

Faculté des sciences

# PARTAGE DES RISQUES DANS LE CADRE DE L'ASSURANCE TAKAFUL

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Année académique 2023 - 2024

Master [120] en sciences actuarielles



Louvain School of Statistics, Biostatistics and Actuarial Science (LSBA)

**Partage des risques dans le cadre de l'assurance Takaful.**

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Thesis submitted for  
the Master degree  
in Actuarial Sciences  
(actuarial sciences orientation)  
by:

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Louvain-La-Neuve  
June 2024

## Abstract

This work concerns an actuarial approach to risk-sharing in Takaful insurance. In a context of financial crises and moral value crises, a growing number of individuals are seeking services that are ethical, such as collaborative insurance or, even more so, religiously-oriented insurance like Takaful. These particular forms of insurance share a common feature: the mutualization of risks among participants who collectively agree to share these risks. The risk-sharing mechanism and the solvency of these forms of insurance are crucial for their sustainability. The specificity of Takaful insurance lies in the interest-free or benevolent loan provided by shareholders to the participants. This is an alternative to the collaborative model with (re-)insurer, but its actuarial approach has not yet been addressed. This dissertation proposes an actuarial modeling of this loan mechanism, drawing a parallel with collaborative insurance. In the same vein, due to its interesting properties, conditional mean risk sharing has been used to define the way contributions are calculated in the pool, ensuring a sharing mechanism that is both equitable and transparent for all participants, as it is based on the concept of the mean. We determine the probability that the loan will be repaid over a fixed horizon to control the level of risk taken by the shareholders. This solvency issue, addressed within the framework of Takaful insurance with benevolent loans, thus strengthens this model and makes it more attractive in the insurance sector.

**Keywords:** Takaful insurance, collaborative insurance, benevolent loan, conditional mean risk sharing.

## Resumé

Ce travail porte sur une approche actuarielle du partage des risques dans l'assurance Takaful. Dans un contexte de crise financière et de crise des valeurs morales, un nombre croissant d'individus cherchent à se tourner vers des services qui se veulent éthiques, en particulier dans le secteur des assurances, comme celle que propose l'assurance collaborative ou, plus encore, l'assurance à orientation religieuse comme le Takaful. Ces formes particulières d'assurance ont une caractéristique commune: la mutualisation des risques entre des participants qui acceptent collectivement de partager ces risques. Le mécanisme de partage des risques et la solvabilité de ces formes d'assurance sont essentiels à leur pérennité. La spécificité de l'assurance Takaful réside dans le prêt sans intérêt ou bénévol accordé par les actionnaires aux participants. Il s'agit d'une alternative au modèle collaboratif avec (ré-)assureur, mais son approche actuarielle n'a pas encore été abordée. Ce mémoire propose une modélisation actuarielle de ce mécanisme de prêt établissant un autre parallèle avec l'assurance collaborative. Dans le même ordre d'idées, en raison de ses propriétés intéressantes, le partage des risques par la moyenne conditionnelle a été utilisé pour définir la manière dont les contributions sont calculées dans le pool, garantissant un mécanisme de partage qui est à la fois équitable et transparent pour tous les participants, puisqu'il est basé sur le concept de la moyenne. Nous déterminons la probabilité pour un horizon fixé que le prêt soit remboursé sur cet horizon afin de contrôler le niveau de risque pris par les actionnaires. Cette question de solvabilité traitée dans le cadre de l'assurance Takaful avec prêt bénévol vient ainsi renforcer ce modèle et le rendre plus attrayant dans le secteur de l'assurance.

## Acknowledgment

There are a number of people without whom this thesis might not have been written, and to whom I am greatly indebted.

A special feeling of gratitude to my promoter, Professor Michel Denuit, I cannot estimate the time and energy you have devoted to supervise this thesis. Your encouragements, advice and expertise have been a decisive contribution to achieve this work. I'm very grateful to you.

I would also like to sincerely thank the faculty of LSBA for their contributions and support throughout this intellectual journey. Their input has added an additional dimension to my research.

Finally, I would especially like to thank my dear mother Louise and my partner D'avila. In general, I also wish to extend my gratitude to my family and friends, whose moral and emotional support has been a solid pillar during these times of intense academic commitment.

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# Introduction

In the context of a financial crisis and a crisis of moral values, an increasing number of individuals are seeking to turn to services that are ethical, or even religious, both in the industry at large and particularly in the insurance sector. Traditional insurance (classical insurance) no longer seems as attractive as it once was, due to the progressive demutualization of the insurance sector, which has severed the link with the ancient mechanism of mutual aid that involved using the contributions of the many to compensate for the losses of the few. In this context, collaborative insurances, also known as peer-to-peer (P2P) insurances, have emerged as a pioneering alternative to traditional insurances. What distinguishes this innovative system is its decentralized approach, where participants come together to collectively share their risks. This new model of risk-sharing among peers offers a fresh and revolutionary perspective for the insurance sector by fundamentally transforming the risk management system and the distribution of risks among participants.

Thanks to this emerging mechanism, we are gradually witnessing the revival of this old concept in a more modern and digital form, eliminating the intermediation required by classical insurance. The value of premiums for classical insurance coverage can sometimes reach significant amounts due to the solvency requirements imposed on insurers by regulators. Consequently, the cost of holding this solvency capital, profit margins, and other expenses related to insurance activities contribute to these high premiums. High-risk individuals often find it challenging to obtain suitable insurance coverage due to prohibitively high premiums. This is particularly evident in health insurance, which often does not account for pre-existing conditions. Beyond the purely financial aspect, there is a social dimension where the sense of belonging and the pursuit of social objectives also drive peers to self-insure.

The pricing of insurance products and the compensation mechanism, often perceived as unclear and inaccessible by the average consumer, have contributed to a climate of mistrust towards the insurance sector. In contrast, collaborative insurance emphasizes the transparency of its cooperation rules to regain consumer trust and bring back to the core of the activity the fundamental principle of mutuality, which is the origin of insurance. More specifically, participants in collaborative insurance can agree to collectively share the initial layer of risk they face, while large losses are transferred to an insurer or a (re-)insurer.

Simultaneously, the Islamic community offers another alternative to insurance that takes into account the religious aspect. Designated by the term "Takaful," which means solidarity in Arabic, Takaful insurance is a model of Islamic insurance based on the principles of mutual

assistance, cooperation, and voluntary contribution. This model involves the separation of funds between shareholders and participants, the distribution of technical profits to participants, compliance of assets with Islamic laws, and certification by an Islamic religious council. The operational models vary depending on how technical profits are distributed between the manager of funds (including shareholders) and its participants. More specifically, Islamic insurance has distinctive characteristics. Uniquely, the insured (participants) are both insurers and insureds, similar to collaborative insurance. Additionally, it has a social and solidarity vocation, as making profits is not the primary goal.

Similar to classical insurance, Islamic insurance covers life insurance, non-life insurance, and even reinsurance operations. In this thesis, we will primarily focus on property and casualty insurance. Within this framework, in addition to transferring large losses to a (re-)insurer, this insurance offers participants the possibility of an interest-free loan called benevolent loan when the contributions paid at the beginning of the period are insufficient to fully cover claims due to losses. This loan, provided by shareholders into a fund, is considered an act of charity, and participants are only required to repay it in the event of profits.

Takaful companies operating in different parts of the world have adopted various models to offer their services. These models are derived from different forms of Islamic finance and comply with the rules and regulations of Islamic law. In Malaysia, the Mudarabah and Wakalah models are practiced, while in Pakistan and South Africa, a combination of the Mudarabah and Wakalah models, known as Hybrid-Takaful, is used. It should be noted that companies implementing these models operate under the assumption of benevolent loans, defined as acts of charity that may eventually not be repaid. The objective of this dissertation is to propose an actuarial modeling of a P2P insurance platform inspired by the Mudarabah-Takaful model, primarily considering the integration of benevolent loans for the pool and studying the solvency aspect to make the platform attractive to both participants and the shareholders who establish the interest-free loan allocation fund. Concretely, the fundamental questions to address are: How can the costs be equitably, understandably, and transparently distributed among the various participants, initially by involving a (re-)insurer to cover a portion of the costs? How should these costs be allocated when the pool includes benevolent loans? What would be the financial impact on participants in each case? Is the proposed model with benevolent loans advantageous compared to P2P insurance with a (re-)insurer?

To achieve this, in the first chapter we introduce a literature review on Takaful, P2P insurance, and classical insurance, highlighting the differences between these three forms of insurance. We present in the second chapter the conditional mean risk sharing, a rule that allows for equitable cost distribution among participants. Specifically, we outline the application of this sharing rule to P2P insurance with a (re-)insurer. We then generalize this application to the dynamic form of P2P insurance, which operates over a fixed multi-period horizon. Finally, we introduce the concept of benevolent loans to this dynamic P2P insurance model, highlighting the unique aspects of Mudarabah-Takaful. For this latter model, we study the solvency to ensure that shareholders are guaranteed the repayment of the loan over the horizon. In the third chapter of this thesis, a numerical example is provided to demonstrate the calculations.

# Chapter 1

## Difference between classical insurance, P2P insurance and Takaful insurance

As human beings, we are exposed to various types of risks. In principle, these risks represent deviations from our expectations, which can cause damage to ourselves and to our valuable possessions. In response, people, over a long period of history, have developed techniques to mitigate, transfer and manage these risks as discussed in this chapter. More specifically, in this chapter, we present three forms of insurance, namely classical insurance, P2P insurance and Takaful insurance by detailing the specific characteristics of each.

### 1.1 Introduction to Takaful insurance

Classical insurance is a mechanism of mutual and legal agreement whereby a customer, called the insured or policyholder, transfers a financial risk to an entity (insurance company or insurer) which provides compensation in the event of misfortune. For such an arrangement of risk cover, insured pays a particular amount of money known as a contribution (premium) to insurer and cover commences when an agreement is affected between the insured and the insurer. Feng et al. (2022) note that the roots of classical insurance date back to many early forms of risk sharing such as burial societies well documented in ancient Rome and Egypt, brotherly associations and religious guilds in Middle Ages Europe, etc.

However, the classical insurance developed in the West does not sit well with Muslim communities, as many Islamic scholars, those who study and interpret the "Quran" (the sacred scripture of Islam) and its "hadiths" (narratives), have deemed that it is unacceptable within the bounds of Islamic law, or sharia. In other words, classical insurance is considered sinful, or haram, as noted by Malik, Ullah, et al. (2019). In fact, there are three main sins outlined in Islam that classical insurance violates: gharar, maysir and riba.

### 1.1.1 Gharar in Insurance

Gharar prohibits any notion of risk or uncertainty in contracts, in contrast to modern insurance which advocates a transfer of risk from the insured to the insurer. In fact, modern insurance is a sale and purchase agreement in which the policyholder (the insured) buys a sum assured from the insurance company or insurer (the seller) for a premium. Neither party knows whether a loss will occur or, if it does, what the extent will be.

**Example 1.1.1.** Assume a person who buys full omnium insurance cover for a car worth Eur 80,000 and pays a single premium of Eur 10,000 to the insurance company. Malik, Ullah, et al. (2019) argue that neither the person nor the insurance company knows whether a loss will occur or, if it does, when the loss will happen and how much it will amount to. If a loss does not occur, the insured will lose their money and the insurer will make a profit. If the car is stolen, the insured will receive compensation of Eur 80,000 and will be in profit (as the insured gets Eur 80,000 more than they paid), while the insurer will suffer a loss. But this is not really the case. In fact, when a loss does not occur in classical insurance, the insurer uses this single premium to cover all claims incurred by other insureds in the portfolio (this method is called risk pooling). In contrary, if the car is stolen, the insurer does not suffer a loss but, uses the premiums paid by other insureds to cover this reported claim.

Moreover, in classical insurance, at the year of the agreement, the policyholder buys the deterministic against the uncertain and pays a premium in return. Therefore, this exchange is not in itself uncertain as the pioneers of Islamic insurance advocate. In other words, at the year of the agreement, the terms of the insurance policy are clearly defined, and the risk that the policy covers is known and deterministic. For example, if someone buys a car insurance policy, they know what risks are covered, such as theft, accidents, or damage to the car, and what events would trigger a payout from the insurance company. This makes the exchange between the policyholder and the insurance company certain and predictable.

Although the insurance mechanism is based on the uncertain, as Malik, Ullah, et al. (2019) point out, their interpretation remains questionable in the sense that the insurer accepts to take the uncertain risk based on risk pooling to return a deterministic guarantee and not the uncertain. Moreover, its main objective is to honour its commitment to the insured in terms of guarantees. This is also possible thanks to the investments that the insurance company makes.

### 1.1.2 Maysir in Insurance

Maysir refers to the prohibition of randomness or speculation. In other words, any game that has the same characteristics as gambling has been banned even if no money is involved. A situation is described in the following example:

**Example 1.1.2.** Consider a policyholder who buys one year of home insurance with Eur 400,000 of cover and pays an annual premium of Eur 50,000 to the insurance company. Malik, Ullah, et al. (2019) argue that if the insured's house is damaged (for case study, by fire) within that year, the company will compensate the insured up to Eur 400,000, depending on the extent of the damage and when a fire breaks out and the house is burned down, the company will pay Eur 400,000. For these authors, if there is no fire, the insured will lose their Eur

50,000. In practise for classical insurance, the insured will not lose, but their Eur 50,000 represent the risk price. They argue also that whether or not a fire breaks out is not within the control of the company or the insured and if it happens, the insured will receive a net benefit of Eur 350,000, and the insurance company will lose that amount. This interpretation is still far from what is commonly done in practice in traditional insurance. Indeed, although both parties do not know whether the loss will occur, nevertheless the insured finds it advantageous to buy a certain event (compensation) against an uncertain event (loss which is transferred to the insurer). Therefore, the insured does not gain EUR 350,000 in case of a claim, but rather remains immune to uncertain unfortunate events. The insurer does not lose this amount because he uses the premiums of the other insureds to indemnify (risk pooling). If the claim does not occur, this money does not remain as profit for the insurer but can be used to compensate other claims.

Thus policyholders make regular payments for an extended year of year to receive an immunization against risks and not for expecting benefits at the end of that year as mentioned by Malik, Ullah, et al. (2019). Then, it would be a mistake to consider classical insurance as a gambling. In fact, In classical insurance, the insured party is paying a premium in exchange for the guarantee that the insurance company will compensate them in case of a loss. The company is not taking a risk for its own gain, but rather providing a service for which it is compensated. This distinguishes it from gambling, which involves placing a bet on an uncertain outcome in the hopes of winning a prize. Also, gambling creates risk for the participants whereas insurance removes it.

Note that, commercial insurance (classical insurance) is offered by for-profit private insurance companies, while mutual insurance is offered by non-profit mutual companies. In addition, insurers in commercial insurance seek to make a profit, while members of a mutual company seek to share risks and help each other in years of need.

### 1.1.3 Riba in Insurance

Riba prohibits transactions leading to usury or the earning of interest, such as bank loans or interest from a savings account. In traditional insurance, once the premium is paid to the insurer, it is often invested in interest-bearing assets. However, these Islamic laws remain flexible regarding the idea of making donations with the interest earned on savings accounts. Let's consider the following example:

**Example 1.1.3.** Suppose an insurance company has received Eur 10 million in the form of premiums. Instead of leaving the premium payment not active until it is needed to pay a claim, the company invests it for a return. The insurance company needs a safe, good return on investments. This may be provided by non-permissible investments, such as offering interest bearing loans to banks and leasing companies or investing in the stocks of companies doing business that is not Shariah-compliant. By lending Eur 7 million to banks and leasing companies in the form of interest-bearing loans, the insurance company earns a return of Eur 1 million. This return is riba al Quran. The company places this amount into the claim fund and uses it to pay insureds when they claim for their insured losses.

In fact, even in Islamic insurance there is investment of part of the funds, although these are

in compliance with the Sharia. It should also be noted that in modern insurance, investments are also in compliance with the rules set by the regulator.

Since commercial insurance transgresses Islamic beliefs according to some Muslim scholars, secularists and other stakeholders, Takaful has emerged as an alternative to provide the same peace of mind and financial stability as modern insurance while complying with Sharia law. However, whether mutual insurance is considered Shariah-compliant or not depends on the specific details of how it is structured and operated. For example, if the mutual insurance company invests its funds in non-Shariah compliant activities or uses interest-based financing, then it would be considered non-compliant with Islamic principles.

#### 1.1.4 History of Takaful insurance

Takaful often referred to as "Islamic insurance", is a mechanism for companies and individuals (participants) to protect themselves from the financial consequences of misfortune. It is a pact between a group of individuals who agree to jointly compensate for loss or damage from a fund to which they collectively contribute and which is based on social solidarity and collaboration.

In practice, this alternative of insurance refers to participants making mutual contributions to a common fund with the purpose of providing mutual indemnity if certain events occur. To avoid conflation with insurers, there exists a fund manager Takaful (company) called Takaful operator. When a loss is incurred, participants use the money in the fund to pay off their losses. Takaful operator manages this fund and all account activities e.g. investment activities, redistribution of the fund to participants, etc (see Feng et al. (2022)). Where classical insurance requires its insureds to pay regularly premiums, Takaful participants make regularly donations to the Takaful fund and how much each participant contributes depends upon the type of cover that they want from the fund and how much risk they bring to the fund. In the event that the Takaful fund is not sufficient in order to pay off all the claims of the participants, the Takaful operator steps in to cover the deficit with an interest-free loan or the *qard hasan*. Translated from Arabic, *qard hasan* means a benevolent loan. This loan comes from the company's shareholders. Although the participants are not legally obligated to pay back this loan, any surplus from future years is used to pay back this loan to reimburse shareholders.

Archer et al. (2011) note that Takaful circumvents the issue with *gharar*, as it is not a bilateral contract between an insurer and a participant, in other words, all the money that goes towards paying losses is not money paid out from the Takaful operators. In addition, donations are made on the basis that nothing is expected in return, just like a donation to charity. These donations are then used to pay participants' claims; the operator does not bear the uncertainty<sup>1</sup> for the participant because each loss is divided among each participant, completing the risk-sharing process. No promises are sold between the Takaful operator and the participants. According to Wahab et al. (2007), Takaful operator just manages the funds and not making a commercial gain from the insurance activities. Profit for operators therefore should be from risk management and not risk taking. Similarly, in classical insurance, the

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<sup>1</sup>Uncertainty (*gharar*) in Takaful remains but is among only the participants whose interests are the same and hence permitted, whereas in modern insurance it is between the insurer and insured, whose interests are different.

insurer's profit comes from efficient risk management, the only difference being that the insurer assumes the risk of the insured, unlike in mutual insurance where the insured bear their own risk as in Takaful.

Since the Takaful fund belongs to all participants, and not to the operator. Also each participant is just donating for the benefit of all the pool members, even if he receives the extra money from his donation to cover his claim, it comes from the donations made by the other participants, and furthermore, the source of the fund is transparent. In this case, Takaful insurance addresses the issues of maysir.

As mentioned by Feng et al. (2022), Takaful addresses the issue of riba by investing in sharia compliant, or halal assets<sup>2</sup>. Sukuk are an example of such halal investment options. Sukuk are actually bonds backed by halal assets, offering fixed income payments at regular intervals. In conventional bonds, people who buy bonds receive a predetermined interest rate in all cases, which is usually paid annually within a fixed year frame from the companies for lending their money. However Ho (2016) explains that in sukuk, the only money someone receives in return comes from the profits generated by the company that takes their loans. Other types of Sharia compliant investments generally do not involve conventional finance, alcohol, non-halal food production, processing or packaging, gambling, adult entertainment, tobacco, or weapons, as described by Morgan (2021). They mentioned that, any the remaining fund after claims and operating fees are paid is returned to participants. This works as an incentive for the group to collectively have less claims, thereby reducing moral hazards in a manner similar that deductibles does in traditional insurance.

Nowadays, people can seek protection against risk through various types of Takaful products, Family Takaful schemes and General Takaful schemes. It is useful to try to differentiate between the two main products being followed in different parts of the world and the concerns that Sharia scholars have raised.

Family Takaful is a kind of life insurance, which is based on Sharia principles. While General Takaful Schemes are divided into various types, such as fire, miscellaneous and accident, marine, engineering and motor Takaful. It is also important to note that the basic variants are based on three fundamental approaches which are the Mudarabah, the Wakalah and the Hybrid model. These models are called For-Profit Takaful models. A blending of models and inclusion of other Sharia permitted contracts and elements also exist in different contracts. Further variations between different models followed in practice by different operators relate primarily to charging of expenses (marketing versus administration) and the basis of the fee structure, especially with regard to sharing in the underwriting profits as an attractive point for the operator.

## 1.2 Profit-sharing Takaful Models

The profit-sharing models are the models have been developed and that retain the provisions of compensation and profit for the Takaful service providers. These models are essentially based on Wakalah and Mudarabah contracts.

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<sup>2</sup>Investment cannot be made into forbidden assets/illicit activity (e.g. drugs, alcohol, pork, gambling, sex).

### 1.2.1 Mudarabah

The Mudarabah model is based on a type of partnership in Islamic business of which one business partner provides the capital and is called the *rab ul maal* (the capital owner or participants) and the other provides the skill and is called the *mudarib* (the skills provider or shareholders of Takaful operator).

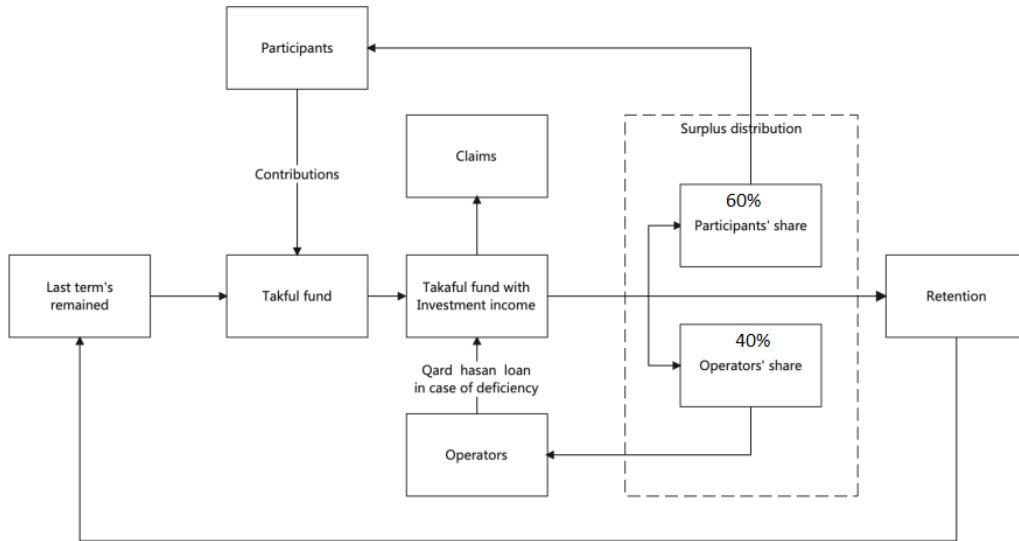


Figure 1.2.1: Mudarabah Takaful model

In case of a profit, the partners divide this between them in accordance with previously agreed profit-sharing ratio. If there is a loss, the participants must bear it through an interest-free loan provided by the shareholders of Takaful operator. In this model, the Takaful operator receives nothing for the skills and services it has provided. The underlying rationale is that the profit sharing as opposed to interest is used to compensate the operator for its willingness to offer an interest-free loan whenever it is needed. According to Wahab et al. (2007), it is commonly a 60 : 40 split, with 60% going to the participants and 40% to shareholders and operators. According to Feng et al. (2022), figure 1.2.1 shows the Mudarabah model and allows one to follow this process from the beginning of the term to the end.

### 1.2.2 Wakalah Takaful model

In the Wakalah model, the process differs slightly from the Mudarabah model in sense that, the Takaful operator receives an early fixed amount (Wakalah fees) from the Takaful fund well before the claims are settled and not in the event of a surplus, in order to provide an interest-free loan when the fund is insufficient. Participants here are entitled to their share of the surplus after claims are paid and the reserve funds are maintained for the next year. This model gives more priority to covering participants than to realizing profits for themselves. Figure 1.2.2 proposed by Feng et al. (2022) shows the Wakalah model and allows one to follow this process from the beginning of the term to the end.

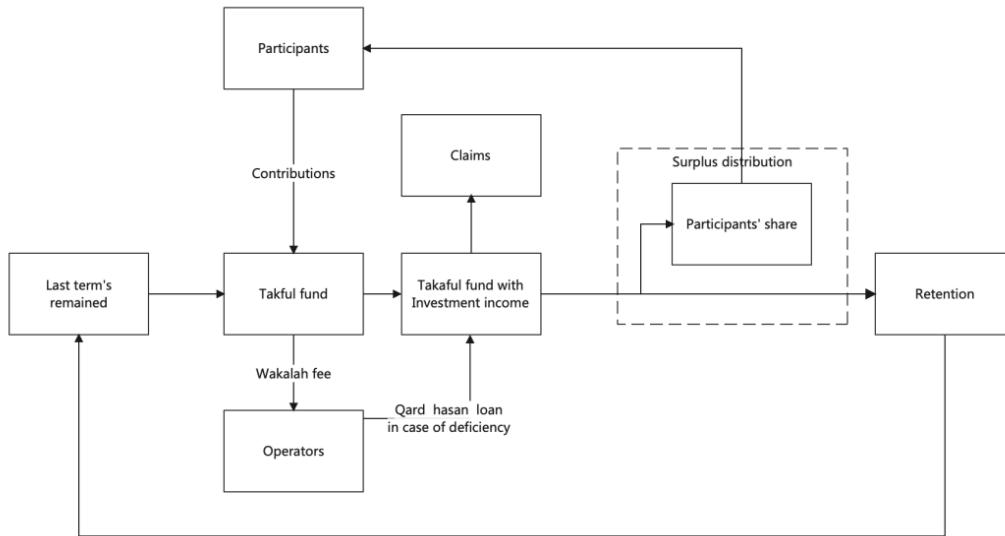


Figure 1.2.2: Wakalah Takaful model

### 1.2.3 Hybrid-Takaful model

The hybrid model is a combination of the two prior business models which utilizes both the profit-sharing ratio and the Wakalah fees. The operator receives both the Wakalah fixed fees and a share of the surplus on a Mudarabah basis. In fact, he receives the Wakalah fee prior to total contribution to provide an interest-free loan (in benevolent loan fund allocated by shareholders) when the Takaful fund becomes insufficient to cover claims. Then, after claim payments and loan repayment the participants receive a share (as a percentage) of the surplus and the balance of this surplus belongs to the shareholders who initiate the Takaful fund. Our hybrid model assume two thresholds respectively for the loan repayment and the surplus distribution between participants and shareholders. The Example 1.2.1 describes well the Hybrid-Takaful model and allows one to follow the process of Hybrid-Takaful model:

**Example 1.2.1.** Consider an insurance that explain the case study:

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Total contributions collected by the operator:	EUR 300 million (A)
Wakalah fee 10% of the Takaful fund:	EUR 30 million (B)
Total left in the Takaful fund:	$(A - B) =$ EUR 270 million (C)
Total of interest-free loan :	EUR 10 million (E)
Profit ratio:	40% to the Shareholders and 60% to the Takaful fund (Participants)
<b>In Case of Surplus</b>	
Total incurred claims:	EUR 175 million (F)
Surplus:	$(C - F) =$ EUR 95 million (G)
loan repayment:	$\min(E; \max(G; 0)) =$ EUR 10 million (H)
Threshold for surplus distribution:	EUR 80 (I) million
Surplus to distribute:	$(G - H - I) =$ EUR 5 million (J)
The surplus belongs to Shareholders	40% of EUR 5 million = 2 million (K)
Surplus belongs to the participants	$(J - K) =$ EUR 3 million (M)
Reserve for Takaful fund:	$\min(I; G - H) =$ 80 million
<b>In Case of Deficit</b>	
Total incurred claims:	EUR 285 million (N)
No Threshold for loan repayment and surplus	
Deficit:	$(C - J) = -15$ million

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In case of deficit, the shareholders of the Takaful operator provide an interest-free loan of EUR 15 million to the Takaful fund, which is used to pay the claims.

