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Internal Categories and Factorisation Systems

Auteur : Sanjiv Ranchod
Promoteur : Professor Marino Gran
Lecteur·rices : Professor Pedro Vaz, Professor Tim Van der Linden
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Sanjiv Ranchod
Advisor: Professor Marino Gran
Readers: Professor Pedro Vaz, Professor Tim Van der Linden

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Abstract

We motivate and define internal categories and related structures. We consider and prove Brown and Spencer's equivalence between internal categories in **Grp** and crossed modules, and Patchkoria's equivalence between Schreier internal categories in **Mon** and crossed semimodules. We provide an overview of the theory of factorisation systems and introduce the notion of an internal factorisation system, as well as prove properties thereof. We consider how internal categories and internal factorisation systems manifest in Mal'tsev varieties.

Introduction

An internal category is a generalisation of a (small) category, produced by considering a small category as a diagram in **Set**. The first instances of this notion were given by Ehresmann in [1], and a comprehensive account of the structure may be found in [2]. Their study provides categorical insight into the notion of a category itself. We motivate and define internal categories and related structures, such as internal functors, natural transformations, groupoids and equivalence relations.

It is of interest to characterise the structure of an internal category in some other, usually algebraic, category. Brown and Spencer [3] did so for **Grp**, showing that they are equivalent to crossed modules. Porter [4] then extends this to groups with additional binary and unary operations. Patchkoria [5] provides an analogous result for **Mon**, with the caveat that the internal categories are required to be Schreier to make up for the lack of inverses that exist in the **Grp** case. Martins-Ferreira, Montoli and Sobral [6] provide the analogous characterisation for monoids with operations. Janelidze [7] then defines an internal crossed module in a general semi-abelian category and shows that they are precisely the internal categories in this context. We provide detailed proofs for both Brown and Spencer, and Patchkoria's results.

On the other hand, factorisation systems, introduced by Freyd and Kelly [8], appear frequently in categorical algebra and have been studied in various general and concrete contexts (see [9], [10]). We provide an overview of factorisation systems, motivating the components of the definition with the archetypal example in **Set**, before considering and proving various properties of this structure.

Internal factorisation systems for internal categories were introduced in [11]. We recount the motivation and definition of this structure as well as establish and prove various the internalised properties of a factorisation system.

We lastly consider how internal categories and internal factorisations manifest within the prominently studied Mal'tsev varieties [18]. In this context, internal categories are always internal groupoids, and we show that internal factorisation systems on internal groupoids are trivial. Note that internal groupoids in Mal'tsev varieties play an important role in commutator theory (see for instance [21] and [17] and the references therein).

Notation. We note that pullback projections will be written as π_1 and π_2 when the pullback for which they are projections is clear. The binary operation of groups and monoids will, in general, be written additively. We present isomorphisms in categories as $\approx$, while $\sim$ is used for categorical equivalences.

Contents

Chapter 1

Internal Categories

The notion of an internal category is derived from the consideration of an small category as a structure within **Set**, the category of sets and functions. Specifically, let C be a small category. We may then consider C_0 as the set of objects of C, C_1 as the set of morphisms of C, $d : C_1 \to C_0$ a map which assigns a morphism to its domain object, $c : C_1 \to C_0$ a map which assigns a morphism to its codomain object, $e : C_0 \to C_1$ a map which assigns each object to its identity morphism, $C^{\leftarrow\leftarrow}$ as the set of composable pairs of morphisms of C and $m : C^{\leftarrow\leftarrow} \to C_1$ which assigns a composable pair to its composition. This data may be expressed as the following diagram in **Set**:

$$C_0 \begin{matrix} \xleftarrow{d} \\ \xrightarrow{e} \\ \xleftarrow{c} \end{matrix} C_1 \xleftarrow{m} C^{\leftarrow\leftarrow}$$

For this structure to fully represent the small category C, we require it to satisfy the axioms of a category, namely:

1. The domains and codomains of identity morphisms are appropriate.
2. The domains and codomains of composed morphisms are appropriate.
3. Identity morphisms behave as identities under composition.
4. Composition is associative.

We make two essential observations. Firstly, the set of composable morphisms is precisely $C^{\leftarrow\leftarrow} = \{(f, g) \in C_1 \times C_1 \mid d(f) = c(g)\}$, which is the following pullback in **Set**:

$$\begin{array}{ccc} C^{\leftarrow\leftarrow} & \xrightarrow{\pi_2} & C_1 \\ {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle c} \\ C_1 & \xrightarrow{d} & C_0 \end{array}$$

Secondly, that the four axioms given above may be expressed as commutative diagrams in **Set** (which are given in 1.0.1).

An internal category is an abstraction of this perspective to another category $\mathbb{C}$, which is not necessarily **Set**. In particular, as we make use of pullbacks, we require $\mathbb{C}$ to *have pullbacks*.

Definition 1.0.1. Let $\mathbb{C}$ be a category with pullbacks. An *internal category*, C, in $\mathbb{C}$, is a diagram

$$C_0 \underset{\overset{\longleftarrow}{c}}{\overset{\overset{d}{\longleftarrow}}{\xrightarrow{e}}} C_1 \xleftarrow{m} C^{\leftarrow\leftarrow} \tag{IC}$$

in $\mathbb{C}$, where $C^{\leftarrow\leftarrow}$ is defined as the pullback:

$$\begin{array}{ccc} C^{\leftarrow\leftarrow} & \xrightarrow{\pi_2} & C_1 \\ {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle c} \\ C_1 & \xrightarrow{d} & C_0 \end{array}$$

such that the following four diagrams commute:

$$\begin{array}{ccc} C_0 & \xrightarrow{e} & C_1 \\ {\scriptstyle e}\downarrow & \diagdown\!\!\diagdown & \downarrow{\scriptstyle d} \\ C_1 & \xrightarrow[c]{} & C_0 \end{array} \qquad \begin{array}{ccccc} C_1 & \xleftarrow{\pi_1} & C^{\leftarrow\leftarrow} & \xrightarrow{\pi_2} & C_1 \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle d} \\ C_0 & \xleftarrow{c} & C_1 & \xrightarrow{d} & C_0 \end{array} \tag{IC1, IC2}$$

$$\begin{array}{ccc} C_1 & \xrightarrow{\langle ec,1\rangle} & C^{\leftarrow\leftarrow} \\ {\scriptstyle \langle 1,ed\rangle}\downarrow & \diagdown\!\!\diagdown & \downarrow{\scriptstyle m} \\ C^{\leftarrow\leftarrow} & \xrightarrow[m]{} & C_1 \end{array} \qquad \begin{array}{ccc} C^{\leftarrow\leftarrow\leftarrow} & \xrightarrow{m\times 1} & C^{\leftarrow\leftarrow} \\ {\scriptstyle 1\times m}\downarrow & & \downarrow{\scriptstyle m} \\ C^{\leftarrow\leftarrow} & \xrightarrow{m} & C_1 \end{array} \tag{IC3, IC4}$$

where $C^{\leftarrow\leftarrow\leftarrow}$ is defined as the pullback

$$
\begin{array}{ccc}
C^{\leftarrow\leftarrow\leftarrow} & \xrightarrow{\pi_1} & C^{\leftarrow\leftarrow} \\
{\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_1} \\
C^{\leftarrow\leftarrow} & \xrightarrow{\pi_2} & C_1
\end{array}
$$

In this definition, $\langle ec, 1\rangle$ is the universal morphism induced by the pullback $C^{\leftarrow\leftarrow}$ and the morphisms $ec : C_1 \to C_1$ and $1_{C_1} : C_1 \to C_1$. Similarly, $\langle 1, ed\rangle$ is the universal morphism induced by $C^{\leftarrow\leftarrow}$ and $1_{C_1} : C_1 \to C_1$ and $ed : C_1 \to C_1$. Then, $1 \times m = \langle \pi_1\pi_1, m\pi_2\rangle$ is the universal morphism induced by $C^{\leftarrow\leftarrow}$ and the morphisms $\pi_1\pi_1 : C^{\leftarrow\leftarrow\leftarrow} \to C_1$ and $m\pi_2 : C^{\leftarrow\leftarrow\leftarrow} \to C_1$. And lastly, $m \times 1 = \langle m\pi_1, \pi_2\pi_2\rangle$ is induced by $C^{\leftarrow\leftarrow}$ and $m\pi_1 : C^{\leftarrow\leftarrow\leftarrow} \to C_1$ and $\pi_2\pi_2 : C^{\leftarrow\leftarrow\leftarrow} \to C_1$.

Let us consider this definition with $\mathbb{C} = \mathbf{Set}$ to understand how the conditions IC1 - IC4 correspond to the axioms 1 - 4 on a category. Firstly, $C^{\leftarrow\leftarrow\leftarrow} = \{(f, g, h) \in C_1 \times C_1 \times C_1 \mid d(f) = c(g), d(g) = c(h)\}$ is the set of composable triples of morphisms. Then, for **IC1**, de and ce map an object to the domain and codomain (repectively) of the identity morphism of that object. This commutative square says that this must be the original object. For **IC2**, consider a pair of composable morphisms (in $C^{\leftarrow\leftarrow}$), $(f : Y \to Z, g : X \to Y)$. Then $c\pi_1$ maps this pair to the codomain of f, Z, and the left square thus asserts that the codomain of the composition fg is Z. On the other hand, $d\pi_2$ maps the pair to X, so the right square is that the domain of fg is X. Therefore **IC2** says that fg is $fg : X \to Z$.

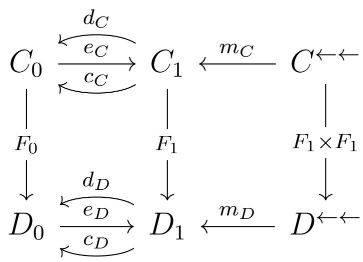
Next, we consider **IC3**. The map $\langle ec, 1\rangle$ maps a morphism $f : X \to Y$ in C_1 to the pair of composable morphism $(1_Y, f)$, while m then maps this pair to its composition $1_Y f$. The equality of this diagram says that this must be f. Similarly, $\langle 1, ed\rangle$ maps f to $(f, 1_X)$, and then m maps this to $f1_X$. Again, the equality says $f1_X = f$. Thus **IC3** says that identity morphisms act as identity under composition. Lastly, for **IC4**, $m \times 1$ maps a composable triple (f, g, h) to the composable pair (fg, h), and then m maps this pair to $(fg)h$. On the other hand, $1 \times m$ maps this pair to (f, gh) and then m maps it to $f(gh)$. The commutativity says that $(fg)h = f(gh)$, which is associativity of composition.

However, the value of this definition is when one considers $\mathbb{C}$ as a category that is not **Set**.

Example 1.0.2. If $\mathbb{C} = \mathbf{Grp}$, the category of groups and group homomorphisms, then an internal category in **Grp** is a small category where the objects and morphisms each form a group, and the maps d, c, e and m are group homomorphisms. On the other hand, if $\mathbb{C} = \mathbf{Top}$, the category of topological spaces and continuous maps, then an internal category is a small category where the objects and morphisms each form a topological space, and the maps d, c, e and m are continuous maps between topological spaces. We note that **Grp** and **Top** both have pullbacks and therefore this consideration is valid.

One may perform this same process on the definition of a functor. Indeed, a functor $F : C \to D$ between two small categories C and D may be considered as a pair of maps $F_0 : C_0 \to D_0$ and $F_1 : C_1 \to D_1$ which respect domains, codomains, identity morphisms and composition, in the usual sense. These latter conditions may be considered as commutative diagrams in **Set**, which leads to the following definition.

Definition 1.0.3. Let $\mathbb{C}$ be a category with pullbacks and let C and D be two internal categories in $\mathbb{C}$. An *internal functor*, $F : C \to D$ from C to D is a pair of morphisms $F = (F_0, F_1)$ as in the following diagram:



where $F_1 \times F_1$ is the universal morphism induced by the pullback $D^{\leftarrow\leftarrow}$ by F_1 (and itself), such that the following four diagrams commute:

$$\begin{array}{ccc} C_1 & \xrightarrow{d_C} & C_0 \\ {\scriptstyle F_1}\downarrow & & \downarrow{\scriptstyle F_0} \\ D_1 & \xrightarrow{d_D} & D_0 \end{array} \qquad \begin{array}{ccc} C_1 & \xrightarrow{c_C} & C_0 \\ {\scriptstyle F_1}\downarrow & & \downarrow{\scriptstyle F_0} \\ D_1 & \xrightarrow{c_D} & D_0 \end{array}$$

$$\begin{array}{ccc} C_0 & \xrightarrow{e_C} & C_1 \\ {\scriptstyle F_0}\downarrow & & \downarrow{\scriptstyle F_1} \\ D_0 & \xrightarrow{e_D} & D_1 \end{array} \qquad \begin{array}{ccc} C^{\leftarrow\leftarrow} & \xrightarrow{m_C} & C_1 \\ {\scriptstyle F_1\times F_1}\downarrow & & \downarrow{\scriptstyle F_1} \\ D^{\leftarrow\leftarrow} & \xrightarrow{m_D} & D_1 \end{array}$$

By definition, an internal functor in **Set** is precisely a functor between small categories.

For an internal category C in a category $\mathbb{C}$ with pullbacks, one may consider $1_C = (1_{C_0}, 1_{C_1}) : C \to C$ as the *internal identity functor* on C, which trivially satisfies the above definition.

If one then considers two internal functors $F : C \to D$ and $G : D \to E$ (for internal categories C, D and E), their *composition* is defined as $GF = (G_0F_0, G_1F_1) : C \to E$, by component wise composition in $\mathbb{C}$. The associativity of this composition follows from the associativity of composition of morphisms in $\mathbb{C}$, and the internal identity functor is an identity for this composition. This allows us to make the next definition.

Definition 1.0.4. Let $\mathbb{C}$ be a category with pullbacks. The *category of internal categories* of $\mathbb{C}$, $\mathrm{Cat}(\mathbb{C})$, has internal categories in $\mathbb{C}$ as objects and internal functors in $\mathbb{C}$ as morphisms.

It should be clear from the above that we have an equivalence of categories $\mathrm{Cat}(\mathbf{Set}) \sim \mathrm{Cat}$, where the latter category has objects as small categories and morphisms as functors between small categories.

While the above structures will be sufficient for our purposes, it is worthwhile considering the internal notion of the third fundamental structure of category theory - the natural transformation.

Let $F, G : C \to D$ be two functors between small categories C and D. A natural transformation $\tau : F \to G$ is classically considered as a family of morphisms in D, indexed by the objects of C. Given that C and D are small categories, it may instead be viewed as a map $\tau : C_0 \to D_1$ from the objects of C to the morphisms of D. For each object in C, the corresponding component of the natural transformation must be a morphism in D with domain and codomain the image of the object under F and G, respectively. That is, for all X in C_0, $\tau(X) = \tau_X : F(X) \to G(X)$. Additionally, these morphisms must satisfy the condition of naturality. As before, these two conditions may be expressed as commutative diagrams in **Set**, and to view this structure in a general $\mathbb{C}$ (with pullbacks) is to give an internal natural transformation.

Definition 1.0.5. Let $\mathbb{C}$ be a category with pullbacks, let C and D be two internal categories in $\mathbb{C}$ and let $F : C \to D$ and $G : C \to D$ be two internal functors from C to D. An *internal natural transformation* $\tau : F \to G$ from F to G is a morphism $\tau : C_0 \to D_1$ in $\mathbb{C}$ such that the following diagrams commute:

$$\begin{array}{ccc} C_0 & \xrightarrow{F_0} & D_0 \\ {\scriptstyle G_0}\downarrow & \searrow^{\tau} & \uparrow{\scriptstyle d_D} \\ D_0 & \xleftarrow[c_D]{} & D_1 \end{array} \qquad \begin{array}{ccc} C_1 & \xrightarrow{\langle \tau c, F_1 \rangle} & D^{\leftarrow\leftarrow} \\ {\scriptstyle \langle G_1, \tau d \rangle}\downarrow & & \downarrow{\scriptstyle m_D} \\ D^{\leftarrow\leftarrow} & \xrightarrow{m_D} & D_1 \end{array}$$

Here, the first diagram provides that the domains and codomains of the components of the natural transformation are appropriate, in an internal sense, while the second diagram is an internalisation of naturality. One may, in a similar fashion, define internal notions of vertical and horizontal composition of natural transformations, identity natural transformations and the interchange law. Therefore, $\mathrm{Cat}(\mathbb{C})$ may be viewed as a 2-category, with 2-cells as the internal natural transformations of $\mathbb{C}$.

We now direct our attention to specific types of internal categories. In particular, we consider internal categories endowed with additional structure. The first, and most commonly occurring example, is that of an internal groupoid.

A *groupoid*, often seen as the categorical generalisation of a group, may be viewed as a category where every morphism is an isomorphism. That is to say, for every morphism, there exists an inverse morphism with opposing domain and codomain, such that composition (on either side) with this inverse yields the appropriate identity morphism. To define this structure internally, one simply imposes this condition on an internal category.

Definition 1.0.6. Let $\mathbb{C}$ be a category with pullbacks, and let C be an internal category in $\mathbb{C}$. C is an *internal groupoid* if there exists a morphism $i : C_1 \to C_1$ such the following diagrams commute:

$$\begin{array}{ccc} C_1 & \xrightarrow{d} & C_0 \\ {\scriptstyle c}\downarrow & \searrow^{i} & \uparrow{\scriptstyle c} \\ C_0 & \xleftarrow[d]{} & C_1 \end{array} \qquad \begin{array}{ccccc} C_1 & \xrightarrow{d} & C_0 & \xleftarrow{c} & C_1 \\ {\scriptstyle \langle i,1\rangle}\downarrow & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle \langle 1,i\rangle} \\ C^{\leftarrow\leftarrow} & \xrightarrow{m} & C_1 & \xleftarrow{m} & C^{\leftarrow\leftarrow} \end{array} \qquad \text{(GP1, GP2)}$$

In the case of $\mathbb{C} = \mathbf{Set}$, the map i assigns a morphism to its inverse, the first diagram asserts the appropriateness of the domain and codomain of the inverse and the second diagram says that composing a morphism with its inverse yields an identity morphism. As expected, an internal groupoid in this context is precisely a small groupoid.

Furthermore, any functor $F : C \to D$, where C and D are groupoids, will preserve the groupoid structure. This is because isomorphisms are uniquely defined by their behaviour under composition, and are therefore preserved by functors. Internally, any internal functor between internal groupoids will be a *morphism of groupoids*, and we may define the category Grpd($\mathbb{C}$) of internal groupoids and internal functors of a category $\mathbb{C}$ with pullbacks.

A groupoid is *connected* if there exists a morphism (that is, an isomorphism) between every two objects. To provide an internal definition of this, consider a small groupoid, C, as an internal groupoid in **Set**, and consider the map $(c, d) : C_1 \to C_0 \times C_0$. This map assigns each morphism $f : X \to Y$ in C to the pair of its codomain and domain (Y, X). To say that this groupoid is connected, is that every such pair is achieved under this mapping, and therefore asks for (c, d) to be a surjection.

We now arrive at an important consideration when *internalising* some mathematical structure. In **Set**, a surjective map is an epimorphism. However, the Axiom of Choice says that every epimorphism in **Set** is split. Therefore, in our internal definition of a connected groupoid, one must ask if (c, d) should be a (usual) epimorphism, a split epimorphism or in fact an epimorphism of strength between these two (eg. *regular*, *strong*, *extremal* etc.). For our purpose, a regular epimorphism is the appropriate choice, and we reserve further discussion on the matter. We define this next, followed by the definition of an *internal connected groupoid*.

Definition 1.0.7. In a category $\mathbb{C}$, a morphism $f : X \to Y$ is a *regular epimorphism* if there exists a parallel pair of morphisms $u, v : Z \to X$ such that f is the coequalizer of u and v.

Note that in our present context, we make use of categorical products and pullbacks, and thus ask for our *ambient category*, $\mathbb{C}$, of the next definition, to be finitely complete.

