, F)\big),$$

and called the *connection 1-form* associated to D_K on the trivialisation (U, φ). We shall from now on always assume to have a connection $K\colon TP \to P$ on our bundle.

We have previously mentioned that a connection $K\colon TP \to P$ separates the tangent bundle TP into the vertical and horizontal subbundles, $T_pP = H_pP \oplus V_pP$. Since we have a vertical lift $\mathrm{Vert}_p\colon F_{\pi_P(p)} \to V_pP$ and $\dim H_pP = \dim \mathcal{N}$, it is natural to ask if one can define a *horizontal lift*. To do this, we begin by considering a notion of *parallelism* [21,24,25,39].

Definition 1.4.23 (Parallel section)**.** *A smooth path* $\tau\colon I = [0, 1] \to P$ *is said to be a* parallel section *along a smooth path* $\gamma\colon [0, 1] \to \mathcal{N}$ *if*

$$\begin{tikzcd} I \arrow[rr, "\tau"] \arrow[dr, "\gamma"'] & & P \arrow[dl, "\pi_P"] \\ & \mathcal{N} & \end{tikzcd}$$

commutes, i.e. $\pi_P \circ \tau = \gamma$*, and, for all* $t \in [0, 1]$*,*

$$\mathrm{D}_K\tau(t) = 0.$$

Lemma 1.4.24. *Let* $\gamma\colon [0, 1] \to \mathcal{N}$ *be a smooth path and consider* $p \in F_{\gamma(0)}$*. Then there exists a unique parallel section* $\tau_\gamma^p\colon [0, 1] \to P$ *along* γ *such that* $\tau_\gamma^p(0) = p$*.*

Proof. We first assume that there exists a local trivialisation (U, φ) of P such that $\gamma\big([0, 1]\big) \subset U$. A parallel section τ along γ must satisfy $\mathrm{D}_K\tau(t)\big[\gamma'(t)\big] = 0$ which one can express in (U, φ) by $\tilde{\tau}'(t) + A\big(\gamma(t)\big)\big[\gamma'(t)\big]\tilde{\tau}(t) = 0$, $\tilde{\tau}$ being the local expression of τ. Since we have the condition $\tau_\gamma^p(0) = p$, $\tilde{\tau}_\gamma^p$ must satisfy the initial value problem

$$\begin{cases} (\tilde{\tau}_\gamma^p)'(t) = -A\big(\gamma(t)\big)\big[\gamma'(t)\big]\tilde{\tau}_\gamma^p & \text{for } t \in [0, 1], \\ \tilde{\tau}_\gamma^p(0) = \varphi(p). \end{cases}$$

The general theory of linear differential equations asserts [22, Thm. 2.1.1] that there exists a unique solution $\tilde{\tau}_\gamma^p \colon [0,1] \to F$ to this problem, and we conclude by setting

$$\tau_\gamma^p(t) = \varphi^{-1}\big(\gamma(t), \tilde{\tau}_\gamma^p(t)\big).$$

When there is no local trivialisation (U, φ) with $\gamma\big([0,1]\big) \subset U$, we may assume by compactness of $\gamma\big([0,1]\big)$ that there exists a finite covering of trivialisations $\big\{(U_i, \varphi_i)\big\}_{1 \leq i \leq k}$ and a partition $0 = t_1 < t_2 < \dots t_k = 1$ such that $\gamma(t) \in U_i$, $t \in [t_i, t_{i+1}]$ and $U_i \cap U_{i+1}$ is non-empty for all $1 \leq i \leq k-1$. By the first part of the proof, we find a solution τ_1 to the problem restricted to $\gamma_{|[0,t_2]}$ with the initial value $\tau_1(0) = p$. By induction, we find a solution τ_i of the problem restricted to $\gamma_{|[t_{i-1},t_i]}$ with the initial value $\tau_i(t_i) = \tau_{i-1}(t_i)$, $2 \leq i \leq k$. The conclusion follows by considering the map τ_γ^p defined by the concatenation of all the τ_i. ☐

Definition 1.4.25 (Horizontal lift)**.** *Let $\gamma \colon [0,1] \to \mathcal{N}$ be a smooth path and let $p \in F_{\gamma(0)}$. The parallel section $\tau_\gamma^p \colon [0,1] \to P$ along γ is called the* horizontal lift *of γ to $p \in F_{\gamma(0)}$.*

The horizontal lift gives a natural way to identify different fibers, which justifies why connections are called *connections*: they connect the different fibers.

Definition 1.4.26 (Parallel transport)**.** *Let $\gamma \colon [0,1] \to \mathcal{N}$ be a smooth path. The* parallel transport Pt_γ *along γ is the operator defined by*

$$\begin{aligned} \mathrm{Pt}_\gamma \colon P &\to C^1\big([0,1], P\big), \\ p &\mapsto \tau_\gamma^p. \end{aligned}$$

For all $t \in [0,1]$, it defines a natural isomorphism

$$\begin{aligned} \mathrm{Pt}_\gamma(t) \colon F_{\gamma(0)} &\to F_{\gamma(t)}, \\ p &\mapsto \mathrm{Pt}_\gamma(t)p = \tau_\gamma^p(t). \end{aligned}$$

The parallel transport Pt_γ along a closed loop γ, $\gamma(0) = \gamma(1) = x$ is however in general not the identity. The isomorphism between fibers is in fact in general highly dependent on the chose path γ. This phenomenon is due to the *curvature* of the connection [39, Sect. 11.1]. This notion of curvature is a generalisation of the curvature of a Riemannian manifold. The fact the the parallel transport is not trivial along a loop is for example visible on the sphere, see Figure 1.4.

Definition 1.4.27 (Curvature)**.** *The* curvature *of the connection K is the 2-form defined for two smooth vector fields $X, Y \colon \mathcal{N} \to T\mathcal{N}$ and a smooth section $s \colon \mathcal{N} \to p$ by*

$$\mathcal{K}_K[X,Y]s(x) = \mathrm{D}_K\big(\mathrm{D}_K s(x)[Y_x]\big)[X_x] - \mathrm{D}_K\big(\mathrm{D}_K s(x)[X_x]\big)[Y_x] - \mathrm{D}_K s(x)\big[[X_x;Y_x]\big].$$

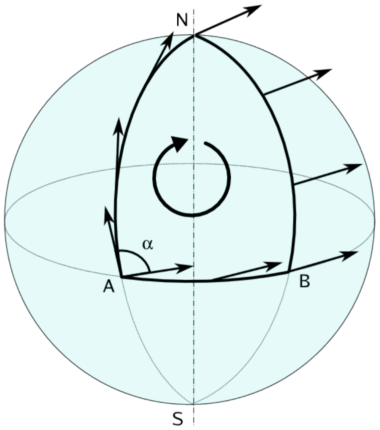


Figure 1.4: Parallel transport of a tangent vector field along a triangle on the sphere (from *Wikipedia : parallel transport*)

The curvature is by definition a kind of commutator as it interchanges the vector fields X and Y. It measures the difference between the parallel transport Pt_{XY} in the direction X_x followed by the direction Y_x and the parallel transport Pt_{YX} in the direction Y_x followed by the direction X_x. As for the covariant derivative, it can be locally expressed in terms of the connection 1-form A.

Proof. Let (U,φ) be a local trivialisation such that the connection $K\colon TP \to P$ is determined by $\mathrm{D}_K = \mathrm{D} + A$. Then $\mathcal{K}_K$ is locally expressed by

$$\mathcal{K}_A(x)[X,Y] = (\mathrm{d}A + A \wedge A)(x)[X,Y]$$

where

$$\mathrm{d}A(x)[X,Y] = \mathrm{D}A(x)[X_x,Y_x] - \mathrm{D}A(x)[Y_x,X_x] - A(x)\big[\mathrm{D}Y(x)[X_x]\big] + A(x)\big[\mathrm{D}X(x)[Y_x]\big]$$

is the exterior derivative of A and

$$(A \wedge A)(x)[X,Y] = A(x)[X_x]A(x)[Y_x] - A(x)[Y_x]A(x)[X_x]$$

is the wedge product. □

Proof. Since locally $\mathrm{D}_K = \mathrm{D} + A$, we have locally, omitting the evaluation point x,

$$\begin{aligned}\mathrm{D}_K\big(\mathrm{D}_K s[Y]\big)[X] &= (\mathrm{D}+A)\big(\mathrm{D}\tilde{s}[Y] + A[Y]\big[X\big]\tilde{s}\big)\\ &= \mathrm{D}^2\tilde{s}[X,Y] + \mathrm{D}\tilde{s}\big[\mathrm{D}Y[X]\big] + \mathrm{D}A[X,Y]\tilde{s} - A[X]A[Y]\tilde{s}.\end{aligned}$$

Doing the same computation for the term $\mathrm{D}_K\big(\mathrm{D}_K s[X]\big)[Y]$ and subtracting, we obtain locally

$$\begin{aligned} \mathrm{D}_K\big(\mathrm{D}_K s[Y]\big)[X] - \mathrm{D}_K\big(\mathrm{D}_K s[X]\big)[Y] \\ = \mathrm{D}A[X,Y]\tilde{s} - \mathrm{D}A[Y,X]\tilde{s} + A[X]A[Y]\tilde{s} - A[Y]A[X]\tilde{s} \\ + \mathrm{D}\tilde{s}\big[\mathrm{D}Y[X]\big] - \mathrm{D}\tilde{s}\big[\mathrm{D}X[Y]\big]. \end{aligned} \tag{1.10}$$

Similarly, we compute locally

$$\mathrm{D}_K s\big[[X;Y]\big] = \mathrm{D}\tilde{s}\big[\mathrm{D}Y[X]\big] - \mathrm{D}\tilde{s}\big[\mathrm{D}X[Y]\big] + A\big[\mathrm{D}Y[X]\big]\tilde{s} - A\big[\mathrm{D}X[Y]\big]\tilde{s}. \tag{1.11}$$

Subtracting (1.11) from (1.10), we obtain

$$\mathcal{K}_A[X,Y]\tilde{s} = \mathrm{d}A[X,Y]\tilde{s} + (A \wedge A)[X,Y]\tilde{s}.$$ □

We conclude this chapter by a characterisation of *metric connections*. For this, we assume that our bundle P is endowed with a bundle metric [39, Sect. 10.5].

Definition 1.4.28 (Metric connection)**.** *Let P be endowed with a bundle metric $\Xi = \langle \cdot \mid \cdot \rangle$. A connection $K\colon TP \longrightarrow P$ is said to be* compatible with the metric Ξ *(or a* metric connection*) if, for all smooth sections $s, t\colon \mathcal{N} \longrightarrow P$,*

$$\mathrm{D}\Xi\big(s(\cdot), t(\cdot)\big)(x) = \big\langle \mathrm{D}_K s(x) \mid t(x)\big\rangle_x + \big\langle s(x) \mid \mathrm{D}_K t(x)\big\rangle_x.$$

We recall that we assume F to be endowed with a scalar product $(\cdot \mid \cdot)$.

Lemma 1.4.29 (Isometric trivialisation)**.** *Let P be a bundle endowed with a bundle metric $\Xi = \langle \cdot \mid \cdot \rangle$. Then there exists a bundle atlas $\big\{(U_\alpha, \varphi_\alpha)\big\}_{\alpha\in\mathcal{A}}$ for P such that φ_α is a fiberwise linear isometry for all $\alpha \in \mathcal{A}$. In other words, for all $\alpha \in \mathcal{A}$ and $x \in U_\alpha$,*

$$\varphi_{\alpha|F_x}\colon F_x \longrightarrow \{x\} \times F$$

satisfies

$$\langle p \mid q \rangle_x = (v \mid w)$$

where $\varphi_\alpha(p) = (x, v)$ and $\varphi_\alpha(q) = (x, w)$. In particular, P has structure group $G \subset O(F)$.

Proof. Let U be a local trivialisation of P. In vertue of Lemma 1.4.21, there exists a frame $(e_i)_{1\le i\le m}$ of P on U. Applying the Gram–Schmidt process, we may assume that $(e_i)_{1\le i\le m}$ is a orthonormal frame, that is, for all $x \in U$,

$$\big\langle e_i(x) \mid e_j(x)\big\rangle_x = \delta_{ij}, \quad 1 \le i, j \le m.$$

Let $(f_i)_{1\le i\le m}$ be an orthonormal basis of F. We define the smooth map

$$\psi\colon U \times F \longrightarrow P_{|U}$$

by mapping (x, f_i) to $e_i(x)$ for all $1 \leq i \leq m$ and extending on each fiber by linearity. In other words, for all $(x, v) \in U \times F$,

$$\psi(x, v) = \sum_{i=1}^{m} v_i e_i(x), \quad v = \sum_{i=1}^{m} v_i f_i.$$

The local trivialisation (U, φ) where $\varphi = \psi^{-1}$ is an isometric trivialisation of P above U. Repeating this construction for a bundle atlas of P yields a bundle atlas $\left\{(U_\alpha, \varphi_\alpha)\right\}_{\alpha \in \mathcal{A}}$ of isometric trivialisations. Since every trivialisation is isometric, the gauge transformations $\{g_{\alpha\beta}\} \subset GL(F)$ all preserve the scalar product of F, i.e. $\{g_{\alpha\beta}\} \subset O(F)$. □

Since the connection 1-form A associated to a connection K depends on the trivialisation (U, φ), it is natural to ask what happens if the connection is metric and the trivialisation φ isometric.

Proposition 1.4.30. *Let P be a bundle endowed with a bundle metric $\Xi = \langle \cdot \mid \cdot \rangle$ and a metric connection $K \colon TP \to P$. Let (U, φ) be an isometric trivialisation. Then, the connection 1-form A is valued in $\mathfrak{o}(F)$, the Lie algebra of $O(F)$. In other words, A is a $\mathfrak{o}(F)$ valued 1-form.*

Proof. Let $s \colon U \to P$ be a smooth section and let $v \in T_x\mathcal{N}$, $x \in U$. By definition of metric connection, we have

$$\mathrm{D}\Xi(s, s)(x)[v] = 2\big\langle \mathrm{D}_K(x)[v] \mid s(x)\big\rangle_x. \tag{1.12}$$

Let us write $\varphi \circ s(x) = \big(x, f(x)\big)$. Since (U, φ) is an isometric trivialisation, we deduce from (1.12) that

$$\mathrm{D}|f|^2(x)[v] = 2\big(\mathrm{D}f(x)[v] \mid f(x)\big) + 2\big(A(x)[v]f(x) \mid f(x)\big) \tag{1.13}$$

but by classical differentiation, we have

$$\mathrm{D}|f|^2(x)[v] = 2\big(\mathrm{D}f(x)[v] \mid f(x)\big) \tag{1.14}$$

and subtracting (1.14) from (1.13), we obtain

$$\big(A(x)[v]f(x) \mid f(x)\big) = 0.$$

Since $f(x) \in F$ is arbitrary, we deduce that

$$\big(A(x)[v]w \mid w\big) = 0$$

for all $w \in F$ which is equivalent to $A(x)[v] \in \mathfrak{o}(F)$. We conclude that A is a $\mathfrak{o}(F)$ valued 1-form,

$$A(x) \in \bigwedge^1\big(T_x^*\mathcal{N}, \mathfrak{o}(F)\big).$$ □

Chapter 2

Bundle valued functional analysis

We introduce in this chapter spaces of sections which generalise well-know spaces from analysis. Precisely, we define in Section 2.1 the *bundle valued Lebesgue space* $L^p_{\pi_P}$ and we briefly present at the beginning of Section 2.2 the *bundle valued Sobolev space* $W^{1,p}_{\pi_P}$, generalising the Lebesgue and Sobolev spaces respectively.

These two spaces set the foundations for our main interest in this thesis : the *gauge covariant Sobolev space* $W^{1,p}_A$. Section 2.2 is mainly dedicated to the gauge covariant Sobolev spaces and some properties of theirs. We then present in Section 2.3 the parallel transport operator which allows to identify different fibers and give geometric meaning to addition between two different fibers.

The connection to Chapter 1 is made by considering that $\mathcal{N}$ is a manifold of dimension $n \geq 1$ and that P is a bundle with projection π_P, base space $\mathcal{N}$ and model fiber F. We furthermore assume that P is endowed with a bundle metric in the sense of Definition 1.4.8 and that F is endowed with a Euclidean scalar product denoted by $(\cdot \mid \cdot)$. We denote by $|\cdot|$ the induced norm on F. Since P is endowed with a bundle metric, we may assume that its structure group is $O(F)$. We consider a connected and isometric local trivialisation (Ω, ω) of P, that is, $\Omega \subset \mathcal{N}$ is connected and $\omega \colon P_{|\Omega} \to \Omega \times F$ is an isometry with respect to the bundle metric of P and the norm of F. We may assume that Ω is a subset of $\mathbb{R}^n$ by using charts for $\mathcal{N}$. Any additional assumptions will be explicitly stated.

It is important to note that we consider this isometric local trivialisation (Ω, ω) as *fixed*. Therefore, we should verify that our definitions and results that follow do not depend on the chosen trivialisation $\omega \colon P_{|\Omega} \to \Omega \times F$. In other words, the analysis we are conducting should remain the same under the transformation

$$v \mapsto \phi v, \quad v \in F$$

for any gauge transformation $\phi \colon \Omega \to O(F)$. This property is called the *gauge invariance* and can be physically interpreted as the invariance under a change of coordinates. The parts that do and do not depend on ω will be made as explicit as possible.

Being familiar with the usual vector valued Lebesgue and Sobolev spaces is strongly recommended. We refer the interested reader to the corresponding chapter(s) in the books [1,6,26,28,43] which contain all the necessary information.

2.1 Bundle valued Lebesgue spaces

The key idea for our work is that a section $s\colon \Omega \to P$ is locally expressed by a vector valued function $f\colon \Omega \to F$. If a notion for vector valued functions is independent of the trivialisation ω, we may transfer it to sections. The following lemma makes this idea rigorous by giving a natural identification between the set of sections and the set of maps to the model fiber above a fixed isometric local trivialisation (Ω, ω).

Lemma 2.1.1. *Let (Ω, ω) be a isometric local trivialisation of P. Then there exists a natural linear isomorphism*

$$\mathrm{Loc}_\omega\colon \{\, s\colon \Omega \to P\colon \pi_P \circ s = \mathrm{id}_\Omega \,\} \to \{\, f\colon \Omega \to F \,\}$$

which preserves the pointwise norm.

Proof. Given a section $s\colon \Omega \to P$,

$$(\omega \circ s)(x) = \big(x, f(x)\big)$$

for some function $f\colon \Omega \to F$. Since ω is a diffeomorphism, the set of sections $s\colon \Omega \to P$ is in one-to-one correspondence with the set of functions $f\colon \Omega \to F$.

Since ω is an isometry and, by definition, restricts to a linear isomorphism on each fiber, the conclusion follows by setting

$$\mathrm{Loc}_\omega(s) = pr_2 \circ \omega \circ s = f,$$

$pr_2\colon \Omega \times F \to F$ being the projection to the second component of $\Omega \times F$. □

Notation 2.1.2. If $\phi\colon \Omega \to O(F)$ is the gauge transformation from a trivialisation (Ω, ω) to a trivialisation $(\Omega, \overline{\omega})$, we write $\overline{\omega} = \phi\omega$ where ϕ acts on the second component of ω, i.e. for $p \in P_{|\Omega}$ with $\omega(p) = (x, v)$

$$\overline{\omega}(p) = \phi\omega(p) = (x, \phi(x)v). \qquad \triangle$$

It follows from the proof of Lemma 2.1.1 that for a gauge transformation $\phi\colon \Omega \to O(F)$ from (Ω, ω) to $(\Omega, \overline{\omega})$, we have

$$\mathrm{Loc}_{\phi\omega} = \phi\, \mathrm{Loc}_\omega$$

which makes Loc_ω, called the *localising map* associated to ω, gauge invariant. For simplicity, we write $\mathrm{Loc} = \mathrm{Loc}_\omega$ as well as $f = \mathrm{Loc}(s)$ and say that f *localises* s in (Ω, ω). In particular, we can define objects on P by localising the sections. For example, a section s will be called measurable if $\mathrm{Loc}(s)$ is measurable — this notion being gauge invariant. This way of localising definitions is nothing new, as it is for example done in the definition of continuity of sections and, more generally, maps between manifolds.

Definition 2.1.3 ($L^p_{\pi_P}$)**.** *Let $p \in [1, +\infty)$. We define the P* valued Lebesgue space *$L^p_{\pi_P}(\Omega, P)$ as the set of all measurable sections $s \colon \Omega \to P$ such that, if $f = \mathrm{Loc}(s)$,*

$$\int_\Omega |f|^p \,\mathrm{d}x < +\infty, \tag{2.1}$$

where two sections $s, \bar{s}$ are identified if $s = \bar{s}$ almost everywhere in Ω.

Similarly to an element of $L^p(\Omega, F)$ which is a *class* of functions equal almost everywhere rather than a single function, an element of $L^p_{\pi_P}(\Omega, P)$ is a *class* of sections equal almost everywhere rather than a single section. Nevertheless, such a class will be referred to as a function or section accordingly. Since (2.1) has the localised map f inside a norm and gauge transformations are isometries, Definition 2.1.3 is gauge invariant. The inequality

$$|a + b|^p \leq 2^{p-1}\big(|a|^p + |b|^p\big)$$

ensures that the vector bundle valued Lebesgue space is a vector space. We endow it with the norm

$$\|s\|_{L^p_{\pi_P}(\Omega,P)} = \left(\int_\Omega |\mathrm{Loc}(s)|^p \,\mathrm{d}x\right)^{1/p}$$

which is also gauge invariant.

Remark 2.1.4. We choose to denote the bundle valued Lebesgue space by $L^p_{\pi_P}(\Omega, P)$ rather than $L^p(\Omega, P)$ for two reasons. The first one being that we do not consider all functions $\Omega \to P$ but only those which are sections, i.e. their composition with the projection π_P is the identity. The second reason is that the notation $L^p(\Omega, P)$ could be misunderstood as the Lebesgue space of maps valued in the *manifold* P. This framework extends to *Sobolev maps into manifolds* which is a currently trending field of research in geometric analysis and partial differential equations (see for example [7] for the particular case of the circle or [11, 41] for a general overview of the field). △

These bundle valued spaces are *not* the usual vector valued Lebesgue spaces. However, the definition is very similar to that of the usual vector valued Lebesgue space $L^p(\Omega, F)$ (see for example [43, Def. 4.2.1]) and entails a priori nothing new. In fact, the localising map Loc restricts to a linear isometry between the two spaces $L^p_{\pi_P}(\Omega, P)$ and $L^p(\Omega, F)$. This allows us to recover many properties of the vector valued Lebesgue spaces such as completeness or Hölder's inequality, see the corresponding chapter(s) in [1,6,26,28,43].

Lemma 2.1.5. *The localising map* Loc *restricts to a linear isometry, again denoted* Loc*,*

$$\mathrm{Loc}\colon L^p_{\pi_P}(\Omega, P) \longrightarrow L^p(\Omega, F)$$

between the bundle valued and vector valued Lebesgue spaces.

The proof essentially follows from inspection of definitions. Since the localising map is gauge invariant, so is the above isometry, which will also be called the localising map.

Notation 2.1.6. When no confusion is possible and with $f = \mathrm{Loc}(s)$, we shall write

$$\|s\|_p = \|s\|_{L^p_{\pi_P}(\Omega,P)} = \|f\|_{L^p(\Omega,F)} = \|f\|_p. \qquad \triangle$$

The key difference between the two spaces is that "$f(x) \pm f(y)$" is always well-defined if $f \in L^p(\Omega, F)$, but if $s \in L^p_{\pi_P}(\Omega, P)$, "$s(x) \pm s(y)$" is well-defined if and only if $x = y$, i.e. $s(x)$ and $s(y)$ lie in the same fiber $F_x = F_y$. This restriction is reflected by the localised map $f = \mathrm{Loc}(s)$ since the quantity $f(x) \pm f(y)$ for $x \neq y$ is in general not gauge invariant. It thus depends on the trivialisation ω and has no real *geometric* meaning. However, this quantity still carries *topological* information such as the continuity of the section s which, in contrast to the quantity itself, is gauge invariant. In order to add or subtract elements of different fibers without losing gauge invariance, that is, geometry, we shall, in Section 2.2, endow P with a connection $K\colon TP \longrightarrow P$. This connection then gives rise to parallel transport and connects the different fibers, allowing a gauge invariant notion of addition and subtraction between fibers, see Section 2.3.

As a consequence, properties such as the continuity of translations in L^p, see for example [6, Lemma 4.3] or [43, Lemma 4.3.8], are not recovered with Lemma 2.1.5. The following definition is, for the same reason, different from the usual definition in the vector valued case.

Definition 2.1.7 ($L^p_{\pi_P,\mathrm{loc}}$)**.** *Let $p \in [1, +\infty)$. We define the set of locally p-integrable sections of P, denoted $L^p_{\pi_P,\mathrm{loc}}(\Omega, P)$, as the set of measurable sections $s\colon \Omega \longrightarrow P$ such that for any compact set $\Sigma \subset \Omega$ one has*

$$\int_\Sigma |f|^p \,\mathrm{d}x < +\infty$$

with $f = \mathrm{Loc}(s)$. As in Definition 2.1.3, we identify sections that are equal almost everywhere.

In a similar manner to $L^p_{\pi_P}$, we note that the above defines a vector space and is again gauge invariant. Correspondingly to Lemma 2.1.5, we have the below result which also follows from inspection of definitions.

Lemma 2.1.8. *For a measurable section* $s\colon \Omega \longrightarrow P$ *and* $f = \mathrm{Loc}(s)$ *we have*

$$s \in L^p_{\pi_P,\mathrm{loc}}(\Omega, P) \iff f \in L^p_{\mathrm{loc}}(\Omega, F).$$

In other words, the localising map Loc from Lemma 2.1.1 restricts to a linear isomorphism from $L^p_{\pi_P,\mathrm{loc}}(\Omega, P)$ to $L^p_{\mathrm{loc}}(\Omega, F)$.

2.2 Gauge covariant Sobolev spaces

In order to define Sobolev type spaces, we need a notion of weak derivation. For a smooth section $s\colon \Omega \longrightarrow P$, we know that its differential, the tangent map $\mathrm{D}s = Ts$ of s, is valued in the tangent bundle TP of P. We could proceed and define weak differentiable sections with the usual differential of maps between manifolds. This definition of weak derivative would come close to that used in the scope of Sobolev maps valued in a manifold [7, 11, 41]. In that framework, one considers that the Riemannian manifold $\mathcal{M}$ is always isometrically embedded in some Euclidean space $\mathbb{R}^\nu$ with $\nu \in \mathbb{N}$ large enough. This may always be assumed by Nash's embedding theorem, see [31, 32]. The $\mathcal{M} \subset \mathbb{R}^\nu$ valued Sobolev space of order k, denoted $W^{k,p}(\Omega, \mathcal{M})$, is then defined as the set of all functions $u \in W^{k,p}(\Omega, \mathbb{R}^\nu)$ such that $u(x) \in \mathcal{M}$ for almost all $x \in \Omega$. In particular, the weak derivative between manifolds is the usual weak derivative. With this definition, one completely avoids the tangent bundle of $\mathcal{M}$, a relatively complicated object. With a similar idea in mind, we will use the local expression of the covariant derivative established in Proposition 1.4.22 and avoid the tangent bundle of P.

Let us recall the definition of the *Hilbert-Schmidt* norm (also called *Frobenius* norm) [1, Def. 6.57; 6, Prob. 40].

Definition 2.2.1 (Hilbert-Schmidt norm)**.** *Let* X, Y *be two finite dimensional Hilbert spaces and denote by* $|\cdot|_Y$ *and* $(\cdot \mid \cdot)_Y$ *the norm and scalar product of* Y*. Let* $\{x_i\}_{i=1}^n$ *and* $\{y_j\}_{j=1}^m$ *be orthonormal bases of* X *and* Y *respectively. Then, the norm*

$$\begin{gathered} \|\cdot\|_{\mathcal{HS}}\colon \mathrm{Lin}(X, Y) \to \mathbb{R}, \\ L \mapsto \|L\|_{\mathcal{HS}} = \left(\sum_{i=1}^n \bigl|L(x_i)\bigr|_Y^2 \right)^{1/2} = \left(\sum_{i=1}^n \sum_{j=1}^m \bigl(y_j \mid L(x_i)\bigr)_Y^2 \right)^{1/2} \end{gathered}$$

on Lin(X, Y) *is called the* Hilbert-Schmidt *norm.*

Remark 2.2.2. The Hilbert-Schmidt norm is induced by the scalar product

$$(L_1 \mid L_2)_{\mathcal{HS}} = \sum_{i=1}^{n} \big((L_1(x_i) \mid L_2(x_i)\big)_Y = \sum_{i=1}^{n} \sum_{j=1}^{m} \big(y_j \mid L_1(x_i)\big)\big(y_j \mid L_2(x_i)\big). \qquad \triangle$$

The definition of the Hilbert-Schmidt norm is independent of the choice of basis and generalises to separable Hilbert spaces (of infinite dimension). For the particular case of this thesis, we need a norm on the space $\mathrm{Lin}(\mathbb{R}^n, F)$ where the weak derivative lives. This space is of finite dimension, which implies that all norms are equivalent and the particular choice of the Hilbert-Schmidt norm is not important. For simplicity and since no confusion is possible, we will write

$$|L| = \|L\|_{\mathcal{HS}}$$

for $L \in \mathrm{Lin}(\mathbb{R}^n, F)$.

We recall the definition of the weak derivative for vector valued mappings. Its main properties can be found in the references [1,6,26,28,43] and will be assumed to be known.