### 1.2.3.1 Mechanism of interest-free loan in Takaful

As Puspita et al. (2020) noted, when the Takaful fund experiences a deficit, two options are available. The first option is to charge participants additional contributions to cover the deficit. However, this practice is not popular due to its lack of commercial feasibility (Tolefat and Asutay (2013)). The second option is the interest-free loan (benevolent loan). The benevolent loan is provided by shareholders as short term solvency resources to the participants' fund. This loan is a part of the shareholders' capital that is specially earmarked to maintain the solvency of the participants' fund. In that option, Puspita et al. (2020) proposed an approach which consists for Takaful operator to provide an interest-free loan. Unlike in classical insurance,

where loans with interest are common, the loan repayment arrangement in Takaful allows the borrower to repay the loan only if they generate a positive surplus in the future. In classical insurance, the borrower is obliged to repay the loan, including interest, when it exceeds a certain level or loan capacity (see Kim and Drekić (2016)). According to regulations from the Islamic Financial Services Board (IFSB)<sup>3</sup> IFSB (2010), if the borrower cannot repay the loan, the lender cannot enforce repayment and must accept the transaction as an act of charity. It is worth noting that in conventional and mutual insurance practice, the reinsurance option is the most widely applied.

For this proposed approach, it is assumed that the benevolent loan will be repaid from future surpluses. Additionally, any undrawn benevolent loan will be invested to enhance the facility.

### 1.2.3.2 Mechanism of surplus distribution

If there is a balance beyond the surplus trigger point in the Takaful fund, the surplus is either returned to participants as cash back or used to reduce contribution amounts for the following term. The reserve generated by the surplus trigger point in the Takaful fund is used to repay the loan. There are generally three different methods for surplus distribution: pro rata, selective, and offsetting. The following descriptions of the three main surplus distribution methods are based on the outline provided by Soualhi (2016).

1. **Pro rata:** Under the pro rata surplus distribution method, each participant has equal rights to the surplus, regardless of whether a claim was made in that term. In the case that all participants make equal contributions, each participant receives the same amount of money back.
2. **Selective:** Under the selective surplus distribution method, only participants who did not make a claim, or survivors, would have the privilege of receiving a share of the surplus. The surplus is shared among survivors based on the proportion each survivor has contributed to the fund.
3. **Offsetting:** It is arguable that participants with more claims should be given less surplus in the following years. With this method, all of the claims that have been made would be taken into account for eligibility of surplus distribution. Only when an individual's total contributions paid is larger than his/her total claim would he/she be eligible for surplus distribution.

## 1.3 P2P insurance

### 1.3.1 Introduction to P2P insurance

A peer-to-peer model (P2P insurance) in the insurance world involves individuals deciding to insure individuals who agree to insure each other without the intermediary of an insurer (without risk transfer). Participants share their respective losses. When claims occur the amount of losses is distributed among participants. As outlined by Denuit and Dhaene

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<sup>3</sup>IFSB: Islamic Financial Services Board Standard on Solvency Requirements for Takaful

(2022), similar risk-sharing mechanisms are prevalent in mutual insurance arrangements and partnerships among professionals such as lawyers, farmers, and physicians, who establish risk pools to protect themselves against professional risks. Risk sharing is also a common strategy for covering natural catastrophes and major industrial risks, such as those associated with nuclear plants, through dedicated funds or pools. In its pure form, P2P insurance does not transfer any risk to a partnering insurer, meaning there is no insurer to absorb the random differences between total premiums and total claims with their own capital. Let's look at the working of the pure model of P2P insurance using Example 1.3.1.

**Example 1.3.1.** Consider a group of ten individuals who decide to insure their home against fire by the other members of the group. By mutual agreement, each of them undertakes to compensate the person who is the victim of a fire. There is no need to pay an initial premium, the compensation would be made at the year of the fire and in proportion to the value of the houses. If all the houses are worth EUR 336,000 each, then each member including the victim will pay 10% or EUR 33,600. If one dwelling is worth EUR 168,000 and the others are worth EUR 336,000 then one individual will pay 5.3% ( $168,000/3,192,000$ ) of the loss and the others will pay 10.5% ( $336,000/3,192,000$ ).

In this context, proportional risk sharing ensures that each member's share of the total risk exposure is determined in proportion to their risk bring into the pool. This risk sharing avoids that pool members with a low hedging risk prefer to find a pool that is homogeneous to the nature of their risk. Although each member contributes according to the coverage they need, they also understand that they are not alone in bearing the potential financial burden of losses. So, it reduces the impact of individual losses.

This model therefore requires a high level of trust between participants. In fact, the contributions to be paid ex post are theoretically unlimited and remain unknown until the end of the year, and some participants may be unable or unwilling to pay their contributions in the event of extreme cost claims over this year. This is why in practice P2P insurance usually involves a transfer to a partner insurer.

As mentioned by Feng et al. (2022), the peer-to-peer insurance began its rise with the launch of an online platform in 2010 by a German start-up. Members form social networks, each with 10 members of family and friends with a common need for insurance coverage. They each contribute to a shared common fund. A portion of the fund is used to buy an excess-of-loss coverage from an insurer. Small claims are paid off the fund with the insurer stepping in to cover larger claims. When there are less claims than the total fund at the end of each year, the surplus would be returned to members. There are three kinds of P2P business models, whose dynamics are shown in the following section.

### 1.3.2 Mechanism of P2P insurance

As discussed in several papers, including Denuit and Dhaene (2021), the three models are known as the self-governing model, the broker model, and the carrier model. The self-governing model operates independently of the insurance market, while the broker and carrier models are integrated within the modern insurance market.

### 1.3.2.1 The self-governing model

This technology-based model allows participants to pool their risks, self-organize and self-administer. In practice, the entirety of the risk is only borne by the members of pool, without transfer to an existing insurance or reinsurance company. This model refers to pure P2P insurance model described in subsection 1.3.1. It succeeds in avoiding conflicts of interest between insurance companies and their policyholders and in reducing moral hazard and fraudulent claims thanks to participants' strong sense of community and of shared assets. A common trait of both self-governing model and reciprocal insurance exchange that differs from mutual insurers is that they have no separate legal personality but only organize indemnity agreements between participants.

As mentioned by Denuit and Robert (2021b), the self-governing model thus builds on the fundamental principle at the origin of insurance without the equity buffer provided by insurer's capital. In line with the sharing economy, the self-governing model exploits technological advances in social networking to implement the mutual aid principle at the heart of insurance. Even if the P2P insurance community can only cope with relatively cheap sums insured because of its generally limited risk-bearing capacity, it is interesting to formalize the self-governing model to make explicitly the move to the other two cases (broker and carrier models). This move can be seen as pure risk sharing progressively supplemented with guarantees. Since this model requires a high level of trust for the remaining unknown contributions to be paid at the end of the year. This is why an alternative approach have been defined.

### 1.3.2.2 The broker model

In the broker model, the organizer of the P2P insurance system acts as an intermediary and buys a protection on behalf of the participants whose liability is limited to the lower layer of the total losses. Whereas pure self-governing model does not include any guarantee, the broker model combines risk sharing of the lower layer with risk transfer of the higher layer. Figure 1.3.1 proposed by Feng et al. (2022) explains the process from the beginning to the end year.

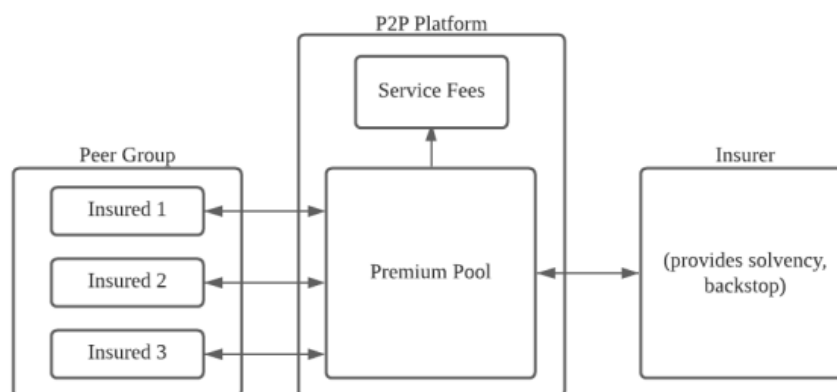


Figure 1.3.1: Broker model

In contrast to the self-governing model, participants 1, 2 and 3 contribute at the beginning

of the year on the basis that part of their contributions will be paid to the insurer as a risk transfer premium for a fixed threshold of claims costs to be borne by the pool. If there is a surplus at the end of the year, it is redistributed among the insured in the form of cash-back or given to a charity. Also, it has been defined a model where each of the participants can have her own insurer or re-insurer.

### 1.3.2.3 The carrier model

Note that in broker model (see Figure 1.3.1), all participants pay their contributions into the same pool which is covered by common insurer. In contrast, for the carrier model, enable participants to cover for each other losses below deductibles of their individual insurance policies. The main difference between the group-covered and individually-covered P2P models is that, members in the former collectively buy an insurance coverage for the funding pool, while members in the latter choose their own insurance policies to cover excessive losses.

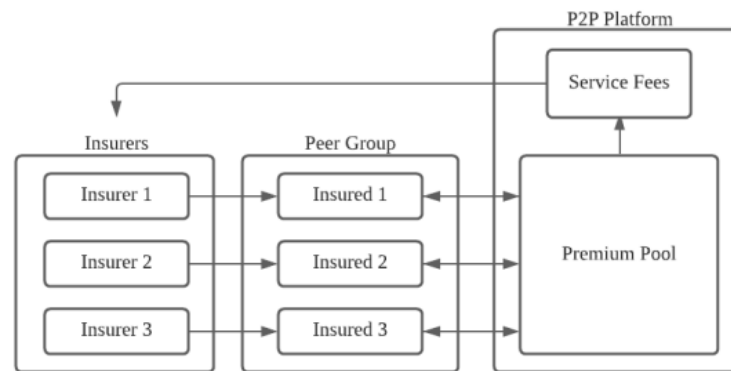


Figure 1.3.2: Carrier model

We observe that, in Figure 1.3.2 proposed by Feng et al. (2022), there could be multiple different insurers that cover their own insureds. There is only a single insurer for the entire pool in Figure 1.3.1. This approach embeds the broker model into a regular insurance policy including a participation to the surplus.

Despite these effectual changes in implementation, these three P2P business models is based on one fundamental idea of sharing risk among peers and it is similarly what Takaful model does among participants of a Takaful fund.

### 1.3.3 Comparative synthesis between Takaful, P2P and classical insurance

Since Takaful means a group of people who insure each other. The operation of Takaful companies is therefore similar to that of a P2P insurance company with some differences. Table 1.3.1 summarizes the points of divergence between the three insurance systems named.

	Takaful	P2P insurance	Classical insurance
Contract	Donation and/or mutual contract	mutual contract	Exchange contract
Responsibility of the company	Management of claims on funds collected; in case of insufficiency funds can borrow without interest	Management of claims on funds collected	Management of claims on funds collected; in case of insufficiency on own funds
Responsibility of the insureds	payment of contributions	payment of contributions	payment of premiums
Shareholders equity	Participant funds	The capital provided by the participants	shareholders capital
Conditions investment	Sharia compliant	No restrictions other than prudential	No restrictions other than prudential

Table 1.3.1: Comparison of models Takaful insurance, P2P insurance and classical insurance

As far as analogies are concerned, we can mention the following points: First, both Takaful and P2P insurance have a non-commercial purpose. Secondly, the status of partner-insured is found in both types of insurance because the participants are both insurers and insured. In addition, they both operate without share capital.

However, there are differences with Takaful insurance, which is based on the mutual insurance model. Firstly, Takaful companies cannot increase their funds in case of financial difficulties, as the only source of equity is retained profits, unlike P2P insurance where the contributions to be paid ex post are theoretically unlimited and remain unknown until the end of the year. Note that, in practice P2P insurance commonly includes some transfer to a partnering insurer or (re-)insurer and for this model the contributions are paid at the beginning of each year as in Takaful insurance. Secondly, if there is a balance beyond the surplus trigger point in the Takaful fund, the surplus is either returned to participants as cash back or used to reduce contribution amounts for the following term. It is the same case with the P2P insurance with (re-)insurer. In contrast, in the case of P2P insurance without (re-)insurer, it is not possible to have a surplus to pay out to participants since contributions are paid at the end of each year based on individual risks knowing the total cost of losses.

## Chapter 2

# Application of conditional mean risk sharing to Takaful insurance

Takaful is a concept of solidarity and cooperation (or a varied form of collaborative insurance) among participants, aiming to collectively mitigate risks and provide financial security. Risk sharing is a fundamental principle aimed at determining how costs and losses associated with claims will be distributed among all participants in the group. In recent years, the exploration of innovative risk-sharing mechanisms, such as conditional mean risk sharing (CMRS), and their application to collaborative insurance have sparked growing interest.

This chapter introduces CMRS as defined by Denuit and Dhaene (2012), which has the advantage of being transparent and relatively easy to communicate to participants as it is based on the familiar concept of the mean. Under this rule, each participant contributes to the pool by paying the conditional expectation of the losses they have incurred, considering the total loss incurred by the entire pool. We explore several key topics in this context, including, in one case, risk sharing within the framework of Peer-to-Peer (P2P) insurance over a fixed horizon, and in another case, the incorporation of benevolent loan options to enhance risk mitigation strategies. In this chapter we model the hybrid Takaful insurance and assume that P2P insurance refers to broker model.

### 2.1 Conditional mean risk sharing

Consider  $n$  economic agents (individuals or entities) with different risks, denoted by  $\mathbf{X} = (X_1, X_2, \dots, X_n)$ . No particular assumption is made about the distribution of the random vector  $\mathbf{X}$ .

Given a portfolio of risks represented by the random vector  $\mathbf{X} = (X_1, X_2, \dots, X_n)$ . A risk-sharing in  $\mathbf{X}$  is a two stage process. In the ex-ante step at year 0, the random losses comprised in the pool  $\mathbf{X}$  are re-allocated by transforming  $\mathbf{X}$  with a correspondence  $h$  into another random vector  $h(\mathbf{X})$  of the same dimension. The ex-post step takes place at year 1, at the moment that the realization of the pool  $\mathbf{X}$  is observed. At that year, each participant  $i$

to the pool  $\mathbf{X}$  receives the corresponding realization of his or her loss  $X_i$  from the pool. In return, each participant  $i$  contributes to the pool the realization of his or her re-allocated loss.

**Definition 2.1.1.** Given a portfolio of risks represented by the random vector  $\mathbf{X} = (X_1, X_2, \dots, X_n)$ . A risk-sharing rule (or risk-sharing allocation) is any mapping  $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$  that transforms a random vector  $\mathbf{X} = (X_1, X_2, \dots, X_n)$  into another of the same dimension  $(h_1(\mathbf{X}), h_2(\mathbf{X}), \dots, h_n(\mathbf{X}))$  where the (measurable) functions  $h_i$  are such that:

$$\sum_{i=1}^n h_i(\mathbf{X}) = \sum_{i=1}^n X_i \quad (2.1.1)$$

In the end, each agent will pay  $(h_1(\mathbf{x}), h_2(\mathbf{x}), \dots, h_n(\mathbf{x}))$  where  $\mathbf{x}$  is the observed realization of  $\mathbf{X}$ . The condition (2.1.1) is called the total risk allocation condition.

Denuit and Dhaene (2012) defined a particular risk-sharing rule that is appealing collaborative insurance.

**Definition 2.1.2. (Denuit and Dhaene)** A risk-sharing rule  $h$  is the **conditional mean risk-sharing rule** in case for any portfolio risks  $\mathbf{X}$  if the contribution for individual  $i$  is given by:

$$h_i(\mathbf{X}) = h_i(S) = \mathbb{E}[X_i|S] \quad i = 1, 2, \dots, n \quad (2.1.2)$$

Where  $S = \sum_{i=1}^n X_i$ .

The conditional mean risk-sharing rule is denoted  $h^*$  for the following Sections. Under  $h^*$ , each participant must contribute the expected value of the loss brought to the pool, given the total loss  $S$  experienced by the pool.

Denuit and Dhaene (2012) and Denuit, Dhaene, et al. (2006), provided several properties of the conditional mean risk-sharing rule that some are listed below:

**Definition 2.1.3. (Actuarial fairness property)** A risk-sharing rule  $h$  satisfies the actuarial fairness property (or is said to be an actuarial fair risk-sharing rule) if for any pool  $\mathbf{X}$  one has that:

$$\mathbb{E}[h_i(\mathbf{X})] = \mathbb{E}[X_i] \quad \text{for all } i \quad (2.1.3)$$

Actuarial fairness property means that on average, participants do neither gain nor lose from risk sharing, in the sense that their expected contribution to the total loss (by joining the pool) is equal to their expected loss (when staying alone). Apart from social insurance, when losses are heterogeneous, actuarial fairness property is generally considered as important requirement (desirable property) in insurance.

**Definition 2.1.4.** Given two random variables  $X$  and  $Y$ .  $X$  is said to be smaller than  $Y$  in the convex order denoted as  $X \preceq_{CX} Y$  if for any convex function  $g : \mathbb{R} \rightarrow \mathbb{R}$  one has that:

$$\mathbb{E}[g(X)] \leq \mathbb{E}[g(Y)] \quad (2.1.4)$$

Intuitively the Definition (2.1.4) means that  $X$  and  $Y$  have the same magnitude (as  $\mathbb{E}[X] = \mathbb{E}[Y]$  holds) but that  $Y$  is more variable than  $X$

**Definition 2.1.5. (Pareto Optimal property)** A risk-sharing rule  $h$  satisfies the Pareto Optimal property for a pool risk  $\mathbf{X}$  if there exists no risk-sharing rule  $h'$  such that:

$$h'_i(\mathbf{X}) \preceq_{CX} h_i(\mathbf{Y}) \quad \text{for all } i = 1, 2, \dots, n \quad (2.1.5)$$

Pareto Optimal property means that no agent can be made strictly better off (in the sense of convex order) without worsening the situation of another agent.

**Definition 2.1.6. (Comonotonicity property)** A  $n$ -dimensional random vector  $\mathbf{X}$  satisfies Comonotonicity property if there exist continuous and non-decreasing functions  $f_i : \mathbb{R} \rightarrow \mathbb{R}$  and a random variable  $Z$  such that:

$$f_1(z) + f_2(z) + \dots + f_n(z) = z \quad \text{for any } z \in \mathbb{R} \quad (2.1.6)$$

and

$$\mathbf{X} = (f_1(Z); f_2(Z); \dots; f_n(Z)) \quad (2.1.7)$$

In that case,  $\mathbf{X}$  is called comonotonic random variable. In particular, a comonotonic portfolio of risks  $\mathbf{X}$  always generates a risk-sharing vector. This explains why comonotonicity is so intimately connected to Pareto-optimality. Denuit and Dhaene (2012) prove the following result.

**Proposition 2.1.7.** *Any comonotonic risk-sharing rule  $(h_1(\mathbf{X}), h_2(\mathbf{X}), \dots, h_n(\mathbf{X}))$  of a portfolio of risks  $\mathbf{X}$  is Pareto-optimal.*

We note that the conditional mean risk-sharing rule  $h^*$  satisfies the actuarial fairness property and the Pareto Optimal property for comonotonic losses. For the comonotonicity property, some sufficient conditions for this property to hold can be provided for pools comprising independent losses, such as the log-concavity of the distributions of the losses  $X_i$  or the large-pool case (See Denuit and Robert (2021a,b)).

Denuit and Dhaene (2012) provided and proved an interesting property of the conditional mean risk sharing rule, which is shown to minimize the sum of the expected squared difference between the individual risks to be shared and the pooling arrangement. It is the following result.

**Proposition 2.1.8.** *The conditional mean risk-sharing rule  $h^*$  for portfolio of risks  $\mathbf{X}$  minimizes*

$$\sum_{i=1}^n \mathbb{E}[(X_i - h_i(S))^2] \quad (2.1.8)$$

over all the risk sharing rule  $h$ .

In other words, the risk share  $h_i^*(S)$  is "as close as possible" to the original risk  $X_i$  taking into account the full risk-sharing rule condition, in the sense of the quadratic distance.

## 2.2 Sharing losses within the P2P insurance

### 2.2.1 Description of the Business model

The model that we use here is similar to Broker model. Let consider an heterogeneous risk pool of  $n$  participants P2P insurance. Assume  $X_i$  to be the risk brought into the pool by participant  $i$ , that represents the individual claim severity (cost of claim). Risks  $(X_1, X_2, \dots, X_n)$  are assumed to be non-negative and independent random variables. Therefore, the total cost of claims is given by  $S = \sum_{i=1}^n X_i$ . According to Denuit (2020), P2P insurance splits the risks into several layers, each managed in a specific way (individually by each participant, collectively at the community's level and transferred to a (re-)insurer). Given a real number  $x$ ,  $[x]_+$  denote the positive part of  $x$ . The organizer of P2P insurance system acts to limit the liability of participants to the lower layer  $(0, \omega)$  of the total losses  $S$  and buys a stop-loss protection transferring  $[S - \omega]_+$  to a (re-)insurer. The model assumes there is a possible surplus  $[\omega - S]_+$  that can be shared among participants or given to a charity at the end of year.  $\omega$  is also called the priority of stop-loss arrangement bought for protection. In other words, the community only retains:

$$S^{P2P} = \min\{S, \omega\} \quad (2.2.1)$$

and the total losses of pool transferred to the (re-)insurer is defined by:

$$L = [S - \omega]_+ \quad (2.2.2)$$

The total amount of contribution  $\omega_i$  paid by participant  $i$  is decomposed into:

$$\omega_i = \omega_i^{P2P} + \omega_i^{SL} \quad (2.2.3)$$

supplemented with management expenses (including possible taxes) where  $\omega_i^{P2P}$  is that part of the total contribution paid at the beginning of each year by participant  $i$  covering the lower layer  $(0, \omega)$  shared within the P2P community which satisfies the relation:

$$\omega = \sum_{i=1}^n \omega_i^{P2P} \quad (2.2.4)$$

and  $\omega_i^{SL}$  is the individual contribution to the price of the stop-loss protection, such that:

$$(1 + \beta_{SL})\mathbb{E}[L] = \sum_{i=1}^n \omega_i^{SL} \quad (2.2.5)$$

where  $\beta_{SL}$  is the corresponding loading to obtain the premium charged by the (re-)insurer covering the upper layer  $(\omega, \infty)$ .

Denuit (2020) argued that, the adjustment of the value of  $\omega$  should satisfy the following conditions:

1.  $\omega$  is determined such a way that,  $P[S < \omega]$  is sufficiently high so that a surplus  $\omega - S$  can be distributed sufficiently often.
2. The value of  $\omega$  also has an impact on  $\omega_i^{P2P}$  and  $\omega_i^{SL}$  so it should be chosen in such a way as to make participation in the P2P insurance scheme affordable and attractive to potential participants.

### 2.2.2 Retention level

A number of strategic and financial considerations need to be taken into account when determining the retention  $\omega$  of the pool. This value can be selected so that the probability  $F_S(\omega)$  defined by:

$$F_S(\omega) = P[S \leq \omega] \quad (2.2.6)$$

is high enough. This is the probability that a surplus  $[\omega - S]_+$  will be granted at the end of the year to the participants. The value of  $\omega$  has an impact on the values of  $\omega_i^{P2P}$  and  $\omega_i^{SL}$ . It must therefore be chosen so that participation in the system is sufficiently attractive to encourage new members to pay  $\omega_i$  rather than the premium demanded by an insurer for classical insurance coverage.

Let  $\varepsilon$  denote the target probability for which a surplus is obtained at the end of the horizon, then the retention  $\omega$  corresponding to this probability  $\varepsilon$  is given by:

$$\varepsilon = F_S(\omega) \iff \omega = F_S^{-1}(\varepsilon) \quad (2.2.7)$$

The chosen retention level is proportional to the quantile of the  $S$  distribution corresponding to the probability  $\varepsilon$ .

### 2.2.3 Contribution to the lower layer

According to Denuit (2020), to determine the amount of individual contribution  $\omega_i^{P2P}$  meant to absorb the pooled part of the losses, the following assumption is made:

$$s \longrightarrow h_i^*(s) = \mathbb{E}[X_i | S = s] \quad \text{non-decreasing for every } i \quad (2.2.8)$$

In theory, this assumption can be checked numerically. It is generally satisfied when the group is large enough, or it satisfies some additional conditions (such as semi-homogeneity). This ensures that the risk sharing is comonotonic and hence Pareto-optimal (see Proposition 2.1.7). In practice, this can be checked graphically, displaying the  $n$  conditional expectations as a function of  $s$ , in case the validity of Assumption (2.2.8) is not supported by theoretical arguments. We define the quantities  $\omega_1^{P2P}, \omega_2^{P2P}, \dots, \omega_n^{P2P}$  such that:

$$\omega_i = F_{h_i^*(S)}^{-1}(F_S(\omega)) = h_i^*(\omega) = \mathbb{E}[X_i | S = \omega] \quad (2.2.9)$$

$\omega_i$  is the individual retention level at each year that must be defined to be able to guarantee a bonus to participants at the end of year with probability  $\varepsilon$ . It is the maximum contribution that a participant can make to support the cover of the lower layer. By using Relation (2.2.9) and the total allocation property of conditional mean risk sharing  $h_i^*$ , we can verify that the set of individual contributions  $\omega_i$  is sufficient to cover the retention  $\omega$  at pool level. That is,

$$\sum_{i=1}^n \omega_i = \sum_{i=1}^n h_i^*(\omega) = \omega \quad (2.2.10)$$

Since the functions  $h_i^*$  are continuous and strictly increasing, we have:

$$S \geq \omega \iff h_i^*(S) \geq h_i^*(\omega) = \omega_i \quad \text{for } i = 1, 2, \dots, n \quad (2.2.11)$$

We can therefore write,

$$[S - \omega]_+ = \left[ \sum_{i=1}^n h_i^*(S) - h_i^*(\omega) \right]_+ \quad (2.2.12)$$

$$= \sum_{i=1}^n [h_i^*(S) - h_i^*(\omega)]_+ \quad (2.2.13)$$

$$= \sum_{i=1}^n [h_i^*(S) - \omega_i^{P2P}]_+ \quad (2.2.14)$$

The similar reasoning applies to the decomposition of surplus  $[\omega - S]_+$  into the sum of random variables  $[\omega_i^{P2P} - h_i^*(S)]_+$ . In this way, we can determine how to distribute the surplus when there is one, as well as the contribution of each participant to cover the retained loss  $\min\{S, \omega\}$ . Indeed,

$$\begin{aligned} \min\{S, \omega\} &= \omega - [\omega - S]_+ \\ &= \sum_{i=1}^n \left( \omega_i^{P2P} - [\omega_i^{P2P} - h_i^*(S)]_+ \right) \\ &= \sum_{i=1}^n \min\{h_i^*(S), \omega_i^{P2P}\} \end{aligned}$$

So, each participant  $i$  contributes to the retained loss  $\min\{S, \omega\}$  an amount  $\min\{h_i^*(S), \omega_i^{P2P}\}$  where the total individual retention levels  $\omega_i^{P2P}$  until the horizon is given by Formula (2.2.9).