Definition 1.0.8. Let $\mathbb{C}$ be a finitely complete category. An *internal connected groupoid* in $\mathbb{C}$ is an internal groupoid, C, in $\mathbb{C}$ such that $(c, d) : C_1 \to C_0 \times C_0$ is a regular epimorphism.

Now, given that the above definition is of interest, one may wonder what such a structure where (c, d) is instead an injective map (or, more generally, a monomorphism) would be characterised as. To answer this question, we introduce the notion of an *internal equivalence relation* or *congruence*.

An equivalence relation R on a set X is a relation which is reflexive, symmetric and transitive. As a relation, R may be viewed as a subset of $X \times X$, and therefore a monomorphism $r : R \to X \times X$ in **Set**. (More accurately, this will be an equivalence class of monomorphisms, called a *subobject*, but we overlook this detail for the moment.) Composing with product projections $\pi_1, \pi_2 : X \times X \to X$, we obtain morphisms $r_1 = \pi_1 r : R \to X$ and $r_2 = \pi_2 r : R \to X$ with $r = (r_1, r_2)$.

That R is reflexive means that there is a map $\rho : X \to R$ defined by $\rho(x) = (x, x)$. This may equivalently be expressed as ρ satisfying the following commutative diagram in **Set**:

$$\begin{array}{ccc} X & \xrightarrow{\rho} & R \\ {\scriptstyle\rho}\downarrow & \diagdown\!\!\diagdown & \downarrow{\scriptstyle r_1} \\ R & \xrightarrow[r_2]{} & X \end{array} \tag{REF}$$

Symmetry of R means that there is a map $\sigma : R \to R$ which assigns $\sigma(x, y) = (y, x)$. As a commutative diagram in **Set**:

$$\begin{array}{ccc} R & \xrightarrow{r_2} & X \\ {\scriptstyle r_1}\downarrow & \searrow^{\sigma} & \uparrow{\scriptstyle r_1} \\ X & \xleftarrow[r_2]{} & R \end{array} \tag{SYM}$$

Finally, for transitivity, we consider the set $R \times_X R = \{((x,y),(x',y')) \in R \times R \mid y = x'\}$, which may be expressed as the pullback:

$$\begin{array}{ccc} R \times_X R & \xrightarrow{\pi_1'} & R \\ {\scriptstyle \pi_2'}\downarrow & \lrcorner & \downarrow{\scriptstyle r_1} \\ R & \xrightarrow{r_2} & X \end{array} \tag{TR1}$$

Then the transitivity condition is that there is a map $\tau : R \times_X R \to R$ such $\tau((x,y),(y,z)) = (x,z)$, which again we express as a commutative diagram in **Set** as:

$$\begin{array}{ccccc} R & \xleftarrow{\pi_1'} & R \times_X R & \xrightarrow{\pi_2'} & R \\ {\scriptstyle r_1}\downarrow & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle r_2} \\ X & \xleftarrow{r_1} & R & \xrightarrow{r_2} & X \end{array} \tag{TR2}$$

We may therefore, by these observations, define an *internal equivalence relation.*

Definition 1.0.9. Let $\mathbb{C}$ be a finitely complete category. An *internal equivalence relation* or *congruence*, R (on X) in $\mathbb{C}$ is a diagram

$$X \underset{\xleftarrow{r_2}}{\overset{\xleftarrow{r_1}}{\xrightarrow{\rho}}} R \circlearrowleft^{\sigma} \xleftarrow{\tau} R \times_X R$$

where $R \times_X R$ is defined as the pullback **TR1** such $(r_1, r_2) : R \to X \times X$ is a monomorphism and the diagrams **REF**, **SYM** and **TR2** commute.

Before expressing the apparent connection between this structure and our previous definitions, we make some observations.

Proposition 1.0.10. *Let R be a congruence in a finitely complete category $\mathbb{C}$. Then the following diagrams commute:*

$$\begin{array}{ccc} R & \xrightarrow{\langle \rho r_1, 1\rangle} & R \times_X R \\ {\scriptstyle \langle 1, \rho r_2\rangle}\downarrow & \searrow\!\!\!\!= & \downarrow{\scriptstyle \tau} \\ R \times_X R & \xrightarrow{\tau} & R \end{array} \qquad \begin{array}{ccc} R \times_X R \times_X R & \xrightarrow{\tau \times 1} & R \times_X R \\ {\scriptstyle 1 \times \tau}\downarrow & & \downarrow{\scriptstyle \tau} \\ R \times_X R & \xrightarrow{\tau} & R \end{array}$$

(CG1, CG2)

$$\begin{array}{ccccc} R & \xrightarrow{r_2} & X & \xleftarrow{r_1} & R \\ {\scriptstyle \langle \sigma, 1\rangle}\downarrow & & {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \langle 1, \sigma\rangle} \\ R \times_X R & \xrightarrow{\tau} & R & \xleftarrow{\tau} & R \times_X R \end{array} \tag{CG3}$$

where $R \times_X R \times_X R$ is defined as the pullback:

$$\begin{array}{ccc} R \times_X R \times_X R \lrcorner & \xrightarrow{\pi_2''} & R \times_X R \\ {\scriptstyle \pi_1''}\downarrow & & {\scriptstyle \pi_1'}\downarrow \\ R \times_X R & \xrightarrow{\pi_2'} & R \end{array}$$

Proof. This follows from straightforward calculation and the fact that (r_1, r_2) is a monomorphism. □

We now observe that by setting $C_0 = X$, $C_1 = R$, $C^{\leftarrow\leftarrow} = R \times_X R$, $d = r_2$, $c = r_1$, $e = \rho$, $m = \tau$ and $i = \sigma$, the diagrams **REF**, **TR2**, **CG1** and **CG2** correspond to **IC1**, **IC2**, **IC3** and **IC4** while **SYM** and **CG3** correspond to **GP1** and **GP2**, respectively. We therefore conclude that a congruence is precisely an internal groupoid for which $(c, d) : C_1 \to C_0 \times C_0$ is a monomorphism.

There is another perspective through which this result may be viewed. If R is a preorder (a reflexive and transitive relation) on a set X, we may view this data as a category where there exists at most one morphism between each pair of objects. That is, we have the elements of X as the objects of the category, and say that if $(x, y) \in R$ then there is a (unique) morphism $x \to y$. The fact that r is reflexive gives identity morphisms, and the transitivity gives composition of morphisms. If we then additionally ask R to be symmetric, this means that for every morphism in the category $x \to y$, there is another morphism $y \to x$. Composing these morphisms will produce the unique morphisms $x \to x$ and $y \to y$, which are identities. Thus, every morphism of this category is an isomorphism, and this is therefore a groupoid. So, an equivalence relation is a groupoid where each pair of objects has at most one morphism between them. Considering this construction internally, we have that an internal equivalence relation is an internally groupoid, with the last condition translating to the fact that (c, d) is a monomorphim.

Chapter 2

Internal Categories in Grp

We direct our attention to a characterisation of an internal category in the category **Grp**. While we have mentioned, in 1.0.2, that this is a small category with group structures on the objects and morphisms where the domain, codomain, identity and composition morphisms are all group homomorphisms, there exists much more useful perspective of this structure.

This characterisation, first published by Brown and Spencer in [3], is given by a categorical equivalence between the category of internal categories in **Grp**, Cat(**Grp**), and the category of *group crossed modules* (or simply *crossed modules*), **XMod**. It is the aim of this chapter to express and prove this equivalence. We begin with the definitions of a *crossed module* and a *crossed module morphism.*

Definition 2.0.1. A *crossed module* is a quadruple $f = (A, B, \alpha, f)$ where A and B are groups, α is a (left) group action of B on A (written as a left superscript) and $f : A \to B$ is a group homomorphism, which satisfy the conditions:

1. $f({}^{b}a) = b + f(a) - b$
2. ${}^{f(a)}a' = a + a' - a$

We note that the first condition is known as *equivariance* while the second condition is known as the *Peiffer identity.* One may see these conditions, with lack of further motivation, as asserting that the interaction between the homomorphism and the action of a crossed module is described by conjugation in each of the constituent groups.

Definition 2.0.2. Let $f = (A, B, \alpha, f)$ and $f' = (A', B', \alpha', f')$ be two crossed modules. A *morphism of crossed modules*, $\xi : f \to f'$ is a pair of group homomorphisms $\xi = (\xi_1 : A \to A', \xi_2 : B \to B')$ such that the following diagrams commute:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \xi_1}\downarrow & & \downarrow{\scriptstyle \xi_2} \\ A' & \xrightarrow{f'} & B' \end{array} \qquad \begin{array}{ccc} B \times A & \xrightarrow{\alpha} & A \\ {\scriptstyle \xi_2 \times \xi_1}\downarrow & & \downarrow{\scriptstyle \xi_1} \\ B' \times A' & \xrightarrow{\alpha'} & A' \end{array}$$

Composition and identity morphisms of crossed modules are defined as expected, and therefore **XMod** is a category.

Observe that for an internal category, C, the morphism d is a split epimorphism with section e, as $de = 1_{C_0}$ by **IC1**. Then, that **Grp** is finitely complete and has a *zero object*, the trivial group, means that we may consider the kernel of d, $(\mathrm{Ker}(d), \kappa)$ as in the following diagram:

$$\mathrm{Ker}(d) \xrightarrow{\kappa} C_1 \underset{e}{\overset{d}{\rightleftarrows}} C_0 \tag{SES}$$

This is, in fact, a *split (short) exact sequence*, and will play an essential role in constructing the equivalence.

Consider some $f \in C_1$. Then f may be expressed as:

$$f = f - ed(f) + ed(f)$$

Now, observe that

$$d(f - ed(f)) = d(f) - ded(f) = d(f) - d(f) = 0$$

where the 0 on the right is the identity element of the group C_0. This means that $f - ed(f)$ is an element of $\mathrm{Ker}(d)$. On the other hand, $d(f)$ is an element of C_0, and thus we obtain a *set-theoretic map* $\varphi : C_1 \to \mathrm{Ker}(d) \times C_0$ defined by:

$$\varphi(f) = (f - ed(f), d(f))$$

However, if we endow $\mathrm{Ker}(d) \times C_0$ with the usual group structure of a product, this structure does not exhibit the desired behaviour. Instead, we consider the notion of a *semi-direct product*.

Definition 2.0.3. Let A and B be two groups and let α be a (left) group action of B on A. Then the *semi-direct product* of A and B (with respect to α), $A \rtimes_\alpha B$, is a group with underlying set $A \times B$, and with group operation:

$$(a, b) + (a', b') = (a + {}^{b}a', b + b')$$

This operation is frequently referred to as component-wise that has been *twisted* by the group action.

Now, returning to our construction, consider the diagram **SES**, and define the (left) group action, α, of C_0 on $\mathrm{Ker}(d)$ by:

$${}^{X}k = e(X) + k - e(X) \tag{ACT}$$

for each $X \in C_0$ and $k \in \mathrm{Ker}(d)$. The right hand side is indeed an element of $\mathrm{Ker}(d)$,

$$d(e(X) + k - e(X)) = de(X) + d(k) - de(X) = X + 0 - X = 0$$

and to show that α satisfies the axioms of a group action is straight forward. We then obtain the semidirect product $\mathrm{Ker}(d) \rtimes_\alpha C_0$. Next, view φ instead as a map $C_1 \to \mathrm{Ker}(d) \rtimes_\alpha C_0$. We show that it is a group homomorphism:

$$\begin{aligned}
&\varphi(f + g) \\
&= \varphi(f - ed(f) + ed(f) + g - ed(g) + ed(g)) \\
&= \varphi(f - ed(f) + ed(f) + g - ed(g) - ed(f) + ed(f) + ed(g)) \\
&= \varphi(f - ed(f) + {}^{d(f)}(g - ed(g)) + e(d(f) + d(g))) \\
&= (f - ed(f) + {}^{d(f)}(g - ed(g)), d(f) + d(g)) \\
&= (f - ed(f), d(f)) + (g - ed(g), d(g)) \\
&= \varphi(f) + \varphi(g)
\end{aligned}$$

On the other hand, consider $\psi : \mathrm{Ker}(d) \rtimes_\alpha C_0 \to C_1$ defined by $\psi(k, X) =$

$k + e(X)$. This is also a homomorphism:

$$\begin{aligned} &\psi((k, X) + (k', X')) \\ &= \psi(k + {}^X k', X + X') \\ &= k + {}^X k' + e(X + X') \\ &= k + e(X) + k' - e(X) + e(X) - e(X) \\ &= k + e(X) + k' + e(X') \\ &= \psi(k, X) + \psi(k', X') \end{aligned}$$

Furthermore, these homomorphisms are inverses:

$$\psi\varphi(f) = \psi(f - ed(f), d(f)) = f - ed(f) + ed(f) = f$$

and

$$\begin{aligned} &\varphi\psi(k, X) \\ &= \varphi(k + e(X)) \\ &= (k + e(X) - ed(k + e(X)), d(k + e(X))) \\ &= (k + e(X) - ed(k) - ede(X), d(k) + de(X)) \\ &= (k + e(X) - 0 - e(X), 0 + X) \\ &= (k, X) \end{aligned}$$

Therefore, $C_1 \approx \mathrm{Ker}(d) \rtimes_\alpha C_0$. We note in passing that this is a special case of a more general correspondence between short exact sequences, actions and semidirect products in an appropriate context. We refer the reader to [12] for further details.

Now, consider $P : \mathrm{Cat}(\mathbf{Grp}) \to \mathbf{XMod}$, which we will show to be a functor, which is defined (on objects) by:

$$P(C) = (\mathrm{Ker}(d), C_0, \alpha, c|_{\mathrm{Ker}(d)})$$

for some internal category C in **Grp**, where $c|_{\mathrm{Ker}(d)} : \mathrm{Ker}(d) \to C_0$ is the restriction of $c : C_1 \to C_0$ to domain $\mathrm{Ker}(d)$ (and thus a homomorphism) and α is the action of C_0 on $\mathrm{Ker}(d)$ defined in **ACT**. Writing simply c for $c|_{\mathrm{Ker}(d)}$, we have

$$c({}^X k) = c(e(X) + k - e(X)) = ce(X) + c(k) - ce(X) = X + c(k) - X$$

and thus $P(C)$ satisfies the equivariance axiom of a crossed module. To obtain the Peiffer identity, we must first make observations on the nature of m, the composition morphism of C. For some $(f, g) \in C^{\leftarrow\leftarrow}$, we have that $d(f) = c(g)$, and thus

$$\begin{aligned}
&m(f, g) \\
&= m(f - ed(f) - ed(f), g) \\
&= m(f - ed(f), 0) + m(ed(f), g) \\
&= m(f - ed(f), ed(f - ed(f))) + m(ec(g), g) \\
&= f - ed(f) + g
\end{aligned}$$

as $f - ed(f) \in \mathrm{Ker}(d)$ and by **IC3**. What this means is that the composition m is uniquely defined by the rest of the structure. With this established, consider some two $k, k' \in \mathrm{Ker}(d)$. Then, recalling that ${}^{c(k)}k' \in \mathrm{Ker}(d)$,

$$d({}^{c(k)}k' + ec(k)) = d({}^{c(k)}k') + dec(k) = 0 + c(k) = c(k)$$

so that $({}^{c(k)}k' + ec(k), k) \in C^{\leftarrow\leftarrow}$. We compute the composition:

$$\begin{aligned}
&m({}^{c(k)}k' + ec(k), k) \\
&= {}^{c(k)}k' + ec(k) - ed({}^{c(k)}k' + ec(k)) + k \\
&= {}^{c(k)}k' + ec(k) - edec(k) - ed({}^{c(k)}k') + k \\
&= {}^{c(k)}k' + ec(k) - ec(k) - e(0) + k \\
&= {}^{c(k)}k' + k
\end{aligned}$$

Therefore,

$$\begin{aligned}
&{}^{c(k)}k' \\
&= m({}^{c(k)}k' + ec(k), k) - k \\
&= m(ec(k) + k' - ec(k) + ec(k), k) - k \\
&= m(ec(k) + k', k) - k \\
&= m(ec(k), k) + m(k', 0) - k \\
&= k + m(k', ed(k')) - k \\
&= k + k' - k
\end{aligned}$$

which is precisely the Peiffer identity. $P(C)$ is thus crossed module. Consider an internal functor in **Grp**, $F = (F_0, F_1) : C \to D$. We show that $P(F) = (F_1|_{\mathrm{Ker}(d)}, F_0) : P(C) \to P(D)$ is a crossed module morphism. From the definition of an internal functor, we have that $F_0 c_C(k) = c_D F_1(k)$, which is first condition, and for the same reason, we have

$$\begin{aligned} & {}^{F_0(X)}F_1(k) \\ &= e_D F_0(X) + F_1(k) - e_D F_0(X) \\ &= F_1 e_C(X) + F_1(k) - F_1 e_C(X) \\ &= F_1(e_C(X) + k - e_C(X)) \\ &= F_1({}^X k) \end{aligned}$$

which is the second condition. We may thus conclude that P is indeed a functor.