Definition 2.2.3 (Weak derivative)**.** *A function $f \in L^1_{\mathrm{loc}}(\Omega, F)$ is* weakly differentiable *if there exists a function $g \in L^1_{\mathrm{loc}}\big(\Omega, \mathrm{Lin}(\mathbb{R}^n, F)\big)$ such that for any smooth and compactly supported vector field $\varphi \in C_c^\infty(\Omega, \mathbb{R}^n)$ we have*

$$\int_\Omega g[\varphi] \, \mathrm{d}x = -\int_\Omega f \operatorname{div} \varphi \, \mathrm{d}x.$$

If such a function g exists, we call it the weak derivative *of f and write $\nabla f = g$.*

When $\varphi(x) = \lambda(x) e_i$ with $\lambda \in C_c^\infty(\Omega, \mathbb{R})$ and $e_i \in \mathbb{R}^n$ the ith vector of the usual basis of $\mathbb{R}^n$, the definition becomes

$$\int_\Omega \nabla f[e_i] \lambda \, \mathrm{d}x = -\int_\Omega f \partial_i \lambda \, \mathrm{d}x.$$

One says that $\nabla f[e_i]$ is the *ith partial derivative* of f and one writes $\partial_i f = \nabla f[e_i]$, which gives the relation between our definition and the more commonly found in classical works, e.g. [43, Def. 6.1.2]. Since we have endowed $\mathrm{Lin}(\mathbb{R}^n, F)$ with the Hilbert-Schmidt norm, we have

$$\big|\nabla f(x)\big|^2 = \sum_{i=1}^{n} \big|\nabla f(x)[e_i]\big|^2 = \sum_{i=1}^{n} \big|\partial_i f(x)\big|^2.$$

We also recall the definition of the first order vector valued Sobolev spaces. We again assume that its main properties, see [1,6,26,28,43], are known. For simplicity, we write

$\nabla f \in L^p$ rather than $\nabla f \in L^p\big(\Omega, \mathrm{Lin}(\mathbb{R}^n, F)\big)$. There is no confusion possible since the codomain of ∇f is determined by that of $f \in L^p(\Omega, F)$. A similar notation is used for L^p_{loc}.

Definition 2.2.4 ($W^{1,p}$ and $W^{1,p}_{\mathrm{loc}}$)**.** *Let $p \in [1, +\infty)$. The first order vector valued* Sobolev space *is defined by*

$$W^{1,p}(\Omega, F) = \{\, f \in L^p(\Omega, F) \mid \nabla f \textit{ exists and } \nabla f \in L^p \,\}$$

and the local vector valued Sobolev space is defined by

$$W^{1,p}_{\mathrm{loc}}(\Omega, F) = \left\{\, f \in L^p_{\mathrm{loc}}(\Omega, F) \;\middle|\; \nabla f \textit{ exists and } \nabla f \in L^p_{\mathrm{loc}} \,\right\}.$$

Let us think of f as the localised map of a smooth section $s\colon \Omega \to P$, i.e. $f = \mathrm{Loc}(s)$ is smooth. Then ∇f localises $\mathrm{D}s$, the differential of s, which lives in the tangent bundle TP. The gauge transformations are thus the differentials of the change of charts of P, see Example 1.2.6. Since those are smooth, the weak derivation of composition rule entails that the following definition is well-defined.

Definition 2.2.5 ($W^{1,1}_{\pi_P,\mathrm{loc}}$)**.** *We define the set $W^{1,1}_{\pi_P,\mathrm{loc}}(\Omega, P)$ as the set of all sections $s\colon \Omega \to P$ such that* $\mathrm{Loc}(s) \in W^{1,1}_{\mathrm{loc}}(\Omega, F)$*. As usually, two sections are identified if they are equal almost everywhere.*

While this definition makes sense, it does not define a notion of weak derivative ∇s for sections $s \in L^p_{\pi_P}$. One could define it by lifting the weak derivative of $f = \mathrm{Loc}(s)$ by the differential of a chart, see [10, Sect. 4.3] for a similar approach. The weak partial derivatives of s would define sections of TP, an object we wish to avoid. Projecting them onto P via a connection $K\colon TP \to P$ solves this problem and defines a *weak covariant derivative*. By Proposition 1.4.22, we know that for any smooth section $s\colon \Omega \to P$ localised by $f = \mathrm{Loc}(s)$, we have

$$\mathrm{Loc}\big(\mathrm{D}^K s\big) = (\mathrm{D} + A)f = \mathrm{D}_A f,$$

where $A\colon \Omega \to \mathrm{Lin}\big(\mathbb{R}^n, \mathfrak{o}(F)\big)$ is the connection 1-form. Strictly speaking, $\mathrm{D}^K s$ is not a section of P, but a bundle map (see Definition 1.3.4) from the tangent bundle $T\Omega = \Omega \times \mathbb{R}^n$ of Ω to P :

$$\begin{aligned} \mathrm{D}^K s\colon T\Omega &\to P, \\ (x, v) &\mapsto \mathrm{D}^K s(x)\big[v\big], \end{aligned}$$

and

$$\begin{tikzcd}
T\Omega \arrow[rr, "\mathrm{D}^K s"] \arrow[dr, "\pi_{T\Omega}"'] & & P \arrow[dl, "\pi_P"] \\
& \Omega &
\end{tikzcd}$$

commutes. When expressed locally, it becomes

$$\begin{gathered}
\operatorname{Loc}(\mathrm{D}^K s) = \mathrm{D}_A f = (\mathrm{D} + A) f = \mathrm{D} f + A \cdot f \colon \Omega \times \mathbb{R}^n \longrightarrow F, \\
(x, v) \longmapsto \mathrm{D}_A f(x)[v] = \mathrm{D} f(x)[v] + A(x)[v] f(x)
\end{gathered}$$

where we introduced the following notation.

Notation 2.2.6. Let $A \colon \Omega \longrightarrow \operatorname{Lin}\big(\mathbb{R}^n, \mathfrak{o}(F)\big)$ and $f \colon \Omega \longrightarrow F$. We denote by $A \cdot f$ the map

$$\begin{gathered}
A \cdot f \colon \Omega \longrightarrow \operatorname{Lin}(\mathbb{R}^n, F), \\
x \longmapsto A(x)[\cdot] f(x),
\end{gathered}$$

where, for all $x \in \Omega$ and $v \in \mathbb{R}^n$ we have

$$(A \cdot f)(x)[v] = A(x)[v] f(x).$$

We also write, for $\varphi \colon \Omega \longrightarrow \mathbb{R}^n$,

$$\begin{gathered}
(A \cdot f)[\varphi] = A[\varphi] f \colon \Omega \longrightarrow F, \\
x \longmapsto A(x)\big[\varphi(x)\big] f(x).
\end{gathered} \qquad \triangle$$

Definition 2.2.7 (Weak Covariant Derivative)**.** *Let $K \colon TP \longrightarrow P$ be a connection expressed in Ω by $A \in L^\infty_{\mathrm{loc}}\big(\Omega, \operatorname{Lin}\big(\mathbb{R}^n, \mathfrak{o}(F)\big)\big)$. The* weak covariant derivative *of a section $s \in W^{1,1}_{\mathrm{loc}}(\Omega, P)$ localised by $f = \operatorname{Loc}(s)$ is the section $\nabla_K s \in L^1_{\mathrm{loc}}$ localised by*

$$\operatorname{Loc}(\nabla_K s) = \nabla_A f = (\nabla + A) f = \nabla f + A \cdot f \in L^1_{\mathrm{loc}}\big(\Omega, \operatorname{Lin}(\mathbb{R}^n, F)\big).$$

Note that with the Hilbert-Schmidt norm, we have

$$\big|\nabla_A f(x)\big|^2 = \sum_{i=1}^n \big|\nabla f(x)[e_i] + A(x)[e_i] f(x)\big|^2 = \sum_{i=1}^n \big|\partial_i f(x) + A(x)[e_i] f(x)\big|^2$$

where $x \in \Omega$ and e_i denotes the ith vector of the usual basis of $\mathbb{R}^n$. As an element of a Lebesgue space, $\nabla_K s$ is not a section but rather a *class* of sections equal almost everywhere. In general, we will not bother talking about the connection K itself, but only the connection 1-form A. We shall thus not talk about $\nabla_K s$, but only its localised form $\nabla_A f$ which is a gauge invariant notion by the following proposition.

Lemma 2.2.8. *If* $A \in L^\infty_{\mathrm{loc}}\big(\Omega, \mathrm{Lin}\big(\mathbb{R}^n, \mathfrak{o}(F)\big)\big)$ *is the connection* 1*-form of a connection* $K\colon TP \to P$ *in a trivialisation* (Ω, ω)*, then the connection* 1*-form* A' *in the local trivialisation* $(\Omega, \phi\omega)$ *for a gauge transformation* $\phi\colon \Omega \to O(F)$ *is expressed as*

$$A' = \Big(\phi A \cdot \phi^{-1} + \phi\nabla(\phi^{-1})\Big) \in L^\infty_{\mathrm{loc}}\big(\Omega, \mathrm{Lin}\big(\mathbb{R}^n, \mathfrak{o}(F)\big)\big).$$

In particular, if $s \in W^{1,1}_{\pi_P,\mathrm{loc}}(\Omega, P)$ *is localised by* $f = \mathrm{Loc}_\omega(s)$ *and* $g = \phi f = \mathrm{Loc}_{\phi\omega}(s)$ *in* (Ω, ω) *and* $(\Omega, \phi\omega)$ *respectively, we have*

$$\phi[\nabla_A f] = \phi\big[(\nabla + A)f\big] = \Big(\nabla + \phi A \cdot \phi^{-1} + \phi\nabla(\phi^{-1})\Big)\, g = \nabla_{A'} g$$

as well as

$$|\nabla_A f| = |\nabla_{A'} g|.$$

An important consequence is that the quantity

$$\int_\Omega |f|^p + |\nabla_A f|^p \,\mathrm{d}x$$

is gauge invariant. This is the analogue of the Sobolev norm

$$\|f\|^p_{W^{1,p}(\Omega,F)} = \int_\Omega |f|^p + |\nabla f|^p \,\mathrm{d}x$$

and will be the norm on the gauge covariant Sobolev space.

Proof of Lemma 2.2.8. We compute

$$\nabla_A(\phi^{-1}g) = \nabla(\phi^{-1})g + \phi^{-1}\nabla g + A \cdot (\phi^{-1}g) = \phi^{-1}\left[\Big(\nabla + \phi A \cdot \phi^{-1} + \phi\nabla(\phi^{-1})\Big)\, g\right]$$

which yields the conclusion. □

When $A = 0$, i.e. the connection is trivial, we recover the usual weak derivative as shown by Proposition 2.2.10, which can be understood as a gauge covariant integration by parts formula. To state this proposition, we introduce the *gauge covariant divergence* operator which is a generalisation of the divergence operator div.

Definition 2.2.9 (Gauge covariant divergence)**.** *Let* $A\colon \Omega \to \mathrm{Lin}\big(\mathbb{R}^n, \mathfrak{o}(F)\big)$*. We define the* gauge covariant divergence *of* $\varphi \in C^\infty_c(\Omega, \mathbb{R}^n)$ *as the map*

$$\begin{aligned} \operatorname{div}_A \varphi = \operatorname{div}\varphi - A[\varphi]\colon \Omega &\to \mathrm{Lin}(F, F), \\ x &\mapsto \operatorname{div}_A \varphi(x) \end{aligned}$$

where, for all $x \in \Omega$,

$$\begin{aligned} \operatorname{div}_A \varphi(x) &\colon F \to F, \\ v \mapsto \operatorname{div}_A \varphi(x)[v] &= \operatorname{div}\big(\varphi(x)\big)v - A(x)\big[\varphi(x)\big]v. \end{aligned}$$

In particular, when $f \colon \Omega \to F$, we define

$$\begin{aligned} \operatorname{div}_A \varphi[f] &\colon \Omega \to F, \\ x &\mapsto \operatorname{div}_A \varphi(x)\big[f(x)\big]. \end{aligned}$$

Proposition 2.2.10. *Let $s \in W^{1,1}_{\pi_P,\mathrm{loc}}(\Omega, P)$ and $A \in L^\infty_{\mathrm{loc}}\big(\Omega, \operatorname{Lin}\big(\mathbb{R}^n, \mathfrak{o}(F)\big)\big)$. For any smooth and compactly supported vector field $\varphi \in C^\infty_c(\Omega, \mathbb{R}^n)$, we have*

$$\int_\Omega \nabla_A f[\varphi] \, \mathrm{d}x = -\int_\Omega \operatorname{div}_A \varphi[f] \, \mathrm{d}x$$

where $f = \operatorname{Loc}(s)$. Conversely, if $g \in L^1_{\mathrm{loc}}\big(\Omega, \operatorname{Lin}(\mathbb{R}^n, F)\big)$ satisfies

$$\int_\Omega g[\varphi] \, \mathrm{d}x = -\int_\Omega \operatorname{div}_A \varphi[f] \, \mathrm{d}x$$

for all $\varphi \in C^\infty_c(\Omega, \mathbb{R}^n)$, then $g = \nabla_A f$.

Proof. By definition of the weak and weak covariant derivatives, we have

$$\begin{aligned} \int_\Omega \nabla_A f[\varphi] \, \mathrm{d}x &= \int_\Omega \nabla f[\varphi] \, \mathrm{d}x + \int_\Omega A[\varphi] f \, \mathrm{d}x \\ &= -\int_\Omega f \operatorname{div} \varphi \, \mathrm{d}x - \int_\Omega -A[\varphi] f \, \mathrm{d}x = -\int_\Omega \operatorname{div}_A \varphi[f] \, \mathrm{d}x \end{aligned}$$

which is the first part.

Now assume that $g \in L^1_{\mathrm{loc}}\big(\Omega, \operatorname{Lin}(\mathbb{R}^n, F)\big)$ satisfies

$$\int_\Omega g[\varphi] \, \mathrm{d}x = -\int_\Omega \operatorname{div}_A \varphi[f] \, \mathrm{d}x$$

for all $\varphi \in C^\infty_c(\Omega, \mathbb{R}^n)$. Then we have

$$\int_\Omega f \operatorname{div} \varphi \, \mathrm{d}x = -\int_\Omega g[\varphi] - A[\varphi] f \, \mathrm{d}x = -\int_\Omega (g - A \cdot f)[\varphi] \, \mathrm{d}x,$$

that is, $g - A \cdot f = \nabla f$ and thus $g = \nabla_A f$. □

We are now able to define the spaces of interest in this thesis: the gauge covariant Sobolev spaces. These spaces appear naturally in various fields of physics. They appear for example when $n = 3$ and $p = 2$ in *electromagnetism* and *quantum physics* in the presence of a magnetic field M. In this setting, the gauge covariant Sobolev space is referred to as a *magnetic Sobolev space*. Such a magnetic field M is divergence-free, i.e. $\operatorname{div} M = 0$, and can therefore be described by the curl of a *vector potential* (sometimes called a *gauge field*) $B\colon \mathbb{R}^3 \to \mathbb{R}^3$, that is, $M = \nabla \times B$ [3, 18, 30, 38, 44, 45]. The curl of B is, in the language of differential geometry, the exterior derivative $\mathrm{d}B$ of B seen as a vector valued 1-form. The magnetic field $M = \nabla \times B = \mathrm{d}B$ (more precisely, the invariant quantity $\mathrm{id}B$) is thus, in terms of geometry, the *curvature* of a connection on a bundle with an abelian structure group (see for example [38, Ch. 11]). In general in this magnetic setting, one considers a $U(1)$-bundle, that is, a bundle P over an open set $\Omega \subset \mathbb{R}^n$ with model fiber $\mathbb{C} \cong \mathbb{R}^2$ and structure group $U(1) \cong SO(2)$. The covariant derivative induced by the vector potential $B\colon \mathbb{R}^n \to \mathbb{R}^n$ is then given by [3, 4, 13, 18, 28, 33, 34, 38]

$$\nabla_{\mathrm{i}B} = \nabla + \mathrm{i}B \cong \nabla + A \tag{2.2}$$

where

$$\mathrm{i}B\colon \Omega \to \operatorname{Lin}\big(\mathbb{R}^n, \mathfrak{u}(1)\big) \quad \text{and} \quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} B\colon \Omega \to \operatorname{Lin}\big(\mathbb{R}^n, \mathfrak{so}(2)\big). \tag{2.3}$$

Gauge covariant Sobolev spaces on non-abelian bundles also appear naturally in physics, a classical example being *Yang–Mills theories* which seek to describe the behaviour of elementary particles using for example $SU(n)$-bundles [2, 16, 18, 20, 30, 36, 38, 44, 45]. It is thus natural to go beyond the initial abelian setting given by electromagnetism and quantum physics and study gauge covariant Sobolev spaces on bundles with non-abelian structure groups, that is, *non-abelian gauge covariant Sobolev spaces*.

Recall that we assume that (Ω, ω) is an isometric local trivialisation of a bundle P endowed with a connection whose local expression is given by the $\mathfrak{o}(F)$ valued 1-form $A\colon \Omega \to \operatorname{Lin}\big(\mathbb{R}^n, \mathfrak{o}(F)\big)$. In order to have a weak covariant derivative, we assume A to be locally bounded, that is, $A \in L^\infty_{\mathrm{loc}}\big(\Omega, \operatorname{Lin}\big(\mathbb{R}^n, \mathfrak{o}(F)\big)\big)$.

Definition 2.2.11 ($W^{1,p}_A$)**.** *Let $p \in [1, +\infty)$. We define the first order* gauge covariant Sobolev space *$W^{1,p}_A(\Omega, P)$ as the set of all sections $s \in W^{1,1}_{\pi_P,\mathrm{loc}}(\Omega, P)$ such that, if $f = \operatorname{Loc}(s)$,*

$$\int_\Omega |f|^p + |\nabla_A f|^p \,\mathrm{d}x < +\infty.$$

In other words, $s \in W^{1,p}_A(\Omega, P)$ if

$$f \in L^p(\Omega, F) \quad \text{and} \quad \nabla_A f \in L^p\big(\Omega, \operatorname{Lin}(\mathbb{R}^n, F)\big).$$

We sometimes refer to gauge covariant Sobolev spaces as *gauge covariant spaces* for short. We use the term *vector valued spaces* in a similar manner to refer to the usual Sobolev spaces. The particular case of *magnetic Sobolev spaces* is the easiest non-trivial case of gauge covariant Sobolev spaces and has already been the subject of thorough research [3, 4, 13, 18, 23, 28, 33, 34, 38, 44, 45]. We shall sometimes refer to them as *magnetic spaces*.

Definition 2.2.12. *Let $B \in L^{\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}^3)$, let P be a bundle with model fiber $\mathbb{C} \cong \mathbb{R}^2$ and structure group $U(1) \cong SO(2)$ and let $p \in [1, +\infty)$. The first order* magnetic Sobolev space *is defined as the gauge covariant Sobolev space*

$$W^{1,p}_A(\Omega, P) = W^{1,p}_{\mathrm{i}B}(\Omega, P)$$

with the notations of (2.2) *and* (2.3).

Similarly to their classical counter-part the Sobolev spaces, gauge covariant Sobolev spaces are used in the study of partial differential equations and variational problems [14, Ch. 8]. In contrast to the usual spaces, the gauge covariant spaces take into account the presence of a magnetic field or, more generally, a curvature acting on the space. The equations studied are thus slightly different and appear for example in settings of quantum physics such as *magnetic Schrödinger equations* and others [3,4,13,18,23,30,33,38,38,44,45].

As for the bundle valued Lebesgue spaces, the inequality

$$|a + b|^p \leq 2^{p-1}\left(|a|^p + |b|^p\right)$$

ensures that $W^{1,p}_A$ is a vector space. We endow it with the norm

$$\|s\|_{W^{1,p}_A(\Omega,P)} = \left(\|f\|_p^p + \|\nabla_A f\|_p^p\right)^{1/p} = \left(\int_\Omega |f|^p + |\nabla_A f|^p \,\mathrm{d}x\right)^{1/p}$$

where $f = \mathrm{Loc}(s)$. When $A = 0$, we obtain the usual Sobolev norm on $W^{1,p}(\Omega, F)$. The gauge covariant Sobolev space and the here defined norm are gauge invariant in virtue of Lemma 2.2.8. The norm for $p = 2$ is induced by a scalar product $(\cdot \mid \cdot)_{W^{1,2}_A(\Omega,P)}$ defined by

$$(s_1 \mid s_2)_{W^{1,2}_A(\Omega,P)} = \int_\Omega (f_1 \mid f_2) + (\nabla_A f_1 \mid \nabla_A f_2)_{\mathcal{HS}} \,\mathrm{d}x \tag{2.4}$$

where $f_i = \mathrm{Loc}(s_i)$, $i \in \{1, 2\}$, $(\cdot \mid \cdot)$ is the scalar product on F and $(\cdot \mid \cdot)_{\mathcal{HS}}$ is the Hilbert-Schmidt scalar product on $\mathrm{Lin}(\mathbb{R}^n, F)$, see Remark 2.2.2. We set $H^1_A(\Omega, P) = W^{1,2}_A(\Omega, P)$ [28, Sect. 7.20].

In order to avoid cumbersome notations as well as lightening the overall text, we shall completely identify a section s and its localised form $f = \mathrm{Loc}(s)$. In other words, we will write

$$f \in W_A^{1,p}(\Omega, P)$$

instead of

$$s \in W_A^{1,p}(\Omega, P) \quad \text{and} \quad f = \mathrm{Loc}(s).$$

In particular, we have

$$\|f\|^p_{W_A^{1,p}(\Omega,P)} = \|f\|_p^p + \|\nabla_A f\|_p^p.$$

Since these notions are gauge invariant, this shortcut does not generate any problems. These notations are commonly used in literature regarding the theory of magnetic Sobolev spaces and related subjects. The therein given definition of $W_A^{1,p}$ is usually

$$W_A^{1,p}(\Omega, F) = \left\{ f \in W^{1,1}_{\mathrm{loc}}(\Omega, F) \,\middle|\, \int_\Omega |f|^p + |\nabla_A f|^p \,\mathrm{d}x < +\infty \right\} \tag{2.5}$$

which completely omits the existence of the bundle P, see for example [4,13,33,34] as well as [3,18,23,28,30,38,44,45].

Now that we have infinite dimensional normed vector space, one might ask whether it is a Banach space or not. The following lemma is a straightforward adaptation of the *closing lemma* [43, Lemma 6.1.5] and is used in the proof of the completeness of $W_A^{1,p}$.

Lemma 2.2.13. *If (f_k) is a sequence in $W_A^{1,p}(\Omega, P)$ such that*

$$f_k \longrightarrow f \quad \text{and} \quad \nabla_A f_k \longrightarrow g$$

in L^p, then $f \in W_A^{1,p}(\Omega, P)$ and $\nabla_A f = g$.

Proof. Let $\varphi \in C_c^\infty(\Omega, \mathbb{R}^n)$. Then

$$\begin{aligned}\int_\Omega \mathrm{div}_A\, \varphi[f] \,\mathrm{d}x = \lim_{k\to+\infty} \int_\Omega \mathrm{div}_A\, \varphi[f_k] \,\mathrm{d}x &= -\lim_{k\to+\infty} \int_\Omega \nabla_A f_k[\varphi] \,\mathrm{d}x \\ &= -\int_\Omega g[\varphi] \,\mathrm{d}x,\end{aligned}$$

which yields $\nabla_A f = g$ by Proposition 2.2.10. □

Proposition 2.2.14. *The gauge covariant Sobolev space $W_A^{1,p}(\Omega, P)$ endowed with the norm $\|\cdot\|_{W_A^{1,p}(\Omega,P)}$ is a Banach space. In particular, $H_A^1(\Omega, P)$ is a Hilbert space for the scalar product defined in (2.4).*

Proof. For a given Cauchy sequence $(f_n) \subset W_A^{1,p}(\Omega, P)$ we have, by definition of $\|\cdot\|_{W_A^{1,p}}$,

$$\|f_n - f_m\|_p \leq \|f_n - f_m\|_{W_A^{1,p}}$$

and

$$\|\nabla_A f_n - \nabla_A f_m\|_p \leq \|f_n - f_m\|_{W_A^{1,p}}$$

which implies that (f_n) and $(\nabla_A f_n)$ are Cauchy sequences in L^p and hence converge in L^p to $f \in L^p(\Omega, F)$ and $g \in L^p\big(\Omega, \mathrm{Lin}(\mathbb{R}^n, F)\big)$ respectively. We conclude by applying the closing lemma. ☐

Let us consider a magnetic Sobolev space (Definition 2.2.12) on a $U(1)$-bundle P induced by a vector potential $B\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ of the form

$$B = \nabla\varphi \in L^\infty_{\mathrm{loc}}$$

for a certain $\varphi \in C^\infty(\Omega, R)$. In other words, we consider the space

$$W^{1,p}_{\mathrm{i}B}(\Omega, P) = W^{1,p}_{\mathrm{i}\nabla\varphi} = \left\{ f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{C}) \;\middle|\; \int_\Omega |f|^p + |\nabla_{\mathrm{i}B} f|^p \,\mathrm{d}x < +\infty \right\}$$

where $\nabla_A = \nabla + \mathrm{i}B = \nabla + \mathrm{i}\nabla\varphi$. The gauge transformations are, under the assumption that Ω is simply connected, of the form

$$\phi = \mathrm{e}^{\mathrm{i}\sigma}$$

for $\sigma\colon \mathbb{R}^n \to \mathbb{R}$. In particular, let us consider the gauge transformation $\phi = \mathrm{e}^{\mathrm{i}\varphi}$. Under this transformation, we obtain a new vector potential B' which, by Lemma 2.2.8, is given by

$$\mathrm{i}B' = \mathrm{e}^{\mathrm{i}\varphi}\mathrm{i}B\mathrm{e}^{-\mathrm{i}\varphi} + \mathrm{e}^{\mathrm{i}\varphi}\nabla(\mathrm{e}^{-\mathrm{i}\varphi}) = \mathrm{i}B - \mathrm{i}\nabla\varphi = 0.$$

This means that, up the a gauge transformation, our here defined space $W^{1,p}_{\mathrm{i}\nabla\varphi}(\Omega, P)$ is the usual Sobolev space $W^{1,p}_0(\Omega, P) = W^{1,p}(\Omega, \mathbb{C})$. More generally, two gauge covariant spaces should be considered the same if they can be obtained one from another by a gauge transformation as such a transformation is merely a change of trivialisation on the bundle P.

Definition 2.2.15 (Gauge equivalence)**.** *Two gauge covariant Sobolev spaces* $W_A^{1,p}(\Omega, P)$ *and* $W_{A'}^{1,p}(\Omega, P)$ *are* gauge equivalent *if there exists a gauge transformation* $\phi\colon \Omega \to O(F)$ *such that*

$$A' = \big(\phi A \phi^{-1} + \phi\nabla(\phi^{-1})\big).$$

In that case, we write $W_A^{1,p}(\Omega, P) \equiv W_{A'}^{1,p}(\Omega, P)$ *and say that* ϕ *is a gauge transformation from* $W_A^{1,p}(\Omega, P)$ *to* $W_{A'}^{1,p}(\Omega, P)$.

This gauge equivalence defines an equivalence relation. Given a bundle P with a fixed connection $K\colon TP \to P$ and a local trivialisation Ω, the equivalence class of gauge equivalent gauge covariant Sobolev spaces is precisely the collection of all possible spaces on Ω obtained from P with the fixed connection K. If we consider a different connection $\overline{K}\colon TP \to P$, we obtain a different equivalence class.

Proposition 2.2.16. *If $W^{1,p}_A(\Omega, P)$ is gauge equivalent to $W^{1,p}_{A'}(\Omega, P)$, then there exists a natural linear isometry between $W^{1,p}_A(\Omega, P)$ and $W^{1,p}_{A'}(\Omega, P)$.*

Proof. Since $W^{1,p}_A(\Omega, P) \equiv W^{1,p}_{A'}(\Omega, P)$, there exists a gauge transformation $\phi\colon \Omega \to O(F)$ from $W^{1,p}_A(\Omega, P)$ to $W^{1,p}_{A'}(\Omega, P)$. We consider the map sending $f \in W^{1,p}_A(\Omega, P)$ to $\phi f \in W^{1,p}_{A'}(\Omega, P)$ and conclude by Lemma 2.2.8. □

Definition 2.2.17 (Trivial space)**.** *A gauge covariant Sobolev space is* trivial *if it is gauge equivalent to $W^{1,p}_0(\Omega, P) = W^{1,p}(\Omega, F)$.*

When A is essentially bounded on Ω, i.e. $A \in L^\infty\big(\Omega, \operatorname{Lin}\big(\mathbb{R}^n, \mathfrak{o}(F)\big)\big)$, we have a linear and continuous isomorphism between the gauge covariant and vector valued spaces [34, Eq. (1.3)]. This isomorphism will be used in Section 3.6 to define traces of gauge covariant Sobolev spaces on smooth and bounded domains when A is continuous up to the boundary.

Proposition 2.2.18. *If $A \in L^\infty\big(\Omega, \operatorname{Lin}\big(\mathbb{R}^n, \mathfrak{o}(F)\big)\big)$, then*

$$f \in W^{1,p}_A(\Omega, P) \iff f \in W^{1,p}(\Omega, F)$$

and there exists a constant $C = C(p) > 0$ depending only on p such that

$$\frac{1}{C \max\big\{1, \|A\|_\infty\big\}} \|f\|_{W^{1,p}(\Omega,F)} \le \|f\|_{W^{1,p}_A(\Omega,P)} \le C \max\big\{1, \|A\|_\infty\big\} \|f\|_{W^{1,p}(\Omega,F)}.$$

If we use the notations $s \in W^{1,p}_A$ and $\operatorname{Loc}(s) = f$, this proposition can be reformulated by stating that the localising map Loc from Lemma 2.1.1 restricts to a linear and continuous isomorphism between $W^{1,p}_A$ and $W^{1,p}$. Note that since the estimate depends on the L^∞ norm of A, it is *not* a gauge invariant.