### 2.2.4 Surplus distribution

The surplus that can be shared among participants or given to a charity is defined by:

$$B^{P2P} = [\omega - S]_+ \quad (2.2.15)$$

If the community decides to distribute  $B^{P2P}$  among participants then the benefit  $B_i^{P2P}$  paid to participant  $i$  is given by:

$$B_i^{P2P} = \begin{cases} 0 & \text{if } S \geq \omega \\ \omega_i^{P2P} - h_i^*(S) & \text{if } S < \omega \end{cases} \quad (2.2.16)$$

The random variables  $B_i^{P2P}$  have a zero-augmented distributions with  $P[B_i^{P2P} = 0] = P[S \geq \omega]$ . The relation (2.2.9) and the properties of the functions  $h_i^*$  ensure that for any total amount of losses  $s$  at the end of the year such that  $s \leq \omega$ , the individual contribution paid by participant  $i$  is always less than or equal to  $\omega_i^{P2P}$ . That is,

$$h_i^*(s) \leq h_i^*(\omega) = \omega_i^{P2P} \quad \text{for all } s \leq \omega \implies B_i^{P2P} \geq 0 \quad \text{for all } i \quad (2.2.17)$$

So, no additional contribution is claimed from the participant  $i$  at the end of the year.

If  $S < \omega$ , then participant  $i$  receives a bonus equal to  $B_i^{P2P} = \omega_i^{P2P} - h_i^*(S)$ . Each participant must then pay at the beginning of each year the maximal amount of contribution  $\omega_i^{P2P}$  he or she is exposed and the cash-back mechanism operates at the end of the year in order to redistribute the surplus between members of the pool.

### 2.2.5 Contribution to the upper layer

The stop-loss premium for  $S$  can be distributed evenly among participants or based on their individual shares in the upper layer of losses. Let us now explain how to determine the amount of individual contribution  $\omega_i^{SL}$  to pay for transferring the stop-loss to (re-)insurer. According to Relation (2.2.14), this stop-loss premium corresponding to the upper layer  $(\omega, \infty)$  can be rewritten as:

$$\mathbb{E}[(S - \omega)_+] = \mathbb{E} \left[ \sum_{i=1}^n [h_i^*(S) - \omega_i^{P2P}]_+ \right] = \sum_{i=1}^n \mathbb{E} [h_i^*(S) - \omega_i^{P2P}]_+ \quad (2.2.18)$$

By using Equations (2.2.14) and (2.2.18), participant  $i$  then contributes:

$$\omega_i^{SL} = (1 + \beta_{SL}) \mathbb{E} [h_i^*(S) - \omega_i^{P2P}]_+ \quad (2.2.19)$$

## 2.3 Sharing losses within the P2P insurance over an horizon

### 2.3.1 Description of the Business model

We consider here a risk model similar to the model of P2P insurance described above. We assume now a dynamic model that includes several years  $t = 1, 2, \dots, \tau$ . Consider an heterogeneous risk pool of  $n$  participants and we assume that the pool is stable over the  $\tau$  years. Formally, we denote as  $\mathbf{X}_t = (X_{1t}, X_{2t}, \dots, X_{nt})$  the experience for year  $t$  and assume that the random vectors are independent and identically distributed (i.i.d). Therefore, the total cost of claims in year  $t$  is given by:

$$S_t = \sum_{i=1}^n X_{it}. \quad (2.3.1)$$

We adopt the damage insurance approach to represent the total cost per participant and we assume that the each individual claim amount  $X_{it}$  over the year  $t$  splits up in  $N_{it}$  partial amounts  $C_{it;1}, C_{it;2}, \dots, C_{it;N_{it}}$ , that is

$$X_{it} = \sum_{k=1}^{N_{it}} C_{it;k}, \quad (2.3.2)$$

where the claims number  $N_{it}$  for year  $t$  is Poisson with parameter  $\lambda_i$ . We suppose that the random variables  $C_{it;k}$  are i.i.d. with the identical distribution to total claim amount  $C_i$  produced by risk  $i$  (when at least one claim has been reported to the insurer). We assume that the random variables  $C_{it;k}$  take discrete values. This is generally the case in actuarial calculations where claims costs are expressed in monetary units after being discretised. All variables  $C_{it;k}, N_{it}$  are independent. Then, the total cost of claims for the pool can be rewritten as:

$$S_t = \sum_{i=1}^n \sum_{k=1}^{N_{it}} C_{it;k}. \quad (2.3.3)$$

We denote

$$Z_t = \sum_{j=1}^t S_j \quad (2.3.4)$$

the total loss of the pool until year  $t$ . The organizer acts to limit the liability of participants for each year to the constant lower layer  $(0, \omega)$  of the total losses  $S_t$  and buys a stop-loss protection transferring  $[S_t - \omega]_+$  to a (re-)insurer. The model assumes there is a possible surplus  $[\omega - S_t]_+$  that can be shared among participants by reducing their individual contributions for the next year. We denote  $B_{i,t}^{P2P}$  the individual surplus for participant  $i$  at the end of year  $t$ . The community only retains:

$$S_t^{P2P} = \min\{S_t, \omega\} \quad (2.3.5)$$

and the total losses of pool transferred to the (re-)insurer is defined by:

$$L_t = [S_t - \omega]_+ \quad (2.3.6)$$

The total amount of contribution  $W_{i,t}$  exactly paid by participant  $i$  at year  $t$  is then decomposed into:

$$W_{i,t} = \omega_i^{P2P} + \omega_{i,t}^{SL} - B_{i,t-1}^{P2P} \quad \text{for } t = 2, 3, \dots, \tau \quad \text{and} \quad \omega_{i,1} = \omega_i^{P2P} + \omega_{i,1}^{SL} \quad (2.3.7)$$

where  $\omega_i^{P2P}$  is that part of the total contribution paid at the beginning of each year  $t$  by participant  $i$  which satisfies the Relation (2.2.4),

$\omega_{i,t}^{SL}$  is the individual contribution to the price of the stop-loss protection for year  $t$ , such that:

$$(1 + \beta_{SL})\mathbb{E}[L_t] = \sum_{i=1}^n \omega_{i,t}^{SL} \quad (2.3.8)$$

where  $\beta_{SL}$  is defined in section 2.2.1 and  $B_{i,t}^{P2P}$  the cash-back redistributed to participant  $i$  at the end of year  $t$ , that satisfies:

$$[\omega - S_t]_+ = \sum_{i=1}^n B_{i,t}^{P2P} \quad (2.3.9)$$

### 2.3.2 Retention level without cash-back before the end of horizon

Firstly, we assume that there is no any cash-back before the year  $\tau$ . We aim to select the constant retention  $\omega$  cover all the years until  $\tau$  such that some surplus emerges at year  $\tau$  and we use the conditional mean risk sharing to determine the amount of individual contribution  $\omega_i^{P2P}$  meant to absorb the pooled part of the losses at each year and/or their each accumulated cash-back  $\hat{B}_{i,\tau}^{P2P}$  at the end of horizon  $\tau$ . We define the probability  $F_{Z_\tau}(\tau\omega)$  that a surplus emerges at the of horizon with a total retention levels  $\tau\omega$  over the  $\tau$  years.

#### 2.3.2.1 Retention level

The choice of the retention level  $\omega$  corresponding to each year must take into account the following strategies and financial considerations: the guarantee of a surplus to be redistributed among the participants at the horizon  $\tau$ . This value can be selected so that the probability  $F_{Z_\tau}(\tau\omega)$  defined by:

$$F_{Z_\tau}(\tau\omega) = P[Z_\tau \leq \tau\omega] \quad (2.3.10)$$

is high enough. This is the probability that a surplus  $[\tau\omega - Z_\tau]_+$  will be granted at the end of the horizon to the participants. The value of  $\omega$  has an impact on the values of  $\omega_i^{P2P}$ ,  $\omega_{i,t}^{SL}$  and  $\hat{B}_{i,\tau}^{P2P}$ . It must therefore be chosen so that participation in the system is sufficiently attractive to encourage new members to pay  $\omega_{i,t}$  rather than the premium demanded by an insurer for classical insurance coverage.

Let  $\varepsilon$  denote the target probability for which a surplus is obtained at the end of the horizon, then the retention  $\omega$  corresponding to this probability  $\varepsilon$  is given by:

$$\varepsilon = F_{Z_\tau}(\tau\omega) \iff \tau\omega = F_{Z_\tau}^{-1}(\varepsilon) \iff \omega = \frac{1}{\tau}F_{Z_\tau}^{-1}(\varepsilon) \quad (2.3.11)$$

The chosen retention level is proportional to the quantile of the  $Z_\tau$  distribution corresponding to the probability  $\varepsilon$ .

**Example 2.3.1.** In particular, for  $\tau = 3$  we obtain;

$$\varepsilon = F_{Z_3}(3\omega) \iff 3\omega = F_{Z_3}^{-1}(\varepsilon) \iff \omega = \frac{1}{3}F_{Z_3}^{-1}(\varepsilon) \quad (2.3.12)$$

Through this example, we tune level of retention for each year in order to obtain a potential surplus in the third year amounts by considering the case of P2P insurance over a total year of three years. The total retention obtained is then shared equally over the three years.

### 2.3.2.2 Contribution to the lower layer

As the retention level  $\omega$  of the pool can be determined according to the pool's objectives, it now remains to define how the individual contributions  $\omega_i$  will be calculated so that the pool can be in a position to amortise this part of the total cost constituting the lower layer. We denote  $\omega_{i,\tau}^{P2P}$  the total individual contribution of participant  $i$  until the year  $\tau$  to cover the lower layer  $(0, \tau\omega)$  and denote  $\omega_{i,\tau}^{SL}$  the total individual contribution of participant  $i$  until the year  $\tau$  to the price of the stop-loss protection  $(1 + \beta_{SL})[Z_\tau - \tau\omega]$ .

We assume that the pool is sufficiently large over the horizon and that at year  $t$  the functions  $s \mapsto \mathbb{E}[Z_{it}|Z_t = s]$  are continuously increasing for all participant  $i$  with  $Z_{it}$  the total claim amount of participant  $i$  until the year  $t$ . The pool's total claim amount can be rewritten as:

$$Z_t = \sum_{i=1}^n Z_{it} \text{ and } Z_{it} = \sum_{j=1}^t X_{ij}. \quad (2.3.13)$$

This requirement guarantees that the conditional mean risk sharing  $h_{i,Z_t}^*(s) = \mathbb{E}[Z_{it}|Z_t = s]$  is comonotonic and therefore Pareto-optimal. We define the quantities  $\omega_{1,\tau}^{P2P}, \omega_{2,\tau}^{P2P}, \dots, \omega_{n,\tau}^{P2P}$  such that:

$$\omega_{i,\tau}^{P2P} = F_{h_{i,Z_\tau}^*(Z_\tau)}^{-1}(F_{Z_\tau}(\tau\omega)) = h_{i,Z_\tau}^*(\tau\omega) = \mathbb{E}[Z_{i\tau}|Z_\tau = \tau\omega] \quad (2.3.14)$$

$\omega_i^{P2P}$  is the individual retention level at each year that must be defined to be able to guarantee a bonus to participants at the end of the year  $\tau$  with probability  $\varepsilon$ . It is the maximum contribution that a participant can make to support the cover of the lower layer. We aim to define the quantities  $\omega_1^{P2P}, \omega_2^{P2P}, \dots, \omega_n^{P2P}$  such that:

$$\sum_{t=1}^{\tau} \omega_i^{P2P} = \omega_{i,\tau}^{P2P} \quad (2.3.15)$$

So, we define the individual retention level  $\omega_i^{P2P}$  at each year for all participant  $i$  by:

$$\omega_i^{P2P} = \frac{1}{\tau} \omega_{i,\tau}^{P2P} = \frac{1}{\tau} \mathbb{E}[Z_{i\tau} | Z_\tau = \tau\omega] \quad (2.3.16)$$

By using Relation (2.3.16) and the total allocation property of conditional mean risk sharing  $h_{i,Z_\tau}^*$ , we can verify that the set of individual contributions  $\omega_i^{P2P}$  is sufficient to cover the retention  $\omega$  at pool level. That is,

$$\sum_{i=1}^n \omega_i^{P2P} = \frac{1}{\tau} \omega_{i,\tau}^{P2P} = \frac{1}{\tau} \sum_{i=1}^n h_{i,Z_\tau}^*(\tau\omega) = \frac{1}{\tau} \tau\omega = \omega \quad (2.3.17)$$

Since the functions  $h_{i,Z_t}^*$  are continuous and strictly increasing, we have:

$$Z_\tau \geq \tau\omega \iff h_{i,Z_\tau}^*(Z_\tau) \geq h_{i,Z_\tau}^*(\tau\omega) = \omega_{i,\tau}^{P2P} \quad \text{for } i = 1, 2, \dots, n \quad (2.3.18)$$

We can therefore write,

$$[Z_\tau - \tau\omega]_+ = \left[ \sum_{i=1}^n h_{i,Z_\tau}^*(Z_\tau) - h_{i,Z_\tau}^*(\tau\omega) \right]_+ \quad (2.3.19)$$

$$= \sum_{i=1}^n \left[ h_{i,Z_\tau}^*(Z_\tau) - h_{i,Z_\tau}^*(\tau\omega) \right]_+ \quad (2.3.20)$$

$$= \sum_{i=1}^n \left[ h_{i,Z_\tau}^*(Z_\tau) - \omega_{i,\tau}^{P2P} \right]_+ \quad (2.3.21)$$

The similar reasoning applies to the decomposition of surplus  $[\tau\omega - Z_\tau]_+$  into the sum of random variables  $[\omega_{i,\tau}^{P2P} - h_{i,Z_\tau}^*(Z_\tau)]_+$ . In this way, we can determine how to distribute the surplus when there is one, as well as the contribution of each participant to cover the retained loss  $\min\{Z_\tau, \tau\omega\}$ . Indeed,

$$\begin{aligned} \min\{Z_\tau, \tau\omega\} &= \tau\omega - [\tau\omega - Z_\tau]_+ \\ &= \sum_{i=1}^n \left( \omega_{i,\tau}^{P2P} - \left[ \omega_{i,\tau}^{P2P} - h_{i,Z_\tau}^*(Z_\tau) \right]_+ \right) \\ &= \sum_{i=1}^n \min\{h_{i,Z_\tau}^*(Z_\tau), \omega_{i,\tau}^{P2P}\} \end{aligned}$$

So, each participant  $i$  contributes to the retained loss  $\min\{Z_\tau, \tau\omega\}$  an amount  $\min\{h_{i,Z_\tau}^*(Z_\tau), \omega_{i,\tau}^{P2P}\}$  where the total individual retention levels  $\omega_{i,\tau}^{P2P}$  until the horizon is given by Formula (2.3.14).

### 2.3.2.3 Surplus distribution

If a surplus  $\widehat{B}_\tau = [\tau\omega - Z_\tau]_+$  is available at the end of the horizon, then the cash-back will be redistributed to the participants by reducing their individual contributions over the next horizons or given to a charity. If we denote  $\widehat{B}_{i,\tau}$  the cash-back redistributed to participant  $i$ , then we have:

$$\widehat{B}_{i,\tau} = \begin{cases} 0 & \text{if } Z_\tau \geq \tau\omega \\ \omega_{i,\tau}^{P2P} - h_{i,Z_\tau}^*(Z_\tau) & \text{if } Z_\tau < \tau\omega \end{cases} \quad (2.3.22)$$

The random variables  $\widehat{B}_{i,\tau}$  have a zero-augmented distributions with  $P[\widehat{B}_{i,\tau} = 0] = P[Z_\tau \geq \tau\omega]$ . Thus the cumulative distribution function of random variabe  $[\widehat{B}_\tau | \widehat{B}_\tau > 0]$  denoted  $F_{\widehat{B}_\tau}$  is given by:

$$F_{\widehat{B}_\tau}(k) = \frac{P[t\omega - Z_t > 0] - P[t\omega - Z_t > k]}{P[\omega - Z_t > 0]} \quad (2.3.23)$$

$$= \frac{F_{Z_t}(\omega) - F_{Z_t}(\omega - k)}{F_{Z_t}(\omega)}, \quad (2.3.24)$$

The relation (2.3.14) and the properties of the functions  $h_{i,Z_\tau}^*$  ensure that for any total amount of losses  $s$  at the end of the year  $\tau$  such that  $s \leq \tau\omega$ , the total contribution at the end of the horizon paid by participant  $i$  is always less than or equal to  $\omega_{i,\tau}^{P2P}$ . That is,

$$h_{i,Z_\tau}^*(s) \leq h_{i,Z_\tau}^*(\tau\omega) = \omega_{i,\tau}^{P2P} \quad \text{for } s \leq \tau\omega \implies \widehat{B}_{i,\tau} \geq 0 \quad \text{for all } i \quad (2.3.25)$$

So, no additional contribution is claimed from the participant  $i$  at the end of the horizon.

If  $Z_\tau < \tau\omega$ , then participant  $i$  receives a bonus equal to  $\widehat{B}_{i,\tau} = \omega_{i,\tau}^{P2P} - h_{i,\tau}^*(Z_\tau)$ . Note that,  $B_{i,t} = 0$  for each year  $t$  until the year  $\tau - 1$  and  $B_{i,\tau} = \widehat{B}_{i,\tau}$ . Each participant must then pay at the beginning of each year  $t = 1, 2, \dots, \tau$  the maximal constant amount of contribution  $\omega_{i,\bullet}^{P2P} = \frac{1}{\tau}\omega_{i,\tau}^{P2P} + \frac{1}{\tau}\omega_{i,\tau}^{SL}$  he or she is exposed to over the horizon and the cash-back mechanism operates at the end of the horizon in order to redistribute the surplus between members of the pool.

### 2.3.2.4 Contribution to the upper layer

By using (2.3.19), the stop-loss premium for  $Z_\tau$  can be rewritten as:

$$\mathbb{E}[Z_\tau - \tau\omega] = \mathbb{E} \left[ \sum_{i=1}^n \left[ h_{i,Z_\tau}^*(Z_\tau) - h_{i,Z_\tau}^*(\omega) \right]_+ \right] = \sum_{i=1}^n \mathbb{E} \left[ h_{i,Z_\tau}^*(Z_\tau) - h_{i,Z_\tau}^*(\omega) \right]_+ \quad (2.3.26)$$

By using Equations (2.3.8) and (2.3.26), participant  $i$  then contributes over the  $\tau$  years

the premium:

$$\omega_{i,\tau}^{SL} = (1 + \beta_{SL}) \mathbb{E} \left[ h_{i,Z_\tau}^*(Z_\tau) - h_{i,Z_\tau}^*(\omega) \right]_+ \quad \text{for } \tau = 1, 2, \dots, \quad (2.3.27)$$

### 2.3.3 Retention level with cash-back for each year

Secondly, we now assume that there is a cash-back for each year. Formally, we aim to select the constant retention  $\omega$  such that a surplus  $[\omega - S_t]_+$  emerges at year  $t$ . We denote  $\sigma_\tau(\omega)$  the probability that a surplus emerges at year  $t = 1, 2, \dots, \tau$ .

#### 2.3.3.1 Retention level

For the case examined in this section, the choice of the retention level  $\omega$  corresponding to each year must take into account the possibility to give a cash-back to the participants at the end of year over the horizon. This value can be selected so that the probability  $\sigma_\tau(\omega)$ :

$$\sigma_\tau(\omega) = P[S_t \leq \omega : \text{for } t = 1, 2, \dots, \tau] \quad (2.3.28)$$

is high enough. Since the random variables are i.i.d., we have:

$$\begin{aligned} \sigma_\tau(\omega) &= P[S_1 \leq \omega, S_2 \leq \omega, \dots, S_\tau \leq \omega] \\ &= P[\max_{j=1}^\tau S_j \leq \omega] \\ &= F_{U_\tau}(\omega) \\ &= [F_{S_1}(\omega)]^\tau \end{aligned}$$

Where  $U_\tau$  is the random variable defined by:

$$U_\tau = \max_{j=1}^\tau S_j \quad (2.3.29)$$

Let  $\varepsilon$  denote the target probability for which a surplus is obtained at the end of the horizon and there is no deficit before the horizon. The retention  $\omega$  corresponding to this probability  $\varepsilon$  is given by:

$$\varepsilon = \sigma_\tau(\omega) = F_{U_\tau}(\omega) \iff \omega = F_{U_\tau}^{-1}(\varepsilon) \quad (2.3.30)$$

The chosen retention level is proportional to the quantile of the max random variable  $U_\tau$  distribution over  $\tau$  years corresponding to the probability  $\varepsilon$ .

#### 2.3.3.2 Contribution to the lower layer

Since the pool is sufficiently large, it follows that the functions  $s \mapsto \mathbb{E}[X_{it}|S_t = s]$  are continuously increasing for all participant  $i$ . This requirement guarantees that the conditional mean risk sharing  $h_{i,S_t}^*(s) = \mathbb{E}[X_{it}|S_t = s]$  is comonotonic and therefore Pareto-optimal.

Recalling that  $X_t$  are i.i.d., we define the quantities  $\tilde{\omega}_1^{P2P}, \tilde{\omega}^{P2P}, \dots, \tilde{\omega}_n^{P2P}$  such that:

$$\tilde{\omega}_i^{P2P} = h_{i,S_t}^*(\omega) = \mathbb{E}[X_{it}|S_t = \omega] = \mathbb{E}[X_{i1}|S_1 = \omega] \quad (2.3.31)$$

By the total allocation property of conditional mean risk sharing, we can verify that the set of individual contributions  $\tilde{\omega}_i^{P2P}$  is sufficient to cover the retention  $\omega$  at pool level. That is,

$$\sum_{i=1}^n \tilde{\omega}_i^{P2P} = \sum_{i=1}^n h_{i,S_1}^*(\omega) = \omega \quad (2.3.32)$$

Since the functions  $h_{i,S_t}^*$  are continuous and strictly increasing, we have:

$$S_t \geq \omega \iff h_{i,S_t}^*(S_t) \geq h_{i,S_t}^*(\omega) = \tilde{\omega}_i^{P2P} \quad \text{for } i = 1, 2, \dots, n \text{ and for all } t \quad (2.3.33)$$

We can therefore write for  $t = 1, 2, \dots, \tau$ ,

$$[S_t - \omega]_+ = \left[ \sum_{i=1}^n h_{i,S_t}^*(S_t) - h_{i,S_t}^*(\omega) \right]_+ \quad (2.3.34)$$

$$= \sum_{i=1}^n \left[ h_{i,S_t}^*(S_t) - h_{i,S_t}^*(\omega) \right]_+ \quad (2.3.35)$$

$$= \sum_{i=1}^n \left[ h_{i,S_t}^*(S_t) - \tilde{\omega}_i^{P2P} \right]_+ \quad (2.3.36)$$

The similar reasoning applies to the decomposition of surplus  $[\omega - S_t]_+$  into the sum of random variables  $[\tilde{\omega}_i^{P2P} - h_{i,S_t}^*(S_t)]_+$  for each year  $t$ . In this way, we can determine how to distribute the surplus when there is one, as well as the contribution of each participant to cover the retained loss  $\min\{S_t, \omega\}$ . Indeed,

$$\begin{aligned} \min\{S_t, \omega\} &= \omega - [\omega - S_t]_+ \\ &= \sum_{i=1}^n \left( \tilde{\omega}_i^{P2P} - \left[ \tilde{\omega}_i^{P2P} - h_{i,S_t}^*(S_t) \right]_+ \right) \\ &= \sum_{i=1}^n \min\{h_{i,S_t}^*(S_t), \tilde{\omega}_i^{P2P}\} \end{aligned}$$

So, each participant  $i$  contributes to the retained loss  $\min\{S_t, \omega\}$  an amount

$$\min\{h_{i,S_t}^*(S_t), \tilde{\omega}_i^{P2P}\}.$$

### 2.3.3.3 Surplus distribution

If a surplus  $B_t = [\omega - S_t]_+$  is available at the end of the year  $t$ , then the cash-back will be redistributed to the participants by reducing their individual contributions for the next year  $t + 1$ . If we denote  $B_{i,t}^{P2P}$  the cash-back redistributed to the participant  $i$ , then we have:

$$B_{i,t}^{P2P} = \left[ \tilde{\omega}_i^{P2P} - h_{i,S_t}^*(S_t) \right]_+ \quad (2.3.37)$$

The random variables  $B_{i,t}^{P2P}$  have a distribution such that  $P[B_{i,t}^{P2P} = 0] = P[S_t \geq \omega]$ . Note here that the chosen retention already guarantees a surplus in years  $t = 1, 2, \dots, \tau$ . The properties of the functions  $h_{i,S_t}^*$  ensure that for any total amount of losses  $s_t$  at the end of year  $t$  such that  $s_t \leq \omega$ , the contribution at the end of each year paid by the participant  $i$  is always less than or equal to  $\tilde{\omega}_i^{P2P}$ . That is,

$$h_{i,S_t}^*(s_t) \leq h_{i,S_t}^*(\omega) = \tilde{\omega}_i^{P2P} \quad \text{for } s_t \leq \omega \implies B_{i,t}^{P2P} \geq 0 \quad \text{for all } i \quad (2.3.38)$$

So, no additional contribution is claimed from the participant  $i$  at the end of each year.

If  $S_t < \omega$ , then participant  $i$  receives a bonus equal to  $B_{i,t}^{P2P} = \tilde{\omega}_i^{P2P} - h_{i,S_t}^*(S_t)$ . Each participant must then pay at the beginning of each year the maximal amount of contribution  $W_{i,t}$  he or she is exposed to over the horizon. Notice here that, this contribution is a random variable in contrary to the case where there is a cash-back only at year  $\tau$ . For this first case the contributions of participants remains deterministic as the P2P insurance model over one year.