The next step is to construct a candidate inverse functor. Define $Q : \mathbf{XMod} \to \mathrm{Cat}(\mathbf{Grp})$ by:

$$Q(A, B, \alpha, f) \quad = \quad B \underset{\gamma}{\overset{\pi_2}{\underset{\iota}{\leftrightarrows}}} A \rtimes_\alpha B \xleftarrow{\mu} (A \rtimes_\alpha B) \times_B (A \rtimes_\alpha B)$$

where $A \rtimes_\alpha B$ is the semidirect product of A and B with respect to α, π_2 is a projection defined by $\pi_2(a, b) = b$, ι is an inclusion defined by $\iota(b) = (0, b)$ and γ is defined by $\gamma(a, b) = f(a) + b$. Both π_2 and ι are trivially group homomorphisms. Then, by equivariance,

$$\begin{aligned} & \gamma((a, b) + (a', b')) \\ &= \gamma(a + {}^b a', b + b') \\ &= f(a + {}^b a') + b + b' \\ &= f(a) + f({}^b a') + b + b' \\ &= f(a) + b + f(a') - b + b + b' \\ &= f(a) + b + f(a') + b' \\ &= \gamma(a, b) + \gamma(a', b') \end{aligned}$$

so γ is also a homomorphism. $(A \rtimes_\alpha B) \times_B (A \rtimes_\alpha B)$ is the pullback of π_2 and γ. More explicitly, its elements $((a, b), (a', b'))$ satisfy $\pi_2(a, b) = \gamma(a', b')$, or $b = f(a') + b'$. So, the elements of $(A \rtimes_\alpha B) \times_B (A \rtimes_\alpha B)$ are precisely $((a, f(a') + b'), (a', b'))$ for all $a, a' \in A$ and $b' \in B$. We will write π_1' and π_2' for the projections of this pullback. μ is then defined by $\mu((a, f(a')+b'), (a', b')) = (a+a', b')$. Noting that the operation in $(A \rtimes_\alpha B) \times_B (A \rtimes_\alpha B)$ is component wise (and not twisted), we use the Peiffer identity to show that μ is a group homomorphism:

$$\begin{aligned}
&\mu(((a_1, f(a_1') + b_1'), (a_1', b_1')) + ((a_2, f(a_2') + b_2'), (a_2', b_2'))) \\
&= \mu(((a_1, f(a_1') + b_1') + (a_2, f(a_2') + b_2')), ((a_1', b_1') + (a_2', b_2'))) \\
&= \mu((a_1 + {}^{(f(a_1')+b_1')}a_2, f(a_1') + b_1' + f(a_2') + b_2'), (a_1' + {}^{b_1'}a_2', b_1' + b_2')) \\
&= (a_1 + {}^{(f(a_1')+b_1')}a_2 + a_1' + {}^{b_1'}a_2', b_1' + b_2') \\
&= (a_1 + {}^{f(a_1')}({}^{b_1'}a_2) + a_1' + {}^{b_1'}a_2', b_1' + b_2') \\
&= (a_1 + a_1' + {}^{b_1'}a_2 - a_1' + a_1' + {}^{b_1'}a_2', b_1' + b_2') \\
&= (a_1 + a_1' + {}^{b_1'}a_2 + {}^{b_1'}a_2', b_1' + b_2') \\
&= (a_1 + a_1' + {}^{b_1'}(a_2 + a_2'), b_1' + b_2') \\
&= (a_1 + a_1', b_1') + (a_2 + a_2', b_2') \\
&= \mu((a_1, f(a_1') + b_1'), (a_1', b_1')) + \mu((a_2, f(a_2') + b_2'), (a_2', b_2'))
\end{aligned}$$

Next, we show that $Q(A, B, \alpha f)$ satisfies the commutative diagrams of an internal category by the following calculations.

$$\pi_2\iota(b) = \pi_2(0, b) = b$$

$$\gamma\iota(b) = \gamma(0, b) = f(0) + b = b$$

$$\begin{aligned}
\gamma\mu((a, f(a') + b'), (a', b')) &= \gamma(a + a', b') = f(a + a') + b' = f(a) + f(a') + b' \\
&= \gamma(a, f(a') + b') = \gamma\pi_1'((a, f(a') + b'), (a', b'))
\end{aligned}$$

$$\pi_2\mu((a, f(a')+b'), (a', b')) = \pi_2(a+a', b') = b' = \pi_2(a', b') = \pi_2\pi_2'((a, f(a')+b'), (a', b'))$$

$$m\langle ec, 1\rangle(a, b) = m(ec(a, b), (a, b)) = m((0, f(a) + b), (a, b)) = (0 + a, b) = (a, b)$$

$$m\langle 1, ed\rangle(a, b) = m((a, b), ed(a, b)) = m((a, b), (0, b)) = (a + 0, b) = (a, b)$$

For **IC4**, we compute each side and show that they agree, paying attention to the form of the first component of $(A \rtimes_\alpha B) \times_B (A \rtimes_\alpha B) \times_B (A \rtimes_\alpha B)$ which plays the role of $C^{\leftarrow\leftarrow\leftarrow}$:

$$\begin{aligned}
&m(m \times 1)((a, f(a') + f(a'') + b''), (a', f(a'') + b''), (a'', b'')) \\
&= m(m((a, f(a') + f(a'') + b''), (a', f(a'') + b'')), (a'', b'')) \\
&= m((a + a', f(a'') + b''), (a'', b'')) \\
&= (a + a' + a'', b'')
\end{aligned}$$

$$\begin{aligned}
&m(1 \times m)((a, f(a') + f(a'') + b''), (a', f(a'') + b''), (a'', b'')) \\
&= m((a, f(a') + f(a'') + b''), m((a', f(a'') + b''), (a'', b''))) \\
&= m((a, f(a') + f(a'') + b''), (a' + a'', b'')) \\
&= (a + a' + a'', b'')
\end{aligned}$$

We thus conclude that $Q(A, B, \alpha, f)$ is indeed an internal category. Consider some morphism of crossed modules $\xi = (\xi_1, \xi_2) : (A, B, \alpha, f) \to (A', B', \alpha', f')$. We define $Q(\xi) = (\xi_2, \xi_1 \times \xi_2) : Q(A, B, \alpha, f) \to Q(A', B', \alpha', f')$, and show that it is an internal functor. ξ_2 is a group homomorphism by definition. That $\xi_1 \times \xi_2$ is a homomorphism follows from the fact that a morphism of crossed modules respects the actions:

$$\begin{aligned}
&(\xi_1 \times \xi_2)((a, b) + (a', b')) \\
&= (\xi_1 \times \xi_2)(a + {}^{b}a', b + b') \\
&= (\xi_1(a + {}^{b}a'), \xi_2(b + b')) \\
&= (\xi_1(a) + \xi_1({}^{b}a'), \xi_2(b) + \xi_2(b')) \\
&= (\xi_1(a) + {}^{\xi_2(b)}\xi_1(a'), \xi_2(b) + \xi_2(b')) \\
&= (\xi_1(a), \xi_2(b)) + (\xi_1(a'), \xi_2(b')) \\
&= (\xi_1 \times \xi_2)(a, b) + (\xi_1 \times \xi_2)(a', b')
\end{aligned}$$

We show now that $Q(\xi)$ satisfies the axioms of an internal functor.

$$\pi_2'(\xi_1 \times \xi_2)(a, b) = \pi_2'(\xi_1(a), \xi_2(b)) = \xi_2(b) = \xi_2\pi_2(a, b)$$

$$\gamma'(\xi_1 \times \xi_2)(a, b) = \gamma'(\xi_1(a), \xi_2(b)) = f'\xi_1(a) + \xi_2(b) = \xi_2 f(a) + \xi_2(b) = \xi_2(f(a) + b) = \xi_2\gamma(a, b)$$

$$(\xi_1 \times \xi_2)\iota(b) = (\xi_1 \times \xi_2)(0, b) = (\xi_1(0), \xi_2(b)) = (0, \xi_2(b)) = \iota'\xi_2(b)$$

$$
\begin{aligned}
&\mu'((\xi_1 \times \xi_2) \times (\xi_1 \times \xi_2))((a, f(a') + b'), (a', b')) \\
&= \mu'((\xi_1(a), \xi_2(f(a') + b')), (\xi_1(a'), \xi_2(b'))) \\
&= (\xi_1(a) + \xi_1(a'), \xi_2(b')) \\
&= (\xi_1(a + a'), \xi_2(b')) \\
&= (\xi_1 \times \xi_2)(a + a', b') \\
&= (\xi_1 \times \xi_2)\mu'((a, f(a') + b'), (a', b'))
\end{aligned}
$$

We therefore conclude that $Q : \mathbf{Xmod} \to \mathrm{Cat}(\mathbf{Grp})$ is a functor. We lastly must show that P and Q form an equivalence. To do this, we note that an isomorphism in **XMod** is a crossed module morphism $\xi = (\xi_1, \xi_2)$ where ξ_1 and ξ_2 are group isomorphisms, and similarly an isomorphism in Cat(**Grp**) is an internal functor $F = (F_0, F_1)$ such that F_0 and F_1 are group isomorphisms.

Now, consider some internal category, C, in **Grp**. Then,

$$QP(C) = Q(\mathrm{Ker}(d), C_0, c_{\mathrm{Ker}(d)}, \alpha)$$

which is the internal category:

$$C_0 \underset{\gamma}{\overset{\pi_2}{\leftleftarrows}}\!\!\xrightarrow{\ \iota\ } \mathrm{Ker}(d) \rtimes_\alpha C_0 \xleftarrow{\ \mu\ } (\mathrm{Ker}(d) \rtimes_\alpha C_0) \times_{C_0} (\mathrm{Ker}(d) \rtimes_\alpha C_0)$$

with $\pi_2(k, X) = X$, $\iota(X) = (0, X)$, $\gamma(k, X) = c(k) + X$ and $\mu((k, c(k') + X), (k', X')) = (k + k', X)$, by the definition of Q. Recalling the isomorphism $\psi : \mathrm{Ker}(d) \rtimes_\alpha C_0 \to C_1$, we show that $(1_{C_0}, \psi) : QP(C) \to C$ is an internal functor (and therefore an isomorphism in Cat(**Grp**), as the constituent morphisms are group isomorphisms). The definition of an internal functor means that we must show that $d\psi = \pi_2$, $\psi\iota = e$, $c\psi = \gamma$ and $m(\psi \times \psi) = \psi\mu$:

$$d\psi(k, X) = d(k + e(X)) = d(k) + de(X) = 0 + X = X = \pi_2(k, X)$$

$$\psi\iota(X) = \psi(0, X) = 0 + e(X) = e(X)$$

$$c\psi(k, X) = c(k + e(X)) = c(k) + ce(X) = c(k) + X = \gamma(k, X)$$

$$
\begin{aligned}
&m(\psi \times \psi)((k, c(k') + X'), (k', X'))\\
&= m(\psi(k, c(k') + X'), \psi(k', X'))\\
&= m(k + e(c(k') + X'), k' + e(X'))\\
&= k + ec(k') + e(X') - ed(k + ec(k') + e(X') + k' + e(X')\\
&= k + ec(k') + e(X') - ede(X') - edec(k') - ed(k) + k' + e(X')\\
&= k + ec(k') + e(X') - e(X') - ec(k') - 0 + k' + e(X')\\
&= k + k' + e(X')\\
&= \psi(k + k', X')\\
&= \psi\mu((k, c(k') + X'), (k', X'))
\end{aligned}
$$

Therefore $QP \approx 1_{\text{Cat}(\mathbf{Grp})}$. Now, consider some crossed module (A, B, α, f). Then $PQ(A, B, \alpha, f) = (\text{Ker}(\pi_2), B, \beta, \gamma|_{\text{Ker}(\pi_2)})$, where $\text{Ker}(\pi_2)$ is the kernel of $\pi_2 : A \rtimes_\alpha B \to B$, which is precisely

$$\text{Ker}(\pi_2) = \{(a, b) \in A \times B \mid \pi_2(a, b) = 0\} = \{(a, 0) \mid a \in A\} \leq A \rtimes_\alpha B$$

In particular, for two $(a, 0), (a', 0) \in \text{Ker}(\pi_2)$, $(a, 0) + (a', 0) = (a + {}^0a', 0 + 0) = (a + a', 0)$. It should be clear that we have an isomorphism of groups $\eta : \text{Ker}(\pi_2) \to A$ defined by $\eta(a, 0) = a$. Then, β is the action of B on $\text{Ker}(\pi_2)$, where $b \in B$ acting on $(a, 0) \in \text{Ker}(\pi_2)$ is

$$
\begin{aligned}
{}^b(a, 0) = &\ \iota(b) + (a, 0) - \iota(b) = (0, b) + (a, 0) + (0, -b) = (0 + {}^ba, b + 0) + (0, -b)\\
= &\ ({}^ba, b) + (0, -b) = ({}^ba + {}^b0, b - b) = ({}^ba + 0, 0) = ({}^ba, 0)
\end{aligned}
$$

Lastly, $\gamma|_{\text{Ker}(\pi_2)}$ is the restriction of $\gamma : A \rtimes_\alpha B \to B$ to domain $\text{Ker}(\pi_2)$. That is, $\gamma(a, 0) = f(a) + 0 = f(a)$.

This means that $\eta({}^b(a, 0)) = \eta({}^ba, 0) = {}^ba = {}^b\eta(a, 0)$ and $f\eta(a, 0) = f(a) = \gamma(a, 0)$, which is precisely to say that $(\eta, 1_B) : PQ(A, B, \alpha, f) \to (A, B, \alpha, f)$ is a crossed module morphism. Because η and 1_B are isomorphisms, $(\eta, 1_B)$ is an isomorphism in **XMod**, and thus $PQ \approx 1_{\mathbf{XMod}}$. We therefore have that P and Q form a categorical equivalence Cat(**Grp**) $\sim$ **XMod**.

Although we have achieved the goal of this chapter, we make one last observation on internal categories in **Grp**: Every such internal category is in fact an internal groupoid. The equivalence we have shown provides a suitable setting to illustrate this.

Consider some internal category C in Cat(**Grp**), and the isomorphic internal category $QP(C)$. A *morphism* in the latter internal category is a pair $(k, X) \in \mathrm{Ker}(d) \rtimes_\alpha C_0$, with *domain* X and *codomain* $c(k)+X$. Composition is defined by the operation in Ker(d), while preserving the second component of the second *morphism* of the composable pair: $m((k, c(k')+X'), (k', X')) = (k+k', X')$. This suggests that a suitable *inverse* to (k, X) would be $(-k, c(k)+X)$. Define $\zeta : \mathrm{Ker}(d) \rtimes_\alpha C_0 \to \mathrm{Ker}(d) \rtimes_\alpha C_0$ by $\zeta(k, X) = (-k, c(k) + X)$. We show that ζ is a homomorphism:

$$
\begin{aligned}
&\zeta((k, X) + (k', X')) \\
&= \zeta(k + {}^X k', X + X') \\
&= (-(k + {}^X k'), c(k + {}^X k') + X + X') \\
&= (-{}^X k' - k, c(k) + c({}^X k') + X + X') \\
&= (-(e(X) + k' - e(X)) - k, c(k) + X + c(k') - X + X + X') \\
&= (e(X) - k' - e(X) - k, c(k) + X + c(k') + X') \\
&= ({}^X(-k') - k, c(k) + X + c(k') + X') \\
&= (-k + k + {}^X(-k') - k, c(k) + X + c(k') + X') \\
&= (-k + {}^{c(k)}({}^X(-k')), c(k) + X + c(k') + X') \\
&= (-k + {}^{(c(k)+X)}(-k'), c(k) + X + c(k') + X') \\
&= (-k, c(k) + X) + (-k', c(k') + X') \\
&= \zeta(k, X) + \zeta(k', X')
\end{aligned}
$$

We show that ζ satisfies the conditions of i in the definition of an internal groupoid, making $QP(C)$ an internal groupoid.

$$
\pi_2\zeta(k, X) = \pi_2(-k, c(k) + X) = c(k) + X = \gamma(k, X)
$$
$$
\gamma\zeta(k, X) = \gamma(-k, c(k) + X) = c(-k) + c(k) + X = X = d(k, X)
$$

$$
\begin{aligned}
&\mu\langle\zeta, 1\rangle(k, X) \\
&= \mu(\zeta(k, X), (k, X)) \\
&= \mu((-k, c(k) + X), (k, X)) \\
&= (-k + k, X) \\
&= (0, X) \\
&= \iota(X) \\
&= \iota\pi_2(k, X)
\end{aligned}
$$

$$
\begin{aligned}
&\mu\langle 1, \zeta\rangle(k, X) \\
&= \mu((k, X), (-k, c(k) + X)) \\
&= (k - k, c(k) + X) \\
&= (0, c(k) + X) \\
&= \iota(c(k) + X) \\
&= \iota\gamma(k, X)
\end{aligned}
$$

Thus $QP(C)$ is an internal groupoid, and so C is too. Specifically, the morphism $\psi\zeta\varphi : C_1 \to C_1$ makes C a groupoid, with the necessary conditions following from the above and the isomorphism $QP(C) \approx C$.

Chapter 3

Internal Categories in Mon

The result of the previous chapter, that $\mathrm{Cat}(\mathbf{Grp}) \sim \mathbf{XMod}$, has since been generalised in various ways. In [4], Porter shows an equivalence between the category of *groups with operations* and the category of some suitably defined crossed module. Here, a group with operations is a group with additional unary and binary operations that are compatible with the group structure in a particular way. Essentially, the previous construction still holds in certain *stronger* contexts. Patchkoria showed in [5] that it in fact holds in the *weaker* context of **Mon**, which is the subject of this chapter.

Observe that the construction in **Grp** heavily relies on expressing an element of C_1 as a pair of elements of the kernel, $\mathrm{Ker}(d)$ and the object of objects C_0. Furthermore, we define a group action from the split exact sequence, that is used to produce the semidirect product $\mathrm{Ker}(d) \rtimes_\alpha C_0$. Both of these constructions make use of the fact that we have inverses within groups, the property specifically absent from a monoid.

Patchkoria's response to this is to define additional structure on an internal category in **Mon**, called the *Schreier condition*, and to proceed to show an equivalence between the category of such internal categories and some other suitably defined crossed module, called a *crossed semimodule*. Note that **Mon** is finitely complete, and has a zero object, the trivial monoid, and thus has kernels.

Definition 3.0.1. A *Schreier internal category* in **Mon** is an internal category, C, in **Mon** such that for all $f \in C_1$, there exists a unique $k \in \mathrm{Ker}(d)$ such that $f = k + ed(f)$.

This definition explicitly requires the existence of this element of the kernel, and we will show that this is enough to mimic the rest of the construction. Of course, every group has an underlying monoid structure, and every internal category in **Grp** has the underlying structure of an internal category in **Mon**. Such internal categories are Schreier, and thus we may view this as a generalisation. A morphism of Schreier internal categories is simply an internal functor between them, and we thus define the category SCat(**Mon**). We now define crossed semimodules, and their morphisms, which will similarly be a generalisation of a group crossed module.

Definition 3.0.2. A *crossed semimodule* is a quadruple (A, B, α, f), where A and B are monoids, α is a monoid action of B on A (written as a left superscript) and $f : A \to B$ is a monoid homomorphism, such that the following hold for all $a, a' \in A$ and $b \in B$:

1. $f({}^{b}a) + b = b + f(a)$
2. ${}^{f(a)}a' + a = a + a'$

As before, the first condition is called equivariance and the second is called the Peiffer identity. We emphasise, besides the similarity to the definition of a crossed module, the way in which this definition accounts for the lack of inverses in a monoid. This represents a general technique used throughout the process of generalising Brown and Spencer's result.

Definition 3.0.3. Let $f = (A, B, \alpha, f)$ and $f' = (A', B', \alpha', f')$ be two crossed semimodules. A *morphism of crossed semimodules*, $\xi : f \to f'$ is a pair of monoid homomorphisms $\xi = \xi_1 : A \to A', \xi_2 : B \to B')$ such that the following diagrams commute:

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
{\scriptstyle \xi_1}\downarrow & & \downarrow{\scriptstyle \xi_2} \\
A' & \xrightarrow{f'} & B'
\end{array}
\qquad\qquad
\begin{array}{ccc}
B \times A & \xrightarrow{\alpha} & A \\
{\scriptstyle \xi_2 \times \xi_1}\downarrow & & \downarrow{\scriptstyle \xi_1} \\
B' \times A' & \xrightarrow{\alpha'} & A'
\end{array}
$$

Of course, every group crossed module is a crossed semimodule. We form the category **XSMod** of crossed semimodules and crossed semimodule morphisms.

Patchkoria's result is then that there is a categorical equivalence SCat(**Mon**) $\sim$ **XSMod**.

As before, we begin with consideration of a split exact sequence, an action and a semidirect product. Let C be a Schreier internal category in **Mon**. Then we may form the following split exact sequence:

$$\mathrm{Ker}(d) \xrightarrow{\kappa} C_1 \underset{e}{\overset{d}{\rightleftarrows}} C_0$$

The Schreier condition allows for the definition of a *set-theoretic map*, $p : C_1 \to \mathrm{Ker}(d)$ by $p(f) = p(k + ed(f)) = k$. In particular p is the unique map such that $\kappa p + ed = 1_{C_0}$. This structure is called a *Schreier split extension* and has been further studied in [13].

We now show that the Schreier condition allows us to give an action of C_0 on $\mathrm{Ker}(d)$. Firstly, we note that for an element $X \in C_0$ and $k \in \mathrm{Ker}(d)$, $k + e(X) \in C_1$. Furthermore,

$$k + e(X) = k + e(d(k) + de(X)) = k + ed(k + e(X))$$

Thus, if there is some other $k' \in \mathrm{Ker}(d)$ and $X' \in C_0$ such that $k + e(X) = k' + e(X')$, we have that:

$$k+ed(k+e(X)) = k+e(X) = k'+e(X') = k'+ed(k'+e(X')) = k'+ed(k+e(X))$$

Then, by the uniqueness in the Schreier condition, we have that $k = k'$. Also, by applying d to $k + e(X) = k' + e(X')$, we obtain that $X = X'$.

