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Supercategorification and Khovanov-like tangle invariants

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Introduction

In 1984, Jones discovered a new invariant of knots, the now well-known Jones polynomial. Since then, it has been described through many different approaches, demonstrating its connection with various different subjects. This thesis is related to two of them: *quantum invariants* and *Khovanov homology*.

Quantum algebras are certain algebraic structures related with Lie algebras. The development of their representation theory lead to the discovery of a new family of knot invariants, called quantum invariants. The Jones polynomial belongs to this family through the representation theory of the quantum algebra $\mathrm{U}_q(\mathfrak{sl}_2)$, a "deformation" of the Lie algebra $\mathfrak{sl}_2$. On the other hand, the Jones polynomial can also be obtained as the graded Euler characteristic of a certain homology theory by assigning to each knot diagram a complex in a certain category. Reidemeister moves on diagrams leave the homotopy classes of the complexes invariant, making its homology an invariant of knots. This invariant is called Khovanov homology, after it was first discovered by Khovanov in 2000 [6]. In the same sense that homology "categorifies" the Euler characteristic, Khovanov homology "categorifies" the Jones polynomial. In general, *categorification* is the process of turning set-theoretic notions into categorical notions. By enriching the structure, we hope to unravel new properties which could not be seen at the uncategorified level. Indeed, similarly to the homology groups of a topological space, Khovanov homology gives a stronger invariant of knots than the Jones polynomial. Moreover, one can assign to each cobordism of knots ("morphism of knots") a morphism between the corresponding homologies, which means it can detect higher dimensions. In other words, Khovanov homology is functorial.

Could one combine these two approaches? That is, would it be possible to categorify (the representation theory of) quantum algebras in order to discover new invariants of knots? In 2008, Webster [19] showed how Khovanov homology could be deduced from a categorification of representations of $\mathrm{U}_q(\mathfrak{sl}_2)$. Later work of Lauda, Queffelec and Rose [8] gave an alternative construction through the categorification of $\mathrm{U}_q(\mathfrak{sl}_m)$. In 2013, Oszváth, Rasmussen and Szabó [15] constructed a different version of Khovanov homology using exterior algebras, called *odd Khovanov homology*. This construction also categorifies the Jones polynomial, and over $\mathbb{Z}/2\mathbb{Z}$ the resulting invariant coincides with Khovanov homology. However, the two homologies are distinct (see [15, Proposition 1.8] or [17] for further details). In march 2020, Naisse and Putyra [11] extended odd Khovanov homology to tangles. At the time being though, there is no representation theory construction similar to the one given by Webster or Lauda, Queffelec and Rose for (even) Khovanov homology. Moreover, it is yet to be shown that odd Khovanov homology is functorial.

In this thesis, we categorify a certain quantum algebra and construct a new invariant of oriented tangles, which we conjecture to coincide with odd Khovanov homology of [11], and of [15] when restricted to knots. We hope that our construction can give the tools to prove its functoriality. The suited framework for this categorification is a *superstructure*. In a nutshell, superstructures are categorical structures whose morphisms are equipped with a *parity*. We mainly focus on two of them: *monoidal supercategories*, the counterpart of monoidal categories (categories with a tensor product), and *2-supercategories*, the counterpart of 2-categories (categories with "morphisms between morphisms"). Chapter 1 describes them in full details. In Chapter 2, we categorify a certain quantum algebra $S_{n,d}$ into a 2-supercategory, denoted $\mathcal{S}(n,d)$. Chapter 3 then assign to each oriented tangle diagram D a certain complex $\mathrm{Kom}(D,n,d)$ in the category of complexes of $\mathcal{S}(n,d)$ and show that it results in an invariant of oriented tangles. In the process, we extend the tensor product of complexes to monoidal supercategories. In Appendix B, we show that this tensor product leaves homotopy classes invariant, an essential property if one wishes to construct an invariant of oriented tangles from $\mathrm{Kom}(D,n,d)$.

My contributions

The definition of the 2-supercategory $\mathcal{S}(n,d)$ in Section 2.2 can be extracted from the work of Vaz in [18]. From there, all results involving $\mathcal{S}(n,d)$ are original (that is, all the results of Chapter 2 and Appendix A). In Appendix B, we give a new extension of the Koszul rule for the tensor product in monoidal supercategories. Such an extension was yet to be defined. All subsequent results are therefore original. Finally, as we define a new invariant of oriented tangles, the proof that our construction does define an invariant is original (that is, almost all the results of Chapter 3). If inspiration has been taken for other works, it is explicitly stated in the text. We refer the reader to the conclusion of the thesis for possible directions in future work.

Prerequisites

This thesis is intended to be readable by any Master student. The only prerequisites are basics in some graduate topics:

- basics in knot theory (Reidemeister moves, the Jones polynomial, the notion of tangles);
- basics in algebraic topology (homology, tensor product of complexes);

- and basics in category theory (functors, natural transformations, adjunctions). Monoidal categories, 2-categories and diagrammatic calculus are recalled in details in Chapter 1.

Acknowledgements

Je remercie mon promoteur Pedro Vaz pour m'avoir guidé tout au long de cette année, pour avoir patiemment répondu à toutes mes questions, pour avoir lu et commenté toutes les versions intermédiaires de ce document et enfin pour m'avoir ouvert les portes de ce magnifique sujet de recherche.

Contents

Chapter 1

Superstructures

Before introducing our invariant, we define the abstract categorical framework in which the invariant will be set. This is the goal of this chapter. We start in Section 1.1 by recalling classical categorical constructions: monoidal categories, which are in essence categories with a tensor product, and 2-categories, which are categories admitting morphisms *between* morphisms. We call these morphisms *2-morphisms*. A good example are natural transformations, which are morphisms between functors. In Section 1.2, we delve deeper into the notion of 2-categories and describe *string diagrams*, a visual and intuitive way of depicting 2-morphisms. The material in these first two sections is basic and well-known, and the accustomed reader may jump directly to Section 1.3, which introduces the "super" counterparts of monoidal categories and 2-categories: *monoidal supercategories* and *2-supercategories*. These superstructures have a special property: their highest level morphisms (that is, morphisms for monoidal supercategories and 2-morphisms for 2-supercategories) have a *parity*. This changes the usual laws of composition, adding signs that depends on the parities of the morphisms involved.

Throughout this chapter we fix $\Bbbk$ a ground field. The reader can safely think of $\Bbbk$ as a unital ring, where our results should work equally well. Moreover, we shall always assume categories are small.

Reference notes

Section 1.1 contains basic notions given for example in the nLab [12] and in Borceux's Handbook [3, Chapter 7]. Additional information can be found in both of them. Globular expressions and string diagrams for 2-morphisms follow conventions given by Lauda [7] (excepted for the orientation of strings, as explained in the text). Except for the term "cartesian superproduct" introduced in Subsection 1.3.2, the superstructures of Section 1.3 stem from the work of Brundan and Ellis [4].

1.1 Monoidal categories and 2-categories

1.1.1 Monoidal categories

Consider $\mathcal{V}ec$ the category of $\Bbbk$-vector spaces and linear maps. Given two vector spaces V and W, one can form their tensor product $V \otimes W$. The tensor product is associative in the sense that there is a *natural* isomorphism between $U \otimes (V \otimes W)$ and $(U \otimes V) \otimes W$. Moreover, the ground field plays the role of the unit in the sense that there are natural isomorphisms $\Bbbk \otimes V \cong V \cong V \otimes \Bbbk$. The general idea of tensor product is encompassed in the following definition:

Definition 1.1 ([3, Definition 6.1.1, 13])**.** *A* monoidal category *is a category $\mathcal{K}$ which admits a functor $\otimes\colon \mathcal{K} \times \mathcal{K} \to \mathcal{K}$, called* tensor product*, such that:*

1. *There exists a natural isomorphism $\alpha_{ABC}\colon (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ satisfying some coherent axiom.*
2. *There exists an object $I \in \mathcal{K}$, called the* unit*, such that there exists a "left unit" natural isomorphism $l_A\colon I \otimes A \to A$ and a "right unit" natural isomorphism $r_A\colon A \otimes I \to A$ satisfying some coherence axioms.*

Using these natural isomorphisms, there are multiple ways of going from $(A \otimes I) \otimes B) \otimes C$ to $((A \otimes B) \otimes I) \otimes (C \otimes I)$, but the coherence axioms ensure that all are equivalent. This means that an ordered sequence of objects (in our example, A, B and C) corresponds to only one isomorphic class, no matter how parenthesis and units are set. In what follows, we shall often avoid parenthesis altogether (e.g. write $A \otimes B \otimes C$), acting as these isomorphisms were equalities. We refer the reader to Borceux's Handbook [3, p. 292, 13] for a complete definition and to Mac Lane [10, Chapter 7] for the details about this fact.

Example 1.2. (a) $\mathcal{V}ec$ is monoidal with the usual tensor product of vector spaces, but also with the direct sum: the way to make a category monoidal is not unique.

(b) Every category $\mathcal{C}$ with finite products is monoidal with tensor product given by the cartesian product and unit given by the empty product [3, Example 6.1.9j]. In particular, the category $\mathcal{C}at$ of small categories is monoidal. Recall that the cartesian product of two categories $\mathcal{A}$ and $\mathcal{B}$ consists in formal pairs (A, B) for $A \in \mathrm{ob}(\mathcal{A})$ and $B \in \mathrm{ob}(\mathcal{B})$, with morphisms being the formal pairs (f, g) where f and g are respectively morphisms in $\mathcal{A}$ and $\mathcal{B}$. The unit is the singleton category $\mathcal{I}$, with only one object I and one morphism, the identity 1_I of I.

Remark 1.3. The functoriality of the tensor product gives a compatibility relation with the composition:

$$(f_1 \otimes g_1) \circ (f_2 \otimes g_2) = (f_1 \circ f_2) \otimes (g_1 \circ g_2). \tag{1.1}$$

This relation is called the *interchange law (for monoidal categories).* We will encounter other interchange laws later, and see that the "super" counterparts of the definitions given here differ mainly in these interchange laws.

1.1.2 Enriched categories

$\mathcal{V}ec$ has another interesting property: the Hom-set $\mathrm{Hom}(V, W)$ is a vector space. In other words, Hom-sets are objects of a category $\mathcal{K} = \mathcal{V}ec$. Moreover, composition is compatible with the vector space structure, that is, it is bilinear. Using the tensor product, it is equivalent to the map

$$\begin{array}{rcl} \mathrm{Hom}(V, W) \otimes \mathrm{Hom}(W, X) & \to & \mathrm{Hom}(V, X) \\ f \otimes g & \mapsto & g \circ f \end{array}$$

being linear: composition is a morphism in $\mathcal{K} = \mathcal{V}ec$. This leads us to the following definition:

Definition 1.4 ([3, Definition 6.2.1])**.** *Let $\mathcal{K}$ be a monoidal category. A $\mathcal{K}$-*enriched category *$\mathcal{C}$ consists in a class of objects* $\mathrm{ob}(\mathcal{C})$ *and objects $\mathcal{C}(A, B)$ in $\mathcal{K}$ for each pair A, B of objects of $\mathcal{C}$. Moreover,*

1. *for each triple A, B, C of objects, there is a* composition morphism *in $\mathcal{K}$*

$$c_{ABC} \colon \mathcal{C}(A, B) \otimes \mathcal{C}(B, C) \to \mathcal{C}(A, C).$$

   which satisfies some associativity axiom.

2. *For each object $A \in \mathcal{C}$, there is a unit morphism $u_A \colon I \to \mathcal{C}(A, A)$ which satisfies some unital axiom.*

The associativity axiom ensures that the composition morphism is coherent with the associativity isomorphism in $\mathcal{K}$, making composition associative. The unital axiom gives a unit for composition through the unit of the tensor product. In particular, a enriched category is a category.

Example 1.5. (a) As one could expect, $\mathcal{V}ec$ is an enriched category in $\mathcal{V}ec$.

(b) A pre-additive category is an $\mathcal{A}b$-enriched category, where $\mathcal{A}b$ is the category of abelian groups.

1.1.3 2-categories

2-categories are categories with "morphisms between morphisms". The first stereotypical example is $\mathcal{C}at$, the category of small categories, since one can define natural transformations between functors. The reader should keep this example in mind throughout the section.

Definition 1.6 ([7, p. 178])**.** *A* 2-category $\mathcal{C}$ *is a* $\mathcal{C}at$*-enriched category, that is, a category where each Hom-set is itself a (small) category and composition induces a bifunctor.*

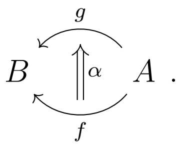
We call the morphisms of a 2-category $\mathcal{C}$ *1-morphisms.* If $f \in \mathcal{C}(A, B)$, we depict it as a leftward arrow:

$$B \xleftarrow{\ f\ } A \ .$$

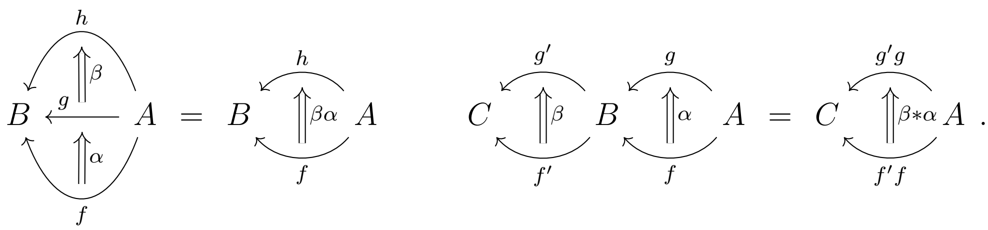
1-morphisms can be composed:

$$C \xleftarrow{\ g\ } B \xleftarrow{\ f\ } A \ = \ C \xleftarrow{\ gf\ } A \ .$$

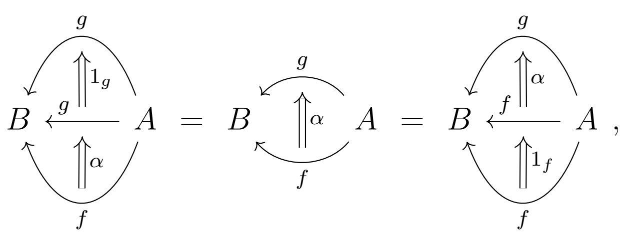
The composition is associative and unital, the unit of A being denoted 1_A. Moreover, the morphisms in each Hom-set are called *2-morphisms.* If $f \overset{\alpha}{\Longrightarrow} g$ is a morphism in $\mathcal{C}(A, B)$, we depict it as a globular diagram:

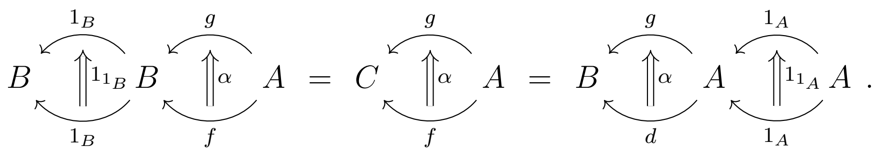


As this notation (and our stereotypical example $\mathcal{C}at$) suggests, 2-morphisms can be composed both vertically and horizontally:



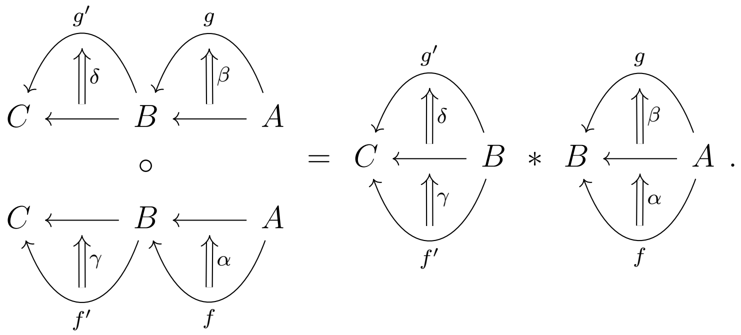
Vertical composition is associative and unital, where the unit of $B \xleftarrow{\ f\ } A$ is denoted 1_f. Horizontal composition is also associative and unital, the units being the 1_{1_f}'s:





Finally, vertical and horizontal compositions are compatible in the sense that the relation

$(\delta * \beta)(\gamma * \alpha) = (\delta\gamma) * (\beta\alpha)$ holds, that is

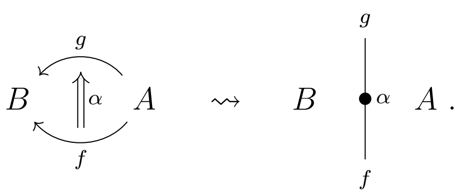


This relation is the interchange law for 2-categories.

Remark 1.7. A monoidal category $\mathcal{C}$ can be seen as a one-object 2-category by considering the objects of $\mathcal{C}$ as the 1-morphisms of a single object, and composition to be given by the tensor product on objects in $\mathcal{C}$ (this is similar to how a monoid can be seen as a one-object category). Then morphisms in $\mathcal{C}$ are 2-morphisms, vertical composition being composition in $\mathcal{C}$ and horizontal composition being the tensor product on morphisms in $\mathcal{C}$. With this identification, the interchange law for monoidal categories is exactly the interchange law for 2-categories.

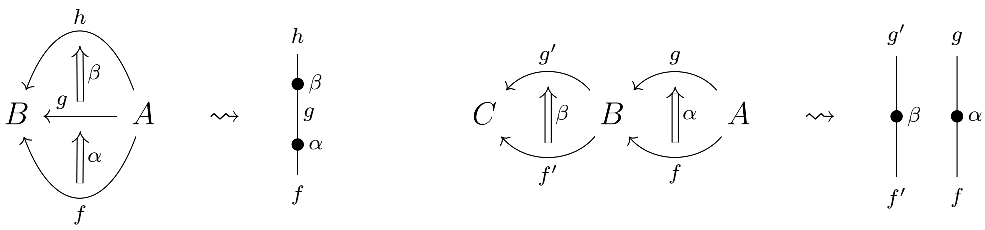
1.2 String diagrams

In 2-categories, calculations have a 2-dimensional aspect as 2-morphisms can be both vertically and horizontally composed. This enlarges the possibilities, but also makes calculations harder. Therefore, we introduce another way of depicting 2-morphisms called *string diagrams*:

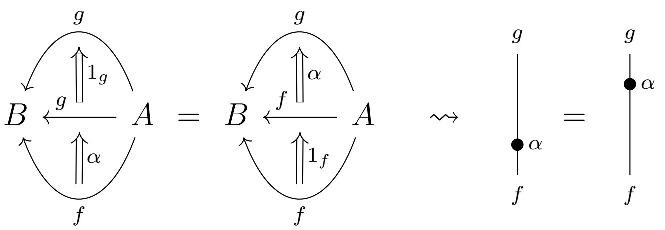


In general, given a globular presentation of a 2-morphism, its corresponding string diagram is its Poincaré dual, that is:

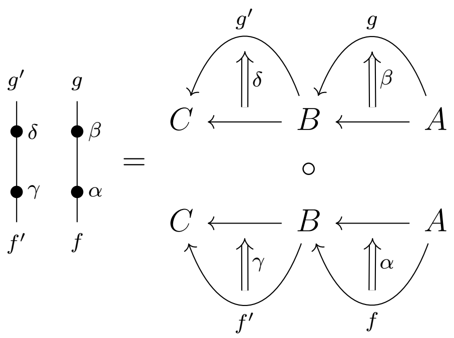
- regions (2-morphisms) become points;
- edges (1-morphisms) become perpendicular egdes;
- and points (objects) become regions.



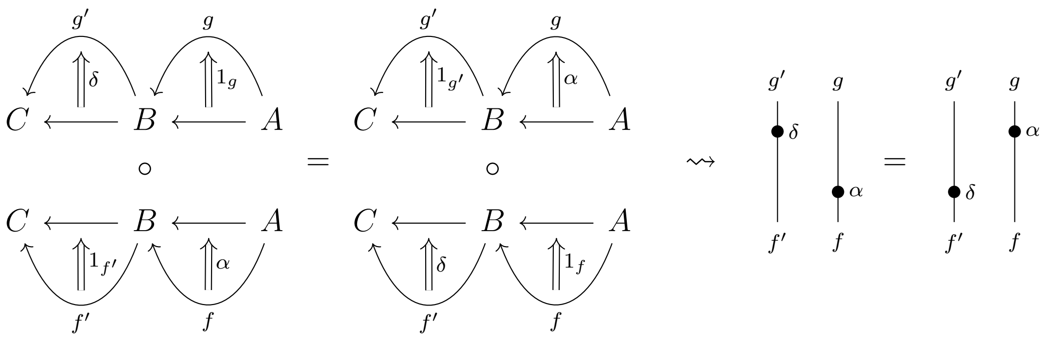
(a) Vertical and horizontal composition as string diagrams.



(b) Dots can slide on the string.



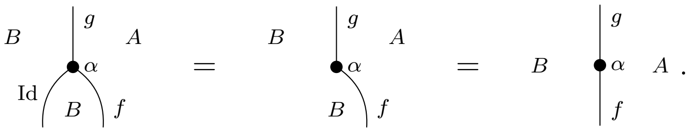
(c) First compose horizontally, then vertically.



(d) Dots can be slided past one another.

Figure 1.1: From globular diagrams to string diagrams.

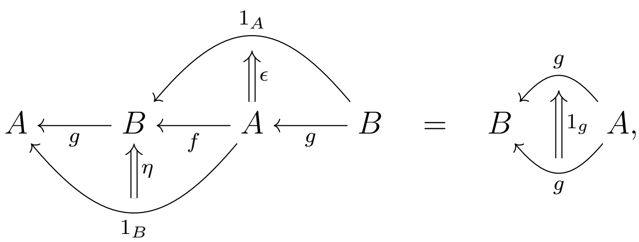
In Fig. 1.1 above we show several typical cases of globular diagrams turned into string diagrams (note that we avoided writing objects in all string diagrams). Figure 1.1a shows that vertical composition translates into two dots on a string, while horizontal composition becomes two strings next to each other. Figure 1.1b introduces the convention that identity 2-morphisms are not drawn: together with the rules of vertical composition, the practical consequence is that *dots can slide along their string.* Figure 1.1c fixes our convention when both vertical and horizontal compositions are involved. The string diagrams should be read as *first compose horizontally, and then vertically.* In 2-categories, this convention turns out to be unnecessary thanks to the interchange law. This won't be the case for 2-supercategories though, as we shall see later in this chapter. Furthermore, Fig. 1.1d shows another consequence of the interchange law and the identity convention: dots can not only slide along a given string, *they can also slide past one another.* Finally, just like we avoid drawing identity 2-morphisms, we also avoid drawing identity 1-morphisms. As a consequence, we can safely *slide strand horizontally.* For example:

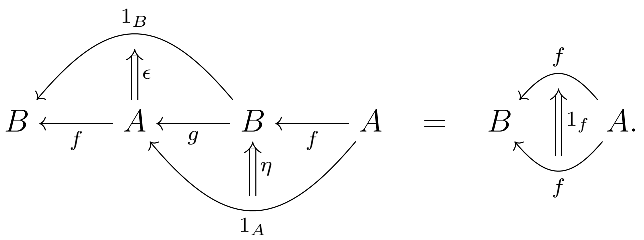


The Poincaré dual helps us focus on what really matters in the calculation: the 2-morphisms (and at a lesser extend, the 1-morphisms). The conventions on identities enable us to think about the strings as topological objects: dots can be moved and strands can be wiggled without changing the actual 2-morphism the diagram represents. It turns out that by adding orientation to the strings, we can strengthen even more this geometric intuition.

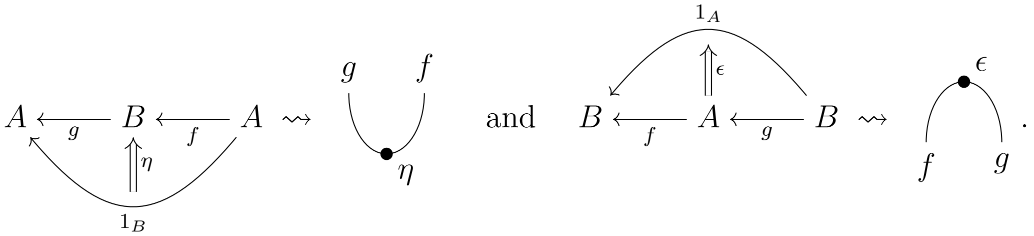
1.2.1 Adjunctions and string diagrams

As $\mathcal{C}at$ is our stereotypical example of a 2-category, we can define adjunction in 2-categories similarly to the typical definition in $\mathcal{C}at$. Then, recall that the 1-morphism $f\colon A \to B$ is *left adjoint* to the 1-morphism $g\colon B \to A$ if there exists 2-morphisms $\eta\colon 1_x \Rightarrow gf$ (the *unit*) and $\epsilon\colon fg \Rightarrow 1_B$ (the *counit*) satisfying the *triangular identities*:

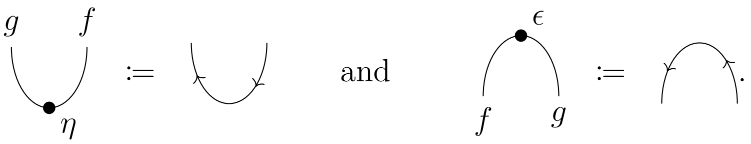




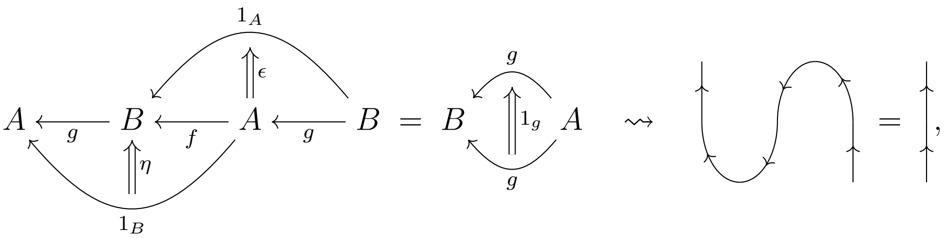
In string diagrams, the unit and the counit become

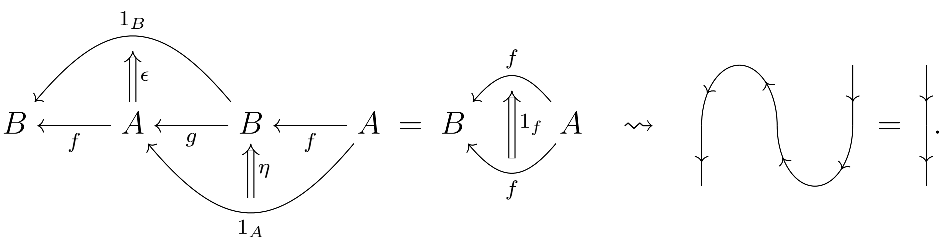


We can get rid of the labels of f and g by assigning an orientation to the string: downward for f and upward for g (beware that this is the opposite of Lauda's convention in [7]). Then explicitly writing the unit and counit can also be avoided if we assume that oriented cup represents the former and oriented cap represents the latter:



With this convention, the triangle identities have a surprising geometrical interpretation:





That is, *oriented zigzags can be "straighten up"*.

1.3 Monoidal supercategories and 2-supercategories

1.3.1 Supercategories

We can now define the "super" counterparts of monoidal categories and 2-categories. We start with the notion of *supercategories*. As vector spaces were the motivating example at the beginning of Section 1.1, the motivating example for this section are vector spaces whose vectors are given a parity:

Definition 1.8. *A* superspace V *is a* $\Bbbk$*-vector space with a* $\mathbb{Z}/2\mathbb{Z}$*-grading, that is a decomposition* $V = V_0 \oplus V_1$*. A vector is* even *if it belongs to* V_0*, and* odd *if it belongs to* V_1*.* Homogeneous vectors *are vectors that are either even or odd. A linear map is* even *if it preserves parity, and* odd *if it switches parity.* Homogeneous maps *are maps that are either even or odd.*

We denote $|v|$ the parity of a non-zero homogeneous vector v (beware that the zero vector is the only homogeneous vector with no defined parity, being both even and odd). As parities are elements of $\mathbb{Z}/2\mathbb{Z}$, $|v|$ should always be understood modulo 2. If we encounter relations involving the parity of an inhomogeneous vector, it should be understood by extending it additively from the homogeneous case. Similar remarks can be made for linear maps.

We can extend the tensor product of vector spaces to superspaces by giving $V \otimes W$ the following $\mathbb{Z}/2\mathbb{Z}$-grading (assuming $V = V_0 \oplus V_1$ and $W = W_0 \oplus W_1$):

$$(V \otimes W)_0 = (V_0 \otimes W_0) \oplus (V_1 \otimes W_1) \quad \text{and} \quad (V \otimes W)_1 = (V_1 \otimes W_0) \oplus (V_0 \otimes W_1).$$

For the ground field $\Bbbk$ to be a unit for the tensor product, we assume it is given a $\mathbb{Z}/2\mathbb{Z}$-grading where all vectors are even. Thanks to the "super" structure, we can define a more interesting tensor product (a *super* tensor product) on linear maps as $(f \otimes g)(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w)$[1]. Note the sign! This forces a different compatibility law between composition and tensor product:

$$(f \otimes g) \circ (h \otimes k) = (-1)^{|g||h|} (f \circ h) \otimes (g \circ k). \tag{1.2}$$

This law is known as the *super interchange law*. Call $\mathcal{SV}ec$ the category of superspaces and linear maps: $\mathcal{SV}ec$ is not a monoidal category! (Compare Eq. (1.2) with Eq. (1.1).) Indeed, we shall see that is is a monoidal *super*category. Nonetheless, we can get a monoidal category by considering only even maps. Call $\underline{\mathcal{SV}ec}$ the category of superspaces and even linear maps: then the sign above disappears and $\underline{\mathcal{SV}ec}$ is a monoidal category. This allows us to categorify the notion of superspace and define the notion of a category whose morphisms have parities:

[1]One should not be confused by the -1: it is simply a scalar and does not depict a change in parity. On the contrary, scalar multiplication does not affect the subspaces, and so does not change parity.

Definition 1.9. *A* supercategory *is a* $\underline{\mathcal{SV}ec}$*-enriched category, that is each Hom-set is a superspace and composition induces an even linear map.*

Since composition is even, it preserves parity and we have the rule $|f \circ g| = |f| \circ |g|$ (assuming $f \circ g$ is non-zero and homogeneous). this also implies that identities must be even, just like it was the case in superspaces. Unsurprisingly, $\mathcal{SV}ec$ is a supercategory. We define $\mathcal{SC}at$ to be the category of small *super*categories. Its morphisms are *superfunctors*, that is functors which preserve parity.

Remark 1.10. Note that in superspaces, the parity of some complicated expression involving any kind of operation (composition, tensor product, or applying a linear map to a vector) is actually easy to compute by taking the sum of the parities of its components (assuming the parity makes sense, that is the expression is homogeneous and non-zero):

- whenever v and f are homogeneous and $f(v) \neq 0$, then $|f(v)| = |f| + |v|$;
- whenever f and g are homogeneous and $f \circ g \neq 0$, then $|f \circ g| = |f| + |g|$;
- whenever $v \in V$ and $w \in W$ are homogeneous and $v \otimes w \neq 0$, then $|v \otimes w| = |v| + |w|$. Similarly, whenever f and g are homogeneous linear maps and $f \circ g \neq 0$, $|f \otimes g| = |f| + |g|$.

With this in mind it is easy to check that the interchange law is indeed compatible with the definition of the tensor product of linear maps. Moreover, note that these laws (and almost all laws in a “super” context) follow the same pattern: whenever elements a are b are switched in an expression, we add the sign $(-1)^{|a||b|}$.