Proof of Proposition 2.2.18. We have

$$\|\nabla f\|_p \le C \left(\int_\Omega |\nabla_A f|^p + \|A\|^p_\infty |f|^p \right)^{1/p} \le C \max\big\{1, \|A\|_\infty\big\} \|f\|_{W^{1,p}_A} \tag{2.6}$$

as well as

$$\|\nabla_A f\|_p \leq C\left(\int_\Omega |\nabla f|^p + \|A\|_\infty^p |f|^p\right)^{1/p} \leq C\max\{1, \|A\|_\infty\}\|f\|_{W^{1,p}}. \tag{2.7}$$

The conclusion follows from (2.6) and (2.7). ☐

We end this section by presenting an important inequality for gauge covariant Sobolev spaces and a simple consequence of it. Recall how the assumption that the connection 1-form A is valued in $\mathfrak{o}(F)$, i.e. $A\colon \Omega \to \mathrm{Lin}\big(\mathbb{R}^n, \mathfrak{o}(F)\big)$, is equivalent to the statement that the the connection is *compatible with the metric* (Definition 1.4.28), that is, for any smooth sections $s_i\colon \Omega \to P$ with $f_i = \mathrm{Loc}(s_i)$, $i \in \{1,2\}$, we have

$$\mathrm{D}(f_1 \mid f_2) = (\mathrm{D}_A f_1 \mid f_2) + (f_1 \mid \mathrm{D}_A f_2) \tag{2.8}$$

where $\mathrm{D}_A = \mathrm{D} + A$ is the local expression of the covariant derivative. In fact, by differentiation of bilinear maps, we have

$$\mathrm{D}(f_1 \mid f_2) = (\mathrm{D}f_1 \mid f_2) + (f_1 \mid \mathrm{D}f_2)$$

and thus

$$\mathrm{D}(f_1 \mid f_2) - \big[(\mathrm{D}_A f_1 \mid f_2) + (f_1 \mid \mathrm{D}_A f_2)\big] = -\big[(A\cdot f_1 \mid f_2) + (f_1 \mid A\cdot f_2)\big]$$

which is zero if and only if A is valued in $\mathfrak{o}(F)$. When $f_1 = f_2 = f$, the equality (2.8) reads

$$\mathrm{D}|f|^2 = 2(\mathrm{D}_A f \mid f)$$

which implies the following inequality, known in the magnetic setting as the *diamagnetic inequality* [28, Thm. 7.21].

Proposition 2.2.19. *Let* $A \in L^\infty_{\mathrm{loc}}\big(\Omega, \mathrm{Lin}\big(\mathbb{R}^n, \mathfrak{o}(F)\big)\big)$ *and* $f \in W^{1,p}_A(\Omega, P)$. *Then* $|f| \in W^{1,p}(\Omega, \mathbb{R})$ *and*

$$\big|\nabla|f|(x)\big| \leq \big|\nabla_A f(x)\big|$$

holds for almost all $x \in \Omega$.

Proof. Since A is valued in $\mathfrak{o}(F)$, we have

$$\nabla|f|^2 = 2(\nabla_A f \mid f). \tag{2.9}$$

The left-hand side of (2.9) rewrites as

$$\nabla|f|^2 = 2|f|\nabla|f|.$$

It then follows by the Cauchy–Schwarz inequality that for almost all $x \in \Omega$

$$\left|f(x)\right|\left|\nabla|f|(x)\right| = \left|\left(\nabla_A f(x) \mid f(x)\right)\right| \leq \left|f(x)\right|\left|\nabla_A f(x)\right|$$

and the conclusion readily follows. □

The diamagnetic inequality, or rather *generalised diamagnetic inequality*, defines a map $W_A^{1,p}(\Omega, P) \longrightarrow W^{1,p}(\Omega, \mathbb{R})$ by $f \mapsto |f|$. This map is not linear but still comes with an estimate of the norm, namely

$$\big\| \, |f| \, \big\|_{W^{1,p}(\Omega,\mathbb{R})} \leq \|f\|_{W_A^{1,p}(\Omega,P)}.$$

It furthermore allows us to recover some properties of the vector valued Sobolev spaces without too much work. For example, let us consider the *Sobolev–Gagliardo–Nirenberg inequality* or *embedding* (see for example [6, Thm. 9.9] or [43, Thm. 6.4.4]).

Lemma 2.2.20 (Sobolev–Gagliardo–Nirenberg). *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain of class C^1 or a product of n open intervals and let $1 \leq p < n$. Then, there exists a positive constant $C > 0$ such that, for all $f \in W^{1,p}(\Omega, F)$,*

$$\|f\|_{L^{p^*}} \leq C\|f\|_{W^{1,p}},$$

where $p^* = \frac{np}{n-p}$. In particular, we have

$$W^{1,p}(\Omega, F) \subset L^q(\Omega, F)$$

for all $p \leq q \leq p^*$ and the canonical injection is continuous.

The diamagnetic inequality directly implies this analogue embedding for the gauge covariant spaces.

Corollary 2.2.21 (Gauge covariant S–G–N). *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain of class C^1 or a product of n open intervals and let $1 \leq p < n$. Then, there exists a positive constant $C > 0$ such that, for all $f \in W_A^{1,p}(\Omega, P)$,*

$$\|f\|_{L_{\pi_P}^{p^*}} \leq C\|f\|_{W_A^{1,p}},$$

where $p^* = \frac{np}{n-p}$. In particular, we have

$$W_A^{1,p}(\Omega, P) \subset L_{\pi_P}^q(\Omega, P)$$

for all $p \leq q \leq p^*$ and the canonical injection is continuous.

Proof. Let $f \in W^{1,p}_A(\Omega, P)$. The diamagnetic inequality implies that $g = |f|$ is in $W^{1,p}(\Omega, R)$, and the Sobolev-Gagliardo-Nirenberg inequality yields $g \in L^{p^*}(\Omega, \mathbb{R})$ with

$$\|g\|_{L^{p^*}(\Omega,\mathbb{R})} \leq C\|g\|_{W^{1,p}(\Omega,\mathbb{R})} \leq C\|f\|_{W^{1,p}_A(\Omega,P)}.$$

Since

$$\|f\|_{L^{p^*}_{\pi_P}(\Omega,P)} = \|g\|_{L^{p^*}(\Omega,\mathbb{R})}$$

by definition, the proof is complete. □

2.3 Connecting the fibers

Now that the bundle P is endowed with a connection (locally encoded by $A\colon \Omega \to \mathrm{Lin}\big(\mathbb{R}^n, \mathfrak{o}(F)\big)$), we obtain a natural way to identify fibers connected by a path via *parallel transport*. To discuss properties of the parallel transport, we recall in this section some notions of *linear differential equations* [22, Ch. 2]. We follow an approach similar to [40].

Recall that a smooth section $s\colon \Omega \to P$ is called *parallel* along the smooth path $\gamma\colon [a,b] \subset \mathbb{R} \to \Omega$ if

$$\mathrm{D}_A s\big(\gamma(t)\big) = 0 \quad \text{for } t \in [a,b]$$

or, in other words, $f = \mathrm{Loc}(s)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} f\big(\gamma(t)\big) = -A\big(\gamma(t)\big)\big[\gamma'(t)\big] f\big(\gamma(t)\big)$$

on the interval $[a,b]$.

Definition 2.3.1. *A function $f \in C^1(\Omega, F)$ is called* parallel *along a path $\gamma\colon [a,b] \to \Omega$ if*

$$\frac{\mathrm{d}}{\mathrm{d}t} f\big(\gamma(t)\big) = -A\big(\gamma(t)\big)\big[\gamma'(t)\big] f\big(\gamma(t)\big) \quad \textit{for } t \in [a,b].$$

In other words, f localises a section $s\colon \Omega \to P$ which is parallel along γ. This definition is therefore in particular gauge invariant. Our wish is to construct the *parallel transport* which creates sections parallel to a path based on an initial value. With this goal in mind, we shift our interest to the linear differential equation

$$u'(t) = -A\big(\gamma(t)\big)\big[\gamma'(t)\big] u(t) \quad \text{for } t \in [a,b]. \tag{2.10}$$

Under the assumption that $A \in C^0\big(\Omega, \mathrm{Lin}\big(\mathbb{R}^n, \mathfrak{o}(F)\big)\big)$, there exists a unique solution of (2.10) on $[a,b]$ with initial condition $u(s) = v \in F$, $s \in [a,b]$, see [22, Thm. 2.1.1]. If we assume γ to be continuous and piecewise C^1, then there exists a unique continuous and piecewise C^1 solution u on $[a,b]$. In fact, if γ is C^1 on the intervals $]t_i, t_{i+1}[$, $0 \leq i \leq N-1$ where $a = t_0 < t_1 < \cdots < t_N = b$, the initial value problem

$$\begin{cases} u'(t) = -A\big(\gamma(t)\big)\big[\gamma'(t)\big]u(t) & \text{for } t \in [a,b] \setminus \{t_i : 0 \leq i \leq N\}, \\ u(s) = v \end{cases} \tag{2.11}$$

can be decomposed into the N differential equations

$$\begin{cases} u_0'(t) = -A\big(\gamma(t)\big)\big[\gamma'(t)\big]u_0(t) & \text{for } t \in [a, t_1], \\ u_1'(t) = -A\big(\gamma(t)\big)\big[\gamma'(t)\big]u_1(t) & \text{for } t \in [t_1, t_2], \\ \dots \\ u_{N-1}'(t) = -A\big(\gamma(t)\big)\big[\gamma'(t)\big]u_{N-1}(t) & \text{for } t \in [t_{N-1}, b]. \end{cases}$$

All of these are defined on subsets of $[a,b]$ where γ is smooth. Given an initial condition $u_i(s_i) = v_i$, $s_i \in [t_i, t_{i+1}]$, there is thus a unique solution on $[t_i, t_{i+1}]$. Since $s \in [t_k, t_{k+1}]$ for some $0 \leq k \leq N-1$, the kth equation has a unique solution u_k on the interval $[t_k, t_{k+1}]$. We then have the initial conditions $u_{k-1}(t_k) = u_k(t_k)$ and $u_{k+1}(t_{k+1}) = u_k(t_{k+1})$ which allow us to find the unique solutions of the $(k-1)$th and $(k+1)$th equations (if those equations exist). Repeating this, we obtain all n unique solutions of the N equations. They satisfy by construction $u_i(t_{i+1}) = u_{i+1}(t_{i+1})$ for all $0 \leq i \leq N-1$. The unique continuous and piecewise C^1 solution of (2.11) is then given by

$$u(t) = \begin{cases} u_0(t) & \text{if } t \in [a, t_1], \\ u_1(t) & \text{if } t \in [t_1, t_2], \\ \dots \\ u_{N-1}(t) & \text{if } t \in [t_{N-1}, b]. \end{cases}$$

The discussion made below thus easily generalises to the case of a merely piecewise smooth but continuous path γ. This is important as we will use parallel transport on paths such as triangles in Chapter 3. In order to lighten this section, we will assume that the path γ is smooth unless otherwise stated.

Assume that u_1, $u_2 \colon [a,b] \to F$ are two solutions of (2.10). Then for any scalars α, $\beta \in \mathbb{R}$, we have

$$\big(\alpha u_1(t) + \beta u_2(t)\big)' = -A\big(\gamma(t)\big)\big[\gamma'(t)\big]\big(\alpha u_1(t) + \beta u_2(t)\big),$$

i.e. the set $\mathcal{S}_\gamma$ of solutions of (2.10) is a vector space.

Lemma 2.3.2. *Let $a,b \in \mathbb{R}$, let $\gamma \colon [a,b] \to \Omega$ be a smooth path and assume that $A \in$*

$C^0\big(\Omega, \mathrm{Lin}\big(\mathbb{R}^n, \mathfrak{o}(F)\big)\big)$. *Then, for all* $s \in [a,b]$, *the* evaluation map

$$\begin{aligned} E_\gamma(s)\colon \mathcal{S}_\gamma &\to F, \\ u &\mapsto u(s) \end{aligned}$$

defines a linear isomorphism.

Proof. Since $E_\gamma(s)$ is linear, injectivity is equivalent to $\ker E_\gamma(s) = \{0\}$. Let $u \in \ker E_\gamma(s)$, then

$$\begin{cases} u'(t) = -A\big(\gamma(t)\big)\big[\gamma'(t)\big]u(t) & \text{for } t \in [a,b], \\ u(s) = 0. \end{cases}$$

This equation is also satisfied by the constant zero function and by uniqueness of solutions of linear initial value problems, we conclude that $u = 0$ i.e. $\ker E_\gamma(s) = \{0\}$.

Given $v \in F$, there exists a unique solution u of

$$\begin{cases} u'(t) = -A\big(\gamma(t)\big)\big[\gamma'(t)\big]u(t) & \text{for } t \in [a,b], \\ u(s) = v. \end{cases}$$

It follows that $E_\gamma(s)u = u(s) = v$ and thus $E_\gamma(s)$ is surjective. □

Definition 2.3.3 (Parallel transport)**.** *Let a,b* $\in \mathbb{R}$, *let* $\gamma\colon [a,b] \to \Omega$ *be a smooth path and let* $A \in C^0\big(\Omega, \mathrm{Lin}\big(\mathbb{R}^n, \mathfrak{o}(F)\big)\big)$. *The* parallel transport *along* γ

$$\mathrm{Pt}_\gamma\colon [a,b] \to GL(F)$$

is defined by

$$t \mapsto \mathrm{Pt}_\gamma(t) = E_\gamma(t) \circ \big(E_\gamma(a)\big)^{-1}.$$

We also define the operator $R\colon [a,b]^2 \to GL(F)$ *by*

$$(t,s) \mapsto R_\gamma(t,s) = \mathrm{Pt}_\gamma(t) \circ \big(\mathrm{Pt}_\gamma(s)\big)^{-1} = E_\gamma(t) \circ \big(E_\gamma(s)\big)^{-1}$$

which we also call parallel transport *along* γ.

If we fix a basis of F and express Pt_γ as a matrix in that basis, it is exactly a fundamental matrix solution of the linear differential equation

$$u'(t) = -A\big(\gamma(t)\big)\big[\gamma'(t)\big] \quad \text{for } t \in [a,b], \tag{2.12}$$

see [22, Def. 2.2.3]. It will therefore appear in formulae such as the *variation of parameters* [22, Thm. 2.3.1], see Chapter 3. For simplicity, we omit the subscript γ when no confusion is possible, and we always assume that A is continuous. We will also omit the composition symbol $\circ$ between linear mappings.

Proposition 2.3.4. *If $\gamma^* \colon [a,b] \to \Omega$ is defined by $\gamma^*(t) = \gamma(a+b-t)$, then for all t, $s \in [a,b]$, we have*

$$R_{\gamma^*}(t,s) = R_\gamma(a+b-t, a+b-s).$$

Essentially, this means that if the paths γ and γ^* parametrise the exact same curve $\Gamma = \gamma\big([a,b]\big) = \gamma^*\big([a,b]\big) \subset \Omega$ but in opposite directions, then the induced parallel transport operators will parametrise the same curve $\Gamma_{\mathrm{Pt}} = \mathrm{Pt}_\gamma\big([a,b]\big) = \mathrm{Pt}_{\gamma^*}\big([a,b]\big) \subset GL(F)$ but in opposite directions. The range Γ_{Pt} of the parallel transport from a point $x \in \Omega$ to a point $y \in \Omega$ depends only on the curve Γ joining these two points and not the precise parametrisation γ of Γ itself. In fact, the parametrisation γ of Γ will simply be inherited by Pt_γ which parametrises Γ_{Pt}. In particular, we can always assume $[a,b] = [0,1]$.

Proof of Proposition 2.3.4. Let $u \in \mathcal{S}_\gamma$. Then, defining $u^* \colon [a,b] \to \Omega$ by

$$u^*(t) = u(a+b-t)$$

for $t \in [a,b]$, we have

$$(u^*)'(t) = -A\big(\gamma(a+b-t)\big)\big[-\gamma'(a+b-t)\big]u^*(t) = -A\big(\gamma^*(t)\big)\big[(\gamma^*)'(t)\big]u^*(t).$$

It follows that $u \in \mathcal{S}_\gamma \mapsto u^* \in \mathcal{S}_{\gamma^*}$ is a linear isomorphism. Given $v_0 \in F$, we have on the one hand

$$R_{\gamma^*}(t,s)v_0 = v^*(t)$$

where $v^* \in \mathcal{S}_{\gamma^*}$ is the unique solution with $v^*(s) = v_0$ and, on the other hand,

$$R_\gamma(a+b-t, a+b-s)v_0 = v(a+b-t)$$

where $v \in \mathcal{S}_\gamma$ is the unique solution with $v(a+b-s) = v_0$. Since $v^*(t) = v(a+b-t)$ for all $t \in [a,b]$ and $v_0 \in F$ is arbitrary, the conclusion follows. □

By definition, for a smooth path $\gamma \colon [0,1] \to \Omega$,

$$\begin{aligned} R(t,t) &= \mathrm{id}_F, \\ R(s,t) &= R(t,s)^{-1}, \\ R(t,s) &= R(t,r)R(r,s) \end{aligned}$$

for all r, t, $s \in [0,1]$. Furthermore, $u \colon [0,1] \to F$ is a solution of (2.12) if and only if

$$u(t) = R(t,s)u(s)$$

for all t, $s \in [0,1]$, in particular $u(t) = \mathrm{Pt}(t)u(0)$. Since $v = u(0) \in F$ is constant and arbitrary, this suggests that the map $t \mapsto \mathrm{Pt}(t)v$ is parallel along γ for any $v \in F$.

Proposition 2.3.5 (Derivation of parallel transport)**.** *The parallel transport operator R is C^1 in both variables. Furthermore*

$$\partial_t R(t,s) = -A\big(\gamma(t)\big)\big[\gamma'(t)\big]R(t,s) \quad \textit{and} \quad \partial_s R(t,s) = R(t,s)A\big(\gamma(t)\big)\big[\gamma'(t)\big]$$

for all $t, s \in [0,1]$. In particular, $\mathrm{Pt} \in C^1\big([0,1], GL(F)\big)$ with

$$\mathrm{Pt}'(t) = -A\big(\gamma(t)\big)\big[\gamma'(t)\big]\,\mathrm{Pt}(t) \quad \textit{and} \quad (\mathrm{Pt}^{-1})'(t) = \mathrm{Pt}^{-1}(t)A\big(\gamma(t)\big)\big[\gamma'(t)\big].$$

Proof. Let u be a solution of (2.12). Then, for all $t, s \in [0,1]$,

$$\begin{aligned}\partial_t R(t,s)u(s) &= \partial_t\big(R(t,s)u(s)\big) = u'(t)\\ &= -A\big(\gamma(t)\big)\big[\gamma'(t)\big]u(t) = -\Big[A\big(\gamma(t)\big)\big[\gamma'(t)\big]R(t,s)\Big]u(s).\end{aligned}$$

Since $v = u(s)$ is arbitrary,

$$\partial_t R(t,s) = -A\big(\gamma(t)\big)\big[\gamma'(t)\big]R(t,s). \tag{2.13}$$

For the partial derivative in s, we compute

$$\partial_s\big[R(t,s)R(s,t)\big] = \partial_s\big[R(t,s)\big]R(s,t) + R(t,s)\partial_s\big[R(s,t)\big] = 0,$$

and therefore

$$\partial_s R(t,s) = -R(t,s)\partial_s\big[R(s,t)\big]R(t,s). \tag{2.14}$$

The conclusion follows from (2.13) and (2.14). □

Corollary 2.3.6. *The element $\mathrm{Pt}(t) \in GL(F)$ is an isometry for all $t \in [0,1]$. In other words, we have*

$$\mathrm{Pt}\colon [0,1] \longrightarrow O(F).$$

In particular, $R\colon [0,1]^2 \longrightarrow O(F)$ as well.

In general, it is possible to give a better characterisation of the range of the parallel transport. If the structure group of our bundle P is assumed to be the Lie subgroup $G \subset O(F)$, the connection 1-form A must take its values in $\mathfrak{g} = \mathrm{Lie}(G)$, the Lie algebra of G. The parallel transport is then the unique solution of

$$\begin{cases}\mathrm{Pt}'(t) = -A\big(\gamma(t)\big)\big[\gamma'(t)\big]\,\mathrm{Pt}(t) & \text{for } t \in [0,1],\\ \mathrm{Pt}(0) = \mathrm{id}_F\end{cases}$$

from which one can deduce that $\mathrm{Pt}(t) \in G$ for all $t \in [0,1]$ (see [42, Prop. 3.22]). In the case of a magnetic Sobolev space for example, $G = U(1) \cong SO(2)$, which implies that $\mathrm{Pt}(t)$ is an element of the unit circle in $\mathbb{C}$ or, equivalently, a rotation of the plane $\mathbb{R}^2$.

This enhanced characterisation will not be useful for us as we only need the isometric property of the parallel transport.

Proof of Corollary 2.3.6. Let v_0 and w_0 be arbitrary elements of F. Define the functions $v, w \colon [0,1] \longrightarrow F$ by

$$v(t) = \mathrm{Pt}(t)v_0 \quad \text{and} \quad w(t) = \mathrm{Pt}(t)w_0.$$

These two functions are parallel along γ by Proposition 2.3.5, i.e. they satisfy $\nabla_A v(t) = \nabla_A w(t) = 0$ for all $t \in [0,1]$. Since the connection is compatible with the metric, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(v(t) \mid w(t)\big) = \big(\nabla_A v(t) \mid w(t)\big) + \big(v(t) \mid \nabla_A w(t)\big) = 0$$

which implies that

$$t \in [0,1] \longmapsto \big(v(t) \mid w(t)\big) = \big(\mathrm{Pt}(t)v_0 \mid \mathrm{Pt}(t)w_0\big)$$

is constant. Since $\mathrm{Pt}(0) = \mathrm{id}_F$,

$$\big(\mathrm{Pt}(t)v_0 \mid \mathrm{Pt}(t)w_0\big) = (v_0 \mid w_0)$$

for all $t \in [0,1]$, i.e. $\mathrm{Pt}(t)$ is an isometry and $\mathrm{Pt} \colon [0,1] \longrightarrow O(F)$. □

The parallel transport $R(t,s)$ thus defines a linear isometry on F with the property that for any smooth map $f \colon \Omega \longrightarrow F$ satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t} f\big(\gamma(t)\big) = -A\big(\gamma(t)\big)\big[\gamma'(t)\big] f\big(\gamma(t)\big),$$

i.e. f localises a parallel section, we have

$$f\big(\gamma(t)\big) = R(t,s) f\big(\gamma(s)\big) \in F_{\gamma(t)}.$$

The parallel transport operator $R(t,s)$ thus transports isometrically vectors from the fiber $F_{\gamma(s)}$ to the fiber $F_{\gamma(t)}$, and therefore defines a natural isomorphism between different fibers. This isomorphism however is in general highly dependent on the path γ, or rather the curve it parametrises. This can be visualised with the tangent bundle of the sphere $\mathbb{S}^2$, in which case a section is a tangent vector field to the sphere, see Figure 2.1.

With this isomorphism, we may now add or subtract the different values of a function $f \colon \Omega \longrightarrow F$ localising a section that is not necessary parallel in a gauge invariant manner. In fact, since $R(t,s) f\big(\gamma(s)\big) \in F_{\gamma(t)}$, the operation

$$R(t,s) f\big(\gamma(s)\big) - f\big(\gamma(t)\big) \tag{2.15}$$

is now well-defined on the bundle P itself (not only in the trivialisation) and is a *geometric* quantity in contrast to $f\big(\gamma(s)\big) - f\big(\gamma(t)\big)$.

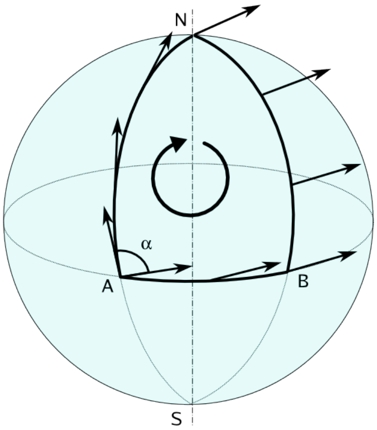


Figure 2.1: Parallel transport of a tangent vector field along a triangle on the sphere (from *Wikipedia : parallel transport*)

It is worth mentioning that a parallel section is in some sense constant (with respect to the connection) along the path since (2.15) is always zero. When the connection is trivial, i.e. the covariant derivative is the usual derivative, parallel sections along a path are *exactly* the constant functions along that path. The translation from a fiber to another given by the parallel transport is thus in a certain manner constant (with respect to the connection).

Before giving an important result involving the geometric quantity (2.15) presented above, we establish the gauge invariance of the parallel transport.

Proposition 2.3.7. *Let* $\phi\colon \Omega \to O(F)$ *be the gauge transformation from* (Ω, ω) *to* $(\Omega, \phi\omega)$*. If* Pt_ω *and* $\mathrm{Pt}_{\phi\omega}$ *are the parallel transports along a path* $\gamma\colon [0,1] \to \Omega$ *in the respective trivialisations, then*

$$\mathrm{Pt}_{\phi\omega}(t) = \phi\big(\gamma(t)\big)\,\mathrm{Pt}_\omega(t)\phi(x)^{-1}$$

for all $t \in [0,1]$*, where* $x = \gamma(0)$ *is the starting point of* γ*. In particular, for all* $v \in F_x$ *expressed in* (Ω, ω)*, we have, for all* $t \in [0,1]$*,*

$$\mathrm{Pt}_{\phi\omega}(t)w = \phi\big(\gamma(t)\big)\,\mathrm{Pt}_\omega(t)v,$$

where $w = \phi(x)v$ *is* v *expressed in the trivialisation* $(\Omega, \phi\omega)$*.*

Proof. By Lemma 2.2.8 and Proposition 2.3.5, $\mathrm{Pt}_{\phi\omega}$ is the unique solution of

$$\begin{cases} \mathrm{Pt}'_{\phi\omega}(t) = -\Big[\phi_\gamma(t)A_\gamma(t)\big[\gamma'(t)\big]\phi_\gamma(t)^{-1} + \phi_\gamma(t)({\phi_\gamma}^{-1})'(t)\Big]\,\mathrm{Pt}_{\phi\omega}(t) & \text{for } t \in [0,1], \\ \mathrm{Pt}_{\phi\omega}(0) = \mathrm{id}_F, \end{cases} \tag{2.16}$$

where $\phi_\gamma = \phi \circ \gamma$ and $A_\gamma = A \circ \gamma$. Since

$$\nabla(\phi)\phi^{-1} = -\phi\nabla(\phi^{-1}),$$

it follows that, with $T(t) = \phi_\gamma(t)\,\mathrm{Pt}_\omega(t)\phi(x)^{-1}$,

$$\begin{aligned} T'(t) &= \Big[(\phi_\gamma)'(t)\phi_\gamma(t)^{-1} - \phi_\gamma(t)A_\gamma(t)\big[\gamma'(t)\big]\phi_\gamma(t)^{-1}\Big]T(t) \\ &= -\Big[\phi_\gamma(t)({\phi_\gamma}^{-1})'(t) + \phi_\gamma(t)A_\gamma(t)\big[\gamma'(t)\big]\phi_\gamma(t)^{-1}\Big]T(t). \end{aligned} \tag{2.17}$$

Finally, since $\mathrm{Pt}_\omega(0) = \mathrm{id}_F$,

$$T(0) = \phi_\gamma(0)\,\mathrm{Pt}_\omega(0)\phi(x)^{-1} = \mathrm{id}_F\,. \tag{2.18}$$

By (2.16), (2.17) and (2.18), we deduce that $T = \phi_\gamma(\cdot)\,\mathrm{Pt}_\omega(\cdot)\phi(x)$ and $\mathrm{Pt}_{\phi\omega}$ satisfy the same initial value problem. The conclusion follows by the uniqueness of the solution, see [22, Thm. 2.1.1]. □

The notation $R(t,s)$ was such that

$$u'(t) = -A\big(\gamma(t)\big)\big[\gamma'(t)\big]u(t) \quad \text{for } t \in [0,1],$$

if and only if

$$u(t) = R(t,s)u(s)$$

for all t, $s \in [0,1]$. When $f\colon \Omega \to F$ is parallel along γ, the latter equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} f\big(\gamma(t)\big) = -A\big(\gamma(t)\big)\big[\gamma'(t)\big]f\big(\gamma(t)\big)$$

if and only if

$$f\big(\gamma(t)\big) = R(t,s)f\big(\gamma(s)\big)$$

which suggests to write $R\big(\gamma(t),\gamma(s)\big) = R(t,s)$ so that

$$f\big(\gamma(t)\big) = R\big(\gamma(t),\gamma(s)\big)f\big(\gamma(s)\big).$$

This notation becomes particularly elegant when $t = 1$ and $s = 0$. Indeed, denoting the endpoints of γ by $y = \gamma(0)$ and $x = \gamma(1)$, we have

$$f(x) = R(x,y)f(y).$$

The operator $R_\gamma(x,y)$ can be read as the parallel transport from y to x along the path γ. This makes it easier to know in which fiber we are working. Note that with this notation we have, by Definition 2.3.3,

$$\begin{aligned} R(z_1, z_1) &= \mathrm{id}_F, \\ R(z_2, z_1) &= R(z_1, z_2)^{-1}, \\ R(z_1, z_3) &= R(z_1, z_2)R(z_2, z_3) \end{aligned}$$

for any $z_i \in \gamma\big([0,1]\big)$, $i \in \{1,2,3\}$. With this notation, we also have an elegant expression for a generalisation of the fundamental theorem of calculus (FTC for short) [34].

Notation 2.3.8. For any $x, y \in \mathbb{R}^n$, we denote by $\gamma_{x,y} \colon [0,1] \longrightarrow \mathbb{R}^n$ the path from x to y defined by

$$\gamma_{x,y}(t) = (1-t)x + ty. \qquad \triangle$$

Proposition 2.3.9 (Gauge covariant FTC). *Let $\Omega \subset \mathbb{R}^n$ be convex and let $f \in C^1(\Omega, F)$. Then for all $x, y \in \Omega$, we have*

$$R(x,y)f(y) - f(x) = \int_0^1 R\big(x, \gamma_{x,y}(t)\big)\nabla_A f\big(\gamma_{x,y}(t)\big)\big[y-x\big]\,\mathrm{d}t$$

where R is the parallel transport along $\gamma_{x,y}$.