On the other hand, for an element $X \in C_0$ and $k \in \mathrm{Ker}(d)$, $e(X) + k$ is also an element of C_1. Then, by the Schreier condition, there exists a unique ${}^{X}k \in \mathrm{Ker}(d)$ such that

$$e(X)+k = {}^{X}k+ed(e(X)+k) = {}^{X}k+ede(X)+ed(k) = {}^{X}k+e(X)+0 = {}^{X}k+e(X)$$

We show that $\alpha : C_0 \times \mathrm{Ker}(d) \to \mathrm{Ker}(d)$ which maps (X, k) to ${}^{X}k$ is a monoid action:

$${}^{X}(k + k') + e(X) = e(X) + k + k' = {}^{X}k + e(X) + k' = {}^{X}k + {}^{X}k' + e(X)$$

$${}^{X}0 + e(X) = e(X) + 0 = e(X) = 0 + e(X)$$

$$\begin{aligned} {}^{(X+X')}k + e(X+X') &= e(X+X') + k = e(X) + e(X') + k \\ &= e(X) + {}^{X'}k + e(X') = {}^{X}({}^{X'}k) + e(X) + e(X') \\ &= {}^{X}({}^{X'}k) + e(X+X') \end{aligned}$$

$${}^{0}k = {}^{0}k + e(0) = e(0) + k = k$$

We therefore have that ${}^{X}(k+k') = {}^{X}k + {}^{X}k'$, ${}^{X}0 = 0$, ${}^{(X+X')}k = {}^{X}({}^{X'}k)$ and ${}^{0}k = k$. A semidirect product in **Mon** is defined almost identically to that in **Grp**.

Definition 3.0.4. Let A and B be two monoids, and let α be a monoid action of B on A. Then the *semidirect product* of A and B (with respect to α), $A \rtimes_\alpha B$, is the monoid with underlying set $A \times B$ and operation:

$$(a, b) + (a', b') = (a + {}^{b}a', b + b')$$

We thus form the semidirect product $\mathrm{Ker}(d) \rtimes_\alpha C_0$. The next step is to show that this is isomorphic to C_1. Define $\varphi : C_1 \to \mathrm{Ker}(d) \rtimes_\alpha C_0$ by $\varphi(f) = \varphi(k + ed(f)) = (k, d(f))$. φ is a monoid homomorphism: For some $f, f' \in C_1$, with $f = k + ed(f)$ and $f' = k' + ed(f')$, we have

$$\begin{aligned} &\varphi(f + f') \\ &= \varphi(k + ed(f) + k' + ed(f')) \\ &= \varphi(k + {}^{d(f)}k' + ed(f) + ed(f')) \\ &= \varphi(k + {}^{d(f)}k' + ed(f + f')) \\ &= (k + {}^{d(f)}k', d(f + f')) \\ &= (k + {}^{d(f)}k', d(f) + d(f')) \\ &= (k, d(f)) + (k', d(f')) \\ &= \varphi(f) + \varphi(f') \end{aligned}$$

On the other hand, define $\psi : \mathrm{Ker}(d) \rtimes_\alpha C_0 \to C_1$ as $\psi(k, X) = k + e(X)$. This is also a monoid homomorphism:

$$
\begin{aligned}
&\psi((k, X), (k', X')) \\
&= \psi(k + {}^{X}k', X + X') \\
&= k + {}^{X}k' + e(X + X') \\
&= k + {}^{X}k' + e(X) + e(X') \\
&= k + e(X) + k' + e(X') \\
&= \psi(k, X) + \psi(k', X')
\end{aligned}
$$

Then, φ, ψ are inverses:

$$\psi\varphi(f) = \psi\varphi(k + ed(f)) = \psi(k, d(f)) = k + ed(f) = f$$

$$\varphi\psi(k, X) = \varphi(k + e(X)) = \varphi(k + ed(k + e(X))) = (k, d(k + e(X)) = (k, X)$$

And thus, we have the isomorphism $C_1 \approx \mathrm{Ker}(d) \rtimes_\alpha C_0$, as was with the case for **Grp**. In this previous case, we found that composition of the internal category C was characterised by the rest of the structure. Let us now show the analogous result for our Schreier internal category in **Mon**. Consider a pair of morphisms (f, f') in $C^{\leftarrow\leftarrow}$, with $f = k + ed(f)$ and $f' = k' + ed(f)$. Then,

$$d(f) = c(f') = c(k' + ed(f')) = c(k') + ced(f') = c(k') + d(f')$$

and so $f = k + ed(f) = k + e(c(k') + d(f'))$. Then,

$$
\begin{aligned}
&m(f, f') \\
&= m(k + e(c(k') + d(f')), k' + ed(f')) \\
&= m(k, 0) + m(e(c(k') + d(f')), k' + ed(f')) \\
&= m(k, ed(k)) + m(e(c(k') + ced(f')), k' + ed(f')) \\
&= k + m(ec(k' + ed(f')), k' + ed(f')) \\
&= k + k' + ed(f')
\end{aligned}
$$

We may now construct the equivalence. As before, let $P : \mathrm{SCat}(\mathbf{Mon}) \to \mathbf{XSMod}$ be defined by

$$P(C) = (\mathrm{Ker}(d), C_0, \alpha, c|_{\mathrm{Ker}(d)})$$

where C is a Schreier internal category in **Mon**, α is the action as we have defined above, and $c|_{\mathrm{Ker}(d)}$ is the restriction of c to domain $\mathrm{Ker}(d)$. To have that $P(C)$ is a crossed semimodule, we need only show that it satisfies equivariance and the Peiffer identity. For $X \in C_0$ and $k \in \mathrm{Ker}(d)$:

$$c({}^{X}k)+X = c({}^{X}k)+ce(X) = c({}^{X}k+e(X)) = c(e(X)+k) = ce(X)+c(k) = X+c(k)$$

Then, for $k, k' \in \mathrm{Ker}(d)$, observe that:

$$d({}^{c(k)}k' + ec(k)) = d({}^{c(k)}k') + dec(k) = 0 + c(k) = c(k)$$

so that $({}^{c(k)}k' + ec(k), k) \in C^{\leftarrow\leftarrow}$. Then, noting that ${}^{c(k)}k' + ec(k) = {}^{c(k)}k' + ed({}^{c(k)}k'+ec(k))$ and $k = k+ed(k)$, we compute the composition:

$$m({}^{c(k)}k' + ec(k), k) = {}^{c(k)}k' + k + ed(k) = {}^{c(k)}k' + k$$

On the other hand, by the definition of the action, we have that:

$${}^{c(k)}k' + ec(k) = ec(k) + k'$$

Putting this together, we obtain the Peiffer identity:

$$\begin{aligned} &{}^{c(k)}k' + k \\ &= m({}^{c(k)}k' + ec(k), k) \\ &= m(ec(k) + k', k) \\ &= m(ec(k), k) + m(k', 0) \\ &= k + m(k', ed(k')) \\ &= k + k' \end{aligned}$$

Next, let $F = (F_0, F_1) : C \to D$ be a morphism in SCat(**Mon**). We show that $(F_1|_{\mathrm{Ker}(d)}, F_0) : P(C) \to P(D)$ is a morphism of crossed semimodules. Of course, $F_1|_{\mathrm{Ker}(d)}$ and F_0 are monoid homomorphism by definition, and

$F_0 c_C(k) = c_D F_1(k)$. Then,

$$\begin{aligned} & {}^{F_0(X)}F_1(k) + e_D F_0(X) \\ &= e_D F_0(X) + F_1(k) \\ &= F_1 e_C(X) + F_1(k) \\ &= F_1(e_C(X) + k) \\ &= F_1({}^X k + e_C(X)) \\ &= F_1({}^X k) + F_1 e_C(X) \\ &= F_1({}^X k) + e_D F_0(X) \end{aligned}$$

And therefore ${}^{F_0(X)}F_1(k) = F_1({}^X k)$. We have that P is a functor. We, as expected, define

$$Q(A, B, \alpha, f) \quad = \quad B \underset{\gamma}{\overset{\pi_2}{\leftleftarrows}}\!\!\xrightarrow{\iota} A \rtimes_\alpha B \xleftarrow{\mu} (A \rtimes_\alpha B) \times_B (A \rtimes_\alpha B)$$

where $\pi_2(a, b) = b$, $\iota(b) = (0, b)$, $\gamma(a, b) = f(a) + b$, $(A \rtimes_\alpha B) \times_B (A \rtimes_\alpha B)$ is the pullback of π_2 and γ, with elements $((a, f(a') + b'), (a', b'))$ and $\mu((a, f(a') + b'), (a', b')) = (a + a', b)$. π_2 and ι are clearly monoid morphisms. For γ and μ it is exactly as with the case of **Grp**:

$$\begin{aligned} & \gamma((a, b) + (a', b')) \\ &= \gamma(a + {}^b a', b + b') \\ &= f(a + {}^b a') + b + b' \\ &= f(a) + f({}^b a') + b + b' \\ &= f(a) + b + f(a') + b' \\ &= \gamma(a, b) + \gamma(a', b') \end{aligned}$$

$$
\begin{aligned}
&\mu(((a_1, f(a'_1) + b'_1), (a'_1, b'_1)) + ((a_2, f(a'_2) + b'_2), (a'_2, b'_2))) \\
&= \mu((a_1, f(a'_1) + b'_1) + (a_2, f(a'_2) + b'_2), (a'_1, b'_1) + (a'_2, b'_2)) \\
&= \mu((a_1 + {}^{(f(a'_1)+b'_1)}a_2, f(a'_1) + b'_1 + f(a'_2) + b'_2), (a'_1 + {}^{b'_1}a'_2, b'_1 + b'_2)) \\
&= (a_1 + {}^{(f(a'_1)+b'_1)}a_2 + a'_1 + {}^{b'_1}a'_2, b'_1 + b'_2) \\
&= (a_1{}^{f(a'_1)}({}^{b'_1}a_2) + a'_1 + {}^{b'_1}a'_2, b'_1 + b'_2) \\
&= (a_1 + a'_1 + {}^{b'_1}a_2 + {}^{b'_1}a'_2, b'_1 + b'_2) \\
&= (a_1 + a'_1 + {}^{b'_1}(a_2 + a'_2), b'_1 + b'_2) \\
&= (a_1 + a'_1, b'_1) + (a_2 + a'_2, b'_2) \\
&= \mu((a_1, f(a'_1) + b'_1), (a'_1, b'_1)) + \mu((a_2, f(a'_2) + b'_2), (a'_2, b'_2))
\end{aligned}
$$

We must show that $Q(A, B, \alpha, f)$ satisfies the conditions **IC1**, **IC2**, **IC3**, **IC4** of an internal category, but this is exactly the same calculation as was done for the case of **Grp** in the previous chapter. In particular, we did not make use of any facts in these previous calculations for which we have not developed analogous results in the **Mon** case. Thus, $Q(A, B\alpha, f)$ is an internal category. We must show that it is Schreier. The kernel of π_2 is:

$$\mathrm{Ker}(\pi_2) = \{(a, b) \in A \rtimes_\alpha B \mid \pi_2(a, b) = 0\} = \{(a, 0) \mid a \in A\}$$

Then, for some $(a, b) \in A \rtimes_\alpha B$

$$(a, b) = (a + {}^0 0, 0 + b) = (a, 0) + (0, b) = (a, 0) + \iota\pi_2(a, b)$$

with $(a, 0) \in \mathrm{Ker}(\pi_2)$, and for some other $(a', 0) \in \mathrm{Ker}(\pi_2)$ with $(a, b) = (a', 0) + \iota\pi_2(a, b)$, we have

$$(a, b) = (a', 0) + \iota\pi_2(a, b) = (a', 0) + (0, b) = (a' + {}^0 0, 0 + b) = (a', b)$$

Applying the *set-theoreric map* $\pi_A : A \times_\alpha B \to A$ defined as $\pi_A(a, b) = a$, we have that $a = a'$, so $(a, 0) = (a', 0)$, which means that $(a, 0)$ is unique, and $Q(A, B, \alpha, f)$ is Schreier. For Q on morphisms, let $\xi = (\xi_1, \xi_2) : (A, B, \alpha, f) \to (A', B', \alpha', f')$ be a morphism of crossed semimodules, and define $Q(\xi) = (\xi_2, \xi_1 \times \xi_2) : Q(A, B, \alpha, f) \to Q(A', B', \alpha', f')$. ξ_2 is a monoid homomorphism, and so is $\xi_1 \times \xi_2$:

$$
\begin{aligned}
&(\xi_1 \times \xi_2)((a,b) + (a',b')) \\
&= (\xi_1 \times \xi_2)(a + {}^{b}a', b + b') \\
&= (\xi_1(a + {}^{b}a'), \xi_2(b + b')) \\
&= (\xi_1(a) + \xi_1({}^{b}a'), \xi_2(b) + \xi_2(b')) \\
&= (\xi_1(a) + {}^{\xi_2(b)}\xi_1(a'), \xi_2(b) + \xi_2(b')) \\
&= (\xi_1(a), \xi_2(b)) + (\xi_1(a'), \xi_2(b')) \\
&= (\xi_1 \times \xi_2)(a,b) + (\xi_1 \times \xi_2)(a',b')
\end{aligned}
$$

The fact that $Q(\xi)$ is an internal functor again follows from the same calculation as in the **Grp** case. Therefore, Q is a functor.

We now show that P and Q from an equivalence of categories. We note here that isomorphisms in **XSMod** and SCat(**Mon**) are, respectively, morphisms of crossed semimodules and internal functors for which the components are isomorphisms in **Mon**.

Let C be a Schreier internal category in **Mon**. Then, $QP(C)$ is

$$
C_0 \underset{\gamma}{\overset{\pi_2}{\underset{\longrightarrow}{\overset{\iota}{\leftleftarrows}}}} \mathrm{Ker}(d) \rtimes_\alpha C_0 \xleftarrow{\ \mu\ } (\mathrm{Ker}(d) \rtimes_\alpha C_0) \times_{C_0} (\mathrm{Ker}(d) \rtimes_\alpha C_0)
$$

where $\mathrm{Ker}(d) \rtimes_\alpha C_0$ is the semidirect product of $\mathrm{Ker}(d)$ and C_0 defined earlier, $\pi_2(k,X) = X$, $\iota(X) = (0,X)$, $\gamma(k,X) = k + e(X)$, $(\mathrm{Ker}(d) \rtimes_\alpha C_0) \times_{C_0} (\mathrm{Ker}(d) \rtimes_\alpha C_0)$ contains all elements $((k, c(k')+X'), (k', X'))$ for $k, k' \in \mathrm{Ker}(d)$ and $X, X' \in C_0$ and $\mu((k, c(k') + X'), (k', X')) = (k + k', X')$. We show that $(1_{C_0}, \psi) : QP(C) \to C$ is a morphism in SCat(**Mon**):

$$d\psi(k,X) = d(k + e(X)) = d(k) + de(X) = 0 + X = X = \pi_2(k,X)$$

$$\psi\iota(X) = \psi(0,X) = 0 + e(X) = e(X)$$

$$c\psi(k,X) = c(k + e(X)) = c(k) + ce(X) = c(k) + X = \gamma(k,X)$$

$$
\begin{aligned}
& m(\psi \times \psi)((k, c(k') + X'), (k', X')) \\
&= m(\psi(k, c(k') + X'), \psi(k', X')) \\
&= m(k + e(c(k') + X'), k' + e(X')) \\
&= m(k + e(c(k') + d(k') + de(X')), k' + e(d(k') + de(X'))) \\
&= m(k + e(c(k') + d(k' + e(X'))), k' + ed(k' + e(X'))) \\
&= k + k' + ed(k' + e(X')) \\
&= k + k' + e(X') \\
&= (k + k', X') \\
&= \mu((k, c(k') + X'), (k', X'))
\end{aligned}
$$

Then, 1_{C_0} and ψ are both isomorphisms of monoids, so $(1_{C_0}, \psi)$ is an isomorphism in SCat(**Mon**), and $QP \approx 1$. On the other hand, for some crossed semimodule, (A, B, α, f), we have that $PQ(A, B, \alpha, f) = (\mathrm{Ker}(\pi_2), B, \beta, \gamma|_{\mathrm{Ker}(\pi_2)})$, where (as above), $\mathrm{Ker}(\pi_2) = \{(a, 0) \mid a \in A\}$ is a subgroup of $A \rtimes_\alpha, B$, with operation $(a, 0) + (a', 0) = (a + a', 0)$. Thus, there is a monoid isomorphism $\eta : \mathrm{Ker}(\pi_2) \to A$ defined as $\eta(a, 0) = a$. β is the action of B on $\mathrm{Ker}(\pi_2)$ defined by:

$$
{}^b(a, 0) + \iota(b) = \iota(b) + (a, 0)
$$

We consider the element $({}^b a, b)$ of $A \rtimes_\alpha B$. Note that $(0, b) = \iota(b) = \iota\pi_2({}^b a, b)$. Then,

$$
({}^b a, b) = ({}^b a + {}^0 0, 0 + b) = ({}^b a, 0) + (0, b) = ({}^b a, 0) + \iota\pi_2({}^b a, b)
$$

While, on the other hand,

$$
({}^b a, b) = (0 + {}^b a, b + 0) = (0, b) + (a, 0) = \iota(b) + (a, 0) = {}^b(a, 0) + \iota(b) = {}^b(a, 0) + \iota\pi_2({}^b a, b)
$$

The uniqueness in the Schreier condition (of $Q(A, B, \alpha, f)$) gives that ${}^b(a, 0) = ({}^b a, 0)$.

And, $\gamma|_{\mathrm{Ker}(\pi_2)}$ is the restriction of γ to domain $\mathrm{Ker}(\pi_2)$, with $\gamma(a, 0) = f(a) + 0 = f(a)$. We show that $(\eta, 1_B) : PQ(A, B, \alpha, f) \to (A, B, \alpha, f)$ is an isomorphism of crossed semimodules. As the two components are monoid isomorphisms, we need only show that they satisfy the two conditions of a crossed semimodule morphism:

$$
f\eta(a, 0) = f(a) = \gamma(a, 0)
$$

$$
\eta({}^b(a, 0)) = \eta({}^b a, 0) = {}^b a = {}^b\eta(a, 0)
$$

We therefore conclude that $PQ \approx 1$, and we have Patchkoria's result that $\mathrm{SCat}(\mathbf{Mon}) \sim \mathbf{XSMod}$.

This is not all that there is to be said for Schreier internal categories in **Mon**. We make an observation that was first given by Lavendhomme and Roisin in [14]. The isomorphism $C_1 \approx \mathrm{Ker}(d) \rtimes_\alpha C_0$ for a Schreier internal category, C, means that every morphism of C may be expressed uniquely as an element of the kernel of d and its domain. In other words, for a fixed object in C, the morphisms with this object as their domain are in correspondence with the elements of the kernel of d. This leads us to the definition of a *homogeneous internal category.*

Definition 3.0.5. Let C be an internal category in **Mon**. Then C is *homogeneous* if for all $X \in C_0$, the set-theoretic map $\sigma_X : \mathrm{Ker}(d) \to d^{-1}(X)$ defined by $\sigma_X(k) = k + e(X)$ is a bijection (where $d^{-1}(X)$ is the preimage of X under d).

It is in fact true that this definition characterises the Schreier condition.

Proposition 3.0.6. *Let C be an internal category in* **Mon**. *Then C is Schreier if and only if it is homogeneous.*

Let us consider some Schreier internal category C, and some $X \in C_0$. Define the map $\tau_X : d^{-1}(X) \to \mathrm{Ker}(d)$ as $\tau_X(f) = \tau_X(k + ed(f)) = k$. Note that by definition, $k \in \mathrm{Ker}(d)$ and this map is well defined. We show that σ_X and τ_X are inverses. On the one hand,

$$\tau_X\sigma_X(k) = \tau_X(k + e(X)) = \tau_X(k + ed(k + e(X))) = k$$

While on the other hand,

$$\sigma_X\tau_X(f) = \sigma_X(k) = k + e(X) = k + ed(f) = f$$

by noting that $d(f) = X$ (as $f \in d^{-1}(X)$). Therefore, σ_X is a bijection (for all X) and C is homogeneous.

Conversely, consider some homogeneous internal category C, and some $f \in C_1$. Then, $\sigma_{d(f)}$ is a bijection, so we have an inverse map $\sigma^{-1}_{d(f)} : d^{-1}(d(f)) \to \mathrm{Ker}(d)$. We observe that,

$$\sigma^{-1}_{d(f)}(f) + ed(f) = \sigma_{d(f)}\sigma^{-1}_{d(f)}(f) = f$$

with $\sigma_{d(f)}^{-1}(f) \in \mathrm{Ker}(d)$, by definition. For the uniqueness of the Schreier condition, assume that there exists some $k \in \mathrm{Ker}(d)$ such that $f = k + ed(f)$. Then,

$$\sigma_{d(f)}^{-1}(f) = \sigma_{d(f)}^{-1}(k + ed(f)) = \sigma_{d(f)}^{-1}\sigma_{d(f)}(k) = k$$

Therefore, C is a Schreier internal category.