1.3.2 Monoidal supercategories

Let $\mathcal{C}$ be a supercategory, and consider the tensor product $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ defined above. Note that it depends on what we mean by $\mathcal{C} \times \mathcal{C}$, that is it depends on the monoidal structure we give to $\mathcal{SC}at$, the category of small supercategories. To get back functoriality of the tensor product, we should define a monoidal structure that suits our setting. Given two supercategories $\mathcal{A}$ and $\mathcal{B}$, their *cartesian superproduct* $\mathcal{A} \boxtimes \mathcal{B}$ is the supercategory whose objects are formal pairs (λ, μ) for $\lambda \in \mathrm{ob}(\mathcal{A})$ and $\mu \in \mathrm{ob}(\mathcal{B})$, and the Hom-sets are superspaces defined using the tensor product:

$$\mathrm{Hom}_{\mathcal{A}\boxtimes\mathcal{B}}((\lambda, \mu), (\sigma, \tau)) := \mathrm{Hom}_{\mathcal{A}}(\lambda, \sigma) \otimes \mathrm{Hom}_{\mathcal{B}}(\mu, \tau).$$

Composition is given by the super interchange law (1.2). One can check that it is indeed associative and unital with $1_{(\lambda,\mu)} = (1_\lambda, 1_\mu)$. This makes $\mathcal{A}\boxtimes\mathcal{B}$ a supercategory. The unit of $\boxtimes$ is the supercategory with only one object and one morphism, namely the ground field $\Bbbk$ with identity. Associativity of $\boxtimes$ can be defined using the associativity of the tensor product. This makes $\mathcal{SC}at$ a monoidal category (we refer the reader to Brundan and Ellis [5, p. 926] for the details). We can now make clear the statement the $\mathcal{SV}ec$ is a monoidal supercategory:

Definition 1.11. *A* monoidal supercategory *is a supercategory $\mathcal{C}$ which admits a superfunctor $\otimes\colon \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C}$ and a unit object, satisfying analogous axioms [see 4, Definition 1.4] to the ones for a monoidal category.*

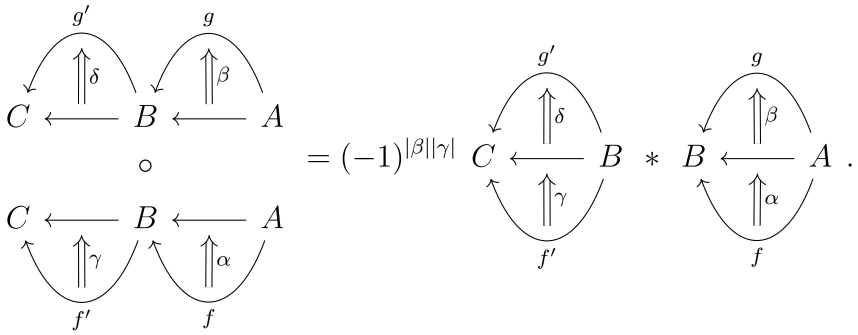
The additional axioms ensure that $\otimes$ is unital and associative up to isomorphism in a coherent way. In what follows, we shall act as all ways of putting parenthesis and tensoring with the identities are equal, just like we do for monoidal categories.

1.3.3 2-supercategories

We define 2-supercategories similarly to 2-categories:

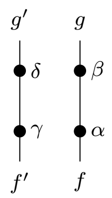
Definition 1.12. *A* 2-supercategory *is a $\mathcal{SC}at$-enriched category, that is, each Hom-set is a supercategory and composition induces a superfunctor.*

This gives us a structure similar to a 2-category, where one can speak of 2-morphisms which can be both vertically and horizontally composed. Unlike 2-categories though, each 2Hom-set (set of 2-morphisms in a given Hom-set) is a superspace, that is 2-morphisms have parities. Since composition preserves parity in a given supercategory, vertical composition of 2-morphisms preserves parity. The same can be said for horizontal composition since it induces a superfunctor. We can depict 2-morphisms as globular diagrams, and all properties found in the case of a 2-category remain the same, except the interchange law. Instead, 2-supercategories follow the super interchange law:

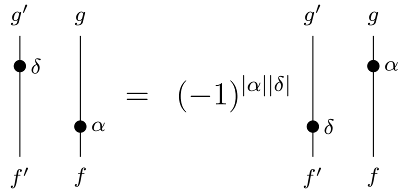


(1.3)

To avoid ambiguity we fix the convention that



is understood *first compose horizontally, then vertically.* Note that unlike 2-categories, this convention is necessary. This leads to a potential additional sign when sliding dots past one another (compare with Fig. 1.1d):



Remark 1.13. Just like monoidal categories are one-object 2-categories, monoidal supercategories are one-object 2-supercategories. In particular, the previous results in monoidal supercategories can be applied to 2-supercategories as long as we restrict ourselves to 2-morphisms whose domain and codomain are 1-morphisms with the same object as domain and codomain. This fact will be important in Chapter 3, as we will define the tensor product of chain complexes in monoidal supercategories, but with applications to 2-supercategories in mind.

Chapter 2

Supercategorification

In Chapter 1, we introduced superstructures, and in particular the notion of 2-supercategories. This chapter describes the 2-supercategory $\mathcal{S}(n,d)$, thanks to which we will define an invariant of oriented tangles in the next chapter. $\mathcal{S}(n,d)$ is the "categorified" (or rather "2-supercategorified") version of a well-known "lower level" notion, the *quantum algebra* $S_{n,d}$. In general, *categorification* is the process of turning a notion into another one of "higher structure"; in our case, it is the process of turning a category into a 2-supercategory (hence the term *super*categorification). Adding structure allow new properties to emerge and leave room for more natural definitions (think of the Euler characteristic of topological spaces: without homology, its existence is rather obscure). In [18], it is shown that one can construct an invariant of oriented tangles solely from a supercategorification of a subalgebra of $S_{n,d}$; in this thesis, we show that the same can be done with $\mathcal{S}(n,d)$. We conjecture that both invariants correspond to odd Khovanov homology. However, our construction will shed more light on it, and in particular should allow us to prove its functoriality in future work [16].

Section 2.1 describes the quantum algebra $S_{n,d}$. As we are only interested in its categorified version $\mathcal{S}(n,d)$, we avoid going into the full details (they will be given in later work). We mainly focus on *ladder diagrams*, the quantum algebra's own diagrammatic approach. In Section 2.2, we define the 2-supercategory $\mathcal{S}(n,d)$. Finally, Section 2.3 computes all the necessary relations between the 2-morphisms of $\mathcal{S}(n,d)$ needed for the proof of invariance in Chapter 3. Appendix A gives more in-depth calculations, given for completeness. They will be needed in later work.

Reference notes

The material described in Section 2.1 follows the exposition in [8] by Lauda, Queffelec and Rose. Section 2.2 and Section 2.3 are new, they contain original results that are part of my work. In particular, while a categorification of $S_{n,d}$ already exists for all levels [9], we give a *super*categorification of $S_{n,d}$ of level 2, which is new. The 2-supercategory $\mathcal{S}(n,d)$ can be extracted from results of Vaz in [18]. The connection will be made clearer in future work.

2.1 The quantum algebra

2.1.1 Formal definition

We describe here the level 2 qSchur algebra, which we will elusively call the *quantum algebra* $S_{n,d}$. We restrict its definition to be purely descriptive, as the background where the $S_{n,d}$ was first defined is not important in what follows, and it is not the subject of this thesis. The interested reader can refer to [8, 9]. $S_{n,d}$ is a category: we shall call it the "uncategorified level" nonetheless, as categories are "one level below" 2-supercategories.

Fix positive integers n and d, and set $\Lambda_{n,d} = \{\lambda \in \{0,1,2\}^n | \lambda_1 + \cdots + \lambda_n = d\}$. The λ's are the objects of $S_{n,d}$. One should think of them as representing vector spaces. Note that only coordinates 0, 1 and 2 are allowed: whenever λ has a different coordinate, its identity morphism 1_λ is set to zero (which implies λ is zero). This is called the (level 2) *Schur quotient*.

Let $\alpha_i = (0,\dots,1,-1,\dots,0)$ with 1 being on the i-th coordinate. For each $1 \le i \le n-1$, define morphisms $\mathsf{E}_i\colon \lambda \to \lambda+\alpha_i$ and $\mathsf{F}_i\colon \lambda \to \lambda-\alpha_i$. Note that they may be zero thanks to the Schur quotient. Denote $\mathbb{C}_q \coloneqq \mathbb{C}(q)$ the field of rational functions in q. Then the Hom-set $\mathrm{Hom}(\lambda,\mu)$ is the $\mathbb{C}_q$-vector space generated by the identity 1_λ (if $\lambda=\mu$), the morphisms E_i and F_i and all their compositions. Furthermore, morphisms are subject to the following relations:

$$\begin{cases} (\mathsf{E}_i\mathsf{F}_j - \mathsf{F}_j\mathsf{E}_i)(\lambda) = \delta_{ij}[\overline{\lambda}_i]_q 1_\lambda & \\ (\mathsf{E}_i\mathsf{E}_j - \mathsf{E}_j\mathsf{E}_i)(\lambda) = 0 & \text{for } |i-j|>1 \\ (\mathsf{F}_i\mathsf{F}_j - \mathsf{F}_j\mathsf{F}_i)(\lambda) = 0 & \text{for } |i-j|>1 \end{cases} \tag{2.1}$$

where $\overline{\lambda}_i = \lambda_i - \lambda_{i+1}$ and $[\cdot]_q$ is the *quantum integer* [see 7, p. 172]:

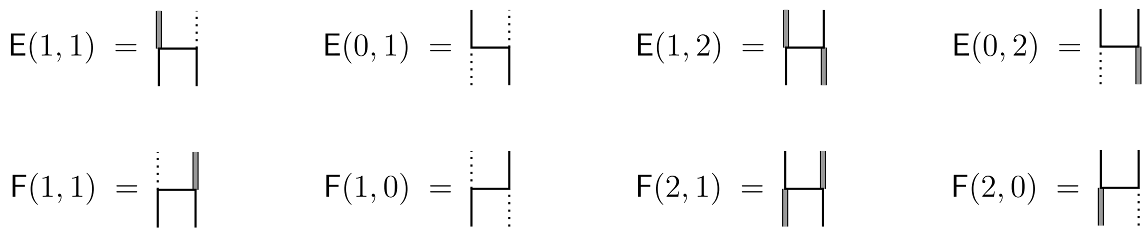
$$[m]_q = \frac{q^m - q^{-m}}{q-q^{-1}} = q^{n-1} + q^{n-3} + \ldots + q^{1-n}.$$

This ends the definition of the quantum algebra. ⋄

To emphasise the codomain, We shall sometimes use the notation $(\lambda+\alpha_i)\mathsf{E}_i(\lambda) \coloneqq \mathsf{E}_i(\lambda)$ and $(\lambda-\alpha_i)\mathsf{F}_i(\lambda) \coloneqq \mathsf{F}_i(\lambda)$, and similarly for all morphisms. As only morphisms with matching domain and codomain can be composed, in this notation λ's are interpreted as *orthogonal idempotents*: $\lambda\mu = \delta_{\lambda,\mu}$.

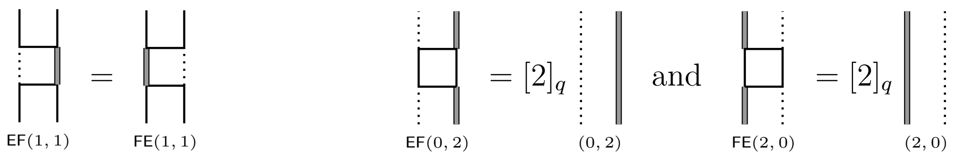
2.1.2 Ladder diagrams

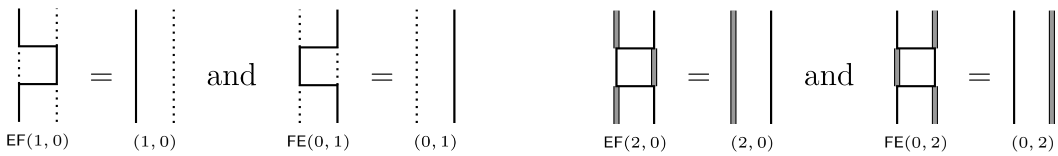
The previous chapter has taught us that finding the right picture is worth a thousand words. For our quantum algebra, the right picture is a *ladder diagram*. Given some λ, write the coordinates 0, 1 and 2 respectively with a dotted line ⋮, a single line | and a double line ‖. Then 1_λ is just a sequence of vertical lines, and if E and F acts on $\lambda = (\lambda_1, \lambda_2)$, they can be pictured as ladders, reading from bottom to top:



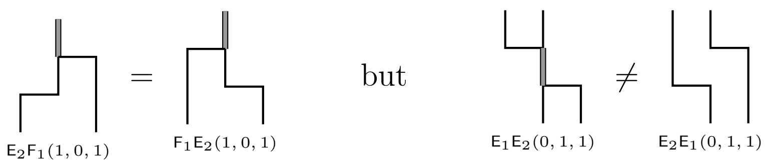
To represent the composition $f \circ g$ of two morphisms f and g, one stacks the ladder representing f on top of the ladder representing g.

Consider the first relation in (2.1) when $i = j$, i.e. $(\mathsf{EF} - \mathsf{FE})(\lambda) = [\overline{\lambda}]_q$, where the subscript i has been removed to simplify the (formulas and) diagrams. In terms of ladders, it corresponds to the following relations (for all the possible values of λ):





In the ladder diagrams above we introduced the notation $1_\lambda = \lambda$. When $|i - j| > 1$, the three relations in (2.1) implies that two ladder's rungs (the vertical bars) can be moved one past the other as long as they don't have any uprights (horizontal bars) in common. The case $|i - j| = 1$ is more subtle: only the first relation of (2.1) applies, telling us that E's and F's can be moved past one another, but not two E's nor two F's. Intuitively, this can be understood as follows: first, avoid writing dotted lines altogether; then, consider the regions delimited by the ladder; the allowed moves are the ones which do not unite two regions nor split a region. For example:



In the remainder of this thesis we shall use extensively the conventions introduced in the text: dotted lines are not drawn and $\lambda \coloneqq 1_\lambda$ (indeed 1_λ will refer to the identity 2-morphism 1_{1_λ}).

2.2 The 2-supercategory $\mathcal{S}(n,d)$

Throughout this section we fix a ground field $\Bbbk$, but the reader can safely think of $\Bbbk$ as a unital ring, where our results should work equally well.

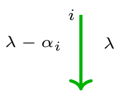
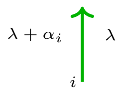
We define a 2-supercategorification of the quantum algebra $S_{n,d}$ described in Section 2.1. Fix a choice of scalars $t_{ij} \in \Bbbk$ (or $\Bbbk^\times$ in the case of a unital ring) for all $i,j \in I := \{1,\dots,n\}$, such that $t_{ii}=1$; $t_{ij}=t_{ji}$ when $|i-j|\neq 1$ and $t_{ij}t_{ji}=-1$ when $|i-j|=1$; and $t_{ij}t_{ik}=1$ for all $|i-j|=1$ and $|i-k|=1$. Let also p_{ij} be defined by $p_{ii}=p_{i+1,i}=1$ and otherwise $p_{ij}=0$, and put $p_{ijk}=p_{ij}p_{ik}+p_{ij}p_{jk}+p_{ik}p_{jk}$. Recall $\Lambda_{(n,d)}$, $\overline{\lambda}_i$ and α_i defined in Subsection 2.1.1. Note that if $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{Z}^n$, then $\overline{\lambda}_i = \langle\lambda,\alpha_i\rangle$ and $p_{ij} \equiv (\alpha_j)_i \mod 2$.

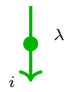
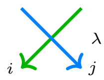
2.2.1 Definition

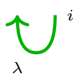
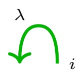
The 2-supercategory $\mathcal{S}(n,d)$ has objects $\lambda \in \Lambda_{n,d}$, generating 1-morphisms $\mathsf{F}_i(\lambda)\colon \lambda \to \lambda-\alpha_i$ and $\mathsf{E}_i(\lambda)\colon \lambda \to \lambda+\alpha_i$ for $i \in I$, and the following generating 2-morphisms below, subject to some relations (we denote by λ the identity 1-morphism of the object λ as before, and by 1_λ the identity 2-morphism of λ).

Primary generating 2-morphisms

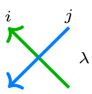
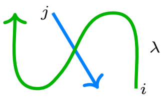
The superscript of Hom and End indicates the parity of the 2-morphisms, 0 denoting "even" and 1 denoting "odd":

 $= \mathrm{Id}_{\mathsf{F}_i(\lambda)}$  $= \mathrm{Id}_{\mathsf{E}_i(\lambda)}$

 $\in \mathrm{End}^1_{\mathcal{S}(n,d)}\big(\mathsf{F}_i(\lambda)\big)$  $\in \mathrm{Hom}^{p_{ij}}_{\mathcal{S}(n,d)}\big(\mathsf{F}_i\mathsf{F}_j(\lambda), \mathsf{F}_j\mathsf{F}_i(\lambda)\big)$

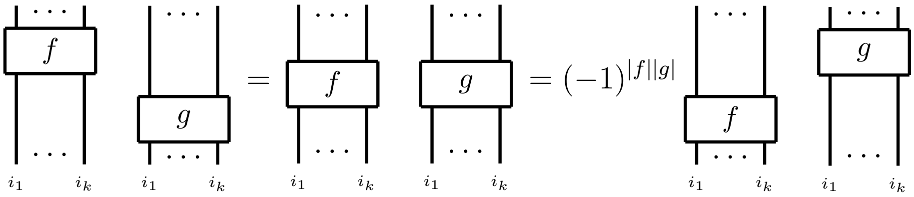
 $\in \mathrm{Hom}^{1+\overline{\lambda}_i}_{\mathcal{S}(n,d)}\big(\lambda, \mathsf{E}_i\mathsf{F}_i(\lambda)\big)$  $\in \mathrm{Hom}^{\overline{\lambda}_i}_{\mathcal{S}(n,d)}\big(\mathsf{F}_i\mathsf{E}_i(\lambda), \lambda\big)$

Secondary generating 2-morphism

 $:=$  $\in \mathrm{Hom}^0_{\mathcal{S}(n,d)}\big(\mathsf{F}_j\mathsf{E}_i(\lambda), \mathsf{E}_i\mathsf{F}_j(\lambda),\big)$ (2.2)

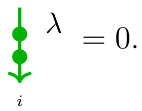
Relations

• Being in a 2-supercategory, the super interchange law (1.3) holds:

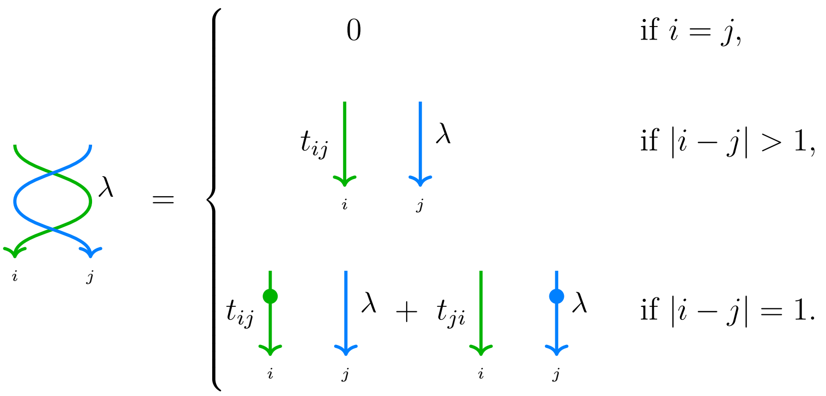


• The (level 2) Schur quotient. The identity 2-morphism 1_λ of the 1-morphism λ is zero if $\lambda \notin \Lambda_{n,d}$. This means that we set to zero all diagrams containing a region with label not in $\Lambda_{n,d}$.

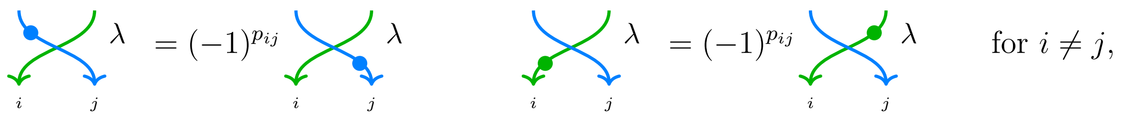
• Skew KLR relations. Downward pointing strands satisfy the skew KLR relations of [18, Definition 2.1] for all $i, j, k \in I$.



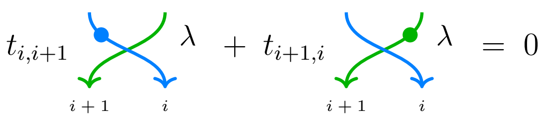
(2.3)



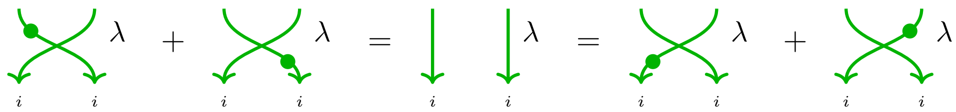
(2.4)

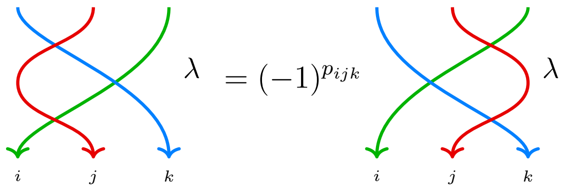


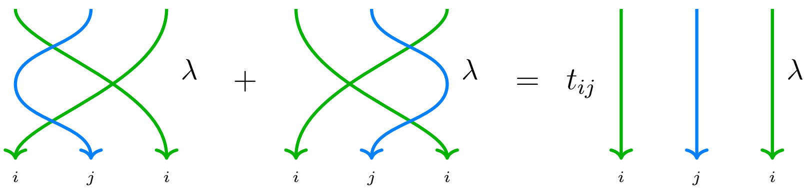
(2.5)



(2.6)

 (2.7)

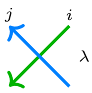
 unless $i = k$ and $|i-j| = 1$, (2.8)

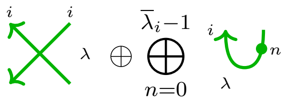
 if $|i-j| = 1$, (2.9)

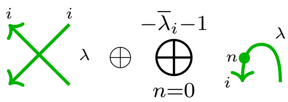
- Adjunction relations. Cups and caps are (almost) adjunctions (see Subsection 1.2.1):

 (2.10)

- Isomorphisms. Whenever non-zero, the following 2-morphisms are isomorphisms:

 $\in \mathrm{Hom}^0_{\mathcal{S}(n,d)}\big(\mathsf{F}_i\mathsf{E}_j(\lambda), \mathsf{E}_j\mathsf{F}_i(\lambda)\big)$ if $i \neq j$, (2.11)

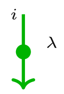
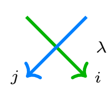
 $\in \mathrm{Hom}_{\mathcal{S}(n,d)}\big(\mathsf{F}_i\mathsf{E}_i(\lambda) \oplus \lambda^{\oplus[\overline{\lambda}_i]}, \mathsf{E}_i\mathsf{F}_i(\lambda)\big)$ if $\overline{\lambda}_i \geq 0$, (2.12)

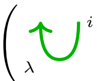
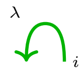
 $\in \mathrm{Hom}_{\mathcal{S}(n,d)}\big(\mathsf{F}_i\mathsf{E}_i(\lambda), \mathsf{E}_i\mathsf{F}_i(\lambda) \oplus \lambda^{\oplus[-\overline{\lambda}_i]}\big)$ if $\overline{\lambda}_i \leq 0$. (2.13)

Here $\oplus[n]$ denotes a shift in the q-grading (see below).

2.2.2 q-Grading

The 2-supercategory $\mathcal{S}(n,d)$ is $\mathbb{Z}$-graded, with

$\deg\Big($  $\Big) = 2,$ $\qquad$ $\deg\Big($  $\Big) = -\langle \alpha_i, \alpha_j \rangle,$

$$\deg\left(\right.$$  $$\left. \right) = 1 - \overline{\lambda}_i, \qquad \deg\left(\right.$$  $$\left. \right) = 1 + \overline{\lambda}_i.$$

This grading is called the *q-grading*. It has the following natural properties:

- Similarly to the $\mathbb{Z}/2\mathbb{Z}$-grading, it is compatible with vertical and horizontal compositions, that is $\deg(\alpha\beta) = \deg(\alpha) + \deg(\beta)$ and $\deg(\alpha * \beta) = \deg(\alpha) + \deg(\beta)$, whenever it makes sense. Moreover, $\deg(1_\lambda) = 0$.

- It induces a $\mathbb{Z}$-action $(n, f) \mapsto f\{n\}$ on the 1-morphisms, leaving the 2Hom-spaces unchanged: $2\mathrm{Hom}(f\{n_1\}, g\{n_2\}) = 2\mathrm{Hom}(f, g)$. On the other hand, the q-grading do change, with the property that if $\alpha \in 2\mathrm{Hom}(f, g)$ and $\deg \alpha = d$, then as an element of $2\mathrm{Hom}(f\{n_1\}, g\{n_2\})$ we have $\deg \alpha = d + n_2 - n_1$. The reader can refer to Bar-Natan [2, Chapter 6] for a more thorough definition of grading in categories.

It is easy to see that all the defining relations are homogeneous with respect to the $\mathbb{Z}$-grading (that is, both sides have the same degree). Furthermore, if f is a 1-morphism we denote $f^{\oplus[n]} = f\{n-1\} \oplus f\{n-3\} \oplus \ldots \oplus f\{1-n\}$. Note that the q-shifts in (2.12) and (2.13) unsure that the isomorphisms are of q-degree 0. Most of the time, we shall use the notation

$$q^n f \coloneqq f\{n\}.$$

Note that this implies $[n]_q f = f^{\oplus[n]}$. Furthermore, q-shifting being an action we have $\deg(f \circ g) = \deg f + \deg g$. In particular, q^n can be understood as a polynomial, composing two 1-morphisms resulting in the product of their polynomial.

2.2.3 Decategorification

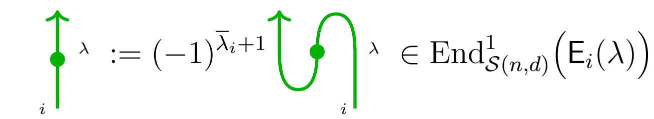
As stated many times, $\mathcal{S}(n, d)$ is intended to categorify the quantum algebra. Indeed, the variable q has turned into a $\mathbb{Z}$-grading, as the notation $q^n f \coloneqq f\{n\}$ suggests. Also, the relations (2.1) have become the isomorphisms (2.4), (2.11), (2.12) and (2.13). Only the relation $(\mathsf{E}_i\mathsf{E}_j - \mathsf{E}_j\mathsf{E}_i)(\lambda) = 0$ for $|i - j| > 1$ does not seem to be categorified, but Lemma 2.7 will show that a relation similar to (2.4) holds for E, that is when all arrows are pointing up. (For the experts: precisely, the quantum algebra is the Grothendieck algebra of $\mathcal{S}(n, d)$. The fact will be established in future work [16].) In other words, equalities at the quantum algebra level have become isomorphisms between 1-morphisms at the level of the 2-supercategories. In particular, as long as we are only interested in the isomorphic class of a 1-morphism, ladder diagrams of Subsection 2.1.2 can also be used at the level of the 2-supercategory.

2.3 Relations between 2-morphisms

In this section we compute relations between the 2-morphisms of $\mathcal{S}(n,d)$ that will be useful in the next chapter.

2.3.1 More secondary generating 2-morphisms

Upward dotted arrow and upward crossing

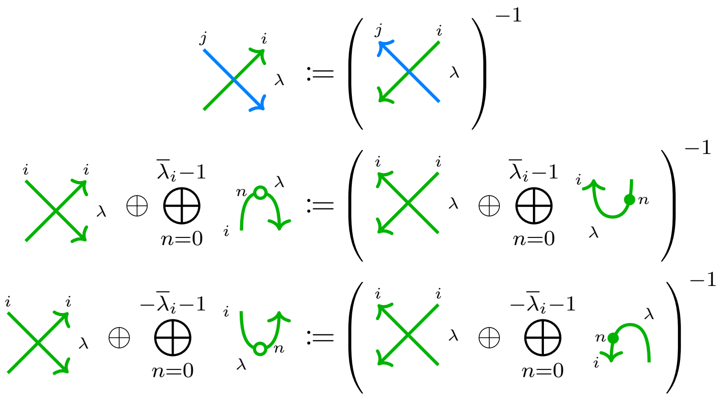


(2.14)

(2.15)

2-morphisms induced by the isomorphisms

Thanks to the isomorphisms (2.11), (2.12) and (2.13), we can define:

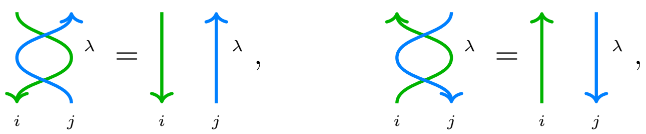


if $i \neq j$ (2.16)

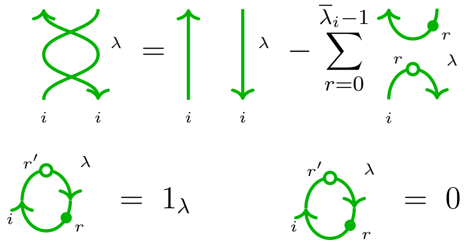
if $\overline{\lambda}_i \geq 0$, (2.17)

if $\overline{\lambda}_i \leq 0$. (2.18)

Explicitly, this implies the following relations:

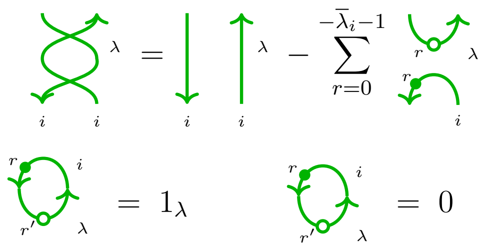


if $i \neq j$, (2.19)



if $\overline{\lambda}_i \geq 0$, (2.20)

if $0 \leq r \leq \overline{\lambda}_i - 1$ and $r' \neq r$, (2.21)

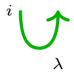
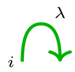


if $\overline{\lambda}_i \leq 0$, (2.22)

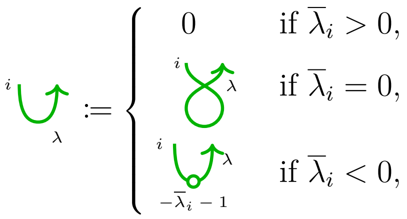
if $0 \leq r \leq -\overline{\lambda}_i - 1$ and $r' \neq r$. (2.23)

Rightward cups and caps

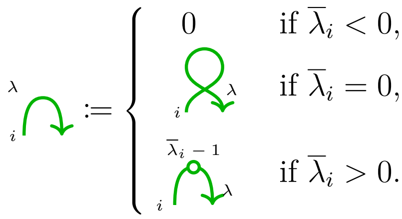
The following 2-morphisms

 $\in \mathrm{Hom}^{1+\lambda_{i+1}}_{\mathcal{S}(n,d)}\big(\lambda, \mathsf{E}_i\mathsf{F}_i(\lambda)\big)$ and  $\in \mathrm{Hom}^{\lambda_{i+1}}_{\mathcal{S}(n,d)}\big(\mathsf{E}_i\mathsf{F}_i(\lambda), \lambda\big)$,

can be defined by declaring that

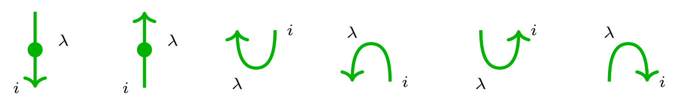


and

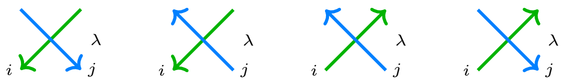


(2.24)

Summary of parities and q-degrees



parity	1	1	$1+\lambda_i$	λ_i	$1+\lambda_{i+1}$	λ_{i+1}
q-degree	2	2	$1-\overline{\lambda}_i$	$1+\overline{\lambda}_i$	$1-\overline{\lambda}_i$	$1+\overline{\lambda}_i$

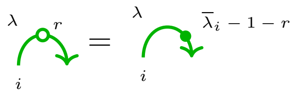


parity	p_{ij}	0	p_{ji}	0
q-degree	$-\langle \alpha_i, \alpha_j \rangle$	0	$-\langle \alpha_i, \alpha_j \rangle$	0

2.3.2 Basic 2-morphism relations

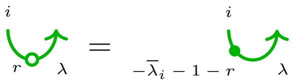
The EF-relations

Lemma 2.1. *We have*



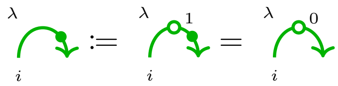
if $\overline{\lambda}_i > 0$,

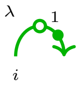
and



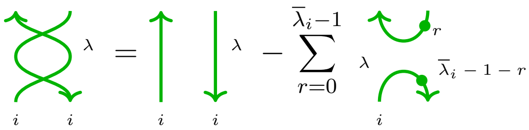
if $\overline{\lambda}_i < 0$.