### 2.3.3.4 Contribution to the upper layer

By using Equation (2.3.36) and recalling that  $S_t$  are identically distributed, the stop-loss premium for  $S_t$  can be rewritten as:

$$\mathbb{E}[L_t] = \mathbb{E}[L_1] = \mathbb{E} \left[ \sum_{i=1}^n \left[ h_{i,S_1}^*(S_1) - \tilde{\omega}_i^{P2P} \right]_+ \right] = \sum_{i=1}^n \mathbb{E} \left[ h_{i,S_1}^*(S_1) - \tilde{\omega}_i^{P2P} \right]_+ \quad (2.3.39)$$

By using Equations (2.3.8) and (2.3.39), participant  $i$  then contributes:

$$\tilde{\omega}_{i,t}^{SL} = \tilde{\omega}_{i,1}^{SL} = (1 + \beta_{SL}) \mathbb{E} \left[ h_{i,S_1}^*(S_1) - \tilde{\omega}_i^{P2P} \right]_+ \quad \text{for } t = 1, 2, \dots, \tau \quad (2.3.40)$$

## 2.3.4 distribution of the total cost of claims over each year

Consider a dynamic risk model of P2P insurance with  $n$  participants observed during an horizon  $t = 1, 2, \dots, \tau$ . We consider the individual model described in Equation (2.3.2). Since for the year  $t$   $X_{it}$  are independent compound Poisson variable, therefore,  $S_t$  is a compound Poisson variable as a sum of independent compound Poisson sums. The expression the total cost of claims in Equation (2.3.3) can be rewritten as:

$$S_t = \sum_{i=1}^n \sum_{k=1}^{N_{it}} C_{it;k} = \sum_{j=1}^{N_t} Y_j \text{ and } Z_t = \sum_{j=1}^t S_t = \sum_{j=1}^{N_t^c} Y_j, \quad (2.3.41)$$

and the distribution of  $S_t$  is obtained from Panjer algorithm. Since the random variables  $\{N_{it}, t = 1, 2, \dots, \tau\}$  (giving the number of claims occurred by year  $t$ ) are assumed to be independent of year and has a common Poisson( $\lambda_i$ ) distribution. We have, the random variable  $N_t$  follows a Poisson( $\lambda_\bullet$ ) distribution with  $\lambda_\bullet = \sum_{i=1}^n \lambda_i$  and  $N_t^c$  denotes the total number of claims until the year  $t$  and obey a Poisson distribution with parameter  $\lambda_\bullet t$ . The random variables  $Y_j$  are independent with the common cumulative distribution function (cdf):

$$F_Y(m) = \frac{1}{\lambda_\bullet} \sum_{i=1}^n \lambda_i P[C_i \leq m] = \frac{1}{\lambda_\bullet} \sum_{i=1}^n \lambda_i G_i(m) \quad \text{for } m = 0, 1, 2, \dots, \quad (2.3.42)$$

where  $G_i$  is the cdf of  $C_i$  the common distribution as  $C_{it;k}$ . Notice that the common cdf of  $Z_{it}$  denoted  $Fz_{it}$  for any year  $t$  is given by:

$$Fz_{it}(m) = \sum_{j=0}^m g_{it,j} \quad \text{for } m = 0, 1, 2, \dots \quad (2.3.43)$$

where  $g_{it,0} = P[Z_{it} = 0] = \exp(-\lambda_i t)$  and the probabilities  $g_{it,j} = P[Z_{it} = j]$  for  $j \geq 1$  are calculated recursively using Panjer's algorithm. That is,

$$Fz_{it}(m) = \sum_{j=0}^m g_{it,j} \quad \text{for } m = 0, 1, 2, \dots \quad (2.3.44)$$

$$g_{it,j} = \frac{\lambda_i t}{j} \sum_{l=1}^j l P[C_i = l] g_{it,j-l} \quad (2.3.45)$$

In particular, for  $t = 1$  we have  $Z_{i1} = X_{i1}$  and the common cdf  $X_{it}$  for any year  $t$  is given by  $Fz_{i1}$  (Recalling that  $X_t$  are i.i.d.). That is,

$$Fz_{i1}(m) = \sum_{j=0}^m g_{i1,j} \quad \text{and } g_{i1,j} = \frac{\lambda_i}{j} \sum_{l=1}^j l P[C_i = l] g_{i1,j-l} \quad (2.3.46)$$

The R package `actuar` has been used to perform the Panjer recursion and obtain the distribution of a compound Poisson variable using the function `aggregateDist`.

### 2.3.5 Formulation of conditional mean risk sharing

Since, we want the sharing rule to respect actuarial fairness property, We have for  $i = 1, 2, \dots, n$ :

$$\omega_i^{P2P} = \frac{1}{\tau} h_{i,\tau}^*(\omega) = \frac{1}{\tau} \mathbb{E}[Z_{i\tau} | Z_\tau = \omega] \quad \text{for all } i \quad (2.3.47)$$

Similarly, the variable  $\sum_{l \neq i} Z_t - Z_{it} = \sum_{l \neq i} Z_{lt}$  ( $t = 1, 2, \dots, \tau$ ) can be rewritten as  $\sum_{k=1}^{M_t^c} T_{i;k}$  with  $M_t^c \sim \text{Poisson}(\lambda_\bullet t - \lambda_i t)$  and  $T_{i;1}, T_{i;2}, \dots$  are independent random variables distributed as

$T_i$ , with

$$P[T_i = m] = \sum_{j \neq i} \frac{\lambda_j}{\lambda_{\bullet} - \lambda_i} G_j(m), \text{ for } m = 0, 1, \dots, \quad (2.3.48)$$

We can now calculate all the distribution functions of the random variables needed to calculate the contributions of the various participants. The calculation of  $\lambda_i$  and the distribution functions of  $Z_t$  and  $Z_t - Z_{it}$  are given in the code in the appendix.

Coming back to Relation (2.3.47), we have for  $t = 1, 2, \dots, \tau$ ,  $\mathbb{E}[Z_{it}|Z_t = 0] = 0$  and for  $s \geq 1$ , we can write:

$$\mathbb{E}[Z_{it}|Z_t = s] = \sum_{k=1}^s k P[Z_{it} = k|Z_t = s] = \sum_{k=1}^s k \frac{P[Z_{it} = k] P[\sum_{l \neq i} Z_{lt} = s - k]}{P[Z_t = s]} \quad (2.3.49)$$

Thus,

$$\omega_i^{P2P} = \frac{1}{\tau} \sum_{k=1}^{\tau\omega} k \frac{P[Z_{i\tau} = k] P[\sum_{l \neq i} Z_{l\tau} = \tau\omega - k]}{P[Z_\tau = \tau\omega]} \text{ for all } i \quad (2.3.50)$$

In particular for  $\tau = 1$ ,  $Z_{i\tau} = X_{i1}$ ,  $Z_\tau = S_1$  and we have:

$$\tilde{\omega}_i^{P2P} = \sum_{k=1}^{\omega} k \frac{P[X_{i1} = k] P[\sum_{l \neq i} X_{l1} = \omega - k]}{P[S_1 = \omega]} \text{ for all } i \quad (2.3.51)$$

Finally, in the next section we assume that there is a possible cash-back at years  $t = 1, 2, \dots, \tau$  and there is benevolent loan fund to cover the upper layer. The cash-back is possible when there is a surplus and there is no more loan to repay. The cash-back is used to reduce the individual contributions. In other words, the surplus is not necessarily redistributed each year.

## 2.4 Sharing losses P2P insurance with option benevolent Loan

### 2.4.1 Description of the Business model

Consider the risk model of P2P insurance described in section 2.3. We assume that there is benevolent loan instead of the (re-)insurer. Given that the pool in this case does not transfer risks but intends to repay this loan later in case of surplus. We assume that in case of surplus at a fixed year and when there is benevolent loan for preceding years to repay, then the priority is to reimburse the benevolent and the available surplus after loan repayment will be redistributed among participants and shareholders. This risk model corresponds to a particular Mudarabah-Takaful risk model. In our Mudarabah-Takaful risk model, the benevolent loan fund contains an unlimited amount of capital needs to cover a part or total losses  $L_t$  in case of deficit at end of each year  $t$  and the following financial account is incorporated:

$F^L$ : **liability account** that gives at the end of each year, the total benevolent loan to be repaid in case of future surplus.

In other words, the benevolent loan fund is a part of shareholders' fund that is specifically allocated to participants in the event of a deficit due to underwriting activities. The liability

account  $F^L$  is introduced to track total benevolent loans borrowed and repaid. The model assumes that all funds are in discrete monetary accounts.

We denote  $F_t^L$  the values of the liability account at the end of year  $t$ . We assume that our Hybrid-Takaful model splits the risks into several layers, each managed in a specific way. Suppose that Takaful participant's only retains a constant lower layer  $(0, \omega)$  of  $S_t$  at the beginning of each year  $t$  and the Takaful operator is obliged to provide an interest-free loan  $[S_t - \omega]_+$  (benevolent loan) to ensure that the total claims reclamation are covered at the end of year. So, the Takaful participant's retains at year  $t = 1, 2, \dots, \tau$ :

$$S_t^{fund} = \min\{S_t, \omega\} \quad (2.4.1)$$

and the excess of losses for the pool covered by benevolent loan is defined for  $t = 1, 2, \dots, \tau$  by:

$$L_t = [S_t - \omega]_+ \quad (2.4.2)$$

So, the liability account  $F_t^L$  defined by:

$$F_t^L = [F_{t-1}^L + L_t - [\omega - S_t]_+]_+ \quad \text{for } t = 1, 2, \dots, \tau, \quad (2.4.3)$$

with

$$F_0^L = 0. \quad (2.4.4)$$

The main assumption is to suppose that, each participant makes a unique contribution of the amount  $\omega_i^{fund}$  to the Takaful fund at the beginning of each year  $t$  that has to cover the lower layer  $(0, \omega)$  and the upper layer is covered by the benevolent loan  $L_t$  that will be reimburse in case of surplus in the future. We denote  $D_t$  the total loan repayment after a surplus  $[\omega - S_t]_+$  at year  $t$ . In case of a surplus at year  $t$  and after the loan repayment, the proportion  $\delta$  of available surplus denoted  $B_t^{fund}$  will be returned to the participants at the beginning of year  $t + 1$  by reducing their unique contribution  $\omega_i^{fund}$  to the individual total contribution denoted  $W_{i,t}$ . We assume that Shareholders receive the proportion  $(1 - \delta)$  of this surplus. The total amount of contribution  $\omega_{i,t}$  exactly paid by participant  $i$  at year  $t$  is then decomposed into:

$$W_{i,t} = \omega_i^{fund} - B_{i,t-1}^{fund} \quad \text{for } t = 2, 3, \dots, \tau \text{ and } \omega_{i,1} = \omega_i^{fund}, \quad (2.4.5)$$

where  $\omega_i^{fund}$  satisfies the relation:

$$\omega = \sum_{i=1}^n \omega_i^{fund} \quad (2.4.6)$$

The surplus  $B_{i,t}^{fund}$  to be returned to the participant  $i$  in the form of a contribution reduction is such that:

$$\sum_{i=1}^n B_{i,t}^{fund} = \delta B_t^{fund} \quad \text{for } t = 1, 2, \dots, \tau, \quad (2.4.7)$$

where

$$B_t^{fund} = [\omega - S_t]_+ - F_{t-1}^L \quad \text{for } t = 1, 2, \dots, \tau, \quad (2.4.8)$$

The total loan repayment  $D_t$  is collective at pool and satisfies the Relation (2.4.9) for  $t = 1, 2, \dots, \tau$ :

$$D_t = [\omega - S_t]_+ - [\omega - S_t]_+ - F_{t-1}^L \quad \text{for } t = 1, 2, \dots, \tau \quad (2.4.9)$$

Note that we have for each year  $t$ :

$$B_t^{fund} + D_t = [\omega - S_t]_+, \quad (2.4.10)$$

and we can also remark that we have in particular for the liability account:

$$F_1^L = \begin{cases} L_1 & \text{if } S_1 > \omega \\ 0 & \text{elsewhere,} \end{cases} \quad (2.4.11)$$

$$F_2^L = \begin{cases} L_1 + L_2 & \text{if } S_1 > \omega \text{ and } S_2 > \omega \\ L_2 & \text{if } S_1 \leq \omega \text{ and } S_2 > \omega \\ L_1 - [\omega - S_2]_+ & \text{if } S_1 > \omega \text{ and } S_2 < \omega \text{ and } Z_2 > 2\omega \\ 0 & \text{elsewhere,} \end{cases} \quad (2.4.12)$$

and

$$F_3^L = \begin{cases} L_1 + L_2 + L_3 & \text{if } S_1 > \omega \text{ and } S_2 > \omega \text{ and } S_3 > \omega \\ L_1 + L_2 - [\omega - S_3]_+ & \text{if } S_1 > \omega \text{ and } S_2 > \omega \text{ and } S_3 < \omega \text{ and } Z_3 > 3\omega \\ L_2 - [\omega - S_3]_+ & \text{if } S_1 \leq \omega \text{ and } S_2 > \omega \text{ and } S_3 < \omega \text{ and } S_2 + S_3 > 2\omega \\ L_2 + L_3 & \text{if } S_1 \leq \omega \text{ and } S_2 > \omega \text{ and } S_3 > \omega \\ L_1 + L_3 - [\omega - S_2]_+ & \text{if } S_1 > \omega \text{ and } S_2 < \omega \text{ and } S_3 > \omega \text{ and } Z_2 > 2\omega \text{ and } Z_3 > 3\omega \\ L_3 & \text{if } S_1 > \omega \text{ and } S_2 < \omega \text{ and } S_3 > \omega \text{ and } Z_2 \leq 2\omega \\ L_1 - [\omega - S_2]_+ - [\omega - S_3]_+ & \text{if } S_1 > \omega \text{ and } S_2 < \omega \text{ and } S_3 < \omega \text{ and } Z_3 > 3\omega \\ L_3 & \text{if } S_1 \leq \omega \text{ and } S_2 \leq \omega \text{ and } S_3 > \omega \\ 0 & \text{elsewhere,} \end{cases} \quad (2.4.13)$$

To better illustrate the model described in this section, let's consider the following example, which presents three scenarios based on the claim costs over a three-year horizon.

**Example 2.4.1.** Consider a Takaful pool over an horizon of  $\tau = 3$  years. We assume that the organiser acts to limit the liability of participants for each year to the constant retention  $\omega = 1500$ . To illustrate our model we consider the following three cases:

- 1) We assume that the total claims costs reclamation at end of year  $t = 1, 2, 3$  are respectively  $S_1 = 2000$ ,  $S_2 = 900$  and  $S_3 = 1800$ . We can compute the expression of  $F_t^L$ ,  $B_t^{fund}$  and  $D_t$  by:

- $F_0^L = 0$  and at year  $t = 1$ ,  $F_1^L = L_1 = 500$ ,  $D_1 = [1500 - 2000]_+ - [0 - 0]_+ = 0$  and  $B_1^{fund} = [0 - 0]_+ = 0$  ;
- at year  $t = 2$ ,  $F_2^L = 0$ ,  
 $D_2 = [1500 - 1000]_+ - [500 - 500]_+ = 500$  and  
 $B_2^{fund} = [[1500 - 900]_+ - 500]_+ = 100$  ;
- at year  $t = 3$ ,  $F_3^L = 500 + 300 - 500 = 300$ ,  
 $D_3 = [1500 - 1800]_+ - [0 - 0]_+ = 0$  and  $B_3^{fund} = [[1500 - 1800]_+ - 0]_+ = 0$ .

2) We assume for this case that the total claims costs reclamation at end of year  $t = 1, 2, 3$  are respectively  $S_1 = 2000$ ,  $S_2 = 800$  and  $S_3 = 500$ . We can compute the expression of  $F_t^L$ ,  $B_t^{fund}$  and  $D_t$  by:

- $F_0^L = 0$  and at year  $t = 1$ ,  $F_1^L = L_1 = 500$ ,  $D_1 = [1500 - 2000]_+ - [0 - 0]_+ = 0$  and  $B_1^{fund} = [0 - 0]_+ = 0$  ;
- at year  $t = 2$ ,  $F_2^L = 0$ ,  
 $D_2 = [1500 - 800]_+ - [700 - 500]_+ = 500$  and  
 $B_2^{fund} = [700 - 500]_+ = 200$  ;
- at year  $t = 3$ ,  $F_3^L = 0$ ,  
 $D_3 = [500 - 1500]_+ - [1000 - 0]_+ = 0$  and  $B_3^{fund} = [[500 - 1500]_+ - 0]_+ = 1000$ .

3) We now assume that the total claims costs reclamation at end of year  $t = 1, 2, 3$  are respectively  $S_1 = 1000$ ,  $S_2 = 2500$  and  $S_3 = 1500$ . Then, we have:

- $F_0^L = 0$  and at year  $t = 1$ ,  $F_1^L = 0$ ,  $D_1 = [1500 - 1000]_+ - [500 - 0]_+ = 0$  and  $B_1^{fund} = [500 - 0]_+ = 500$  ;
- at year  $t = 2$ ,  $F_2^L = L_2 = 1000$ ,  
 $D_2 = [1500 - 2500]_+ - [0 - 0]_+ = 0$  and  
 $B_2^{fund} = [0 - 0]_+ = 0$  ;
- at year  $t = 3$ ,  $F_3^L = 1000$ ,  
 $D_3 = [1500 - 1500]_+ - [0 - 1000]_+ = 0$  and  $B_3^{fund} = [[1500 - 1500]_+ - 1000]_+ = 0$ .

The value of retention level  $\omega$  for the Takaful hybrid model should be calibrated to satisfy the following conditions:

1. the contributions  $\omega_{i,t}$  contributions are as small as possible to make participation in the Takaful insurance scheme affordable and attractive to potential participants.
2. The value of retention level  $\omega$  can be tuned to ensure that there is a possible surplus at each year that is used to repay benevolent loan or to reduce individual contributions.

3. The probability that benevolent loan fund will be totally repay (or the liability account is positive) at the end of horizon is high enough.

The calibration of retention level  $\omega$  is the aim of the next section.

### 2.4.2 Retention level

For the case examined in this section, the choice of the retention level  $\omega$  corresponding to each year must take into account the following strategies and financial considerations: there is no more benevolent loan at year  $\tau$  to reimburse and guarantees a surplus to be redistributed among the participants for some years  $t = 1, 2, \dots, \tau$ . The retention level  $\omega$  can be selected such that the probability  $\Phi_\tau(\omega)$  defined by:

$$\Phi_\tau(\omega) = P[F_\tau^L = 0], \quad (2.4.14)$$

is high enough. That means the probability that the benevolent loan will be entirely repay over the  $\tau$  years and a surplus will be granted to participants some years. To compute an explicit formula of  $\Phi_\tau(\omega)$ , we first compute the probability  $P[F_t^L > 0]$  that there is benevolent loan at year  $t = 1, 2, \dots, \tau$ . In particular, we have:

$$P[F_0^L > 0] = 0 \text{ and } P[F_1^L > 0] = P[S_1 > \omega] = 1 - F_{S_1}(\omega). \quad (2.4.15)$$

It Follows that:

$$P[F_0^L = 0] = 1 \text{ and } P[F_1^L = 0] = F_{S_1}(\omega). \quad (2.4.16)$$

By using expression of  $F_2^L$  in Equation (2.4.12) and recalling that  $S_t$  are i.i.d., we can write:

$$\begin{aligned} P[F_2^L > 0] &= F_{S_1}(\omega)(1 - F_{S_1}(\omega)) + P[S_1 > \omega \text{ and } S_2 < \omega \text{ and } Z_2 > 2\omega] + [1 - F_{S_1}(\omega)]^2 \\ &= 1 - F_{S_1}(\omega) + P[S_1 > \omega]P[S_2 < \omega \text{ and } Z_2 > 2\omega | S_1 > \omega] \\ &= 1 - F_{S_1}(\omega) + \sum_{k>\omega} P[S_1 = k]P[S_2 < \omega \text{ and } S_2 > 2\omega - k] \\ &= 1 - F_{S_1}(\omega) + \sum_{k>\omega} P[S_1 = k](F_{S_1}(\omega) - P[S_1 = \omega] - F_{S_1}(2\omega - k)) \\ &= 1 - F_{S_1}(\omega) + (F_{S_1}(\omega) - P[S_1 = \omega])(1 - F_{S_1}(\omega)) - \sum_{k>\omega} P[S_1 = k]F_{S_1}(2\omega - k) \\ &= (1 + F_{S_1}(\omega) - P[S_1 = \omega])(1 - F_{S_1}(\omega)) + \sum_{k \leq \omega} P[S_1 = k]F_{S_1}(2\omega - k) - F_{S_1}^{*(2)}(2\omega) \end{aligned}$$

where  $F_{S_1}^{*(j)}$  the  $j^{th}$ -order convolution product of  $F_{S_1}$  given by:

$$F_{S_1}^{*(j)}(s) = P[S_1 + S_2 + \dots + S_j < s] \quad (2.4.17)$$

Thus,

$$P[F_2^L = 0] = 1 - P[F_2^L > 0] \quad (2.4.18)$$

$$= F_{S_1}(\omega) - (F_{S_1}(\omega) - P[S_1 = \omega])(1 - F_{S_1}(\omega)) - \sum_{k \leq \omega} P[S_1 = k]F_{S_1}(2\omega - k) + F_{S_1}^{*(2)}(2\omega) \quad (2.4.19)$$

Similarly, in order to compute the probability  $P[F_3^L > 0]$ , we use Formula 2.4.13 and calculate the probabilities for the following cases:

(a)  $P_1 = P[S_1 > \omega, S_2 > \omega, S_3 < \omega, Z_3 > 3\omega]$  is given by:

$$P_1 = \sum_{\substack{m > \omega \\ k > \omega}} P[S_1 = k]P[S_1 = m]P[S_3 < \omega, S_3 > 3\omega - k - m] \quad (2.4.20)$$

$$= \sum_{\substack{m > \omega \\ k > \omega}} P[S_1 = k]P[S_1 = m](F_{S_1}(\omega) - P[S_1 = \omega] - F_{S_1}(3\omega - k - m)) \quad (2.4.21)$$

$$= (1 - F_{S_1}(\omega))^2(F_{S_1}(\omega) - P[S_1 = \omega]) - \sum_{\substack{m > \omega \\ k > \omega}} P[S_1 = k]P[S_1 = m]F_{S_1}(3\omega - k - m) \quad (2.4.22)$$

Coming to  $E_1 = \sum_{\substack{m > \omega \\ k > \omega}} P[S_1 = k]P[S_1 = m]F_{S_1}(3\omega - k - m)$ , we have:

$$\begin{aligned} E_1 &= \sum_{m > \omega} P[S_1 = m] \left( \sum_{k > \omega} P[S_1 = k]F_{S_1}(3\omega - k - m) \right) \\ &= \sum_{m > \omega} P[S_1 = m] \left( F_{S_1}^{*(2)}(3\omega - m) - \sum_{k \leq \omega} P[S_1 = k]F_{S_1}(3\omega - k - m) \right) \\ &= \sum_{m > \omega} P[S_1 = m]F_{S_1}^{*(2)}(3\omega - m) - \sum_{k \leq \omega} P[S_1 = k] \left( \sum_{m > \omega} P[S_1 = m]F_{S_1}(3\omega - k - m) \right) \\ &= F_{S_1}^{*(3)}(3\omega) - \sum_{m \leq \omega} P[S_1 = m]F_{S_1}^{*(2)}(3\omega - m) - \sum_{k \leq \omega} P[S_1 = k] \left( \sum_{m > \omega} P[S_1 = m]F_{S_1}(3\omega - k - m) \right) \\ &= F_{S_1}^{*(3)}(3\omega) - \sum_{m \leq \omega} P[S_1 = m]F_{S_1}^{*(2)}(3\omega - m) - \\ &\quad \left( F_{S_1}^{*(3)}(3\omega) - \sum_{\substack{m \leq \omega \\ k \leq \omega}} P[S_1 = k]P[S_1 = m]F_{S_1}(3\omega - k - m) \right) \\ &= \sum_{\substack{m \leq \omega \\ k \leq \omega}} P[S_1 = k]P[S_1 = m]F_{S_1}(3\omega - k - m) - \sum_{m \leq \omega} P[S_1 = m]F_{S_1}^{*(2)}(3\omega - m) \end{aligned}$$

(b) We also have for  $P_2 = P[S_1 \leq \omega, S_2 > \omega, S_3 < \omega, S_2 + S_3 > 2\omega]$ :

$$P_2 = P[S_2 > \omega]P[S_1 \leq \omega, S_3 < \omega, S_2 + S_3 > 2\omega | S_2 > \omega] \quad (2.4.23)$$

$$= \sum_{k>\omega} P[S_2 = k]P[S_1 \leq \omega, S_3 < \omega, S_3 > 2\omega - k] \quad (2.4.24)$$

$$= P[S_1 \leq \omega] \sum_{k>\omega} P[S_2 = k]P[S_3 < \omega, S_3 > 2\omega - k] \quad (2.4.25)$$

$$= P[S_1 \leq \omega] \sum_{k>\omega} P[S_1 = k](F_{S_1}(\omega) - P[S_1 = \omega] - F_{S_1}(2\omega - k)) \quad (2.4.26)$$

$$= F_{S_1}(\omega)(1 - F_{S_1}(\omega))(F_{S_1}(\omega) - P[S_1 = \omega]) - F_{S_1}(\omega) \sum_{k>\omega} P[S_1 = k]F_{S_1}(2\omega - k) \quad (2.4.27)$$

$$= F_{S_1}(\omega)(1 - F_{S_1}(\omega))(F_{S_1}(\omega) - P[S_1 = \omega]) + F_{S_1}(\omega) \left( \sum_{k \leq \omega} P[S_1 = k]F_{S_1}(2\omega - k) - F_{S_1}^{*(2)}(2\omega) \right) \quad (2.4.28)$$

(c) For the probability  $P_3 = P[S_1 > \omega, S_2 < \omega, S_3 > \omega, Z_2 > 2\omega, Z_3 > 3\omega]$  we have:

$$P_3 = P[S_1 > \omega]P[S_3 > \omega]P[S_2 < \omega, S_1 + S_2 > 2\omega, S_1 + S_2 + S_3 > 3\omega | S_1 > \omega, S_3 > \omega] \quad (2.4.29)$$

$$= \sum_{\substack{m>\omega \\ k>\omega}} P[S_1 = k]P[S_1 = m]P[S_2 < \omega, S_2 > 2\omega - k, S_2 > 3\omega - k - m] \quad (2.4.30)$$

$$= \sum_{\substack{m>\omega \\ k>\omega}} P[S_1 = k]P[S_1 = m](F_{S_1}(\omega) - P[S_1 = \omega] - F_{S_1}(\max(2\omega - k, 3\omega - k - m))) \quad (2.4.31)$$

$$= (1 - F_{S_1}(\omega))^2(F_{S_1}(\omega) - P[S_1 = \omega]) - \sum_{\substack{m>\omega \\ k>\omega}} P[S_1 = k]P[S_1 = m]F_{S_1}(\max(2\omega - k, 3\omega - k - m)), \quad (2.4.32)$$

(d) We now compute  $P_4 = P[S_1 > \omega, S_2 < \omega, S_3 > \omega, Z_2 \leq 2\omega]$ . That is,

$$P_4 = P[S_1 > \omega]P[S_3 > \omega, S_2 < \omega, S_1 + S_2 > 2\omega | S_1 > \omega] \quad (2.4.33)$$

$$= \sum_{k>\omega} P[S_1 = k]P[S_2 < \omega, S_2 > 2\omega - k, S_3 > \omega] \quad (2.4.34)$$

$$= P[S_3 > \omega] \sum_{k>\omega} P[S_1 = k](F_{S_1}(\omega) - P[S_1 = \omega] - F_{S_1}(2\omega - k)) \quad (2.4.35)$$

$$= (1 - F_{S_1}(\omega))^2(F_{S_1}(\omega) - P[S_1 = \omega]) + (1 - F_{S_1}(\omega)) \left( \sum_{k \leq \omega} P[S_1 = k]F_{S_1}(2\omega - k) - F_{S_1}^{*(2)}(2\omega) \right), \quad (2.4.36)$$