At the end of the previous chapter, we showed that every internal category in **Grp** is an internal groupoid. We now provide a counterexample to show that even a Schreier internal category in **Mon** is not necessarily a groupoid.

Let $\mathbb{N} = (\mathbb{N}, +, 0)$ be the monoid of natural numbers (including zero), with usual addition and let $1 = \{*\}$ be the trivial monoid. Recall that 1 is the zero object of **Mon**, and consider the following diagram:

$$1 \; \overset{!_d}{\longleftarrow} \xrightarrow{\;!_e\;} \underset{!_c}{\longleftarrow} \; \mathbb{N} \xleftarrow{\;m\;} \mathbb{N} \times \mathbb{N}$$

where $\mathbb{N} \times \mathbb{N}$ is the product of $\mathbb{N}$ with itself (with component-wise addition), $!_d$, $!_e$ and $!_c$ are the unique morphisms to or from the zero object, and m is defined by $m(a, b) = a + b$. Note that because $\mathbb{N}$ is a commutative monoid, m is a homomorphism:

$$\begin{aligned}
&m((a, b) + (a', b')) \\
&= m(a + a', b + b') \\
&= a + a' + b + b' \\
&= a + b + a' + b \\
&= m(a, b) + m(a', b')
\end{aligned}$$

It is straightforward to show that this is an internal category, as the **IC1** and **IC2** are trivial, while **IC3** and **IC4** follow from the identity and associativity axioms of a monoid, respectively. Furthermore, for all $a \in \mathbb{N}$, $!_d(a) = *$, and so $\mathrm{Ker}(!_d) = \mathbb{N}$, while $!_e(*) = 0$. Thus, a may be uniquely expressed as $a = a + !_e!_d(a)$, and thus this is a Schreier internal category.

Assume now that there exists $i : \mathbb{N} \to \mathbb{N}$ satisfying the conditions of an internal groupoid. The second of these conditions means that for all $a \in \mathbb{N}$, $i(a) + a = 0 = a + i(a)$, which is to ask $\mathbb{N}$ to have additive inverses, which is not the case. We therefore have that this Schreier internal category is not an internal groupoid.

Chapter 4

Factorisation Systems

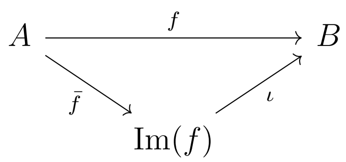
In this chapter we introduce the second main topic of this work: the *factorisation system*. A factorisation system is a categorical generalisation of the set-theoretic notion of the *canonical factorisation of a map* and, therefore, this will be a point of departure.

Let A and B be two sets, and $f : A \to B$ a map between them. We may consider the *image* of this map:

$$\mathrm{Im}(f) = \{b \in B \mid (\exists\, a \in A)\ f(a) = b\} = \{f(a) \in B \mid a \in A\}$$

With this, we obtain two canonical maps. Firstly, $\bar{f} : A \to \mathrm{Im}(f)$ defined by $\bar{f}(a) = f(a)$. From the definition of the image, it follows that $\bar{f}$ is a surjection. Secondly, we have $\iota : \mathrm{Im}(f) \to B$ defined by $\iota(b) = b$, by viewing $\mathrm{Im}(f)$ as a subset of B. This is clearly an injection. Furthermore, we have that for all $a \in A$, $\iota\bar{f}(a) = \iota(f(a)) = f(a)$, and thus $\iota\bar{f} = f$.

Putting this result in the context of the category **Set**, we observe that every morphism of **Set** (that is, map), may be written as the composition of an epimorphism (a surjection) and a monomorphism (an injection). We write this as a commutative diagram:



Let us write Surj and Inj for the class of surjections and injections, respectively. In order to generalise this notion to a general category, we must examine certain desirable properties of surjections and injections, which allow us to consider Surj and Inj as *well behaved*.

Firstly, both of these classes are closed under composition: The composition of two surjections is a surjection and the composition of two injections is an injection.

Next, every isomorphism is both a surjection and an injection. Of course, the converse of this is statement is also true: if a map is both surjective and injective then it is bijective, an isomorphism in **Set**. However, for our purposes, we consider a *stronger* interaction between these two types of maps, for which we introduce the following core definition.

Definition 4.0.1. Let $f : A \to B$ and $g : A' \to B'$ be two morphisms in a category $\mathbb{C}$. We say that f is *orthogonal* to g, written $f \downarrow g$, if for all pairs of morphisms $u : A \to A'$ and $v : B \to B'$ such that $vf = gu$, there exists a unique morphism $z : B \to A'$ such that $zf = u$ and $gz = v$, as in the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle u}\downarrow & \swarrow_{z} & \downarrow{\scriptstyle v} \\ A' & \xrightarrow[g]{} & B' \end{array}$$

Indeed, we have that,

Proposition 4.0.2. *In* **Set**, *every surjection is orthogonal to every injection.*

To see this, consider the following commutative diagram in **Set** where e is a surjection and m is an injection:

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ A' & \xrightarrow{m} & B' \end{array}$$

Then, for all $b \in B$, there exists some $a \in A$ such that $e(a) = b$, as e is surjective. Then, we may obtain $u(a) \in A'$. Define $z : B \to A'$ as $z(b) = u(a)$. We note in passing that we are invoking the Axiom of Choice. z is well defined, as if there is some other $a' \in A$ such that $e(a') = b$, then $e(a) = e(a')$, so $ve(a) = ve(a')$, and thus $mu(a) = mu(a')$ and m is injective, so $u(a) = u(a')$. Then, $ze(a) = z(b) = u(a)$, and $mz(b) = mu(a) = ve(a) = v(b)$, and so $ze = u$ and $mz = v$. Then the the uniqueness of z follows from m being an injection: If there exists some $z' : B \to A'$ such that $z'e = u$ and $mz = v$, we have that for all $b \in B$, $mz(b) = v(b) = mz'(b)$. m is injective, so $z(b) = z'(b)$. Thus $z = z'$, and z is unique.

We remark, through the fact that injections are precisely the monomorphisms of **Set**, that this frames surjections as the *strong epimorphisms* of **Set**, which are defined to be the morphisms which are orthogonal to all monomorphisms of the category in question.

Definition 4.0.3. Let $\mathbb{C}$ be a category, and let $\mathcal{E}$ and $\mathcal{M}$ be two classes of morphisms of $\mathbb{C}$. Then $\mathcal{E}$ is *orthogonal* to $\mathcal{M}$, written $\mathcal{E} \downarrow \mathcal{M}$ if for all $e \in \mathcal{E}$ and $m \in \mathcal{M}$, we have that $e \downarrow m$.

We therefore conclude that in **Set**, $\mathrm{Surj} \downarrow \mathrm{Inj}$. Combining these observations, we say that $(\mathrm{Surj}, \mathrm{Inj})$ forms a *factorisation system* on **Set**, according to the following definition.

Definition 4.0.4. Let $\mathbb{C}$ be a category and let $\mathcal{E}$ and $\mathcal{M}$ be two classes of morphisms of $\mathbb{C}$. Then the pair $(\mathcal{E}, \mathcal{M})$ forms a *factorisation system* on $\mathbb{C}$ if the following four conditions are met:

FS1. $\mathcal{E}$ and $\mathcal{M}$ contain all the isomorphisms of $\mathbb{C}$.

FS2. $\mathcal{E}$ and $\mathcal{M}$ are closed under composition.

FS3. $\mathcal{E} \downarrow \mathcal{M}$.

FS4. $\mathbb{C}$ has $(\mathcal{E}, \mathcal{M})$-factorisations: For all morphisms $f \in \mathbb{C}$, there exists an $e \in \mathcal{E}$ and $m \in \mathcal{M}$ such that $f = me$.

The archetypal example of the canonical factorisation of maps in **Set** extends to varieties of algebras. For example, in the category **Grp**, if $\mathrm{Surj}(\mathbf{Grp})$ is the class of surjective group homomorphisms and $\mathrm{Inj}(\mathbf{Grp})$ is the class of injective group homomorphisms, then $(\mathrm{Surj}(\mathbf{Grp}), \mathrm{Inj}(\mathbf{Grp}))$ forms a factorisation system on **Grp**.

Before moving on to examining general properties of a factorisation system, we make observations on the property of orthogonality that we will make use of later.

Consider some $f : A \to B$ and $g : A' \to B'$ with $f \downarrow g$ in a category $\mathbb{C}$. Then, by orthogonality, for each commutative diagram on the left, we obtain the (trivially commutative) diagram on the right.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ A' & \xrightarrow{g} & B' \end{array} \qquad \mapsto \qquad \begin{array}{ccc} A & \xrightarrow{f} & B \\ & \swarrow{\scriptstyle z} & \\ A' & \xrightarrow{g} & B' \end{array}$$

On the other hand, for any morphism $z' : B \to A'$, the diagram on the left, by composition, gives the commutative diagram on the right:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \swarrow{\scriptstyle z'} & \\ A' & \xrightarrow{g} & B' \end{array} \qquad \mapsto \qquad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle z'f}\downarrow & & \downarrow{\scriptstyle gz'} \\ A' & \xrightarrow{g} & B' \end{array}$$

The definition of orthogonality implies that these two processes are inverse, providing a correspondence between diagrams of the following forms:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ A' & \xrightarrow{g} & B' \end{array} \qquad\qquad \begin{array}{ccc} A & \xrightarrow{f} & B \\ & \swarrow & \\ A' & \xrightarrow{g} & B' \end{array}$$

This observation allows us to characterise orthogonality as pullback of hom-sets, as was done by Kelly in [15].

Proposition 4.0.5. *Let $f : A \to B$ and $g : A' \to B'$ be two morphisms in a category $\mathbb{C}$. Then $f \downarrow g$ if and only if the following square is a pullback in* **Set***:*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{C}}(B, A') & \xrightarrow{\mathrm{Hom}_{\mathbb{C}}(f, A')} & \mathrm{Hom}_{\mathbb{C}}(A, A') \\ {\scriptstyle \mathrm{Hom}_{\mathbb{C}}(B,g)}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}_{\mathbb{C}}(A,g)} \\ \mathrm{Hom}_{\mathbb{C}}(B, B') & \xrightarrow[\mathrm{Hom}_{\mathbb{C}}(f, B')]{} & \mathrm{Hom}_{\mathbb{C}}(A, B') \end{array}$$

We mentioned before that orthogonality presented a stronger relationship between surjections and injections than the fact that to be both is to be a bijection. We now show that for any factorisation system, to be in both classes of morphisms is to be an isomorphism in the category.

Proposition 4.0.6. *Let $(\mathcal{E}, \mathcal{M})$ be a factorisation system on a category $\mathbb{C}$. Let $f : A \to B$ be a morphism of $\mathbb{C}$ such that $f \in \mathcal{E}$ and $f \in \mathcal{M}$. Then f is an isomorphism*

Proof. Consider the following commutative diagram in $\mathbb{C}$:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle 1_A}\downarrow & & \downarrow{\scriptstyle 1_B} \\ A & \xrightarrow{f} & B \end{array}$$

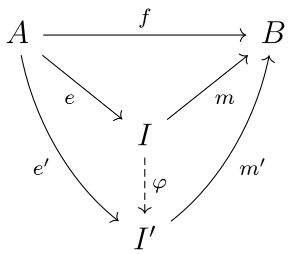
$f \downarrow f$, so there exists a unique morphism $g : B \to A$ such that $gf = 1_A$ and $fg = 1_B$. Therefore f is an isomorphism. ☐

Because both $\mathcal{E}$ and $\mathcal{M}$ contain all isomorphism, if we write $\mathrm{Iso}(\mathbb{C})$ for the class of isomorphisms of $\mathbb{C}$, we have

$$\mathcal{E} \cap \mathcal{M} = \mathrm{Iso}(\mathbb{C})$$

Note now, that the definition of a factorisation system requires only that the *factorisation* of a morphism exists and does not provide any uniqueness condition. We show that (as a consequence of orthogonality) factorisations are *essentially unique* - unique up to isomorphism.

Proposition 4.0.7. *Let $(\mathcal{E}, \mathcal{M})$ be a factorisation system on a category $\mathbb{C}$, and let $f : A \to B$ be a morphism in $\mathbb{C}$ such that there exist $e, e' \in \mathcal{E}$ and $m, m' \in \mathcal{M}$ such that $f = me$ and $f = m'e'$ are factorisations of f. Then, there exists a unique isomorphism φ making the following diagram commute:*



Proof. Consider the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{e} & I \\ {\scriptstyle e'}\downarrow & & \downarrow{\scriptstyle m} \\ I' & \xrightarrow{m'} & B \end{array}$$

Then $e \downarrow m'$, so there exists a unique morphism $\varphi : I \to I'$ such that $\varphi e = e'$ and $m'\varphi = m$. On the other hand, we may write this same diagram instead as

$$\begin{array}{ccc} A & \xrightarrow{e'} & I' \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle m'} \\ I & \xrightarrow{m} & B \end{array}$$

which, as $e' \downarrow m$, induces the isomorphism $\psi : I' \to I$ such that $\psi e' = e$ and $m\psi = m'$. We need only show that φ is an isomorphism. To do this, consider the following two diagrams:

$$\begin{array}{ccc} A & \xrightarrow{e} & I \\ {\scriptstyle e}\downarrow & \overset{\psi\varphi}{\swarrow} & \downarrow{\scriptstyle m} \\ I & \xrightarrow[m]{} & B \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{e} & I \\ {\scriptstyle e}\downarrow & \overset{1_I}{\swarrow} & \downarrow{\scriptstyle m} \\ I & \xrightarrow[m]{} & B \end{array}$$

The left diagram commutes as $\psi\varphi e = \psi e' = e$ and $m\psi\varphi = m'\varphi = m$, while the right diagram commutes trivially. However, $e \downarrow m$, and the *diagonal morphism* is necessarily unique. Thus, $\psi\varphi = 1_I$. We similarly consider the two diagrams

$$\begin{array}{ccc} A & \xrightarrow{e'} & I' \\ {\scriptstyle e'}\downarrow & \overset{\varphi\psi}{\swarrow} & \downarrow{\scriptstyle m'} \\ I' & \xrightarrow[m']{} & B \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{e'} & I' \\ {\scriptstyle e'}\downarrow & \overset{1_{I'}}{\swarrow} & \downarrow{\scriptstyle m'} \\ I' & \xrightarrow[m']{} & B \end{array}$$

The left commutes as $\varphi\psi e' = \varphi e = e'$ and $m'\varphi\psi = m\psi = m'$, and the right trivially so. $e' \downarrow m'$ implies that $\varphi\psi = 1_{I'}$, and thus φ is an isomorphism. □

It is worth noting that this proof manifests as repeatedly applying the orthogonality condition. In fact, it is true that under the assumption of the other axioms of a factorisation system (FS1, FS2 and FS4), the essential uniqueness of factorisations is equivalent to orthogonality and may therefore replace the orthogonality condition in the definition. We show this now.

Proposition 4.0.8. *Let* $(\mathcal{E}, \mathcal{M})$ *be a pair of classes of morphisms of a category* $\mathbb{C}$ *which satisfy* FS1*,* FS2 *and* FS4*. Then the following are equivalent:*

1. $(\mathcal{E}, \mathcal{M})$ *forms a factorisation system.*
2. $(\mathcal{E}, \mathcal{M})$*-factorisations are essentially unique.*

Proof. By 4.0.7, 1 implies 2. Assume now that $(\mathcal{E}, \mathcal{M})$-factorisations are unique up to isomorphism. We must show that $\mathcal{E} \downarrow \mathcal{M}$. Let $e : A \to B \in \mathcal{E}$ and $m : A' \to B' \in \mathcal{M}$, and consider two morphisms u and v in $\mathbb{C}$ such that the following square commutes:

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ {\scriptstyle u}\big\downarrow & & \big\downarrow{\scriptstyle v} \\ A' & \xrightarrow{m} & B' \end{array}$$

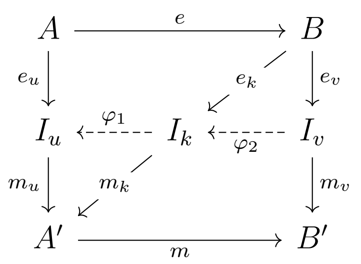
We consider $(\mathcal{E}, \mathcal{M})$-factorisations of u and v, $u = m_u e_u$ and $v = m_v e_v$. Then, the following diagram commutes:

$$\begin{array}{ccccc} A & \xrightarrow{e} & B & \xrightarrow{e_v} & I_v \\ {\scriptstyle e_u}\big\downarrow & & \overset{\varphi}{\swarrow} & & \big\downarrow{\scriptstyle m_v} \\ I_u & \xrightarrow[m_u]{} & A' & \xrightarrow[m]{} & B' \end{array}$$

We observe that, as $\mathcal{E}$ and $\mathcal{M}$ are both closed under composition, this diagram provides two equal $(\mathcal{E}, \mathcal{M})$-factorisations, which, by essential uniqueness, induces a unique isomorphism $\varphi : I_v \to I_u$ such that $\varphi e_v e = e_u$ and $m m_u \varphi = m_v$. We consider the composite $m_u \varphi e_v : B \to A'$, which makes the next diagram commute

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ {\scriptstyle u}\big\downarrow & \overset{m_u\varphi e_v}{\swarrow} & \big\downarrow{\scriptstyle v} \\ A' & \xrightarrow[m]{} & B' \end{array}$$

as $m_u \varphi e_v e = m_u e_u = u$ and $m m_u \varphi e_v = m_v e_v = m$. We need to show that $m_u \varphi e_v$ is the unique such morphism. Consider some $k : B \to A'$ with $ke = u$ and $mk = v$. Then k has an $(\mathcal{E}, \mathcal{M})$-factorisation, $k = m_k e_k$. We have the following commutative diagram:



The upper triangle and the lower triangle each exhibit two equal $(\mathcal{E}, \mathcal{M})$-factorisations, which induce the isomorphisms $\varphi_1 : I_k \to I_u$ and $\varphi_2 : I_v \to I_k$ such that $\varphi_1 e_k e = e_u$, $m_u \varphi_1 = m_k$, $\varphi_2 e_v = e_k$ and $m m_k \varphi_2 = m_v$. Consider the composite $\varphi_1 \varphi_2 : I_v \to I_u$ and observe that

$$\varphi_1 \varphi_2 e_v e = \varphi_1 e_k e = e_u \quad \text{and} \quad m m_u \varphi_1 \varphi_2 = m m_k \varphi_2 = m_v$$

which is the defining property of φ, whose uniqueness yields $\varphi = \varphi_1 \varphi_2$. We therefore have

$$m_u \varphi e_v = m_u \varphi_1 \varphi_2 e_v = m_k e_k = k$$

so $m_u \varphi e_v$ is unique, and $\mathcal{E} \downarrow \mathcal{M}$, which completes the proof. □

Before we consider the next property of factorisation systems, we make a remark on duality. If $(\mathcal{E}, \mathcal{M})$ forms a factorisation system on a category $\mathbb{C}$, then $(\mathcal{M}^{\text{op}}, \mathcal{E}^{\text{op}})$ forms a factorisation system on $\mathbb{C}^{\text{op}}$, where $\mathbb{C}^{\text{op}}$ is the opposite category of $\mathbb{C}$, and $\mathcal{M}^{\text{op}}$ and $\mathcal{E}^{\text{op}}$ are defined as expected.