Proof. We focus on the case $\overline{\lambda}_i > 0$. The relation follows directly from the definition (2.24), except when $r \neq \overline{\lambda}_i - 1$. This only happens when $\overline{\lambda}_i = 2$ and $r = 0$. Then we must show that:

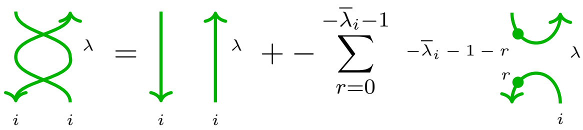


It suffices to show that  satisfies the relation (2.21) for $r = 0$. It is easy to see using the same relation for $r = 1$ and Eq. (2.3). ☐

Corollary 2.2.



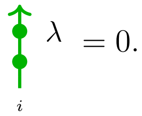
if $\overline{\lambda}_i \geq 0$,



if $\overline{\lambda}_i \leq 0$.

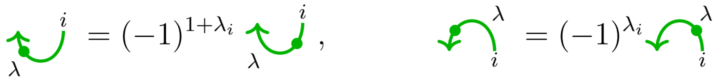
Dot relations

Lemma 2.3.



Proof. It is an easy consequence of the adjunction relations (2.10) and the definition (2.14). ☐

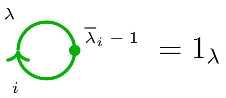
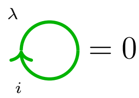
Lemma 2.4.

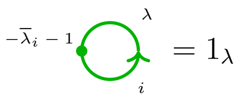
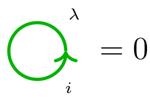


Proof. It follows from the definition (2.14) and the adjunction relations (2.10). ☐

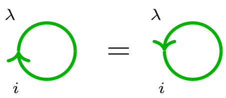
Bubbles

Bubbles are formed by composing cups, dots and caps. It follows directly from (2.21) and (2.23) and the definitions of rightward cup and cap (2.24) that:

 if $\overline{\lambda}_i > 0$,  if $\overline{\lambda}_i = 2$, (2.25)

 if $\overline{\lambda}_i < 0$,  if $\overline{\lambda}_i = -2$. (2.26)

Moreover it follows from Corollary 2.2 and (2.24) that when $\overline{\lambda}_i = 0$:

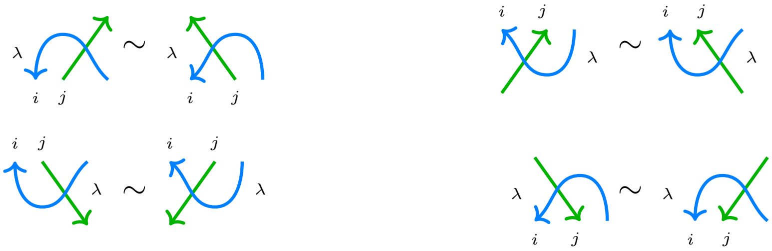


2.3.3 Relations up to a scalar multiplication

We develop here the bear minimum for our purpose, that is, proving that we can derive an invariant of oriented tangles from $\mathcal{S}(n, d)$. In 2-supercategories, the main difficulty in calculations is to get the signs right. Thankfully, for our purpose only the relations up to a non-zero scalar multiplication matter. Therefore, in relations below (and only for them) we introduce the notation $\sim$ to denote "equal up to a non-zero scalar multiplication". Similarly, $\overset{+}{\sim}$ denotes addition up to a non-zero scalar multiplication. The exact computations (and other relations) can be found in Appendix A.

Pitchforks and kinks

Lemma 2.5 (Left pitchforks)**.** *We have*

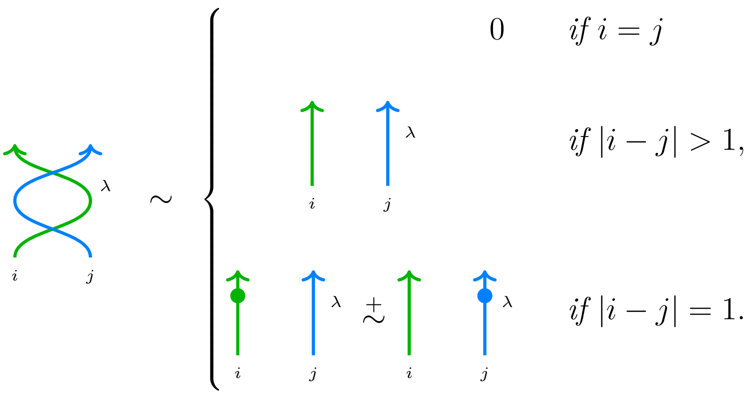


Proof. Use the definition of the leftward (2.2) and upward (2.15) crossings and the adjunction relations (2.10). □

Lemma 2.6 (Kinks)**.** *For* $\overline{\lambda}_i = 0$ *we have*

Proof. Use Lemma 2.5 and the adjunction relations (2.10). □

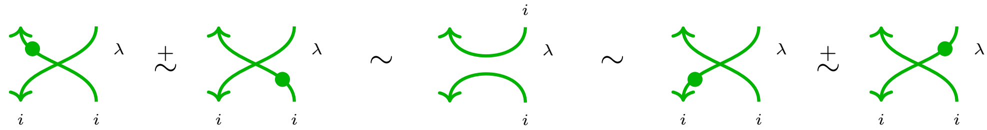
Lemma 2.7 (KLR Reidemeister 2 relation for E)**.**



Proof. Use the definition (2.15) and the adjunction relations (2.10). □

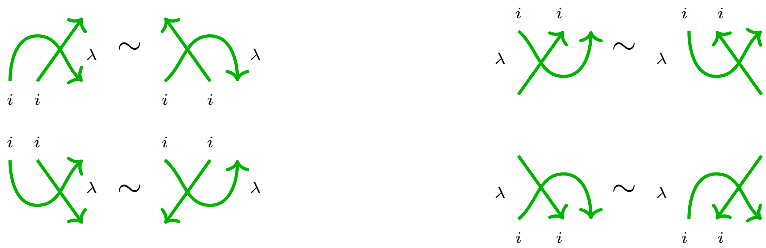
Second adjunctions

Lemma 2.8 (Dot slide)**.**

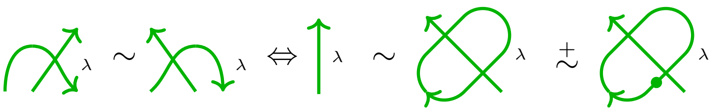


Proof. It is a consequence of the definition of the leftward crossing (2.2) and of (2.7). □

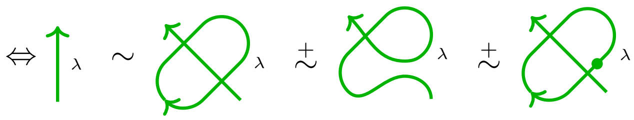
Lemma 2.9 (Right pitchforks)**.** *We have*



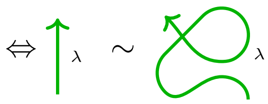
Proof. We show the first relation. The other relations are similar. We assume $\lambda_i = (1,1)$ since all other cases are zero thanks to the Schur quotient. All strings are labelled i.



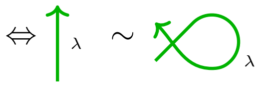
compose with isomorphisms in Corollary 2.2



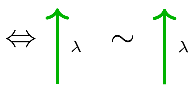
dot slide (Lemma 2.8)



left pitchforks (Lemma 2.5) and KLR Reidemeister 2 for E (Lemma 2.7)



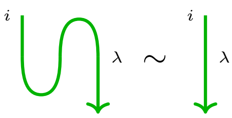
adjunction relation (2.10)

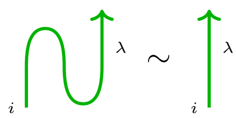


kink (Lemma 2.6)

□

Lemma 2.10 (rightward zigzags)**.** *We have:*



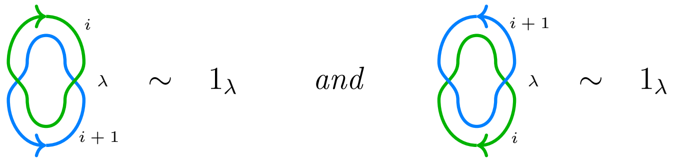


In particular, the right zigzags are isomorphisms.

Proof. When $\overline{\lambda}_i = 1$, it a consequence of Corollary 2.2 and the adjunction relations (2.10), since then cups and caps are isomorphisms. If $\overline{\lambda}_i \neq 1$, it is a consequence of Lemma 2.9 and Lemma 2.6. □

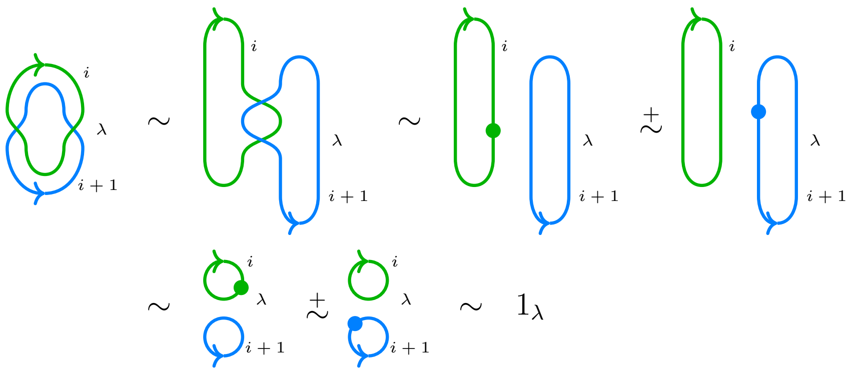
Double bubble

Lemma 2.11 (double bubble case A)**.** *If $\lambda = (\dots, 1, 0, 2, \dots)$, 1 being the i-coordinate:*



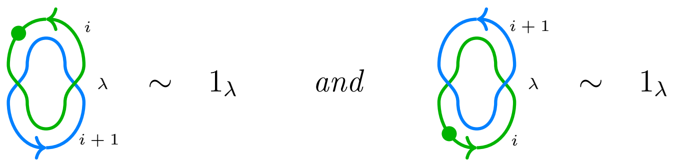
In particular, these 2-morphisms are isomorphisms.

Proof. We focus on the first case, the other case being similar. Using the left pitchforks (Lemma 2.5), Reidemeister 2 for F (2.4) and the bubble relations (2.25) and (2.26), we get:



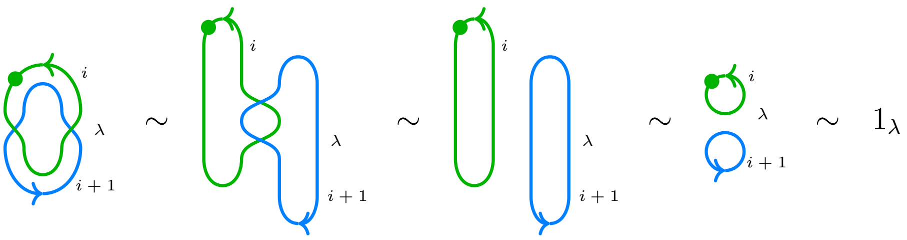
□

Lemma 2.12 (double bubble case B)**.** *If $\lambda = (\dots, 0, 2, 1, \dots)$, 0 being the i-coordinate:*



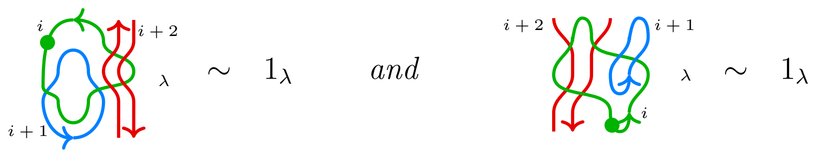
In particular, these 2-morphisms are isomorphisms.

Proof. We focus on the first case, the other case being similar. Using the pitchforks (Lemma 2.5 and Lemma 2.9), the isomorphism (2.19) and the bubble relations (2.25) and (2.26), we get:



□

Corollary 2.13. *If* $\lambda = (\ldots, 0, 2, 1, 1, \ldots)$*, 0 being the* i*-coordinate:*



In particular, these 2-morphisms are isomorphisms.

Proof. In the first of these two 2-morphisms, the rightmost red arrow can be “separated” from the rest using the isomorphism (2.19). The same can be done for the other red arrow using the KLR Reidemeister 2 relation for E (Lemma 2.7) (indeed, $|i - (i+2)| > 1$). The first case of Lemma 2.12 concludes. A similar argument holds for the second 2-morphism, using the second case of Lemma 2.12. □

Chapter 3

An invariant of oriented tangles

This final chapter introduces our new invariant of oriented tangles. For each diagram D, we define $\mathrm{Kom}(D, n, d)$ as the horizontal product of some chain complexes in the category $\mathcal{S}(n, d)$, described in Chapter 2. It is fully described in Section 3.2. Before that though, the meaning of "horizontal product of chain complexes" in the context of 2-supercategories needs to be clarified. We deal with this question in Section 3.1, where we define the tensor product of chain complexes in monoidal supercategories. Indeed, as we have seen in Remark 1.13 2-supercategories are closely related to monoidal supercategories, and under this relation the horizontal product corresponds to the tensor product.

To get an invariant of oriented tangles from $\mathrm{Kom}(D, n, d)$, we show in Section 3.3 that if two diagrams D and D' represent the same tangle, then $\mathrm{Kom}(D, n, d)$ and $\mathrm{Kom}(D', n, d)$ belong to the same homotopy class. For T an oriented tangle, $\mathrm{Kom}(T, n, d)$ is then the homotopy class of $\mathrm{Kom}(D, n, d)$, where D is any diagram representing T. An important step in this proof is to show that homotopy classes are invariant under the horizontal product of chain complexes (or the tensor product in the context of monoidal supercategories). This step is quite technical and therefore, despite its importance, it is postponed to Appendix B to facilitate reading.

Reference notes

Except for the Homological Lemmas (3.15) and (3.19), all results in this chapter are original, as well as the results of Appendix B. However, some approaches are inspired by other works: the combinatorial approach of Subsection 3.1.1 is similar to Oszváth, Rasmussen and Szabó in [15], although the proof of Lemma 3.5 is original; the definition of $\mathrm{Kom}(D, n, d)$ in Section 3.2 is almost identical to the one given in [8, Part 4] by Lauda, Queffelec and Rose; gaussian elimination is already used in [1, Lemma 4.2] by Bar-Natan in the context of (even) Khovanov homology; and the importance of cones in the proof of Reidemeister III is pointed out in [2] by Bar-Natan. These references are recalled in the text.

3.1 Chain complexes in monoidal supercategories

Let $\mathcal{C}$ be a monoidal supercategory (we shall always assume categories are pre-additive). Our goal is to define all the usual constructions related to chain complexes in this new framework, and to check that some typical results still hold. First of all, basic definitions remain the same:

- *a (chain) complex* is a sequence of composable morphisms

$$\dots \longrightarrow A^r \xrightarrow{\alpha^r} A^{r+1} \xrightarrow{\alpha^{r+1}} A^{r+2} \longrightarrow \dots$$

  such that $\alpha^{r+1} \circ \alpha^r = 0$ for all r. the A^r's are the *chain spaces*, and the α_r's are the *differentials*. We usually denote such a complex (A, α) or simply A. A complex is *of length* n if there exists some r such that $A^i = 0$ for all $i < r$ and $i + r \geq n$. *Bounded complexes* are complexes of finite length. In what follows, we will only consider such complexes, and from now on all complexes are assumed to be bounded.

- a *morphism* $f\colon A \to B$ between the complexes (A, α) and (B, β) is a set of morphisms $f^r\colon A^r \to B^r$ which commute with the differentials, that is

$$\begin{array}{ccc} A^r & \xrightarrow{\alpha^r} & A^{r+1} \\ \downarrow{\scriptstyle f^r} & & \downarrow{\scriptstyle f^{r+1}} \\ B^r & \xrightarrow{\beta^r} & B^{r+1} \end{array}$$

  is a commuting square for all r. This defines the category $\mathrm{Ch}_\bullet(\mathcal{C})$ whose objects are complexes and whose morphisms are morphisms of complexes.

- A *homotopy* $h\colon f \to g$ of morphisms of complexes $f\colon A \to B$ and $g\colon A \to B$ is a set of morphisms $h^r\colon A^r \to B^{r+1}$ such that $f^r - g^r = \alpha^r h^{r+1} + h^r \beta^{r-1}$ for all r. If there exists such a homotopy, we say that f and g are *homotopic* and write $f \simeq g$. Moreover, if $f\colon A \to B$ and $g\colon B \to A$ are morphisms such that $fg \simeq \mathrm{Id}_B$ and $gf \simeq \mathrm{Id}_A$, we say that f and g are *homotopy equivalences* and that A and B are *homotopic* (or *homotopy equivalent*) and write $A \simeq B$. This defines an equivalence relation, and therefore we can talk about *homotopy classes*.

None of the above depends either on the monoidal structure or the super structure, and the same is true for any typical construction or result that do not involve the tensor product (for instance, both Lemma 3.15 and Lemma 3.19 are "non-super" results, but are still applicable in a super context). In particular, homology can be safely defined and homotopic complexes have isomorphic homologies[1].

[1]Note that we speak of homology: indeed, (even) Khovanov homology is covariant with respect to cobordisms [2]. Nonetheless, we use chain complexes with increasing degrees, which usually refers to *co*homology. This is an unfortunate historical consequence, and we shall not try to remake history here.

Difficulties arise when we try to define the tensor product of two complexes: compared to the even case, new signs arise when composing the tensor products of the differentials. This means that we need to adjust the definition, adding some signs to the right differentials. We restrict ourselves to *homogeneous* complexes:

Definition 3.1. *A complex is* homogeneous *if all its differentials are homogeneous (note that it includes the zero map).*

The *tensor product* (A, α) of n homogeneous complexes $A_i^{r_i} \xrightarrow{\alpha_i^{r_i}} A_i^{r_i+1}$ is the complex given by

$$A^r := \bigoplus_{\vec{r}:|\vec{r}|=r} A^{\vec{r}} \qquad \text{where } A^{\vec{r}} := A_1^{r_1} \otimes \ldots \otimes A_n^{r_n}$$

$$\text{and } \alpha|_{A^{\vec{r}}} := \sum_{1\leq i\leq n} \alpha_i^{\vec{r}} \qquad \text{where } \alpha_i^{\vec{r}} := \epsilon_A^{\vec{r},i} \mathrm{Id}_{A_1^{r_1}} \otimes \ldots \otimes \alpha_i^{r_i} \otimes \ldots \otimes \mathrm{Id}_{A_n^{r_n}}.$$

Here $\epsilon_{\vec{r},i}^A$ is a choice of signs such that A is indeed a complex, that is such that $\alpha^{r+1} \circ \alpha^r = 0$. For this to be true, a sufficient condition is that *all squares anti-commute*: $\alpha_j^{\vec{r}+\vec{e}_i} \circ \alpha_i^{\vec{r}} = -\alpha_i^{\vec{r}+\vec{e}_j} \circ \alpha_j^{\vec{r}}$ for all $\vec{r}$ and $i \neq j$ (here $(e_i)_i$ denotes the canonical basis of $\mathbb{Z}^n$). In the even case, this is done with the *Koszul sign rule* $\epsilon_{\vec{r},i}^A = (-1)^{\sum_{j<i} r_j}$, which ensures that in every square there is an odd number of signs, making all squares anti-commute. In Appendix B, we give a choice that works in the super case:

$$\epsilon_{\vec{r},i}^A = (-1)^{\sum_{j<i} r_j + \alpha_i^{r_i} \cdot |\alpha|(\vec{r},i)}$$

where $|\alpha|(\vec{r},i)$ is a quantity such that $|\alpha|(\vec{r}+\vec{e}_j,i)+|\alpha|(\vec{r},i) \equiv \delta_{j\leq i}\alpha_j^{r_j}$ for all $1 \leq i,j \leq n$. This defines the following category:

Definition 3.2. *The category* $\mathrm{TCh}_\bullet(\mathcal{C})$ *is the category*[2] *of tensor products of chain complexes in* $\mathcal{C}$ *whose factors are homogeneous complexes. The morphisms of* $\mathrm{TCh}_\bullet(\mathcal{C})$ *are the usual morphisms of complexes, making* $\mathrm{TCh}_\bullet(\mathcal{C})$ *a full subcategory of* $\mathrm{Ch}_\bullet(\mathcal{C})$. $\mathrm{TCh}_\bullet^n(\mathcal{C})$ *is the subcategory of* $\mathrm{TCh}_\bullet(\mathcal{C})$ *consisting only in tensor products whose factors are of length* n.

To facilitate reading, the proofs of the following important facts have been postponed to Appendix B:

- Since the tensor product was defined for an arbitrary number of factors, we can define a tensor product on $\mathrm{TCh}_\bullet(\mathcal{C})$ (Definition B.7). From the definition, it is associative.
- Given two morphisms in $\mathrm{TCh}_\bullet(\mathcal{C})$, we can induce a morphism on the tensor product, such that the induced morphism of a pair of identities is the identity (Proposition B.9).

[2] Note that morphisms of complexes do not have a naturally defined parity, and thus $\mathrm{TCh}_\bullet(\mathcal{C})$ is in general not a supercategory.

- Given two homotopies in $\mathrm{TCh}_\bullet(\mathcal{C})$, we can induce a homotopy on the induced morphisms (Proposition B.12).

As a direct consequence, we have that homotopy classes are invariant under taking tensor products:

Theorem 3.3 (Theorem B.13)**.** *Let A_1, A_2, B_1 and B_2 be complexes of $\mathrm{TCh}_\bullet(\mathcal{C})$. If $A_1 \simeq B_1$ and $A_2 \simeq B_2$, then $A_1 \otimes A_2 \simeq B_1 \simeq B_2$.*

This final result is sufficient for the purpose of this chapter. Moreover, we shall see that we can actually restrict our attention to the category $\mathrm{TCh}^2_\bullet(\mathcal{C})$. As the factors of the objects of $\mathrm{TCh}^2_\bullet(\mathcal{C})$ only have (at most) one non-trivial differential, we introduce the following suggestive definition:

Definition 3.4. *A complex (A, α) is said to be a* hypercubic complex *if it is isomorphic to a tensor product whose factors are all of length 2.*

If (A, α) is a hypercubic complex with n factors, we can associated to it a hypercube T of dimension n:

- the *vertices* of T are the chain spaces $A^{\vec{r}}$.
- the *edges* of T are the differentials $\alpha_i^{\vec{r}}$. We call an *i-edge* any edge in direction i (that is, an edge such that the degrees of its vertices only differ at coordinate i).
- the *faces* of T are the anti-commuting squares

$$\begin{array}{ccc} A^{\vec{r}} & \xrightarrow{\alpha_i^{\vec{r}}} & A^{\vec{r}+\vec{e}_i} \\ \downarrow{\scriptstyle \alpha_j^{\vec{r}}} & & \downarrow{\scriptstyle \alpha_j^{\vec{r}+\vec{e}_i}} \\ A^{\vec{r}+\vec{e}_j} & \xrightarrow{\alpha_i^{\vec{r}+\vec{e}_j}} & A^{\vec{r}+\vec{e}_i+\vec{e}_j} \end{array}$$

  where $i \neq j$ and $(\vec{e}_i)_i$ is the canonical basis of $\mathbb{Z}^n$. We call an *i-face* any face containing an i-edge.

The hypercubic complex (A, α) can be recovered by "smashing" T, that is by summing up all vertices with the same degree together. The hypercubic complex and its associated hypercube are essentially the same thing, and we often won't distinguish the two in what follows. By using the combinatorial picture of the hypercube, we show in the next subsection that in the specific case of hypercubic complexes, all choice of signs in the definition of the tensor product are actually equivalent; that is, two different choices of signs result in two isomorphic complexes. Since the proof of Section 3.3 only cares about the homotopy classes, this will allow us to forget about the definition given in Appendix B and set signs as we wish as long as the hypercube represents a complex.

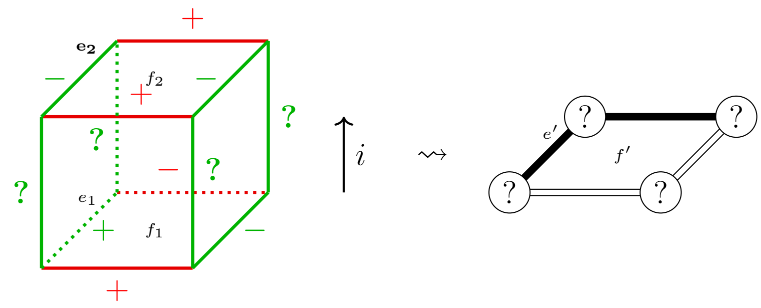


Figure 3.1: Proof of Lemma 3.5: example with $n = 2$. Even and odd edges are respectively pictured green and red.

3.1.1 Equivalence under different choices of signs

The combinatorial approach of this section is similar to Oszváth, Rasmussen and Szabó in [15], although the proof of Lemma 3.5 is original.

Let (A, α) be a hypercubic complex. We wish to use the corresponding hypercube T as a proper combinatorial model for A, containing only the necessary information for our purpose. Therefore, each edge of T must have a sign and a parity, such that opposite edges have the same parity (since $\left|\alpha_i^{\vec{r}}\right| \equiv \left|\alpha_i^{\vec{s}}\right|$ for all $\vec{r}$, $\vec{s}$). We also give signs and parities to faces:

> a face is *odd* if all its edges are odd, and *even* if at least one pair of opposite edges is even. The (global) *sign* of a face is the product of the signs of its edges. A face *commutes* if the pair $(\mathit{parity}, \mathit{sign})$ is $(\mathit{even}, \mathit{positive})$ or $(\mathit{odd}, \mathit{negative})$, and *anti-commutes* otherwise. This definition of commutation matches with the special case of hypercubic complexes.

Given two choices of signs for (A, α) such that all the faces anti-commute (that is, such that (A, α) is a complex), we can always consider the identity morphisms between the corresponding vertices. This creates a hypercube of dimension $n+1$ whose k-faces anti-commute for all $k \neq n+1$. To get a proper morphism of complexes (and therefore an isomorphism, since we are dealing with identities), we need to set the signs of the $n+1$-edges such that the $n+1$-faces all commute. This is done by the following (purely combinatorial) lemma.

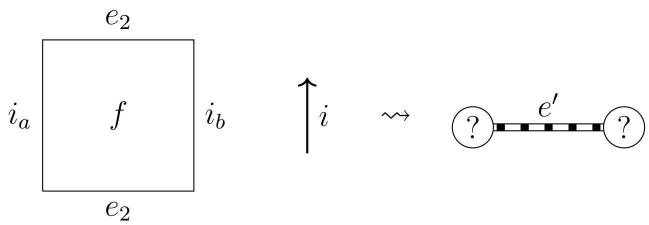
Lemma 3.5. *Let T be a hypercube of dimension $n+1$ and let $1 \leq i \leq n+1$ be some direction. Assume the edges of T are given signs and parities such that opposite edges have opposite parities. If all k-faces for $k \neq i$ anti-commute, then there exists a choice of signs for the i-edges such that the i-faces commute.*

Proof. By removing i-edges in T, we get two hypercubes T_1 and T_2 of dimension n with the same parities but different signs. Let T' be another hypercube of dimension n.

To every edge e' in T' corresponds a pair of edges (e_1, e_2) where $e_1 \in T_1$ and $e_2 \in T_2$ (one could think of T' as being obtained from T by "smashing" it along direction i, therefore contracting T_1 and T_2 into one another). We rephrase the problem in T' (see the Fig. 3.1):

- Vertices of T' are given signs, representing the signs of the i-edges in T.
- Edges are given a black and white colouring. Let $e' \in T'$ and (e_1, e_2) the corresponding pair. Let s be the product of the signs of e_1 and e_2: the edge is white if $s = +1$ and black if $s = -1$. In other words, white says that e_1 and e_2 have the same sign, and black that they have opposite sign.
- Let f' be a face in T' and (f_1, f_2) the corresponding pair of faces. T_1 and T_2 have the same parities and f_1 and f_2 both anti-commute, so they have the same global sign. The product of these global signs must then be $+1$. This sign being the product of the signs of the edges of f_1 and f_2, one checks that this implies that there is an even number of white (or black) edges in f'.

Let f be a i-face in T and call its edges e_1, e_2, i_a and i_b where $e_1 \in T_1$ and $e_2 \in T_2$ (with corresponding edge $e' \in T'$) and i_a and i_b are i-edges:



We look for the conditions on T' under which f commutes. If f is even, then its global sign must be $+1$, that is f must have an even number of -1 in its edges: if e_1 and e_2 have the same sign (e' is white), then i_a and i_b must have the same sign (the vertices of e' have the same sign), and vice versa. We end up with the following condition:

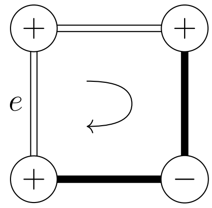
> the vertices of white edges must have the same sign and the vertices of black edges must have opposite signs.

Now assume f is odd: then its global sign must be -1 and the opposite condition must hold! Therefore, we modify the definition of our colouring: if the i-face in T corresponding to e' is odd, switch the colouring of e'. Note that since opposite edges have the same parities, this does not affect the property that there is an even number of white (or black) edges in each face.

The problem is thus rephrased in this terms: find a choice of signs for the vertices of T' such that white edges have vertices with the same sign, and black edges have vertices with opposite signs. We proceed by induction. Small cases are easily checked, so we focus on the inductive step. Note the following fact:

if in a face three edges are *well-signed* (that is, the signs of the vertices are such that the condition is verified for the edge), then so is the last edge.

Indeed, call this last edge e and walk along the three well-signed edges, starting at a vertex of e. While moving, the sign of the vertex will switch every time we cross a black edge, and remain the same when we cross a white edge. But since there is an even number of black edges, either e is black and when we reach the other vertex of e the sign has switched, or e is white and the sign has remained the same. Either case the condition is verified and e is well-signed (see example below).