Since for all $t \in [0,1]$

$$\big(x, \gamma_{x,y}(t)\big)\nabla_A f\big(\gamma_{x,y}(t)\big)\big[y-x\big] \in F_x$$

lies in F_x, this integral makes sense as an integral on P itself, not just in the trivialisation. In other words, it gauge invariant, it is a geometric quantity. Without the parallel transport, we would be integrating

$$\nabla_A f\big(\gamma_{x,y}(t)\big)\big[y-x\big] \in F_{\gamma_{x,y}(t)}$$

which are in general elements of different fibers, i.e. different vector spaces. Addition between different fibers is not geometrically defined, that is, on P. In particular, one cannot integrate between different fibers without losing the gauge invariance. Similarly, the usual convolution

$$(\varphi * f)(x) = \int_\Omega \varphi(x-y)f(y)\,\mathrm{d}y, \quad \varphi \in C_c^\infty(\Omega, \mathbb{R})$$

does not define a section as it is not gauge invariant. One can however still define a gauge invariant convolution-like operation by setting

$$(\varphi *_A f)(x) = \int_\Omega \varphi(x-y)R_{\gamma_{x,y}}(x,y)f(y)\,\mathrm{d}y, \quad \varphi \in C_c^\infty(\Omega, \mathbb{R}).$$

This kind of operation will be used for example for extensions of gauge covariant Sobolev spaces mappings, see Section 3.4 as well as [34].

Proof of Proposition 2.3.9. Let us define the function $v\colon [0,1] \to F$ by setting

$$v(t) = R\big(x, \gamma_{x,y}(t)\big) f\big(\gamma_{x,y}(t)\big) \tag{2.19}$$

for all $t \in [0,1]$. By the chain rule and Proposition 2.3.5, we have

$$\begin{aligned} v'(t) = R\big(x, \gamma_{x,y}(t)\big) A\big(\gamma_{x,y}(t)\big)[y-x] f\big(\gamma_{x,y}(t)\big) + R\big(x, \gamma_{x,y}(t)\big) \nabla f\big(\gamma_{x,y}\big)[y-x] \\ = R\big(x, \gamma_{x,y}(t)\big) \nabla_A f\big(\gamma_{x,y}(t)\big)[y-x]. \end{aligned} \tag{2.20}$$

By the fundamental theorem of calculus, we have

$$v(1) - v(0) = \int_0^1 v'(t)\,\mathrm{d}t, \tag{2.21}$$

and the conclusion follows from (2.19), (2.20) and (2.21). □

The parallel transport in general depends on the path γ and not only on the endpoints. In particular, parallel transport along a closed curve has no reason to be the identity. Given a closed curve $\gamma\colon [0,1] \to \Omega$, $\gamma(0) = \gamma(1) = x$, the parallel transport $\mathrm{Pt}_\gamma(1)$ defines an isomorphism from the fiber above x to itself. This isomorphism $\mathrm{Pt}_\gamma(1)$ is called the *holonomy* of the connection along γ. The set of all the holonomies at a point x forms a subgroup of $O(F_x)$ which is called the *holonomy group* at x, see [21, Ch. II] as well as [37, Def. 1.7.6]. An important problem is finding estimates for the difference between a vector $v \in F$ and its transported self $\mathrm{Pt}_\gamma(1)v$, i.e. finding estimates for

$$\big|\mathrm{Pt}_\gamma(1)v - v\big| \le \big|\mathrm{Pt}_\gamma(1) - \mathrm{id}_F\big|\,|v|.$$

When $A \in C^1\big(\Omega, \mathrm{Lin}\big(\mathbb{R}^n, \mathfrak{o}(F)\big)\big)$ estimates involving the *curvature* of the connection can be made, see Section 3.2 as well as [8].

Definition 2.3.10 (Curvature). *Let $A \in C^1\big(\Omega, \mathrm{Lin}\big(\mathbb{R}^n, \mathfrak{o}(F)\big)\big)$. The* curvature *of the connection ∇_A is defined as the $\mathfrak{o}(F)$ valued 2-form*

$$\mathcal{K}_A\colon \Omega \to \mathrm{Bilin}\big(\mathbb{R}^n \times \mathbb{R}^n, \mathfrak{o}(F)\big)$$

defined by $\mathcal{K}_A = \mathrm{d}A + A \wedge A$ where $\mathrm{d}A$ is the exterior derivative of A and $\wedge$ denotes the the wedge product. In other words, for $(x, v, w) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^n$, we have

$$\begin{aligned} \mathcal{K}_A(x)[v,w] &= \mathrm{d}A(x)[v,w] + A(x)[v] \wedge A(x)[w] \\ &= \mathrm{D}A(x)[v,w] - \mathrm{D}A(x)[w,v] + A(x)[v]A(x)[w] - A(x)[w]A(x)[v] \end{aligned}$$

where

$$\mathrm{D}A(x)[v,w] = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} A(x+tv)[w].$$

In the particular case of a magnetic Sobolev space or, more generally, when the structure group G is abelian, the term $A \wedge A$ is always identically zero as the Lie algebra is abelian. For a magnetic Sobolev space, the curvature is therefore given by

$$\mathcal{K}_A = \mathrm{d}A$$

as we have already mentioned in Section 2.2. From now on, we assume the connection 1-form A to be C^1 in order to have curvature.

The gauge invariance of the curvature is easily established with a simple yet lengthy computation using Lemma 2.2.8 and the fact that for a gauge transformation ϕ one has $\phi\nabla(\phi^{-1}) = -\nabla(\phi)\phi^{-1}$.

Proposition 2.3.11. *Let* $\phi\colon \Omega \to O(F)$ *be the gauge transformation from* (Ω, ω) *to* $(\Omega, \phi\omega)$. *If* $\mathcal{K}_A$ *and* $\mathcal{K}_{A'}$ *denote the curvature* 2*-forms in* (Ω, ω) *and* $(\Omega, \phi\omega)$ *respectively, then, for all* $x \in \Omega$,

$$\mathcal{K}_{A'}(x) = \phi(x)\mathcal{K}_A\phi(x)^{-1}.$$

The idea of curvature is that it controls how much a vector $v \in F_x$ is different from itself after being transported along a small closed loop $\gamma\colon [0,1] \to \Omega$, $\gamma(0) = \gamma(1) = x$. For example, if γ is the parametrisation of the sides of a closed diamond $Q \subset \Omega$ of area $\epsilon > 0$, then

$$\left|\mathrm{Pt}_\gamma(1) - \mathrm{id}_F\right| \le \epsilon \sup_{x\in Q} |\mathcal{K}_A(x)|.$$

It follows that when the curvature is bounded on Ω, i.e. $\|\mathcal{K}_A\|_\infty < +\infty$, one can obtain upper bounds which are invariant under translations. To establish these bounds in Section 3.2, we need the notion of *amplitude of holonomy* [8].

Definition 2.3.12 (Amplitude of holonomy)**.** *Let* $\gamma\colon [0,1] \to \Omega$ *be a closed loop,* $x = \gamma(0) = \gamma(1)$. *The* amplitude of holonomy $\langle\!\langle\gamma\rangle\!\rangle$ *of* γ *is defined by*

$$\langle\!\langle\gamma\rangle\!\rangle = \inf\left\{ \mathcal{E}_A(g) : g \in C^1\big([0,1], O(F)\big) \text{ and } g \sim_* \mathrm{Pt}_\gamma \right\}$$

with the gauge covariant energy

$$\mathcal{E}_A(g) = \int_0^1 \left|\frac{\mathrm{d}}{\mathrm{d}t} g(t) + A\big(\gamma(t)\big)\big(g(t)\big)\big[\mathrm{D}\pi_{O(F)}\big(g(t)\big)[g'(t)]\big]g(t)\right| \mathrm{d}t$$

where $\pi_{O(F)}$ *is the projection of the bundle* $\Omega \times O(F)$ *and* $g_1 \sim_* g_2$ *means that the two paths* $g_1, g_2\colon [0,1] \to O(F)$ *are homotopic with respect to their endpoints.*

The path g in the energy $\mathcal{E}_A$ also has a base point $\pi_{O(F)}\big(g(t)\big) \in \Omega$, but it is not necessary a lift of the path γ, meaning that it is not required that $\pi_{O(F)} \circ g$ is the same as γ. The only condition on $\pi_{O(F)} \circ \gamma$ is essentially inherited by the condition $g \sim_* \mathrm{Pt}_\gamma$ and is that $\pi_{O(F)}$ is homotopic to γ in Ω. The condition that g and Pt_γ are homotopic with respect to their endpoints in particular implies that $g(0) = \mathrm{id}_F$ and $g(1) = \mathrm{Pt}_\gamma(1)$. If the structure group of the bundle P is specified to be $G \subset O(F)$, one can replace $O(F)$ in the definition by G [8]. When the group is simply connected, the amplitude of holonomy corresponds to the geodesic distance in G between id_F and $\mathrm{Pt}_\gamma(1)$. In particular, we have

$$\left|\mathrm{Pt}_\gamma(1) - \mathrm{id}_F\right| \leq \langle\!\langle \gamma \rangle\!\rangle$$

where the left hand-side is the operator norm, the metric on $\mathrm{Lin}(F, F)$. If the structure group is abelian, such as in the case of magnetic Sobolev spaces, the amplitude of holonomy can be computed by the integral formula [8]

$$\langle\!\langle \gamma \rangle\!\rangle = \left| \int_0^1 A\big(\gamma(t)\big)\big[\gamma'(t)\big] \,\mathrm{d}t \right|.$$

This was in particular used in [34] to establish a key estimate in the proof of the trace theorem for magnetic Sobolev spaces, see [34, Lemma 2.2]. Section 3.2 is dedicated to establishing the equivalent of that key estimate in order to prove the trace theorem for non-abelian gauge covariant Sobolev spaces.

We conclude this chapter with the proof that the amplitude of holonomy is independent of the particular choice of gauge.

Proposition 2.3.13. *The amplitude of holonomy is gauge invariant.*

Proof. Let (Ω, ω) and $(\Omega, \phi\omega)$ be two local trivialisations of P, $\phi\colon \Omega \to O(F)$ being the gauge transformation. We denote by Pt^ω and $\mathrm{Pt}^{\phi\omega}$ the parallel transports along γ in their respective trivialisation and we label $x = \gamma(0)$. Let $g \in C^1\big([0,1], O(F)\big)$ such that $g \sim_* \mathrm{Pt}^\omega_\gamma$. In vertue of Proposition 2.3.7, this implies that

$$\tilde{g}(\cdot) = \phi\big(\gamma(\cdot)\big) g(\cdot) \phi(x)^{-1} \sim_* \mathrm{Pt}^{\phi\omega}, \quad \tilde{g} \in C^1\big([0,1], O(F)\big)$$

and we compute

$$\tilde{g}'(t) = \nabla\phi\big(\gamma(t)\big)\big[\gamma'(t)\big] g(t)\phi(x)^{-1} + \phi\big(\gamma(t)\big) g'(t) \phi(x)^{-1}.$$

In vertue of Lemma 2.2.8, we then have, since $\pi_O(F) \circ g = \pi_{O(F)} \circ \tilde{g}$

$$\begin{aligned}
\mathcal{E}_{A'}(\tilde{g}) = \int_0^1 \Big| & \phi'_\gamma(t) g(t) \phi(x)^{-1} + \phi_\gamma(t) g'(t) \phi(x)^{-1} \\
& + \phi_\gamma(t) A_\gamma(t) \big[\mathrm{D}\pi_{O(F)}[g'(t)] \big] \phi_\gamma(t)^{-1} \phi_\gamma(t) g(t) \phi(x)^{-1} \\
& - \phi'_\gamma(t) \phi_\gamma(t)^{-1} \phi_\gamma(t) g(t) \phi(x)^{-1} \Big| \, \mathrm{d}t \\
= \int_0^1 & \Big| \phi_\gamma(t) g' \phi(x)^{-1} + \phi_\gamma(t) A_\gamma(t) \big[\mathrm{D}\pi_{O(F)}[g'(t)] \big] g(t) \phi(x)^{-1} \Big| \, \mathrm{d}t \\
& = \int_0^1 \Big| \phi_\gamma(t) \big(g'(t) + A_\gamma(t) \big[\mathrm{D}\pi_{O(F)}[g'(t)] \big] \big) \phi(x)^{-1} \Big| \, \mathrm{d}t
\end{aligned}$$

where the γ subscript means composition, i.e. $\phi_\gamma = \phi \circ \gamma$. Since the gauge transformations are isometries, the conclusion follows. □

Chapter 3

Trace theory of gauge covariant Sobolev spaces

The study of partial differential equations [6, 14, 28] often involves *boundary value problems*, that is, a partial differential equation on a smooth and bounded domain $\Omega \subset \mathbb{R}^n$ with a condition on the value of the solution on the boundary $\partial\Omega$ of Ω. For example, consider the *Poisson equation* : find $u \in C^2(\Omega) \cap C^0\left(\overline{\Omega}\right)$ such that

$$\begin{cases} -\Delta u = f & \text{on } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

When $f \in C^0(\Omega)$ and $g \in C^0(\partial\Omega)$, there exists at most one solution $u \in C^2(\Omega) \cap C^0\left(\overline{\Omega}\right)$ to (3.1) [14, Thm. 5]. Constructing an explicit solution can however be tricky as the regularity conditions are very robust. This problem, although seemingly simple, is already complicated. An approach which we believe to be particularly elegant is by weakening the problem. First of all, the equation

$$-\Delta u = f \quad \text{on } \Omega$$

with $u \in C^2(\Omega)$ is weakened to

$$\int_\Omega (\nabla u \mid \nabla \varphi)\, \mathrm{d}x = \int_\Omega f\varphi\, \mathrm{d}x, \quad \text{for all } \varphi \in C_c^\infty(\Omega) \tag{3.2}$$

with $u \in H^1(\Omega)$. By integration by parts, we see that a solution of the strong problem (3.1) is always a solution of the weak problem (3.2). However, since solutions of the latter are merely $H^1(\Omega)$ functions, how does one interpret the equation

$$u = g \quad \text{on } \partial\Omega \tag{3.3}$$

in the weak sense ? In fact, a mapping $u \in H^1(\Omega) \subset L^2(\Omega)$ is defined almost everywhere and the boundary $\partial\Omega$ has measure zero. The equation (3.3) is thus not well-defined. The goal of *trace theory* [1, 6, 14, 43] of Sobolev spaces is exactly that : giving a weak meaning to equality (3.3).

We recall that the fractional Sobolev space $W^{s,p}(\partial\Omega, F)$ for $0 < s < 1$ is defined as the set of all functions $u \in L^p(\partial\Omega, F)$ such that the Gagliardo seminorm

$$[u]_{s,p} = \left(\int_{\partial\Omega} \int_{\partial\Omega} \frac{\left| u(x) - u(y) \right|^p}{|x - y|^{n+sp}} \, \mathrm{d}x \, \mathrm{d}y \right)^{1/p}$$

is finite. It is endowed with the norm

$$\|u\|_{W^{s,p}(\Omega,F)} = \left(\|u\|_p^p + [u]_{s,p}^p \right)^{1/p}.$$

Note that the measure on $L^p(\partial\Omega, F)$ and $W^{s,p}(\partial\Omega, F)$ is not the Lebesgue measure of $\mathbb{R}^n$ but the natural surface measure on $\partial\Omega$.

Theorem 3.0.1 (Trace theorem)**.** *Let $n \geq 1$, let $1 < p < +\infty$ and let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain of class C^1 or $\Omega = \mathbb{R}^n_+ = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n > 0 \right\}$. Then there exists a unique linear and continuous operator*

$$\mathrm{Tr}\colon W^{1,p}(\Omega, F) \longrightarrow W^{1-1/p,p}(\partial\Omega, F)$$

called the trace operator*, which satisfies* $\mathrm{Tr}\, u = u_{|\partial\Omega}$ *for all* $u \in C^0\left(\overline{\Omega}, F\right)$*. Furthermore, there exists a linear and continuous operator*

$$\mathrm{Ext}\colon W^{1-1/p}(\partial\Omega, F) \longrightarrow W^{1,p}(\Omega, F)$$

such that $\mathrm{Tr} \circ \mathrm{Ext}$ *is the identity map on* $W^{1-1/p,p}(\partial\Omega, F)$*.*

The equation $u = g$ on $\partial\Omega$ for $u \in H^1(\Omega)$ can thus be interpreted as $\mathrm{Tr}\, u = g$ in $H^{1/2}(\partial\Omega)$. The trace theorem states that the trace operator is surjective. However, it is not injective since any function $u \in C^0\left(\overline{\Omega}, F\right)$ which vanishes on the boundary is mapped to $0 \in W^{1-1/p,p}(\partial\Omega, F)$.

The aim of this chapter is to prove an equivalent to the trace theorem for gauge covariant Sobolev spaces. A summary of the results and the structure of the chapter is given in Section 3.1.

3.1 Overview of the results

The general idea when working on a smooth and bounded domain Ω is first to study simple domains such the half-space $\mathbb{R}^n_+ = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n > 0 \right\}$ whose boundary is $\mathbb{R}^{n-1} \times \{0\}$. Once the result has been proven for this particular case, one uses charts of the form $\mathbb{R}^n_+$ for the manifold with boundary $\overline{\Omega}$ and a partition of unity allowing to obtain a global result on Ω. Therefore, we begin by studying the half-space.

Definition 3.1.1. *Let $n \geq 1$, let $1 \leq p < +\infty$ and let $0 < s < 1$. Furthermore, let $A' \in C^0\big(\mathbb{R}^n, \mathrm{Lin}\big(\mathbb{R}^n, \mathfrak{o}(F)\big)\big)$. We define the* fractional gauge covariant Sobolev space *$W^{s,p}_{A'}(\mathbb{R}^n, P)$ as the set of all functions $u \in L^p(\mathbb{R}^n, F)$ such that the* gauge covariant Gagliardo seminorm

$$[u]_{W^{s,p}_{A'}(\mathbb{R}^n,P)} = \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\big|R(x,y)u(y) - u(x)\big|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/p} \tag{3.4}$$

is finite, where $R(x,y)$ is the parallel transport from y to x along $\gamma_{y,x}(t) = (1-t)y + tx$.

The gauge covariant Gagliardo seminorm is gauge invariant in vertue of Proposition 2.3.7. We endow it with the gauge invariant norm

$$\|u\|_{W^{s,p}_{A'}(\mathbb{R}^n,P)} = \big(\|u\|_p^p + [u]^p_{W^{s,p}_{A'}(\mathbb{R}^n,P)}\big)^{1/p}.$$

A standard proof shows that this defines a Banach space.

We prove the following theorem, based on [34, Thm. 1.1], in two parts. The first part will be established in Section 3.3. The proof needs some fundamental estimates on the amplitude of holonomy which we obtain in Section 3.2 based on [8]. The second part of the theorem is proven in Section 3.4.

Theorem 3.1.2. *Let $n \geq 1$ and $1 < p < +\infty$. If $A \in C^1\big(\overline{\mathbb{R}^{n+1}_+}, \mathrm{Lin}\big(\mathbb{R}^{n+1}, \mathfrak{o}(F)\big)\big)$ and $\|\mathcal{K}_A\|_\infty \leq \beta < +\infty$, then there exists a positive constant $C = C(n,p) > 0$ depending only on n and p such that, with $A'(\cdot)[*] = A(\cdot,0)[*,0] \in C^1\big(\mathbb{R}^n, \mathrm{Lin}\big(\mathbb{R}^n, \mathfrak{o}(F)\big)\big)$,*

(i) For all $u \in C^1_c\big(\overline{\mathbb{R}^{n+1}_+}, F\big)$

$$\big|u(\cdot,0)\big|_{W^{1-1/p,p}_{A'}(\mathbb{R}^n,P)} + \beta^{(p-1)/2p}\big\|u(\cdot,0)\big\|_{L^p(\mathbb{R}^n,F)} \leq C\Big(\|\nabla_A u\|_{L^p(\mathbb{R}^{n+1}_+,\mathrm{Lin}(\mathbb{R}^{n+1},F))} + \beta^{1/2}\|u\|_{L^p(\mathbb{R}^{n+1}_+,F)}\Big), \tag{3.5}$$

(ii) For all $U \in C^1_c(\mathbb{R}^n, F)$, there exists $u \in C^1_c\big(\overline{\mathbb{R}^{n+1}_+}, F\big)$ depending linearly on U such that $u(x,0) = U(x)$ on $\mathbb{R}^n$ and

$$\|\nabla_A u\|_{L^p(\mathbb{R}^{n+1}_+,\mathrm{Lin}(\mathbb{R}^{n+1},F))} + \beta^{1/2}\|u\|_{L^p(\mathbb{R}^{n+1}_+,F)} \leq C\left(|U|_{W^{1-1/p}_{A}(\mathbb{R}^n,P)} + \beta^{(p-1)/2p}\|U\|_{L^p(\mathbb{R}^n,F)}\right). \tag{3.6}$$

In Section 3.5, we show that the space $C_c^\infty\big(\overline{\mathbb{R}_+^{n+1}}, F\big)$ is dense in the gauge covariant Sobolev space $W_A^{1,p}(\mathbb{R}_+^{n+1}, P)$ and that the space $C^\infty(\mathbb{R}^n, F)$ is dense in the fractional gauge covariant Sobolev space $W_{A'}^{s,p}(\mathbb{R}^n, P)$. By a standard density argument, we obtain the following result generalising [34, Thm. 1.2].

Theorem 3.1.3. *Let $n \geq 1$ and $1 < p < +\infty$. Assume that $A \in C^1\big(\overline{\mathbb{R}_+^{n+1}}, \operatorname{Lin}\big(\mathbb{R}^{n+1}, \mathfrak{o}(F)\big)\big)$ and $\|\mathcal{K}_A\|_{L^\infty(\mathbb{R}_+^{n+1})} < +\infty$. Set $A'(\cdot)[*] = A(\cdot, 0)[*, 0] \in C^1\big(\mathbb{R}^n, \operatorname{Lin}\big(\mathbb{R}^n, \mathfrak{o}(F)\big)\big)$. Then there exists a linear and continuous operator*

$$\operatorname{Tr}\colon W_A^{1,p}(\mathbb{R}_+^{n+1}, P) \to W_{A'}^{1-1/p,p}(\mathbb{R}^n, P)$$

called the trace operator*, which satisfies $\operatorname{Tr} u(\cdot) = u(\cdot, 0)$ for all $u \in C_c^0(\overline{\mathbb{R}_+^{n+1}}, F)$. Furthermore, there exists a linear and continuous operator*

$$\operatorname{Ext}\colon W_{A'}^{1-1/p,p}(\mathbb{R}^n, P) \to W_A^{1,p}(\mathbb{R}_+^{n+1}, P)$$

such that $\operatorname{Tr} \circ \operatorname{Ext}$ is the identity map on $W_{A'}^{1-1/p,p}(\mathbb{R}^n, P)$ and the estimates of Theorem 3.1.2 with $U = \operatorname{Tr} u$ and $u = \operatorname{Ext} U$ hold.

With a localised version of Theorem 3.1.3, we obtain in Section 3.6 a gauge covariant version of the trace theorem. As a little bit of work is needed to define the fractional gauge covariant space on a non-convex domain, we leave the details for later.

3.2 Estimating the amplitude of holonomy

The aim of this section is to establish Proposition 3.2.1 which gives a gauge invariant upper bound on the amplitude of holonomy along a triangle depending on the curvature $\mathcal{K}_A = \mathrm{d}A + A \wedge A$ of the connection ∇_A.

Let x, y and z be arbitrary points of $\mathbb{R}^n$. We denote by $\Delta_{x,y,z} \subset \mathbb{R}^n$ the open triangle with vertices x, y and z, and we denote by $|\Delta_{x,y,z}|$ its area. We define the path $\gamma\colon [0,3] \to \mathbb{R}^n$ as the parametrisation of $\partial\Delta_{x,y,z}$, the boundary of $\Delta_{x,y,z}$, defined by

$$\gamma(t) = \begin{cases} ty + (1-t)x & \text{if } 0 \leq t < 1, \\ (t-1)z + (2-t)y & \text{if } 1 \leq t < 2, \\ (t-2)x + (3-t)z & \text{if } 2 \leq t \leq 3. \end{cases} \tag{3.7}$$

The path γ is the concatenation of the paths $\gamma_{x,y}$, $\gamma_{y,z}$ and $\gamma_{z,x}$ from Notation 2.3.8. It is continuous and piecewise smooth with

$$\gamma'(t) = \begin{cases} y - x & \text{if } 0 \leq t < 1, \\ z - y & \text{if } 1 < t < 2, \\ x - z & \text{if } 2 < t \leq 3. \end{cases}$$

Proposition 3.2.1. *Let $A \in C^1\big(\Delta_{x,y,z}, \operatorname{Lin}\big(\mathbb{R}^n, \mathfrak{o}(F)\big)\big)$ and assume that the curvature $\mathcal{K}_A$ is bounded, i.e. $\|\mathcal{K}_A\|_\infty < +\infty$. Then*

$$\langle\!\langle \gamma \rangle\!\rangle \leq \|\mathcal{K}_A\|_\infty |\Delta_{x,y,z}|.$$

By definition, we have

$$\langle\!\langle \gamma \rangle\!\rangle = \inf\left\{ \int_0^3 \big|g'(t)\big| \,\mathrm{d}t : g \in C^1\big([0,3], O(F)\big) \text{ and } g \sim_* \mathrm{Pt}_\gamma \right\}$$

and it is thus sufficient to construct a function $g \in C^1\big([0,3], O(F)\big)$ homotopic to Pt_γ relatively to their endpoints which satisfies

$$\int_0^3 \big|g'(t)\big| \,\mathrm{d}t \leq \|\mathcal{K}_A\|_\infty |\Delta_{x,y,z}|.$$

With the change of variable $t = 3r$, we have

$$\int_0^3 \big|g'(t)\big| \,\mathrm{d}t = \int_0^1 \big|g'(3r)\big| 3 \,\mathrm{d}r = \int_0^1 \big|h'(r)\big| \,\mathrm{d}r$$

where $h(r) = g(3r)$ for all $r \in [0,1]$. It follows that the particular interval on which we construct an homotopic map g is not important.

For $r \in [0,1]$, we set

$$y_r = ry + (1-r)x \quad \text{and} \quad z_r = rz + (1-r)x.$$

The triangle Δ_{x,y_r,z_r} is a scaled copy of $\Delta_{x,y,z}$ with the same vertex x. We parametrise $\partial\Delta_{x,y_r,z_r}$ by $\gamma_r \colon [0,3] \longrightarrow \mathbb{R}^n$ as in (3.7), replacing y and z by y_r and z_r respectively, see Figure 3.1 for a visual representation. In particular, $\gamma_1 = \gamma$ and $\gamma_0 = x$, i.e. γ_0 is constant. A rapid computation shows that the function $(r,t) \in [0,1] \times [0,3] \mapsto \gamma(r,t) = \gamma_r(t)$ is smooth and piecewise smooth in the variables r and t respectively. In fact, we

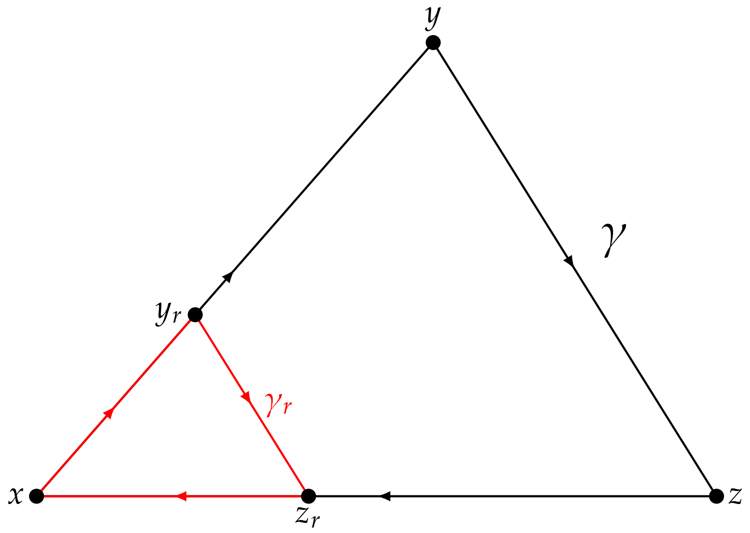


Figure 3.1: Representation of the parametrisations γ and γ_r.

have

$$\begin{gathered}\gamma_r'(t) = r\gamma(t),\\ \partial_r\gamma_r(t) = \gamma(t) - x,\\ \partial_r\partial_t\gamma_r(t) = \partial_t\partial_r\gamma_r(t) = \gamma'(t)\end{gathered} \tag{3.8}$$

for all $r \in [0,1]$ and $t \in [0,3] \setminus \{1,2\}$.

For each path γ_r we define the continuous and piecewise C^1 function $g_r\colon [0,3] \to O(F)$ as the solution of the initial value problem

$$\begin{cases} g_r'(t) + A\big(\gamma_r(t)\big)\big[\gamma_r'(t)\big]g_r(t) = 0 & \text{for } t \in [0,3] \setminus \{1,2\},\\ g_r(0) = \mathrm{id}\,. \end{cases} \tag{3.9}$$

In other words, $g_r = \mathrm{Pt}_{\gamma_r}$ is the parallel transport along the path γ_r. In particular,

$$g_r(3) = R(x_r, z_r)R(z_r, y_r)R(y_r, x_r)$$

is the holonomy at x along that path. Here and in what follows, $R(x,y) = R_{\gamma_{y,x}}(1,0)$ is the parallel transport from y to x along $\gamma_{y,x}(t) = tx + (1-t)y$, see the comments preceding Proposition 2.3.9 in Section 2.3.

The initial value problem (3.9) is linear and homogeneous with a parameter $r \in [0,1]$. The general theory of ordinary differential equations with parameters [22, Thm. 1.5.3] implies the following differentiation of the parallel transport with parameters formula.