We now move on to the cancellation properties, which say that for a pair of composable morphisms such that one of the morphisms and their composition are in a particular class of morphisms, we have that the other morphism is also in the class. Stated more precisely:

Definition 4.0.9. Let $\mathbb{C}$ be a category, and let $\mathcal{A}$ be a class of morphisms of $\mathbb{C}$. Then $\mathcal{A}$ has the *right cancellation property* if for all composable pairs of morphisms:

$$A \xrightarrow{f} B \xrightarrow{g} C$$

with gf and f in $\mathcal{A}$, we have that g is in $\mathcal{A}$.

And dually,

Definition 4.0.10. Let $\mathbb{C}$ be a category, and let $\mathcal{A}$ be a class of morphisms of $\mathbb{C}$. Then $\mathcal{A}$ has the *left cancellation property* if for all composable pairs of morphisms:

$$A \xrightarrow{f} B \xrightarrow{g} C$$

with gf and g in $\mathcal{A}$, we have that f is in $\mathcal{A}$.

Each of the classes of a factorisation system satisfy a cancellation property.

Proposition 4.0.11. *Let $(\mathcal{E}, \mathcal{M})$ form a factorisation system on a category $\mathbb{C}$. Then $\mathcal{E}$ has the right cancellation property and $\mathcal{M}$ has the left cancellation property.*

Proof. By duality, we need only show that $\mathcal{E}$ has the right cancellation property. Consider two composable morphisms $f : A \to B$ and $g : B \to C$ with gf and f in $\mathcal{E}$. g has an $(\mathcal{E}, \mathcal{M})$-factorisation $g = m_g e_g$. We consider the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{gf} & C \\ {\scriptstyle e_g f}\downarrow & & \downarrow{\scriptstyle 1_C} \\ I_g & \xrightarrow{m_g} & C \end{array}$$

Observe that, as an isomorphism, 1_C is in $\mathcal{M}$ and gf is in $\mathcal{E}$ by assumption. Also, e_g and f are in $\mathcal{E}$, so their composition $e_g f$ is too. The commutativity is by $m_g e_g f = gf = 1_c gf$. We thus have that this diagram presents two equal $(\mathcal{E}, \mathcal{M})$-factorisations. We therefore have a unique isomorphism $\varphi : C \to I_g$ such that $m_g \varphi = 1_C$ and $\varphi gf = e_g f$. Precomposing the first of these equations with the inverse of φ, φ^{-1}, we get $m_g = \varphi^{-1}$, which means that m_g is an isomorphism. As an isomorphism, m_g is then in $\mathcal{E}$, and thus the compostion $m_g e_g$ is in $\mathcal{E}$. But $g = m_g e_g$, so g is in $\mathcal{E}$. Therefore, $\mathcal{E}$ satisfies the right cancellation property. $\square$

The next property is arguably one of the most important of a factorisation system. It frames a factorisation system as a *prefactorisation system.* In categorical algebra, one usually starts with a prefactorisation system, and develops a factorisation system as this structure, with additional conditions. See, for example [16]. We have deviated from this convention for the usefulness of our present (equivalent) definition, which will become apparent in the next chapter.

This property says that for a factorisation system $(\mathcal{E}, \mathcal{M})$, each class contains all morphisms orthogonal to all morphisms of the other.

Proposition 4.0.12. *Let* $(\mathcal{E}, \mathcal{M})$ *form a factorisation system on a category* $\mathbb{C}$*. Then,*

1. $e \in \mathcal{E}$ *if and only if* $e \downarrow m$ *for all* $m \in \mathcal{M}$
2. $m \in \mathcal{M}$ *if and only if* $e \downarrow m$ *for all* $e \in \mathcal{E}$

Proof. As before, by duality, we need only show 1. The forward direction follows from the fact that $\mathcal{E} \downarrow \mathcal{M}$: if $e \in \mathcal{E}$, then for all $m \in \mathcal{M}$, $e \downarrow m$. Let us conversely assume that there is some morphism $e : A \to B$ in $\mathbb{C}$ such that for all $m \in \mathcal{M}$, $e \downarrow m$. Then e has an $(\mathcal{E}, \mathcal{M})$-factorisation, $e = m_e e_e$. Consider the following commutative square:

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ {\scriptstyle e_e}\downarrow & & \downarrow{\scriptstyle 1_B} \\ I_e & \xrightarrow{m_e} & B \end{array}$$

$m_e \in \mathcal{M}$, so $e \downarrow m_e$, thus there exists a unique morphism $z : B \to I_e$ such that $ze = e_e$ and $m_e z = 1_B$. Then, consider the following two diagrams:

$$\begin{array}{ccc} A & \xrightarrow{e_e} & I_e \\ {\scriptstyle e_e}\downarrow & {\scriptstyle 1_{I_e}} \swarrow & \downarrow{\scriptstyle m_e} \\ I_e & \xrightarrow[m_e]{} & B \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{e_e} & I_e \\ {\scriptstyle e_e}\downarrow & {\scriptstyle zm_e} \swarrow & \downarrow{\scriptstyle m_e} \\ I_e & \xrightarrow[m_e]{} & B \end{array}$$

The left diagram trivially commutes, while the right diagram commutes because $zm_e e_e = ze = e_e$ and $m_e z m_e = m_e$. Then, because $e_e \downarrow m_e$, the diagonal morphism is unique, so $1_{I_e} = zm_e$. We conclude that m_e is an isomorphism, and therefore in $\mathcal{E}$, and by closure under composition, $e = m_e e_e$ is in $\mathcal{E}$. $\square$

We may phrase this same proposition in terms of of orthogonality of classes of morphisms.

Proposition 4.0.13. *Let* $(\mathcal{E}, \mathcal{M})$ *form a factorisation system on a category* $\mathbb{C}$*. Let* $\mathcal{A}$ *be a class of morphisms of* $\mathbb{C}$*. Then:*

1. $\mathcal{A} \subseteq \mathcal{E}$ *if and only if* $\mathcal{A} \downarrow \mathcal{M}$

2. *$\mathcal{A} \subseteq \mathcal{M}$ if and only if $\mathcal{E} \downarrow \mathcal{A}$*

This perspective provides two further observations. Firstly, that the classes of a factorisation system determine each other and secondly, that we may construct order on the factorisation systems of a particular category.

Proposition 4.0.14. *Let $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{E}', \mathcal{M}')$ be two factorisations systems on a category $\mathbb{C}$. Then $\mathcal{E} = \mathcal{E}'$ if and only if $\mathcal{M} = \mathcal{M}'$*

Proof. Assume $\mathcal{E} = \mathcal{E}'$. Then $\mathcal{E}' \downarrow \mathcal{M}$, so $\mathcal{M} \subseteq \mathcal{M}'$. On the other hand, $\mathcal{E} \downarrow \mathcal{M}'$, so $\mathcal{M}' \subseteq \mathcal{M}$. Therefore, $\mathcal{M} = \mathcal{M}'$. The converse is dual. $\square$

Definition 4.0.15. The *order on factorisation systems* for a category $\mathbb{C}$ is defined as follows:
For two factorisation systems $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{E}', \mathcal{M}')$ on $\mathbb{C}$, we have

$$(\mathcal{E}, \mathcal{M}) \leq (\mathcal{E}', \mathcal{M}') \text{ if and only if } \mathcal{M} \subseteq \mathcal{M}'$$

Note that we may equivalently defined this relation by $\mathcal{E}' \subseteq \mathcal{E}$. Furthermore, the reflexivity and transitivity of this relation follows from that of the inclusion relation on classes, while the antisymmetry is a consequence of the previous Proposition 4.0.14.

To conclude this chapter, we provide a notion of a *trivial factorisation system* that exists for any given category.

Proposition 4.0.16. *Let $\mathbb{C}$ be a category, let $\mathrm{Iso}(\mathbb{C})$ be the class of isomorphisms of $\mathbb{C}$ and let $\mathbb{C}_1$ be the class of all morphisms of $\mathbb{C}$. Then $(\mathrm{Iso}(\mathbb{C}), \mathbb{C}_1)$ forms a factorisation system on $\mathbb{C}$. Furthermore, this is the top element of the order on factorisation systems on $\mathbb{C}$.*

Proof. Both $\mathrm{Iso}(\mathbb{C})$ and $\mathbb{C}_1$ trivially contain all isomorphisms of $\mathbb{C}$. It is straightforward to see that $\mathrm{Iso}(\mathbb{C})$ is closed under composition, while it is trivially true for $\mathbb{C}_1$.

Let us show that $\mathrm{Iso}(\mathbb{C}) \downarrow \mathbb{C}_1$. Consider the following diagram in $\mathbb{C}$ such that f is an isomorphism:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle u}\big\downarrow & & \big\downarrow{\scriptstyle v} \\ A' & \xrightarrow{g} & B' \end{array}$$

Consider the inverse of f, $f^{-1} : B \to A$. We show that $uf^{-1} : B \to A'$ is the unique diagonal morphism. Indeed, $uf^{-1}f = u$ and $guf^{-1} = vff^{-1} = v$. For uniqueness, assume that there is some $z : B \to A'$ such that $zf = u$ and $gz = v$. Precomposing the first equation with f^{-1}, we have that $z = uf^{-1}$.

Lastly, we must show that $\mathbb{C}$ has $(\mathrm{Iso}(\mathbb{C}), \mathbb{C}_1)$-factorisations. But for all morphisms $f : A \to B$ in $\mathbb{C}$, $f = f1_A$ and 1_A is an isomorphism. Therefore $(\mathrm{Iso}(\mathbb{C}), \mathbb{C}_1)$ is a factorisation system on $\mathbb{C}$.

For any other factorisation system $(\mathcal{E}, \mathcal{M})$ on $\mathbb{C}$, we have that $\mathcal{M} \subseteq \mathbb{C}_1$, and thus $(\mathcal{E}, \mathcal{M}) \leq (\mathrm{Iso}(\mathbb{C}), \mathbb{C}_1)$. $\square$

Chapter 5

Internal Factorisation Systems

The aim of this chapter is to motivate and provide an internal definition for factorisation systems and their various properties, essentially combining the notions of the first and fourth chapters. We will in general omit proofs of propositions, and note that full details may be found in [11].

A factorisation system is defined as a structure on a category. Thus, an *internal factorisation system* is defined as a structure on an internal category. As was done with internal categories themselves, we will be modelling this general definition off the case of **Set**, relying on the notion of a factorisation system on a small category (that is, an internal category in **Set**). This is done is such a way that an internal factorisation system on an internal category in **Set** is exactly a usual factorisation system on the corresponding small category.

The data of a factorisation system is a pair of classes of morphisms $(\mathcal{E}, \mathcal{M})$. For a small category, C, these will be subsets of C_1. Moving to a general internal category, one uses a *subobject* of C_1. We have already mentioned this notion in passing, and now provide further details.

For any (set-theoretic) map $f : A \to B$, the image of f, $\mathrm{Im}(f)$ is a subset of B. If f is injective (a monomorphism in **Set**) then there is a bijection (an isomorphism) between A and $\mathrm{Im}(f)$. Moreover, the subset defined by the image does not take into account exactly what A is, and another set, A', with another monomorphism $f' : A' \to B$ that has the same image as f, will define the same subset. To say that f and f' have the same image, is precisely to ask for a bijection (isomorphism) $\varphi : A \to A'$, making the following diagram commute:

$$\begin{array}{ccc} A' & \xrightarrow{f'} & B \\ {\scriptstyle\varphi}\uparrow & \nearrow_{f} & \\ A & & \end{array}$$

It makes sense to consider all injective maps with such bijections as together defining the subset of B. Conversely, a subset of B, $B' \subseteq B$ produces the injective map of the the inclusion $b : B' \to B$, with, of course, $\mathrm{Im}(b) \approx B'$. It turns out that is a suitable general description of *subobjects* of structures, for example, subgroups of groups, submonoids of monoids and subspaces of topological spaces. We now formally develop this general notion.

Let $\mathbb{C}$ be a category, and fix an object B of $\mathbb{C}$. Consider two monomorphism $f : A \to B$ and $f' : A' \to B$. We say that $f \leq f'$ if there exists a morphism $\varphi : A \to A'$ such that the following diagram commutes:

$$\begin{array}{ccc} A' & \xrightarrow{f'} & B \\ {\scriptstyle\varphi}\uparrow & \nearrow_{f} & \\ A & & \end{array}$$

It should be clear that this is a reflexive and transitive relation on the class $\mathrm{Mono}(B)$ of monomorphism of $\mathbb{C}$ with codomain B. We note that in this case φ will be a monomorphism.

We say that $f \sim f'$ if φ is an isomorphism. This relation, in contrast, is an equivalence relation on $\mathrm{Mono}(B)$. Furthermore, it is straightforward to show that:

Proposition 5.0.1. *For an object B in a category $\mathbb{C}$, and two $f, f' \in \mathrm{Mono}(B)$, $f \sim f'$ if and only if $f \leq f'$ and $f' \leq f$.*

We are then able to define a *subobject* as follows.

Definition 5.0.2. Let $\mathbb{C}$ be a category and let B be an object of $\mathbb{C}$. A *subobject* of B is a $\sim$-equivalence class in $\mathrm{Mono}(B)$. We write $\mathrm{Sub}(B)$ for the class of all subobjects of B.

Then proposition 5.0.1 means that $\leq$ will be a partial order on $\mathrm{Sub}(B)$. In the case of $\mathbb{C} = \mathbf{Set}$, $\mathrm{Sub}(B)$ is the set of all subsets of B, that is, the powerset of

B, while $\leq$ correseponds to the inclusion relation $\subseteq$ on the powerset.

Despite these formalities, we adopt the common convention of identifying a subobject with some representative monomorphism of the equivalence class. That is, for some monomorphism $f : A \to B$, we will call f a subobject of B.

Returning now to factorisation systems, for an internal category C, (in a category $\mathbb{C}$ with pullbacks) we will call a subobject of C_1 a *subobject of morphisms* of C, and the internal factorisation system will be made up of a pair of subobjects of morphisms:

$$\varepsilon : E \to C_1 \quad \text{and} \quad \mu : M \to C_1$$

Observe that the identity morphism $1_{C_1} : C_1 \to C_1$ is a monomorphism, and thus a subobject of morphisms. In the case that $\mathbb{C} = \mathbf{Set}$, this will be the subset containing all morphisms. We therefore call 1_{C_1} the *subobject of all morphisms* of C. Also, $e : C_0 \to C_1$ is a split monomorphism by **IC1** and may also be considered as a subobject of morphisms. From the case of **Set**, we call e the *subobject of identity morphisms* of C.

The next step to to define internal notions of the conditions **FS1**, **FS2**, **FS3** and **FS4**, and we do so in order.

Because we are working internally, we are unable to speak about individual isomorphisms in the context of **FS1**. Instead, we consider the subclass of all isomorphism of a category $\mathbb{C}$, Iso($\mathbb{C}$), and phrase **FS1** as: Iso($\mathbb{C}$) $\subseteq \mathcal{E}$ and Iso($\mathbb{C}$) $\subseteq \mathcal{M}$. We therefore need to construct the *subobject of isomorphisms* of the internal category C.

To do this, we observe that an isomorphism may be equivalently defined to be a morphism which is both a split epimorphism and a split monomorphism. We also remark that a pair of morphisms ($r : B \to A$, $s : A \to B$) with $rs = 1_A$ (so that r is a split epimorphism and s is a split monomorphism) is called a *point.*

Definition 5.0.3. Let C be an internal category in a category $\mathbb{C}$ with pullbacks. The *object of points* of C is defined as the pullback:

$$\begin{array}{ccc} \mathrm{Pt}(C) & \xrightarrow{\pi_2} & C^{\leftarrow\leftarrow} \\ {\scriptstyle \pi_1}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle m} \\ C_0 & \xrightarrow{\quad e \quad} & C_1 \end{array} \tag{PT}$$

For $\mathbb{C} = \mathbf{Set}$, this is the set of triples (X, r, s) for $(r, s) \in C^{\leftarrow\leftarrow}$ and $X \in C_0$ such that $m(r, s) = e(X)$. This is only possible when $c(r) = d(s) = X$, and is precisely the condition that $rs = 1_X$. For an element of C_1 to appear as both the second and third component of (possible different) such triples is for it to be both a split epimorphism and a split monomorphism, and thus an isomorphism. We therefore obtain the following definition.

Definition 5.0.4. Let C be an internal category in a category $\mathbb{C}$ with pullbacks. The *object of isomorphisms* of C is defined as the pullback:

$$\begin{array}{ccccc} \mathrm{Iso}(C) & & \xrightarrow{\qquad\quad \pi_2 \qquad\quad} & & \mathrm{Pt}(C) \\ & \lrcorner & & & \big\downarrow{\scriptstyle \pi_2} \\ {\scriptstyle \pi_1}\Big\downarrow & & & & C^{\leftarrow\leftarrow} \\ & & & & \big\downarrow{\scriptstyle \pi_1} \\ \mathrm{Pt}(C) & \xrightarrow{\pi_2} & C^{\leftarrow\leftarrow} & \xrightarrow{\pi_2} & C_1 \end{array}$$

Furthermore, in the case $\mathbb{C} = \mathbf{Set}$, an element of $\mathrm{Iso}(C)$, $((X, r, s), (X', r', s'))$ is defined to have $s = r'$. However, because inverses of isomorphisms are unique, it is also true that $r = s'$. This is true in the general internal case, and may be phrased as the following pullback. Note that the the pullback projections are the same as those of the above definition, but are interchanged.

Proposition 5.0.5. *Let C be an internal category in a category $\mathbb{C}$ with pull-*

backs. Then the following is a pullback:

$$\begin{array}{ccccc} \mathrm{Iso}(C) & & \xrightarrow{\pi_1} & & \mathrm{Pt}(C) \\ & \lrcorner & & & \downarrow{\scriptstyle \pi_2} \\ \downarrow{\scriptstyle \pi_2} & & & & C^{\leftarrow\leftarrow} \\ & & & & \downarrow{\scriptstyle \pi_1} \\ \mathrm{Pt}(C) & \xrightarrow{\pi_2} & C^{\leftarrow\leftarrow} & \xrightarrow{\pi_2} & C_1 \end{array}$$

Having now the notion of an object of isomorphisms, we need to define the *subobject of isomorphisms*. However, the above Definition 5.0.4 and Proposition 5.0.5 suggest two canonical choices for this subobject in the form of the composite of each square.

Definition 5.0.6. Let C be an internal category in a category $\mathbb{C}$ with pullbacks. The *subobjects of isomorphisms* are the composites

$$\sigma : \qquad \mathrm{Iso}(C) \xrightarrow{\pi_1} \mathrm{Pt}(C) \xrightarrow{\pi_2} C^{\leftarrow\leftarrow} \xrightarrow{\pi_1} C_1$$

$$\sigma' : \qquad \mathrm{Iso}(C) \xrightarrow{\pi_1} \mathrm{Pt}(C) \xrightarrow{\pi_2} C^{\leftarrow\leftarrow} \xrightarrow{\pi_2} C_1$$

And, it may be shown that σ and σ' are indeed subobject of morphisms of C.

Proposition 5.0.7. *Let C be an internal category in a category $\mathbb{C}$ with pullbacks. Then $\sigma : \mathrm{Iso}(C) \to C_1$ and $\sigma' : \mathrm{Iso}(C) \to C_1$ are monomorphisms.*

Perhaps unsurprisingly, $\sigma \sim \sigma'$ as elements of $\mathrm{Mono}(C_1)$, with the relevant isomorphism being $\langle \pi_2, \pi_1 \rangle : \mathrm{Iso}(C) \to \mathrm{Iso}(C)$, and are therefore the same subobject of morphisms. However, we treat them separately. The reason for this is that in $\mathbb{C} = \mathbf{Set}$, while σ may be understood as including an isomorphism into the set of all morphisms of the small category, σ' maps that same isomorphism to its inverse.

Of course, composing an isomorphism with its inverse (on either side) will yield the appropriate identity morphism. Furthermore, this is a characterising property of isomorphisms. We obtain the following theorem. Note that we now make use of products as well as pullbacks, and thus $\mathbb{C}$ is required to be finitely complete.