(e) We denote  $P_5 = P[S_1 > \omega, S_2 < \omega, S_3 < \omega, Z_3 > 3\omega]$  that is given by:

$$P_5 = P[S_1 > \omega]P[S_3 < \omega]P[S_2 < \omega, S_1 + S_2 + S_3 > 3\omega | S_1 > \omega, S_3 < \omega] \quad (2.4.37)$$

$$= \sum_{\substack{m < \omega \\ k > \omega}} P[S_1 = k]P[S_1 = m]P[S_2 < \omega, S_2 > 3\omega - k - m] \quad (2.4.38)$$

$$= \sum_{\substack{m < \omega \\ k > \omega}} P[S_1 = k]P[S_1 = m](F_{S_1}(\omega) - P[S_1 = \omega] - F_{S_1}(3\omega - k - m)) \quad (2.4.39)$$

$$= F_{S_1}(\omega)(1 - F_{S_1}(\omega))(F_{S_1}(\omega) - P[S_1 = \omega]) - \sum_{\substack{m < \omega \\ k > \omega}} P[S_1 = k]P[S_1 = m]F_{S_1}(3\omega - k - m), \quad (2.4.40)$$

Coming to  $E_2 = \sum_{\substack{m < \omega \\ k > \omega}} P[S_1 = k]P[S_1 = m]F_{S_1}(3\omega - k - m)$ , we have:

$$\begin{aligned} E_2 &= \sum_{m < \omega} P[S_1 = m] \left( \sum_{k > \omega} P[S_1 = k]F_{S_1}(3\omega - k - m) \right) \\ &= \sum_{m < \omega} P[S_1 = m] \left( F_{S_1}^{*(2)}(3\omega - m) - \sum_{k \leq \omega} P[S_1 = k]F_{S_1}(3\omega - k - m) \right) \\ &= \sum_{m < \omega} P[S_1 = m]F_{S_1}^{*(2)}(3\omega - m) - \sum_{\substack{m < \omega \\ k \leq \omega}} P[S_1 = k]P[S_1 = m]F_{S_1}(3\omega - k - m) \\ &= -E_1 - P[S_1 = \omega]F_{S_1}^{*(2)}(2\omega) \end{aligned}$$

The probabilities for other cases in Formula 2.4.12 are easily compute and by using the computed probabilities in cases (a)-(b), we have:

$$\begin{aligned} q_2 &= P_2 + P_3 + P_4 = (1 - F_{S_1}(\omega))(F_{S_1}(\omega) - P[S_1 = \omega])(2 - F_{S_1}(\omega)) + \\ &\quad F_{S_1}(\omega) \left( \sum_{k \leq \omega} P[S_1 = k]F_{S_1}(2\omega - k) - F_{S_1}^{*(2)}(2\omega) \right) - \\ &\quad \sum_{\substack{m > \omega \\ k > \omega}} P[S_1 = k]P[S_1 = m]F_{S_1}(\max(2\omega - k, 3\omega - k - m)) + \\ &\quad (1 - F_{S_1}(\omega)) \left( \sum_{k \leq \omega} P[S_1 = k]F_{S_1}(2\omega - k) - F_{S_1}^{*(2)}(2\omega) \right) \\ &= (1 - F_{S_1}(\omega))(F_{S_1}(\omega) - P[S_1 = \omega])(2 - F_{S_1}(\omega)) + \\ &\quad \sum_{k \leq \omega} P[S_1 = k]F_{S_1}(2\omega - k) - F_{S_1}^{*(2)}(2\omega) - \\ &\quad \sum_{\substack{m > \omega \\ k > \omega}} P[S_1 = k]P[S_1 = m]F_{S_1}(\max(2\omega - k, 3\omega - k - m)) \end{aligned}$$

We have also,

$$q_1 = P_1 + P_5 = (1 - F_{S_1}(\omega))(F_{S_1}(\omega) - P[S_1 = \omega]) + P[S_1 = \omega]F_{S_1}^{*(2)}(2\omega) \quad (2.4.41)$$

and

$$\begin{aligned} q_0 &= P[S_1 > \omega, S_2 > \omega, S_3 > \omega] + P[S_1 \leq \omega, S_2 > \omega, S_3 > \omega] + P[S_1 \leq \omega, S_2 \leq \omega, S_3 > \omega] \\ &= (1 - F_{S_1}(\omega))^3 + F_{S_1}(\omega)(1 - F_{S_1}(\omega))^2 + (1 - F_{S_1}(\omega))(F_{S_1}(\omega))^2 \\ &= (1 - F_{S_1}(\omega)) \left( F_{S_1}(\omega) + (1 - F_{S_1}(\omega))^2 \right) \end{aligned}$$

We conclude that the probability of having no benevolent loan to repay at the end of year 3 for the pool is given by:

$$P[F_2^L = 0] = 1 - P[F_2^L > 0] = 1 - q_1 - q_2 - q_0 \quad (2.4.42)$$

The value of  $\omega$  has an impact on the values of  $\omega_i^{P2P}$ ,  $B_{i,t}^{fund}$  and  $F_t^L$ . It must therefore be chosen so that participation in the system is sufficiently attractive to encourage shareholders to provide benevolent loan and to encourage participants to contribute  $\omega_{i,t}$  rather than the premium demanded by an insurer for classical insurance coverage.

In particular, for  $\tau = 3$  and  $\varepsilon$  the target probability for which shareholders are guaranteed full repayment over three years in the event of a loan. Then the retention level  $\omega$  corresponding to this probability  $\varepsilon$  is given by  $\Phi_3(\omega)$  in Formula (2.4.42).

### 2.4.3 Contribution to the lower layer

As the retention level  $\omega$  of the pool can be determined according to the pool's objectives, it now remains to define how the individual contributions  $\omega_i^{fund}$  will be calculated so that the pool can be in a position to amortise this part of the total cost constituting the lower layer. In according to Section 2.3 the conditional mean risk sharing  $h_{i,S_t}^*(s) = \mathbb{E}[X_{it}|S_t = s]$  is comonotonic and therefore Pareto-optimal. We define the quantities  $\omega_1^{fund}, \omega_2^{fund}, \dots, \omega_n^{fund}$  such that:

$$\omega_i^{fund} = h_{i,S_t}^*(\omega) = \mathbb{E}[X_{it}|S_t = \omega] = \mathbb{E}[X_{i1}|S_1 = \omega] \quad (2.4.43)$$

Since  $\omega_i^{fund} = \tilde{\omega}_i^{P2P}$ , by using Relation (2.3.32) the set of individual contributions  $\omega_i^{fund}$  is sufficient to cover the retention level  $\omega$  at pool.

According to Relation (2.3.36), the surplus  $[\omega - S_t]_+$  can be decomposed into the sum of random variables  $[\tilde{\omega}_i^{P2P} - h_{i,S_t}^*(S_t)]_+$  for each year  $t$  and similarly the benevolent loan  $L_t$  is decomposed into the sum of random variable  $[h_{i,S_t}^*(S_t) - \tilde{\omega}_i^{P2P}]_+$ . In this way, each participant  $i$  contributes to the retained loss  $\min\{S_t, \omega\}$  an amount  $\min\{h_{i,S_t}^*(S_t), \tilde{\omega}_i^{P2P}\}$ . We can now determine how to distribute the surplus when there is one.

### 2.4.4 Surplus distribution

When a surplus  $B_t^{fund}$  defined in Formula (2.4.8) is available at the end of the year  $t$ , then the cash-back will be redistributed to the participants by reducing their individual contributions

for the next year  $t + 1$ .

In particular at year  $t = 1$ , since  $F_0^L$  then the surplus  $B_1^{fund}$  is given by:

$$B_1^{fund} = [\omega - S_1]_+ = \sum_{i=1}^n [\tilde{\omega}_i^{P2P} - h_{i,S_1}^*(S_1)]_+ \quad (2.4.44)$$

By using Relation (2.4.7), the cash-back  $B_{i,1}^{fund}$  received by participant  $i$  is given by:

$$B_{i,1}^{fund} = \delta [\tilde{\omega}_i^{P2P} - h_{i,S_1}^*(S_1)]_+ \quad (2.4.45)$$

By using Equations (2.4.44), the random variables  $B_{i,1}^{fund}$  have a distribution such that  $P[B_1^{fund} = 0] = P[S_1 > \omega]$ . Thus the cumulative distribution function of random variable  $[B_1^{fund} | B_1^{fund} > 0]$  denoted  $F_{B_1}$  is given by:

$$F_{B_1}(k) = \frac{P[\omega - S_t > 0] - P[\omega - S_t > k]}{P[\omega - S_t > 0]} \quad (2.4.46)$$

$$= \frac{F_{S_1}(\omega) - F_{S_1}(\omega - k)}{F_{S_1}(\omega)}, \quad (2.4.47)$$

At year  $t = 2$ , the surplus  $B_2^{fund}$  is given by:

$$B_2^{fund} = [[\omega - S_2]_+ - F_1^L]_+ = \begin{cases} [\omega - S_2]_+ - L_1 & \text{if } S_1 > \omega \text{ and } S_2 < \omega \text{ and } Z_2 < 2\omega \\ [\omega - S_2]_+ & \text{if } S_1 < \omega \text{ and } S_2 < \omega \\ 0 & \text{elsewhere,} \end{cases} \quad (2.4.48)$$

Thus, the cash-back  $B_{i,2}^{fund}$  received by participant  $i$  is given by:

$$B_{i,2}^{fund} = \begin{cases} \delta [\tilde{\omega}_i^{P2P} - h_{i,S_2}^*(S_2)]_+ - \delta [h_{i,S_1}^*(S_1) - \tilde{\omega}_i^{P2P}]_+ & \text{if } S_1 > \omega \text{ and } S_2 < \omega \text{ and } Z_2 < 2\omega \\ \delta [\tilde{\omega}_i^{P2P} - h_{i,S_2}^*(S_2)]_+ & \text{if } S_1 < \omega \text{ and } S_2 < \omega \\ 0 & \text{elsewhere,} \end{cases} \quad (2.4.49)$$

At this step, the random variables  $B_{i,2}^{fund}$  have a distribution such that:

$$P[B_2^{fund} = 0] = P[F_2^L > 0] = 1 - P[F_2^L = 0] \quad (2.4.50)$$

similarly, by using Relation (2.4.12) at year  $t = 3$  the surplus  $B_3^{fund}$  is defined by:

$$B_3^{fund} = \begin{cases} [\omega - S_3]_+ - L_1 - L_2 & \text{if } S_1 > \omega \text{ and } S_2 > \omega \text{ and } S_3 < \omega \text{ and } Z_3 < 3\omega \\ [\omega - S_3]_+ - L_2 & \text{if } S_1 < \omega \text{ and } S_2 > \omega \text{ and } S_3 < \omega \text{ and } Z_2 < 2\omega \\ [\omega - S_3]_+ - [\omega - S_2]_+ - L_1 & \text{if } S_1 > \omega \text{ and } S_2 < \omega \text{ and } S_3 < \omega \text{ and } Z_3 < 3\omega \\ [\omega - S_3]_+ & \text{if } S_1 < \omega \text{ and } S_2 < \omega \text{ and } S_3 < \omega \\ 0 & \text{elsewhere,} \end{cases} \quad (2.4.51)$$

So, by using the decomposition of  $[\omega - S_t]_+$  and  $L_t$  then the cash-back  $B_{i,3}^{fund}$  received by participant  $i$  can be written as a sum of random variables  $\left[\tilde{\omega}_i^{P2P} - h_{i,S_3}^*(S_3)\right]_+$  and random variables  $\left[h_{i,S_3}^*(S_3) - \tilde{\omega}_i^{P2P}\right]_+$ . Moreover, the random variables  $B_{i,3}^{fund}$  have a distribution such that:

$$P[B_3^{fund} = 0] = P[F_3^L > 0] = 1 - P[F_3^L = 0] \quad (2.4.52)$$

Note here that the retention chosen already guarantees a possible surplus in years  $t = 1, 2, \dots, \tau$ . Each participant must then pay at the beginning of each year the maximal amount of contribution  $\omega_{i,t}$  he or she is exposed at year  $t$ .

The application of conditional mean risk sharing in as a sharing rule revealed its potential for establishing an equitable distribution of the total claim costs within the Mudarabah-Takaful pool and dynamic P2P with (re-)insurer, considering what each participant individually contributes in terms of claims to the pool. This rule could be intuitively appealing because it is based on the familiar concept of the mean, and it is easy to implement. Transparency is complete since participants can be informed from the outset about the reimbursement mechanism for benevolent loans and/or the reduction of individual contributions at the beginning of each year, according to the total observed cost at the end of the previous year. In the next chapter we illustrate a numerical example to compare the model with (re-)insurer and the model with benevolent loan.

# Chapter 3

## Numerical illustration

### 3.1 Descriptive analysis of the data

In our simulation study, we apply the set of input parameters of example described in Denuit (2020). The following section presents the risk profile of the participants in the Takaful pool.

#### 3.1.1 Data Presentation

We consider a Takaful insurance pool with  $n$  participants composed of three risk profiles. The participants agree to join the pool over an horizon of three years ( $\tau = 3$ ) in order to pool their risks. Each participant  $i$  brings a compound Poisson loss to the pool that are i.i.d. over the three years as the random variable  $X_i = \sum_{k=1}^{N_i} C_{ik}$  with  $N_i$  has Poisson( $\lambda_i$ ) distribution, where severities  $\frac{C_{ik}}{1000}$  have the Beta( $a_i, 5$ ) distribution. all random variables being independent. The Table 3.1.1 shows the values of the parameters we will use in the numerical illustrations below. The data from the Takaful pool are summarized in Figure 1, which presents the

<b>Group</b>	1 (low risks)	2 (medium risks)	3 (high risks)
<i>Size</i>	$n_1 = 0.5n$	$n_2 = 0.3n$	$n_3 = 0.2n$
$\lambda_i$	0.05	0.1	0.2
$a_i$	2	3	4
$\mathbb{E}[C_{i1}]$	286	375	444
$\mathbb{E}[X_i]$	14.3	37.5	88.8

Table 3.1.1: Takaful pool parameters

probability distributions of the frequency and severity components of claims based on the selected parameters for the three described groups. It can be observed that participants in Group 1 and Group 2 have a lower frequency of experiencing at least one claim compared to participants in Group 3. Additionally, the probability density of severities gradually shifts towards higher values. Participants in Group 3 have a comparatively higher severity than

those in Group 2, who themselves have a comparatively higher severity than those in Group 1. The groups are thus ranked according to the increasing magnitude of expected frequencies and severity degrees. In our example, for  $n = 500$  participants, the expected value of  $S_t$  at each year  $t$  is:

$$\mathbb{E}[S_t] = n_1\mathbb{E}[X_{i,G_1}] + n_2\mathbb{E}[X_{i,G_2}] + n_3\mathbb{E}[X_{i,G_3}] = 18080 \quad (3.1.1)$$

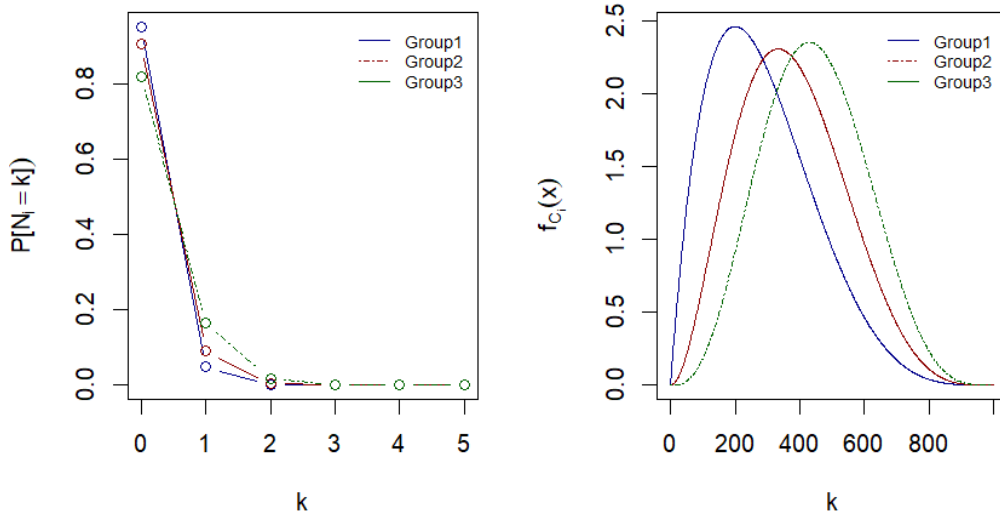


Figure 3.1.1: Distribution of the number of claims  $N_i$  and probability density of severities  $C_{ij}$ .

### 3.1.2 Discretization of severities

The total cost per participant  $X_i$  follows a compound Poisson distribution, with severities drawn from a  $\text{Beta}(a_i, 5)$  distribution. However, the Panjer algorithm requires severity values to be discrete, i.e.,  $C_{ik}$  taking values in  $\{1, 2, 3, \dots\}$ . From the parameters of the Beta distributions in Table 3.1.1, one of the initial steps for our illustrations involves discretizing the Beta distributions of severities.

Let  $F_{i,k}(x)$  denote the cumulative distribution function (cdf),  $\mathbb{E}[\min(X, x)]$  the limited expected value at  $x$ ,  $h$  the step,  $f_{i,k}(x)$  the probability mass at  $x$  in the discretized distribution. Then, the discretization of  $F_{i,k}(x)$  from  $a$  to  $b$  by step  $h$  can be performed using various methods:

1. Method "upper" is the forward difference of the cdf  $F$ :

$$f_{i,k}(x) = F_{i,k}(x + h) - F_{i,k}(x)$$

for  $x = a, a + h, \dots, b - h$ . The discretized cdf computed using this method always lies above the true cdf.

2. Method "lower" is the backward difference of the cdf  $F$ :

$$f_{i,k}(x) = \begin{cases} F_{i,k}(a) & \text{if } x = a \\ F_{i,k}(x) - F_{i,k}(x-h) & \text{if } x = a+h, \dots, b \end{cases}$$

The discretized cdf is always under the true cdf.

3. Method "rounding" has the true cdf pass through the midpoints of the intervals  $[x - h/2, x + h/2)$ :

$$f_{i,k}(x) = \begin{cases} F_{i,k}(a + h/2) & \text{if } x = a \\ F_{i,k}(x + h/2) - F_{i,k}(x - h/2) & \text{if } x = a + h, \dots, b - h \end{cases}$$

The true cdf passes precisely halfway between the steps of the discretized cdf.

4. Method "unbiased" matches the first moment of the discretized and the true distributions. The probabilities are as follows:

$$f_{i,k}(x) = \begin{cases} \frac{\mathbb{E}[\min(X,a)] - \mathbb{E}[\min(X,a+h)]}{h} + 1 - F_{i,k}(a) & \text{if } x = a \\ \frac{2\mathbb{E}[\min(X,x)] - \mathbb{E}[\min(X,x+h)] - \mathbb{E}[\min(X,x-h)]}{h} & \text{if } a < x < b \\ \frac{\mathbb{E}[\min(X,b)] - \mathbb{E}[\min(X,b-h)]}{h} - 1 + F_{i,k}(b) & \text{if } x = b \end{cases}$$

The discretization of the cdf  $F$  by this method ensures that the discretized cdf and the true cdf have identical total probability and expectation over the interval  $[a, b]$ .

In order to maintain consistency with the calculations in previous sections, which preserved the mean unchanged, we will use this latter discretization method going forward. These discretization methods were developed and implemented by Dutang et al. (2008) within the R package `actuar`, which also contains numerous very interesting and practical functions for actuarial calculations. Firstly, in Figure 3.1.2, we represented the probability distributions of severity components of claims based on the selected parameters, for each of the groups. In the figure, the red curve represents the true probability density function of the Beta distribution, while the vertical lines represent the probability mass at  $k$ , obtained after discretizing the cdf.

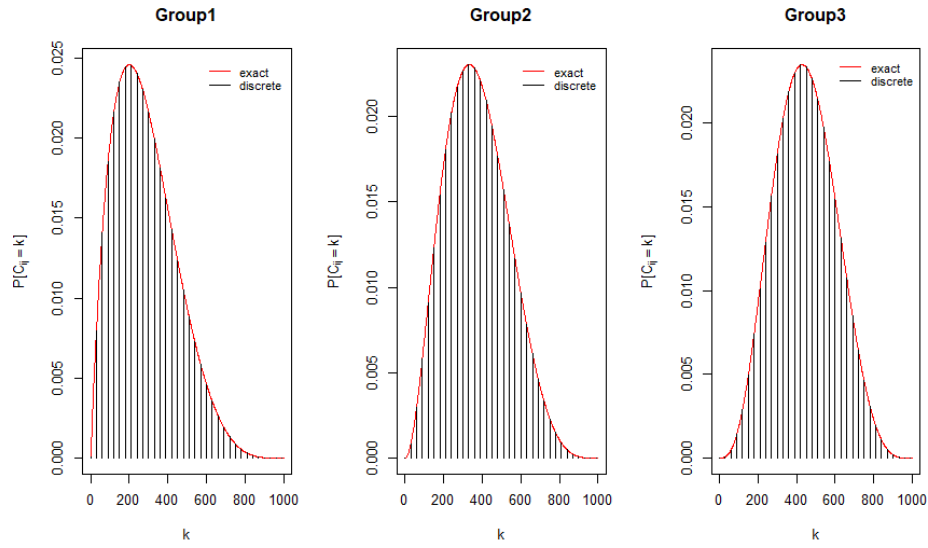


Figure 3.1.2: probability mass functions of discretized severities  $C_{ij}$ .

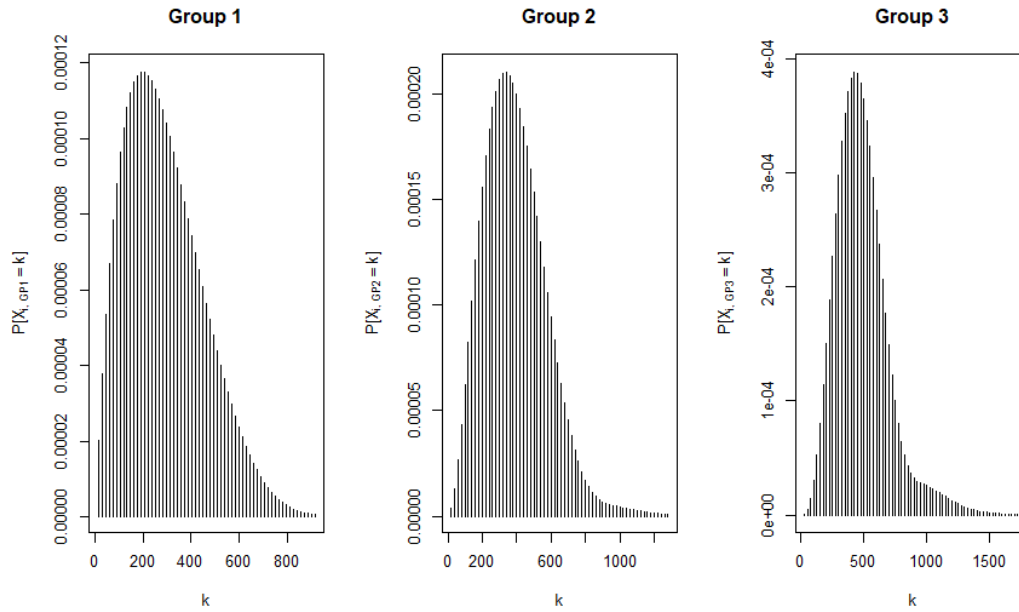


Figure 3.1.3: Probability mass functions of the annual total cost  $X_i$  according to the participant's Group membership.

The probability mass function of the random variable  $Z_{i1} = X_i$  representing the total claim cost (compound Poisson losses) for each participant  $i$  at the end of each year, according to their Group membership, is depicted in Figure 3.1.3. It is clearly observed that this cost is higher for a participant in Group 3 than for a participant in Group 1 or Group 2. This

observation is a result of the descriptions we have regarding the frequencies and severities of the claims. Notice that the probability mass equal to  $\exp(-\lambda_{i,G_l})$ ,  $l = 1, 2, 3$  at zero for each participant  $i$  is not displayed.

### 3.1.3 Distribution of the total pool cost claims $Z_t$ until year $t$

By using R package actuar, we have performed the Panjer recursion and obtain the distribution of the compound Poisson variable  $Z_t$  using the function aggregateDist (see Appendix in Section A.5). The probability mass functions of  $S_1$  and  $Z_3$  are depicted (without the probability mass equal  $\exp(-\lambda_{\bullet}t)$  at zero  $t = 1$  and  $t = 3$ ) in Figure 3.1.4 for  $n = 500$  participants. The figure presents the results obtained by the exact calculation of the distribution of  $Z_t$ . Aggregating cost claims over three years results in a distribution that is more predictable and less variable relative to the one-year distribution. This is evident from the more pronounced peak and narrower spread in the three-year curve. Overall, while the total cost of claims increases over three years, the distribution of these costs becomes more concentrated and predictable, reflecting the stabilizing effect of aggregating over a longer year period.

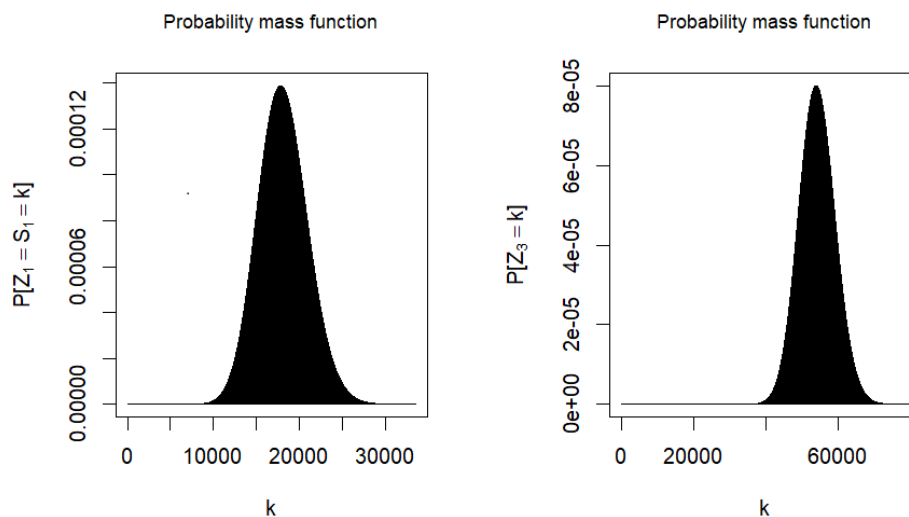


Figure 3.1.4: The distribution of the total pool cost  $Z_1$  and  $Z_3$  for  $n = 500$  participants.