Lemma 3.2.2. *Let* $A \in C^1\big(\Omega, \mathrm{Lin}\big(\mathbb{R}^n, \mathfrak{o}(F)\big)\big)$ *and let* $H \in C^1\big([0,1] \times J, \Omega\big)$ *where* $J \subset \mathbb{R}^q$

is an open set of parameters. If $\gamma_\alpha = H(\cdot,\alpha) \in C^1\big([0,1],\Omega\big)$ *for* $\alpha \in J$, *then* $\operatorname{Pt}_{\gamma_\alpha} \in C^1\big([0,1]\times J, O(F)\big)$ *and* $h_\alpha = \partial_\alpha \operatorname{Pt}_{\gamma_\alpha}$ *is solution of the non-homogeneous linear initial value problem*

$$\begin{cases} h'_\alpha(t) + A\big(\gamma_\alpha(t)\big)\big[\gamma'_\alpha(t)\big]h_\alpha(t) \\ \qquad\qquad = -\Big(\mathrm{D}A\big(\gamma_\alpha(t)\big)\big[\partial_\alpha\gamma_\alpha(t),\gamma'_\alpha(t)\big] + A\big(\gamma_\alpha(t)\big)\big[\partial_\alpha\gamma'_\alpha(t)\big]\Big)\operatorname{Pt}_{\gamma_\alpha}(t), \\ h_\alpha(0) = 0 \end{cases}$$

for $t \in [0,1]$.

Combining this lemma with the variation of parameters formula for solutions of non-homogeneous linear initial value problems, see [22, Thm. 2.3.1], we obtain the following expression for h_α.

Lemma 3.2.3. *Let* $A \in C^1\big(\Omega, \operatorname{Lin}\big(\mathbb{R}^n, \mathfrak{o}(F)\big)\big)$ *and let* $H \in C^1\big([0,1]\times J,\Omega\big)$ *where* $J \subset \mathbb{R}^q$ *is an open set of parameters. Let* $\gamma_\alpha = H(\cdot,\alpha) \in C^1\big([0,1],\Omega\big)$ *for* $\alpha \in J$. *Then* $h_\alpha = \partial_\alpha \operatorname{Pt}_{\gamma_\alpha}$ *is given by*

$$\begin{aligned} &h_\alpha(t) \\ &\quad = -\int_0^t R_{\gamma_\alpha}(t,\tau)\Big(\mathrm{D}A\big(\gamma_\alpha(\tau)\big)\big[\partial_\alpha\gamma_\alpha(\tau),\gamma'_\alpha(\tau)\big] + A\big(\gamma_\alpha(\tau)\big)\big[\partial_\alpha\gamma'_\alpha(\tau)\big]\Big)\operatorname{Pt}_{\gamma_\alpha}(\tau)\,\mathrm{d}\tau. \end{aligned}$$

Lemma 3.2.2 and Lemma 3.2.3 will be used multiple times during this chapter. The first instance of their use will be in the proof of Lemma 3.2.4 and the other instances of their use will be very similar, see Proposition 3.4.1 and Lemma 3.6.6.

We now establish Lemma 3.2.4 which is the core element in the proof of Proposition 3.2.1. It is strongly inspired by [8, Lemma 3.1] which is the analogue result when γ parametrises a circle.

Lemma 3.2.4. *Let* $\Delta_{x,y,z} \subset \mathbb{R}^n$ *be a local trivialisation of a vector bundle* P *and assume that* $A \in C^1\big(\Delta_{x,y,z}, \operatorname{Lin}\big(\mathbb{R}^n, \mathfrak{o}(F)\big)\big)$. *Then, for all* $r \in (0,1)$, *we have*

$$\frac{\mathrm{d}}{\mathrm{d}r} g_r(3) = r\int_1^2 g_r(3)g_r(t)^{-1}\mathcal{K}_A\big(\gamma_r(t)\big)[z-y,y-x]g_r(t)\,\mathrm{d}t. \tag{3.10}$$

Proof. We begin by setting $h_r = \partial_r g_r$. Since g_r satisfies (3.9) and is piecewise C^1, Lemma 3.2.2 applies on the intervals $[0,1]$, $[1,2]$ and $[2,3]$. Together with (3.2), it implies that h_r is continuous and piecewise C^1 with

$$\begin{cases} h_r'(t) + A\big(\gamma_r(t)\big)\big[r\gamma'(t)\big]h_r(t) \\ \qquad\qquad = -\Big(\mathrm{D}A\big(\gamma_r(t)\big)\big[\gamma(t)-x, r\gamma'(t)\big] + A\big(\gamma_r(t)\big)\big[\gamma'(t)\big]\Big)g_r(t), \\ h_\alpha(0) = 0 \end{cases}$$

for $t \in [0,3] \setminus \{1,2\}$. Applying Lemma 3.2.3, we obtain

$$h_r(t) = -\int_0^t g_r(t)g_r(\tau)^{-1}\Big(\mathrm{D}A\big(\gamma_r(\tau)\big)\big[\gamma(\tau)-x, r\gamma'(\tau)\big] + A\big(\gamma_r(\tau)\big)\big[\gamma'(\tau)\big]\Big)g_r(\tau)\,\mathrm{d}\tau$$

and in particular

$$h_r(3) = -\int_0^3 g_r(3)g_r(t)^{-1}\Big(\mathrm{D}A\big(\gamma_r(t)\big)\big[\gamma(t)-x, r\gamma'(t)\big] + A\big(\gamma_r(t)\big)\big[\gamma'(t)\big]\Big)g_r(t)\,\mathrm{d}t.$$

We now decompose the integral (3.11) into the integrals on the intervals $[0,1]$, $[1,2]$ and $[2,3]$ which we analyse individually,

$$\begin{aligned} h_r(3) = &-\int_0^1 g_r(3)g_r(t)^{-1}\big(tr\mathrm{D}A\big(\gamma_r(t)\big)\big[(y-x),(y-x)\big] + A\big(\gamma_r(t)\big)[y-x]\big)g_r(t)\,\mathrm{d}t \\ &-\int_1^2 g_r(3)g_r(t)^{-1}\big(r\mathrm{D}A\big(\gamma_r(t)\big)\big[y-x+(t-1)(z-y),(z-y)\big] + A\big(\gamma_r(t)\big)[z-y]\big)g_r(t)\,\mathrm{d}t \\ &+\int_2^3 g_r(3)g_r(t)^{-1}\big((t-3)r\mathrm{D}A\big(\gamma_r(t)\big)\big[(x-z),(x-z)\big] + A\big(\gamma_r(t)\big)[x-z]\big)g_r(t)\,\mathrm{d}t. \end{aligned} \tag{3.11}$$

In the first place, since for $t \in [0,1]$ $y - x = \big(t(y-x)\big)'$, an integration by parts with Proposition 2.3.5 yields

$$\begin{aligned} -\int_0^1 & g_r(3)g_r(t)^{-1}A\big(\gamma_r(t)\big)\big[y-x\big]g_r(t)\,\mathrm{d}t \\ &= -g_r(3)g_r(1)^{-1}A(y_r)[y-x]g_r(1) + g_r(3)A(x)[0] \\ &\quad + \int_0^1 g_r(3)g_r(t)^{-1}A\big(\gamma_r(t)\big)\big[r(y-x)\big]A\big(\gamma_r(t)\big)\big[t(y-x)\big]g_r(t)\,\mathrm{d}t \\ &\quad + \int_0^1 g_r(3)g_r(t)^{-1}\mathrm{D}A\big(\gamma_r(t)\big)\big[r(y-x), t(y-x)\big]g_r(t)\,\mathrm{d}t \\ &\quad - \int_0^1 g_r(3)g_r(t)^{-1}A\big(\gamma_r(t)\big)\big[t(y-x)\big]A\big(\gamma_r(t)\big)\big[r(y-x)\big]g_r(t)\,\mathrm{d}t \end{aligned}$$

which reduces by linearity to

$$-\int_0^1 g_r(3)g_r(t)^{-1}A\big(\gamma_r(t)\big)\big[\gamma'(t)\big]g_r(t)\,\mathrm{d}t = -g_r(3)g_r(1)^{-1}A(y_r)[y-x]g_r(1) + \int_0^1 rt\,g_r(3)g_r(t)^{-1}\mathrm{D}A\big(\gamma_r(t)\big)[y-x,y-x]g_r(t)\,\mathrm{d}t. \tag{3.12}$$

Similarly, we obtain

$$-\int_2^3 g_r(3)g_r(t)^{-1}A\big(\gamma_r(t)\big)\big[x-z\big]g_r(t)\,\mathrm{d}t = g_r(3)g_r(2)^{-1}A(z_r)[z-x]g_r(2) + \int_2^3 r(t-3)g_r(3)g_r(t)^{-1}\mathrm{D}A\big(\gamma_r(t)\big)[x-z,x-z]g_r(t)\,\mathrm{d}t. \tag{3.13}$$

Finally, we reach with the same arguments

$$\begin{aligned}-\int_1^2 & g_r(3)g_r(t)^{-1}A\big(\gamma_r(t)\big)[z-y]g_r(t)\,\mathrm{d}t \\ &= -g_r(3)g_r(2)^{-1}A(z_r)[z-x]g_r(2) + g_r(3)g_r(1)^{-1}A(y_r)[y-x]g_r(1) \\ &\quad + r\int_1^2 g_r(3)g_r(t)^{-1}\mathrm{D}A\big(\gamma_r(t)\big)\big[z-y,y-x+(t-1)(z-y)\big]g_r(t)\,\mathrm{d}t \\ &\quad + r\int_1^2 g_r(3)g_r(t)^{-1}(A\wedge A)\big(\gamma_r(t)\big)\big[z-y,y-x+(t-1)(z-y)\big]g_r(t)\,\mathrm{d}t. \end{aligned} \tag{3.14}$$

Injecting (3.12), (3.13) and (3.14) into (3.11), we obtain

$$h_r(3) = r\int_1^2 g_r(3)g_r(t)^{-1}\mathcal{K}_A\big(\gamma_r(t)\big)\big[z-y,y-x+(t-1)(z-y)\big]g_r(t)\,\mathrm{d}t. \tag{3.15}$$

Since the curvature $\mathcal{K}_A$ is bilinear and alternating, we have, for all $t \in [1,2]$,

$$\mathcal{K}_A\big(\gamma_r(t)\big)\big[z-y,y-x+(t-1)(z-y)\big] = \mathcal{K}_A\big(\gamma_r(t)\big)[z-y,y-x].$$

It follows that (3.15) reduces to

$$h_r(3) = r\int_1^2 g_r(3)g_r(t)^{-1}\mathcal{K}_A\big(\gamma_r(t)\big)[z-y,y-x]g_r(t)\,\mathrm{d}t.$$

which is the announced result (3.10). □

The path

$$g\colon [0,1] \longrightarrow O(F),$$

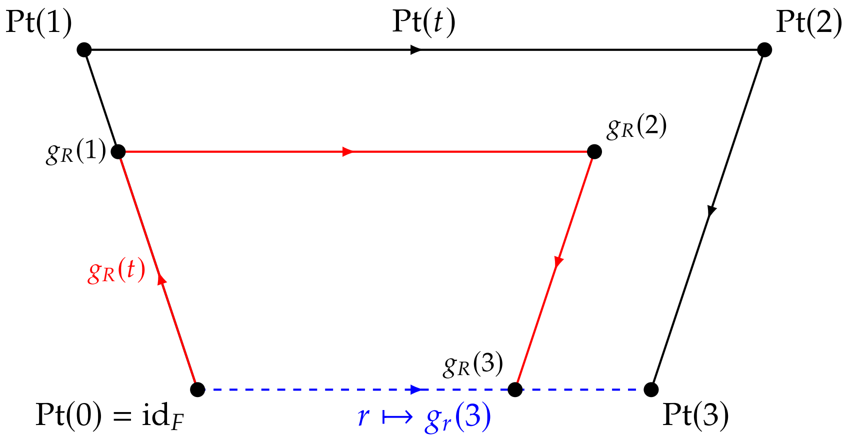


Figure 3.2: Curves in $O(F)$ drawn by the parallel transports $\mathrm{Pt} = g_1$ and $g_R = \mathrm{Pt}_{\gamma_R}$ with $R = 7/10$ as well as the map $r \in [0,1] \mapsto g_r(3)$ homotopic to Pt w.r.t. their endpoints.

$$r \mapsto g_r(3)$$

is homotopic to Pt_γ with respect to their endpoints, see Figure 3.2 for a representation of the paths involved in the above proof and the homotopy. Lemma 3.2.4 gives an explicit formula for its derivative, $\mathrm{d}/\mathrm{d}r g_r(3)$, which can thus be plugged into the definition of the amplitude of holonomy along the path γ.

Proof of Proposition 3.2.1. If there exists $c \in \mathbb{R}_*$ such that $y - x = c(z - y)$, then

$$\langle\!\langle \gamma \rangle\!\rangle = \mathcal{K}_A(\cdot)[z - y, y - x] = |\Delta_{x,y,z}| = 0,$$

and the result follows from Lemma 3.2.4. If $y - x = 0$ or $z - y = 0$, the same phenomenon produces. Therefore, we may assume without loss of generality that

$$X = \operatorname{span}\{y - z, y - x\} \subset \mathbb{R}^n$$

is a two-dimensional subspace of $\mathbb{R}^n$. Let x_1, $x_2 \in X$ be an orthonormal basis of X. Since the curvature $\mathcal{K}_A$ is a differential 2-form, its restriction to the two-dimensional space X is completely determined by the formula

$$\mathcal{K}_A(p)[v, w] = \det(L)\mathcal{K}_A(p)[x_1, x_2]$$

where $p \in \Omega$ and $v, w \in X$, and $L\colon X \to X$ is linear with $L(x_1) = v$ and $L(x_2) = w$, see for example [25, Ch. 14]. We therefore obtain that, for all $t \in [1,2]$ and $r \in (0,1)$,

$$\left|\mathcal{K}_A\big(\gamma_r(t)\big)[y - z, y - x]\right| \leq \|\mathcal{K}_A\|_\infty \left|(y - x) \wedge (y - z)\right| = 2\|\mathcal{K}_A\|_\infty |\Delta_{x,y,z}|. \tag{3.16}$$

We deduce from Lemma 3.2.4 and (3.16) that

$$\begin{aligned}\langle\!\langle\gamma\rangle\!\rangle \le \int_0^1 \left|\frac{\mathrm{d}}{\mathrm{d}r} g_r(3)\right| \mathrm{d}r = \int_0^1 r \left|\int_1^2 g_r(3) g_r(t)^{-1} \mathcal{K}_A\big(\gamma_r(t)\big)[y-z, y-x] g_r(t)\, \mathrm{d}t\right| \mathrm{d}r \\ \le 2\|\mathcal{K}_A\|_\infty |\Delta_{x,y,z}| \int_0^1 r\, \mathrm{d}r = \|\mathcal{K}_A\|_\infty |\Delta_{x,y,z}|\end{aligned}$$

which concludes the proof. □

Since the amplitude of holonomy $\langle\!\langle\gamma\rangle\!\rangle$ represents the geodesic distance between id_F and $\mathrm{Pt}_\gamma(1)$ in the simply connected component of $\mathrm{Pt}_\gamma(\cdot)$, a consequence of Proposition 3.2.1 is that if $\|\mathcal{K}_A\|_\infty < +\infty$ on a convex set Ω, then for all x, y, $z \in \Omega$ we have

$$\big|R(x,z)R(z,y)R(y,x) - \mathrm{id}_F\big| \le \|\mathcal{K}_A\|_\infty |\Delta_{x,y,z}|.$$

3.3 Trace estimates on the half-space

In this section we derive the following trace estimates on the half-space when the curvature of the connection is bounded. This generalises the result [34, Prop. 2.1] to the non-abelian case. Our proof follows a structure similar to the one presented in the article [34].

Proposition 3.3.1. *Let* $n \ge 1$, $0 < s < 1$ *and* $1 \le p < +\infty$. *Furthermore, assume that* $A \in C^1\big(\overline{\mathbb{R}^{n+1}_+}, \mathrm{Lin}\big(\mathbb{R}^{n+1}, \mathfrak{o}(F)\big)\big)$ *and* $\|\mathcal{K}_A\|_\infty \le \beta$. *Then there exists a positive constant* $C = C(n,s,p) > 0$ *depending only on* n, s *and* p *such that for all* $u \in C^1_c\big(\overline{\mathbb{R}^{n+1}_+}, F\big)$

$$\big|u(\cdot,0)\big|^p_{W^{s,p}_{A'}(\mathbb{R}^n,P)} \le C \int_{\mathbb{R}^n} \int_0^{+\infty} \frac{\big|\nabla_A u(z,t)\big|^p + \beta^{p/2}\big|u(z,t)\big|^p}{t^{1-(1-s)p}}\, \mathrm{d}t\, \mathrm{d}z \tag{3.17}$$

and

$$\begin{aligned}&\big\|u(\cdot,0)\big\|^p_{L^p(\mathbb{R}^n,F)} \\ &\quad\le C \left(\int_{\mathbb{R}^n} \int_0^{+\infty} \frac{\big|\nabla_A u(z,t)\big|^p}{t^{1-(1-s)p}}\, \mathrm{d}t\, \mathrm{d}z\right)^{1-s} \left(\int_{\mathbb{R}^n} \int_0^{+\infty} \frac{\big|u(z,t)\big|^p}{t^{1-(1-s)p}}\, \mathrm{d}t\, \mathrm{d}z\right)^{s}, \end{aligned}\tag{3.18}$$

where $A'(\cdot)[*] = A(\cdot,0)[*,0] \in C^1\big(\mathbb{R}^n, \mathrm{Lin}\big(\mathbb{R}^n, \mathfrak{o}(F)\big)\big)$.

By choosing a suitable $0 < s < 1$, it implies (i) in Theorem 3.1.2.

Proof of Theorem 3.1.2 (i). Since $1 < p < +\infty$, we may consider $0 < s = 1 - \frac{1}{p} < 1$ in Proposition 3.3.1. The inequality (3.17) then reads

$$\begin{aligned} \left|u(\cdot,0)\right|^p_{W^{1-1/p,p}_{A'}(\mathbb{R}^n,P)} &\leq C \int_{\mathbb{R}^n} \int_0^{+\infty} \left|\nabla_A u(z,t)\right|^p + \beta^{p/2}\left|u(z,t)\right|^p \,\mathrm{d}t\,\mathrm{d}z \\ &= C\left(\|\nabla_A u\|^p_{L^p(\mathbb{R}^{n+1}_+,\mathrm{Lin}(\mathbb{R}^{n+1},F))} + \beta^{p/2}\|u\|^p_{L^p(\mathbb{R}^{n+1}_+,F)}\right) \end{aligned} \tag{3.19}$$

while (3.18) reads

$$\begin{aligned} &\left\|u(\cdot,0)\right\|^p_{L^p(\mathbb{R}^n,F)} \\ &\quad\leq C\left(\int_{\mathbb{R}^n}\int_0^{+\infty}\left|\nabla_A u(z,t)\right|^p \,\mathrm{d}t\,\mathrm{d}z\right)^{1/p}\left(\int_{\mathbb{R}^n}\int_0^{+\infty}\left|u(z,t)\right|^p\,\mathrm{d}t\,\mathrm{d}z\right)^{1-1/p} \\ &\quad= C\|\nabla_A u\|_{L^p(\mathbb{R}^{n+1}_+,\mathrm{Lin}(\mathbb{R}^{n+1},F))}\|u\|^{p-1}_{L^p(\mathbb{R}^{n+1}_+,F)} \end{aligned} \tag{3.20}$$

for some constant $C > 0$ depending only on n and p. Applying Young's inequality with $\frac{1}{p} + \frac{1}{p'} = 1$ to (3.20) yields $\bar{C} > 0$ depending only on n and p such that

$$\beta^{(p-1)/2}\left\|u(\cdot,0)\right\|^p_{L^p(\mathbb{R}^n,F)} \leq \bar{C}\left(\|\nabla_A u\|^p_{L^p(\mathbb{R}^{n+1}_+,\mathrm{Lin}(\mathbb{R}^{n+1},F))} + \beta^{p/2}\|u\|^p_{L^p(\mathbb{R}^{n+1}_+,F)}\right). \tag{3.21}$$

Applying the triangle inequality to (3.19) and (3.21) yields

$$\begin{aligned} \left|u(\cdot,0)\right|_{W^{1-1/p,p}_{A'}(\mathbb{R}^n,P)} &\leq C^{1/p}\left(\|\nabla_A u\|^p_{L^p(\mathbb{R}^{n+1}_+,\mathrm{Lin}(\mathbb{R}^{n+1},F))} + \beta^{p/2}\|u\|^p_{L^p(\mathbb{R}^{n+1}_+,F)}\right)^{1/p} \\ &\leq C^{1/p}\left(\|\nabla_A u\|_{L^p(\mathbb{R}^{n+1}_+,\mathrm{Lin}(\mathbb{R}^{n+1},F))} + \beta^{1/2}\|u\|_{L^p(\mathbb{R}^{n+1}_+,F)}\right) \end{aligned} \tag{3.22}$$

as well as

$$\beta^{(p-1)/p2}\left\|u(\cdot,0)\right\|_{L^p(\mathbb{R}^n,F)} \leq \bar{C}^{1/p}\left(\|\nabla_A u\|_{L^p(\mathbb{R}^{n+1}_+,\mathrm{Lin}(\mathbb{R}^{n+1},F))} + \beta^{1/2}\|u\|_{L^p(\mathbb{R}^{n+1}_+,F)}\right). \tag{3.23}$$

Summing the inequalities (3.22) and (3.23) completes the proof. □

Before proving Proposition 3.3.1 we present the following lemma generalising the result [34, Lemma 2.3]. The proof is essentially the same since the core of the proof is the estimate of the amplitude of holonomy which was made in Proposition 3.2.1. The estimate in the abelian case was easier to obtain as it essentially follows from Stoke's theorem, see [34, Lemma 2.2] for details. The idea of the lemma is to estimate the impact of parallel transport along a straight line by the parallel transport along two other sides of a triangle. In particular, we obtain a bound depending on the area of the triangle delimited by the two paths.

Lemma 3.3.2. *If $n \geq 1$, $u \in C^1\big(\overline{\mathbb{R}^{n+1}}_+, F\big)$ and $A \in C^1\big(\overline{\mathbb{R}^{n+1}_+}, \mathrm{Lin}\big(\mathbb{R}^{n+1}, \mathfrak{o}(F)\big)\big)$, then for every $x, y, z \in \overline{\mathbb{R}^{n+1}_+}$, we have*

$$\begin{aligned}\big|R(x,y)u(y) - u(x)\big| \leq \big|R(z,y)u(y) - u(z)\big| \\ + \big|R(z,x)u(x) - u(z)\big| + \big|u(z)\big| \min\big\{2, \|\mathcal{K}_A\|_\infty |\Delta_{x,y,z}|\big\}.\end{aligned} \tag{3.24}$$

Proof. Since $R(v,w) \in O(F)$ is an isometry for all $v, w \in \overline{\mathbb{R}^{n+1}_+}$, it follows that

$$\begin{aligned}&\big|R(x,y)u(y) - R(x,y)R(y,z)u(z)\big| \\ &\quad = \big|R(z,y)R(y,x)R(x,y)u(y) - R(z,y)R(y,x)R(x,y)R(y,z)u(z)\big| \\ &\quad = \big|R(z,y)u(y) - u(z)\big|\end{aligned} \tag{3.25}$$

and

$$\big|u(x) - R(x,z)u(z)\big| = \big|R(z,x)u(x) - u(z)\big|. \tag{3.26}$$

We have

$$\begin{aligned}R(x,y)u(y) - u(x) = R(x,y)u(y) - R(x,y)R(y,z)u(z) - u(x) \\ + R(x,z)u(z) + R(x,y)R(y,z)u(z) - R(x,z)u(z)\end{aligned} \tag{3.27}$$

and thus, applying the triangle inequality as well as (3.25) and (3.26), we obtain

$$\begin{aligned}\big|R(x,y)u(y) - u(x)\big| \leq \big|R(z,y)u(y) - u(z)\big| \\ + \big|R(z,x)u(x) - u(z)\big| + \big|R(x,y)R(y,z)u(z) - R(x,z)u(z)\big|.\end{aligned} \tag{3.28}$$

Finally, by Proposition 3.2.1

$$\begin{aligned}&\big|R(x,y)R(y,z)u(z) - R(x,z)u(z)\big| \\ &\quad \leq \big|R(x,y)R(y,z)R(z,x) - \mathrm{id}_F\big|\big|u(z)\big| \\ &\quad \leq \big|u(z)\big| \min\big\{2, \|\mathcal{K}_A\|_\infty |\Delta_{x,y,z}|\big\}.\end{aligned} \tag{3.29}$$

Combining (3.28) and (3.29) yields the desired result. □

We may now prove Proposition 3.3.1. In this proof, we denote by $C > 0$ multiple different positive constants that depend at most on n, s and p. The constants may vary in-between lines. If that is the case, a short sentence or inequality will make that change explicit. We also identify $\mathbb{R}^n$ with $\partial\big(\mathbb{R}^{n+1}_+\big) = \mathbb{R}^n \times \{0\}$ so that $x \in \mathbb{R}^n$ is identified with $(x, 0) \in \mathbb{R}^{n+1}$.

Proof of Proposition 3.3.1. For each $x, y \in \mathbb{R}^n$, setting $z = \left(\frac{x+y}{2}, |y-x|\right)$, we have

$$\min\left\{2, \|\mathcal{K}_A\|_\infty |\Delta_{x,y,z}|\right\} \le \sqrt{2\|\mathcal{K}_A\|_\infty |\Delta_{x,y,z}|} = |y-x|\sqrt{\|\mathcal{K}_A\|_\infty} \tag{3.30}$$

and thus Lemma 3.3.2 with (3.30) yields

$$\begin{aligned}\left|R(x,y)u(y,0) - u(x,0)\right| \le{}& \left|R(z,y)u(y) - u(z)\right| \\ &+ \left|R(z,x)u(x) - u(z)\right| + \left|u(z)\right||y-x|\sqrt{\|\mathcal{K}_A\|_\infty}.\end{aligned} \tag{3.31}$$

The gauge covariant fundamental theorem of calculus, see Proposition 2.3.9, applied to (3.31) yields

$$\begin{aligned}\left|R(x,y)u(y,0) - u(x,0)\right| \le{}& \int_0^1 \left|\nabla_A u\big(\gamma_{x,z}(t)\big)\left[\gamma'_{x,z}(t)\right]\right| \mathrm{d}t \\ &+ \int_0^1 \left|\nabla_A u\big(\gamma_{y,z}(t)\big)\left[\gamma'_{y,z}(t)\right]\right| \mathrm{d}t + \left|u(z)\right||y-x|\beta^{1/2}.\end{aligned} \tag{3.32}$$

Since $z = \left(\frac{x+y}{2}, |y-x|\right)$, we have the estimates

$$\left|\gamma'_{x,z}(t)\right| \le C|y-x| \quad \text{and} \quad \left|\gamma'_{y,z}(t)\right| \le C|y-x|$$

for some positive constant $C > 1$. Using these estimates in (3.32) we obtain

$$\begin{aligned}&\left|R(x,y)u(y,0) - u(x,0)\right| \\ &\quad\le C|y-x|\left(\int_0^1 \left|\nabla_A u\big(\gamma_{x,z}(t)\big)\right| \mathrm{d}t + \int_0^1 \left|\nabla_A u\big(\gamma_{y,z}(t)\big)\right| \mathrm{d}t + \beta^{1/2}\left|u(z)\right|\right).\end{aligned} \tag{3.33}$$

By assumption $0 < s < 1$ and Hölder's inequality with $\frac{1}{p} + \frac{1}{p'} = 1$ then implies that there exists $C > 0$ such that for any measurable function $f\colon [0,1] \to F$

$$\left(\int_0^1 \left|f(t)\right| \mathrm{d}t\right)^p \le C \int_0^1 t^{(1-s)(p-1)}\left|f(t)\right|^p \mathrm{d}t. \tag{3.34}$$

Combining the inequalities (3.34) and $|a+b+c|^p \le 3^{p-1}\left(|a|^p + |b|^p + |c|^p\right)$, we derive from (3.33) that for some $C > 0$

$$\begin{aligned}\frac{\left|R(x,y)u(y,0) - u(x,0)\right|^p}{|y-x|^p} \le{}& C\left(\int_0^1 t^{(1-s)(p-1)}\left|\nabla_A u\big(\gamma_{x,z}(t)\big)\right|^p \mathrm{d}t\right. \\ &\left.+ \int_0^1 t^{(1-s)(p-1)}\left|\nabla_A u\big(\gamma_{y,z}(t)\big)\right|^p \mathrm{d}t + \beta^{p/2}\left|u(z)\right|\right)\end{aligned} \tag{3.35}$$

and dividing both sides of (3.35) by $|y-x|^{n-(1-s)p}$ yields, since $z = \big(\frac{x+y}{2}, |y-x|\big)$,

$$\begin{aligned} \frac{\big|R(x,y)u(y,0) - u(x,0)\big|^p}{|y-x|^{n+sp}} \leq C \Bigg(& \int_0^1 t^{(1-s)(p-1)} \frac{\big|\nabla_A u\big((1-\frac{t}{2})x + \frac{t}{2}y, \frac{t}{2}|y-x|\big)\big|^p}{|y-x|^{n-(1-s)p}} \,\mathrm{d}t \\ & + \int_0^1 t^{(1-s)(p-1)} \frac{\big|\nabla_A u\big((1-\frac{t}{2})y + \frac{t}{2}x, \frac{t}{2}|y-x|\big)\big|^p}{|y-x|^{n-(1-s)p}} \,\mathrm{d}t \\ & + \beta^{p/2} \frac{\big|u\big(\frac{x+y}{2}, |y-x|\big)\big|^p}{|y-x|^{n-(1-s)p}} \Bigg). \end{aligned} \tag{3.36}$$

The integral of the left-hand side of (3.36) with respect to x and y will now be estimated by the integrals of the three terms on the right-hand side. For $t \in (0,1)$, we perform the change of variable $\eta = (1-\frac{1}{t})x + \frac{t}{2}y$ and $\xi = t(x-y)$ which has the $2n \times 2n$ block matrix

$$\begin{pmatrix} \mathrm{id}_n & \frac{1}{2}\,\mathrm{id}_n \\ \mathrm{id}_n & \frac{t-2}{2t}\,\mathrm{id}_n \end{pmatrix}$$

as Jacobian matrix, id_n denoting the $n \times n$ identity matrix. The formula

$$\det \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} = \det(\mathcal{A}) \det(\mathcal{D} - \mathcal{C}\mathcal{A}^{-1}\mathcal{B})$$

for block matrices yields a Jacobian determinant of t^{-d} for this change of variable. For all $t \in (0,1)$, we thus obtain

$$\begin{aligned} & t^{(1-s)(p-1)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\big|\nabla_A u\big((1-\frac{t}{2})x + \frac{t}{2}y, \frac{t}{2}|y-x|\big)\big|^p}{|y-x|^{n-(1-s)p}} \,\mathrm{d}x\,\mathrm{d}y \\ & = \frac{1}{t^{1-s}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\big|\nabla_A u(\eta, |\xi|)\big|^p}{|\xi|^{n-(1-s)p}} \,\mathrm{d}\xi\,\mathrm{d}\eta = \frac{|\mathbb{S}^{n-1}|}{t^{1-s}} \int_{\mathbb{R}^n} \int_0^{+\infty} \frac{\big|\nabla_A u(\eta, r)\big|^p}{r^{1-(1-s)p}} \,\mathrm{d}r\,\mathrm{d}\eta, \end{aligned} \tag{3.37}$$

where the last equality follows from integration of radial functions. Since $0 < s < 1$, Tonelli's theorem and (3.37) imply that there exists $C > 0$ such that

$$\begin{aligned} & \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_0^1 t^{(1-s)(p-1)} \frac{\big|\nabla_A u\big((1-\frac{t}{2})x + \frac{t}{2}y, \frac{t}{2}|y-x|\big)\big|^p}{|y-x|^{n-(1-s)p}} \,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y \\ & \qquad \leq C \int_{\mathbb{R}^n} \int_0^{+\infty} \frac{\big|\nabla_A u(x,r)\big|^p}{r^{1-(1-s)p}} \,\mathrm{d}r\,\mathrm{d}x \end{aligned} \tag{3.38}$$

holds. This gives us the estimate for the first term of the right-hand side of (3.36). For the second term we obtain in a similar manner a constant $C > 0$ such that

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_0^1 t^{(1-s)(p-1)}\frac{\left|\nabla_A u\left((1-\frac{t}{2})y+\frac{t}{2}x,\frac{t}{2}|y-x|\right)\right|^p}{|y-x|^{n-(1-s)p}}\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y&\\ \le C\int_{\mathbb{R}^n}\int_0^{+\infty}\frac{\left|\nabla_A u(x,r)\right|^p}{r^{1-(1-s)p}}\,\mathrm{d}r\,\mathrm{d}x.&\quad(3.39)\end{aligned}$$

Finally for the last term, a computation as in (3.37) with the change of variable $\eta=\frac{x+y}{2}$ and $\xi=y-x$ as well as polar coordinates yields

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\left|\nabla_A u\left(\frac{x+y}{2},|y-x|\right)\right|^p}{|y-x|^{n-(1-s)p}}\,\mathrm{d}x\,\mathrm{d}y=|\mathbb{S}^{n-1}|\int_{\mathbb{R}^n}\int_0^{+\infty}\frac{\left|u(\eta,r)\right|^p}{r^{1-(1-s)p}}\,\mathrm{d}r\,\mathrm{d}\eta. \tag{3.40}$$

Integrating both sides of (3.36) with respect to x and y and using the estimates (3.38), (3.39) and (3.40), we reach

$$\left|u(\cdot,0)\right|^p_{W^{s,p}_{A'}(\mathbb{R}^n,P)}\le C\int_{\mathbb{R}^n}\int_0^{+\infty}\frac{\left|\nabla_A u(z,t)\right|^p+\beta^{p/2}\left|u(z,t)\right|^p}{t^{1-(1-s)p}}\,\mathrm{d}t\,\mathrm{d}z$$

which is (3.17). The proof of the first part of Proposition 3.3.1 is thus complete.