Proposition 5.0.8. *Let C be an internal category in a finitely complete category $\mathbb{C}$. Then the following square is a pullback:*

$$\begin{array}{ccc} \mathrm{Iso}(C) & \xrightarrow{\langle\langle\sigma',\sigma\rangle,\langle\sigma,\sigma'\rangle\rangle} & C^{\leftrightarrows} \\ {\scriptstyle (d,c)\sigma}\downarrow & \lrcorner & \downarrow{\scriptstyle m\times m} \\ C_0\times C_0 & \xrightarrow{e\times e} & C_1\times C_1 \end{array} \tag{ISO}$$

where $C^{\leftrightarrows}$ is defined as the following pullback

$$\begin{array}{ccc} C^{\leftrightarrows} & \xrightarrow{\pi_2} & C^{\leftarrow\leftarrow} \\ {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle (\pi_2,\pi_1)} \\ C^{\leftarrow\leftarrow} & \xrightarrow{(\pi_1,\pi_2)} & C_1\times C_1 \end{array} \tag{BAF}$$

With this notion well established, we may now define what it means for a class of morphisms to contain all the isomorphisms of the category, internally.

Definition 5.0.9. Let C be an internal category in a finitely complete category $\mathbb{C}$, and let $\alpha : A \to C_1$ be a subobject of morphisms of C. Then α *contains all isomorphisms* of C if $\sigma \leq \alpha$. That is, if there exists a morphism $\sigma_\alpha : \mathrm{Iso}(C) \to A$ such that the following commutes:

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & C_1 \\ {\scriptstyle \sigma_\alpha}\uparrow & \nearrow{\scriptstyle \sigma} & \\ \mathrm{Iso}(C) & & \end{array}$$

We now move on to **FS2**, that $\mathcal{E}$ and $\mathcal{M}$ are closed under composition. In order to internalise this notion, we need to first consider what it means for morphisms of particular (possibly different) classes of morphisms to be composable. We, in fact, require this for **FS3** and **FS4** too.

Generally, two morphisms f and g are composable if the domain of f is the same as the codomain of g, and the set of all such pairs is given by $C^{\leftarrow\leftarrow}$. Because, in the internal case, the domain and codomain morphisms, d and c, are defined on C_1, if we are speaking of the domain or codomain of a morphism that belongs to a specified class of morphisms, we must first include it into C_1. We thus have the following.

Definition 5.0.10. Let C be an internal category in a category $\mathbb{C}$ with pullbacks, and let $\alpha : A \to C_1$ and $\beta : B \to C_1$ be two subobjects of morphisms of C. The *object of composable morphism* of α and β is the object part of the pullback:

$$\begin{array}{ccc} A^{\leftarrow}B^{\leftarrow} & \xrightarrow{\pi_2} & B \\ {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle c\beta} \\ A & \xrightarrow{d\alpha} & C_0 \end{array}$$

We emphasise the notion used in the above definition. In the case $\mathbb{C} = \mathbf{Set}$, where α and β are subsets of morphisms of the small category C, $A^{\leftarrow}B^{\leftarrow} = \{(a, b) \in A \times B \mid d(a) = c(b)\}$.

In the case that $A = B$ and $\alpha = \beta$, we write $A^{\leftarrow\leftarrow} = A^{\leftarrow}A^{\leftarrow}$. In particular, by considering the subobject of all morphisms, $C_1^{\leftarrow\leftarrow} = C^{\leftarrow\leftarrow}$. We will later need to speak of composable triples, and therefore make the following definition.

Definition 5.0.11. Let C be an internal category in a category $\mathbb{C}$ with pullbacks, and let $\alpha : A \to C_1$, $\beta : B \to C_1$ and $\delta : D \to C_1$ be three subobjects of morphisms of C. Then the *object of composable triples* of α, β and δ is the pullback:

$$\begin{array}{ccc} A^{\leftarrow}B^{\leftarrow}D^{\leftarrow} & \xrightarrow{\pi_2} & B^{\leftarrow}D^{\leftarrow} \\ {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_1} \\ A^{\leftarrow}B^{\leftarrow} & \xrightarrow{\pi_2} & B \end{array}$$

Again, we will write $A^{\leftarrow\leftarrow}D^{\leftarrow}$ if $\alpha = \beta$, $A^{\leftarrow}B^{\leftarrow\leftarrow}$ if $\beta = \delta$, and $A^{\leftarrow\leftarrow\leftarrow}$ if $\alpha = \beta = \delta$. We have that $C_1^{\leftarrow\leftarrow\leftarrow} = C^{\leftarrow\leftarrow\leftarrow}$.

We now provide an internal definition for closure under composition, followed by its motivation.

Definition 5.0.12. Let C be an internal category in a category $\mathbb{C}$ with pullbacks, and let $\alpha : A \to C_1$ be a subobject of morphisms of C. Then α is *closed* *under composition* if there exists a morphism $m_\alpha : A^{\leftarrow\leftarrow} \to A$ such that the following diagram commutes:

$$\begin{array}{ccc} A^{\leftarrow\leftarrow} & \xrightarrow{m_\alpha} & A \\ {\scriptstyle \alpha\times\alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ C^{\leftarrow\leftarrow} & \xrightarrow{m} & C_1 \end{array} \tag{CL}$$

To see that that this is a reasonable definition, we consider the case $\mathbb{C} = \mathbf{Set}$. Then, C is a small category and A is a subset of morphisms in C, with inclusion α. A being closed under composition means that for a pair of composable morphisms (a_1, a_2) in A, $m(\alpha \times \alpha)(a_1, a_2) = \alpha m_\alpha(a_1, a_2)$. As α and $\alpha \times \alpha$ are inclusions, this means that $m_\alpha(a_1, a_2) = m(a_1, a_2) = a_1 a_2$, making the m_α the restriction of m to domain $A^{\leftarrow\leftarrow}$, a subset of $C^{\leftarrow\leftarrow}$. The fact that m_α has codomain A then means that every composition of morphisms in A in again in A. This agrees with the usual definition of closure under composition on a subset of morphisms of a small category.

Furthermore, in this special case, as m_α is the restriction of m, it will inherit the associativity of m. This is true in the general case, and we have the following.

Proposition 5.0.13. *Let C be an internal category in a category $\mathbb{C}$ with pullbacks, and let $\alpha : A \to C_1$ be a subobject of morphisms that is closed under composition. Then the following diagram commutes:*

$$\begin{array}{ccc} A^{\leftarrow\leftarrow\leftarrow} & \xrightarrow{1\times m_\alpha} & A^{\leftarrow\leftarrow} \\ {\scriptstyle m_\alpha\times 1}\downarrow & & \downarrow{\scriptstyle m_\alpha} \\ A^{\leftarrow\leftarrow} & \xrightarrow{m_\alpha} & A \end{array} \tag{ASC}$$

The composition of two isomorphisms is again an isomorphism. This is also true internally:

Proposition 5.0.14. *Let C be an internal category in a finitely complete category $\mathbb{C}$. Then the subobjects of isomorphisms σ and σ' are closed under composition.*

Our next consideration is orthogonality. To provide an internal definition, we will use the correspondence leading up to Proposition 4.0.5. We consider the case of an internal category in **Set** to motivate the definition.

Let C be an internal category in **Set**, and let $\varepsilon : E \to C_1$ and $\mu : M \to C_1$ be two subobjects of morphisms of C. Thus, C is a small category and E and M are subsets of morphisms. Consider $M^{\leftarrow}C_1^{\leftarrow}$. This is the set of pairs of composable morphisms (f, u) of C such that $f \in M$. On the other hand, $C_1^{\leftarrow}E^{\leftarrow}$ is the set of pairs of morphism (v, g) with $g \in E$. The map $m(\mu \times 1) : M^{\leftarrow}C_1^{\leftarrow} \to C_1$ maps a pair to its composition: $m(\mu \times 1)(f, u) = fu$, and the same is true for the map $m(1 \times \varepsilon) : C_1^{\leftarrow}E^{\leftarrow} \to C_1$, which maps $m(1 \times \varepsilon)(v, g) = vg$. We now consider the pullback of these two maps:

$$\begin{array}{ccc} (M^{\leftarrow}C_1^{\leftarrow}) \times_{C_1} (C_1^{\leftarrow}E^{\leftarrow}) & \xrightarrow{\pi_2} & C_1^{\leftarrow}E^{\leftarrow} \\ {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle m(1\times\varepsilon)} \\ M^{\leftarrow}C_1^{\leftarrow} & \xrightarrow{m(\mu\times 1)} & C_1 \end{array}$$

This pullback will be the set

$$(M^{\leftarrow}C_1^{\leftarrow}) \times_{C_1} (C_1^{\leftarrow}E^{\leftarrow}) = \{((f, u), (v, g)) \in (M \times C_1) \times (C_1 \times E) \mid fu = vg\}$$

which may be understood to be the set of all commutative diagrams of C of the form:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ A' & \xrightarrow{g} & B' \end{array}$$

where $f \in M$ and $g \in E$. We have noted that to say that $E \downarrow M$ is to say that this set is in bijective correspondence with the diagrams of the form

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \swarrow{\scriptstyle z} & \\ A' & \xrightarrow[g]{} & B' \end{array}$$

again with $f \in E$ and $g \in M$. But the set of such diagrams may be expressed as the object of composable triples $M^{\leftarrow}C_1^{\leftarrow}E^{\leftarrow}$. The correspondence may be given by the existence of a *canonical* isomorphism (that is, bijection) $(M^{\leftarrow}C_1^{\leftarrow}) \times_{C_1} (C_1^{\leftarrow}E^{\leftarrow}) \approx M^{\leftarrow}C_1^{\leftarrow}E^{\leftarrow}$.

To complete construction, we must understand what we mean here by canonical. The correspondence we have given says that while z is obtained by the commutative square implicitly, given an element of $M^{\leftarrow}C_1^{\leftarrow}E^{\leftarrow}$, one obtains the commutative square through composition. Specifically, the map $1 \times m(1 \times \varepsilon) : M^{\leftarrow}C_1^{\leftarrow}E^{\leftarrow} \to M^{\leftarrow}C^{\leftarrow}$ maps $(1 \times m(1 \times \varepsilon))(g, z, f) = (g, zf)$ and the map $m(\mu \times 1) \times 1 : M^{\leftarrow}C_1^{\leftarrow}E^{\leftarrow} \to C_1^{\leftarrow}E^{\leftarrow}$ maps $(m(\mu \times 1) \times 1)(g, z, f) = (gz, f)$. Thus, canonical means that the pullback of $m(\mu \times 1)$ and $m(1 \times \varepsilon)$ is in fact $M^{\leftarrow}C_1^{\leftarrow}E^{\leftarrow}$, with pullback projections $1 \times m(1 \times \varepsilon)$ and $m(\mu \times 1) \times 1$. We thus arrive at our definition.

Definition 5.0.15. Let C be an internal category in a category $\mathbb{C}$ with pullbacks, and let $\varepsilon : E \to C_1$ and $\mu : M \to C_1$ be two subobjects of morphisms of C. Then ε is *orthogonal* to μ, written $\varepsilon \downarrow \mu$, if the following square is a pullback:

$$\begin{array}{ccc} M^{\leftarrow}C_1^{\leftarrow}E^{\leftarrow} & \xrightarrow{m(\mu\times 1)\times 1} & C_1^{\leftarrow}E^{\leftarrow} \\ {\scriptstyle 1\times m(1\times\varepsilon)}\downarrow & \lrcorner & \downarrow{\scriptstyle m(1\times\varepsilon)} \\ M^{\leftarrow}C_1^{\leftarrow} & \xrightarrow{m(\mu\times 1)} & C_1 \end{array}$$

We now consider the last of the conditions of a factorisation system, **FS4**, that the category has $(\mathcal{E}, \mathcal{M})$-factorisations. For an internal category C in **Set**, and the subobjects of morphisms $\varepsilon : E \to C_1$ and $\mu : M \to C_1$, this is to say that for each $f \in C_1$, there exists an element of $M^{\leftarrow}E^{\leftarrow}$, (m_f, e_f) such that $f = m_f e_f$. We may define the map $\tau : C_1 \to M^{\leftarrow}E^{\leftarrow}$ by $\tau(f) = (m_f, e_f)$, and phrase the condition as $m(\mu \times \varepsilon)\tau = 1_{C_1}$, noting that $m(\mu \times \varepsilon)$ returns the composition of the pair of morphisms of M and E.

Requiring the existence of such a τ is precisely to ask that $m(\mu \times \varepsilon)$ is a split epimorphism. However, in **Set**, every epimorphism is such, and it would be equivalent to ask $m(\mu \times \varepsilon)$ to be a (usual) epimorphism. We again face the problem of choosing an appropriate strength of epimorphism for our general definition. Requiring the full strength of a split epimorphism allows for the explicit consideration of factorisations of morphisms, in an internal sense, in the general contexts of a finitely complete category. We thus opt for this in our definition.

Definition 5.0.16. Let C be an internal category in a category $\mathbb{C}$ with pullbacks, and let $\varepsilon : E \to C_1$ and $\mu : M \to C_1$ be two subobjects of morphisms of C. Then C *has* (ε, μ)*-factorisations* if there exists a morphism $\tau : C_1 \to M^{\leftarrow}E^{\leftarrow}$ such that $m(\mu \times \varepsilon)\tau = 1_{C_1}$.

We may now combine these definitions to introduce the notion of an *internal factorisation system.*

Definition 5.0.17. Let C be an internal category in a finitely complete category $\mathbb{C}$. Let $\varepsilon : E \to C_1$ and $\mu : M \to C_1$ be two subobjects of morphisms of C. Then the pair (ε, μ) forms an *internal factorisation system* on C if the following four conditions are met:

IFS1. ε and μ contain all isomorphisms of C: There exist morphisms σ_ε and σ_μ such the following triangles commute

$$\begin{array}{ccc} E & \xrightarrow{\varepsilon} & C_1 \\ {\scriptstyle\sigma_\varepsilon}\uparrow & \nearrow_{\sigma} & \\ \mathrm{Iso}(C) & & \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\mu} & C_1 \\ {\scriptstyle\sigma_\mu}\uparrow & \nearrow_{\sigma} & \\ \mathrm{Iso}(C) & & \end{array}$$

IFS2. ε and μ are closed under composition: There exist morphisms m_ε and m_μ such that the following squares commute:

$$\begin{array}{ccc} E^{\leftarrow\leftarrow} & \xrightarrow{m_\varepsilon} & E \\ {\scriptstyle\varepsilon\times\varepsilon}\downarrow & & \downarrow{\scriptstyle\varepsilon} \\ C^{\leftarrow\leftarrow} & \xrightarrow{m} & C_1 \end{array} \qquad \begin{array}{ccc} M^{\leftarrow\leftarrow} & \xrightarrow{m_\mu} & M \\ {\scriptstyle\mu\times\mu}\downarrow & & \downarrow{\scriptstyle\mu} \\ C^{\leftarrow\leftarrow} & \xrightarrow{m} & C_1 \end{array}$$

IFS3. $\varepsilon \downarrow \mu$: The following square is a pullback:

$$\begin{array}{ccc} M^{\leftarrow}C_1^{\leftarrow}E^{\leftarrow} & \xrightarrow{m(\mu\times 1)\times 1} & C^{\leftarrow}E^{\leftarrow} \\ {\scriptstyle 1\times m(1\times\varepsilon)}\downarrow & \lrcorner & \downarrow{\scriptstyle m(1\times\varepsilon)} \\ M^{\leftarrow}C_1^{\leftarrow} & \xrightarrow{m(\mu\times 1)} & C_1 \end{array}$$

IFS4. C has (ε, μ)-factorisations: There exists a morphism τ such that $m(\mu \times \varepsilon)\tau = 1_{C_1}$

With this definition established, we may now consider the various properties of factorisation systems, provide internal definitions of them and show that they are satisfied by internal factorisation systems. The first of which is the fact that the intersection of the classes of a factorisation system is the class of isomorphisms.

Consider some set X and two subsets of X, $A \subseteq X$ and $B \subseteq X$. Let $\alpha : A \to X$ and $\beta : B \to X$ be the inclusions of the subsets into X. Note that α and β are subobjects of X in **Set**. Consider the pullback:

$$\begin{array}{ccc} A \cap B & \xrightarrow{\pi_2} & B \\ {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow{\alpha} & X \end{array}$$

The notation suggests that this pullback is the intersection of A and B. Indeed, noting that α and β are inclusions,

$$A \cap B = \{(a, b) \in A \times B \mid a = b\} = \{(x, x) \in X \times X \mid x \in A \ \wedge \ x \in B\}$$

which is isomorphic to the set-theoretic intersection of A and B. In particular, note that π_1 and π_2 are respectively the inclusions of the intersection $A \cap B$ into A and B. We use this observation to define the intersection of subobjects in a general category. That is,

Definition 5.0.18. Let $\alpha : A \to X$ and $\beta : B \to X$ be two subobjects of an object X in a category $\mathbb{C}$. The *intersection* of α and β is the pullbacks:

$$\begin{array}{ccc} A \cap B & \xrightarrow{\pi_2} & B \\ {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow{\alpha} & X \end{array}$$

Because pullbacks preserve monomorphisms, π_1 and π_2 will be subobjects, as was in the case of **Set**. This allows us to provide the first internal property. Note that we will often not require the assumption that the pair (ε, μ) form an internal factorisation system, but rather only some of the defining conditions.

Proposition 5.0.19. *Let $(\varepsilon : E \to C_1, \mu : M \to C_1)$ be a pair of subobjects of morphisms of an internal category C in a finitely complete category $\mathbb{C}$, that* *satisfy* **IFS1**, **IFS2** *and* **IFS3**. *Then the following square is a pullback:*

$$\begin{array}{ccc} \mathrm{Iso}(C) & \xrightarrow{\sigma_\varepsilon} & E \\ {\scriptstyle \sigma_\mu}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \varepsilon} \\ M & \xrightarrow{\mu} & C_1 \end{array} \tag{IEM}$$

The next property we showed was that factorisations are unique up to isomorphism. To internalise this, we make an observation on this property which is similar to that made on the nature of the orthogonality axiom.

Firstly, to say that $(\mathcal{E}, \mathcal{M})$-factorisations are unique up to isomorphism does not necessarily require $(\mathcal{E}, \mathcal{M})$-factorisations to exist in a category $\mathbb{C}$. It is equivalent to say that for morphisms $e, e' \in \mathcal{E}$ and $m, m' \in \mathcal{M}$ (as in the following diagram), with $me = m'e'$, there exists a unique isomorphisms φ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ {\scriptstyle e'}\big\downarrow & \overset{\varphi}{\swarrow} & \big\downarrow{\scriptstyle m} \\ A' & \xrightarrow[m']{} & B' \end{array}$$

That is, we do not speak of some f for which (m, e) and (m', e') are factorisations. In fact, the composite me is such an f. Then, we observe that this means that for each commutative square on the left, we obtain the diagram on the right,

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ {\scriptstyle e'}\big\downarrow & & \big\downarrow{\scriptstyle m} \\ A' & \xrightarrow{m'} & B' \end{array} \quad \mapsto \quad \begin{array}{ccc} A & \xrightarrow{e} & B \\ & \overset{\varphi}{\swarrow} & \\ A' & \xrightarrow{m'} & B' \end{array}$$

where $e, e' \in \mathcal{E}$, $m, m' \in \mathcal{M}$ and $\varphi \in \mathrm{Iso}(\mathbb{C})$. On the the other hand, given the following diagram on the left, we obtain the commutative diagram on the right:

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ & \overset{\varphi'}{\swarrow} & \\ A' & \xrightarrow{m'} & B' \end{array} \quad \mapsto \quad \begin{array}{ccc} A & \xrightarrow{e} & B \\ {\scriptstyle \varphi' e}\big\downarrow & & \big\downarrow{\scriptstyle m'\varphi'} \\ A' & \xrightarrow{m'} & B' \end{array}$$

Where $e \in \mathcal{E}$, $m' \in \mathcal{M}$ and $\varphi' \in \mathrm{Iso}(\mathbb{C})$. In particular, φ' is also in both $\mathcal{E}$ and $\mathcal{M}$, which are both also closed under composition. Therefore $\varphi' e \in \mathcal{E}$ and $m'\varphi \in \mathcal{M}$. This is a similar correspondence to the one used to internalise orthogonality, and thus we expect the internalisation of the essential uniqueness of factorisations to be similar to that internalisation.