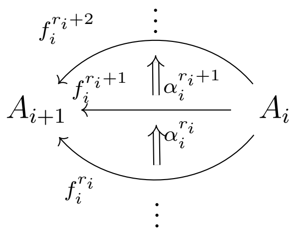
Assume now that we can give signs to hypercubes of dimension $n-1$ so that all edges are well-signed, and consider a hypercube A of dimension n. Let A_1 and A_2 be the two hypercubes of dimension $n-1$ obtained by removing n-edges in A. We can give signs to the vertices of A such that the edges in A_1 and in A_2 (that is, the edges of A which are not n-edges) are well-signed. Assume at least one n-edge is well-signed: then by the previous fact, all n-edges sharing a face with this edge are well-signed. We can now re-apply the fact to these edges to find new well-signed edges, and eventually conclude that all n-edges are well-signed. On the other hand, if no n-edges are well-signed it suffices to switch the signs of all the edges of A_2: all the n-edges will then be well-signed, and the edges of A_2 will remain well-signed. This concludes the inductive step, and ends the proof of the lemma. □

Remark 3.6. In the previous proof, we could equally have chosen to make the i-faces anti-commute, adding signs according to the Koszul sign rule. This gives the inductive step necessary to prove the existence of a choice of signs in the specific case of hypercubic complex. We thus have another proof of its existence, in addition to the explicit one given in Appendix B.

3.1.2 Horizontal product of chain complexes in 2-supercategories

Until now we have been working exclusively with monoidal supercategories. Indeed, they carry less data than 2-supercategories and are closer to constructions we are already familiar with, which makes them better suited for introducing the product of chain complexes. However, Chapter 2 defined a 2-supercategory: we need to extend the tensor product of complexes in monoidal supercategories to a suitable product in 2-supercategories. The identification of monoidal supercategories as one-object 2-supercategories pointed out in Remark 1.13 makes it straightforward.

Under this identification, composition of morphisms become vertical composition of 2-morphisms, so that a complex in a monoidal category becomes a sequence of vertically composable 2-morphisms. Moreover, tensor product on objects become composition of 1-morphisms, and tensor product on morphisms become horizontal composition. In other words, we can define a *horizontal product* on complexes in a 2-supercategories for horizontally composable complexes. More precisely, if we are given n sequences of horizontally composable 2-morphisms:



such that $\alpha_i^{r_i+1}\alpha_i^{r_i} = 0$, their horizontal product is a sequence (f, α) given by:

$$f^r := \bigoplus_{\vec{r}:|\vec{r}|=r} f^{\vec{r}} \qquad \text{where } f^{\vec{r}} := f_1^{r_1} \circ \ldots \circ f_n^{r_n}$$

$$\text{and } \alpha|_{f^{\vec{r}}} := \sum_{1 \leq i \leq n} \alpha_i^{\vec{r}} \qquad \text{where } \alpha_i^{\vec{r}} := \epsilon_f^{\vec{r},i} \mathrm{Id}_{f_1^{r_1}} * \ldots * \alpha_i^{r_i} * \ldots * \mathrm{Id}_{f_n^{r_n}}$$

with the same choice of signs defined previously. We extend all definitions to this context, and in particular the definitions of $\mathrm{TCh}_\bullet(\mathcal{C})$ and $\mathrm{TCh}_\bullet^n(\mathcal{C})$ for $\mathcal{C}$ a 2-supercategory. All results developed in this section remain true, and in particular Theorem 3.3.

3.2 Definition of the invariant

Our construction is inspired by the work of Lauda, Queffelec and Rose in [8]: our definition is almost identical to theirs.

In this section, we associate to any oriented tangle diagram D a complex $\mathrm{Kom}(D, n, d)$ in $\mathrm{TCh}_\bullet^2(\mathcal{S}(n, d))$ for n sufficiently large and d restricted to some values depending on n. The next section will then show how one can induce an invariant of oriented tangles from $\mathrm{Kom}(D, n, d)$. In what follows, if no orientation is given to a diagram then it is assumed that the definition or the discussion does not depend on the orientation.

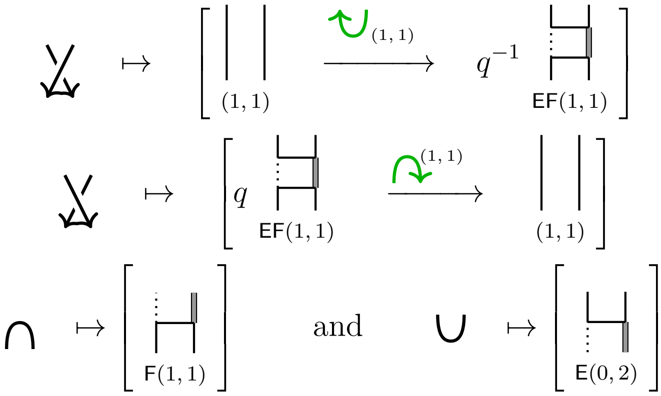
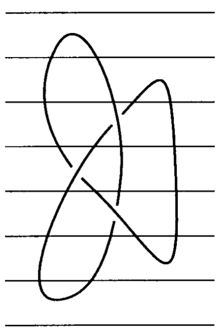


Figure 3.2: Local definition of Kom(D, n, d). The bracket notation emphasises that the inside is a complex. In all cases, the chain space of homological degree 0 is the one of q-degree 0.

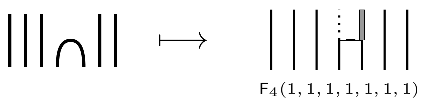


Any tangle diagram can be decomposed in slices containing only one basic element: a crossing, a cup or a cap (this is explained for example in [14, Chapter 3]). The picture on the right illustrates this idea (it is taken from [14, Figure 3.7]). We proceed as follows: first, associate a basic homogeneous complex of length 2 to each basic element of oriented tangle diagrams; second, extend this definition to a whole slice; third, take the horizontal product of the complexes associated to the slices (reading from bottom to top); last, normalise the complex by q-shifting it according to the writhe.

The first step is given by Fig. 3.2. Note that all differentials have q-degree 0. For the second step, assume there are i strands before the basic element and j strands after it. The 1-morphism(s) in the basic complex can be extended such that it (they) act(s) on

$$(\underbrace{1, \ldots, 1}_{i}, a, b, \underbrace{1, \ldots, 1}_{j})$$

instead of (a, b) (where $(a, b) = (1, 1)$ for crossings and caps, and $(a, b) = (0, 2)$ for cups). For example (here $i = 3$ and $j = 2$):

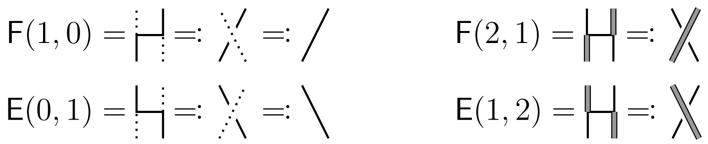


(3.1)

Before proceeding to the third step (taking the horizontal product), we must ensure that the complexes are horizontally composable. In other words, the λ's of consecutive slides must match. First, we restrict them to a specific subset of $\Lambda_{n,d}$:

Definition 3.7. *A* canonical sequence *is a $\lambda \in \Lambda_{n,d}$ whose coordinates are in increasing order. Given a $\lambda \in \Lambda_{n,d}$, there is a unique canonical sequence associated to λ obtained by reordering the coordinates.*

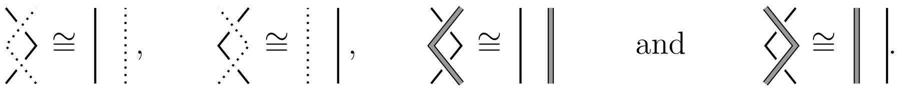
The domain and codomain of the 1-morphisms can be set to their corresponding canonical sequence by composing with the following 1-morphisms:



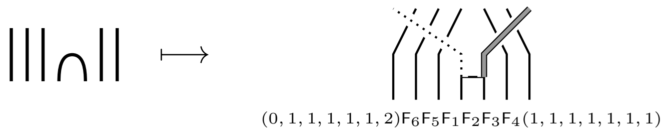
(3.2)

where the new notations are intended to make clearer the idea that "we send all dotted lines to the left and all double lines to the right". Thanks to the following remark, all ways of applying isomorphisms (3.2) to get from one λ to another result in isomorphic 1-morphisms, so that the process of turning a λ into its corresponding canonical sequence is well-defined.

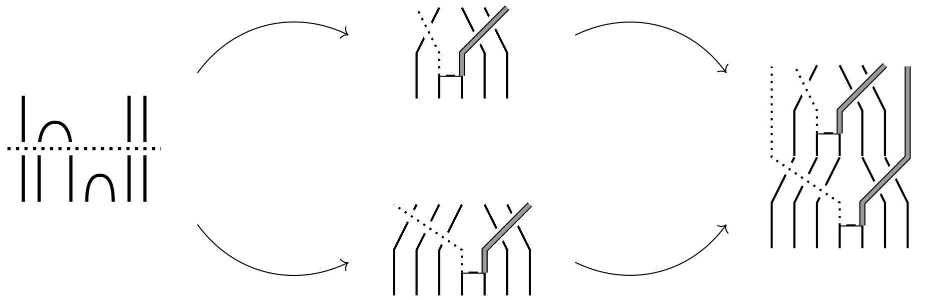
Remark 3.8. The relations (2.1) implies the following relations, matching the diagrammatic intuition:



As an example, the codomain of the 1-morphism (3.1) is $(1,1,1,0,2,1,1)$. Its corresponding canonical sequence is $(0,1,1,1,1,1,2)$ and we can use the 1-morphisms $\mathsf{F}_3(1,0)$, $\mathsf{F}_2(1,0)$, $\mathsf{F}_1(1,0)$, $\mathsf{F}_5(2,1)$ and $\mathsf{F}_6(2,1)$ to turn $(1,1,1,0,2,1,1)$ into $(0,1,1,1,1,1,2)$:



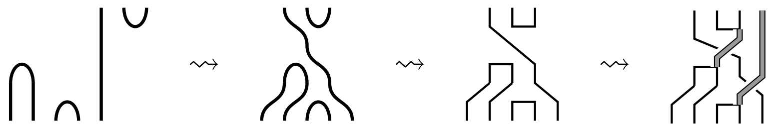
From now on, we assume all 1-morphisms have canonical sequences as domain and codomain. Note that the number of 1's in the λ's corresponds to the number of strands. This means that the only difference between the λ's of consecutive slides is the number of 0's on the left and the number of 2's on the right. Thus, it suffices to add enough 0's and 2's to the λ's to ensure that all consecutive λ's match up. In other words, if we allow n (the length of the λ's) to be sufficiently large, the complexes can be made horizontally composable. Here is an example of the full procedure with two slices:



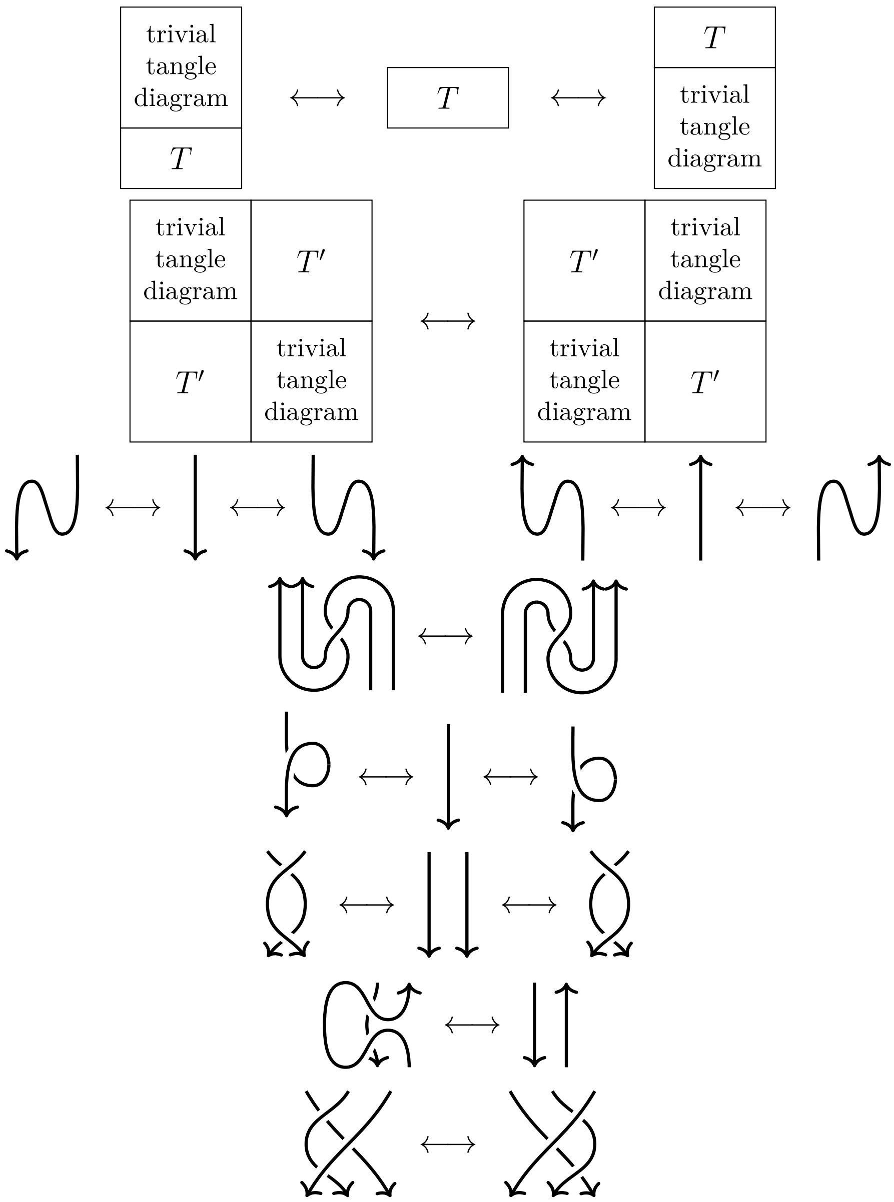
Finally, Kom(D, n, d) is normalised by a global q-shift of $-w(D)$, where $w(D)$ denotes the writhe of the diagram D (that is, all 1-morphisms are shifted by $q^{-w(D)}$). This normalisation is similar to how one must normalise the Kauffman bracket to define the Jones polynomial.

Remark 3.9. It is clear that the minimal n for which Kom(D, n, d) is defined is $n = N$, where N is the maximum number of strands in a given slice of D. In that case, the only possibility for d is $d = d_N = N$. By extending the canonical sequences with 0's and 2's, Kom(D, n, d) can be defined for any $n \geq N$ and $d = N + 2k$ for $0 \leq k \leq n - N$ ($n - N - k$ and k being respectively the number of 0's and the number of 2's added). However, doing so doesn't bring anything new to the complex. In particular, if Kom$(D, m, d) \simeq$ Kom(D', m, d) for two diagrams D and D', then Kom$(D, \tilde{m}, \tilde{d}) \simeq$ Kom$(D', \tilde{m}, \tilde{d})$ for all $\tilde{m}$ and $\tilde{d}$ such that the complexes are well-defined.

Remark 3.10. There is an alternative way of proceeding (steps one to three). First, align all the slices such that the tangle is in a "centered position". That is, if one only looks at the *gluing slices* (the set of points connecting two slices), then all the center points are aligned (or the pair of center points, recalling that the parity of the number of strands in the slices is invariant in a given diagram). Then, *rigidify* the tangle so that the strands are piecewise segments. Finally, add double lines by starting on the right corners of caps and cups, only travelling through diagonals and reaching another right corners of a cap/cup or an endpoint of the diagram (and if you wish, similarly for dotted lines).



Remark 3.11. The local definition of Kom(D, n, d) for the crossings mimics the Kauffman bracket. Indeed, the non-trivial 1-morphism is really the composition of the contribution of a cap and a cup. In some sense, our definition is a supercategorification of the Kauffman bracket.



(3.3)

(3.4)

(3.5)

(3.6)

(3.7)

(3.8)

(3.9)

(3.10)

Figure 3.3: The Reidemeister-Turaev moves for oriented tangles.

3.3 Proof of invariance

This section is devoted to the proof of the following theorem:

Theorem 3.12. *Let D and D' be two diagrams representing the same oriented tangle T, and let n and d be such that* $\mathrm{Kom}(D,n,d)$ *and* $\mathrm{Kom}(D',n,d)$ *are well-defined (such n and d always exist). Then*

$$\mathrm{Kom}(D,n,d) \simeq \mathrm{Kom}(D',n,d)$$

in $\mathrm{TCh}^2_\bullet(\mathcal{S}(n,d))$.

Call $\mathrm{TCh}^2_{/h}(\mathcal{S}(n,d))$ the quotient of $\mathrm{TCh}^2_\bullet(\mathcal{S}(n,d))$ under the homotopy equivalence relation (that is, the objects are the same but homotopic maps are identified). Thanks to the previous theorem, we can define an invariant of oriented tangles:

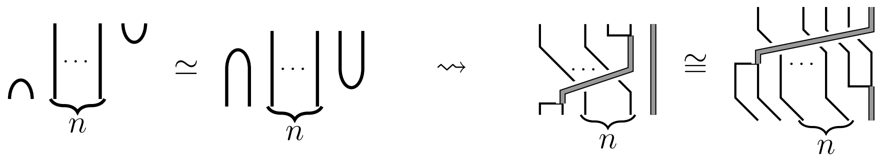
Definition 3.13. *Let T be an oriented tangle, D a diagram of T and* $\mathrm{Kom}_{/h}(D,n,d)$ *the image of* $\mathrm{Kom}(D,n,d)$ *in* $\mathrm{TCh}^2_{/h}(\mathcal{S}(n,d))$*. By Theorem 3.12, the homotopy class of* $\mathrm{Kom}(D,n,d)$ *does not depend on the choice of D. Therefore we define*

$$\mathrm{Kom}_{/h}(T,n,d) \coloneqq \mathrm{Kom}_{/h}(D,n,d).$$

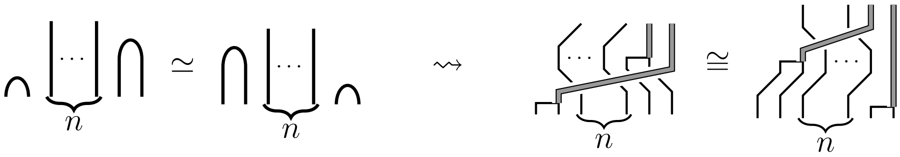
To prove Theorem 3.12, we verify all Reidemeister-Turaev moves of oriented tangle diagrams (Fig. 3.3). Indeed, it is shown for example in [14, p. 47] that all the diagrams representing a given oriented tangle are related by a sequence of these moves. Thanks to the invariance of homotopy classes under the tensor product (Theorem 3.3), we can proceed locally, that is without considering the moves as being applied to a wider diagram. Finally, except for the Reidemeister I moves (3.7), all moves in Fig. 3.3 leave the writhe invariant. Thus normalisation can be safely forgotten in all cases except Reidemeister I.

3.3.1 Invariance under planar isotopies of diagrams

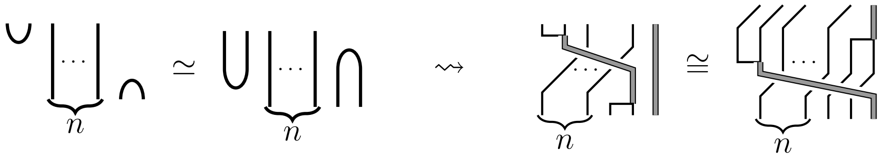
Relation (3.3) is trivial. We focus on (3.4). It is actually already verified at the uncategorified level of $\mathcal{S}(n,d)$ (this is an abuse of language: to be precise, the relations are satisfied on the Grothendieck group of $\mathcal{S}(n,d)$. See Subsection 2.2.3). In particular, all calculations can be done using ladder diagrams (see Subsection 2.2.3): we will then prove that the complexes are not merely, homotopic but are isomorphic. To show (3.4), it suffices to show the invariance under the switch of any two basic elements of tangle:



(3.11)



(3.12)



(3.13)

(3.14)

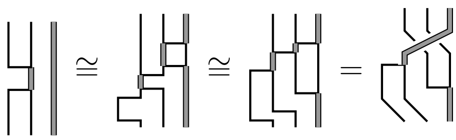
Note that we don't need to consider the cases involving the two crossings. Thanks to Remark 3.11, the non-trivial 1-morphism in the complexes corresponding to the two crossings is the composition of the 1-morphisms assigned to a cap and cup, and therefore the cases involving the two crossings are deduced by horizontally composing the isomorphisms above. We focus on (3.11) and (3.12), the two relations (3.13) and (3.14) being analogous.

Switching a cap and a cup (3.11)

We have underlined position 1 in the calculation below:

$$\begin{aligned}
\mathsf{E}_{n+1}\dots\mathsf{E}_1\mathsf{F}_{n+1}\dots\mathsf{F}_1(\underline{1},\dots,1,2) &\cong \mathsf{E}_{n+1}\mathsf{F}_{n+1}\dots\mathsf{E}_1\mathsf{F}_1(\underline{1},\dots,1,2)\\
&\cong \mathsf{E}_{n+2}\mathsf{F}_{n+2}\mathsf{E}_{n+1}\mathsf{F}_{n+1}\dots\mathsf{E}_1\mathsf{F}_1\mathsf{E}_0\mathsf{F}_0(0,\underline{1},\dots,1,2)\\
&\cong \mathsf{F}_{n+2}\mathsf{F}_{n+1}\dots\mathsf{F}_1\mathsf{F}_0\mathsf{E}_{n+2}\mathsf{E}_{n+1}\dots\mathsf{E}_1\mathsf{E}_0(0,\underline{1},\dots,1,2)
\end{aligned}$$

Here is a graphical proof of the calculation above in the case $n = 0$:



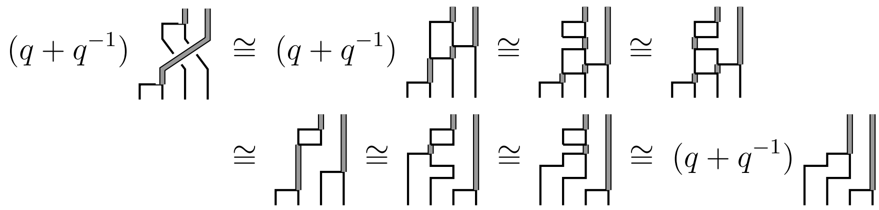
Switching a cap and a cap (3.12)

We must show that

$$\mathsf{F}_1\dots\mathsf{F}_{n+1}\mathsf{F}_{n+2}\mathsf{F}_{n+1}\dots\mathsf{F}_1\mathsf{F}_0(1,\underline{1},\dots,1) \cong \mathsf{F}_{n+1}\dots\mathsf{F}_1\mathsf{F}_0\mathsf{F}_1\dots\mathsf{F}_{n+1}\mathsf{F}_{n+2}(1,\underline{1},\dots,1)$$

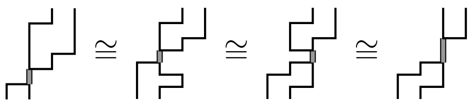
We first check the case $n = 0$ ($\lambda = (1, \underline{1}, 1, 1)$):

$$\begin{aligned}(q+q^{-1})\mathsf{F}_1\mathsf{F}_2\mathsf{F}_1\mathsf{F}_0(1,\underline{1},1,1) &\cong \mathsf{F}_1\mathsf{E}_1\mathsf{F}_1\mathsf{F}_2\mathsf{F}_1\mathsf{F}_0(\lambda) \cong \mathsf{F}_1^2\mathsf{E}_1\mathsf{F}_2\mathsf{F}_1\mathsf{F}_0(\lambda)\\ &\cong \mathsf{F}_1^2\mathsf{F}_2\mathsf{E}_1\mathsf{F}_1\mathsf{F}_0(\lambda) \cong \mathsf{F}_1^2\mathsf{F}_2\mathsf{F}_0(\lambda)\\ &\cong \mathsf{F}_1^2\mathsf{F}_0\mathsf{E}_1\mathsf{F}_1\mathsf{F}_2(\lambda) \cong \mathsf{F}_1^2\mathsf{E}_1\mathsf{F}_0\mathsf{F}_1\mathsf{F}_2(\lambda)\\ &\cong (q+q^{-1})\mathsf{F}_1\mathsf{F}_0\mathsf{F}_1\mathsf{F}_2(1,\underline{1},1,1)\end{aligned}$$



We conclude that $\mathsf{F}_1\mathsf{F}_2\mathsf{F}_1\mathsf{F}_0(1,\underline{1},1,1) \cong \mathsf{F}_1\mathsf{F}_0\mathsf{F}_1\mathsf{F}_2(1,\underline{1},1,1)$. To get the general case, we use the following simple relation (the *lockswitch*; here $\lambda = (1, \underline{1}, 0, 0)$):

$$\begin{aligned}\mathsf{F}_1\mathsf{F}_2\mathsf{F}_1\mathsf{F}_0(1,\underline{1},0,0) &\cong \mathsf{F}_1\mathsf{F}_2\mathsf{F}_1\mathsf{F}_0\mathsf{E}_1\mathsf{F}_1(\lambda) \cong \mathsf{F}_1\mathsf{F}_2\mathsf{F}_1\mathsf{E}_1\mathsf{F}_0\mathsf{F}_1(\lambda) \cong \mathsf{F}_1\mathsf{F}_2\mathsf{E}_1\mathsf{F}_1\mathsf{F}_0\mathsf{F}_1(\lambda)\\ &\cong \mathsf{F}_1\mathsf{E}_1\mathsf{F}_2\mathsf{F}_1\mathsf{F}_0\mathsf{F}_1(\lambda) \cong \mathsf{F}_2\mathsf{F}_1\mathsf{F}_0\mathsf{F}_1(1,\underline{1},0,0).\end{aligned}$$



Note that the proof would have work just as well if we had $\lambda_0 = 2$ or/and $\lambda_3 = 1$. Therefore we have the general lockswitch given by

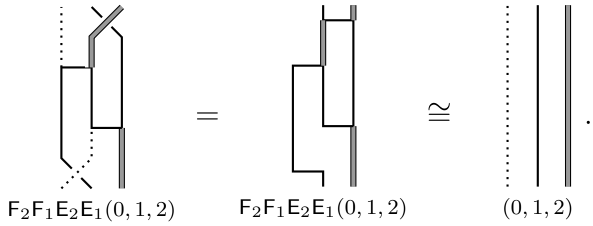
$$\mathsf{F}_1\mathsf{F}_2\mathsf{F}_1\mathsf{F}_0(1/2,\underline{1},0,0/1) \cong \mathsf{F}_2\mathsf{F}_1\mathsf{F}_0\mathsf{F}_1(1/2,\underline{1},0,0/1)$$

where $\cdot/\cdot$ denotes different possible choices. We can now prove the general case:

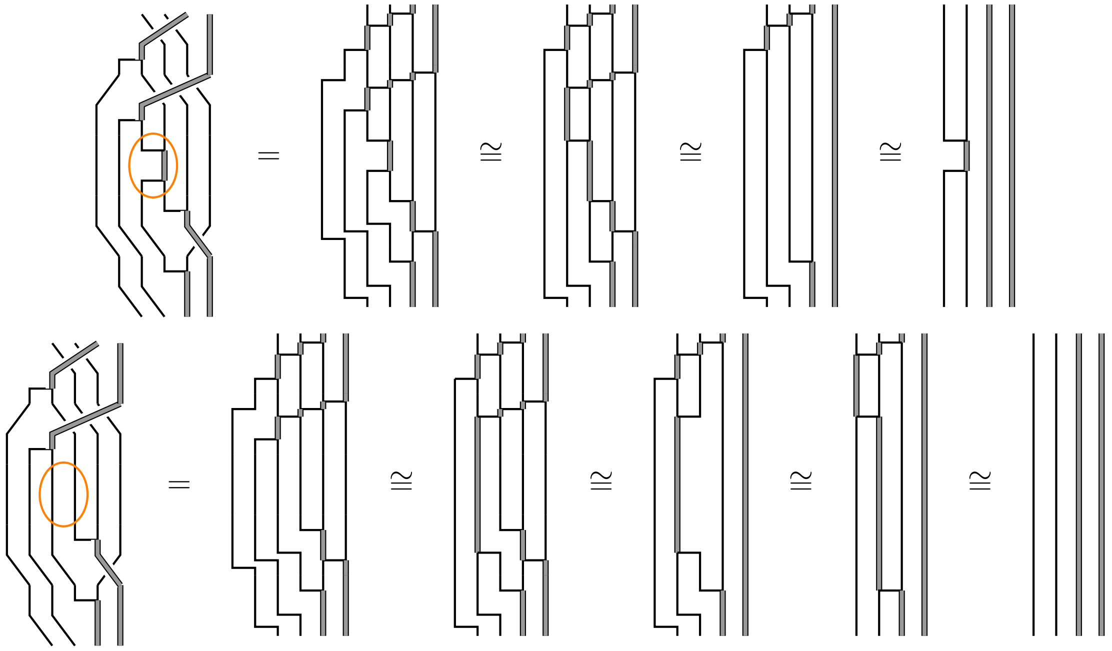
$$\begin{aligned}&\mathsf{F}_1\ldots\mathsf{F}_n\mathsf{F}_{n+1}\mathsf{F}_{n+2}\mathsf{F}_{n+1}\mathsf{F}_n\mathsf{F}_{n-1}\ldots\mathsf{F}_1\mathsf{F}_0(1,\underline{1},\ldots,1)\\ &\quad\cong \mathsf{F}_1\ldots\mathsf{F}_n\mathsf{F}_{n+1}\mathsf{F}_n\mathsf{F}_{n+1}\mathsf{F}_{n+2}\mathsf{F}_{n-1}\ldots\mathsf{F}_1\mathsf{F}_0(\lambda) && n=0\\ &\quad\cong \mathsf{F}_1\ldots\mathsf{F}_{n-1}\mathsf{F}_n\mathsf{F}_{n+1}\mathsf{F}_n\mathsf{F}_{n-1}\mathsf{F}_{n-2}\ldots\mathsf{F}_1\mathsf{F}_0\mathsf{F}_{n+1}\mathsf{F}_{n+2}(\lambda)\\ &\quad\cong \mathsf{F}_1\ldots\mathsf{F}_{n-1}\mathsf{F}_{n+1}\mathsf{F}_n\mathsf{F}_{n-1}\mathsf{F}_n\mathsf{F}_{n-2}\ldots\mathsf{F}_1\mathsf{F}_0\mathsf{F}_{n+1}\mathsf{F}_{n+2}(\lambda) && \text{lockswitch}\\ &\quad\cong \mathsf{F}_{n+1}\mathsf{F}_1\ldots\mathsf{F}_{n-2}\mathsf{F}_{n-1}\mathsf{F}_n\mathsf{F}_{n-1}\mathsf{F}_{n-2}\mathsf{F}_{n-3}\ldots\mathsf{F}_1\mathsf{F}_0\mathsf{F}_n\mathsf{F}_{n+1}\mathsf{F}_{n+2}(\lambda)\\ &\qquad\qquad\qquad\ldots && \text{lockswitches}\\ &\quad\cong \mathsf{F}_{n+1}\ldots\mathsf{F}_3\mathsf{F}_1\mathsf{F}_2\mathsf{F}_1\mathsf{F}_0\mathsf{F}_2\ldots\mathsf{F}_{n+2}(\lambda)\\ &\quad\cong \mathsf{F}_{n+1}\ldots\mathsf{F}_1\mathsf{F}_0\mathsf{F}_1\ldots\mathsf{F}_{n+1}\mathsf{F}_{n+2}(1,\underline{1},\ldots,1) && \text{lockswitch}\end{aligned}$$

3.3.2 Invariance under zigzags and the crossing twist

Invariance under the relations (3.5) and (3.6) is already checked at the uncategorified level, so that we can use ladder diagrams to check invariance. The first relation of (3.5) gives

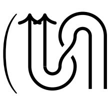


The other relation is similar. The two 1-morphisms of the complex associated to the left-and side of (3.6) can be simplified as follows:



Here the orange circle indicates the difference between the two 1-morphisms, that is, the two resolvings of the crossing. A similar calculation can be done with the right-hand side, giving the same 1-morphisms. We conclude that the complexes associated to the two sides of the relation are isomorphic.