We now work towards (3.18). By the diamagnetic inequality we have $|\nabla|u||\le|\nabla_A u|$ everywhere in $\overline{\mathbb{R}^{n+1}_+}$ and therefore we have

$$\left|u(x,0)\right|\le\left|u(x,t)\right|+\int_0^t\left|\nabla_A u(x,\tau)\right|\,\mathrm{d}\tau$$

for all $x\in\mathbb{R}^n$ and $t\ge 0$. It follows that for $\lambda>0$ and $x\in\mathbb{R}^n$ we have

$$\begin{aligned}\left|u(x,0)\right|\le\frac{2}{\lambda}\int_{\frac{\lambda}{2}}^{\lambda}\int_0^t\left|\nabla_A u(x,\tau)\right|\,\mathrm{d}\tau\,\mathrm{d}t+\frac{2}{\lambda}\int_{\frac{\lambda}{2}}^{\lambda}\left|u(x,t)\right|\,\mathrm{d}t&\\ \le\int_0^{\lambda}\left|\nabla_A u(x,t)\right|\,\mathrm{d}t+\frac{2}{\lambda}\int_{\frac{\lambda}{2}}^{\lambda}\left|u(x,t)\right|\,\mathrm{d}t.&\quad(3.41)\end{aligned}$$

Since $0<s<1$, by applying Hölder's inequality with $\frac{1}{p}+\frac{1}{p'}=1$, we obtain $C>0$ such that

$$\begin{aligned}\left(\int_0^{\lambda}\left|\nabla_A u(x,t)\right|\,\mathrm{d}t\right)^p\le\left(\int_0^{\lambda}t^{sp/(p-1)-1}\,\mathrm{d}t\right)^{p-1}\int_0^{+\infty}\frac{\left|\nabla_A u(x,t)\right|^p}{t^{1-(1-s)p}}\,\mathrm{d}t&\\ \le C\lambda^{sp}\int_0^{+\infty}\frac{\left|\nabla_A u(x,t)\right|^p}{t^{1-(1-s)p}}\,\mathrm{d}t&\quad(3.42)\end{aligned}$$

as well as

$$\frac{\lambda^p}{2^p}\left(\int_{\frac{\lambda}{2}}^{\lambda}\left|u(x,t)\right|\,\mathrm{d}t\right)^p \leq \frac{C}{\lambda^{(1-s)p}}\int_0^{+\infty}\frac{\left|u(x,t)\right|^p}{t^{1-(1-s)p}}\,\mathrm{d}t. \tag{3.43}$$

Injecting the estimates (3.42) and (3.43) into (3.41) we obtain $C > 0$ such that

$$\int_{\mathbb{R}^n}\left|u(x,0)\right|^p\,\mathrm{d}x \leq C\left(\lambda^{sp}\int_{\mathbb{R}^n}\int_0^{+\infty}\frac{\left|\nabla_A u(x,t)\right|^p}{t^{1-(1-s)p}}\,\mathrm{d}t\,\mathrm{d}x\right. \\ \left. +\frac{1}{\lambda^{(1-s)p}}\int_{\mathbb{R}^n}\int_0^{+\infty}\frac{\left|u(x,t)\right|^p}{t^{1-(1-s)p}}\,\mathrm{d}t\,\mathrm{d}x\right). \tag{3.44}$$

Finally, optimising (3.44) with respect to $\lambda > 0$ yields $C > 0$ such that

$$\int_{\mathbb{R}^n}\left|u(x,0)\right|^p\,\mathrm{d}x \leq C\left(\int_{\mathbb{R}^n}\int_0^{+\infty}\frac{\left|\nabla_A u(x,t)\right|^p}{t^{1-(1-s)p}}\,\mathrm{d}t\,\mathrm{d}x\right)^{1-s}\left(\int_{\mathbb{R}^n}\int_0^{+\infty}\frac{\left|u(x,t)\right|^p}{t^{1-(1-s)p}}\,\mathrm{d}t\,\mathrm{d}x\right)^{s}$$

which is (3.18) and thus concludes the proof. ☐

With the same arguments, we obtain a localised version of Proposition 3.3.1 which is the equivalent of [34, Prop. 2.4] in the non-abelian case. The localised version will be used in Section 3.6 where we approximate a smooth and bounded domain Ω via local charts. Here and in what follows, $B(x,R)$ denotes the open ball in $\mathbb{R}^n$ centred in x with radius R. When $x = 0$, we also write $B_R = B(0,R)$.

Proposition 3.3.3. *Let $n \geq 1$, $0 < s < 1$, $1 \leq p < +\infty$ and $0 < R < +\infty$. Furthermore, assume that $A \in C^1\big(\overline{B_R}\times[0,R], \mathrm{Lin}(\mathbb{R}^{n+1},\mathfrak{o}(F))\big)$, $\|\mathcal{K}_A\|_\infty < \beta$. Then there exists a positive constant $C = C(n,s,p) > 0$ depending only on n, s and p such that for all $u \in C^1(B_R\times[0,R],F)$*

$$\int_{B_R}\int_{B_R}\frac{\left|R(x,y)u(y,0)-u(x,0)\right|^p}{|y-x|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y \\ \leq C\int_{B_R}\int_0^R\frac{\left|\nabla_A u(z,t)\right|^p+\beta^{p/2}\left|u(z,t)\right|^p}{t^{1-(1-s)p}}\,\mathrm{d}t\,\mathrm{d}z.$$

3.4 Extension to the half-space

We establish in this section an extension result from the boundary of the half-space to the entire half-space under the assumption of a connection with bounded curvature. The proof of this proposition is similar to the proof of [34, Prop. 3.1] which it generalises beyond the abelian case.

Proposition 3.4.1. *Let $n \geq 1$, $0 < s < 1$ and $1 \leq p < +\infty$. Furthermore, assume that $A \in C^1\big(\overline{\mathbb{R}^{n+1}_+}, \operatorname{Lin}\big(\mathbb{R}^{n+1}, \mathfrak{o}(F)\big)\big)$ and $\|\mathcal{K}_A\|_\infty \leq \beta$. Then there exists a constant $C = C(n,s,p) > 0$ depending only on n, s and p such that for all $U \in C^1_c(\mathbb{R}^n, F)$ there exists*

$$u \in C^1_c\big(\mathbb{R}^{n+1}_+, F\big) \cap C^0_c\big(\overline{\mathbb{R}^{n+1}_+}, F\big)$$

depending linearly on U and depending on β such that for all $x \in \mathbb{R}^n$ we have $u(x,0) = U(x)$,

$$\int_{\mathbb{R}^n} \int_0^{+\infty} \frac{\big|\nabla_A u(x,t)\big|^p}{t^{1-(1-s)p}} \,\mathrm{d}t\,\mathrm{d}x \leq C\Big(|U|^p_{W^{s,p}_{A'}(\mathbb{R}^n,P)} + \beta^{sp/2}\|U\|^p_{L^p(\mathbb{R}^n,F)}\Big) \tag{3.45}$$

and

$$\int_{\mathbb{R}^n} \int_0^{+\infty} \frac{\big|u(x,t)\big|^p}{t^{1-(1-s)p}} \,\mathrm{d}t\,\mathrm{d}x \leq \frac{C}{\beta^{(1-s)p/2}}\|U\|^p_{L^p(\mathbb{R}^n,F)}, \tag{3.46}$$

where $A'(\cdot)[*] = A(\cdot,0)[*,0] \in C^1\big(\mathbb{R}^n, \operatorname{Lin}\big(\mathbb{R}^n, \mathfrak{o}(F)\big)\big)$

Choosing the appropriate $0 < s < 1$ gives us (ii) in Theorem 3.1.2.

Proof of Theorem 3.1.2 (ii). Since $1 < p < +\infty$, we may consider $0 < s = 1 - \frac{1}{p} < 1$ in Proposition 3.4.1. The first inequality (3.45) then states that

$$\begin{aligned}\|\nabla_A u\|_{L^p(\mathbb{R}^{n+1}_+, \operatorname{Lin}(\mathbb{R}^{n+1},F))} &\leq C\left(|U|^p_{W^{1-1/p,p}_{A'}(\mathbb{R}^n,P)} + \beta^{(p-1)/2}\|U\|^p_{L^p(\mathbb{R}^n,F)}\right)^{1/p} \\ &\leq C\left(|U|_{W^{1-1/p,p}_{A'}(\mathbb{R}^n,P)} + \beta^{(p-1)/2p}\|U\|_{L^p(\mathbb{R}^n,F)}\right)\end{aligned} \tag{3.47}$$

while the inequality (3.46) yields

$$\beta^{1/2}\|u\|_{L^p(\mathbb{R}^{n+1}_+,F)} \leq C\|U\|_{L^p(\mathbb{R}^n,F)} \tag{3.48}$$

for a constant $C > 0$ depending only on n and p and thus summing (3.47) with (3.48) yields $C' > 0$ depending only on n and p such that

$$\|\nabla_A u\|_{L^p(\mathbb{R}^{n+1}_+, \operatorname{Lin}(\mathbb{R}^{n+1},F))} + \beta^{1/2}\|u\|_{L^p(\mathbb{R}^{n+1}_+,F)} \leq C'\left(|U|_{W^{1-1/p,p}_{A'}(\mathbb{R}^n,P)} + \beta^{(p-1)/2p}\|U\|_{L^p(\mathbb{R}^n,F)}\right)$$

which concludes the proof. □

We already have all elements needed to prove Proposition 3.4.1. The main difficulty is finding the function $u \in C^1_c\big(\overline{\mathbb{R}^{n+1}_+}, F\big)$ with the right properties. Fortunately, the candidate given by Nguyen and Van Schaftingen in their proof of [34, Prop. 3.1] works here as well. The main differences in the proof appear in the estimations of the functions L_2 and L_3 where we use results established in Section 3.2.

As in the proof of Proposition 3.3.1, we denote by $C', C > 0$ multiple different positive constants depending at most on n, s and p. Any changes in between lines will be made explicit. Finally, we again identify $\mathbb{R}^n$ with $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$.

Proof of Proposition 3.4.1. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be a positive function such that

$$\int_{\mathbb{R}^n} \varphi \, \mathrm{d}x = 1 \quad \text{and} \quad \varphi = 0 \text{ on } \mathbb{R}^n \setminus B(0,1) \tag{3.49}$$

and let $\theta \in C^\infty(\mathbb{R})$ be a function such that

$$\theta = 1 \text{ in } \big(-a/2, a/2\big), \quad \theta = 0 \text{ on } \mathbb{R} \setminus (-a, a) \quad \text{and } |\theta'| \leq \frac{C}{a} \text{ on } \mathbb{R} \tag{3.50}$$

where $a = \beta^{-1/2}$ and $C > 0$ is positive constant independent of a. For all $t > 0$, we set

$$\varphi_t(\cdot) = t^{-n}\varphi(\cdot/t) \colon \mathbb{R}^n \to \mathbb{R}$$

which defines a sequence of mollifiers. Given $U \in C_c^1(\mathbb{R}^n, F)$, we define the function $u \colon \mathbb{R}_+^{n+1} \to F$ by setting

$$u(x,t) = \theta(t) \int_{\mathbb{R}^n} \varphi_t(x-y) R\big((x,t), y\big) U(y) \, \mathrm{d}y \tag{3.51}$$

for all $(x,t) \in \mathbb{R}_+^{n+1}$. We begin by showing that u takes the right values on the boundary, i.e. we need to show that, for all $x_0 \in \mathbb{R}^n$,

$$\lim_{\substack{(x,t)\to(x_0,0)\\ t>0}} u(x,t) = U(x_0).$$

By (3.50), we have for $0 < t < \frac{a}{2}$

$$u(x,t) = \int_{\mathbb{R}^n} \varphi_t(x-y) R\big((x,t), y\big) U(y) \, \mathrm{d}y,$$

and the change of variable $y = x - tz$ the yields

$$u(x,t) = \int_{\mathbb{R}^n} \varphi(z) R\big((x,t), x - tz\big) U(x - tz) \, \mathrm{d}z \tag{3.52}$$

and we deduce from (3.52) and Lebesgue's dominated convergence theorem that

$$\lim_{\substack{(x,t)\to(x_0,0)\\ t>0}} u(x,t) = R(x_0, x_0) U(x_0) = U(x_0) \tag{3.53}$$

which shows that $u \in C^0(\overline{\mathbb{R}^{n+1}}, F)$ and $u(x_0, 0) = U(x_0)$.

We now work towards the estimates (3.45) and (3.46). Let $e_1, \dots, e_{n+1}$ denote the usual basis of $\mathbb{R}^{n+1}$. For all $1 \leq j \leq n$, differentiating under the integral sign in (3.51) gives us

$$\begin{aligned} &\partial_j u(x,t) + A(x,t)[e_j]u(x,t) \\ &\quad = \theta(t) \int_{\mathbb{R}^n} \Big(\partial_j \varphi_t(x-y) R\big((x,t),y\big) + \varphi_t(x-y)\partial_j R\big((x,t),y\big)\Big) U(y)\,\mathrm{d}y \\ &\qquad\qquad + A(x,t)[e_j]u(x,t) \end{aligned} \tag{3.54}$$

and, with the understanding that t is the $n+1$th variable in $\mathbb{R}^{n+1}$,

$$\begin{aligned} \partial_{n+1} u(x,t) + A(x,t)[e_{n+1}] &= \theta'(t) \int_{\mathbb{R}^n} \varphi_t(x-y) R\big((x,t),y\big) U(y)\,\mathrm{d}y \\ &- \theta(t) \int_{\mathbb{R}^n} \left(\frac{n}{t}\varphi_t(x-y) + \frac{1}{t^{n+2}} \mathrm{D}\varphi\Big(\frac{x-y}{t}\Big)[x-y]\right) R\big((x,t),y\big)U(y)\,\mathrm{d}y \\ &+ \theta(t) \int_{\mathbb{R}^n} \varphi_t(x-y)\partial_{n+1} R\big((x,t),y\big)U(y)\,\mathrm{d}y + A(x,t)[e_{n+1}]u(x,t). \end{aligned} \tag{3.55}$$

Let us introduce the notations $\Phi_t^j(z) = \partial_j \varphi_t(z)$ if $1 \leq j \leq n$ and $\Phi_t^{n+1}(z) = \frac{n}{t}\varphi_t(z) + t^{-(n+2)}\mathrm{D}\varphi(z/t)[z]$ for $z \in \mathbb{R}^n$. Then we have for all $1 \leq j \leq n+1$, since $\int_{\mathbb{R}^n} \Phi_t^j(z)\,\mathrm{d}z = 0$,

$$\begin{aligned} &\int_{\mathbb{R}^n} \Phi_t^j(x-y) R\big((x,t),y\big) U(y)\,\mathrm{d}y \\ &\qquad \int_{\mathbb{R}^n} \Phi_t^j(x-y) \Big(R\big((x,t),y\big) - R\big((x,t),y\big)R(x,y)\Big) U(y)\,\mathrm{d}y \\ &\qquad\qquad + \int_{\mathbb{R}^n} \Phi_t^j(x-y) R\big((x,t),y\big) \Big(R(x,y)U(y) - U(x)\Big)\,\mathrm{d}y. \end{aligned} \tag{3.56}$$

Using the fact that $R(v,w)$ is an isometry for any $v, w \in \overline{\mathbb{R}^{n+1}_+}$, we deduce from (3.49), (3.50), (3.54), (3.55) and (3.56) that there exists $C > 0$ such that for all $(x,t) \in \mathbb{R}^{n+1}_+$

$$\big|\nabla_A u(x,t)\big| \leq C\big(L_1(x,t) + L_2(x,t) + L_3(x,t) + L_4(x,t)\big) \tag{3.57}$$

where for all $(x,t) \in \mathbb{R}^{n+1}_+$ we set

$$\begin{aligned} L_1(x,t) &= \frac{\mathbb{1}_{(0,a)}(t)}{t^{n+1}} \int_{B(x,t)} \big|R(x,y)U(y) - U(x)\big|\,\mathrm{d}y, \\ L_2(x,t) &= \frac{\mathbb{1}_{(0,a)}(t)}{t^{n+1}} \int_{B(x,t)} \big|R\big((x,t),y\big)R(y,x)R\big(x,(x,t)\big) - \mathrm{id}_F\big|\big|U(y)\big|\,\mathrm{d}y, \end{aligned}$$

$$L_3(x,t) = \frac{\mathbb{1}_{(0,a)}(t)}{t^n} \int_{B(x,t)} \sum_{j=1}^{n+1} \left|\partial_j R\big((x,t),y\big) + A(x,t)[e_j] R\big((x,t),y\big)\right| \left|U(y)\right| \,\mathrm{d}y,$$

$$L_4(x,t) = \frac{\mathbb{1}_{(0,a)}(t)}{at^n} \int_{B(x,t)} \left|U(y)\right| \,\mathrm{d}y.$$

In order to estimate the integral of $\nabla_A u$ in (3.45), we will now estimate the related integrals of the functions L_1 to L_4.

First, using Hölder's inequality with $\frac{1}{p} + \frac{1}{p'} = 1$ we have for all $(x,t) \in \mathbb{R}^{n+1}_+$

$$\begin{aligned}\left|L_1(x,t)\right|^p &\leq \frac{\left|B(x,t)\right|^{p-1}}{t^{(n+1)p}} \int_{B(x,t)} \left|R(x,y)U(y) - U(x)\right|^p \,\mathrm{d}y \\ &= \frac{1}{t^{p+n}} \int_{B(x,t)} \left|R(x,y)U(y) - U(x)\right|^p \,\mathrm{d}y \qquad (3.58)\end{aligned}$$

which together with Tonelli's theorem implies our first estimate

$$\begin{aligned}&\int_{\mathbb{R}^n} \int_0^{+\infty} \frac{\left|L_1(x,t)\right|^p}{t^{1-(1-s)p}} \,\mathrm{d}t \,\mathrm{d}x \\ &\qquad \leq \int_{\mathbb{R}^n} \int_0^{+\infty} \frac{1}{t^{1+n+sp}} \left(\int_{B(x,t)} \left|R(x,y)U(y) - U(x)\right|^p \,\mathrm{d}y\right) \mathrm{d}t \,\mathrm{d}x \\ &\qquad = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left|R(x,y)U(y) - U(x)\right|^p \left(\int_{|y-x|}^{+\infty} \frac{1}{t^{1+n+sp}} \,\mathrm{d}t\right) \mathrm{d}y \,\mathrm{d}x \\ &\qquad \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\left|R(x,y)U(y) - U(x)\right|^p}{|y-x|^{n+sp}} \,\mathrm{d}x \,\mathrm{d}y. \qquad (3.59)\end{aligned}$$

Next, we have for all $y \in B(x,t)$ in vertue of Proposition 3.2.1

$$\left|R\big((x,t),y\big) R(y,x) R\big(x,(x,t)\big) - \mathrm{id}_F\right| \leq \sqrt{\beta t |y-x|} \leq \beta^{1/2} t.$$

Therefore, Hölder's inequality as in (3.58) together with Tonelli's theorem yields, since $a = \beta^{-1/2}$,

$$\begin{aligned}\int_{\mathbb{R}^n} \int_0^{+\infty} \frac{\left|L_2(x,t)\right|^p}{t^{1-(1-s)p}} \,\mathrm{d}t \,\mathrm{d}x &\leq \int_{\mathbb{R}^n} \int_0^{a} \frac{\beta^{p/2}}{t^{d+1-(1-s)p}} \left(\int_{B(x,t)} \left|U(y)\right|^p \,\mathrm{d}y\right) \mathrm{d}t \,\mathrm{d}x \\ &\leq C' \int_0^{a} \frac{\beta^{p/2}}{t^{1-(1-s)p}} \int_{\mathbb{R}^n} \left|U(y)\right|^p \,\mathrm{d}y \,\mathrm{d}t = C\beta^{sp/2} \int_{\mathbb{R}^n} \left|U(y)\right|^p \,\mathrm{d}y \qquad (3.60)\end{aligned}$$

for some $C', C > 0$, giving us our second estimate.

To compute $\partial_j R\big((x,t),y\big)$ for all $1 \le j \le n+1$, we recall that $R(v,w)$ is a notation for $R_{\gamma_{w,v}}(1,0) = \mathrm{Pt}_{\gamma_{w,v}}(1)$ which is the solution of the initial value problem

$$\begin{cases} \partial_\tau R_{\gamma_{w,v}}(\tau,0) = -A\big(\gamma_{w,v}(\tau)\big)[\gamma'_{w,v}(\tau)]R_{\gamma_{w,v}}(\tau,0) & \text{on } [0,1], \\ R_{\gamma_{w,v}}(0,0) = \mathrm{id}_F, \end{cases}$$

with parameter v and w, where $\gamma_{w,v}(\tau) = (1-\tau)w + \tau v$ for $\tau \in [0,1]$. Applying Lemma 3.2.2 together with Lemma 3.2.3 and integrating by parts as in the proof of Lemma 3.2.4, we obtain

$$\begin{aligned} \partial_j R\big((x,t),y\big) &= -A(x,t)[e_j]R\big((x,t),y\big) \\ &\qquad + \int_0^1 R\big((x,t),\gamma(\tau)\big)\mathcal{K}_A\big(\gamma(\tau)\big)\big[(x-y,t),\tau e_j\big]R\big(\gamma(\tau),y\big)\,\mathrm{d}\tau \end{aligned}$$

where $\gamma\colon [0,1] \to \overline{\mathbb{R}^{n+1}_+}$ is defined by $\gamma(\tau) = \big((1-\tau)y + \tau x, \tau t\big)$. It follows that there exists $C > 0$ such that for all $1 \le j \le n+1$ and $y \in B(x,t)$

$$\big|\partial_j R\big((x,t),y\big) + A(x,t)[e_j]R\big((x,t),y\big)\big| \le C\|\mathcal{K}_A\|_\infty t \le C\beta t.$$

We then reach as in (3.60) the third estimate

$$\begin{aligned} &\int_{\mathbb{R}^n}\int_0^{+\infty} \frac{\big|L_3(x,t)\big|^p}{t^{1-(1-s)p}}\,\mathrm{d}t\,\mathrm{d}x \\ &\quad \le C'\int_{\mathbb{R}^n}\int_0^a \frac{\beta^p t^p}{t^{n+1-(1-s)p}}\left(\int_{B(x,t)}\big|U(y)\big|^p\,\mathrm{d}y\right)\mathrm{d}t\,\mathrm{d}x = C\beta^{sp/2}\int_{\mathbb{R}^n}\big|U(y)\big|^p\,\mathrm{d}y \end{aligned} \tag{3.61}$$

where $C', C > 0$ are as usually.

Finally we argue as previously to obtain $C' > 0$ and $C > 0$ such that

$$\begin{aligned} \int_{\mathbb{R}^n}\int_0^{+\infty} \frac{\big|L_4(x,t)\big|^p}{t^{1-(1-s)p}}\,\mathrm{d}t\,\mathrm{d}x &\le C'\int_{\mathbb{R}^n}\int_0^a \frac{1}{a^p t^{1+n-(1-s)p}}\left(\int_{B(x,t)}\big|U(y)\big|^p\,\mathrm{d}y\right)\mathrm{d}t\,\mathrm{d}x \\ &= C\beta^{sp/2}\int_{\mathbb{R}^n}\big|U(y)\big|^p\,\mathrm{d}y. \end{aligned} \tag{3.62}$$

Combining (3.57), (3.59), (3.60), (3.61) and (3.62), we reach

$$\int_{\mathbb{R}^n}\int_0^{+\infty} \frac{\big|\nabla_A u(x,t)\big|^p}{t^{1-(1-s)p}}\,\mathrm{d}t\,\mathrm{d}x \le C\Big(|U|_{W^{s,p}_{A'}(\mathbb{R}^n,P)} + \beta^{sp/2}\|U\|^p_{L^p(\mathbb{R}^n,F)}\Big) \tag{3.63}$$

which is precisely (3.45), the first estimate of Proposition 3.4.1.

Similarly to (3.62) and using (3.49), (3.50) as well as (3.51), we reach

$$\int_{\mathbb{R}^n}\int_0^{+\infty} \frac{\left|u(x,t)\right|^p}{t^{1-(1-s)p}}\,\mathrm{d}t\,\mathrm{d}x \le \frac{C}{\beta^{(1-s)p/2}}\|U\|^p_{L^p(\mathbb{R}^n,F)}. \tag{3.64}$$

The conclusion follows from (3.63) and (3.64). □

As in Section 3.3, we also have a local version of Proposition 3.4.1 which will be used in Section 3.6. The proof uses the same arguments as the proof of Proposition 3.4.2.

Proposition 3.4.2. *Let $n \ge 1$, $0 < s < 1$, $1 \le p < +\infty$ and $0 < R < +\infty$. Furthermore, assume that $A \in C^1\big(\overline{B_R}\times[0,R], \mathrm{Lin}\big(\mathbb{R}^{n+1}, \mathfrak{o}(F)\big)\big)$ and $\|\mathcal{K}_A\|_{L^\infty(B_{2R}\times[0,R])} + \frac{1}{R^2} \le \beta$. Then there exists a constant $C = C(n,s,p) > 0$ depending only on n, s and p such that for all $U \in C^1(B_{2R},F)$ there exists*

$$u \in C^1\big(B_R\times(0,R),F\big)\cap C^0\big(\overline{B_R}\times[0,R],F\big)$$

depending linearly on U such that $u(\cdot,0) = U(\cdot)$ on B_R,

$$\int_{B_R}\int_0^R \frac{\left|\nabla_A u(x,t)\right|^p}{t^{1-(1-s)p}}\,\mathrm{d}t\,\mathrm{d}x \\ \le C\left(\int_{B_{2R}}\int_{B_{2R}}\frac{\left|R(x,t)U(y)-U(x)\right|^p}{|y-x|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y + \beta^{sp/2}\int_{B_{2R}}\left|U(x)\right|^p\,\mathrm{d}x\right)$$

and

$$\int_{B_R}\int_0^R \frac{\left|u(z,t)\right|^p}{t^{1-(1-s)p}}\,\mathrm{d}t\,\mathrm{d}z \le \frac{C}{\beta^{(1-s)p/2}}\int_{B_{2R}}\left|U(x)\right|^p\,\mathrm{d}x.$$

3.5 Characterisation of trace spaces on the half-space

We prove in this section a density result concerning the usual gauge covariant Sobolev space on the half-space $\mathbb{R}^{n+1}_+$ and the fractional gauge Covariant Sobolev space on $\mathbb{R}^n = \partial(\mathbb{R}^{n+1}_+)$. Together with Theorem 3.1.2, they imply Theorem 3.1.3 by a standard density argument.