Now working on an internal category C in a finitely complete category $\mathbb{C}$, with subobjects of morphisms $\varepsilon : E \to C_1$ and $\mu : M \to C_1$ which necessarily satisfy **IFS1** and **IFS2**, recall that $M^{\leftarrow}E^{\leftarrow}$ is the pairs of composable morphisms for which the first is in μ and the second is in ε, internally. The morphism $m(\mu \times \varepsilon) : M^{\leftarrow}E^{\leftarrow} \to C_1$ composes such morphisms. The pullback of this morphism with itself will be the object of all commutative squares of the above correspondence. On the other hand, the "z shaped diagrams" of the correspondence are represented by the object of composable triples $M^{\leftarrow}\mathrm{Iso}(C)^{\leftarrow}E^{\leftarrow}$. We will assert that this is the pullback of $m(\mu \times \varepsilon)$ with itself. Then, the *canonical projections*, in this case, will be $1 \times m_\varepsilon(1 \times \sigma_\varepsilon) : M^{\leftarrow}\mathrm{Iso}(C)^{\leftarrow}E^{\leftarrow} \to M^{\leftarrow}E^{\leftarrow}$ and $m_\mu(1 \times \sigma_\mu) \times 1 : M^{\leftarrow}\mathrm{Iso}(C)^{\leftarrow}E^{\leftarrow} \to M^{\leftarrow}E^{\leftarrow}$.

Definition 5.0.20. Let $(\varepsilon : E \to C_1, \mu : M \to C_1)$ be a pair of subobjects of morphisms of an internal category C in a finitely complete category $\mathbb{C}$, that satisfy **IFS1** and **IFS2**. Then *(ε, μ)-factorisations are unique up to isomorphism* if the following diagram is a pullback:

$$\begin{array}{ccc} M^{\leftarrow}\mathrm{Iso}(C)^{\leftarrow}E^{\leftarrow} & \xrightarrow{m_\mu(\sigma_\mu \times 1)\times 1} & M^{\leftarrow}E^{\leftarrow} \\ {\scriptstyle 1\times m_\varepsilon(\sigma_\varepsilon\times 1)}\downarrow \quad \lrcorner & & \downarrow{\scriptstyle m(\mu\times\varepsilon)} \\ M^{\leftarrow}E^{\leftarrow} & \xrightarrow[m(\mu\times\varepsilon)]{} & C_1 \end{array} \tag{FUI}$$

And, indeed, if (ε, μ) also satisfies **IFS3**, then this is always the case. If we instead assume that (ε, μ) also satisfies **IFS4** and that factorisations are unique up to isomorphism, then it satisfies **IFS3**. This is the internalisation of 4.0.8.

Proposition 5.0.21. *Let $(\varepsilon : E \to C_1, \mu : M \to C_1)$ be a pair of subobjects of morphisms of an internal category C in a finitely complete category $\mathbb{C}$, that satisfy* **IFS1**, **IFS2** *and* **IFS4**. *Then the following are equivalent:*

1. *(ε, μ) forms an internal factorisation system on C.*

2. (ε, μ)*-factorisations are unique up to isomorphism.*

To internalise the cancellation properties, we again phrase them in terms of a correspondence. Consider some class of morphisms $\mathcal{A}$ of a category $\mathbb{C}$, which we assume to be closed under composition. Then the right cancellation property may be phrased as:

A pair of morphisms f and g is a composable pair of morphisms of $\mathcal{A}$ if and only if f and g are a pair of composable morphisms in $\mathbb{C}$ with fg in $\mathcal{A}$ and g in $\mathcal{A}$.

The forward direction is true by closure under composition while the reverse direction is the cancellation property. The left cancellation may be phrased dually.

Now, consider an internal category C in a category $\mathbb{C}$ with pullbacks, and let $\alpha : A \to C_1$ be a subobject of morphisms of C that is closed under composition. Consider the pullback:

$$\begin{array}{ccc} (C_1^{\leftarrow} A^{\leftarrow}) \times_{C_1} A & \xrightarrow{\pi_2} & A \\ {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle \alpha} \\ C_1^{\leftarrow} A^{\leftarrow} & \xrightarrow{m(1\times\alpha)} & C_1 \end{array}$$

To understand this, consider the case $\mathbb{C} = \mathbf{Set}$, then the elements of $C_1^{\leftarrow} A^{\leftarrow}$ are pairs of composable morphisms (f, a), with $a \in A$ and $m(1 \times \alpha)$ maps this pair to its composition fa. Then, $(C_1^{\leftarrow} A^{\leftarrow}) \times_{C_1} A = \{(f, a) \in C_1^{\leftarrow} A^{\leftarrow} \mid fa \in A\}$, which is the right hand side of our above correspondence.

Returning to the general internal case, the left hand side of our correspondence is simply $A^{\leftarrow\leftarrow}$. In a similar fashion to before, the *canonical projections* obtained by our correspondence are the *inclusion* $\alpha \times 1 : A^{\leftarrow\leftarrow} \to C_1^{\leftarrow} A^{\leftarrow}$ and the composition in α, $m_\alpha : A^{\leftarrow\leftarrow} \to A$.

Definition 5.0.22. Let $\alpha : A \to C_1$ be a subobject of morphisms of an internal category C in a category $\mathbb{C}$ with pullbacks, that is closed under composition.

Then α *has the right cancellation property* if the following square is a pullback:

$$\begin{array}{ccc} A{\leftarrow}{\leftarrow} & \xrightarrow{m_\alpha} & A \\ {\scriptstyle \alpha\times 1}\downarrow & \lrcorner & \downarrow{\scriptstyle \alpha} \\ C_1^{\leftarrow} A^{\leftarrow} & \xrightarrow{m(1\times\alpha)} & C_1 \end{array} \tag{RCP}$$

We may, by a similar argument, obtain the left cancellation property.

Definition 5.0.23. Let $\alpha : A \to C_1$ be a subobject of morphisms of an internal category C in a category $\mathbb{C}$ with pullbacks, that is closed under composition. Then α *has the left cancellation property* if the following square is a pullback:

$$\begin{array}{ccc} A{\leftarrow}{\leftarrow} & \xrightarrow{m_\alpha} & A \\ {\scriptstyle 1\times \alpha}\downarrow & \lrcorner & \downarrow{\scriptstyle \alpha} \\ A^{\leftarrow} C_1^{\leftarrow} & \xrightarrow{m(\alpha\times 1)} & C_1 \end{array} \tag{LCP}$$

We remark that the fact that these squares commute follows directly from the fact that α is closed under composition. As expected, for an internal factorisation system we have the following.

Proposition 5.0.24. *Let (ε, μ) be an internal factorisation system on an internal category C in a finitely complete category $\mathbb{C}$. Then ε has the right cancellation property and μ has the left cancellation property.*

The last of our properties, pertaining to the order on factorisation systems are readily internalised with the structures we have thus far developed. For 4.0.13, we have:

Proposition 5.0.25. *Let (ε, μ) be an internal factorisation system on an internal category C in a finitely complete category $\mathbb{C}$. Let $\alpha : A \to C_1$ be a subobject of morphisms of C. Then:*

1. *$\alpha \leq \varepsilon$ if and only if $\alpha \downarrow \mu$*
2. *$\alpha \leq \mu$ if and only if $\varepsilon \downarrow \alpha$*

And then for 4.0.14, we have:

Proposition 5.0.26. *Let (ε, μ) and (ε', μ') be two internal factorisation systems on an internal category C in a finitely complete category $\mathbb{C}$. Then $\varepsilon \sim \varepsilon'$ if and only if $\mu \sim \mu'$.*

And we may similarly define an order on the internal factorisation systems of an internal category.

Definition 5.0.27. The *order on internal factorisation systems* for an internal category C in a finitely complete category $\mathbb{C}$ is defined as follows: For two internal factorisation systems (ε, μ) and (ε', μ') on C, we have

$$(\varepsilon, \mu) \leq (\varepsilon', \mu') \text{ if and only if } \mu \leq \mu'$$

where the relation on the right hand side is the order on $\mathrm{Sub}(C_1)$.

To conclude this chapter, we note that as with the non-internal case, $(\sigma, 1_{C_1})$ is an internal factorisation system, called the *trivial internal factorisation system*, on any internal category C in a finitely complete category $\mathbb{C}$, and indeed it is the top element of the above order.

Chapter 6

Mal'tsev Varieties

To *be a Mal'tsev variety* is a property that is shared by various varieties of algebras, such as **Grp**, **Ab** (the category of abelian groups), $\mathbf{Vect}_K$ (the category of vector spaces over K) and **Heyt** (the category of Heyting algebras). Its various characterisations make this property of interest in categorical algebra, and the notion has been generalised to and studied in the more general context of categories that do not necessarily form a variety. See [17] for further details. We, however, will focus on the case of varieties, and note that the following definition is due to Smith [18].

Definition 6.0.1. A *Mal'tsev variety* is a variety of universal algebra for which there exists a ternary term, t, called the *Mal'tsev operation*, which satisfies the following two equations:

1. $t(x, x, y) = y$
2. $t(x, y, y) = x$

To understand this definition, we show how **Grp** is a Mal'tsev variety. Consider some group G, and define the term t as $t(x, y, z) = x - y + z$. Then, for all $x, y \in G$,

$$t(x, x, y) = x - x + y = y$$
$$t(x, y, y) = x - y + y = x$$

In fact, we may define the Mal'tsev operation similarly for an abelian group or a vector space, and for any variety that has an underlying group structure.

On the other hand, the category of monoids is not a Mal'tsev variety. To show this, we need to consider one of the characterising Mal'tsev properties.

Recall from Chapter 1 that for a category $\mathbb{C}$, a relation is a monomorphism $r = (r_1, r_2) : R \to X \times X$ in $\mathbb{C}$. For clarity, we make the following definition which was previously overlooked.

Definition 6.0.2. An *internal reflexive relation* on an object X in a category $\mathbb{C}$ is a monomorphism $r = (r_1, r_2) : R \to X \times X$ such that there exists a morphism $\rho : X \to R$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\rho} & R \\ {\scriptstyle\rho}\downarrow & \diagdown\!\!\diagdown & \downarrow{\scriptstyle r_1} \\ R & \xrightarrow[r_2]{} & X \end{array} \tag{REF}$$

Recall also Definition 1.0.9 of an internal equivalence relation.

We will be working within a variety, and so we note that an internal relation in a variety $\mathbb{C}$ is precisely a relation on the underlying sets of the objects of the variety, with the additional condition that they are *homomorphic*. This means that if ω is an n-ary operation of the variety, and $r : R \to X \times X$ is an internal relation in $\mathbb{C}$, then:

$$(x_1, x_1'), ..., (x_n, x_n') \in R \implies (\omega(x_1, ..., x_n), \omega(x_1', ..., x_n')) \in R$$

Then, to be reflexive, symmetric or transitive is exactly what it means to be such on the underlying set.

We only require the one direction of the following equivalence, and will thus omit the the proof for the other direction, directing the reader to [19].

Proposition 6.0.3. *Let $\mathbb{C}$ be a variety. Then the following are equivalent:*

1. *$\mathbb{C}$ is a Mal'tsev variety.*
2. *Every internal reflexive relation in $\mathbb{C}$ is an internal equivalence relation.*

Proof. Assume that $\mathbb{C}$ is a Mal'tsev variety, and consider some internal reflexive relation $r = (r_1, r_2) : R \to X \times X$ in $\mathbb{C}$. We show that R is symmetric and transitive. Assume that $(x, y) \in R$, for $x, y \in X$. Then R is reflexive, so

$(x,x) \in R$ and $(y,y) \in R$. Recalling that we have a Mal'tsev operation, t, and that R is homomorphic, we observe that

$$(x,x),(x,y),(y,y) \in R \implies (t(x,x,y),t(x,y,y)) \in R$$

But $t(x,x,y) = y$ and $t(x,y,y) = x$. Thus $(y,x) \in R$, and so R is symmetric. Assume now that $(x,y),(y,z) \in R$ for $x,y,z \in X$. Again, $(y,y) \in R$ because R is reflexive, and so:

$$(x,y),(y,y),(y,z) \in R \implies (t(x,y,y),t(y,y,z)) \in R$$

But, $t(x,y,y) = x$ and $t(y,y,z) = z$, so $(x,z) \in R$, and R is therefore transitive and thus an equivalence relation. We omit that 2 implies 1. □

Now, to show that **Mon** is not a Mal'tsev variety, we simply provide an example of an internal reflexive relation in **Mon** that is not an internal equivalence relation. Consider $\mathbb{N} = (\mathbb{N},+,0)$, the natural numbers (including 0) as a monoid with usual addition as the binary operation, and consider the usual order on $\mathbb{N}$, $\leq$. Of course, for natural numbers $a,b,c,d \in \mathbb{N}$, if $a \leq b$ and $c \leq d$, then $a+c \leq b+d$, so $\leq$ is homomorphic. Also, for every natural number $a \in \mathbb{N}$, $a \leq a$, so $\leq$ is reflexive. Thus, we have an internal reflexive relation in **Mon**. Furthermore, this relation is not symmetric because while $1 \leq 2$, clearly $2 \nleq 1$. We thus have an internal reflexive relation that is not an equivalence relation, so **Mon** is not a Mal'tsev variety.

With this definition, example and non-example established, we now approach the key theorem of this chapter. In Chapter 2, we showed that every internal category in **Grp** is an internal groupoid. We have shown now that **Grp** is a Mal'tsev variety. It is in fact true that in any Mal'tsev variety, every internal category is an internal groupoid. The following proof is due to Janelidze [20].

Proposition 6.0.4. *Let $\mathbb{C}$ be a Mal'tsev variety, and let C be an internal category in $\mathbb{C}$. Then C is an internal groupoid.*

Proof. Let $f \in C_1$. Then $ed(f)$ and $ec(f)$ are also in C_1. $\mathbb{C}$ is a Mal'tsev variety, so we have a Mal'tsev operation t, and thus $t(ed(f),f,ec(f)) \in C_1$. Define the morphism $i : C_1 \to C_1$ by $i(f) = t(ed(f),f,ec(f))$. We show that i satisfies the axioms making C an internal groupoid. Firstly,

$$di(f) = dt(ed(f),f,ec(f)) = t(ded(f),d(f),dec(f)) = t(d(f),d(f),c(f)) = c(f)$$

$$ci(f) = ct(ed(f), f, ec(f)) = t(ced(f), c(f), cec(f)) = t(d(f), c(f), c(f)) = d(f)$$

by the definition of i, the fact that d and c are homomorphisms, **IC1**, and the definition of the Mal'tsev operation. This means that for all $f \in C_1$, $(f, i(f))$ and $(i(f), f)$ are both in $C^{\leftarrow\leftarrow}$. Secondly, by noting that $f = t(f, ec(f), ec(f)) = t(ed(f), ed(f), f)$ by the definition of t, we have:

$$\begin{aligned}
&m(f, i(f))\\
&= m(t(f, ec(f), ec(f)), t(ed(f), f, ec(f)))\\
&= t(m(f, ed(f)), m(ec(f), f), m(ec(f), ec(f)))\\
&= t(f, f, ec(f))\\
&= ec(f)
\end{aligned}$$

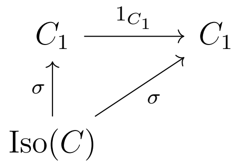
$$\begin{aligned}
&m(i(f), f)\\
&= m(t(ed(f), f, ec(f)), t(ed(f), ed(f), f))\\
&= t(m(ed(f), ed(f)), m(f, ed(f)), m(ec(f), f))\\
&= t(ed(f), f, f)\\
&= ed(f)
\end{aligned}$$

We conclude therefore that C is an internal groupoid in $\mathbb{C}$. □

Our next theorem is one which in the non-internal context is trivial. Of course, a groupoid is defined to have every morphism an isomorphism, and thus the class of all morphisms is precisely the class of isomorphisms of the groupoid.

Proposition 6.0.5. *Let C be an internal groupoid in a finitely complete category $\mathbb{C}$. Then $\sigma \sim 1_{C_1}$ as subobjects of morphisms.*

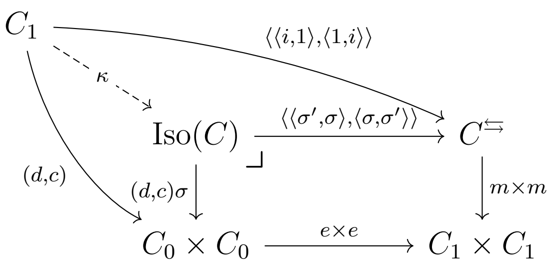
Proof. Firstly, note that we have that $\sigma \leq 1_{C_1}$ by the following trivially commutative diagram:



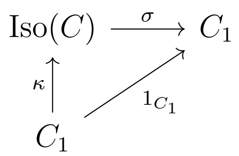
Next, observe that

$$(\pi_2, \pi_1)\langle 1, i\rangle = (\pi_2\langle 1, i\rangle, \pi_1\langle 1, i\rangle) = (i, 1) = (\pi_1, \pi_2)\langle i, 1\rangle$$

So by the universal property of $C^{\leftrightarrows}$, we obtain the morphism $\langle\langle i, 1\rangle, \langle 1, i\rangle\rangle : C_1 \to C^{\leftrightarrows}$. Now, consider the following diagram:



Now, the square of this diagram is a pullback by 5.0.8. The outside of the diagram commutes from **GP2**. Then, the pullback induces the universal morphism κ making the whole diagram commute. Considering the upper triangle, we have that $\langle\langle\sigma', \sigma\rangle, \langle\sigma, \sigma'\rangle\rangle\kappa = \langle\langle i, 1\rangle, \langle 1, i\rangle\rangle$. Postcomposing with the projections $\pi_1 : C^{\leftrightarrows} \to C^{\leftarrow\leftarrow}$ and then $\pi_2 : C^{\leftarrow\leftarrow} \to C_1$, we obtain the following commutative diagram:



And so $1_{C_1} \leq \sigma$. By 5.0.26 we have that $\sigma \sim 1_{C_1}$. □

The final observation we make links internal factorisation systems to our present considerations.

Proposition 6.0.6. *The only internal factorisation system on an internal groupoid C in a finitely complete category $\mathbb{C}$ is $(\sigma, 1_{C_1})$.*

Proof. Let (ε, μ) be an internal factorisation system on C. Then, $\sigma \leq \varepsilon \leq 1_{C_1} \sim \sigma$. Thus $\varepsilon \sim \sigma$. On the other hand, $1_{C_1} \sim \sigma \leq \mu \leq 1_{C_1}$, so $\mu \sim 1_{C_1}$. □

We conclude that there are no non-trivial (that is, not equivalent to $(\sigma, 1_{C_1})$) internal factorisation systems in any Mal'tsev variety. In particular, this is the case for **Grp**. From the equivalence Cat(**Grp**) $\sim$ **XMod**, we see that there are no non-trivial factorisation systems for crossed modules induced by internal factorisation systems.

As a closing remark, we reiterate that **Mon** is not a Mal'tsev variety, and more specifically, we have an example of a (Schreier) internal category in **Mon** which is not a groupoid. We may then have non-trivial internal factorisation systems in this category and indeed factorisation systems for crossed semimodules. We leave this as future research on this topic.

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UNIVERSITÉ CATHOLIQUE DE LOUVAIN
Faculté des sciences
Place des sciences, 2 bte L6.06.01, 1348 Louvain-la-Neuve, Belgique | **www.uclouvain.be/sc**