Remark 3.14. In the definition of $\mathrm{Kom}(D,n,d)$ in Section 3.2, we didn't give an explicit definition of the complex associated to an upward positive crossing. That is because it can be derived from the definitions given for the downward positive crossing and for cups and caps. One possible way is through the *counter-clockwise* twist:

$\mathrm{Kom}\Big($  $, n, d\Big) := \mathrm{Kom}\Big($  $, n, d\Big).$

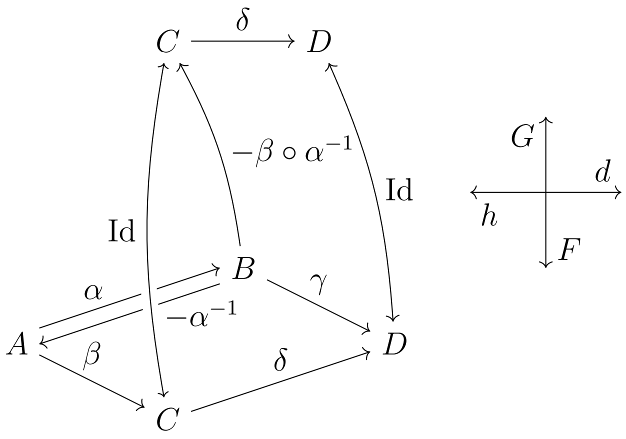


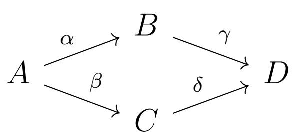
Figure 3.4: Proof of Lemma 3.15.

The other possibility is through the *clockwise* twist. Thanks to the previously proven isomorphism, both choices result in the same complex (up to isomorphism). Actually, the explicit isomorphisms given above show that the *same* complex is associated to both the downward positive crossing and the upward positive crossing (again, up to isomorphism). The same argument holds for negative crossings.

3.3.3 A useful tool: gaussian elimination

Until now, all relations were already verified at the uncategorified level, such that we proved the complexes to be isomorphic, and not merely homotopic. This won't be the case for the three Reidemeister moves, hence we will need to deal with homotopies. To make the task easier, we introduce here a useful tool in homological algebra: *gaussian elimination* (note that since it doesn't involve the tensor product, it applies to our setting).

Lemma 3.15. *Consider the complex*

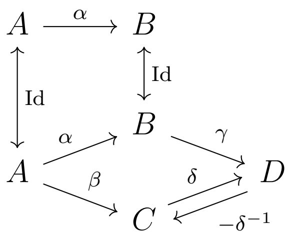


If α is an isomorphism, then this complex is homotopic to the complex $C \xrightarrow{\delta} D$ *. More generally, if one of the edge is an isomorphism, the complex is homotopic to the opposite edge. (The reader can find a more detailed version of this result in [1].)*

Proof. Assume α is an isomorphism. Consider the diagram in Remark 3.14 and call d the differentials of both complexes, F and G the morphisms of complexes respectively going from top to bottom and from bottom to top and h the homotopy. No arrow means the zero arrow.

It is straightforward to check that F is a morphism of complexes and that $GF = \mathrm{Id} + dh + hd = \mathrm{Id}$. Since the square $ABCD$ is a complex, we have $\gamma \circ \alpha = -\delta \circ \beta$. But α is an isomorphism, therefore $\gamma = -\delta \circ \beta \circ \alpha^{-1}$. With this in mind, one easily checks that G is a morphism of complexes and that $FG = \mathrm{Id} + dh + hd \Leftrightarrow \mathrm{Id}_C - \beta \circ \alpha^{-1} = \mathrm{Id}_{B\oplus C} - \mathrm{Id}_B - \beta \circ \alpha^{-1}$.

The case where β is an isomorphic is similar. The cases where γ and δ are even simpler, as the following diagram shows when we assume that δ is an isomorphism:



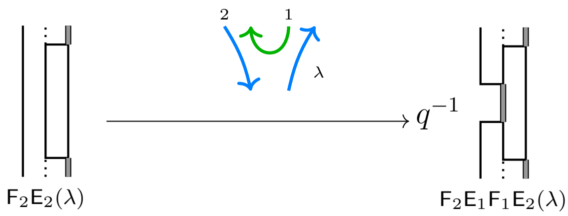
□

Remark 3.16. G is actually a bit more than a homotopy: it is a strong deformation retract (in the sense of the following definition). This fact will be important in the proof of the invariance under Reidemeister III (Subsection 3.3.6).

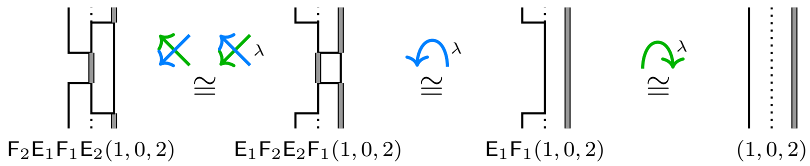
Definition 3.17 (Strong deformation retract [2])**.** *A morphism of complex* $G\colon (A,\alpha) \to (B,\beta)$ *is said to be a* strong deformation retract *if there is a morphism* $F\colon (B,\beta) \to (A,\alpha)$ *and homotopy maps* h *from* A *to itself so that* $GF = \mathrm{Id}$, $FG = \mathrm{Id} + \beta h + h\alpha$ *and* $hF = 0$. *In this case we say that* F *is the* inclusion in a strong deformation retract*. Note that a strong deformation retract is in particular a homotopy equivalence.*

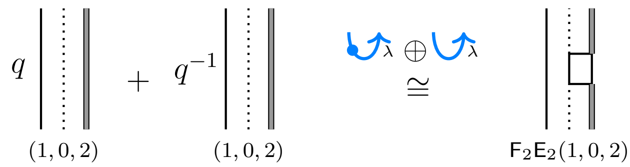
3.3.4 Invariance under Reidemeister I

The complex associated to the left-hand side of (3.7) is, up to some q-shifting (here $\lambda = (1,0,2)$),

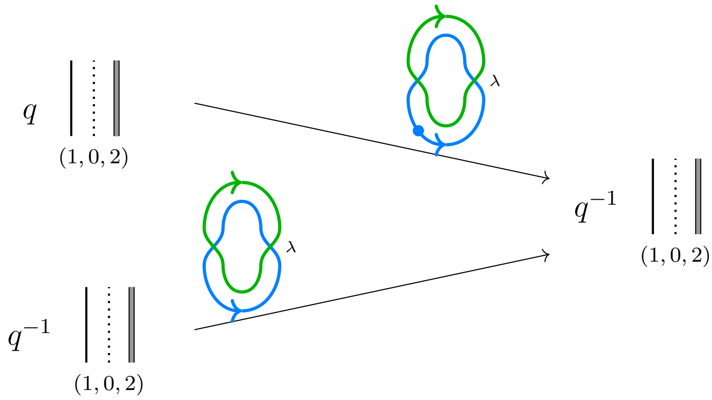


We show that it is homotopy equivalent to the single-object complex $[\lambda]$. Note that we didn't pre-compose with $\mathsf{E}_1(0,1)$ and post-compose with $\mathsf{F}_1(1,0)$ to get 1-morphisms with canonical sequences as domain and codomain. Indeed, thanks to Remark 3.8 if this complex is equivalent to $[\lambda]$ then so is the same complex pre- and post-composed respectively with $\mathsf{E}_1(0,1)$ and $\mathsf{F}_1(1,0)$. We start by simplifying the 1-morphisms:

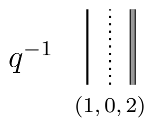
- 

- 

Therefore the above complex is isomorphic to the complex



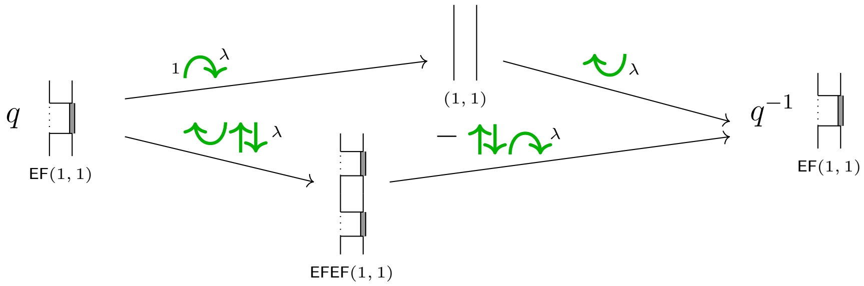
Thanks to the first case of Lemma 2.11, the bottom 2-morphism is an isomorphism. Applying Lemma 3.15, the complex is homotopic to the following single object complex:



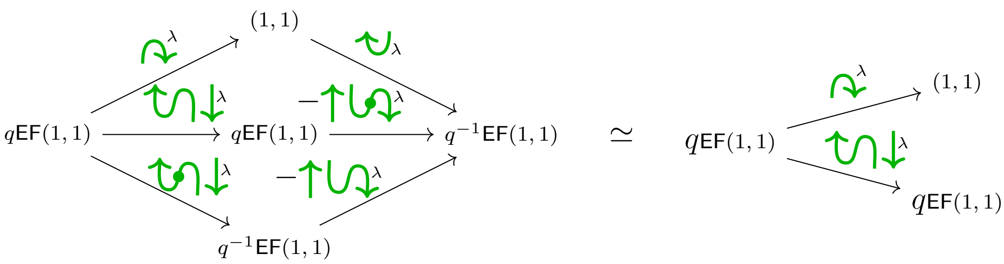
Finally, recall the normalisation by $q^{-w(D)}$: since $w(D) = -1$, this ends the proof of invariance of homotopy classes under the first Reidemeister I move of (3.7). The invariance under the other Reidemeister I move is similar, only using the second case of Lemma 2.11.

3.3.5 Invariance under Reidemeister II

Consider the first of the two Reidemeister II moves in (3.8). The left-hand side is assigned the complex ($\lambda = (1, 1)$):

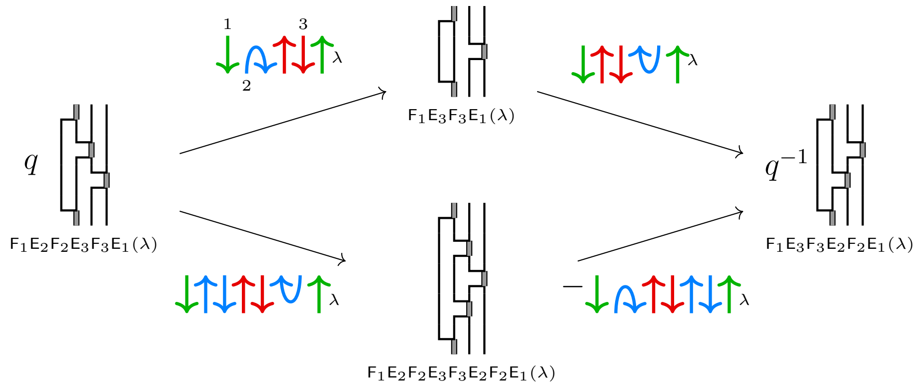


By applying the isomorphism  on the bottom 1-morphism we get the complex below. Thanks to the rightward adjunction relations (Lemma 2.10), the bottom right 2-morphism is an isomorphism and we can apply Lemma 3.15. Applying this lemma once again on the bottom 2-morphism of this new complex (thanks to the leftward adjunction relations (2.10)), we conclude that it is homotopic to the single-object complex $[\lambda]$.

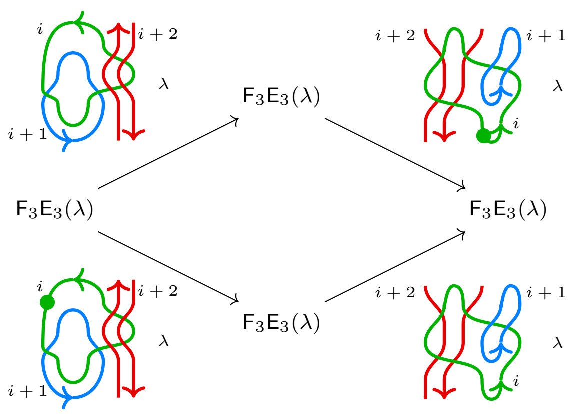


The same argument holds for the second relation, as the right-hand is assigned the same complex.

We prove the invariance under the move (3.9) in a similar fashion. The left-hand side is assigned the following complex, where $\lambda = (0, 2, 1, 1)$ (note that analogously to Subsection 3.3.4, we discarded the 1-morphisms used to make λ a canonical sequence):



Note that the middle bottom 1-morphism is isomorphic to the identity. Moreover, we can apply the isomorphism  on the middle top 1-morphism to split the top two differentials into a square. Therefore, we can proceed similarly to the previous Reidemeister II move: find two opposite edges in this square that correspond to isomorphisms and conclude with Lemma 3.15. The square is given below, where in addition 1-morphisms have been simplified through 2-isomorphisms:



The bottom left and top right edges are isomorphisms, thanks to Corollary 2.13. This concludes the proof.

3.3.6 Invariance under Reidemeister III

We start by introducing a typical construction in homological algebra:

Definition 3.18 (Cone [2])**.** *Let $\psi\colon (A,\alpha) \to (B,\beta)$ be a morphism of complexes. The* cone $\Gamma(\psi)$ *of ψ is the complex with chain spaces $\Gamma(\psi) = A^{r+1} \otimes A^r$ and with differentials $d^r = \begin{pmatrix} -\alpha^{r+1} & 0 \\ \psi^{r+1} & \beta^r \end{pmatrix}$. One should have in mind the following picture:*

$$\begin{array}{ccccc} A^r & \xrightarrow{-\alpha^r} & A^{r+1} & \xrightarrow{-\alpha^{r+1}} & A^{r+2} \\ \downarrow \psi^r & \oplus & \downarrow \psi^{r+1} & \oplus & \downarrow \psi^{r+2} \\ B^r & \xrightarrow[\beta^r]{} & B^{r+1} & \xrightarrow[\beta^{r+1}]{} & B^{r+2} \end{array}$$

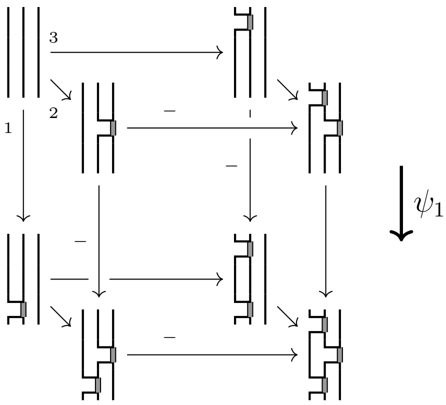
Any hypercube T of dimension n can be seen as cone. Indeed, choose a direction k in the hypercube. Ignoring the k-edges, T breaks in two hypercubes T_1 and T_2 of dimension $n-1$. Then switch the signs of all differentials in T_1. T_1 is still a complex, but the k-faces of T now commute instead of anti-commuting: the k-edges form a morphism $\psi\colon T_1 \to T_2$. It is then easy to see that $\Gamma(\psi) = T$. The importance of cones comes from the following lemma:

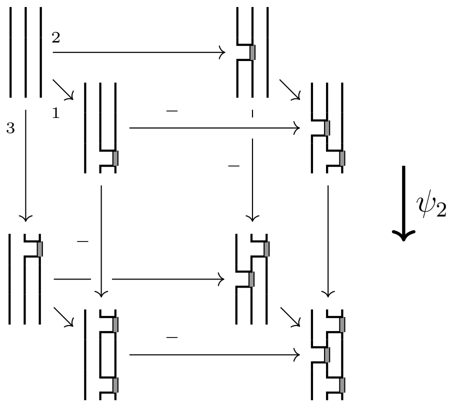
Lemma 3.19 (Invariance of cone under strong deformation retract [2])**.** *The cone construction is invariant up to homotopy under compositions with the inclusions in strong deformation retracts. That is, consider the diagram of complexes and morphisms*

$$\begin{array}{ccc} A_0 & \underset{F_0}{\overset{G_0}{\rightleftarrows}} & B_0 \\ \downarrow \psi & & \\ A_1 & \underset{G_1}{\overset{F_1}{\rightleftarrows}} & B_1 \end{array}$$

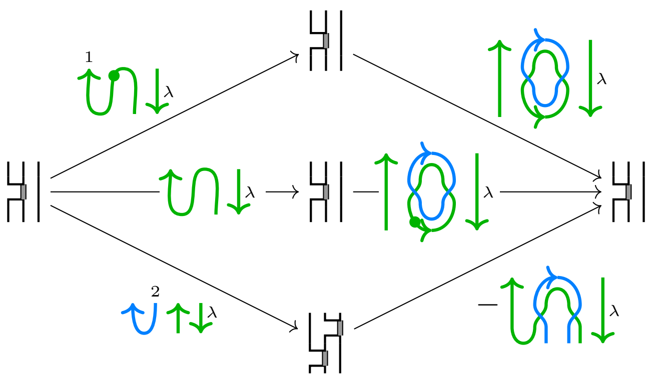
If in that diagram G_0 is a strong deformation retract with inclusion F_0, then the cones $\Gamma(\psi)$ and $\Gamma(\psi F_0)$ are homotopy equivalent, and if G_1 is a strong deformation retract with inclusion F_1, then the cones $\Gamma(\psi)$ and $\Gamma(F_1\psi)$ are homotopy equivalent.

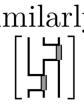
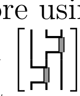
The reader can refer to [2, Lemma 4.5] for a proof of Lemma 3.19 (where the idea of using cones to prove Reidemeister III has been taken). We are now ready to prove invariance under Reidemeister III. The two sides of (3.10) are respectively assigned the cubes below (where the differentials have been omitted). The choice of signs and the ordering of the differentials are such that similarities are apparent. The ordering is given by the numbers on the edges, assuming the crossings are numbered reading from bottom to top.

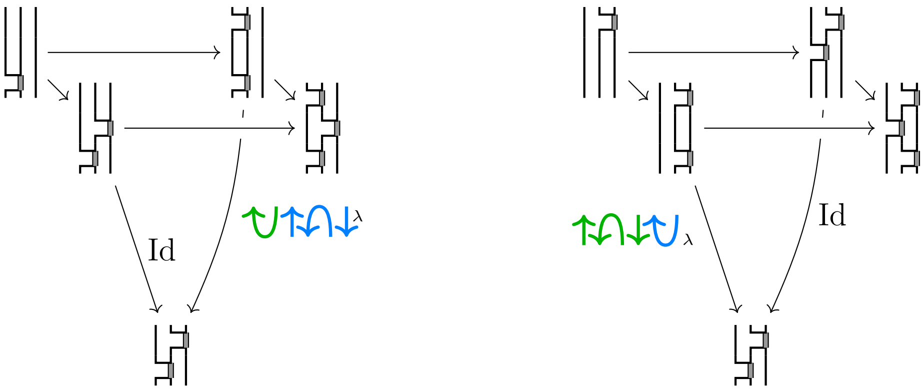




Note that the top faces are the same. Consider the cubes to be respectively cones of morphisms $\psi_1\colon A_1 \to B_1$ and $\psi_2\colon A_2 \to B_2$ as defined in the picture, where $A_1 = A_2$. Focusing on B_1, the middle top 1-morphism can be split using cups and caps (similarly to the proof of Reidemeister II) and the rightmost 1-morphism can be simplified through 2-isomorphisms, giving the following complex:



Two opposite edges are isomorphisms: the middle left edge thanks to the adjunction relations (2.10), and the top right edge thanks to Lemma 2.11. Applying twice Lemma 3.15 similarly to the proof of Reidemeister II, there exists a morphism of complex $F_1\colon B_1 \to$  inducing a homotopy equivalence. Thanks to Remark 3.16, F is the inclusion of a strong deformation retract, and therefore using Lemma 3.19 we find that $\Gamma(\psi_1) \simeq \Gamma(F_1\psi_1)$. A similar argument gives $F_2\colon B_2 \to$  such that $\Gamma(\psi_2) \simeq \Gamma(F_2\psi_2)$. To finish the proof we need to check that $F_1\psi_1 = F_2\psi_2$, and therefore we need an exact expression for F_1 and F_2. This turns out not to be that difficult, delving into the proof of Lemma 3.15. We find:



Using the adjunction relations (2.10), it is not difficult to check whether $F_1\psi_1 = F_2\psi_2$. It turns out that some signs appear, and one finds $F_1\psi_1 = -F_2\psi_2$. But this not problematic: $-\mathrm{Id}_\lambda$ is an isomorphism of complex, and in particular the inclusion of a deformation retract. Therefore,

$$\Gamma(\psi_2) \simeq \Gamma(F_2\psi_2) \simeq \Gamma(-F_2\psi_2) = \Gamma(F_1\psi_1) \simeq \Gamma(F_1).$$

This concludes the proof of invariance under Reidemeister III, and ends the proof of Theorem 3.12. ☐

Conclusion

We have fulfil our original goal, which was to construct an invariant of tangles through a supercategorification of the quantum algebra $S_{n,d}$, and thus give a good candidate to explain odd Khovanov homology. As usual though, we have raised more questions than we answered. We detail a few of them. Some will be answered in future work [16].

- *Is $\mathcal{S}(n,d)$ well-defined?*

  In Chapter 2, we only gave a presentation of $\mathcal{S}(n,d)$ by generators and relations: at first glance, it is not clear that $\mathcal{S}(n,d)$ is non-zero. This will be shown by constructing a faithful 2-representation of $\mathcal{S}(n,d)$. This 2-representation provides an isomorphism between our invariant and the invariant constructed in [18], which is distinct from even Khovanov homology. In particular, it implies that our invariant is distinct from even Khovanov homology.

- *Does our invariant correspond to odd Khovanov homology?*

  This was our original motivation. As hinted in [18], we conjecture it does.

- *What is the value of our invariant in specific examples?*

  In [18], some examples have been computed. However, in this uncategorified framework calculations are significantly harder. Our 2-supercategorical construction is more natural, and give an easier method to compute examples. (Notice that the differentials in Fig. 3.2 are not isomorphisms, and thus do not have a clear correspondence at the uncategorified level of the quantum algebra.) To do so, explicit relations between the 2-morphisms of $\mathcal{S}(n,d)$ (such as the ones in Appendix A) and standard computation shortcuts in even Khovanov homology [1] (such as gaussian elimination) will be needed.

- *Is our invariant functorial?*

  As already pointed out, the functoriality of odd Khovanov homology is still an open question. Our construction could give the necessary tools to prove it.

- *Can other quantum algebras be supercategorified?*

  The Temperley-Lieb quantum algebra TL_n controls the representation theory of $\mathrm{U}_q(\mathfrak{sl}_2)$. Indeed, if U_i are the generators of TL_n, setting $\tau(U_i) = (1,1)\mathsf{E}_i\mathsf{F}_i(1,1)$ gives an embedding of algebras $\mathrm{TL}_n \subset S_{n,d}$ (for all $d \geq n$). Can we then similarly categorify TL_n?

- *What are the properties of the tensor product?*

  As pointed out in Remark B.10, it is not clear whether the tensor product is functorial. Also, we only showed equivalence between different choices of signs in the case of hypercubic complexes. What about tensor products with longer factors? What about the choice of signs made to define induced morphisms and induced homotopies? Moreover, when defining the tensor product we had to restrict to homogeneous factors: what about the general case? If such a definition is possible, a different approach should probably be needed, as our hypothesis (homogeneity of the factors but no conditions on the morphisms and the homotopies) seem to be the most general case where choosing parity is applicable.

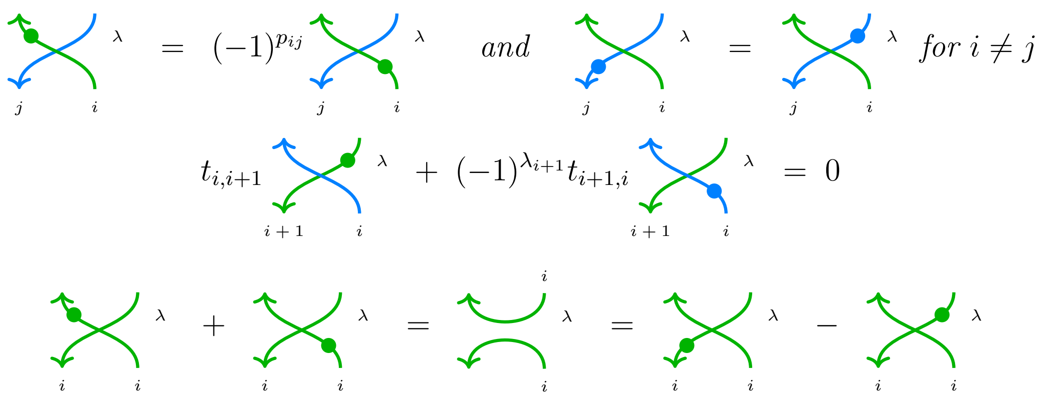
Appendix A

Additional relations between 2-morphisms in $\mathcal{S}(n,d)$

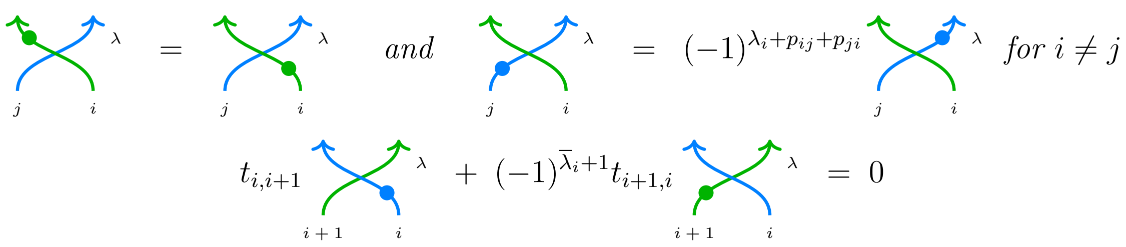
This appendix gives additional relations between the 2-morphisms of $\mathcal{S}(n,d)$. In particular, it gives exact calculations for most of the results given in Section 2.3. Such relations are not needed in this thesis, but will be needed in later work (for example, to compute the invariant for particular tangles). We give them for completeness.

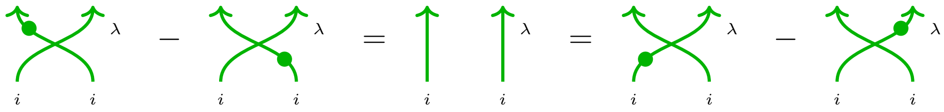
Dot slides

Lemma A.1. *We have*

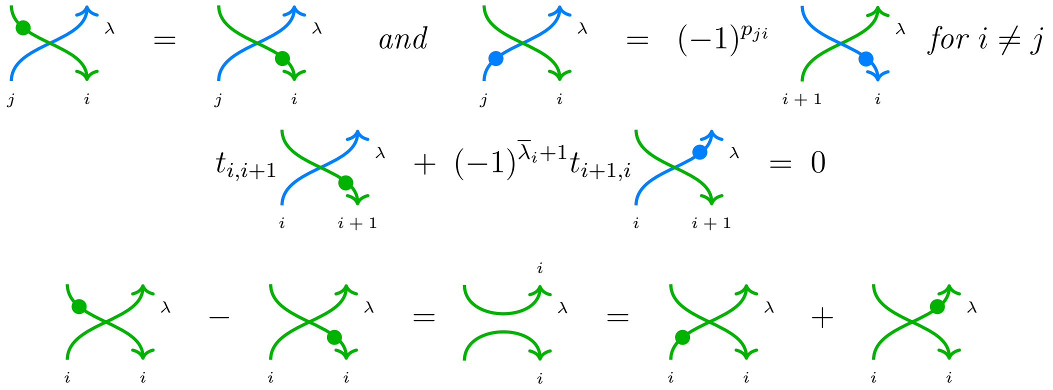


Lemma A.2. *We have*



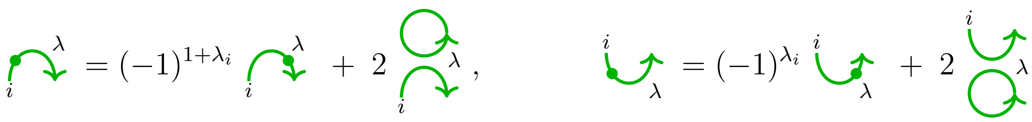


Lemma A.3. *We have*



Proof of three previous lemmas. Lemma A.1 and Lemma A.2 are consequences of the definitions and Lemma 2.4. Lemma A.3 is a direct consequence of Lemma A.1, where all relations are obtained by composing both upward and downward by a right crossing. □

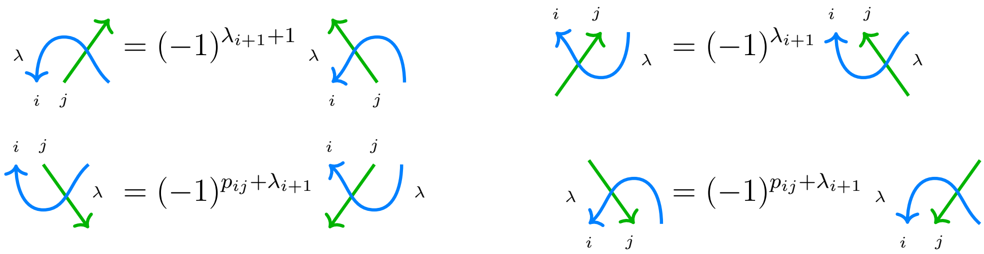
Lemma A.4. *We have*



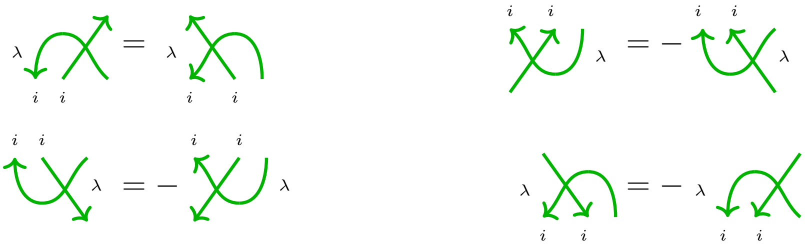
Proof. It follows from Lemma 2.4, Corollary 2.2 and Lemma A.3 for the case $\overline{\lambda}_i = 0$. □

Pitchforks and kinks

Lemma A.5 (Left pitchforks)**.** *We have*



In particular,



Proof. Use the definition of the leftward (2.2) and upward (2.15) crossings and the adjunction relations (2.10). □

Lemma A.6 (Kinks)**.** *For* $\overline{\lambda}_i = 0$ *we have*

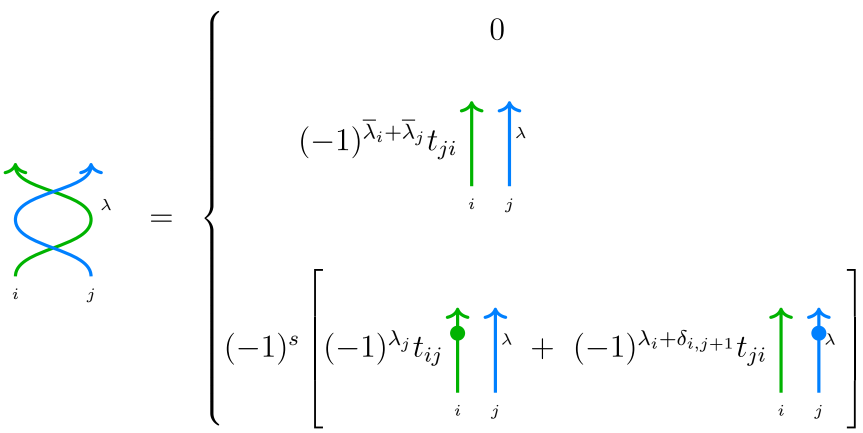
(A.1)

(A.2)

Proof. Use Lemma A.5 and the adjunction relations (2.10). □

Second adjunctions

Lemma A.7 (KLR Reidemeister 2 relation for E)**.** *We have*



if $i = j$,

if $|i-j| > 1$,

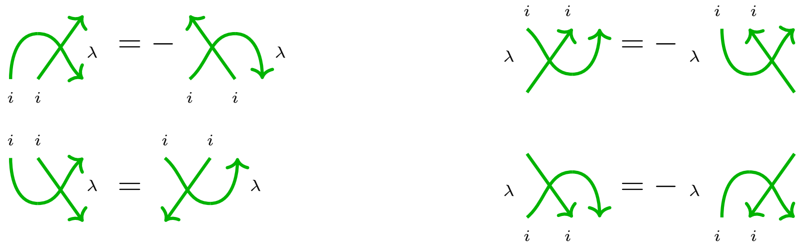
if $|i-j| = 1$,

(A.3)

where $s = \delta_{i,j+1}(\lambda_j + 1) + \delta_{j,i+1}(\lambda_i + 1) + \overline{\lambda}_i + \overline{\lambda}_j$.