## 3.2 Contribution of participants in each Group with (re-)insurer

### 3.2.1 Case of dynamic P2P insurance with cash-back at $t = 1, 2$

To illustrate the layer decomposition and to get an idea of the amounts involved, we have performed the calculations for a pool consisting of  $n = 500$  participants. We assume a safety loading value for the (re-)insurer  $\beta_{SL}$  to be applied to the stop-loss premium, set at  $\beta_{SL} = 0.5$ . We also consider the target probability  $\varepsilon = 0.985$  corresponding the retention level  $\omega = 25963$ . The application of the conditional mean risk sharing in Formula (2.3.31) to the

pool is illustrated in Figure 3.2.1, showing the evolution of contributions from members of each Group based on the realization  $s$  up to  $\omega$  of the observed total cost  $S_1$  at the end of each year. The contributions  $h_{i,S_t}^*(s)$  are strictly increasing, implying an increase in contributions with the total cost. We can deduce that the rule is Pareto optimal, as an increase in the total cost necessarily leads to an increase in the value of contributions for all members. Members would thus have an interest in adopting behavior aimed at reducing the total cost. Due to a higher average frequency of claims  $\lambda_i$  for the participants in Group 3, it is observed, as expected, that the value of contributions is higher for participants in that Group compared to those in Group 1 or Group 2. The contributions of participants  $Wi.2 = Wi.3$  in year  $t = 2, 3$  are reduced by using cash-back. All the results of our calculations are summarized in Table 3.2.1. Figure ?? illustrates the probability distribution of cash-back  $B_{i,t}^{P2P} = B_{i,1}^{P2P}$  for the

Group	Ni	Wi.P2P	$\mathbb{E}[X_{i1}]$	Wi.SL	Wi.1	Wi.2	$\mathbb{E}[Bi.P2P]$
Group 1	250	19.256	14.284	0.005	19.260	14.284	4.976
Group 2	150	53.263	37.498	0.017	53.280	37.496	15.784
Group 3	100	131.590	88.886	0.047	131.637	88.878	42.759

Table 3.2.1: Results of the layer decomposition:  $n = 500$ ,  $\beta_{SL} = 0.5$  and  $\varepsilon = 0.985$

different groups in the pool. This distribution is typical of a zero-augmented distribution, characterized by a significant probability mass at zero. This probability mass is clearly higher for low-risk profiles (Group1), meaning those profiles for which the average losses are low. The probability mass at zero decreases as we approach profiles with higher  $\mathbb{E}[X_{i1}]$  (Group2) for these profiles, it becomes more likely to receive cash-back at the end of the period. This could be explained by the fact that these profiles contribute more to the total cost, due to having a higher retention, and thus, when a surplus is available, they are more likely to benefit from it. The application of layer decomposition was implemented using the code provided in the Appendix A.6.1.

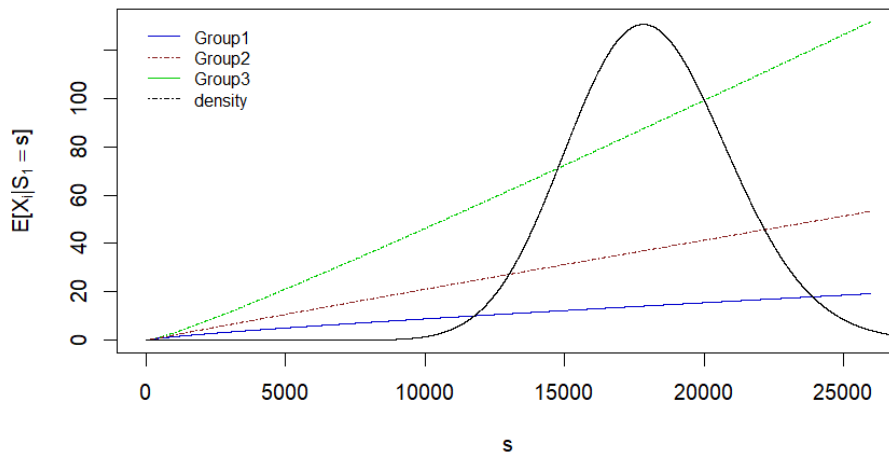


Figure 3.2.1: Respective contributions  $s \mapsto \mathbb{E}[X_i | S_1 = s]$  in Group1, Group2 and Group3

As observed in Figure 3.2.2, the conditional mean risk sharing almost perfectly satisfies the total allocation condition 2.1.1 as stated in theory. This means that the sum of the contributions collected is sufficient to cover the total cost of the claims reported in the pool at the end of the year.

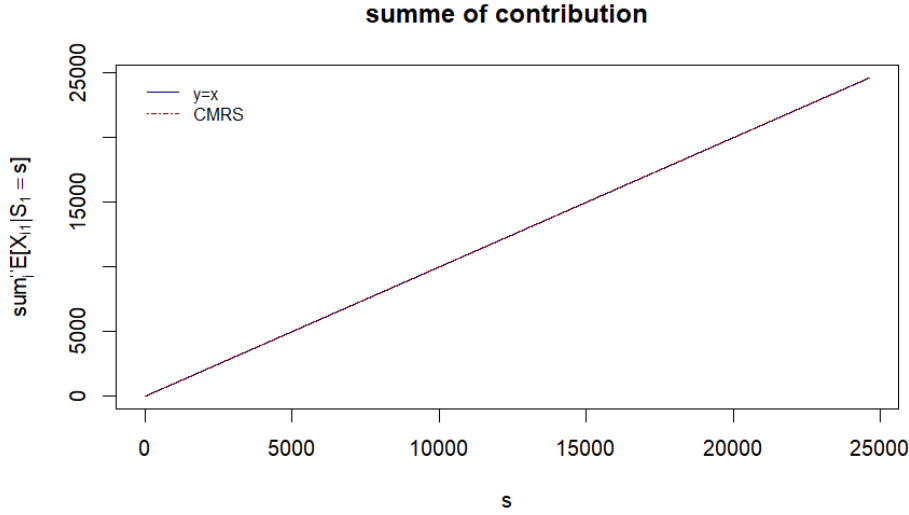


Figure 3.2.2: Total contributions at pool

### 3.2.2 Case of dynamic P2P insurance without cash-back at $t = 1, 2$

We consider the same the pool of  $n = 500$  participants with the set of parameters above. The retention level  $\omega = 65388$  covering the three years corresponds to the fixed target probability  $\varepsilon = 0.985$ . The application of the conditional mean risk sharing of Formula (2.3.14) to the pool is illustrated in Figure 3.2.3. Similarly, the conditional mean risk sharing is Pareto optimal. We can also see that the costs are representative of each participant’s profile.

Table 3.2.2 summarizes all the results of our calculations. We observe that the individual contributions to cover the retention at the pool level each year in this case are lower than the contributions obtained when the model guarantees surpluses in years  $t = 1$  and  $t = 2$ . This can be explained by the fact that, in the model without surpluses in these years, the organizer requires less contributions from the participants due to fewer constraints. On the other hand, by aggregating the claim costs over three years, these costs become more predictable and require a lower level of retention to ensure a surplus.

Group	Ni	$W_{i,3}^{P2P}$	Wi.P2P	$\mathbb{E}[Z_{i3}]$	Wi.SL	$W_{i,3}$	$\mathbb{E}[Bi.P2P]$
Group 1	250	49.973	16.668	42.855	0.031	50.066	7.151
Group 2	150	134.840	44.980	112.497	0.099	135.137	22.450
Group 3	100	326.684	108.984	266.663	0.270	327.492	60.314

Table 3.2.2: Results of the layer decomposition:  $n = 500$ ,  $\beta_{SL} = 0.5$  and  $\varepsilon = 0.985$

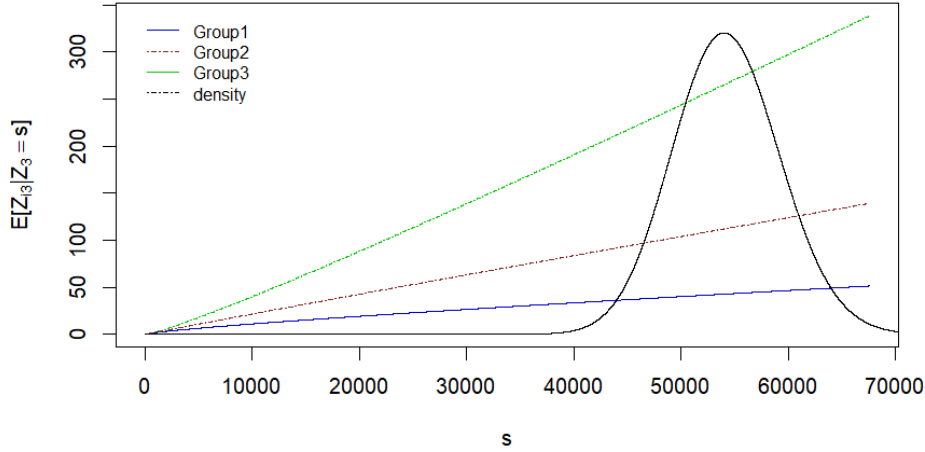


Figure 3.2.3: Respective contributions  $s \mapsto \mathbb{E}[Z_{i3}|Z_3 = s]$  in Group1, Group2 and Group3

### 3.3 Contribution of participants with benevolent loan

#### 3.3.1 Comparison groups contributions in the P2P model with (re-)insurer

When the system uses the benevolent loan and has a strategy of repaying all loans over three years and then redistributing any surplus to participants, we can expect a reduction in contributions for the same target probability, compared with the model with a (re-)insurer who guarantees the surplus for each year. In fact, the retention for the pool in the first case is lower than in the model with (re-)insurer. For the purposes of illustration in this section, we consider the same pool with the same assumptions as in the case with (re-)insurer. Table 3.3.1 summarizes our results obtained with the benevolent loan and compares the individual contributions to cover lower layer each year obtained with a (re-)insurer. We also compare the obtained contribution with the benevolent loan over an horizon of two years.

Group	Ni	Wi.fund.3Y	Wi.fund.2Y	$\mathbb{E}[X_{i1}]$	$\mathbb{E}[B_{i,1}^{fund}].3Y$	$\mathbb{E}[B_{i,1}^{fund}].2Y$	Wi.P2P
Group 1	250	18.958	18.405	14.284	2.808	2.480	19.259
Group 2	150	52.297	50.510	37.498	8.894	7.834	53.275
Group 3	100	128.925	124.009	88.886	24.066	21.150	131.623

Table 3.3.1: Results of the layer decomposition:  $n = 500$  and  $\varepsilon = 0.985$ ,  $\delta = 0.6$

For the target probability  $\varepsilon = 0.985$ , the corresponding retention level is  $\omega = 25477$  over an horizon of three years. Recall that the surplus for year  $t = 1$  matches for the risk model with benevolent loan and the model with (re-)insurer by considering a same retention level. Since we first chose the target probability, then the objectives of the pool lead to different retention levels. We observe a more advantageous individual cash-back in the model with (re-)insurer. These results are a consequence of the mechanisms of both models. Figure 3.3.1 displays

the probability distribution for cash-back in the first year of each participant for model over three years. When varying the pool's retention level  $\omega$  through the target probability  $\varepsilon$  with which shareholders are guaranteed repayment over three years a bonus is granted to the group participants, it is observed that the volatility of contributions gradually increases and then tends to stabilize at a certain level. This can be seen in Figure 3.3.2, where the variability of contributions in the lower layer is illustrated for each group.

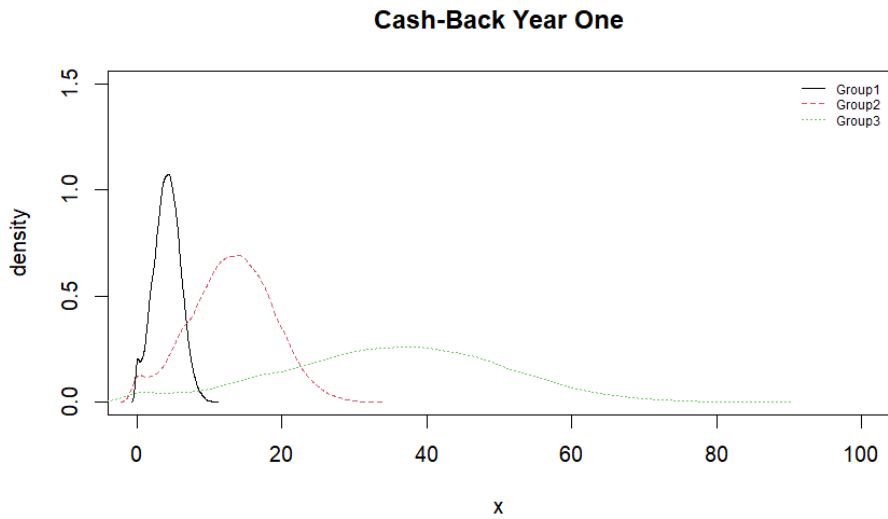


Figure 3.3.1: Probability distribution of cash-back  $B_{i,1}^{fund}$

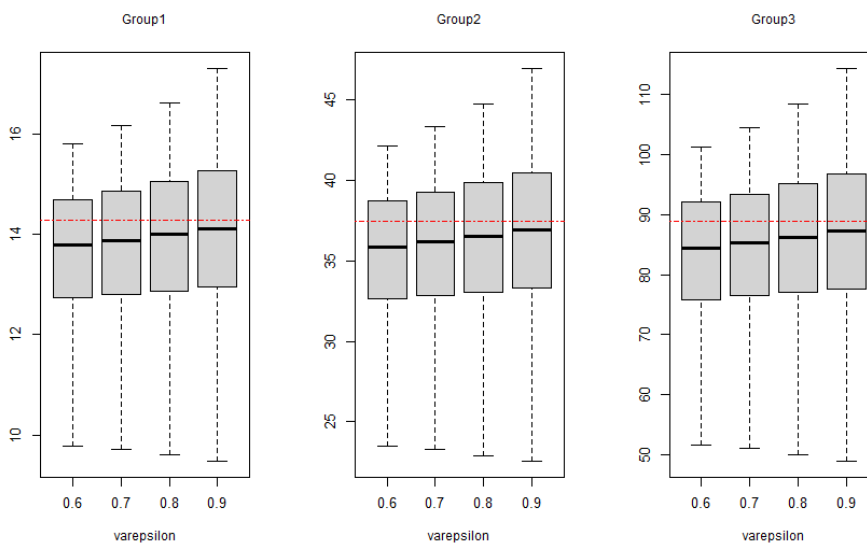


Figure 3.3.2: Evolution in retention and effect on individual contributions

# Conclusion

At the end of this work, it is worth noticing that our aim was the actuarial modeling of Takaful insurance with its particular feature of benevolent loans. We particularly study the solvency of such a mechanism over the finite horizon of two years or three years. This model offers an alternative to collaborative insurance with the intervention of a (re-)insurer. The study of the solvency of this model in this dissertation strengthens the commitment of the participants within the pool and motivates the Takaful fund shareholders to provide benevolent loans. To achieve this, we first conducted a comparative analysis of different forms of insurance: traditional insurance, P2P insurance, and Takaful insurance. In this sense, we drew a parallel with P2P insurance including the (re-)insurer, which already exists and also has the advantage of having a mechanism similar to that of Takaful insurance, namely the mutualization of risks for a group of individuals to mitigate the individual costs of claims and the guarantee of a surplus for participants at the end of each year.

We have thus presented the P2P model with (re-)insurer over one year, then considered the dynamic model over a finite horizon of several years, and finally proposed a Mudarabah-Takaful model. This latter model distributes profits to participants in the form of reduced individual contributions and to shareholders in the case of a surplus at the end of each year. It is noteworthy that this model is inspired by the dynamic P2P insurance model, with the key difference being that the (re-)insurer is replaced by the benevolent loan.

As with the P2P model with (re-)insurer over one year, the study of conditional mean risk sharing in chapter 2 as a sharing rule revealed its potential for establishing an equitable distribution of the total claim costs within the Mudarabah-Takaful pool and dynamic P2P with (re-)insurer, considering what each participant individually contributes in terms of claims to the pool. This rule could be intuitively appealing because it is based on the familiar concept of the mean, and it is easy to implement. Transparency is complete since participants can be informed from the outset about the reimbursement mechanism for benevolent loans and/or the reduction of individual contributions at the beginning of each year, according to the total observed cost at the end of the previous year. Through the numerical illustrations in chapter 3, we observed that individual contributions for the same target probability to achieve the pool's specific objectives (retention level) are more advantageous for the Mudarabah-Takaful model compared to the dynamic P2P model with (re-)insurer.

Key parameters of the system have proven to be significantly important in controlling the stability of contributions within the Takaful pool, particularly the number of participants and the time horizon. The group can leverage these parameters to achieve specific objectives regarding contribution stability. The illustrations in Section 3.2.2 revealed that, in the dynamic P2P model without surplus before the horizon, the organizer demands fewer contributions from participants due to reduced constraints. On the other hand, by aggregating the claim costs over the horizon before deducting the benefits, these costs become more predictable and require a lower retention level to ensure a surplus. On the other hand, we observed that the pool's retention level can also be used to increase contributions and decrease the risk for shareholders of providing a benevolent loan fund for the pool. It was noted that by increasing this retention, the contributions tend to stabilize.

We specifically addressed the solvency aspect for the cases of two-year and three-year horizons. Concretely, we control the risk taken by the shareholders for these fixed horizons by setting the target probability according to the objectives of both shareholders and participants. It should be noted that from the three-year horizon onward, the mechanism of benevolent loans becomes quite complex to control this probability, and a general formula as a function of the horizon has not yet been determined.

Observing the advantage of this model with benevolent loans over other P2P models, it would be interesting to explore this model in its pure form as described in the literature. Indeed, the pure Takaful model includes a threshold for the repayment of the benevolent loan in case of surplus or a threshold for the redistribution of benefits to the participants. The idea is to limit the volatility of participants' contributions over the horizon, making the pool even more attractive.

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# Appendix A

## Code R

### A.1 Package loading

```
Requirepackages<-c("actuar","distrEx","distr","latex2Cond.exp","ggplot2","gridExtra",
  "dplyr","plyr","plotrix","rgdal","rgeos","xtable","maptools",
  "scales","readr","questionr","labelled","ggpubr","purrr",
  "conflicted","reshape2","melt")

for (i in Requirepackages) {
  if(!require(i,character.only = T)) install.packages(i)
  require(i,character.only = T)
}
```

### A.2 Setting group parameters

```
n<-500
Param_Gr=data.frame(Group= seq(1,3,1),
  Lambda = c(5,10,20)/100,
  MeanSev = c(286,375,444),
  Alpha = c(2,3,4),
  beta = rep(5,3),
  Prop=c(0.5,0.30,0.20))

row.names(Param_Gr)<-as.character(Param_Gr$Group)
```

The following function returns the param parameter following the Group variable of the participant.

```
feat<-function(param=c("Lambda","MeanSev","Alpha","beta","Prop"),Group){
  res<-Param_Gr[as.character(Group),param]
  return(res)
}
```

#### A.2.1 test

```

grouptest=c(1,2,3)
#number of year
t<-1
# claim frequency of Group1
feat("Lambda",grouptest[1])

#Cond.expected total loss of pool
Mean_Z_t<-n*feat("Prop",grouptest[1])*t*feat("Lambda",grouptest[1])*
  feat("MeanSev",grouptest[1])+n*feat("Prop",grouptest[2])*t*
  feat("Lambda",grouptest[2])*feat("MeanSev",grouptest[2])+
  n*feat("Prop",grouptest[3])*t*feat("Lambda",grouptest[3])*
  feat("MeanSev",grouptest[3])
Mean_Z_t

```

## A.3 exact Frequency distribution and severities

### A.3.1 Representation for each Group = 1, 2, 3

We display the profiles of risk for the pool

```

# exact Frequency distribution
par(mfrow=c(1,2))
u=seq(0,5,1)
plot(u,dpois(u,feat("Lambda",grouptest[1])),type="b",lty=1,xlab="k",
      ylab=list(TeX('$\\P[N_i=k]$')),col="darkblue")

lines(u,dpois(u,feat("Lambda",grouptest[2])),type="b",col="darkred")

lines(u,dpois(u,feat("Lambda",grouptest[3])),type="b",lty=4,col="darkgreen")

legend("topright",inset = 0.02,cex = 0.7,legend = c("Group1","Group2","Group3"),
      col = c("darkblue","darkred","darkgreen"),lty = c(1,4),box.lty=0)

# exact severities distribution
y=seq(0,1000,1)

dGP1<-dbeta(y/1000,shape1=feat("Alpha",grouptest[1]),
            shape2= feat("beta",grouptest[1]))
dGP2<-dbeta(y/1000,shape1=feat("Alpha",grouptest[2]),
            shape2=feat("beta",grouptest[2]))
dGP3<-dbeta(y/1000,shape1=feat("Alpha",grouptest[3]),
            shape2= feat("beta",grouptest[3]))
plot(y,dGP1,type="l",lty=1,xlab="k",
      ylab=list(TeX('$f_{C_i}(x)$')),col="darkblue")

lines(y,dGP2,type="l",col="darkred")

lines(y,dGP3,type="l",lty=4,col="darkgreen")

legend("topright",inset = 0.02,cex = 0.7,legend = c("Group1","Group2","Group3"),
      col = c("darkblue","darkred","darkgreen"),lty = c(1,4),box.lty=0)

```

### A.3.2 Obtained severities by discretizing the beta distribution

```
## Severities for each Group and Representation
```

```

# Generate a sequence of value for Ci
a=0
b=1
step=0.001
x <- seq(a, b, step)

##### Group1#####

# Generate distribution for discrete severity values for each claim
discr_G1 <- discretize(pbeta(x,shape1=feat("Alpha",grouptest[1]),
                             shape2= feat("beta",grouptest[1])),
                      method = "unbiased",
                      lev =levbeta(x,shape1=feat("Alpha",
grouptest[1]),shape2= feat("beta",grouptest[1])),
                      from = a, to = b, step = step)
G1<-data.frame(k=seq(a,b*1000,step*1000),G1=discr_G1*10)

##### Group2 #####

# Generate distribution for discrete severity values for each claim
discr_G2 <- discretize(pbeta(x,shape1=feat("Alpha",grouptest[2]),
                             shape2= feat("beta",grouptest[2])),
                      method = "unbiased",
                      lev =levbeta(x,shape1=feat("Alpha",grouptest[2]),
                             shape2= feat("beta",grouptest[2])),
                      from = a, to = b, step = step)
G2<-data.frame(k=seq(a,b*1000,step*1000),G2=discr_G2*10)

##### Group3 #####

# Generate distribution for discrete severity values for each claim
discr_G3 <- discretize(pbeta(x,shape1=feat("Alpha",grouptest[3]),
                             shape2= feat("beta",grouptest[3])),
                      method = "unbiased",
                      lev =levbeta(x,shape1=feat("Alpha",grouptest[3]),
                             shape2= feat("beta",grouptest[3])),
                      from = a, to = b, step = step)
G3<-data.frame(k=seq(a,b*1000,step*1000),G3=discr_G3*10)

```

### A.3.3 Representation of Probability mass functions of the discretized sever-

```

...
par(mfrow=c(1,3))

#discrete severity distribution
pas=30
G1<-subset(G1,k %% pas==0)
G2<-subset(G2,k %% pas==0)
G3<-subset(G3,k %% pas==0)

plot(y,dGP1/100,
     type="l",xlab="k",ylab=list(TeX('$P[C_{ij}=k]$')),main="Group1",col="red")

lines(G1$k,G1$G1,type="h",col="black")

```

```

legend("topright",inset = 0.02,cex = 0.8,legend = c("exact","discrete"),
      col=c("red","black"),lty=c(1,1),box.lty=0)

plot(y,dGP2/100,
     type="l",xlab="k",ylab=list(TeX('$P[C_{ij}=k]$')),main="Group2",col="red")

lines(G2$k,G2$G2,type="h",col="black")

legend("topright",inset = 0.02,cex = 0.8,legend = c("exact","discrete"),
      col=c("red","black"),lty=c(1,1),box.lty=0)

plot(y,dGP3/100,
     type="l",xlab="k",ylab=list(TeX('$P[C_{ij}=k]$')),main="Group3",col="red")

lines(G3$k,G3$G3,type="h",col="black")

legend("topright",inset = 0.01,cex = 0.8,legend = c("exact","discrete"),
      col=c("red","black"),lty=c(1,1),box.lty=0)

# Plot the histogram
#ggplot(G1, aes(x =k, y =G1)) +
# geom_bar(stat = "identity") +
# labs(title = "Group1", x = "k", y =("$P[C_{1}=k]$"))

#ggplot(G2, aes(x =k, y =G2)) +
# geom_bar(stat = "identity") +
# labs(title = "Group1", x = "k", y =Cond.expression(P[C_2=k]))

#ggplot(G3, aes(x =k, y =G3)) +
# geom_bar(stat = "identity") +
# labs(title = "Group1", x = "k", y =Cond.expression(P[C_3=k]))

```

## A.4 Distribution of total individual costs $Z_{it}$ with $Z_{i1} = X_{i1} = X_i$

### A.4.1 Calculation of cumulative distribution functions for all groups

```

### Choice of the horizon \tau=3
t<-1
FZ_it<-function(Group,t){
  ## Function for calculating the cumulative distribution function of the total
  ## individual cost according to the participant's group
  a=0
  b=10*qbeta(0.9999,shape1=feat("Alpha",Group),shape2= feat("beta",Group))
  step=0.001
  x <- seq(a, b, step)

  discr<-discretize(pbeta(x,shape1=feat("Alpha",Group),
                        shape2= feat("beta",Group)),
                  method = "unbiased",
                  lev =levbeta(x,shape1=feat("Alpha",
                                             Group),shape2= feat("beta",Group)),
                  from = a, to = b, step = step)
  Fx<- aggregateDist("recursive", model.freq = "poisson", model.sev=discr,
                    maxit=200000,lambda = t*feat("Lambda",Group))
  return(Fx)
}

```

#### A.4.2 Calculation of cumulative distribution functions for a participant of Group = 1, 2, 3.

```
## Calculation of cumulative distribution functions for Group1
FZ_it.GP1<-FZ_it(grouptest[1],t)
## Calculation of cumulative distribution functions for Group2
FZ_it.GP2<-FZ_it(grouptest[2],t)
## Calculation of cumulative distribution functions for Group3
FZ_it.GP3<-FZ_it(grouptest[3],t)
summary(FZ_it.GP3)
```

#### A.4.3 Representation of cdfs for a participant of Group = 1, 2, 3.

```
##mat<-matrix(c(1,2,3,3),nrow = 2,ncol=2,byrow = TRUE)
#layout(mat)
par(mfrow=c(1,3), family = "sans", font.lab = 2, font.axis = 1, cex.lab=0.9,
    cex.main=0.9, font.main=1,font.lab=1,font.axis=1)

plot(FZ_it.GP1, main = list(TeX('Distribution of  $X_{i,GP1}$  Group1')), xlab="k",
    sub = "",ylab = list(TeX('$F_{X_{i,GP1}}(k)$')),
    xlim=c(0,quantile(FZ_it.GP1,0.9999)),
    ylim=c(FZ_it.GP1(0),1))

plot(FZ_it.GP2, main = list(TeX('Distribution of  $X_{i,GP2}$  Group2')), xlab="k",
    sub = "",ylab = list(TeX('$F_{X_{i,GP2}}(k)$')),
    xlim=c(0,quantile(FZ_it.GP2,0.9999)),
    ylim=c(FZ_it.GP2(0),1))

plot(FZ_it.GP3, main = list(TeX('Distribution of  $X_{i,GP3}$  Group3')), xlab="k",
    sub = "",ylab = list(TeX('$F_{X_{i,GP3}}(k)$')),
    xlim=c(0,quantile(FZ_it.GP3,0.9999)),
    ylim=c(FZ_it.GP3(0),1))
```