Our results are the analogue of [34, Lemma 4.1] and [34, Lemma 4.2] which state these results in the case of magnetic Sobolev spaces. The here given proofs are analogue to the proofs given in [34] which we follow.

Lemma 3.5.1. *Let* $1 \leq p < +\infty$ *and* $A \in C^0\big(\mathbb{R}^{n+1}_+, \operatorname{Lin}\big(\mathbb{R}^{n+1}, \mathfrak{o}(F)\big)\big)$. *Then the space* $C^\infty_c\big(\overline{\mathbb{R}^{n+1}_+}, F\big)$ *is dense in* $W^{1,p}_A(\mathbb{R}^{n+1}_+, P)$.

Proof. Let $\chi \in C^\infty_c(\mathbb{R}^{n+1}, \mathbb{R})$ such that $0 \leq \chi \leq 1$ and $\chi(x) = 1$ for all $|x| < 1$. For all $\lambda > 0$, we set

$$\begin{aligned} \chi_\lambda \colon \mathbb{R}^{n+1} &\to \mathbb{R}, \\ x &\mapsto \chi_\lambda(x) = \chi(x/\lambda). \end{aligned}$$

We then compute, for $u \in W^{1,p}_A(\mathbb{R}^{n+1}_+, P)$,

$$\nabla_A(u - \chi_\lambda u) = -u\nabla\chi_\lambda + (1 - \chi_\lambda)\nabla_A u.$$

Therefore, we have

$$\begin{aligned} &\|u - \chi_\lambda u\|^p_{W^{1,p}_A(\mathbb{R}^{n+1},P)} \\ &\qquad \leq C \int_{\mathbb{R}^{n+1}} (1 - \chi_\lambda(x))^p \left(\big|u(x)\big|^p + \big|\nabla_A u(x)\big|^p\right) + \big|\nabla\chi_\lambda(x)\big|^p \big|u(x)\big|^p \,\mathrm{d}x \to 0 \end{aligned}$$

as $\lambda \to 0$ since $|\nabla\chi_\lambda| \leq C'/\lambda$ for some constant $C' > 0$. It follows that any function in $W^{1,p}_A(\mathbb{R}^{n+1}_+, P)$ can be approximated by functions in $W^{1,p}_A(\mathbb{R}^{n+1}, P)$ whose support is bounded.

A classical result in extensions of Sobolev mappings asserts that any function in the Sobolev space $W^{1,p}_0(\mathbb{R}^{n+1}_+, P) = W^{1,p}(\mathbb{R}^{n+1}_+, F)$ can be extended by even reflection to the whole space $\mathbb{R}^{n+1}$, see [43, Lemma 6.2.1] (see also [1, Sect. 5.4] and [6, Sect. 9.2]). In particular, $C^\infty_c\big(\overline{\mathbb{R}^{n+1}_+}, F\big)$ is dense in $W^{1,p}_0(\mathbb{R}^{n+1}_+, P) = W^{1,p}(\mathbb{R}^{n+1}_+, F)$ by restricting smooth and compactly supported functions on $\mathbb{R}^{n+1}$ to the half space, see [6, Thm. 9.2] or [43, Thm. 6.1.10].

Since A is locally bounded on $\mathbb{R}^{n+1}_+$, any function $u \in W^{1,p}_A(\mathbb{R}^{n+1}_+, P)$ with bounded support is also in $W^{1,p}_0(\mathbb{R}^{n+1}_+, P)$. In fact, for such a function $u \in W^{1,p}_A(\mathbb{R}^{n+1}_+, P)$ with bounded support, we have

$$\|u\|_{W^{1,p}_0} \leq \big(1 + \|A\|_{L^\infty(\operatorname{supp} u)}\big)\|u\|_{W^{1,p}_A}.$$

Since the space $C^\infty_c\big(\overline{\mathbb{R}^{n+1}_+}, F\big)$ is dense in $W^{1,p}_0(\mathbb{R}^{n+1}_+, P) = W^{1,p}(\mathbb{R}^{n+1}_+, F)$, a function $u \in W^{1,p}_A(\mathbb{R}^{n+1}_+, P)$ with bounded support can be approximated in $W^{1,p}_0(\mathbb{R}^{n+1}, P)$ by $C^\infty_c\big(\overline{\mathbb{R}^{n+1}_+}, F\big)$ functions. Indeed, for a mapping $u \in W^{1,p}_A(\mathbb{R}^{n+1}_+, P)$ with bounded support, we have

$$\|u\|_{W^{1,p}_A} \leq \big(1 + \|A\|_{L^\infty(\operatorname{supp} u)}\big)\|u\|_{W^{1,p}_0}.$$

It follows that $W_A^{1,p}(\mathbb{R}^{n+1}, P)$ functions with bounded support can be approximated in $W_A^{1,p}(\mathbb{R}^{n+1}, P)$ by smooth and compactly supported functions from $C_c^\infty\big(\overline{\mathbb{R}_+^{n+1}}, F\big)$. The conclusion follows by a diagonal argument. ◻

Lemma 3.5.2. *Let $0 < s < 1$ and $1 \leq p < +\infty$. Furthermore, assume that $A \in C^0\big(\mathbb{R}^n, \mathrm{Lin}(\mathbb{R}^n, F)\big)$. Then the space $C_c^\infty(\mathbb{R}^n, F)$ is dense in $W_A^{s,p}(\mathbb{R}^n, P)$.*

Proof. Consider $\chi \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$ such that $0 \leq \chi \leq 1$ and $\chi = 1$ on the unit ball B_1. For $\lambda > 0$, we set

$$\begin{aligned} \chi_\lambda \colon \mathbb{R}^n &\longrightarrow \mathbb{R}, \\ x &\longmapsto \chi_\lambda(x) = \chi(x/\lambda). \end{aligned}$$

Let $u \in W_A^{s,p}(\mathbb{R}^n, P)$. Then, for all $\lambda > 0$ and for all $x, y \in \mathbb{R}^n$, we have

$$\begin{aligned} &\big(1 - \chi_\lambda(y)\big)R(x,y)u(y) - \big(1 - \chi_\lambda(x)\big)u(x) \\ &\quad = \left(1 - \frac{\chi_\lambda(x) + \chi_\lambda(y)}{2}\right)\big(R(x,y)u(y) - u(x)\big) + \frac{\chi_\lambda(x) - \chi_\lambda(y)}{2}\big(R(x,y)(y) + u(x)\big). \end{aligned} \tag{3.65}$$

Furthermore, for all $y \in \mathbb{R}^n$ and $\lambda > 0$, we have

$$\int_{\mathbb{R}^n} \frac{\big|\chi_\lambda(x) - \chi_\lambda(y)\big|^p}{|y - x|^{n+sp}} \,\mathrm{d}x \leq \frac{C}{\lambda^{sp}} \tag{3.66}$$

for some $C > 0$. It follows from (3.65) and (3.66) that, for every $\lambda > 0$,

$$\begin{aligned} &|\chi_\lambda u - u|^p_{W_A^{s,p}(\mathbb{R}^n, P)} \\ &\quad \leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(1 - \frac{\chi_\lambda(x) + \chi_\lambda(y)}{2}\right)^p \frac{\big|R(x,y) - u(x)\big|^p}{|x - y|^{n+sp}} \,\mathrm{d}y \,\mathrm{d}x + C' \int_{\mathbb{R}^n} \frac{\big|u(y)\big|^p}{\lambda^{sp}} \,\mathrm{d}y \end{aligned} \tag{3.67}$$

where $C, C' > 0$. We deduce from (3.67) and Lebesgue's dominated convergence theorem that

$$|u - \chi_\lambda u|_{W_A^{s,p}} \longrightarrow 0, \quad \lambda \longrightarrow +\infty$$

which implies that $\|u - \chi_\lambda u\|_{W_A^{s,p}} \longrightarrow 0$ when $\lambda \longrightarrow +\infty$. In other words, $W_A^{s,p}(\mathbb{R}^n, P)$ functions with bounded support are dense in $W_A^{s,p}(\mathbb{R}^n, P)$.

Since A is locally bounded on $\mathbb{R}^n$, any function in $W_A^{s,p}(\mathbb{R}^n, P)$ whose support is bounded is also in $W_0^{s,p}(\mathbb{R}^n, P) = W^{s,p}(\mathbb{R}^n, F)$. Indeed, if $u \in L^p(\mathbb{R}^n, F)$ has bounded support, then, by the triangle inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\left|R(x,y)u(y)-u(x)\right|^p}{|y-x|^{n+sp}} - \frac{\left|u(y)-u(x)\right|^p}{|y-x|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y \\ \leq C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\left|R(x,y)u(y)-u(y)\right|^p}{|y-x|^{n+sp}} \\ \leq \sup_{y\in\operatorname{supp} u}\left(\int_{\mathbb{R}^n}\frac{\left|R(x,y)-\mathrm{id}_F\right|^p}{|y-x|^{n+sp}}\,\mathrm{d}x\right)\int_{\mathbb{R}^n}\left|u(y)\right|^p\,\mathrm{d}y. \quad (3.68)\end{aligned}$$

Since the parallel transport is an isometry, we have

$$\left|R(x,y)-\mathrm{id}_F\right| \leq 2. \quad (3.69)$$

By Proposition 2.3.5, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}R\big(\gamma_{y,x}(t),y\big) = -A\big(\gamma_{y,x}(t)\big)[x-y]R\big(\gamma_{y,x}(t),y\big)$$

and by the fundamental theorem of calculus, we have

$$\left|R(x,y)-\mathrm{id}_F\right| = \left|\int_0^1 -A\big(\gamma_{y,x}(t)\big)[x-y]R\big(\gamma_{y,x}(t),y\big)\,\mathrm{d}t\right| \leq \|A\|_{L^\infty(\gamma_{y,x}([0,1]))}|x-y|. \quad (3.70)$$

It follows from (3.69) and (3.70) that

$$\begin{aligned}\sup_{y\in\operatorname{supp} u}\int_{\mathbb{R}^n}\frac{\left|R(x,y)-\mathrm{id}_F\right|^p}{|y-x|^{n+sp}}\,\mathrm{d}x \\ \leq C\sup_{y\in\operatorname{supp} u}\int_{\mathbb{R}^n}\frac{\min\big\{1,|y-x|^p\|A\|^p_{L^\infty(\operatorname{supp} u)}\big\}}{|y-x|^{n+sp}}\,\mathrm{d}x < +\infty. \quad (3.71)\end{aligned}$$

Combining (3.68) and (3.71), we deduce that if $u \in L^p(\mathbb{R}^n, F)$ has compact support, then

$$u \in W^{s,p}_A(\mathbb{R}^n P) \iff u \in W^{s,p}_0(\mathbb{R}^n, P).$$

Since $C^\infty(\mathbb{R}^n, F)$ is dense in $W^{s,p}(\mathbb{R}^n, P)$, see [27, Thm. 6.66], any function $u \in W^{s,p}_A(\mathbb{R}^n, P)$ can be approximated in $W^{s,p}_0(\mathbb{R}^n, P)$ by functions with uniformly compact support. Since A is locally bounded, the above calculation shows that the approximating sequence also converges in $W^{s,p}_A(\mathbb{R}^n, P)$, and the conclusion follows by a diagonal argument. □

3.6 Traces and extensions on a domain

In this section, we consider the trace problem on a general open and bounded domain $\Omega \subset \mathbb{R}^{n+1}$ of class C^1, that is, its boundary $\partial\Omega$ is a C^1 manifold of dimension n. We then use the fact that Ω and its boundary can locally be approximated by subsets of the Euclidean space such as half-balls $B(0,R) \cap \mathbb{R}^{n+1}_+$ via local charts up to the boundary. The results of Chapter 3 on the trace and extension are then used locally on the charts, and the local results are glued back together. The structure is inspired by [34, Sect. 6] which studies the same problem in the case of magnetic Sobolev spaces.

Proposition 2.2.18 asserts that, assuming that the connection 1-form A is bounded, the gauge covariant Sobolev space is characterised by

$$W^{1,p}_A(\Omega, P) = W^{1,p}(\Omega, F).$$

Therefore, the trace space of $W^{1,p}_A(\Omega, P)$ for a bounded A is simply the fractional Sobolev space $W^{1-1/p,p}(\partial\Omega, F)$. Although this already characterises the trace space of $W^{1,p}_A(\Omega, P)$ when

$$A \in C^1\big(\overline{\Omega}, \operatorname{Lin}\big(\mathbb{R}^{n+1}, \mathfrak{o}(F)\big)\big),$$

this does not yield a gauge invariant estimate. Since in that case A is continuous up to the boundary, the trace space of $W^{1,p}_A(\Omega, P)$ should be characterised with a gauge invariant dependence in A. We achieve this at the end of this section.

We begin by introducing some geometric tools needed for our study. We shall denote by W a chart of Ω and by V a chart of $\partial\Omega$. When $\psi\colon \overline{W} \to \overline{\Omega}$ is a chart up to the boundary such that $\psi(\overline{W}) \cap \partial\Omega$ is not empty, one should think of V as a subset of ∂W with $\psi(V) = \psi(\overline{W}) \cap \partial\Omega$.

The pull-back [21, 24, 25, 38] is a tool that allows us to transport a 1-form from a domain to another diffeomorphic domain. In particular, we shall use it to transport a connection from Ω to a local chart (W, ψ) of Ω.

Definition 3.6.1 (Pull-back). *Let X be a finite dimensional vector space and $W \subset \mathbb{R}^n$ open. Assume that $\psi\colon W \to \mathbb{R}^n$ is a diffeomorphism on its image and consider a map $L\colon \psi(W) \to \operatorname{Lin}(\mathbb{R}^n, X)$. Then, the* pull-back

$$\psi^* L\colon W \to \operatorname{Lin}(\mathbb{R}^n, X)$$

of L by ψ is defined for all $x \in W$ by

$$\psi^* L(x)[\cdot] = L\big(\psi(x)\big) \circ \mathrm{D}\psi(x)[\cdot] \in \operatorname{Lin}(\mathbb{R}^n, X).$$

When $L = A\colon \Omega \to \mathrm{Lin}\big(\mathbb{R}^n, \mathfrak{o}(F)\big)$ is a connection form on Ω, the pull-back $\psi^* A\colon W \to \mathrm{Lin}\big(\mathbb{R}^n, \mathfrak{o}(F)\big)$ defines a natural connection form on W, and thus also a covariant derivative $\nabla_{\psi^* A} = \nabla + \psi^* A$. When $L = \nabla_A u\colon \Omega \to \mathrm{Lin}(\mathbb{R}^n, F)$ is the weak covariant derivative of some $u \in W^{1,p}_A(\Omega, P)$, the pull-back $\psi^*(\nabla_A u)$ also defines a weak covariant derivative of a certain map on W. The following classical lemma shows that these two induced connections are one and the same.

Lemma 3.6.2. *Let $W \subset \mathbb{R}^n$ be an open and bounded domain and let $\psi\colon \Omega \to \mathbb{R}^n$ be a diffeomorphism up to the boundary on its image. Then*

$$u \circ \psi \in W^{1,p}_{\psi^* A}(W, P) \iff u \in W^{1,p}_A\big(\psi(W), P\big)$$

and for almost all $x \in W$

$$\nabla_{\psi^* A}(u \circ \psi)(x) = \big(\psi^*(\nabla_A u)\big)(x).$$

Consequently, we have

$$\frac{\big|\nabla_A u\big(\psi(x)\big)\big|}{\big\|\mathrm{D}(\psi^{-1})\big\|_\infty} \le \big|\nabla_{\psi^* A}(u \circ \psi)(x)\big| \le \|\mathrm{D}\psi\|_\infty \big|\nabla_A u\big(\psi(x)\big)\big|$$

for almost all $x \in W$.

Proof. By Definition 3.6.1 and the chain rule, we have for almost all $x \in W$

$$\begin{aligned}\nabla_{\psi^* A}&(u \circ \psi)(x)\\ &= \nabla u\big(\psi(x)\big) + A\big(\psi(x)\big) \circ \mathrm{D}\psi(x) \cdot u\big(\psi(x)\big) = \nabla_A u\big(\psi(x)\big) \circ \mathrm{D}\psi(x)\\ &= \psi^*(\nabla_A u)(x). \qquad (3.72)\end{aligned}$$

Since ψ is a diffeomorphism on its image up to the boundary, it follows that

$$\big|\nabla_{\psi^* A}(u \circ \psi)(x)\big| \le \|\mathrm{D}\psi\|_\infty \big|\nabla_A u\big(\psi(x)\big)\big| \qquad (3.73)$$

and

$$\big|\nabla_A u\big(\psi(x)\big)\big| = \big|\nabla_{\psi^* A}(u \circ \psi)(x) \circ \mathrm{D}(\psi^{-1})\big(\psi(x)\big)\big| \le \big\|\mathrm{D}(\psi^{-1})\big\|_\infty \big|\nabla_{\psi^* A}(u \circ \psi)(x)\big|. \qquad (3.74)$$

The conclusion follows from (3.72), (3.73) and (3.74). □

Given a smooth path $\gamma \in C^1\big([0,1], W\big)$, the connection $\nabla_{\psi^* A}$ gives rise to a parallel transport $\mathrm{Pt}^{\psi^* A}_\gamma$ along γ which is defined as in Definition 2.3.3 with respect to the linear differential equation

$$u'(t) = -\psi^* A\big(\gamma(t)\big)\big[\gamma'(t)\big] u(t) \quad \text{for } t \in [0,1]. \qquad (3.75)$$

We recall that we established in Section 2.3 that $\mathrm{Pt}_\gamma^{\psi^*A}\colon [0,1] \to O(F)$ satisfies the linear initial value problem

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} \mathrm{Pt}_\gamma^{\psi^*A}(t) = -\psi^*A\big(\gamma(t)\big)\big[\gamma'(t)\big]\,\mathrm{Pt}_\gamma^{\psi^*A}(t) & \text{for } t \in [0,1], \\ \mathrm{Pt}_\gamma^{\psi^*A}(0) = \mathrm{id}_F \end{cases}$$

and $u \in C^1\big([0,1], F\big)$ satisfies (3.75) if and only if, for all $t \in [0,1]$,

$$u(t) = \mathrm{Pt}_\gamma^{\psi^*A}(t)u(0).$$

For such a path $\gamma \in C^1\big([0,1], W\big)$, we define, as in Definition 2.3.3, the operator $R_\gamma^{\psi^*A}\colon [0,1]^2 \to O(F)$ by

$$R_\gamma^{\psi^*A}(t,s) = \mathrm{Pt}_\gamma^{\psi^*A}(t)\left(\mathrm{Pt}_\gamma^{\psi^*A}(s)\right)^{-1}$$

where $(t,s) \in [0,1]^2$. We also write

$$R_\gamma^{\psi^*A}\big(\gamma(s), \gamma(t)\big) = R_\gamma^{\psi^*A}(s,t)$$

so that $u\colon \gamma\big([0,1]\big) \to F$ is parallel along γ, that is, $\nabla_{\psi^*A} u\big(\gamma(t)\big)\big[\gamma'(t)\big] = 0$ for all $t \in [0,1]$, if and only if

$$u\big(\gamma(t)\big) = R_\gamma^{\psi^*A}\big(\gamma(t), \gamma(s)\big)u\big(\gamma(s)\big).$$

When W is convex, which we may assume by taking a smaller convex chart of Ω, we denote by $\mathrm{Pt}_{x,y}^{\psi^*A}$ the parallel transport along the path

$$\begin{gathered} \gamma_{x,y}\colon [0,1] \to W, \\ \gamma_{x,y}(t) = (1-t)x + ty \end{gathered}$$

where $x, y \in W$. As above, we set

$$R_{x,y}^{\psi^*A}(s,t) = \mathrm{Pt}_{x,y}^{\psi^*A}(s)\left(\mathrm{Pt}_{x,y}^{\psi^*A}(t)\right)^{-1} = \mathrm{Pt}_{x,y}^{\psi^*A}(s)\,\mathrm{Pt}_{y,x}^{\psi^*A}(t)$$

as well as

$$R^{\psi^*A}\big(\gamma_{x,y}(t), \gamma_{x,y}(s)\big) = R_{x,y}^{\psi^*A}(t,s)$$

for all $(s,t) \in [0,1]^2$. In particular, we have

$$R^{\psi^*A}(y,x) = R_{x,y}^{\psi^*A}(1,0) = \mathrm{Pt}_{x,y}^{\psi^*A}(1), \tag{3.76}$$

the parallel transport from $x \in W$ to $y \in W$ along $\gamma_{x,y}$. We then have $\nabla_{\psi^*A} u = 0$ in W

if and only if

$$u(x) = R^{\psi^*A}(x, y)u(y)$$

for all $x, y \in W$. The general discussion made in Section 2.3 holds with respect to the connection ∇_{ψ^*A}. In particular, the path γ may only be piecewise C^1.

We next present a result similar to Lemma 3.6.2 regarding the gauge covariant Gagliardo seminorm with charts $\psi\colon V \mapsto \partial\Omega$. The boundary $\partial\Omega$ of a bounded domain Ω of class C^1 is not convex and therefore, parallel transport along $\gamma_{x,y}$ does not make sense in general. However, since $\partial\Omega$ is a compact C^1 manifold by assumption, it has a positive injectivity radius $\mathrm{inj}_{\partial\Omega} > 0$, and if $x, y \in \partial\Omega$ are such that $\mathrm{dist}_{\partial\Omega}(x, y) < \mathrm{inj}_{\partial\Omega}$, then there exists a unique distance minimising geodesic

$$\zeta_{x,y}\colon [0, 1] \longrightarrow \partial\Omega$$

with $\zeta_{x,y}(0) = x$ and $\zeta_{x,y}(1) = y$, see [12, Ch. 13]. Given a continuous connection 1-form

$$A' \in C^0\big(\partial\Omega, \mathrm{Lin}\big(\mathbb{R}^n, \mathfrak{o}(F)\big)\big)$$

on the boundary, we define for such points $x, y \in \partial\Omega$ with $\mathrm{dist}_{\partial\Omega}(x, y) < \mathrm{inj}_{\partial\Omega}$ the parallel transport operator

$$\mathrm{Pt}^{A'}_{\partial\Omega,x,y}\colon [0, 1] \longrightarrow O(F)$$

as previously with respect to the linear differential equation

$$u'(t) = -A'\big(\zeta_{x,y}(t)\big)\big[\zeta'_{x,y}(t)\big]u(t) \quad \text{for } t \in [0, 1].$$

We also write, for $x, y \in \partial\Omega$ with $\mathrm{dist}_{\partial\Omega}(x, y) < \mathrm{inj}_{\partial\Omega}$,

$$R^{A'}_{\partial\Omega}(x, y) = \mathrm{Pt}^{A'}_{\partial\Omega,y,x}(1),$$

the parallel transport from y to x along the geodesic $\zeta_{y,x}\colon [0, 1] \longrightarrow \partial\Omega$, $\zeta_{y,x}(0) = y$ and $\zeta_{y,x}(1) = x$. Here again, the general discussion made in Section 2.3 holds with respect to the connection $A'\colon \partial\Omega \longrightarrow \mathrm{Lin}\big(\mathbb{R}^n, \mathfrak{o}(F)\big)$. Pulling back the connection A' by $\psi\colon V \longrightarrow \partial\Omega$ induces the connection 1-form ψ^*A' which in return gives rise to another parallel transport. Assuming that V is convex, we denote the parallel transport from $y \in V$ to $x \in V$ along $\gamma_{y,x}$ by $R^{\psi^*A'}(x, y)$. Its general definition along a path $\gamma\colon [0, 1] \longrightarrow V$ is similar to that of R^{ψ^*A}, see (3.76).

Definition 3.6.3. *Let $n \geq 1$, let $1 \leq p < +\infty$, let $0 < s < 1$ and let $\Omega \subset \mathbb{R}^{n+1}$ be an open bounded domain of class C^1. Furthermore, let $A' \in C^0\big(\partial\Omega, \mathrm{Lin}\big(\mathbb{R}^n, \mathfrak{o}(F)\big)\big)$. We define the* fractional gauge covariant Sobolev space $W^{s,p}_{A'}(\partial\Omega, P)$ *as the set of all functions*

$u \in L^p(\mathbb{R}^n, F)$ *such that the* gauge covariant Gagliardo seminorm

$$[u]^p_{W^{s,p}_{A'}(\partial\Omega,P)} = \iint\limits_{\substack{(x,y)\in\partial\Omega\times\partial\Omega\\ \operatorname{dist}_{\partial\Omega}(x,y)<\operatorname{inj}_{\partial\Omega}}} \frac{\left|R^{A'}_{\partial\Omega}(x,y)u(y) - u(x)\right|^p}{|x-y|^{n+sp}} \,\mathrm{d}x\,\mathrm{d}y \tag{3.77}$$

is finite.

The gauge covariant Gagliardo seminorm is gauge invariant in vertue of Proposition 2.3.7. We endow it with the gauge invariant norm

$$\|u\|_{W^{s,p}_{A'}(\partial\Omega,P)} = \left(\|u\|^p_p + [u]^p_{W^{s,p}_{A'}(\partial\Omega,P)}\right)^{1/p}.$$

A standard proof shows that this defines a Banach space.

Lemma 3.6.4. *Let* $x, y \in \partial\Omega$ *with* $\operatorname{dist}_{\partial\Omega}(x,y) < \operatorname{inj}_{\partial\Omega}$. *Then, for all* $t \in [0,1]$,

$$\mathrm{Pt}^{\psi^*A'}_{\tilde{\zeta}_{x,y}}(t) = \mathrm{Pt}^{A'}_{\partial\Omega,x,y}(t)$$

where $\tilde{\zeta}_{x,y} = \psi^{-1} \circ \zeta_{x,y}$.

Proof. On one hand

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} \mathrm{Pt}^{A'}_{\partial\Omega,x,y}(t) = -A'\big(\zeta_{x,y}(t)\big)\left[\zeta'_{x,y}(t)\right] \mathrm{Pt}^{A'}_{\partial\Omega,x,y}(t) & \text{for } t \in [0,1], \\ \mathrm{Pt}^{A'}_{\partial\Omega,x,y}(0) = \mathrm{id}_F \end{cases} \tag{3.78}$$

and on the other hand,

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} \mathrm{Pt}^{\psi^*A'}_{\tilde{\zeta}_{x,y}}(t) = -(\psi^*A')\big(\tilde{\zeta}_{x,y}(t)\big)\left[\tilde{\zeta}'_{x,y}(t)\right] \mathrm{Pt}^{\psi^*A'}_{\tilde{\zeta}_{x,y}}(t) & \text{for } t \in [0,1], \\ \mathrm{Pt}^{\psi^*A'}_{\tilde{\zeta}_{x,y}}(0) = \mathrm{id}_F\,. \end{cases} \tag{3.79}$$

By the chain rule and Definition 3.6.1, we have, for all $t \in [0,1]$

$$-(\psi^*A')\big(\tilde{\zeta}_{x,y}(t)\big)\left[\tilde{\zeta}'_{x,y}(t)\right] \mathrm{Pt}^{\psi^*A'}_{\tilde{\zeta}_{x,y}}(t) = A'\big(\zeta_{x,y}(t)\big)\left[\zeta'_{x,y}(t)\right]. \tag{3.80}$$

We deduce from (3.78), (3.79) and (3.80) that $\mathrm{Pt}^{\psi^*A'}_{\tilde{\zeta}_{x,y}}$ and $\mathrm{Pt}^{A'}_{\partial\Omega,x,y}$ satisfy the same initial value problem and conclude by uniqueness of the solution. ☐

Lemma 3.6.4 in particular tells us that parallel transport in $\partial\Omega$ with respect to the connection A' boils down to the same as parallel transport in a chart (V, ψ) of $\partial\Omega$ with respect to the pull-back connection $\psi^* A'$. In particular, we may observe how parallel transport along a path γ in $\partial\Omega$ behaves by analysing the parallel transport along the path $\tilde{\gamma} = \psi^{-1} \circ \gamma$ for in V behaves. We shall use this to find estimates on the difference between the parallel transport along the path $\gamma_{x,y}$ in V, $x,y \in V$, and the parallel transport along the unique geodesic $\zeta_{\psi(x),\psi(y)}$ in $\partial\Omega$ by computing the amplitude of holonomy along the concatenation $\gamma \colon [0,2] \to V$,

$$\gamma(t) = \begin{cases} \tilde{\zeta}_{x,y}(t) = (\psi^{-1} \circ \zeta_{\psi(x),\psi(y)})(t) & \text{for } 0 \leq t < 1, \\ \gamma_{y,x}(t-1) = (t-1)x + (2-t)y & \text{for } 1 \leq t \leq 2. \end{cases} \tag{3.81}$$

We obtain the following relation for the gauge covariant Gagliardo seminorms with the connections $\psi^* A'$ and A'.