Proof. It derives directly from the definition of the upward crossing (2.15) and the adjunction relations (2.10). □

Lemma A.8 (Right pitchforks)**.** *We have*



(A.4)

(A.5)

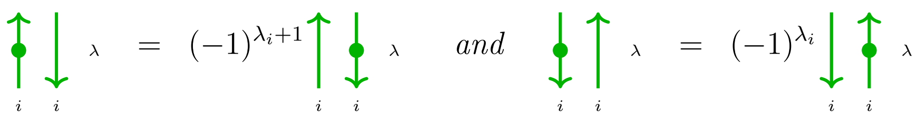
Proof. Use Corollary 2.2 to compose with the right isomorphisms. Then use Lemma A.5 and Lemma A.1 to get three terms, two of them being zero thanks to KLR R2 (either for F, (2.4), or E, Lemma A.7). Conclude using adjunction relations (2.10) and Lemma A.6. □

(A.6)

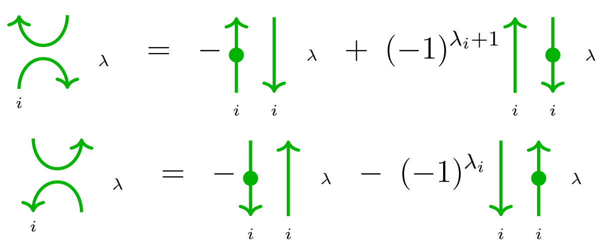
Proof. When $\overline{\lambda}_i = 1$, it a consequence of Corollary 2.2 and the adjunction relations (2.10), since then cups and caps are isomorphisms. If $\overline{\lambda}_i \neq 1$, it is a consequence of Lemma A.8 and Lemma A.6. □

Dot interchanges

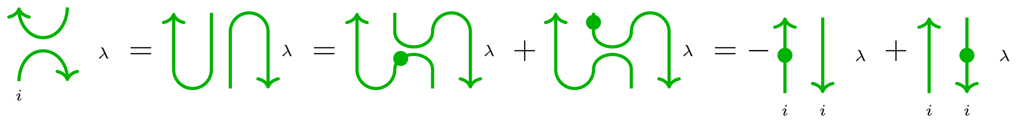
Lemma A.9 (Dot interchange)**.** *When* $\overline{\lambda}_i \neq 0$, *we have*



When $\overline{\lambda}_i = 0$, *we have*

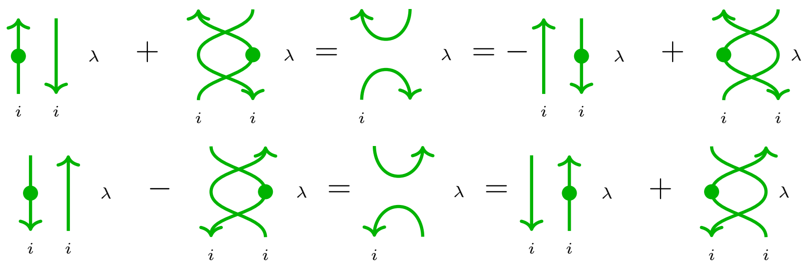


Proof. The first relations are consequences of Corollary 2.2 and Lemma 2.4. When $\overline{\lambda}_i = 0$, the first relation follows from



where we used Corollary 2.2 and adjunction relations (2.10) and (A.6). A similar argument holds for the other one. $\square$

Lemma A.10. *If* $\overline{\lambda}_i = 0$ *:*



Proof. Consequence of Corollary 2.2 and Lemma A.1. $\square$

Appendix B

Tensor product of chain complexes in monoidal supercategories

This appendix shows all the results stated in Section 3.1. In particular, we show that the Koszul rule can be extended to monoidal supercategories in a sensible way, such that the tensor product leaves homotopy classes invariant. This last result is central if one wishes to define an invariant of oriented tangles in the context of 2-supercategories. Nonetheless, the proofs are gathered here to facilitate reading, as they are quite technical. Every definition and result in this appendix is original.

B.1 Further aspects of working with monoidal supercategories

Scalars depending on parities

Let $\mathcal{C}$ be a monoidal supercategory. Each morphisms space is a superspace, that is, any morphism F can be uniquely decomposed into an even and an odd part:

$$F = f_e + f_o$$

where f_e and f_o are respectively the projections on the even and the odd subspaces.

Definition B.1. *A* choice of parity *for F is a choice between the two projections f_e and f_o. In what follows, we always denote a (not necessary homogeneous) morphism with an uppercase letter (e.g. F) and a choice of parity for this morphism as the corresponding lowercase letter (e.g. f).*

More generally a *choice of parities* for morphisms $F_1, \ldots, F_n$ is a choice of parity for each of them. Recall that in a monoidal supercategory, the interchange law

$$(F_1 \otimes F_2) \circ (G_1 \otimes G_2) = (-1)^{|F_2||G_1|}(F_1 \circ G_1) \otimes (F_2 \circ G_2)$$

is understood for inhomogeneous morphisms by extending it additively. Using the notion of choice of parity, this amounts to *summing up over every choice of parities*:

$$\begin{aligned}(F_1 \otimes F_2) \circ (G_1 \otimes G_2) &= (F_1 \circ g_{1e}) \otimes (f_{2e} \circ G_2) + (F_1 \circ g_{1e}) \otimes (f_{2o} \circ G_2) \\ &\quad + (F_1 \circ g_{1o}) \otimes (f_{2e} \circ G_2) - (F_1 \circ g_{1o}) \otimes (f_{2o} \circ G_2) \\ &= \sum_{f_2, g_1} (-1)^{|f_2||g_1|} (F_1 \circ g_1) \otimes (f_2 \circ G_2)\end{aligned}$$

More generally, an expression e involving inhomogeneous morphisms and scalars depending on the parities of these morphisms should be understood as *the sum over every choice of parities for the morphisms of e, each time setting the scalars accordingly.*

Remark B.2. Note that an expression doesn't make sense if one of the scalars depends on the parity of an inhomogeneous morphism which doesn't belong to the expression. Nonetheless, when encountering an equality between such expressions we shall mean that the equality holds whatever the choice of parity for this morphism.

Choice of parities for equations

Since the decomposition into even and odd parts is unique, any equality between two morphisms should really be understood as two equalities:

$$F = G \qquad \Leftrightarrow \qquad f_e = g_e \quad \text{and} \quad f_o = g_o.$$

This leads to the following definition:

Definition B.3. *A* choice of parity *for an equation $F = G$ is a choice of one of the projection $f_e = g_e$ or $f_o = g_o$. An equation is verified if and only if both its choices of parity are verified.*

We would like to relate a choice of parity for an equation to a choice of parities for all the morphisms involved. In general, this is not possible, as the quite simple example below shows:

$$F \circ G = H \qquad \Leftrightarrow \qquad f_e \circ g_e + f_o \circ g_o = h_e \quad \text{and} \quad f_e \circ g_o + f_o \circ g_e = h_o.$$

The first of the two projections is a single choice of parity for $F \circ G = H$, namely the choice "even". If both f and g are inhomogeneous, both $f_e \circ g_e$ and $f_o \circ g_o$ are non-zero and both choices of parity for f and g appear in this equation. In other words, a choice of parity for $F \circ G = H$ *does not* determine a choice of parities for the morphisms involved. On the other hand, if we now assume that g is homogeneous, e.g. odd, we get:

$$F \circ G = H \qquad \Leftrightarrow \qquad f_o \circ g_o = h_e \quad \text{and} \quad f_e \circ g_o = h_o.$$

Here a choice of parity for $F \circ G = H$ *does* determine the choices of parities for the morphisms involved. Notice that now, equations before and after the choice of parity both look the same. This encourages us to write:

$$F \circ G = H \qquad \Leftrightarrow \qquad f \circ g = h.$$

The right-hand side should be understood as "$f \circ g = h$ is verified for all choice of parities such that $|f \circ g| = |h|$" (here f_o is considered to be odd even if it is zero, and similarly for the other choices of parity). The benefit of this trick is that we can work with essentially the same equation as $F \circ G = H$, but with the additional assumptions that the morphisms are homogeneous. In particular, scalars depending on parities do not induce an intricate summation over all choices of parities. We call this trick ***choosing parity***.

In general, we can choose parity if a choice of parity for the equation determines a choice of parities for the morphisms involved. Call a *term* a sequence of compositions and/or tensor products of morphisms: the choice of a projection for this term determines a choice of parities for the morphisms if and only if all the morphisms are homogeneous, except perhaps one. Then if both sides of an equation are sums of such terms, a choice of parity for the equation determines a choice of parities for the morphisms involved if and only if for each term, all morphisms are homogeneous, except perhaps one.

We will extensively use the trick of choosing parity in the proofs. The attentive reader will notice that if we often encounter cases where choosing parity is applicable, it is essentially due to the fact that the factors of the tensor product are homogeneous.

Sign conventions

Whenever the context makes it clear, we usually avoid using $|.|$ for the parity of a morphism. This should not be a problem for the reader, since parities appear either as an exponent of (-1), or in equations where we use the symbol $\equiv$ to denote equalities modulo 2, which only happens when dealing with parities.

Composition of tensor products

Recall the rules of parities detailed in Remark 1.10.

As already mentioned, the composition of two tensor products of morphisms follows *the interchange law*:

$$(F_1 \otimes F_2) \circ (G_1 \otimes G_2) = (-1)^{F_2 G_1} (F_1 \circ G_2) \otimes (F_2 \circ G_2).$$

We can deduce the general rule of composition of tensor products:

$$\begin{aligned}
&(F_1 \otimes \ldots \otimes F_n) \circ (G_1 \otimes \ldots \otimes G_n) \\
&\quad = (-1)^{(F_2\otimes\ldots\otimes F_n)G_1}(F_1 \circ G_1) \otimes ((F_2 \otimes \ldots \otimes F_n) \circ (G_2 \otimes \ldots \otimes G_n)) \\
&\quad = (-1)^{(F_2\otimes\ldots\otimes F_n)G_1+(F_3\otimes\ldots\otimes F_n)G_2} \\
&\qquad\qquad (F_1 \circ G_1) \otimes (F_2 \circ G_2) \otimes ((F_3 \otimes \ldots \otimes F_n) \circ (G_3 \otimes \ldots \otimes G_n)) \\
&\quad = \ldots \\
&\quad = (-1)^{\sum_{k=1}^{n-1}(F_{k+1}\otimes\ldots\otimes F_n)G_k}(F_1 \circ G_1) \otimes \ldots \otimes (F_n \circ G_n) \\
&\quad = (-1)^{\sum_{k=1}^{n-1}(F_{k+1}+\ldots+F_n)G_k}(F_1 \circ G_1) \otimes \ldots \otimes (F_n \circ G_n)
\end{aligned}$$

In particular, if only F_i and G_j are (possibly) not even:

$$\sum_{k=1}^{n-1}(F_{k+1} + \ldots + F_n)G_k \equiv (F_{j+1} + \ldots + F_n)G_j \equiv \delta_{i>j} F_i \cdot G_j.$$

In short, add $(-1)^{F \cdot G}$ whenever you need to "switch" F and G when composing.

Notations

The proofs in the rest of this appendix are quite technical, therefore a good deal of simplified notations is welcome. The reader may refer to this list whenever a notation seems unclear.

- As mentioned, uppercase letters denote general morphisms while the corresponding lowercase letters denote some choice of parity. This convention does not apply to Greek letters as they most of the time already denote homogeneous morphisms (see the next entry).
- α always denote the differential of the complex A, and β the differential of the complex B (with an additional subscript if more than one complex of the kind is needed). Moreover, we shall always use the letter r (resp. s) to denote the degree in the complex A (resp. B), and the letters i or n (resp. j or m) when an iterator is needed. Again, we may sometimes use an additional subscript.
- $(e_i)_i$ always denote the canonical basis of $\mathbb{N}^n$, where n denotes whatever dimension makes sense in the context.
- We use $\vec{r}$ as the vector of all the "r's" in the context at hand: it may either mean $\vec{r} = (r_1, \ldots, r_n)$ or $\vec{r} = (\vec{r}_1, \vec{r}_2)$. Similarly for $\vec{s}$.
- Since we will be dealing a lot (if not only) with signs, we use throughout the paper the notation $\epsilon = (-1)^\sigma$, with additional subscripts depending on context.

B.2 Summary of the results

Recall the definition of homogeneous complexes (Definition 3.1).

Tensor product of homogeneous chain complexes

Let (A_i, α_i) be n homogeneous complexes and $\vec{r} = (r_1, \ldots, r_n)$. We set

$$A^{\vec{r}} = A_1^{r_1} \otimes \ldots \otimes A_n^{r_n} \quad \text{ and } \quad \alpha_i^{\vec{r}} = \epsilon_A^{\vec{r},i} \mathrm{Id}_{A_1^{r_1}} \otimes \ldots \otimes \alpha_i^{r_i} \otimes \ldots \otimes \mathrm{Id}_{A_n^{r_n}},$$

where $\epsilon_A^{\vec{r},i}$ is a choice of signs given below. Note that $\left|\alpha_i^{\vec{r}}\right| \equiv |\alpha_i^{r_i}|$.

Lemma B.4. *There exists a quantity* $|\alpha|(\vec{r}, i)$ *for all* $\vec{r}$ *and* i *such that*

$$|\alpha|(\vec{r} + \vec{e}_j, i) + |\alpha|(\vec{r}, i) \equiv \delta_{j \le i} \alpha_j^{r_j} \qquad \forall 1 \le i, j \le n,$$

where $\delta_{j \le i}$ *is the Kronecker symbol* ($\delta_{j \le i} = 1$ *if* $j \le i$, 0 *otherwise).*

Recall our convention that we don't use $|\cdot|$ to denote parities: here $\alpha_j^{r_j}$ should really be understood as the parity of $\alpha_j^{r_j}$. We also define $|\alpha|(\vec{r}) := |\alpha|(\vec{r}, n)$, which is such that for all $1 \le 1 \le n$, $|\alpha|(\vec{r} + \vec{e}_i) \equiv \alpha_i^{r_i}$. Similar definitions are given for m homogeneous complexes (B_j, β_j). Assuming this lemma, we can define the tensor product of homogeneous complexes:

Definition B.5. *The* tensor product *of n homogeneous complexes (A_i, α_i) is the complex (A, α) with chain spaces*

$$A^r := \bigoplus_{\vec{r}:|\vec{r}|=r} A^{\vec{r}}$$

and with differentials

$$\alpha^r|_{A^{\vec{r}}} := \alpha^{\vec{r}} := \sum_{1 \le i \le n} \alpha_i^{\vec{r}}.$$

where the choice of signs $\epsilon_A^{\vec{r},i} = (-1)^{\sigma_A^{\vec{r},i}}$ *is given by*

$$\sigma_A^{\vec{r},i} \equiv \sum_{j<i} r_j + \alpha_i^{r_i} \cdot |\alpha|(\vec{r}, i).$$

This extends the Koszul rule to monoidal supercategories.

By our convention on parities, $\alpha_i^{r_i}$ in the definition of $\sigma_A^{\vec{r},i}$ denotes the parity of $\alpha_i^{r_i}$: $\alpha_i^{r_i}$ and $|\alpha|(\vec{r}, i)$ are elements of $\mathbb{Z}/2\mathbb{Z}$ and $\cdot$ merely denotes the product of two numbers. Obviously, we must check that the tensor product is a complex:

Lemma B.6. *If* $A = \otimes A_i$ *is a tensor product, then* A *is a chain complex.*

The previous definition induces a tensor product on the objects of $\mathrm{TCh}_\bullet(\mathcal{C})$ by splitting each object into its factors and then use the definition above. This gives the following definition:

Definition B.7. *Let $A = \otimes_i A_i$ and $B = \otimes_j B_j$ be objects of* $\mathrm{TCh}_\bullet(\mathcal{C})$. *Their* tensor product *is the tensor product of their factors. In other words, $A \otimes B$ is the complex with chain spaces:*

$$(A \otimes B)^t := \bigoplus_{\vec{r},\vec{s}:|\vec{r}|+|\vec{s}|=t} A^{\vec{r}} \otimes B^{\vec{s}}$$

and with differentials $d_{A\otimes B}$, acting on $A^{\vec{r}} \otimes B^{\vec{s}}$ as

$$d^{\vec{r},\vec{s}}_{A\otimes B} := \sum_{1\leq i\leq n} \epsilon^{\vec{r},i}_A \alpha^{\vec{r}}_i \otimes \mathrm{Id}_{B^{\vec{s}}} + \sum_{1\leq j\leq m} \epsilon^{\vec{r},\vec{s},j}_{A\otimes B} \epsilon^{\vec{s},j}_B \mathrm{Id}_{A^{\vec{r}}} \otimes \beta^{\vec{s}}_j.$$

Here $\epsilon^{\vec{r},\vec{s},j}_{A\otimes B} = (-1)^{\sigma^{\vec{r},\vec{s},j}_{A\otimes B}}$ is a choice of signs such that

$$\sigma^{\vec{r},\vec{s},j}_{A\otimes B} \equiv |\vec{r}| + \beta^{s_j}_j \cdot |\alpha|\,(\vec{r}).$$

By definition, the tensor product is associative.

Induced morphisms on the tensor product

A morphism F between the complexes A and B is a set of morphisms between objects of the same degree which commute with the differentials: $\beta \circ F = F \circ \alpha$. If the complexes are tensor products, $A = \otimes_i A_i$ and $B = \otimes_j B_j$, this is equivalent to the data of morphisms

$$F^{\vec{r},\vec{s}} \colon A^{\vec{r}} \to B^{\vec{s}}, \quad \forall \vec{r}, \vec{s} : |\vec{r}| = |\vec{s}|\,,$$

such that

$$\sum_{1\leq j\leq m} \beta^{\vec{s}-\vec{e}_j}_j \circ F^{\vec{r},\vec{s}-\vec{e}_j} = \sum_{1\leq i\leq n} F^{\vec{r}+\vec{e}_i,\vec{s}} \circ \alpha^{\vec{r}}_i.$$

Definition B.8. *if $F^{\vec{r}_1,\vec{s}_1}_1 \colon A^{\vec{r}_1}_1 \to B^{\vec{s}_1}_1$ and $F^{\vec{r}_2,\vec{s}_2}_2 \colon A^{\vec{r}_2}_2 \to B^{\vec{s}_2}_2$ are two morphisms of tensor products, their* induced morphism *is the morphism $F \colon A_1 \otimes A_2 \to B_1 \otimes B_2$ given by the data of morphisms (here $\vec{r} = (\vec{r}_1, \vec{r}_2)$ and $\vec{s} = (\vec{s}_1, \vec{s}_2)$)*

$$F^{\vec{r},\vec{s}} = (-1)^{\sigma^{\vec{r},\vec{s}}_F} F^{\vec{r}_1,\vec{s}_1}_1 \otimes F^{\vec{r}_2,\vec{s}_2}_2,$$

setting

$$\sigma^{\vec{r},\vec{s}}_F \equiv |\beta_2|\,(\vec{s}_2) \cdot (F^{\vec{r}_1,\vec{s}_1}_1 + |\alpha_1|\,(\vec{r}_1) + |\beta_1|\,(\vec{s}_1)) + F^{\vec{r}_2,\vec{s}_2}_2 \cdot |\alpha_1|\,(\vec{r}_1).$$

Proposition B.9. *$F^{\vec{r},\vec{s}}$ defined above is a morphism of complexes. Moreover, if $F^{\vec{r}_1,\vec{s}_1}_1$ and $F^{\vec{r}_2,\vec{s}_2}_2$ are identities, then $F^{\vec{r},\vec{s}}$ is the identity.*

Remark B.10. Even though we can assign a morphism to a pair of morphisms through the tensor product, it is not clear whether the tensor product is functorial. With the usual cartesian product of categories, it seems not to be. Nonetheless, it is possible that similarly to the case of monoidal supercategories, a suitable product of categories would turn the tensor product into a functor (although it won't be that easy, as for example it is not clear whether we can assign a parity to morphisms of complexes). We leave this question to be investigated in future work, as it is not important for our purpose.

Induced homotopies on the tensor product

A homotopy H between morphisms $F\colon A \to B$ and $G\colon A \to B$ is a set of morphisms going one down in degree such that $F - G = H \circ \alpha + \beta \circ H$. If the complexes are tensor products $A = \otimes_i A_i$ and $B = \otimes_j$, this is equivalent to the data of morphisms

$$H^{\vec{r},\vec{s}}\colon A^{\vec{r}} \to B^{\vec{s}} \quad \forall \vec{r}, \vec{s} : |\vec{r}| = |\vec{s}| + 1$$

such that

$$F^{\vec{r},\vec{s}} - G^{\vec{r},\vec{s}} = \sum_{1\le i\le n} H^{\vec{r}+\vec{e}_i,\vec{s}} \circ \alpha_i^{\vec{r}} + \sum_{1\le j\le m} \beta_j^{\vec{s}-\vec{e}_j} \circ H^{\vec{r},\vec{s}-\vec{e}_j}.$$

Definition B.11. *If F_1 and G_1 (resp. F_2 and G_2) are morphisms of tensor products A_1 and B_1 (resp. A_2 and B_2) and H_1 (resp. H_2) is a homotopy between F_1 and G_1 (resp. between F_2 and G_2), their* induced homotopy *is the homotopy $H\colon F \to G$ (where F and G are the morphisms induced by (F_1, G_1) and (F_2, G_2)) given by the data of morphisms*

$$H^{\vec{r},\vec{s}} = \frac{1}{2}\Big[(-1)^{\sigma_{H,w}^{\vec{r},\vec{s}}} H_1^{\vec{r}_1,\vec{s}_1} \otimes F_2^{\vec{r}_2,\vec{s}_2} + (-1)^{\sigma_{H,x}^{\vec{r},\vec{s}}} H_1^{\vec{r}_1,\vec{s}_1} \otimes G_2^{\vec{r}_2,\vec{s}_2} + (-1)^{\sigma_{H,y}^{\vec{r},\vec{s}}} F_1^{\vec{r}_1,\vec{s}_1} \otimes H_2^{\vec{r}_2,\vec{s}_2} + (-1)^{\sigma_{H,z}^{\vec{r},\vec{s}}} G_1^{\vec{r}_1,\vec{s}_1} \otimes H_2^{\vec{r}_2,\vec{s}_2}\Big]$$

setting

$$\begin{aligned}
\sigma_{H,w}^{\vec{r},\vec{s}} &\equiv |\beta_2|\,(\vec{s}_2) \cdot (H_1^{\vec{r}_1,\vec{s}_1} + |\alpha_1|\,(\vec{r}_1) + |\beta_1|\,(\vec{s}_1)) + F_2^{\vec{r}_2,\vec{s}_2} \cdot |\alpha_1|\,(\vec{r}_1),\\
\sigma_{H,x}^{\vec{r},\vec{s}} &\equiv |\beta_2|\,(\vec{s}_2) \cdot (H_1^{\vec{r}_1,\vec{s}_1} + |\alpha_1|\,(\vec{r}_1) + |\beta_1|\,(\vec{s}_1)) + G_2^{\vec{r}_2,\vec{s}_2} \cdot |\alpha_1|\,(\vec{r}_1),\\
\sigma_{H,y}^{\vec{r},\vec{s}} &\equiv |\vec{r}_1| + |\beta_2|\,(\vec{s}_2) \cdot (F_1^{\vec{r}_1,\vec{s}_1} + |\alpha_1|\,(\vec{r}_1) + |\beta_1|\,(\vec{s}_1)) + H_2^{\vec{r}_2,\vec{s}_2} \cdot |\alpha_1|\,(\vec{r}_1),\\
\sigma_{H,z}^{\vec{r},\vec{s}} &\equiv |\vec{r}_1| + |\beta_2|\,(\vec{s}_2) \cdot (G_1^{\vec{r}_1,\vec{s}_1} + |\alpha_1|\,(\vec{r}_1) + |\beta_1|\,(\vec{s}_1)) + H_2^{\vec{r}_2,\vec{s}_2} \cdot |\alpha_1|\,(\vec{r}_1).
\end{aligned}$$

Proposition B.12. *H is a homotopy between F and G.*

Invariance of homotopy classes under the tensor product

Theorem B.13. *Let A_1, A_2, B_1 and B_2 be tensor products of homogeneous complexes. If $A_1 \simeq B_1$ and $A_2 \simeq B_2$, we have*

$$A_1 \otimes A_2 \simeq B_1 \otimes B_2.$$

Proof. Let $F_1 \colon A_1 \to B_1$ and $G_1 \colon B_1 \to A_1$ be morphisms and $H_{A_1} \colon G_1 F_1 \to \mathrm{Id}_{A_1}$ and $H_{B_1} \colon F_1 G_1 \to \mathrm{Id}_{B_1}$ homotopies. Define similarly F_2, G_2, H_{A_2} and H_{B_2}. By Proposition B.9, the pairs (F_1, F_2) and (G_1, G_2) induce morphisms $F \colon A_1 \otimes A_2 \to B_1 \otimes B_2$ and $G \colon B_1 \otimes B_2 \to A_1 \otimes A_2$. By Proposition B.12, the pairs (H_{A_1}, H_{A_2}) and (H_{B_1}, H_{B_2}) induce homotopies $H_A \colon GF \to \mathrm{Id}_{A_1 \otimes A_2}$ and $H_B \colon FG \to \mathrm{Id}_{B_1 \otimes B_2}$, recalling that the pairs of identity morphisms induce an identity morphism on the tensor product. This concludes. □

B.3 Proofs of the results

Tensor product of homogeneous chain complexes

Proof of Lemma B.4. The following definition gives the desired property:

$$|\alpha|\,(\vec{r}, i) \equiv \sum_{j \leq i} \sum_{\substack{s < r_j \\ \text{non-zero}}} \alpha_j^s$$

where "non-zero" should be understood as "start summation at $r_j - 1$ and continue until running into a zero map. Then stop". It is well-defined since the complexes are bounded. □

Proof of Lemma B.6. We must show that for all $\vec{r}$ and $\vec{s}$ such that $|\vec{s}| = |\vec{r}| + 2$, the morphisms going from $A^{\vec{r}}$ to $A^{\vec{s}}$ sum to zero. Since there are only two such paths, this amounts to show that all squares anti-commute:

$$(\epsilon_A^{\vec{r}+\vec{e}_i, j} \alpha_j^{\vec{r}+\vec{e}_i}) \circ (\epsilon_A^{\vec{r}, i} \alpha_i^{\vec{r}}) = -(\epsilon_A^{\vec{r}+\vec{e}_j, i} \alpha_i^{\vec{r}+\vec{e}_j}) \circ (\epsilon_A^{\vec{r}, j} \alpha_j^{\vec{r}}).$$

When $i = j$, both sides are zero since (A_i, α_i) is a complex. When $i \neq j$, the condition is satisfied if, whenever $\alpha_i^{r_i}$ and $\alpha_j^{r_j}$ are non-zero:

$$\begin{aligned} \epsilon_A^{\vec{r}+\vec{e}_i, j} \epsilon_A^{\vec{r}, i} &= -(-1)^{\alpha_i^{r_i} \alpha_j^{r_j}} \epsilon_A^{\vec{r}+\vec{e}_j, i} \epsilon_A^{\vec{r}, j} \\ \Leftrightarrow \sigma_A^{\vec{r}+\vec{e}_i, j} + \sigma_A^{\vec{r}, i} &\equiv 1 + \alpha_i^{r_i} \alpha_j^{r_j} + \sigma_A^{\vec{r}+\vec{e}_j, i} + \sigma_A^{\vec{r}, j}. \end{aligned}$$

It is easily check using the property of $|\alpha|\,(\vec{r}, i)$ (here, we assume $i < j$, the other case is analogous):

$$\begin{aligned} &\sum_{k<j} r_k + 1 + \alpha_j^{r_j} \cdot |\alpha|\,(\vec{r} + \vec{e}_i, j) + \sum_{k<i} r_k + \alpha_i^{r_i} \cdot |\alpha|\,(\vec{r}, i) \\ &\qquad \equiv 1 + \alpha_i^{r_i} \alpha_j^{r_j} + \sum_{k<i} r_k + \alpha_i^{r_i} \cdot |\alpha|\,(\vec{r} + \vec{e}_j, i) + \sum_{k<j} r_k + \alpha_j^{r_j} \cdot |\alpha|\,(\vec{r}, j) \\ &\Leftrightarrow \alpha_j^{r_j} \cdot (|\alpha|\,(\vec{r}, j) + \alpha_i^{r_i}) + \alpha_i^{r_i} \cdot |\alpha|\,(\vec{r}, i) \equiv \alpha_i^{r_i} \alpha_j^{r_j} + \alpha_i^{r_i} \cdot |\alpha|\,(\vec{r}, i) + \alpha_j^{r_j} \cdot |\alpha|\,(\vec{r}, j). \end{aligned}$$

□

Induced morphisms on the tensor product

Let $F_1^{\vec{r}_1,\vec{s}_1}\colon A_1^{\vec{r}_1} \to B_1^{\vec{s}_1}$ and $F_2^{\vec{r}_2,\vec{s}_2}\colon A_2^{\vec{r}_2} \to B_2^{\vec{s}_2}$ be two morphisms of tensor products. Our goal is to define a morphism F from $A = A_1 \otimes A_2$ to $B = B_1 \otimes B_2$. The only natural choice is to set

$$F^{\vec{r},\vec{s}} = \epsilon_F^{\vec{r},\vec{s}} F_1^{\vec{r}_1,\vec{s}_1} \otimes F_2^{\vec{r}_2,\vec{s}_2}$$

where $\epsilon_F^{\vec{r},\vec{s}}$ is a choice of signs. We will prove that

$$\sigma_F^{\vec{r},\vec{s}} \equiv |\beta_2|(\vec{s}_2) \cdot (F_1^{\vec{r}_1,\vec{s}_1} + |\alpha_1|(\vec{r}_1) + |\beta_1|(\vec{s}_1)) + F_2^{\vec{r}_2,\vec{s}_2} \cdot |\alpha_1|(\vec{r}_1)$$

is the right choice for F to be a morphism (as usual, we write $\epsilon_F^{\vec{r},\vec{s}} = (-1)^{\sigma_F^{\vec{r},\vec{s}}}$). To avoid unnecessary heavy notations, we will sometimes abuse notation for the canonical basis $(e_i)_i$, e.g. writing $\vec{e}_{i_2}$ for $\vec{e}_{n_1+i_2}$. The meaning should be clear by the context and the nature of the subscript (e.g. $\vec{s}_2 + \vec{e}_{j_2}$ really means $\vec{s}_2 + \vec{e}_{j_2}$, while $\vec{s} + \vec{e}_{j_2}$ means $\vec{s} + \vec{e}_{m_1+j_2}$).