##Representation of the probability mass function of the individual cost  $Z_{it}$ . ##

```
sup1<-quantile(FZ_it.GP1,0.9999)
u1=seq(0,sup1,1)

sup2<-quantile(FZ_it.GP2,0.9999)
u2=seq(0,sup2,1)

sup3<-quantile(FZ_it.GP3,0.9999)
u3=seq(0,sup3,1)
##probability distribution

## pdf for Group1
FZ.GP1<-DiscreteDistribution(supp=knots(FZ_it.GP1), prob =diff(FZ_it.GP1))
## pdf for Group2
FZ.GP2<-DiscreteDistribution(supp =knots(FZ_it.GP2), prob =diff(FZ_it.GP2))
## pdf for Group3
FZ.GP3<-DiscreteDistribution(supp =knots(FZ_it.GP3), prob =diff(FZ_it.GP3))
```

```

par(mfrow=c(1,3))
valX_GP1=list(valX_GP1=u1[1:sup1],probX_GP1=diff(FZ_it.GP1(u1)))
valX_GP2=list(valX_GP2=u2[1:sup2],probX_GP2=diff(FZ_it.GP2(u2)))
valX_GP3=list(valX_GP3=u3[1:sup3],probX_GP3=diff(FZ_it.GP3(u3)))

#Group1
k1=15#number of discretization steps
plot(valX_GP1$valX_GP1[seq(0,sup1,k1)],valX_GP1$probX_GP1[seq(0,sup1,k1)],xlab="k",
      ylab=TeX(r"${P}[X_{i,GP1} = k]${}"), type="h")

#lines(valX_GP1$valX_GP1[seq(0,sup1,k1)],valX_GP1$probX_GP1[seq(0,sup1,k1)],xlab="x",
#      ylab=TeX(r"${P}[X_{i,GP2} = k]${}"), type="l",col="blue3")
title(main = "Group 1")

#Group2
k2=20#number of discretization steps
plot(valX_GP2$valX_GP2[seq(0,sup2,k2)],valX_GP2$probX_GP2[seq(0,sup2,k2)],xlab="k",
      ylab=TeX(r"${P}[X_{i,GP2} = k]${}"), type="h")

#lines(valX_GP2$valX_GP2[seq(0,sup2,k2)],valX_GP2$probX_GP2[seq(0,sup2,k2)],xlab="k",
#      ylab=TeX(r"${P}[X_{i,GP2} = k]${}"), type="l",col="blue3")
title(main = "Group 2")

#Group3
k3=25#number of discretization steps
plot(valX_GP3$valX_GP3[seq(0,sup3,k3)],valX_GP3$probX_GP3[seq(0,sup3,k3)],xlab="k",
      ylab=TeX(r"${P}[X_{i,GP3} = k]${}"), type="h")

#lines(valX_GP3$valX_GP3[seq(0,sup3,k3)],valX_GP3$probX_GP3[seq(0,sup3,k3)],xlab="x",
#      ylab=TeX(r"${P}[X_{i,GP3} = k]${}"), type="l",col="blue3")
title(main = "Group 3")

```

```

Gx <- function(m, Group) {
  # Discretized severities
  a <- 0
  b <- qbeta(1,shape1=feat("Alpha",Group),shape2= feat("beta",Group))
  step <- 0.001
  x <- seq(a, b, step)

  discr <- discretize(pbeta(x, shape1 = feat("Alpha", Group),
                           shape2 = feat("beta", Group)),
                    method = "unbiased",
                    lev = levbeta(x, shape1 = feat("Alpha", Group),
                                   shape2 = feat("beta", Group)),
                    from = a, to = b, step = step)

  discr <- data.frame(k = seq(a, b * 1000, step * 1000), discr = discr)

  # Calculate the cdf of Ci for discrete values
  prob <- sapply(m, function(mi) sum(discr$discr[discr$k < mi]))

  return(prob)
}

```

```
}

```

## A.5 Exact Calculation distribution of $Z_t$

### A.5.1 Calculation of the required claims costs cdfs for $Z_t$ and $Z_t - Z_{it}$ at time $t$

The following function can be used to calculate the cumulative distribution function of the formula (2.3.42) and for a given participant the cumulative distribution function of its severities for  $Z_t - Z_{it}$  that the formula is given in Equation (??). The cumulative distribution function is calculated for each of the groups to which  $Z_{it}$  can belong. The LZ() function is used to calculate the cumulative distribution functions.

```
LZ <- function(m,t,n){
  # x= value of x for which we calculate L(x)=FY=P[Y'=x] or L'(x)=FT=P[T=x]
  ## Lambda total
  lambda_dot <- n*feat("Prop",grouptest[1])*t*feat("Lambda",grouptest[1])+
    n*feat("Prop",grouptest[2])*t*feat("Lambda",grouptest[2])+
    t*feat("Lambda",grouptest[3])*n*feat("Prop",grouptest[3])

  ##### Calculation cumulative distribution function L(x)
  ## (collective model for Stop-loss)
  Val <-n*feat("Prop",grouptest[1])*t*feat("Lambda",grouptest[1])*Gx(m ,grouptest[1])+
n*feat("Prop",grouptest[2])*t*feat("Lambda",grouptest[2])*Gx(m ,grouptest[2]) +
n*feat("Prop",grouptest[3])*t*feat("Lambda",grouptest[3])*Gx(m ,grouptest[3])
  Value =Val /lambda_dot

  ##### Removing a Group1 participant: function L'(x)
  Val_GP1 =(n*feat("Prop",grouptest[1])-1)*t*feat("Lambda",grouptest[1])*
Gx(m,grouptest[1]) +n*feat("Prop",grouptest[2])*t*feat("Lambda",grouptest[2])*
Gx(m,grouptest[2]) +
n*feat("Prop",grouptest[3])*t*feat("Lambda",grouptest[3])*Gx(m,grouptest[3])
  Value_GP1 =Val_GP1 /(lambda_dot-t*feat("Lambda",grouptest[1]))

  ##### Removing a Group2 participant: function L'(x)
  Val_GP2 =n*feat("Prop",grouptest[1])*t*feat("Lambda",grouptest[1])*Gx(m,grouptest[1])+
(n*feat("Prop",grouptest[2])-1)*t*feat("Lambda",grouptest[2])*Gx(m,grouptest[2]) +
n*feat("Prop",grouptest[3])*t*feat("Lambda",grouptest[3])*Gx(m,grouptest[3])
  Value_GP2 =Val_GP2 /(lambda_dot-t*feat("Lambda",grouptest[2]))

  ##### Removing a Group3 participant: function L'(x)
  Val_GP3 =n*feat("Prop",grouptest[1])*t*feat("Lambda",grouptest[1])*Gx(m,grouptest[1]) +
n*feat("Prop",grouptest[2])*t*feat("Lambda",grouptest[2])*Gx(m,grouptest[2]) +
(n*feat("Prop",grouptest[3])-1)*t*feat("Lambda",grouptest[3])*Gx(m,grouptest[3])
  Value_GP3 =Val_GP3 /(lambda_dot-t*feat("Lambda",grouptest[3]))

  ### Result
  result<-list(LZ= Value, LZ_GP1=Value_GP1,
    LZ_GP2 = Value_GP2,LZ_GP3 = Value_GP3,
    Lda = lambda_dot, Lda_GP1= lambda_dot-t*feat("Lambda",grouptest[1]),
    Lda_GP2=lambda_dot-t*feat("Lambda",grouptest[2]),
    Lda_GP3= lambda_dot-t*feat("Lambda",grouptest[3]))
  return(result)
}
```

```
}

```

### A.5.2 Calculation of the distribution function of $Z_t$ : Panjer's algorithm

The following function AGGRe uses the result of the previous function to calculate the cumulative distribution functions of  $Z_t$  and  $Z_t - Z_{it}$  for  $Z_{it}$  belonging to all the groups.

```
timer<- Sys.time()
u<-seq(0,6000,1)
## Compute the cdf for the severities
F.all<-LZ(u,t,n)

## cdf of Z_{t} for stop-loss
FZ_t<-aggregateDist("recursive", model.freq = "poisson",
                    model.sev = diff(F.all$LZ),
                    lambda = F.all$Lda, maxit=200000)
FZ_t=DiscreteDistribution(supp=knots(FZ_t),prob=diff(FZ_t))

## cumulative distribution function of Zt-Zit where Zit belongs
##to a Group1 participant
ZZit_GP1<- aggregateDist("recursive", model.freq = "poisson",
                        model.sev = diff(F.all$LZ_GP1),
                        lambda = F.all$Lda_GP1, maxit=200000)
ZZit_GP1=DiscreteDistribution(supp=knots(ZZit_GP1),prob=diff(ZZit_GP1))

## cumulative distribution function of Zt-Zit where Zit belongs
##to a Group2 participant
ZZit_GP2<- aggregateDist("recursive", model.freq = "poisson",
                        model.sev = diff(F.all$LZ_GP2),
                        lambda = F.all$Lda_GP2, maxit=200000)
ZZit_GP2=DiscreteDistribution(supp=knots(ZZit_GP2),prob=diff(ZZit_GP2))

## cumulative distribution function of Zt-Zit where Zit
##belongs to a Group3 participant
ZZit_GP3<- aggregateDist("recursive", model.freq = "poisson",
                        model.sev = diff(F.all$LZ_GP3),
                        lambda = F.all$Lda_GP3, maxit=200000)
ZZit_GP3=DiscreteDistribution(supp=knots(ZZit_GP3),prob=diff(ZZit_GP3))
print(Sys.time()-timer)
summary(FZ_t)

```

### A 5 3 Representation of distribution for $Z_t$

```
plf<-q(FZ_t)(1)
u<-seq(1,plf,1)
v<-seq(0,1,1/plf)
### Display FZ_t
par(mfrow=c(1,3),font.main=1,cex.main = 0.9, cex.sub = 0.8)

plot(u,d(FZ_t)(u),type="h",xlab="k",ylab=TeX(r"${P}[Z_{t}=k]$"),
     main="Probability mass function")

plot(u,p(FZ_t)(u),type="l",xlab="q",ylab="p(q)",main="CDF")

```

```
plot(v,q(FZ_t)(v),type="l",xlab="p",ylab="q(p)",main="Quantile")
```

```

A.5.4 Display cumulative distribution function of  $Z_t$  at  $Z_t = 7$ 

### Display FZ_t
par(mfrow=c(1,3),font.main=1,cex.main = 0.9, cex.sub = 0.8)

plot(u,d(FZ_t)(u),type="l",xlab="x",ylab="d(x)",
     main="Probability mass function")
lines(u,d(ZZit_GP1)(u),lty=2, col="darkblue")
lines(u,d(ZZit_GP2)(u),lty=4, col="darkred")
lines(u,d(ZZit_GP3)(u),lty=1, col="darkgreen")
legend("topleft",inset = 0.02,cex = 0.8,legend = c("Group1","Group2","Group3"),
      col=c("darkblue","darkred","darkgreen"),lty=c(1,4),box.lty=0)

plot(u,p(FZ_t)(u),type="l",xlab="q",ylab="p(q)",main="CDF")
lines(u,p(ZZit_GP1)(u),lty=2, col="darkblue")
lines(u,p(ZZit_GP2)(u),lty=4, col="darkred")
lines(u,p(ZZit_GP3)(u),lty=1, col="darkgreen")
legend("topleft",inset = 0.02,cex = 0.8,legend = c("Group1","Group2","Group3"),
      col=c("darkblue","darkred","darkgreen"),lty=c(1,4),box.lty=0)

plot(v,q(FZ_t)(v),type="l",xlab="p",ylab="q(p)",main="Quantile")
lines(v,q(ZZit_GP1)(v),lty=2, col="darkblue")
lines(v,q(ZZit_GP2)(v),lty=4, col="darkred")
lines(v,q(ZZit_GP3)(v),lty=1, col="darkgreen")
legend("topleft",inset = 0.02,cex = 0.8,legend = c("Group1","Group2","Group3"),
      col=c("darkblue","darkred","darkgreen"),lty=c(1,4),box.lty=0)

```

## A.6 Conditional Cond. expectation: CMRS.

```

Cond.exp<-function(s,FZ,ZZit_GP1,ZZit_GP2,ZZit_GP3){
  ## Function to compute the conditional Cond. expectation
  ## of  $X_{GP1}$  or  $X_{GP2}$  or  $X_{GP3}$  given  $S_1=s$ .
  ## s: realization of  $Z_{\{t\}}$ 
  ## FZ: cdf of  $Z_{\{t\}}$ 
  ## ZZit_GP1: cdf of  $\$Z_t-Z_{\{it,GP1\}}\$$ 
  ## ZZit_GP2: cdf of  $\$Z_t-Z_{\{it,GP2\}}\$$ 
  ## ZZit_GP3: cdf of  $\$Z_t-Z_{\{it,GP3\}}\$$ 
  ## Returns  $E[X_{GP1}|S_1=s]$ ,  $E[X_{GP2}|S_1=s]$  and  $E[X_{GP3}|S_1=s]$ 
  if(s==0){
    p_GP1=0
    p_GP2=0
    p_GP3=0
  } else{
    ps=p(FZ)(s)-p(FZ)(s-1) #Pr[S=s]
    ## Group1
    pz_GP1=FZ_it.GP1(1:s)-FZ_it.GP1(1:s-1) # Pr[Xi=k] k from 1 to s
    pzz_GP1=p(ZZit_GP1)(s-1:s)-p(ZZit_GP1)(s-1:s-1) # Pr[S-Xi=s-k]
    p_GP1=sum((1:s)* pz_GP1*pzz_GP1/ps) # E[Xi|S=s]

    ## Group2
    pz_GP2=FZ_it.GP2(1:s)-FZ_it.GP2(1:s-1) # Pr[Xi=k] k from 1 to s

```

```

pzz_GP2=p(ZZit_GP2)(s-1:s)-p(ZZit_GP2)(s-1:s-1) # Pr[S-Xi=s-k]
p_GP2=sum((1:s)* pz_GP2*pzz_GP2/ps) # E[Xi|S=s]

## Group1
pz_GP3=FZ_it.GP3(1:s)-FZ_it.GP3(1:s-1) # Pr[Xi=k] k from 1 to s
pzz_GP3=p(ZZit_GP3)(s-1:s)-p(ZZit_GP3)(s-1:s-1) # Pr[S-Xi=s-k]
p_GP3=sum((1:s)* pz_GP3*pzz_GP3/ps) # E[Xi|S=s]

}
return(list(E_GP1=p_GP1,E_GP2= p_GP2,E_GP3= p_GP3))
}

timer<-Sys.time()
set.seed(1234)
A<-Cond.exp(s=1500,FZ_t,ZZit_GP1,ZZit_GP2,ZZit_GP3)

Sys.time()-timer
A
total.contri<-(n*feat("Prop",grouptest[1])*
               A$E_GP1+n*feat("Prop",grouptest[2])*A$E_GP2+
               n*feat("Prop",grouptest[3])*A$E_GP3)
total.contri
rm(A)

```

### A.6.1 Application of CMRS for $n = 500$ participants

We compute the value of CMRS of each participant by group

```

timer=Sys.time()
s=seq(0,q(FZ_t)(0.985^(1/3)),1)

result=data.frame(s=s,CMRG1=rep(NA,length(s)),CMRG2=rep(NA,length(s)),
                  CMRG3=rep(NA,length(s)),Total=rep(NA,length(s)))
for (i in 1:length(s)) {
  contri=Cond.exp(result$s[i],FZ_t,ZZit_GP1,ZZit_GP2,ZZit_GP3)
  result$CMRG1[i]= contri$E_GP1
  result$CMRG2[i]= contri$E_GP2
  result$CMRG3[i]= contri$E_GP3
  result$Total[i]= n*feat("Prop",grouptest[1])*
  contri$E_GP1+n*feat("Prop",grouptest[2])*
  contri$E_GP2+n*feat("Prop",grouptest[3])*contri$E_GP3
}
Sys.time()-timer

##Representation of the different individuals contributions from the groups.
# contribution in Group 1, Group 2 and Group 3 for n=500
plot(s,result$CMRG3,col="green3",type="l",lty=4,
     ylab=list(TeX('$E[Z_{it}|Z_{t}=s]$'))
     ,xlab = "s", ylim=c(0,max(result$CMRG3,result$CMRG1,result$CMRG2)))

lines(s,result$CMRG1,type="l",lty=1,col="blue3")

lines(s,result$CMRG2,col="brown4",lty=4)

```

```

lines(u,d(FZ_t)(u)*(9e5),col="black",type="l",xlab="x",ylab="d(x)")

legend("topleft",inset = 0.02,cex = 0.8,
      legend=c("Group1","Group2","Group3","density"),
      col = c("blue3","brown4","green3","black"),lty=c(1,4),box.lty = 0)

##Verification of the total allocation property of conditional mean risk sharing.
plot(result$s,type="l",xlab="s",ylab=list(TeX('$sum_{i}^{n}E[X_{i1}|S_{1}=s]$')),
      col="darkblue",main="summe of contribution")

lines(result$Total,type="l",lty=4,col="darkred")

legend("topleft",inset = 0.02,cex = 0.8,
      legend=c("y=x","CMRS"),
      col = c("darkblue","darkred"),lty=c(1,4),box.lty = 0)

```

## A.7 System with (re-)insurer

### A.7.1 Sensitivity study of contributions: Size

Size of pool

```

timer=Sys.time()
t<-1
n=c(100,300,500,1000,1500,2500)
cdf.Z_t.n=list()
cdf.ZZit.n_GP1=list()
cdf.ZZit.n_GP2=list()
cdf.ZZit.n_GP3=list()
quant.Z_t.n=rep(NA,length(n))

for (i in 1:length(n)){
  U=seq(0,6000,1)

  ##Z_t
  fZ_t.n= LZ(U,t,n[i])
  FZ_t.n=aggregateDist("recursive", model.freq = "poisson",
                      model.sev=diff(fZ_t.n$ LZ),
                      lambda=fZ_t.n$Lda,
                      maxit = 1500000)
  FZ_t.n=DiscreteDistribution(supp = knots(FZ_t.n),prob = diff(FZ_t.n))

  ##Group1: Zt.n-Zit_GP1
  fZZit.n_GP1=LZ(U,t,n[i])
  FZZit.n_GP1=aggregateDist("recursive", model.freq = "poisson",
                          model.sev=diff( fZZit.n_GP1$ LZ_GP1),
                          lambda=fZZit.n_GP1$Lda_GP1,maxit =1500000)
  FZZit.n_GP1=DiscreteDistribution(supp = knots(FZZit.n_GP1),prob = diff(FZZit.n_GP1))

  ##Group2: Zt.n-Zit_GP2
  fZZit.n_GP2=LZ(U,t,n[i])
  FZZit.n_GP2=aggregateDist("recursive", model.freq = "poisson",
                          model.sev=diff( fZZit.n_GP2$ LZ_GP2),
                          lambda=fZZit.n_GP2$Lda_GP2,maxit =1500000)
  FZZit.n_GP2=DiscreteDistribution(supp = knots(FZZit.n_GP2),prob = diff(FZZit.n_GP2))

```

```

##Group3: Zt.n-Zit_GP3
  fZZit.n_GP3=LZ(U,t,n[i])
FZZit.n_GP3=aggregateDist("recursive", model.freq ="poisson",
                          model.sev=diff( fZZit.n_GP3$LZ_GP3),
                          lambda=fZZit.n_GP3$Lda_GP3,maxit =1500000)
FZZit.n_GP3=DiscreteDistribution(supp = knots(FZZit.n_GP3),prob = diff(FZZit.n_GP3))

quant.Z_t.n[i]=min(q(FZ_t.n)(0.9999),q(FZZit.n_GP1)(0.9999),
                  q(FZZit.n_GP2)(0.9999),q(FZZit.n_GP3)(0.9999))

cdf.Z_t.n[[length(cdf.Z_t.n)+1]]=FZ_t.n
cdf.ZZit.n_GP1[[length(cdf.ZZit.n_GP1)+1]]=FZZit.n_GP1
cdf.ZZit.n_GP2[[length(cdf.ZZit.n_GP2)+1]]=FZZit.n_GP2
cdf.ZZit.n_GP3[[length(cdf.ZZit.n_GP3)+1]]=FZZit.n_GP3
  rm(fZ_t.n,FZ_t.n,FZZit.n_GP1,fZZit.n_GP1,FZZit.n_GP2,fZZit.n_GP2,FZZit.n_GP3,fZZit.n_GP3)
}
print(Sys.time()-timer)
names(cdf.Z_t.n)=paste("N",as.character(n),sep="")
names(cdf.ZZit.n_GP1)=paste("N",as.character(n),sep="")
names(cdf.ZZit.n_GP2)=paste("N",as.character(n),sep="")
names(cdf.ZZit.n_GP3)=paste("N",as.character(n),sep="")

##Contribution based on number of participants
n=c(100,300,500,1000,1500,2500)
cmrs.GP1=list()
cmrs.GP2=list()
cmrs.GP3=list()
timer=Sys.time()
  for (i in 1:length(n)) {
S=seq(0,quant.Z_t.n[i],10)
S.quant=S/quant.Z_t.n[i]
esp.GP1=data.frame(S=S,S.quant=S.quant,C.exp=rep(NA,length(S)))
esp.GP2=data.frame(S=S,S.quant=S.quant,C.exp=rep(NA,length(S)))
esp.GP3=data.frame(S=S,S.quant=S.quant,C.exp=rep(NA,length(S)))

for (j in 1:length(S)) {

  #Conditional mean risk sharing
  contri=Cond.exp(s=S[j],cdf.Z_t.n[[i]],cdf.ZZit.n_GP1[[i]],
                 cdf.ZZit.n_GP2[[i]],cdf.ZZit.n_GP3[[i]])
  esp.GP1$C.exp[j]=contri$E_GP1
  esp.GP2$C.exp[j]=contri$E_GP2
  esp.GP3$C.exp[j]=contri$E_GP3
}
cmrs.GP1[[length(cmrs.GP1)+1]]=esp.GP1
cmrs.GP2[[length(cmrs.GP2)+1]]=esp.GP2
cmrs.GP3[[length(cmrs.GP3)+1]]=esp.GP3
rm(S,contri,esp.GP1,esp.GP2,esp.GP3)
}
print(Sys.time()-timer)
names(cmrs.GP1)=paste("N",as.character(n),sep="")
names(cmrs.GP2)=paste("N",as.character(n),sep="")
names(cmrs.GP3)=paste("N",as.character(n),sep="")

```

## A.8 System with cash-back at each year and re(-)insurer

```
##parameter
N=500
grouptest=c(1,2,3)

Creating pools
pool=function(grouptest,prop1=0.5,prop2=0.3,prop3=0.2,N){
  L=length(grouptest)
  gr1=ceiling(prop1*N)

  gr2=ceiling(prop2*N)

  gr3=N-(gr1+gr2)

  Ni=c(gr1,gr2,gr3)
  Grp=data.frame(Group=c("Group 1","Group 2","Group 3"),Ni=Ni)

  return(Grp)
}
Pool=pool(grouptest,0.5,0.3,0.2,N)

##Target probability for having a surplus
varepsilon<-0.985
##safety loading
beta_SL<-0.5
##Retention level
w=distr::q(cdf.Z_t.n[["N500"]])(varepsilon^(1/3))
w
##individual contributions to lower layer
wi.P2P<-Cond.exp(s=w,cdf.Z_t.n[["N500"]],cdf.ZZit.n_GP1[["N500"]],
                 cdf.ZZit.n_GP2[["N500"]],cdf.ZZit.n_GP3[["N500"]])
Pool$Wi.P2P<-t(as.data.frame(wi.P2P))[1:length(wi.P2P)]
sum(Pool$Ni[1:3]*Pool$Wi.P2P[1:3])
```

### A.8.1 Probability distribution of individual contributions

```
timer<-Sys.time()
n<-c(100,300,500,1000,1500,2500)
# number of values to generate
num<-15000
Law.N<-list()

for (i in 1:length(n)) {
  S<-r(cdf.Z_t.n[[i]])(num)
  esp<-data.frame(S=S,expM.GP1=rep(NA,num),expM.GP2=rep(NA,num),
                 expM.GP3=rep(NA,num),prob=distr::p(cdf.Z_t.n[[i]])(S))
  for (j in 1:length(S)) {
    contri=Cond.exp(s=S[j],cdf.Z_t.n[[i]],cdf.ZZit.n_GP1[[i]],
                   cdf.ZZit.n_GP2[[i]],cdf.ZZit.n_GP3[[i]])
    esp$expM.GP1[j]=contri$E_GP1
    esp$expM.GP2[j]=contri$E_GP2
    esp$expM.GP3[j]=contri$E_GP3
  }
  Law.N[[length(Law.N)+1]]=esp
}
```

```

  rm(S,esp,contri)
}
print(Sys.time()-timer)
names(Law.N)=paste("N",as.character(n),sep = "")
## save(Law.N,file="CMRS_Law.Rdata")
##Law.N <- load(file="CMRS_Law.Rdata")

```

## A.9 Application 1: Decomposition for a pool of $n = 500$

In this first illustration, we apply the layer decomposition to a pool of  $n = 500$  participants.

```

## Previously simulated values of  $E[X_i \mathbb{1}_{S=s}]$ 
CMRS.Layer<-Law.N[["N500"]][,-5]

timer = Sys.time()
## Vector containing the values of  $\min\{S,W\}$ 
Charge.P2P<-CMRS.Layer
Charge.P2P[1:nrow(CMRS.Layer),2:ncol(CMRS.Layer)] <- 0

## Vector containing the values of the premium SL
Wi.SL<-CMRS.Layer
Wi.SL[1:nrow(CMRS.Layer),2:ncol(CMRS.Layer)] <- 0

## Vector containing the values of cash-back
BI.t <-CMRS.Layer
BI.t[1:nrow(CMRS.Layer),2:ncol(CMRS.Layer)] <- 0

Pool$EZit<- rep(NA,nrow(Pool))
Pool

#cdf of different groups
FZ.t = list()
FZ.t[[1]] = FZ.GP1
FZ.t[[2]] = FZ.GP2
FZ.t[[3]] = FZ.GP3

for (i in 2:ncol(CMRS.Layer)) {
  # $E[Z_{it}]$ 
  Pool$EZit[i-1] <- E(FZ.t[[i-1]])

  # $S_{P2P}=\min\{h,w_i\}$ 
  Charge.P2P[,i] = ifelse(
    test = CMRS.Layer[,i] < Pool$wi[i-1],
    yes = CMRS.Layer[,i],
    no = Pool$wi[i-1])

  # Calcul de la prime stop-loss
  Wi.SL[,i] <- ifelse(
    test = CMRS.Layer[,i] - Pool$Wi[i-1] > 0,
    yes = CMRS.Layer[,i] - Pool$Wi[i-1],
    no = 0
  )

  # Cash-back
  BI.t[,i] <- ifelse(
    test = Pool$Wi.P2P[i-1]-CMRS.Layer[,i] > 0,
    yes = Pool$Wi.P2P[i-1]-CMRS.Layer[,i],