Lemma 3.6.5. *Let $n \geq 1$, $1 \leq p < +\infty$ and $0 < s < 1$. Let $V \subset \mathbb{R}^n$ be open and bounded and let $\psi \colon V \to \mathbb{R}^{n+1}$ be a diffeomorphism up to the boundary on its image as a subset of the manifold $\partial\Omega$ such that $\operatorname{diam}(\psi(V)) < \operatorname{inj}_{\partial\Omega}$. Furthermore, assume that $A' \in C^1(\partial\Omega, \operatorname{Lin}(\mathbb{R}^n, \mathfrak{o}(F)))$. If V is convex and $\psi(V)$ is geodesically convex, then there exists a positive constant C such that for every measurable function $U \colon \psi(V) \to F$,*

$$\begin{aligned} \int_{\psi(V)} \int_{\psi(V)} & \frac{\left|R^{A'}_{\partial\Omega}(x,y)U(y) - U(x)\right|^p}{|y-x|^{n+sp}} \,\mathrm{d}y \,\mathrm{d}x \\ & \leq C \Biggl(\int_V \int_V \frac{\left|R^{\psi^* A'}(v,w)U\bigl(\psi(w)\bigr) - U\bigl(\psi(v)\bigr)\right|^p}{|w-v|^{n+sp}} \,\mathrm{d}w \,\mathrm{d}v \\ & \qquad + \min\bigl\{\|\mathcal{K}_{A'}\|_\infty^p, \|\mathcal{K}_{A'}\|_\infty^{sp/3}\bigr\} \int_V \left|U\bigl(\psi(v)\bigr)\right|^p \,\mathrm{d}v \Biggr) \end{aligned}$$

and

$$\begin{aligned} \int_V \int_V & \frac{\left|R^{\psi^* A'}(v,w)U\bigl(\psi(w)\bigr) - U\bigl(\psi(v)\bigr)\right|^p}{|w-v|^{n+sp}} \,\mathrm{d}w \,\mathrm{d}v \\ & \leq C \Biggl(\int_{\psi(V)} \int_{\psi(V)} \frac{\left|R^{A'}_{\partial\Omega}(x,y)U(y) - U(x)\right|^p}{|y-x|^{n+sp}} \,\mathrm{d}y \,\mathrm{d}x \\ & \qquad + \min\bigl\{\|\mathcal{K}_{A'}\|_\infty^p, \|\mathcal{K}_{A'}\|_\infty^{sp/3}\bigr\} \int_{\psi(V)} \left|U(x)\right|^p \,\mathrm{d}x \Biggr). \end{aligned}$$

To prove this result, we first establish an estimate comparing the two parallel transport operators via the amplitude of holonomy along the concatenation γ defined in (3.81).

Lemma 3.6.6. *Let $n \geq 1$, let $V \subset \mathbb{R}^n$ be open and bounded and let $\psi \colon V \to \partial\Omega$ be a diffeomorphism up to the boundary on its image as a subset of the manifold $\partial\Omega$ such that* $\operatorname{diam}\big(\psi(V)\big) < \operatorname{inj}_{\partial\Omega}$. *Furthermore, assume that $A \in C^1\big(\partial\Omega, \operatorname{Lin}(\mathbb{R}^n, \mathfrak{o}(F))\big)$. If V is convex and $\psi(V)$ is geodesically convex, then there exists $C > 0$ such that for all $x, y \in V$*

$$\langle\!\langle \gamma \rangle\!\rangle \leq C \min\big\{1, \|\mathcal{K}_{A'}\|_{L^\infty(\psi(V))}|y-x|^3\big\}$$

where γ is the concatenation (3.81). In particular,

$$\Big|R^{\psi^* A'}(x,y) R^{A'}_{\partial\Omega}\big(\psi(y), \psi(x)\big) - \operatorname{id}_F\Big| \leq C \min\big\{2, \|\mathcal{K}_{A'}\|_{L^\infty(\psi(V))}|y-x|^3\big\}.$$

Proof. Since $\zeta = \zeta_{\psi(y),\psi(x)}$ is a geodesic, its local form

$$\tilde{\zeta} = \tilde{\zeta}_{x,y} = \psi^{-1} \circ \zeta_{\psi(x),\psi(y)}$$

satisfies the local geodesic equation

$$\tilde{\zeta}''(t) = \Gamma\big(\tilde{\zeta}(t)\big)\big[\tilde{\zeta}'(t), \tilde{\zeta}'(t)\big]$$

where Γ is a symmetric 2-form (defined with the Christofell symbols), see for example [12, Ch. 3]. Since geodesic have constant speed proportional to the Euclidean distance between their endpoints, it follows that there exists $C_1, C_2 > 0$ such that

$$\big|\tilde{\zeta}''(t)\big| \leq C_1 \big|\tilde{\zeta}'(t)\big|^2 \leq C_2 |y-x|^2. \tag{3.82}$$

Let us define $\mu \in C^2\big([0,1], V\big)$ by

$$\mu(t) = \tilde{\zeta}(t) - \gamma_{x,y}(t).$$

By the Taylor–Lagrange theorem, we have

$$\mu(0) = \mu(t) - \mu'(t)t + \frac{\mu''(c)}{2}t^2$$

for some $0 < c < t$ and

$$\mu(1) = \mu(t) + \mu'(t)(1-t) + \frac{\mu''(d)}{2}(1-t)^2$$

for some $t < d < 1$. Since $\mu(0) = \mu(1) = 0$, we obtain

$$\mu(t) = (1-t)\mu(t) + t\mu(t) = \frac{t(1-t)}{2}\big((1-t)\mu''(d) - t\mu''(c)\big) \tag{3.83}$$

for all $t \in [0,1]$. Since $\mu'' = \tilde{\zeta}''$, it follows from (3.82) and (3.83) that, for all $t \in [0,1]$ and some $C > 0$,

$$\big|\mu(t)\big| \leq Ct(1-t)|y-x|^2. \tag{3.84}$$

Let $H\colon [0,2]\times[0,1] \longrightarrow V$ be the continuous and piecewise differentiable homotopy operator defined by

$$H(t,s) = \begin{cases} (1-s)\big((1-t)x + ty\big) + s\tilde{\zeta}(t) & \text{for } 0 \leq t < 1, \\ (t-1)x + (2-t)y & \text{for } 1 \leq t \leq 2. \end{cases}$$

We have $H(t,1) = \gamma(t)$ and $H(t,0)$ is the concatenation of $\gamma_{x,y}$ and $\gamma_{y,x}$. We consider the parallel transport $\mathrm{Pt}_s^{\psi^*A'}(\cdot)$ along $\gamma_s(\cdot) = H(\cdot,s)$. In particular, $\mathrm{Pt}_0^{\psi^*A'}(2) = \mathrm{id}_F$ and

$$\mathrm{Pt}_1^{\psi^*A'}(2) = \mathrm{Pt}_\gamma^{\psi^*A'}(2) = R^{\psi^*A'}(x,y)R^{A'}_{\partial\Omega}\big(\psi(y),\psi(x)\big)$$

is the holonomy along γ. We compute, for all $t \in [0,1)$ and for all $s \in [0,1]$,

$$\begin{aligned} \gamma_s'(t) &= (1-s)(y-x) + s\tilde{\zeta}'(t), \\ \partial_s\gamma_s(t) &= \tilde{\zeta}(t) - \big((1-t)x + ty\big) = \mu(t), \\ \partial_t\partial_s\gamma_s(t) &= \partial_s\gamma_s'(t) = \tilde{\zeta}'(t) + x - y = \mu'(t) \end{aligned} \tag{3.85}$$

while for $t \in (1,2]$ and $s \in [0,1]$ $\partial_t H(t,s) = y - x$ and $\partial_s H(t,s) = 0$. In vertue of Lemma 3.2.2, Lemma 3.2.3 and (3.85), we have

$$\begin{aligned} &\partial_s \mathrm{Pt}_s^{\psi^*A'}(2) \\ &\quad = -\int_0^1 \mathrm{Pt}_s^{\psi^*A'}(2)\,\mathrm{Pt}_s^{\psi^*A'}(t)^{-1}\mathrm{D}(\psi^*A')\big(\gamma_s(t)\big)\big[\mu(t),\gamma_s'(t)\big]\,\mathrm{Pt}_s^{\psi^*A'}(t)\,\mathrm{d}t \\ &\qquad\qquad - \int_0^1 \mathrm{Pt}_s^{\psi^*A'}(2)\,\mathrm{Pt}_s^{\psi^*A'}(t)^{-1}(\psi^*A')\big(\gamma_s(t)\big)\big[\mu'(t)\big]\,\mathrm{Pt}_s^{\psi^*A'}(t)\,\mathrm{d}t. \end{aligned} \tag{3.86}$$

Integrating by parts, we obtain

$$\begin{aligned} &-\int_0^1 \mathrm{Pt}_s^{\psi^*A'}(2)\,\mathrm{Pt}_s^{\psi^*A'}(t)^{-1}(\psi^*A')\big(\gamma_s(t)\big)\big[\mu'(t)\big]\,\mathrm{Pt}_s^{\psi^*A'}(t)\,\mathrm{d}t \\ &\quad = \int_0^1 \mathrm{Pt}_s^{\psi^*A'}(2)\,\mathrm{Pt}_s^{\psi^*A'}(t)^{-1}(\psi^*A')\big(\gamma_s(t)\big)\big[\gamma_s'(t)\big](\psi^*A')\big(\gamma_s(t)\big)\big[\mu(t)\big]\,\mathrm{Pt}_s^{\psi^*A'}(t)\,\mathrm{d}t \\ &\qquad + \int_0^1 \mathrm{Pt}_s^{\psi^*A'}(2)\,\mathrm{Pt}_s^{\psi^*A'}(t)^{-1}\mathrm{D}(\psi^*A')\big(\gamma_s(t)\big)\big[\gamma_s'(t),\mu(t)\big]\,\mathrm{Pt}_s^{\psi^*A'}(t)\,\mathrm{d}t \\ &-\int_0^1 \mathrm{Pt}_s^{\psi^*A'}(2)\,\mathrm{Pt}_s^{\psi^*A'}(t)^{-1}(\psi^*A')\big(\gamma_s(t)\big)\big[\gamma_s'(t)\big](\psi^*A')\big(\gamma_s(t)\big)\big[\mu(t)\big]\,\mathrm{Pt}_s^{\psi^*A'}(t)\,\mathrm{d}t. \end{aligned} \tag{3.87}$$

Injecting (3.87) into (3.86), we reach

$$\partial_s \mathrm{Pt}^{\psi^*A'}(2) = \int_0^1 \mathrm{Pt}_s^{\psi^*A'}(2)\,\mathrm{Pt}_s^{\psi^*A'}(t)^{-1}\mathcal{K}_{\psi^*A'}\big(\gamma_s(t)\big)\big[\gamma_s'(t),\mu(t)\big]\,\mathrm{Pt}_s^{\psi^*A'}(t)\,\mathrm{d}t$$

and therefore, by (3.84) and (3.85), we have

$$\left|\mathrm{Pt}_s^{\psi^* A'}(2)\right| \leq C\|\mathcal{K}_{\psi^* A'}\|_\infty |y-x|^3.$$

We conclude that

$$\langle\!\langle \gamma \rangle\!\rangle \leq \int_0^1 \left|\partial_s \mathrm{Pt}_s^{\psi^* A'}(2)\right| \,\mathrm{d}s \leq C\|\mathcal{K}_{\psi^* A'}\|_{L^\infty(V)}|y-x|^3 = C\|\mathcal{K}_{A'}\|_{L^\infty(\psi(V))}|y-x|^3. \qquad \square$$

Proof of Lemma 3.6.5. By the change of variable formula, we have

$$\begin{aligned}\int_{\psi(V)}\int_{\psi(V)} &\frac{\left|R_{\partial\Omega}^{A'}(x,y)U(y)-U(x)\right|^p}{|y-x|^{n+sp}}\,\mathrm{d}y\,\mathrm{d}x\\ &= \int_V\int_V \frac{\left|R_{\partial\Omega}^{A'}\big(\psi(v),\psi(w)\big)U\big(\psi(w)\big)-U\big(\psi(v)\big)\right|^p}{\left|\psi(w)-\psi(v)\right|^{n+sp}} \operatorname{Jac}\psi(v)\operatorname{Jac}\psi(w)\,\mathrm{d}w\,\mathrm{d}v.\end{aligned}$$

Since ψ is a diffeomorphism up to the boundary, we obtain $C>0$ such that

$$\begin{aligned}\int_{\psi(V)}\int_{\psi(V)} &\frac{\left|R_{\partial\Omega}^{A'}(x,y)U(y)-U(x)\right|^p}{|y-x|^{n+sp}}\,\mathrm{d}y\,\mathrm{d}x\\ &\leq C\Bigg(\int_V\int_V \frac{\left|R^{\psi^* A'}(w,v)U\big(\psi(w)\big)-U\big(\psi(v)\big)\right|^p}{|w-v|^{n+sp}}\,\mathrm{d}w\,\mathrm{d}v\\ &\quad+\int_V\int_V \frac{\left|R_{\partial\Omega}^{A'}\big(\psi(v),\psi(w)\big)-R^{\psi^* A'}(v,w)\right|^p\left|U\big(\psi(w)\big)\right|^p}{|w-v|^{n+sp}}\,\mathrm{d}w\,\mathrm{d}v\Bigg).\end{aligned} \tag{3.88}$$

It follows from Lemma 3.6.6 that, for all $v,w\in V$, we have

$$\frac{\left|R_{\partial\Omega}^{A'}\big(\psi(v),\psi(w)\big)-R^{\psi^* A'}(v,w)\right|^p}{|w-v|^{n+sp}} \leq C_1\frac{\min\left\{1,\|\mathcal{K}_{A'}\|_{L^\infty(\psi(V))}^p|v-w|^{3p}\right\}}{|w-v|^{n+sp}}. \tag{3.89}$$

The conclusion for the first estimate is then obtained from (3.88) and (3.89) together with the estimates

$$\int_{\mathbb{R}^n}\frac{\min\left\{1,\|\mathcal{K}_{A'}\|_{L^\infty(\psi(V))}^p|z|^{3p}\right\}}{|z|^{n+sp}}\,\mathrm{d}z \leq C_2\|\mathcal{K}_{A'}\|_{L^\infty(\psi(V))}^{sp/3}$$

and

$$\int_{B(0,\operatorname{diam}(V))}\frac{\|\mathcal{K}_{A'}\|_{L^\infty(\psi(V))}^p|z|^{3p}}{|z|^{n+sp}}\,\mathrm{d}z \leq C_3\|\mathcal{K}_{A'}\|_{L^\infty(\psi(V))}^p$$

which follow from direct computation. The proof for the second estimate is analogue. $\square$

We are now ready to characterise traces and extensions of gauge covariant Sobolev spaces on open bounded domains $\Omega \subset \mathbb{R}^{n+1}$ of class C^1 with connection 1-form $A \in C^1\big(\overline{\Omega}, \operatorname{Lin}\big(\mathbb{R}^{n+1}, \mathfrak{o}(F)\big)\big)$. Under this assumption on A, the gauge covariant Sobolev space $W^{1,p}_A(\Omega, P)$ is the same as the usual vector valued Sobolev space $W^{1,p}(\Omega, F)$ since the connection 1-form A is bounded, see Proposition 2.2.18. Therefore, the usual trace theorem applies and yields a linear and continuous trace operator

$$\operatorname{Tr}\colon W^{1,p}_A(\Omega, P) \to W^{1-1/p}(\partial\Omega, F).$$

However, the norm estimates are not gauge invariant, and the trace space should depend on A in a gauge invariant manner. The results that follow achieve this enhanced characterisation.

We obtain the following estimates on the traces of gauge covariant Sobolev spaces, generalising the particular case of [34, Prop. 6.4] regarding magnetic Sobolev spaces. In this and the following proof, we denote by $C_1, C_2, \ldots$ positive constants independent of A, u or U.

Proposition 3.6.7. *Let $n \geq 1$, let $1 \leq p < +\infty$ and let $\Omega \subset \mathbb{R}^{n+1}$ be an open and bounded domain of class C^1. Furthermore, assume that $A \in C^1\big(\overline{\Omega}, \operatorname{Lin}\big(\mathbb{R}^{n+1}, \mathfrak{o}(F)\big)\big)$ and that $\|\mathcal{K}_A\|_\infty \leq \beta$. Then there exists a constant $C = C(\Omega, p) > 0$ depending only on Ω and p such that for all $u \in W^{1,p}_A(\Omega, P)$ with $U = \operatorname{Tr} u$, we have*

$$\iint\limits_{\substack{(x,y)\in\partial\Omega\times\partial\Omega\\ \operatorname{dist}_{\partial\Omega}(y,x)<\operatorname{inj}_{\partial\Omega}}} \frac{\big|R^{A'}_{\partial\Omega}(x,y)U(y) - U(x)\big|^p}{|y-x|^{n+p-1}}\,\mathrm{d}x\,\mathrm{d}y \leq C \int_\Omega |\nabla_A u(x)|^p + \big(1+\beta^{p/2}\big)|u(x)|^p\,\mathrm{d}x.$$

Here and in what follows, we write, for $z \in \partial\Omega$,

$$A'(z) = A(z) - \big(A(z) \mid \nu(z)\big)\nu(z)$$

where $\nu(z)$ denotes a unit normal vector of $\partial\Omega$ at z.

Proof of Proposition 3.6.7. Since $C^\infty\big(\overline{\Omega}, F\big)$ is dense in $W^{1,p}(\Omega, F) = W^{1,p}_A(\Omega, P)$, see for example [6, Cor. 9.8], we may assume $u \in C^\infty\big(\overline{\Omega}, F\big)$. By divergence theorem and the diamagnetic inequality, we have

$$\int_{\partial\Omega} \big|U(x)\big|^p\,\mathrm{d}x \leq\leq C_1 \left(\int_\Omega \big|\nabla_A u(x)\big|^p + \big|u(x)\big|^p\,\mathrm{d}x\right)^{1/p} \left(\int_\Omega \big|u(x)\big|^p\,\mathrm{d}x\right)^{1-1/p}. \tag{3.90}$$

Since $\partial\Omega$ is compact and Ω of class C^1, there exists a finite number of maps, say $j \in \{1, \dots, k\}$,

$$\psi_j \colon B_1 \times (-1, 1) \to \mathbb{R}^{n+1}$$

that are diffeomorphisms on their image such that

$$\begin{gathered}
\psi_j\big(B_1 \times (0,1)\big) = \psi_j\big(B_1 \times (-1,1)\big) \cap \Omega, \\
\psi_j\big(B_1 \times \{0\}\big) = \psi\big(B_1 \times (-1,1)\big) \cap \partial\Omega, \\
\partial\Omega \subset \bigcup_{j=1}^{k} \psi_j\big(B_{1/2} \times \{0\}\big)
\end{gathered}$$

and, for every $j \in \{1, \dots, k\}$, $\psi_j\big(B_1 \times \{0\}\big)$ is geodesically convex with

$$\operatorname{diam}\big(\psi_j\big(B_1 \times \{0\}\big)\big) < \operatorname{inj}_{\partial\Omega}.$$

In vertue of Proposition 3.3.3 with $s = 1 - \frac{1}{p}$, we have, for $j \in \{1, \dots, k\}$,

$$\begin{aligned}
\int_{B_1} \int_{B_1} & \frac{\big|R^{\psi^* A'}(x,y) U\big(\psi_j(y,0)\big) - U\big(\psi_j(x,0)\big)\big|^p}{|x-y|^{n+p-1}} \,\mathrm{d}x \,\mathrm{d}y \\
& \leq C_2 \int_{B_1} \int_0^1 \big|\nabla_{\psi^* A'} (u \circ \psi)(z,t)\big|^p + \beta^{p/2} \big|(u \circ \psi_j)(z,t)\big|^p \,\mathrm{d}t \,\mathrm{d}z. \qquad (3.91)
\end{aligned}$$

Since β is an arbitrary upper bound on $\|\mathcal{K}_A\|_\infty$, we may assume without loss of generality that $\beta \geq 1$. Lemma 3.6.2 and Lemma 3.6.5 then imply together with (3.91) that

$$\begin{aligned}
\iint\limits_{\psi_j(B_1 \times \{0\}) \times \psi_j(B_1 \times \{0\})} & \frac{\big|R^{A'}_{\partial\Omega}(x,y) U(y) - U(x)\big|^p}{|y-x|^{n+p-1}} \,\mathrm{d}x \,\mathrm{d}y \\
& \leq C_3 \Bigg(\int\limits_{\psi_j(B_1 \times (0,1))} \big|\nabla_A u(x)\big|^p + \beta^{p/2} \big|u(x)\big|^p \,\mathrm{d}x + \beta^{(p-1)/3} \int\limits_{\psi_j(B_1 \times \{0\})} \big|U(x)\big|^p \,\mathrm{d}x \Bigg). \qquad (3.92)
\end{aligned}$$

Since $\beta^{(p-1)/3} \leq \beta^{(p-1)/2}$, using (3.90) and with Young's inequality we obtain

$$\begin{aligned}
\beta^{(p-1)/3} & \int\limits_{\psi_j(B_1 \times \{0\})} \big|U(x)\big|^p \,\mathrm{d}x \\
& \leq C_1 \beta^{(p-1)/2} \left(\int_\Omega \big|\nabla_A u(z)\big|^p + \big|u(x)\big|^p \,\mathrm{d}x \right)^{1/p} \left(\int_\Omega \big|u(x)\big|^p \,\mathrm{d}x \right)^{1-1/p}
\end{aligned}$$

$$\leq C_4 \int\limits_{\psi_j(B_1\times\{0\})} \bigl|\nabla_A u(x)\bigr|^p + \bigl(1+\beta^{p/2}\bigr)\bigl|u(x)\bigr|^p \,\mathrm{d}x. \tag{3.93}$$

The conclusion follows by summing over $j \in \{1,\dots,k\}$ and combining the estimate (3.92) with the estimate (3.93). □

Regarding extensions, we have the following result generalising its analogue for magnetic Sobolev spaces [34, Prop. 6.5].

Proposition 3.6.8. *Let $n \geq 1$, $1 \leq p < +\infty$ and let $\Omega \subset \mathbb{R}^{n+1}$ be an open and bounded domain of class C^1. Furthermore, assume that $A \in C^1\bigl(\overline{\Omega}, \mathrm{Lin}\bigl(\mathbb{R}^{n+1}, \mathfrak{o}(F)\bigr)\bigr)$ and that $\|\mathcal{K}_A\|_\infty \leq \beta$. Then there exists a constant $C = C(\Omega,p) > 0$ depending only on Ω and p such that for all $U \in W^{1-1/p,p}_{A'}(\partial\Omega, P)$, there exists $u \in W^{1,p}_A(\Omega,P) \cap C^1(\Omega,F)$ such that $\mathrm{Tr}\,u = U$ and*

$$\int_\Omega \bigl|\nabla u(x)\bigr|^p \,\mathrm{d}x \leq C\Biggl(\iint\limits_{\substack{(x,y)\in\partial\Omega\times\partial\Omega\\ \mathrm{dist}_{\partial\Omega}(y,x)<\mathrm{inj}_{\partial\Omega}}} \frac{\bigl|R^{A'}_{\partial\Omega}(x,y)U(y) - U(x)\bigr|^p}{|y-x|^{n+p-1}} \,\mathrm{d}y\,\mathrm{d}x + \bigl(1+\beta^{(p-1)/2}\bigr)\int_{\partial\Omega}\bigl|U(x)\bigr|^p \,\mathrm{d}x\Biggr)$$

and

$$\int_\Omega \bigl|u(x)\bigr|^p \,\mathrm{d}x \leq \frac{C}{1+\beta^{p/2}} \int_{\partial\Omega}\bigl|U(x)\bigr|^p \,\mathrm{d}x.$$

Proof. Since $\partial\Omega$ is compact and Ω of class C^1, there exists a finite number of maps, say $j \in \{1,\dots,k\}$,

$$\psi_j \colon B_1 \times (-1,1) \to \mathbb{R}^{n+1}$$

that are diffeomorphisms on their image such that

$$\psi_j\bigl(B_1\times(0,1)\bigr) = \psi_j\bigl(B_1\times(-1,1)\bigr)\cap\Omega,$$
$$\psi_j\bigl(B_1\times\{0\}\bigr) = \psi\bigl(B_1\times(-1,1)\bigr)\cap\partial\Omega,$$
$$\partial\Omega \subset \bigcup_{j=1}^k \psi_j\bigl(B_{1/2}\times\{0\}\bigr)$$

and, for every $j \in \{1,\dots,k\}$, $\psi_j\bigl(B_1\times\{0\}\bigr)$ is geodesically convex with

$$\mathrm{diam}\bigl(\psi\bigl(B_1\times\{0\}\bigr)\bigr) < \mathrm{inj}_{\partial\Omega}.$$

Moreover, there exists a a collection of smooth functions $\{\eta_j\}_{j=1}^{k} \subset C^\infty(\overline{\Omega}, \mathbb{R})$ such that for all $j \in \{1, \dots, k\}$

$$\operatorname{supp} \psi_j \subset \psi_j\big(B_{1/2} \times [0, 1/2]\big)$$

and $\sum_{j=1}^{k} \eta_j = 1$ on Ω.

In vertue of Proposition 3.4.2 with $s = 1 - \frac{1}{p}$, for every $j \in \{1, \dots, k\}$, there exists

$$u_j \in W^{1,p}_{\psi^* A}\big(B_{1/2} \cap [0, 1/2]\big) \times C^1\big(B_{1/2} \times [0, 1/2]\big)$$

such that $\operatorname{Tr} u_j = U \circ \psi_j$ on $B_{1/2} \times \{0\}$ and furthermore u_j satisfies

$$\begin{aligned} &\int_{B_{1/2}} \int_0^{\frac{1}{2}} \big|\nabla_{\psi_j^* A} u_j(x,t)\big|^p \,\mathrm{d}t \,\mathrm{d}x \\ &\quad \le C_1 \bigg(\int_{B_1} \int_{B_1} \frac{\big|R^{\psi_j^* A'}(x,y) U\big(\psi_j(y,0)\big) - U\big(\psi_j(x,0)\big)\big|^p}{|y-x|^{n+p-1}} \,\mathrm{d}x \,\mathrm{d}y \\ &\qquad\qquad + \beta^{(p-1)/2} \int_{B_1} \big|U\big(\psi_j(x,0)\big)\big|^p \,\mathrm{d}x \bigg) \end{aligned} \tag{3.94}$$

and, assuming without loss of generality that $\beta \ge 1$,

$$\int_{B_{1/2}} \int_0^{\frac{1}{2}} \big|\nabla_{\psi_j^* A'} u_j(x)\big|^p \,\mathrm{d}x \le C_2 \int_{B_1} \big|(U \circ \psi)(x)\big|^p \,\mathrm{d}x. \tag{3.95}$$

Applying Lemma 3.6.2 and Lemma 3.6.5 to (3.94) and (3.95), we obtain

$$\begin{aligned} &\int\limits_{\psi_j(B_{1/2} \times [0,1/2])} \big|\nabla_A\big(u_j \circ \psi_j^{-1}\big)(x)\big|^p \,\mathrm{d}x \\ &\quad \le C_3 \bigg(\iint\limits_{\psi_j(B_1 \times \{0\}) \times \psi_j(B_1 \times \{0\})} \frac{\big|R^{A'}_{\partial\Omega}(x,y) U(y) - U(x)\big|^p}{|y-x|^{n+p-1}} \,\mathrm{d}x \,\mathrm{d}y \\ &\qquad\qquad + \beta^{(p-1)/2} \int\limits_{\psi_j(B_1 \times \{0\})} \big|U(x)\big| \,\mathrm{d}x \bigg) \end{aligned} \tag{3.96}$$

and

$$\int\limits_{\psi_j(B_{1/2} \times [0,1/2])} \big|u_j(x)\big|^p \,\mathrm{d}x \le C_4 \int\limits_{\psi_j(B_1 \times \{0\})} \big|U(x)\big|^p \,\mathrm{d}x. \tag{3.97}$$

We define on Ω

$$u = \sum_{j=1}^{k} \eta_j \big(u_j \circ \psi_j^{-1}\big).$$

By the Leibniz rule for covariant derivatives, see Definition 1.4.10, we have

$$\nabla_A u = \sum_{j=1}^{k} \Big(\big(u_j \circ \psi^{-1}\big) \mathrm{D}\eta_j + \eta_j \nabla_A \big(u_j \circ \psi_j^{-1}\big) \Big). \tag{3.98}$$

The conclusion follows from (3.96), (3.97) and (3.98). □

In vertue of Proposition 3.6.7 and Proposition 3.6.8, we conclude that if the curvature is bounded on Ω, the trace space of the Gauge covariant Sobolev space $W_A^{1,p}(\Omega, P)$ is the fractional gauge covariant Sobolev space $W_{A'}^{1-1/p,p}(\partial\Omega, P)$.

Conclusion

This concludes our exploration of gauge covariant Sobolev spaces. Although this thesis places itself in mathematical analysis, our study was intertwined with the beautiful branch of mathematics that is differential geometry. The different nature of sections and covariant derivatives together with parallel transport yields many initial obstructions which one has to overcome.

Our aim was to give a new and gauge invariant characterisation of the trace space of a gauge covariant Sobolev space. We reached this goal in Chapter 3, and concluded that the trace space of $W^{1,p}_A(\Omega, P)$ is the fractional gauge covariant Sobolev space $W^{1-1/p,p}_{A'}(\partial\Omega, P)$ with the gauge covariant Gagliardo seminorm

$$\iint\limits_{\substack{(x,y)\in\partial\Omega\times\partial\Omega \\ \mathrm{dist}_{\partial\Omega}(y,x)<\mathrm{inj}_{\partial\Omega}}} \frac{\left|R^{A'}_{\partial\Omega}(x,y)U(y) - U(x)\right|^p}{|y-x|^{n+p-1}} \,\mathrm{d}y\,\mathrm{d}x.$$

Although new, this result is not surprising as it was already known in the classical case, due to Gagliardo [17], and the case of abelian gauge covariant Sobolev spaces, due to Ngyuen and Van Schaftingen [34]. Most propositions we proved in Chapter 3 are better than needed. In fact, the presence of the $0 < s < 1$ was not needed as we could simply chose $s = 1 - \frac{1}{p}$. However, this sets a first step towards another result, namely characterising fractional gauge covariant Sobolev spaces as interpolation spaces between L^p and $W^{1,p}_A$. This problem was already approached and solved by Ngyuen and Van Schaftingen in the case of magnetic Sobolev spaces [34]. It is expected that the proof adapts without much effort.

In Chapter 2, we constructed piece by piece these gauge covariant Sobolev spaces and established some basic properties. Remarkably, we were able to obtain an equivalent to the Gagliardo-Nirenberg-Sobolev embedding without much effort. A continuation of this thesis could be to establish the Morrey-Sobolev embedding, as the definition of the Hölder spaces should involve parallel transport similarly to the fractional gauge covariant Sobolev spaces.

Another possible continuation of this thesis is trying to work with L^p connections or L^p curvatures rather than C^1 connections like we did. Such connections appear in Yang-Mills type theories, where the connection itself is an unknown and part of the problem.

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