Proof of Proposition B.9. We must check that $\beta \circ F = F \circ \alpha$. First, we unfold both sides of the equation:

$$\begin{aligned}
&\sum_{1\le j\le m} \beta_j^{\vec{s}-\vec{e}_j} \circ F^{\vec{r},\vec{s}-\vec{e}_j}\\
&\quad= \sum_{1\le j_1\le m_1} \left[\beta_{j_1}^{\vec{s}_1-\vec{e}_{j_1}} \otimes \mathrm{Id}\right] \circ \left[\epsilon_F^{\vec{r},\vec{s}-\vec{e}_{j_1}} F_1^{\vec{r}_1,\vec{s}_1-\vec{e}_{j_1}} \otimes F_2^{\vec{r}_2,\vec{s}_2}\right]\\
&\qquad+ \sum_{1\le j_2\le m_2} \left[\epsilon_{B_1\otimes B_2}^{\vec{s}_1,\vec{s}_2-\vec{e}_{j_2},j_2} \mathrm{Id} \otimes \beta_{j_2}^{\vec{s}_2-\vec{e}_{j_2}}\right] \circ \left[\epsilon_F^{\vec{r},\vec{s}-\vec{e}_{j_2}} F_1^{\vec{r}_1,\vec{s}_1} \otimes F_2^{\vec{r}_2,\vec{s}_2-\vec{e}_{j_2}}\right]\\
&\quad= \left[\sum_{1\le j_1\le m_1} \epsilon_F^{\vec{r},\vec{s}-\vec{e}_{j_1}} \beta_{j_1}^{\vec{s}_1-\vec{e}_{j_1}} \circ F_1^{\vec{r}_1,\vec{s}_1-\vec{e}_{j_1}}\right] \otimes F_2^{\vec{r}_2,\vec{s}_2}\\
&\qquad+ F_1^{\vec{r}_1,\vec{s}_1} \otimes \left[\sum_{1\le j_2\le m_2} (-1)^{\beta_{j_2}^{\vec{s}_2-\vec{e}_{j_2}}\cdot F_1^{\vec{r}_1,\vec{s}_1}} \epsilon_{B_1\otimes B_2}^{\vec{s}_1,\vec{s}_2-\vec{e}_{j_2},j_2} \epsilon_F^{\vec{r},\vec{s}-\vec{e}_{j_2}} \beta_{j_2}^{\vec{s}_2-\vec{e}_{j_2}} \circ F_2^{\vec{r}_2,\vec{s}_2-\vec{e}_{j_2}}\right]
\end{aligned}$$

$$\begin{aligned}
&\sum_{1\le i\le n} F^{\vec{r}+\vec{e}_i,\vec{s}} \circ \alpha_i^{\vec{r}}\\
&\quad= \sum_{1\le i_1\le n_1} \left[\epsilon_F^{\vec{r}+\vec{e}_{i_1},\vec{s}} F_1^{\vec{r}_1+\vec{e}_{i_1},\vec{s}_1} \otimes F_2^{\vec{r}_2,\vec{s}_2}\right] \circ \left[\alpha_{i_1}^{\vec{r}_1} \otimes \mathrm{Id}\right]\\
&\qquad+ \sum_{1\le i_2\le n_2} \left[\epsilon_F^{\vec{r}+\vec{e}_{i_2},\vec{s}} F_1^{\vec{r}_1,\vec{s}_1} \otimes F_2^{\vec{r}_2+\vec{e}_{i_2},\vec{s}_2}\right] \circ \left[\epsilon_{A_1\otimes A_2}^{\vec{r}_1,\vec{r}_2,i_2} \mathrm{Id} \otimes \alpha_{i_2}^{\vec{r}_2}\right]\\
&\quad= \left[\sum_{1\le i_1\le n_1} (-1)^{F_2^{\vec{r}_2,\vec{s}_2}\cdot \alpha_{i_1}^{\vec{r}_1}} \epsilon_F^{\vec{r}+\vec{e}_{i_1},\vec{s}} F_1^{\vec{r}_1+\vec{e}_{i_1},\vec{s}_1} \circ \alpha_{i_1}^{\vec{r}_1}\right] \otimes F_2^{\vec{r}_2,\vec{s}_2}\\
&\qquad+ F_1^{\vec{r}_1,\vec{s}_1} \otimes \left[\sum_{1\le i_2\le n_2} \epsilon_F^{\vec{r}+\vec{e}_{i_2},\vec{s}} \epsilon_{A_1\otimes A_2}^{\vec{r}_1,\vec{r}_2,i_2} F_2^{\vec{r}_2+\vec{e}_{i_2},\vec{s}_2} \circ \alpha_{i_2}^{\vec{r}_2}\right]
\end{aligned}$$

Since $F_1^{\vec{r}_1,\vec{s}_1}$ is zero whenever $|\vec{r_1}| \neq |\vec{s}_1|$, and similarly $F_2^{\vec{r}_2,\vec{s}_2}$ is zero whenever $|\vec{r}_2| \neq |\vec{s}_2|$, we can split the equation $\beta \circ F = F \circ \alpha$ in two cases.

Case 1. When $|\vec{s_1}| = |\vec{r_1}|$ and $|\vec{s_2}| = |\vec{r_2}| + 1$, the equation holds if:

$$\sum_{1 \leq j_2 \leq m_2} (-1)^{\beta_{j_2}^{\vec{s}_2-\vec{e}_{j_2}} \cdot F_1^{\vec{r}_1,\vec{s}_1}} \epsilon_{B_1 \otimes B_2}^{\vec{s}_1,\vec{s}_2-\vec{e}_{j_2},j_2} \epsilon_F^{\vec{r},\vec{s}-\vec{e}_{j_2}} \left(\beta_{j_2}^{\vec{s}_2-\vec{e}_{j_2}} \circ F_2^{\vec{r}_2,\vec{s}_2-\vec{e}_{j_2}}\right)$$
$$= \sum_{1 \leq i_2 \leq n_2} \epsilon_F^{\vec{r}+\vec{e}_{i_2},\vec{s}} \epsilon_{A_1 \otimes A_2}^{\vec{r}_1,\vec{r}_2,i_2} \left(F_2^{\vec{r}_2+\vec{e}_{i_2},\vec{s}_2} \circ \alpha_{i_2}^{\vec{r}_2}\right).$$

Choosing parity, we can assume $s \equiv \beta_{j_2}^{s_{j_2}-1} + f_2^{\vec{r}_2,\vec{s}_2-\vec{e}_{j_2}} \equiv f_2^{\vec{r}_2+\vec{e}_{i_2},\vec{s}_2} + \alpha_{i_2}^{r_{i_2}}$ is constant. Moreover, using the fact that F_2 is a morphism, the equation holds if:

- $C_1 = (-1)^{\beta_{j_2}^{\vec{s}_2-\vec{e}_{j_2}} \cdot f_1^{\vec{r}_1,\vec{s}_1}} \epsilon_{B_1 \otimes B_2}^{\vec{s}_1,\vec{s}_2-\vec{e}_{j_2},j_2} \epsilon_f^{\vec{r},\vec{s}-\vec{e}_{j_2}}$ does not depend on j_2 for all $1 \leq j_2 \leq m_2$ such that $\beta_{j_2}^{\vec{s}_2-\vec{e}_{j_2}} \circ f_2^{\vec{r}_2,\vec{s}_2-\vec{e}_{j_2}} \neq 0$;
- $C_2 = \epsilon_f^{\vec{r}+\vec{e}_{i_2},\vec{s}} \epsilon_{A_1 \otimes A_2}^{\vec{r}_1,\vec{r}_2,i_2}$ does not depend on i_2 for all $1 \leq i_2 \leq n_2$ such that $f_2^{\vec{r}_2+\vec{e}_{i_2},\vec{s}_2} \circ \alpha_{i_2}^{\vec{r}_{i_2}} \neq 0$;

and $C_1 = C_2$. Using respectively $s \equiv \beta_{j_2}^{s_{j_2}-1} + f_2^{\vec{r}_2,\vec{s}_2-\vec{e}_{j_2}}$ and $s \equiv f_2^{\vec{r}_2+\vec{e}_{i_2},\vec{s}_2} + \alpha_{i_2}^{r_{i_2}}$:

$$\begin{aligned}
\sigma_{B_1 \otimes B_2}^{\vec{s}_1,\vec{s}_2-\vec{e}_{j_2},j_2} &+ \sigma_f^{\vec{r},\vec{s}-\vec{e}_{j_2}} + \beta_{j_2}^{s_{j_2}-1} \cdot f_1^{\vec{r}_1,\vec{s}_1} \\
&\equiv |\vec{s}_1| + \beta_{j_2}^{s_{j_2}-1} |\beta_1| (\vec{s}_1) + \beta_{j_2}^{s_{j_2}-1} \cdot f_1^{\vec{r}_1,\vec{s}_1} \\
&\quad + |\beta_2| (\vec{s}_2 - \vec{e}_{j_2}) \cdot (f_1^{\vec{r}_1,\vec{s}_1} + |\alpha_1| (\vec{r}_1) + |\beta_1| (\vec{s}_1)) + f_2^{\vec{r}_2,\vec{s}_2-\vec{e}_{j_2}} \cdot |\alpha_1| (\vec{r}_1) \\
&\equiv |\vec{s}_1| + |\beta_2| (\vec{s}_2) \cdot (f_1^{\vec{r}_1,\vec{s}_1} + |\alpha_1| (\vec{r}_1) + |\beta_1| (\vec{s}_1)) + s \cdot |\alpha_1| (\vec{r}_1)
\end{aligned}$$

$$\begin{aligned}
\epsilon_f^{\vec{r}+\vec{e}_{i_2},\vec{s}} \epsilon_{A_1 \otimes A_2}^{\vec{r}_1,\vec{r}_2,i_2} &\equiv |\vec{r}_1| + \alpha_{i_2}^{r_{i_2}} |\alpha_1| (\vec{r}_1) \\
&\quad + |\beta_2| (\vec{s}_2) \cdot (f_1^{\vec{r}_1,\vec{s}_1} + |\alpha_1| (\vec{r}_1) + |\beta_1| (\vec{s}_1)) + f_2^{\vec{r}_2+\vec{e}_{i_2},\vec{s}_2} \cdot |\alpha_1| (\vec{r}_1) \\
&\equiv |\vec{r}_1| + |\beta_2| (\vec{s}_2) \cdot (f_1^{\vec{r}_1,\vec{s}_1} + |\alpha_1| (\vec{r}_1) + |\beta_1| (\vec{s}_1)) + s \cdot |\alpha_1| (\vec{r}_1)
\end{aligned}$$

and the two are equal since $|\vec{r_1}| = |\vec{s}_1|$.

Case 2. when $|\vec{s_1}| = |\vec{r_1}| + 1$ and $|\vec{s_2}| = |\vec{r_2}|$, the equation holds if :

$$\sum_{1 \leq j_1 \leq m_1} \epsilon_F^{\vec{r},\vec{s}-\vec{e}_{j_1}} \left[\beta_{j_1}^{\vec{s}_1-\vec{e}_{j_1}} \circ F_1^{\vec{r}_1,\vec{s}_1-\vec{e}_{j_1}}\right] = \sum_{1 \leq i_1 \leq n_1} (-1)^{F_2^{\vec{r}_2,\vec{s}_2} \cdot \alpha_{i_1}^{\vec{r}_1}} \epsilon_F^{\vec{r}+\vec{e}_{i_1},\vec{s}} \left[F_1^{\vec{r}_1+\vec{e}_{i_1},\vec{s}_1} \circ \alpha_{i_1}^{\vec{r}_1}\right].$$

Choosing parity, we can assume $s \equiv \beta_{j_1}^{s_{j_1}-1} + f_1^{\vec{r}_1,\vec{s}_1-\vec{e}_{j_1}} \equiv f_1^{\vec{r}_1+\vec{e}_{i_1},\vec{s}_1} + \alpha_{i_1}^{r_{i_1}}$ is constant. Moreover, using the fact that F_1 is a morphism, the equation holds if:

- $C_1 = \epsilon_f^{\vec{r},\vec{s}-\vec{e}_{j_1}}$ is constant for all $1 \leq j_1 \leq m_1$ such that $\beta_{j_1}^{\vec{s}_1-\vec{e}_{j_1}} \circ f_1^{\vec{r}_1,\vec{s}_1-\vec{e}_{j_1}} \neq 0$;
- $C_2 = (-1)^{f_2^{\vec{r}_2,\vec{s}_2} \cdot \alpha_{i_1}^{\vec{r}_1}} \epsilon_f^{\vec{r}+\vec{e}_{i_1},\vec{s}}$ is constant for all $1 \leq i_1 \leq n_1$ such that $f_1^{\vec{r}_1+\vec{e}_{i_1},\vec{s}_1} \circ \alpha_{i_1}^{\vec{r}_1} \neq 0$;

and $C_1 = C_2$. Using respectively $s \equiv \beta_{j_1}^{s_{j_1}-1} + f_1^{\vec{r}_1,\vec{s}_1-\vec{e}_{j_1}}$ and $s \equiv f_1^{\vec{r}_1+\vec{e}_{i_1},\vec{s}_1} + \alpha_{i_1}^{r_{i_1}}$:

$$\begin{aligned}\sigma_f^{\vec{r},\vec{s}-\vec{e}_{j_1}} &\equiv |\beta_2|(\vec{s}_2)\cdot(f_1^{\vec{r}_1,\vec{s}_1-\vec{e}_{j_1}} + |\alpha_1|(\vec{r}_1) + |\beta_1|(\vec{s}_1-\vec{e}_{j_1})) + f_2^{\vec{r}_2,\vec{s}_2}\cdot|\alpha_1|(\vec{r}_1)\\ &\equiv |\beta_2|(\vec{s}_2)\cdot(s + |\alpha_1|(\vec{r}_1) + |\beta_1|(\vec{s}_1)) + f_2^{\vec{r}_2,\vec{s}_2}\cdot|\alpha_1|(\vec{r}_1)\end{aligned}$$

$$\begin{aligned}f_2^{\vec{r}_2,\vec{s}_2}\cdot\alpha_{i_1}^{\vec{r}_1} + \sigma_f^{\vec{r}+\vec{e}_{i_1},\vec{s}} &\equiv f_2^{\vec{r}_2,\vec{s}_2}\cdot\alpha_{i_1}^{\vec{r}_1} + f_2^{\vec{r}_2,\vec{s}_2}\cdot|\alpha_1|(\vec{r}_1+\vec{e}_{i_1})\\ &\quad + |\beta_2|(\vec{s}_2)\cdot(f_1^{\vec{r}_1+\vec{e}_{i_1},\vec{s}_1} + |\alpha_1|(\vec{r}_1+\vec{e}_{i_1}) + |\beta_1|(\vec{s}_1))\\ &\equiv f_2^{\vec{r}_2,\vec{s}_2}\cdot|\alpha_1|(\vec{r}_1) + |\beta_2|(\vec{s}_2)\cdot(s + |\alpha_1|(\vec{r}_1) + |\beta_1|(\vec{s}_1))\end{aligned}$$

and the two are indeed equal.

We conclude that F is a morphism of complexes. Moreover, if both F_1 and F_2 are identities, then F is the identity ($\sigma_F^{\vec{r},\vec{s}} \equiv 0$ in all cases). □

Induced homotopies on the tensor product

Let F_1 and G_1 (resp. F_2 and G_2) be morphisms of tensor products A_1 and B_1 (resp. A_2 and B_2). Denote F (resp. G) the morphism between $A_1 \otimes A_2$ and $B_1 \otimes B_2$ induced by F_1 and F_2 (resp. G_1 and G_2), that is,

$$F^{\vec{r},\vec{s}} = \epsilon_F^{\vec{r}_1,\vec{r}_2} F_1^{\vec{r}_1,\vec{s}_1} \otimes F_2^{\vec{r}_2,\vec{s}_2} \quad \text{and} \quad G^{\vec{r},\vec{s}} = \epsilon_G^{\vec{r}_1,\vec{r}_2} G_1^{\vec{r}_1,\vec{s}_1} \otimes G_2^{\vec{r}_2,\vec{s}_2}$$

where

$$\begin{aligned}\sigma_F^{\vec{r}_1,\vec{s}_2} &\equiv |\beta_2|(\vec{s}_2)\cdot(F_1^{\vec{r}_1,\vec{s}_1} + |\alpha_1|(\vec{r}_1) + |\beta_1|(\vec{s}_1)) + F_2^{\vec{r}_2,\vec{s}_2}\cdot|\alpha_1|(\vec{r}_1),\\ \sigma_G^{\vec{r}_1,\vec{s}_2} &\equiv |\beta_2|(\vec{s}_2)\cdot(G_1^{\vec{r}_1,\vec{s}_1} + |\alpha_1|(\vec{r}_1) + |\beta_1|(\vec{s}_1)) + G_2^{\vec{r}_2,\vec{s}_2}\cdot|\alpha_1|(\vec{r}_1).\end{aligned}$$

Let also H_1 (resp. H_2) be a homotopy between F_1 and G_1 (resp. between F_2 and G_2). Our goal is to define an induced homotopy H between F and G. A natural choice for H is

$$\begin{aligned}H^{\vec{r},\vec{s}} = &(\epsilon_{H,w}^{\vec{r},\vec{s}} H_1^{\vec{r}_1,\vec{s}_1} \otimes F_2^{\vec{r}_2,\vec{s}_2}) + (\epsilon_{H,x}^{\vec{r},\vec{s}} H_1^{\vec{r}_1,\vec{s}_1} \otimes G_2^{\vec{r}_2,\vec{s}_2})\\ &+ (\epsilon_{H,y}^{\vec{r},\vec{s}} F_1^{\vec{r}_1,\vec{s}_1} \otimes H_2^{\vec{r}_2,\vec{s}_2}) + (\epsilon_{H,z}^{\vec{r},\vec{s}} G_1^{\vec{r}_1,\vec{s}_1} \otimes H_2^{\vec{r}_2,\vec{s}_2})\end{aligned}$$

where we define $\epsilon_{H,w}^{\vec{r},\vec{s}} = \frac{1}{2}(-1)^{\sigma_{H,w}^{\vec{r},\vec{s}}}$ (similarly for x, y and z). Note the factor $\frac{1}{2}$. We show that under the following definitions:

$$\begin{aligned}\sigma_{H,w}^{\vec{r},\vec{s}} &\equiv |\beta_2|(\vec{s}_2)\cdot(H_1^{\vec{r}_1,\vec{s}_1} + |\alpha_1|(\vec{r}_1) + |\beta_1|(\vec{s}_1)) + F_2^{\vec{r}_2,\vec{s}_2}\cdot|\alpha_1|(\vec{r}_1),\\ \sigma_{H,x}^{\vec{r},\vec{s}} &\equiv |\beta_2|(\vec{s}_2)\cdot(H_1^{\vec{r}_1,\vec{s}_1} + |\alpha_1|(\vec{r}_1) + |\beta_1|(\vec{s}_1)) + G_2^{\vec{r}_2,\vec{s}_2}\cdot|\alpha_1|(\vec{r}_1),\\ \sigma_{H,y}^{\vec{r},\vec{s}} &\equiv |\vec{r}_1| + |\beta_2|(\vec{s}_2)\cdot(F_1^{\vec{r}_1,\vec{s}_1} + |\alpha_1|(\vec{r}_1) + |\beta_1|(\vec{s}_1)) + H_2^{\vec{r}_2,\vec{s}_2}\cdot|\alpha_1|(\vec{r}_1),\\ \sigma_{H,z}^{\vec{r},\vec{s}} &\equiv |\vec{r}_1| + |\beta_2|(\vec{s}_2)\cdot(G_1^{\vec{r}_1,\vec{s}_1} + |\alpha_1|(\vec{r}_1) + |\beta_1|(\vec{s}_1)) + H_2^{\vec{r}_2,\vec{s}_2}\cdot|\alpha_1|(\vec{r}_1),\end{aligned}$$

H is indeed a homotopy.

Proof of Proposition B.12. We must check that $F - G = \beta \circ H + H \circ \alpha$. First, we unfold the terms of this equation, focusing on the paths from $A^{\vec{r}}$ to $B^{\vec{s}}$:

$$(F-G)|_{A^{\vec{r}}}^{B^{\vec{s}}} = (\epsilon_F^{\vec{r}_1,\vec{r}_2} F_1^{\vec{r}_1,\vec{s}_1} \otimes F_2^{\vec{r}_2,\vec{s}_2}) - (\epsilon_G^{\vec{r}_1,\vec{r}_2} G_1^{\vec{r}_1,\vec{s}_1} \otimes G_2^{\vec{r}_2,\vec{s}_2})$$

$$\begin{aligned}
(H\circ\alpha)|_{A^{\vec{r}}}^{B^{\vec{s}}} &= \sum_{1\le i\le n} H^{\vec{r}+\vec{e}_i,\vec{s}} \circ \alpha_i^{\vec{r}} \\
&= \sum_{1\le i_1\le n_1} \Big[(\epsilon_{H,w}^{\vec{r}+\vec{e}_{i_1},\vec{s}} H_1^{\vec{r}_1+\vec{e}_{i_1},\vec{s}_1} \otimes F_2^{\vec{r}_2,\vec{s}_2}) + (\epsilon_{H,x}^{\vec{r}+\vec{e}_{i_1},\vec{s}} H_1^{\vec{r}_1+\vec{e}_{i_1},\vec{s}_1} \otimes G_2^{\vec{r}_2,\vec{s}_2}) \\
&\qquad + (\epsilon_{H,y}^{\vec{r}+\vec{e}_{i_1},\vec{s}} F_1^{\vec{r}_1+\vec{e}_{i_1},\vec{s}_1} \otimes H_2^{\vec{r}_2,\vec{s}_2}) + (\epsilon_{H,z}^{\vec{r}+\vec{e}_{i_1},\vec{s}} G_1^{\vec{r}_1+\vec{e}_{i_1},\vec{s}_1} \otimes H_2^{\vec{r}_2,\vec{s}_2}) \Big] \circ \Big[\alpha_{i_1}^{\vec{r}_1} \otimes \mathrm{Id} \Big] \\
&\quad + \sum_{1\le i_2\le n_2} \Big[(\epsilon_{H,w}^{\vec{r}+\vec{e}_{i_2},\vec{s}} H_1^{\vec{r}_1,\vec{s}_1} \otimes F_2^{\vec{r}_2+\vec{e}_{i_2},\vec{s}_2}) + (\epsilon_{H,x}^{\vec{r}+\vec{e}_{i_2},\vec{s}} H_1^{\vec{r}_1,\vec{s}_1} \otimes G_2^{\vec{r}_2+\vec{e}_{i_2},\vec{s}_2}) \\
&\qquad + (\epsilon_{H,y}^{\vec{r}+\vec{e}_{i_2},\vec{s}} F_1^{\vec{r}_1,\vec{s}_1} \otimes H_2^{\vec{r}_2+\vec{e}_{i_2},\vec{s}_2}) + (\epsilon_{H,z}^{\vec{r}+\vec{e}_{i_2},\vec{s}} G_1^{\vec{r}_1,\vec{s}_1} \otimes H_2^{\vec{r}_2+\vec{e}_{i_2},\vec{s}_2}) \Big] \circ \Big[\epsilon_{A_1\otimes A_2}^{\vec{r}_1,\vec{r}_2,i_2} \mathrm{Id} \otimes \alpha_{i_2}^{\vec{r}_2} \Big] \\
&= \sum_{1\le i_1\le n_1} (-1)^{\alpha_{i_1}^{\vec{r}_1} F_2^{\vec{r}_2,\vec{s}_2}} \epsilon_{H,w}^{\vec{r}+\vec{e}_{i_1},\vec{s}} \Big[H_1^{\vec{r}_1+\vec{e}_{i_1},\vec{s}_1} \circ \alpha_{i_1}^{\vec{r}_1} \Big] \otimes F_2^{\vec{r}_2,\vec{s}_2} \\
&\qquad + (-1)^{\alpha_{i_1}^{\vec{r}_1} G_2^{\vec{r}_2,\vec{s}_2}} \epsilon_{H,x}^{\vec{r}+\vec{e}_{i_1},\vec{s}} \Big[H_1^{\vec{r}_1+\vec{e}_{i_1},\vec{s}_1} \circ \alpha_{i_1}^{\vec{r}_1} \Big] \otimes G_2^{\vec{r}_2,\vec{s}_2} \\
&\qquad + (-1)^{\alpha_{i_1}^{\vec{r}_1} H_2^{\vec{r}_2,\vec{s}_2}} \epsilon_{H,y}^{\vec{r}+\vec{e}_{i_1},\vec{s}} \Big[F_1^{\vec{r}_1+\vec{e}_{i_1},\vec{s}_1} \circ \alpha_{i_1}^{\vec{r}_1} \Big] \otimes H_2^{\vec{r}_2,\vec{s}_2} \\
&\qquad + (-1)^{\alpha_{i_1}^{\vec{r}_1} H_2^{\vec{r}_2,\vec{s}_2}} \epsilon_{H,z}^{\vec{r}+\vec{e}_{i_1},\vec{s}} \Big[G_1^{\vec{r}_1+\vec{e}_{i_1},\vec{s}_1} \circ \alpha_{i_1}^{\vec{r}_1} \Big] \otimes H_2^{\vec{r}_2,\vec{s}_2} \\
&\quad + \sum_{1\le i_2\le n_2} \epsilon_{A_1\otimes A_2}^{\vec{r}_1,\vec{r}_2,i_2} \epsilon_{H,w}^{\vec{r}+\vec{e}_{i_2},\vec{s}} \quad H_1^{\vec{r}_1,\vec{s}_1} \otimes \Big[F_2^{\vec{r}_2+\vec{e}_{i_2},\vec{s}_2} \circ \alpha_{i_2}^{\vec{r}_2} \Big] \\
&\qquad + \epsilon_{A_1\otimes A_2}^{\vec{r}_1,\vec{r}_2,i_2} \epsilon_{H,x}^{\vec{r}+\vec{e}_{i_2},\vec{s}} \quad H_1^{\vec{r}_1,\vec{s}_1} \otimes \Big[G_2^{\vec{r}_2+\vec{e}_{i_2},\vec{s}_2} \circ \alpha_{i_2}^{\vec{r}_2} \Big] \\
&\qquad + \epsilon_{A_1\otimes A_2}^{\vec{r}_1,\vec{r}_2,i_2} \epsilon_{H,y}^{\vec{r}+\vec{e}_{i_2},\vec{s}} \quad F_1^{\vec{r}_1,\vec{s}_1} \otimes \Big[H_2^{\vec{r}_2+\vec{e}_{i_2},\vec{s}_2} \circ \alpha_{i_2}^{\vec{r}_2} \Big] \\
&\qquad + \epsilon_{A_1\otimes A_2}^{\vec{r}_1,\vec{r}_2,i_2} \epsilon_{H,z}^{\vec{r}+\vec{e}_{i_2},\vec{s}} \quad G_1^{\vec{r}_1,\vec{s}_1} \otimes \Big[H_2^{\vec{r}_2+\vec{e}_{i_2},\vec{s}_2} \circ \alpha_{i_2}^{\vec{r}_2} \Big]
\end{aligned}$$

$$\begin{aligned}
(\beta\circ H)|_{A^{\vec{r}}}^{B^{\vec{s}}} &= \sum_{1\le j\le m} \beta_j^{\vec{s}-\vec{e}_j} \circ H^{\vec{r},\vec{s}-\vec{e}_j} \\
&= \sum_{1\le j_1\le m_1} \Big[\beta_{j_1}^{\vec{s}_1-\vec{e}_{j_1}} \otimes \mathrm{Id} \Big] \circ \\
&\qquad \Big[(\epsilon_{H,w}^{\vec{r},\vec{s}-\vec{e}_{j_1}} H_1^{\vec{r}_1,\vec{s}_1-\vec{e}_{j_1}} \otimes F_2^{\vec{r}_2,\vec{s}_2}) + (\epsilon_{H,x}^{\vec{r},\vec{s}-\vec{e}_{j_1}} H_1^{\vec{r}_1,\vec{s}_1-\vec{e}_{j_1}} \otimes G_2^{\vec{r}_2,\vec{s}_2}) \\
&\qquad + (\epsilon_{H,y}^{\vec{r},\vec{s}-\vec{e}_{j_1}} F_1^{\vec{r}_1,\vec{s}_1-\vec{e}_{j_1}} \otimes H_2^{\vec{r}_2,\vec{s}_2}) + (\epsilon_{H,z}^{\vec{r},\vec{s}-\vec{e}_{j_1}} G_1^{\vec{r}_1,\vec{s}_1-\vec{e}_{j_1}} \otimes H_2^{\vec{r}_2,\vec{s}_2}) \Big]
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{1\le j_2\le m_2} \left[\epsilon_{B_1\otimes B_2}^{\vec{s}_1,\vec{s}_2-\vec{e}_{j_2},j_2} \mathrm{Id}\otimes \beta_{j_2}^{\vec{s}_2-\vec{e}_{j_2}}\right]\circ \\
&\quad \Big[(\epsilon_{H,w}^{\vec{r},\vec{s}-\vec{e}_{j_2}} H_1^{\vec{r}_1,\vec{s}_1}\otimes F_2^{\vec{r}_2,\vec{s}_2-\vec{e}_{j_2}}) + (\epsilon_{H,x}^{\vec{r},\vec{s}-\vec{e}_{j_2}} H_1^{\vec{r}_1,\vec{s}_1}\otimes G_2^{\vec{r}_2,\vec{s}_2-\vec{e}_{j_2}}) \\
&\qquad + (\epsilon_{H,y}^{\vec{r},\vec{s}-\vec{e}_{j_2}} F_1^{\vec{r}_1,\vec{s}_1}\otimes H_2^{\vec{r}_2,\vec{s}_2-\vec{e}_{j_2}}) + (\epsilon_{H,z}^{\vec{r},\vec{s}-\vec{e}_{j_2}} G_1^{\vec{r}_1,\vec{s}_1}\otimes H_2^{\vec{r}_2,\vec{s}_2-\vec{e}_{j_2}})\Big]
\end{aligned}
$$

$$
\begin{aligned}
= &\sum_{1\le j_1\le m_1} \epsilon_{H,w}^{\vec{r},\vec{s}-\vec{e}_{j_1}}\left[\beta_{j_1}^{\vec{s}_1-\vec{e}_{j_1}}\circ H_1^{\vec{r}_1,\vec{s}_1-\vec{e}_{j_1}}\right]\otimes F_2^{\vec{r}_2,\vec{s}_2} \\
&\quad + \epsilon_{H,x}^{\vec{r},\vec{s}-\vec{e}_{j_1}}\left[\beta_{j_1}^{\vec{s}_1-\vec{e}_{j_1}}\circ H_1^{\vec{r}_1,\vec{s}_1-\vec{e}_{j_1}}\right]\otimes G_2^{\vec{r}_2,\vec{s}_2} \\
&\quad + \epsilon_{H,y}^{\vec{r},\vec{s}-\vec{e}_{j_1}}\left[\beta_{j_1}^{\vec{s}_1-\vec{e}_{j_1}}\circ F_1^{\vec{r}_1,\vec{s}_1-\vec{e}_{j_1}}\right]\otimes H_2^{\vec{r}_2,\vec{s}_2} \\
&\quad + \epsilon_{H,z}^{\vec{r},\vec{s}-\vec{e}_{j_1}}\left[\beta_{j_1}^{\vec{s}_1-\vec{e}_{j_1}}\circ G_1^{\vec{r}_1,\vec{s}_1-\vec{e}_{j_1}}\right]\otimes H_2^{\vec{r}_2,\vec{s}_2}
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{1\le j_2\le m_2} (-1)^{\beta_{j_2}^{\vec{s}_2-\vec{e}_{j_2}} H_1^{\vec{r}_1,\vec{s}_1}} \epsilon_{B_1\otimes B_2}^{\vec{s}_1,\vec{s}_2-\vec{e}_{j_2},j_2}\epsilon_{H,w}^{\vec{r},\vec{s}-\vec{e}_{j_2}} \quad H_1^{\vec{r}_1,\vec{s}_1}\otimes\left[\beta_{j_2}^{\vec{s}_2-\vec{e}_{j_2}}\circ F_2^{\vec{r}_2,\vec{s}_2-\vec{e}_{j_2}}\right] \\
&\quad + (-1)^{\beta_{j_2}^{\vec{s}_2-\vec{e}_{j_2}} H_1^{\vec{r}_1,\vec{s}_1}} \epsilon_{B_1\otimes B_2}^{\vec{s}_1,\vec{s}_2-\vec{e}_{j_2},j_2}\epsilon_{H,x}^{\vec{r},\vec{s}-\vec{e}_{j_2}} \quad H_1^{\vec{r}_1,\vec{s}_1}\otimes\left[\beta_{j_2}^{\vec{s}_2-\vec{e}_{j_2}}\circ G_2^{\vec{r}_2,\vec{s}_2-\vec{e}_{j_2}}\right] \\
&\quad + (-1)^{\beta_{j_2}^{\vec{s}_2-\vec{e}_{j_2}} F_1^{\vec{r}_1,\vec{s}_1}} \epsilon_{B_1\otimes B_2}^{\vec{s}_1,\vec{s}_2-\vec{e}_{j_2},j_2}\epsilon_{H,y}^{\vec{r},\vec{s}-\vec{e}_{j_2}} \quad F_1^{\vec{r}_1,\vec{s}_1}\otimes\left[\beta_{j_2}^{\vec{s}_2-\vec{e}_{j_2}}\circ H_2^{\vec{r}_2,\vec{s}_2-\vec{e}_{j_2}}\right] \\
&\quad + (-1)^{\beta_{j_2}^{\vec{s}_2-\vec{e}_{j_2}} G_1^{\vec{r}_1,\vec{s}_1}} \epsilon_{B_1\otimes B_2}^{\vec{s}_1,\vec{s}_2-\vec{e}_{j_2},j_2}\epsilon_{H,z}^{\vec{r},\vec{s}-\vec{e}_{j_2}} \quad G_1^{\vec{r}_1,\vec{s}_1}\otimes\left[\beta_{j_2}^{\vec{s}_2-\vec{e}_{j_2}}\circ H_2^{\vec{r}_2,\vec{s}_2-\vec{e}_{j_2}})\right]
\end{aligned}
$$

Similarly to the proof in the case of induced morphisms, we split the equation $F-G=\beta\circ H+H\circ\alpha$ into cases depending on the values of $\vec{r}$ and $\vec{s}$, using the fact that morphisms of complexes preserve the degree and homotopies reduce it by one.