```

```

    no = 0
  )
}

print(Sys.time()-timer)
Pool$prime.SL <- (1+beta_SL)*colMeans(Wi.SL[,-1],na.rm = TRUE)
Pool$Cotis_tot <- Pool$prime.SL + Pool$Wi.P2P
Pool$Cotis_1Y<-Pool$Cotis_tot/t
Pool$Ex.Bi.P2P <- colMeans(BI.t[,-1],na.rm = TRUE)
#Pool$Cotis.red <- Pool$Cotis_tot-Pool$Ex.Bi.P2P
Grp.table <- Pool
Pool
## gÃ©nÃ©re le script latex pour le tableau Grp.table
#print( xtable(Grp.table,type="latex",digits = 3))

```

### A.9.1 Distribution of Cash-Back $B_t$

Since  $B_t$  can be rewritten as the random variable  $tw - Z_t | tw - Z_t > 0$ , we can obtain the distribution of  $B_t | B_t > 0$ . The following block allows us to determine the distribution of  $B_t$  and the distribution of cash-back.

```

# cdf of cash-back for the Pool
F_Bi <- function(x,FZ,omega){
  func <-( p(FZ)(omega)-p(FZ)(omega-x) )/p(FZ)(omega)
  return(func)
}

supp=seq(0,w,1)
prob=diff(F_Bi(supp,FZ_t,w))
cash.back<- DiscreteDistribution(supp=supp[-1],prob=prob)
rm(supp,prob)

plf<-q(cash.back)(0.999)
u<-seq(1,plf,1)
v<-seq(0,1,1/plf)
### Representation of FB_t
par(mfrow=c(1,3),font.main=1,cex.main = 0.9, cex.sub = 0.8)

plot(u,d(cash.back)(u),type="l",xlab="x",ylab="d(x)", main="Density")
plot(u,p(cash.back)(u),type="l",xlab="q",ylab="p(q)",main="CDF")
plot(v,q(cash.back)(v),type="l",xlab="p",ylab="q(p)",main="Quantile")

```

### A.9.2 Representation 3: Distribution of cash-back $B_{i,t} = B_{i,1}$ .

Representation of density probability for cash-back  $B_{i,1}$  for all participants by Group.

```

BI.t <- na.omit(BI.t)
pdf.Cash <- list()
for (i in colnames(BI.t[-1])) {
  den.Bi <- density(BI.t[,i])
  A <- data.frame(val=den.Bi$x, prob=den.Bi$y)
  pdf.Cash[[length(pdf.Cash)+1]] <- A
  rm(den.Bi, A)
}

```

```

}
#par(mfrow=c(1,2),family = "sans", font.lab = 2, font.axis = 1, cex.lab=0.9,
#   cex.main=0.9, font.main=1,font.lab=1,font.axis=1)
plot(pdf.Cash[[1]]$val, pdf.Cash[[1]]$prob*5, type = "l", lty = 1,
     col = 1, ylim = c(0,1.5), xlim = c(0,100),
     xlab = "x", ylab = "density", main = "Cash-Back with (re-)insurer")
for (i in 2:3) {
  lines(pdf.Cash[[i]]$val, pdf.Cash[[i]]$prob*10, type = "l", lty = i,
        col = i)
}
legend("topright", cex = 0.6, legend = c("Group1", "Group2", "Group3"), col = 1:3, lty=1:3,
      box.lty=0, inset = 0.005)

```

## A.10 Variability of contributions in lower layer $[0, \omega]$ .

### A.10.1 Comparison with a Variation in the Number of Participants

Study of variability in the lower layer Based on the number of participants

```

timer <- Sys.time()
#number of participants.
n<-c(100,300,500,1000,1500,2500)

##Target probability for having a surplus
varepsilon<-0.985
##safety loading
beta_SL<-0.5
GRP=list()

down.level.N = list()
up.level.N = list()
SL.prem = data.frame(Grp=seq(1,3,1))

for (i in 1:length(n)) {
  #Creation of a Pool with the Three Groups
  GRP[[i]]=pool(grouptest,0.5,0.3,0.2,n[i])
  Grp.level = pool(grouptest,0.5,0.3,0.2,n[i])

  # cdf of Z_t.n
  Fpart = cdf.Z_t.n[[i]]

  # retention level for pool
  w_n = distr::q(cdf.Z_t.n[[i]])(varepsilon^(1/3))

  # individual contributions Wi
  wi = Cond.exp(s=w_n,FZ=cdf.Z_t.n[[i]],cdf.ZZit.n_GP1[[i]],
               cdf.ZZit.n_GP2[[i]],cdf.ZZit.n_GP3[[i]])
  Grp.level$Wi = t(as.data.frame(wi))[1:length(wi)]

  GRP[[i]]$Wi = Grp.level$Wi
  # Recovering E[Zit/Zt.n] values for different values of n
  cmrs.level = Law.N[[i]][,-5]

  # Vector containing the values of min{E[Zit/Zt],Wi}
  S.P2P = cmrs.level
}

```

```

S.P2P[1:nrow(cmrs.level),2:ncol(cmrs.level)] <- 0
#
# Vector containing the values of  $\min\{E[Z_{it}/Z_t], W_i\}$ 
S.UP <- cmrs.level
S.UP[1:nrow(cmrs.level),2:ncol(cmrs.level)] <- 0
#
prem <- data.frame(SL=rep(NA,nrow(Grp.level)))
for (i in 2:ncol(cmrs.level)) {
  for (j in 1:nrow(cmrs.level)) {
    #lower layer:  $S.P2P=\min\{h, w_i\}$ 
    S.P2P[j,i] <- ifelse(
      test = cmrs.level[j,i] < Grp.level$Wi[i-1],
      yes = cmrs.level[j,i],
      no = Grp.level$Wi[i-1])

    # upper layer:  $S.UP=S-\min\{S, w_n\}$ 
    S.UP[j,i] <- ifelse(
      test = cmrs.level[j,i] - Grp.level$Wi[i-1] > 0,
      yes = cmrs.level[j,i] - Grp.level $Wi[i-1],
      no = 0)
  }
  prem[i-1,1] <- (1+beta_SL)*mean(S.UP[,i])
}
SL.prem <- cbind(SL.prem,prem,Grp.level$Wi)
down.level.N[[length(down.level.N)+1]] <- S.P2P
up.level.N[[length(up.level.N)+1]] <- S.UP
rm(Grp.level,w_n,wi,S.P2P,S.UP)
}
colnames(SL.prem)[1] <- "Grp"
names(down.level.N)<-paste("N", as.character(n), sep = "")
names(up.level.N)<-paste("N", as.character(n), sep = "")
print(Sys.time()-timer)

##distribution of contributions in lower layer (0,w) Highlighting the variability of contributions
around the pure premium in the lower layer
Level.plot = list()

for (j in 2:ncol(down.level.N[[3]])) {
  Bx_plt=list()
  for (i in 1:length(down.level.N)) {
    tmp = down.level.N[[i]][[j]]
    tmp = subset(tmp,tmp < GRP[[i]]$Wi[j-1])
    Bx_plt[[length(Bx_plt)+1]]=tmp
  }
  names(Bx_plt)=as.character(n)
  Bx_plt_num=melt(do.call(rbind,Bx_plt))
  Level.plot[[length(Level.plot)+1]] = Bx_plt_num
}

rm(Bx_plt_num,Bx_plt)

names(Level.plot) = colnames(down.level.N[[1]][2:ncol(down.level.N[[1]])])

par(mfrow=c(1,3), family = "sans", font.lab = 2, font.axis = 1, cex.lab=0.9,
    cex.main=0.9, font.main=1,font.lab=1,font.axis=1)

```

```

for (i in 1:length(Level.plot)) {
  boxplot(value~Var1,Level.plot[[i]],xlab = "",ylab = "",
          main = names(Level.plot)[i],
          outline = FALSE)
  abline(h=E(FZ.t[[i]]),lwd = 1,lty=4,col="red")
}

```

### A.10.2 Representation of contribution variability in the upper layer

Contribution variability in the upper layer.

*#number of participants.*

```
n<-c(100,300,500,1000,1500,2500)
```

```

Level.Up <- list()
for (j in 2:ncol(up.level.N[[3]])) {
  Bx_plt<-list()
  for (i in 1:length(up.level.N)) {
    tmp <- up.level.N[[i]][[j]]
    tmp <- subset(tmp,tmp > 0)
    Bx_plt[[length(Bx_plt)+1]]<-tmp
    rm(tmp)
  }
  names(Bx_plt)<-as.character(n)
  Bx_plt_num<-melt(do.call(rbind,Bx_plt))
  Level.Up[[length(Level.Up)+1]] <- Bx_plt_num
}

```

```

rm(Bx_plt_num,Bx_plt)
names(Level.Up) <- colnames(up.level.N[[1]][2:ncol(up.level.N[[1]])])

```

```

par(mfrow=c(1,3), family = "sans", font.lab = 2, font.axis = 1, cex.lab=0.9,
     cex.main=0.9, font.main=1,font.lab=1,font.axis=1)
for (i in 1:length(Level.Up)) {
  boxplot(value~Var1,Level.Up[[i]],xlab = "",ylab = "",
          main = names(Level.Up)[i],
          outline = FALSE)
  #abline(h=E(FX.all[[i]]),lwd = 1,lty=4,col="red")
}

```

### A.10.3 Representation 5: Box plot retention level + number of participants

*#number of participants.*

```
n<-c(100,300,500,1000,1500,2500)
```

```

Box.Vs <- list()
for (j in 2:ncol(down.level.N[[1]])) {
  Bx.vs<-list()
  for (i in 1:length(down.level.N)) {
    tmp <- down.level.N[[i]][[j]]
    tmp <- subset(tmp,tmp < GRP[[i]]$Wi[j-1])
    tmp <- as.data.frame(tmp)
    tmp$Hin <- Law.N[[i]][[j]][1:nrow(tmp)]
    tmp$N <- n[i]
    colnames(tmp) <- c("SP2P","Hin","N")
    Bx.vs[[length(Bx.vs)+1]]<-tmp
  }
}

```

```

names(Bx.vs)<-as.character(n)
Bx.vs_long<-melt(do.call(rbind,Bx.vs),id="N")
Bx.vs_long$N <- as.factor(Bx.vs_long$N)
Box.Vs[[length(Box.Vs)+1]] <- Bx.vs_long
}

names(Box.Vs) <- colnames(down.level.N[[1]][2:ncol(down.level.N[[1]])])

par(mfrow=c(1,3), family = "sans", font.lab = 2, font.axis = 1, cex.lab=0.9,
    cex.main=0.9, font.main=1,font.lab=1,font.axis=1)
for (i in 1:length(Level.plot)) {
  boxplot(value~variable+N,Box.Vs[[i]],xlab = "N",ylab = "",
    at = c(1:2, 4:5, 7:8, 10:11, 13:14, 16:17),
    main = names(Level.plot)[i], col = c("chocolate1","cadetblue"),
    names = c("100","300","500","1000","1500","","","","","","2500",""),
    outline = FALSE)
  abline(h=E(FZ.t[[i]]),lwd = 1,lty=4,col="red")
  legend("topright",box.lty=0,inset = 0.02,
    fill = c("chocolate1","cadetblue"),
    legend = c(TeX('$S^{P2P}_{t,n}$'),TeX('$S_{t,n}$')), horiz = T)
}

```

## A.11 System with benevolent loan

### A.11.1 Sensitivity study of contributions: Size

Size of pool

```

timer=Sys.time()
#Calculation of cdf of S1, Z2 and Z3
n=c(100,300,500)
cdf.S_1.n=list()
cdfS.conv_2.n=list()
cdfS.conv_3.n=list()
quant.Z.n=rep(NA,length(n))

for (i in 1:length(n)){
  U=seq(0,4000,1)

##S1
  fS_1.n= LZ(U,1,n[i])
  FS_1.n=aggregateDist("recursive", model.freq="poisson",
    model.sev=diff(fS_1.n$LZ),
    lambda=fS_1.n$Lda,
    maxit = 100000)
  FS_1.n=DiscreteDistribution(supp = knots(FS_1.n),prob = diff(FS_1.n))

##Convolution S1 order two
  Conv_2.fS.n=FS_1.n+FS_1.n

##Convolution S1 order three
  # Conv_3.fS.n= convpow(FS_1.n,3)

  cdf.S_1.n[[length(cdf.S_1.n)+1]]=FS_1.n
}

```

```

cdfS.conv_2.n[[length(cdfS.conv_2.n)+1]]=Conv_2.fS.n
#cdfS.conv_3.n[[length(cdfS.conv_3.n)+1]]=Conv_3.fS.n
rm(fS_1.n,FS_1.n,Conv_2.fS.n)
}
print(Sys.time()-timer)
names(cdf.S_1.n)=paste("N",as.character(n),sep="")
names(cdfS.conv_2.n)=paste("N",as.character(n),sep="")
#names(cdfS.conv_3.n)=paste("N",as.character(n),sep="")

```

### A.11.2 # Calculation of probability $\phi_\tau(\omega)$

In this section, we first calculate the probability that there is no benevolent loan at year  $\tau$ .

```

Phi_tau<-function(tau,w,FS,cdfS.conv2){
  #function return the probability that there is no loan at year tau
  # w: retention level
  # FS: cdf for claims cost only on one year
  #M maximal value of S
  #cdfS.conv2

  M<-distr::q(FS)(1)
  if(tau==1){
    prob.surv=p(FS)(w)
  }
  else if(tau==2){

    proba1<-d(FS)((1:(w+1))-1)*p(FS)(2*w-(1:(w+1))+1)
    proba1<-sum(proba1)
    prob.surv=p(FS)(w)-(p(FS)(w)-d(FS)(w))*(1-p(FS)(w))-proba1+
      p(cdfS.conv2)(w*tau)

  }

  else if(tau==3){
    q_0<-(1-p(FS)(w))*(p(FS)(w)+(1-p(FS)(w))^2)

    q_1=(p(FS)(w)-d(FS)(w))*(1-p(FS)(w))+d(FS)(w)*p(cdfS.conv2)(w*tau-w)

    V1<-d(FS)((w+1):M)*d(FS)((w+1):M)*p(FS)(max(2*w-((w+1):M),3*w-((w+1):M)-((w+1):M)))
    V1<-sum(V1)

    U1<-d(FS)(1:w)*p(FS)(2*w-(1:w))
    U1<-sum(U1)
    q_2=(2-p(FS)(w))*(1-p(FS)(w))*(p(FS)(w)-d(FS)(w))-V1+U1-p(cdfS.conv2)(w*tau-w)

    prob.surv<-1-q_1-q_2-q_0
  }
  return(prob.surv)
}

find_w <- function(tau, varepsilon,FS,cdfS.conv2, tol = 1e-5, max_iter = 4000) {
  Quant1<-distr::q(FS)(1)
  Quant2<-distr::q(cdfS.conv2)(1)

```

```

lower <- 0
upper <- 2*max(Quant1,Quant2)
iter <- 0
while (iter < max_iter) {
  w <- (lower + upper) / 2
  phi <- Phi_tau(tau,w,Fs,cdfS.conv2)
  if (abs(phi - varepsilon) < tol) {
    return(w)
  } else if (phi < varepsilon) {
    lower <- w
  } else {
    upper <- w
  }
  iter <- iter + 1
}
return(w) # return the best estimate after max_iter
}

#Phi_tau(3,33000,cdf.S_1.n[["N500"]],cdfS.conv_2.n[["N500"]])
#find_w(1,0.9999,cdf.S_1.n[["N500"]],cdfS.conv_2.n[["N500"]],tol = 1e-5, max_iter = 1000)
#q(cdfS.conv_2.n[["N500"]])(50000)

```

```

##parameter
N=500
grouptest=c(1,2,3)

```

```

Creating pools
Pool.tak=pool(grouptest,0.5,0.3,0.2,N)

```

### A.11.3 # Application 1: Decomposition for a pool of $n = 500$

In this first illustration, we apply the layer decomposition to a pool of  $n = 500$  participants.

```

timer = Sys.time()
##Target probability for having a surplus
varepsilon<-0.985
## ratio distribution shareholders and participants 40%:60%
delta<-0.6
##horizon
tau<-3
##safety loading
beta_SL<-0.5
##Retention level
w=ceiling(find_w(tau,varepsilon,cdf.S_1.n[["N500"]],cdfS.conv_2.n[["N500"]],
               tol = 1e-5, max_iter = 1000))
w
#individual contributions to lower layer
wi.fund<-Cond.exp(s=w,cdf.Z_t.n[["N500"]],cdf.ZZit.n_GP1[["N500"]],
                 cdf.ZZit.n_GP2[["N500"]],cdf.ZZit.n_GP3[["N500"]])
Pool.tak$Wi.fund<-t(as.data.frame(wi.fund))[1:length(wi.fund)]
sum(Pool.tak$Ni[1:3]*Pool.tak$Wi.fund[1:3])

## Previously simulated values of  $E[Xi\hat{S}=s]$ 
CMRS.Layer<-Law.N[["N500"]][,-5]

```

```

timer = Sys.time()
## Vector containing the values of min{S,W}
Charge.tak<-CMRS.Layer
Charge.tak[1:nrow(CMRS.Layer),2:ncol(CMRS.Layer)] <- 0

## Vector containing the values of the premium SL
Wi.SL<-CMRS.Layer
Wi.SL[1:nrow(CMRS.Layer),2:ncol(CMRS.Layer)] <- 0

## Vector containing the values of cash-back
BI.t <-CMRS.Layer
BI.t[1:nrow(CMRS.Layer),2:ncol(CMRS.Layer)] <- 0

Pool.tak$EZit<- rep(NA,nrow(Pool))
Pool.tak

for (i in 2:ncol(CMRS.Layer)) {
  #E[Z_{it}]
  Pool.tak$EZit[i-1] <- E(FZ.t[[i-1]])

  #S_tak=min{h,wi}
  Charge.tak[,i] = ifelse(
    test = CMRS.Layer[,i] < Pool.tak$wi[i-1],
    yes = CMRS.Layer[,i],
    no = Pool.tak$wi[i-1])

  # Calcul de la prime stop-loss
  Wi.SL[,i] <- ifelse(
    test = CMRS.Layer[,i] - Pool.tak$Wi[i-1] > 0,
    yes = CMRS.Layer[,i] - Pool.tak$Wi[i-1],
    no = 0
  )

  # Cash-back
  BI.t[,i] <- ifelse(
    test =Pool.tak$Wi.fund[i-1]-CMRS.Layer[,i] > 0,
    yes = Pool.tak$Wi.fund[i-1]-CMRS.Layer[,i],
    no = 0
  )
}

print(Sys.time()-timer)
#Pool.tak$prime.SL <- (1+beta_SL)*colMeans(Wi.SL[,-1],na.rm = TRUE)
Pool.tak$Cotis_tot <- Pool.tak$Wi.fund
Pool.tak$Cotis_P2P<-Pool$Cotis_tot/t
Pool.tak$Ex.Bi.fund <- colMeans(BI.t[,-1],na.rm = TRUE)*delta
#Pool$Cotis.red <- Pool$Cotis_tot-Pool$Ex.Bi.tak
Grp.table <- Pool.tak
Pool.tak
## gÃ©nÃ©rer le script latex pour le tableau Grp.table
print( xtable(Grp.table,type="latex",digits = 3))

```

#### A.11.4 Distribution of Cash-Back $B_1.fund$

Since  $B_t$  can be rewritten as the random variable  $tw - Z_t | tw - Z_t > 0$ , we can obtain the distribution of  $B_1 | B_1 > 0$ . The following block allows us to determine the distribution of  $B_t$  and the distribution of cash-back.

```

# cdf of cash-back for the Pool at year t=1
F_B1.fund <- function(x,FZ,omega){
  func <-( p(FZ)(omega)-p(FZ)(omega-x) )/p(FZ)(omega)
  return(func)
}

supp=seq(0,w,1)
prob=diff(F_B1.fund(supp,cdf.S_1.n[["N500"]],w))
cash.back<- DiscreteDistribution(supp=supp[-1],prob=prob)
rm(supp,prob)

plf<-q(cash.back)(0.999)
u<-seq(1,plf,1)
v<-seq(0,1,1/plf)
### Representation of FB_t
par(mfrow=c(1,3),font.main=1,cex.main = 0.9, cex.sub = 0.8)

plot(u,d(cash.back)(u),type="l",xlab="x",ylab="d(x)", main="Density")

plot(u,p(cash.back)(u),type="l",xlab="q",ylab="p(q)",main="CDF")

plot(v,q(cash.back)(v),type="l",xlab="p",ylab="q(p)",main="Quantile")

```

### A.11.5 Representation 3: Distribution of cash-back $B_{i,1}$ .

Representation of density probability for cash-back  $B_{i,1}$  for all participants by Group.

```

BI.t <- na.omit(BI.t)
pdf.Cash <- list()
for (i in colnames(BI.t[-1])) {
  den.Bi <- density(BI.t[,i])
  A <- data.frame(val=den.Bi$x, prob=den.Bi$y)
  pdf.Cash[[length(pdf.Cash)+1]] <- A
  rm(den.Bi, A)
}
#par(mfrow=c(1,2),family = "sans", font.lab = 2, font.axis = 1, cex.lab=0.9,
#    cex.main=0.9, font.main=1,font.lab=1,font.axis=1)
plot(pdf.Cash[[1]]$val, pdf.Cash[[1]]$prob*5, type = "l", lty = 1,
     col = 1, ylim = c(0,1.5), xlim = c(0,100),
     xlab = "x", ylab = "density", main = "Cash-Back Year One")
for (i in 2:3) {
  lines(pdf.Cash[[i]]$val, pdf.Cash[[i]]$prob*10, type = "l", lty = i,
       col = i)
}
legend("topright", cex = 0.6,legend =c("Group1","Group2","Group3"),col = 1:3,lty=1:3,
     box.lty=0,inset = 0.005)

```

## A.12 Variability of contributions in lower layer $[0, \omega]$ .

### A.12.1 Comparison with variation in retention level

Study of variability in the lower band as a function of retention level.

```

timer <- Sys.time()
#number of participants.
n=500

## target probability vector
varepsilon=seq(0.6,0.985,0.1)

## loading target
beta_SL=0.5
GRP=list()

down.level.N = list()
up.level.N = list()
SL.prem = data.frame(Grp=seq(1,3,1))
  Grp.level = pool(grouptest,0.5,0.3,0.2,n)

for (i in 1:length(varepsilon)) {
  #pool creating

  # retention level
  w = ceiling(find_w(3,varepsilon[i],cdf.S_1.n[["N500"]],cdfS.conv_2.n[["N500"]],
    tol = 1e-5, max_iter = 1000))

  # individual contributions Wi
  Wi = Cond.exp(s=w,cdf.S_1.n[["N500"]],cdf.ZZit.n_GP1[["N500"]],
    cdf.ZZit.n_GP2[["N500"]],cdf.ZZit.n_GP3[["N500"]])
  Grp.level$Wi = t(as.data.frame(Wi))[1:length(Wi)]

  GRP[[length(GRP)+1]] = Grp.level
# values of  $E[X_i|S]$ 
  cmrs.level = Law.N[["N500"]][,-5]

# values of  $\min\{E[X_i|S],W_i\}$ 
  S.P2P = cmrs.level
  S.P2P[1:nrow(cmrs.level),2:ncol(cmrs.level)] <- 0
#
# values of  $\min\{E[X_i|S],W_i\}$ 
  S.UP <- cmrs.level
  S.UP[1:nrow(cmrs.level),2:ncol(cmrs.level)] <- 0
#
  prem <- data.frame(SL=rep(NA,nrow(Grp.level)))
  for (i in 2:ncol(cmrs.level)) {

    for (j in 1:nrow(cmrs.level)) {

      #lower layer:  $S.P2P=\min\{h,w_i\}$ 
      S.P2P[j,i] <- ifelse(
        test = cmrs.level[j,i] < Grp.level$Wi[i-1],
        yes = cmrs.level[j,i],
        no = Grp.level$Wi[i-1])

# # upper layer= $S-\min\{S,w_i\}$ 
      S.UP[j,i] <- ifelse(
        test = cmrs.level[j,i] - Grp.level$Wi[i-1] > 0,
        yes = cmrs.level[j,i] - Grp.level$Wi[i-1],
        no = 0)
    }
  }
}

```

```

    }
    prem[i-1,1] <- (1+beta_SL)*mean(S.UP[,i])
  }
  SL.prem <- cbind(SL.prem,pre,Grp.level$Wi)
  down.level.N[[length(down.level.N)+1]] <- S.P2P
  up.level.N[[length(up.level.N)+1]] <- S.UP
  rm(w,wi,S.P2P,S.UP)
}
colnames(SL.prem)[1] <- "Grp"
names(down.level.N) <- paste("Var", as.character(varepsilon), sep = "")
names(up.level.N) <- paste("Var", as.character(varepsilon), sep = "")
print(Sys.time()-timer)

```

### A.12.2 Representation : Effect of retention level on contributions

Highlighting the variability of contributions around the pure premium.

```

level.plot <- list()
for (j in 2:ncol(down.level.N[[1]])) {
  Bx_plt <- list()
  for (i in 1:length(down.level.N)) {
    tmp <- down.level.N[[i]][[j]]
    tmp <- subset(tmp,tmp < GRP[[i]]$Wi[j-1])
    Bx_plt[[length(Bx_plt)+1]] <- tmp
    rm(tmp)
  }
  names(Bx_plt) <- as.character(varepsilon)
  Bx_plt_long <- melt(do.call(rbind,Bx_plt))
  level.plot[[length(level.plot)+1]] <- Bx_plt_long
}

rm(Bx_plt_long,Bx_plt)
names(level.plot) <- colnames(down.level.N[[1]])[2:ncol(down.level.N[[1]])]

par(mfrow=c(1,3), family = "sans", font.lab = 2, font.axis = 1, cex.lab=0.9,
    cex.main=0.9, font.main=1,font.lab=1,font.axis=1)
for (i in 1:length(level.plot)) {
  boxplot(value~Var1,level.plot[[i]],xlab = "",ylab = "",
    main = names(level.plot)[i],
    outline = FALSE)
  #abline(h=E(FX.all[[i]]),lwd = 1,lty=4,col="red")
}

par(mfrow=c(1,3), family = "sans", font.lab = 2, font.axis = 1, cex.lab=0.9,
    cex.main=0.9, font.main=1,font.lab=1,font.axis=1)
boxplot(value~Var1,level.plot[[1]],xlab =TeX('$\\varepsilon$'),ylab = "",
  main = "Group1",
  outline = FALSE)

abline(h=E(FZ.t[[1]]),lwd = 1,lty=4,col="red")

boxplot(value~Var1,level.plot[[2]],xlab =TeX('$\\varepsilon$'),ylab = "",
  main = "Group2",
  outline = FALSE)

abline(h=E(FZ.t[[2]]),lwd = 1,lty=4,col="red")

```

```
boxplot(value~Var1,level.plot[[3]],xlab =TeX('$\\varepsilon$'),ylab = "",  
        main = "Group3",  
        outline = FALSE)
```

```
abline(h=E(FZ.t[[3]]),lwd = 1,lty=4,col="red")
```

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