Case 1. When $|\vec{r}_1|=|\vec{s}_1|$ and $|\vec{r}_2|=|\vec{s}_2|$, we need to check that the following expression is equal to $(\epsilon_F^{\vec{r}_1,\vec{r}_2}F_1^{\vec{r}_1,\vec{s}_1}\otimes F_2^{\vec{r}_2,\vec{s}_2})-(\epsilon_G^{\vec{r}_1,\vec{r}_2}G_1^{\vec{r}_1,\vec{s}_1}\otimes G_2^{\vec{r}_2,\vec{s}_2})$:

$$
\begin{aligned}
(*) = F_1^{\vec{r}_1,\vec{s}_1}\otimes &\Bigg[\sum_{1\le i_2\le n_2}\epsilon_{H,y}^{\vec{r}+\vec{e}_{i_2},\vec{s}}\epsilon_{A_1\otimes A_2}^{\vec{r}_1,\vec{r}_2,i_2}(H_2^{\vec{r}_2+\vec{e}_{i_2},\vec{s}_2}\circ\alpha_{i_2}^{\vec{r}_2}) \\
&+\sum_{1\le j_2\le m_2}(-1)^{\beta_{j_2}^{\vec{s}_2-\vec{e}_{j_2}}\cdot F_1^{\vec{r}_1,\vec{s}_1}}\epsilon_{B_1\otimes B_2}^{\vec{s}_1,\vec{s}_2-\vec{e}_{j_2},j_2}\epsilon_{H,y}^{\vec{r},\vec{s}-\vec{e}_{j_2}}(\beta_{j_2}^{\vec{s}_2-\vec{e}_{j_2}}\circ H_2^{\vec{r}_2,\vec{s}_2-\vec{e}_2})\Bigg] \\
+\Bigg[&\sum_{1\le i_1\le n_1}(-1)^{F_2^{\vec{r}_2,\vec{s}_2}\cdot\alpha_{i_1}^{\vec{r}_1}}\epsilon_{H,w}^{\vec{r}+\vec{e}_{i_1},\vec{s}}(H_1^{\vec{r}_1+\vec{e}_{i_1},\vec{s}_1}\circ\alpha_{i_1}^{\vec{r}_1}) \\
&+\sum_{1\le j_1\le m_1}\epsilon_{H,w}^{\vec{r},\vec{s}-\vec{e}_{j_1}}(\beta_{j_1}^{\vec{s}_1-\vec{e}_{j_1}}\circ H_1^{\vec{r}_1,\vec{s}_1-\vec{e}_{j_1}})\Bigg]\otimes F_2^{\vec{r}_2,\vec{s}_2}
\end{aligned}
$$

$$
\begin{aligned}
&+G_1^{\vec{r}_1,\vec{s}_1} \otimes \Bigg[\sum_{1\le i_2\le n_2} \epsilon_{H,z}^{\vec{r}+\vec{e}_{i_2},\vec{s}} \epsilon_{A_1\otimes A_2}^{\vec{r}_1,\vec{r}_2,i_2} (H_2^{\vec{r}_2+\vec{e}_{i_2},\vec{s}_2} \circ \alpha_{i_2}^{\vec{r}_2}) \\
&\qquad + \sum_{1\le j_2\le m_2} (-1)^{\beta_{j_2}^{\vec{s}_2-\vec{e}_{j_2}} \cdot G_1^{\vec{r}_1,\vec{s}_1}} \epsilon_{B_1\otimes B_2}^{\vec{s}_1,\vec{s}_2-\vec{e}_{j_2},j_2} \epsilon_{H,z}^{\vec{r},\vec{s}-\vec{e}_{j_2}} (\beta_{j_2}^{\vec{s}_2-\vec{e}_{j_2}} \circ H_2^{\vec{r}_2,\vec{s}_2-\vec{e}_{j_2}}) \Bigg] \\
&+\Bigg[\sum_{1\le j_1\le m_1} \epsilon_{H,x}^{\vec{r},\vec{s}-\vec{e}_{j_1}} (\beta_{j_1}^{\vec{s}_1-\vec{e}_{j_1}} \circ H_1^{\vec{r}_1,\vec{s}_1-\vec{e}_{j_1}}) \\
&\qquad + \sum_{1\le i_1\le n_1} (-1)^{G_2^{\vec{r}_2,\vec{s}_2}\cdot \alpha_{i_1}^{\vec{r}_1}} \epsilon_{H,x}^{\vec{r}+\vec{e}_{i_1},\vec{s}} (H_1^{\vec{r}_1+\vec{e}_{i_1},\vec{s}_1} \circ \alpha_{i_1}^{\vec{r}_1}) \Bigg] \otimes G_2^{\vec{r}_2,\vec{s}_2}
\end{aligned}
$$

We can simplify each bracket using homotopy relations, assuming the signs are equal when the corresponding morphisms are non-zero. To check that, we can choose parity similarly to the proof of induced morphisms, and assume equality of the parities of the morphisms involved.

- Consider the first bracket $[\ldots]$. We can choose parity and set

$$
s_y \equiv h_2^{\vec{r}_2+\vec{e}_{i_2},\vec{s}_2} + \alpha_{i_2}^{\vec{r}_2} \equiv \beta_{j_2}^{\vec{s}_2-\vec{e}_{j_2}} + h_2^{\vec{r}_2,\vec{s}_2-\vec{e}_{j_2}}.
$$

Then:

$$
\begin{aligned}
\sigma_{h,y}^{\vec{r}+\vec{e}_{i_2},\vec{s}} + \sigma_{A_1\otimes A_2}^{\vec{r}_1,\vec{r}_2,i_2} &\equiv |\vec{r}_1| + |\beta_2|\,(\vec{s}_2)\cdot(f_1^{\vec{r}_1,\vec{s}_1} + |\alpha_1|\,(\vec{r}_1) + |\beta_1|\,(\vec{s}_1)) \\
&\quad + h_2^{\vec{r}_2+\vec{e}_{i_2},\vec{s}_2}\cdot|\alpha_1|\,(\vec{r}_1) + |\vec{r}_1| + \alpha_{i_2}^{r_{i_2}}\cdot|\alpha_1|\,(\vec{r}_1) \\
&\equiv |\beta_2|\,(\vec{s}_2)\cdot(f_1^{\vec{r}_1,\vec{s}_1} + |\alpha_1|\,(\vec{r}_1) + |\beta_1|\,(\vec{s}_1)) + s_y\cdot|\alpha_1|\,(\vec{r}_1),
\end{aligned}
$$

$$
\begin{aligned}
&\beta_{j_2}^{\vec{s}_2-\vec{e}_{j_2}}\cdot f_1^{\vec{r}_1,\vec{s}_1} + \sigma_{B_1\otimes B_2}^{\vec{s}_1,\vec{s}_2-\vec{e}_{j_2},j_2} + \sigma_{h,y}^{\vec{r},\vec{s}-\vec{e}_{j_2}} \\
&\quad \equiv \beta_{j_2}^{s_{j_2}-1}\cdot f_1^{\vec{r}_1,\vec{s}_1} + |\vec{s}_1| + \beta_{j_2}^{s_{j_2}-1}\cdot|\beta_1|\,(\vec{s}_1) \\
&\qquad + |\vec{r}_1| + |\beta_2|\,(\vec{s}_2-\vec{e}_{j_2})\cdot(f_1^{\vec{r}_1,\vec{s}_1} + |\alpha_1|\,(\vec{r}_1) + |\beta_1|\,(\vec{s}_1)) + h_2^{\vec{r}_2,\vec{s}_2-\vec{e}_{j_2}}\cdot|\alpha_1|\,(\vec{r}_1) \\
&\quad \equiv |\beta_2|\,(\vec{s}_2)\cdot(f_1^{\vec{r}_1,\vec{s}_1} + |\alpha_1|\,(\vec{r}_1) + |\beta_1|\,(\vec{s}_1)) + s_y\cdot|\alpha_1|\,(\vec{r}_1).
\end{aligned}
$$

The two quantities are equal and independent of i_2/j_2. We call it λ_y. Observe that since we chose parity, by the homotopy relation of h_2 we get $s_y \equiv f_2^{\vec{r}_2,\vec{s}_2} \equiv g_2^{\vec{r}_2,\vec{s}_2}$, that is,

$$
\begin{aligned}
\lambda_y &\equiv |\beta_2|\,(\vec{s}_2)\cdot(f_1^{\vec{r}_1,\vec{s}_1} + |\alpha_1|\,(\vec{r}_1) + |\beta_1|\,(\vec{s}_1)) + f_2^{\vec{r}_2,\vec{s}_2}\cdot|\alpha_1|\,(\vec{r}_1) \\
&\equiv |\beta_2|\,(\vec{s}_2)\cdot(f_1^{\vec{r}_1,\vec{s}_1} + |\alpha_1|\,(\vec{r}_1) + |\beta_1|\,(\vec{s}_1)) + g_2^{\vec{r}_2,\vec{s}_2}\cdot|\alpha_1|\,(\vec{r}_1).
\end{aligned}
$$

This shows that the first bracket is

$$
[\ldots] = (-1)^{\lambda_y}(f_2^{\vec{r}_2,\vec{s}_2} - g_2^{\vec{r}_2,\vec{s}_2}) = (-1)^{\lambda_y(f_2^{\vec{r}_2,\vec{s}_2})} f_2^{\vec{r}_2,\vec{s}_2} - (-1)^{\lambda_y(g_2^{\vec{r}_2,\vec{s}_2})} g_2^{\vec{r}_2,\vec{s}_2}.
$$

Similar argument can be given for the three other brackets. We only give the calculations.

- Setting $s_w \equiv h_1^{\vec{r}_1+\vec{e}_{i_1},\vec{s}_1} + \alpha_{i_1}^{\vec{r}_1} \equiv \beta_{j_1}^{\vec{s}_1-\vec{e}_{j_1}} + h_1^{\vec{r}_1,\vec{s}_1-\vec{e}_{j_1}}$:

$$\begin{aligned} f_2^{\vec{r}_2,\vec{s}_2} \cdot \alpha_{i_1}^{\vec{r}_1} + \sigma_{h,w}^{\vec{r}+\vec{e}_{i_1},\vec{s}} &\equiv f_2^{\vec{r}_2,\vec{s}_2} \cdot \alpha_{i_1}^{r_{i_1}} + f_2^{\vec{r}_2,\vec{s}_2} \cdot |\alpha_1|(\vec{r}_1 + \vec{e}_{i_1}) \\ &\quad + |\beta_2|(\vec{s}_2) \cdot (h_1^{\vec{r}_1+\vec{e}_{i_1},\vec{s}_1} + |\alpha_1|(\vec{r}_1 + \vec{e}_{i_1}) + |\beta_1|(\vec{s}_1)) \\ &\equiv f_2^{\vec{r}_2,\vec{s}_2} \cdot |\alpha_1|(\vec{r}_1) + |\beta_2|(\vec{s}_2) \cdot (s_w + |\alpha_1|(\vec{r}_1) + |\beta_1|(\vec{s}_1)), \end{aligned}$$

$$\begin{aligned} \sigma_{h,w}^{\vec{r},\vec{s}-\vec{e}_{j_1}} &\equiv f_2^{\vec{r}_2,\vec{s}_2} \cdot |\alpha_1|(\vec{r}_1) + |\beta_2|(\vec{s}_2) \cdot (h_1^{\vec{r}_1,\vec{s}_1-\vec{e}_{j_1}} + |\alpha_1|(\vec{r}_1) + |\beta_1|(\vec{s}_1 - \vec{e}_{j_1})) \\ &\equiv f_2^{\vec{r}_2,\vec{s}_2} \cdot |\alpha_1|(\vec{r}_1) + |\beta_2|(\vec{s}_2) \cdot (s_w + |\alpha_1|(\vec{r}_1) + |\beta_1|(\vec{s}_1)). \end{aligned}$$

  The two quantities are equal and we call it λ_w, where $s_w \equiv f_1^{\vec{r}_1,\vec{s}_1} \equiv g_1^{\vec{r}_1,\vec{s}_1}$.

- Setting $s_z \equiv h_2^{\vec{r}_2+\vec{e}_{i_2},\vec{s}_2} + \alpha_{i_2}^{\vec{r}_2} \equiv \beta_{j_2}^{\vec{s}_2-\vec{e}_{j_2}} + h_2^{\vec{r}_2,\vec{s}_2-\vec{e}_{j_2}}$:

$$\begin{aligned} \sigma_{h,z}^{\vec{r}+\vec{e}_{i_2},\vec{s}} + \sigma_{A_1\otimes A_2}^{\vec{r}_1,\vec{r}_2,i_2} &\equiv |\vec{r}_1| + |\beta_2|(\vec{s}_2) \cdot (g_1^{\vec{r}_1,\vec{s}_1} + |\alpha_1|(\vec{r}_1) + |\beta_1|(\vec{s}_1)) \\ &\quad + h_2^{\vec{r}_2+\vec{e}_{i_2},\vec{s}_2} \cdot |\alpha_1|(\vec{r}_1) + |\vec{r}_1| + \alpha_{i_2}^{r_{i_2}} \cdot |\alpha_1|(\vec{r}_1) \\ &\equiv |\beta_2|(\vec{s}_2) \cdot (g_1^{\vec{r}_1,\vec{s}_1} + |\alpha_1|(\vec{r}_1) + |\beta_1|(\vec{s}_1)) + s_z \cdot |\alpha_1|(\vec{r}_1) \end{aligned}$$

$$\begin{aligned} &\beta_{j_2}^{\vec{s}_2-\vec{e}_{j_2}} \cdot g_1^{\vec{r}_1,\vec{s}_1} + \sigma_{B_1\otimes B_2}^{\vec{s}_1,\vec{s}_2-\vec{e}_{j_2},j_2} + \sigma_{h,z}^{\vec{r},\vec{s}-\vec{e}_{j_2}} \\ &\quad \equiv \beta_{j_2}^{s_{j_2}-1} \cdot g_1^{\vec{r}_1,\vec{s}_1} + |\vec{s}_1| + \beta_{j_2}^{s_{j_2}-1} \cdot |\beta_1|(\vec{s}_1) + |\vec{r}_1| \\ &\qquad + |\beta_2|(\vec{s}_2 - \vec{e}_{j_2}) \cdot (g_1^{\vec{r}_1,\vec{s}_1} + |\alpha_1|(\vec{r}_1) + |\beta_1|(\vec{s}_1)) + h_2^{\vec{r}_2,\vec{s}_2-\vec{e}_{j_2}} \cdot |\alpha_1|(\vec{r}_1) \\ &\quad \equiv + |\beta_2|(\vec{s}_2) \cdot (g_1^{\vec{r}_1,\vec{s}_1} + |\alpha_1|(\vec{r}_1) + |\beta_1|(\vec{s}_1)) + s_z \cdot |\alpha_1|(\vec{r}_1) \end{aligned}$$

  The two quantities are equal and we call it λ_z, where $s_z \equiv f_2^{\vec{r}_2,\vec{s}_2} \equiv g_2^{\vec{r}_2,\vec{s}_2}$.

- Setting $s_x \equiv \beta_{j_1}^{\vec{s}_1-\vec{e}_{j_1}} + h_1^{\vec{r}_1,\vec{s}_1-\vec{e}_{j_1}} \equiv h_1^{\vec{r}_1+\vec{e}_{i_1},\vec{s}_1} + \alpha_{i_1}^{\vec{r}_1}$:

$$\begin{aligned} \sigma_{h,x}^{\vec{r},\vec{s}-\vec{e}_{j_1}} &\equiv |\beta_2|(\vec{s}_2) \cdot (h_1^{\vec{r}_1,\vec{s}_1-\vec{e}_{j_1}} + |\alpha_1|(\vec{r}_1) + |\beta_1|(\vec{s}_1 - \vec{e}_{j_1})) + g_2^{\vec{r}_2,\vec{s}_2} \cdot |\alpha_1|(\vec{r}_1) \\ &\equiv |\beta_2|(\vec{s}_2) \cdot (s_x + |\alpha_1|(\vec{r}_1) + |\beta_1|(\vec{s}_1)) + g_2^{\vec{r}_2,\vec{s}_2} \cdot |\alpha_1|(\vec{r}_1), \end{aligned}$$

$$\begin{aligned} g_2^{\vec{r}_2,\vec{s}_2} \cdot \alpha_{i_1}^{\vec{r}_1} + \sigma_{h,x}^{\vec{r}+\vec{e}_{i_1},\vec{s}} &\equiv g_2^{\vec{r}_2,\vec{s}_2} \cdot \alpha_{i_1}^{r_{i_1}} + g_2^{\vec{r}_2,\vec{s}_2} \cdot |\alpha_1|(\vec{r}_1 + \vec{e}_{i_1}) \\ &\quad + |\beta_2|(\vec{s}_2) \cdot (h_1^{\vec{r}_1+\vec{e}_{i_1},\vec{s}_1} + |\alpha_1|(\vec{r}_1 + \vec{e}_{i_1}) + |\beta_1|(\vec{s}_1)) \\ &\equiv g_2^{\vec{r}_2,\vec{s}_2} \cdot |\alpha_1|(\vec{r}_1) + |\beta_2|(\vec{s}_2) \cdot (s_x + |\alpha_1|(\vec{r}_1) + |\beta_1|(\vec{s}_1)). \end{aligned}$$

  The two quantities are equal and we call it λ_x, where $s_x \equiv f_1^{\vec{r}_1,\vec{s}_1} \equiv g_1^{\vec{r}_1,\vec{s}_1}$.

We see that $\lambda :\equiv \lambda_w \equiv \lambda_x \equiv \lambda_y \equiv \lambda_z \equiv \sigma_f^{\vec{r}_1,\vec{s}_2} \equiv \sigma_g^{\vec{r}_1,\vec{s}_2}$! Thus

$$\begin{aligned}(*) &= \frac{1}{2}(-1)^{\lambda_w}(f_1^{\vec{r}_1,\vec{s}_1} - g_1^{\vec{r}_1,\vec{s}_1}) \otimes f_2^{\vec{r}_2,\vec{s}_2} + \frac{1}{2}(-1)^{\lambda_x}(f_1^{\vec{r}_1,\vec{s}_1} - g_1^{\vec{r}_1,\vec{s}_1}) \otimes g_2^{\vec{r}_2,\vec{s}_2} \\ &\quad + \frac{1}{2}(-1)^{\lambda_y} f_1^{\vec{r}_1,\vec{s}_1} \otimes (f_2^{\vec{r}_2,\vec{s}_2} - g_2^{\vec{r}_2,\vec{s}_2}) + \frac{1}{2}(-1)^{\lambda_z} g_1^{\vec{r}_1,\vec{s}_1} \otimes (f_2^{\vec{r}_2,\vec{s}_2} - g_2^{\vec{r}_2,\vec{s}_2}) \\ &= (\epsilon_f^{\vec{r}_1,\vec{r}_2} f_1^{\vec{r}_1,\vec{s}_1} \otimes f_2^{\vec{r}_2,\vec{s}_2}) - (\epsilon_g^{\vec{r}_1,\vec{r}_2} g_1^{\vec{r}_1,\vec{s}_1} \otimes g_2^{\vec{r}_2,\vec{s}_2}).\end{aligned}$$

Case 2. When $|\vec{r}_1| = |\vec{s}_1| - 1$ and $|\vec{r}_2| = |\vec{s}_2| + 1$, we need to prove that

$$\begin{aligned}0 = &\Bigg[\sum_{1\le i_1\le n_1} (-1)^{H_2^{\vec{r}_2,\vec{s}_2}\cdot\alpha_{i_1}^{\vec{r}_1}} \epsilon_{H,y}^{\vec{r}+\vec{e}_{i_1},\vec{s}} (F_1^{\vec{r}_1+\vec{e}_{i_1},\vec{s}_1} \circ \alpha_{i_1}^{\vec{r}_1}) \\ &\quad + \sum_{1\le j_1\le m_1} \epsilon_{H,y}^{\vec{r},\vec{s}-\vec{e}_{j_1}} (\beta_{j_1}^{\vec{s}_1-\vec{e}_{j_1}} \circ F_1^{\vec{r}_1,\vec{s}_1-\vec{e}_{j_1}})\Bigg] \otimes H_2^{\vec{r}_2,\vec{s}_2} \\ &+ \Bigg[\sum_{1\le i_1\le n_1} (-1)^{H_2^{\vec{r}_2,\vec{s}_2}\cdot\alpha_{i_1}^{\vec{r}_1}} \epsilon_{H,z}^{\vec{r}+\vec{e}_{i_1},\vec{s}} (G_1^{\vec{r}_1+\vec{e}_{i_1},\vec{s}_1} \circ \alpha_{i_1}^{\vec{r}_1}) \\ &\quad + \sum_{1\le j_1\le m_1} \epsilon_{H,z}^{\vec{r},\vec{s}-\vec{e}_{j_1}} (\beta_{j_1}^{\vec{s}_1-\vec{e}_{j_1}} \circ G_1^{\vec{r}_1,\vec{s}_1-\vec{e}_{j_1}})\Bigg] \otimes H_2^{\vec{r}_2,\vec{s}_2}\end{aligned}$$

Again, we can use the morphisms relations of F_1 and G_1 and prove that the right signs are equal by choosing parity:

- Consider the first bracket. We need to prove that
  - $(-1)^{H_2^{\vec{r}_2,\vec{s}_2}\cdot\alpha_{i_1}^{\vec{r}_1}} \epsilon_{H,y}^{\vec{r}+\vec{e}_{i_1},\vec{s}}$ is independent of i_1 whenever $F_1^{\vec{r}_1+\vec{e}_{i_1},\vec{s}_1} \circ \alpha_{i_1}^{\vec{r}_1} \neq 0$;
  - $\epsilon_{H,y}^{\vec{r},\vec{s}-\vec{e}_{j_1}}$ is independent of j_1 whenever $\beta_{j_1}^{\vec{s}_1-\vec{e}_{j_1}} \circ F_1^{\vec{r}_1,\vec{s}_1-\vec{e}_{j_1}} \neq 0$;
  - and $(-1)^{H_2^{\vec{r}_2,\vec{s}_2}\cdot\alpha_{i_1}^{\vec{r}_1}} \epsilon_{H,y}^{\vec{r}+\vec{e}_{i_1},\vec{s}} = -\epsilon_{H,y}^{\vec{r},\vec{s}-\vec{e}_{j_1}}$ whenever $G_1^{\vec{r}_1+\vec{e}_{i_1},\vec{s}_1} \circ \alpha_{i_1}^{\vec{r}_1} \neq 0$ and $\beta_{j_1}^{\vec{s}_1-\vec{e}_{j_1}} \circ F_1^{\vec{r}_1,\vec{s}_1-\vec{e}_{j_1}} \neq 0$.

  Choosing parity and setting $s \equiv f_1^{\vec{r}_1+\vec{e}_{i_1},\vec{s}_1} + \alpha_{i_1}^{\vec{r}_1} \equiv \beta_{j_1}^{\vec{s}_1-\vec{e}_{j_1}} + f_1^{\vec{r}_1,\vec{s}_1-\vec{e}_{j_1}}$, we indeed check that:

$$\begin{aligned}h_2^{\vec{r}_2,\vec{s}_2}\cdot\alpha_{i_1}^{\vec{r}_1} + \sigma_{h,y}^{\vec{r}+\vec{e}_{i_1},\vec{s}} &\equiv h_2^{\vec{r}_2,\vec{s}_2}\cdot\alpha_{i_1}^{r_{i_1}} + |\vec{r}_1+\vec{e}_{i_1}| + h_2^{\vec{r}_2,\vec{s}_2}\cdot|\alpha_1|(\vec{r}_1+\vec{e}_{i_1}) \\ &\quad + |\beta_2|(\vec{s}_2)\cdot(f_1^{\vec{r}_1+\vec{e}_{i_1},\vec{s}_1} + |\alpha_1|(\vec{r}_1+\vec{e}_{i_1}) + |\beta_1|(\vec{s}_1)) \\ &\equiv +|\vec{r}_1| + 1 + h_2^{\vec{r}_2,\vec{s}_2}\cdot|\alpha_1|(\vec{r}_1) \\ &\quad + |\beta_2|(\vec{s}_2)\cdot(s + |\alpha_1|(\vec{r}_1) + |\beta_1|(\vec{s}_1)),\end{aligned}$$

$$\sigma_{h,y}^{\vec{r},\vec{s}-\vec{e}_{j_1}} \equiv |\vec{r}_1| + h_2^{\vec{r}_2,\vec{s}_2}\cdot|\alpha_1|(\vec{r}_1) + |\beta_2|(\vec{s}_2)\cdot(f_1^{\vec{r}_1,\vec{s}_1-\vec{e}_{j_1}} + |\alpha_1|(\vec{r}_1) + |\beta_1|(\vec{s}_1-\vec{e}_{j_1}))$$

$$\equiv |\vec{r}_1| + h_2^{\vec{r}_2,\vec{s}_2} \cdot |\alpha_1|(\vec{r}_1) + |\beta_2|(\vec{s}_2) \cdot (s + |\alpha_1|(\vec{r}_1) + |\beta_1|(\vec{s}_1)).$$

The other bracket being similar, we only give the calculations.

- Setting $s \equiv g_1^{\vec{r}_1+\vec{e}_{i_1},\vec{s}_1} + \alpha_{i_1}^{\vec{r}_1} \equiv \beta_{j_1}^{\vec{s}_1-\vec{e}_{j_1}} \circ g_1^{\vec{r}_1,\vec{s}_1-\vec{e}_{j_1}}$:

$$\begin{aligned} h_2^{\vec{r}_2,\vec{s}_2} \cdot \alpha_{i_1}^{\vec{r}_1} + \sigma_{h,z}^{\vec{r}+\vec{e}_{i_1},\vec{s}} &\equiv h_2^{\vec{r}_2,\vec{s}_2} \cdot \alpha_{i_1}^{r_{i_1}} + |\vec{r}_1 + \vec{e}_{i_1}| + h_2^{\vec{r}_2,\vec{s}_2} \cdot |\alpha_1|(\vec{r}_1 + \vec{e}_{i_1}) \\ &\quad + |\beta_2|(\vec{s}_2) \cdot (g_1^{\vec{r}_1+\vec{e}_{i_1},\vec{s}_1} + |\alpha_1|(\vec{r}_1 + \vec{e}_{i_1}) + |\beta_1|(\vec{s}_1)) \\ &\equiv + |\vec{r}_1| + 1 + h_2^{\vec{r}_2,\vec{s}_2} \cdot |\alpha_1|(\vec{r}_1) \\ &\quad + |\beta_2|(\vec{s}_2) \cdot (s + |\alpha_1|(\vec{r}_1) + |\beta_1|(\vec{s}_1)) \end{aligned}$$

$$\begin{aligned} \sigma_{h,z}^{\vec{r},\vec{s}-\vec{e}_{j_1}} &\equiv |\vec{r}_1| + |\beta_2|(\vec{s}_2) \cdot (g_1^{\vec{r}_1,\vec{s}_1-\vec{e}_{j_1}} + |\alpha_1|(\vec{r}_1) + |\beta_1|(\vec{s}_1 - \vec{e}_{j_1})) + h_2^{\vec{r}_2,\vec{s}_2} \cdot |\alpha_1|(\vec{r}_1) \\ &\equiv |\vec{r}_1| + |\beta_2|(\vec{s}_2) \cdot (s + |\alpha_1|(\vec{r}_1) + |\beta_1|(\vec{s}_1)) + h_2^{\vec{r}_2,\vec{s}_2} \cdot |\alpha_1|(\vec{r}_1) \end{aligned}$$

Case 3. When $|\vec{r}_1| = |\vec{s}_1| + 1$ and $|\vec{r}_2| = |\vec{s}_2| - 1$, we need to prove that

$$\begin{aligned} 0 = H_1^{\vec{r}_1,\vec{s}_1} \otimes \Bigg[&\sum_{1\le i_2\le n_2} \epsilon_{H,w}^{\vec{r}+\vec{e}_{i_2},\vec{s}} \epsilon_{A_1\otimes A_2}^{\vec{r}_1,\vec{r}_2,i_2} (F_2^{\vec{r}_2+\vec{e}_{i_2},\vec{s}_2} \circ \alpha_{i_2}^{\vec{r}_2}) \\ &+ \sum_{1\le j_2\le m_2} (-1)^{\beta_{i_2}^{s_{j_2}-1} \cdot H_1^{\vec{r}_1,\vec{s}_1}} \epsilon_{B_1\otimes B_2}^{\vec{s}_1,\vec{s}_2-\vec{e}_{j_2},j_2} \epsilon_{H,w}^{\vec{r},\vec{s}-\vec{e}_{j_2}} (\beta_{j_2}^{\vec{s}_2-\vec{e}_{j_2}} \circ F_2^{\vec{r}_2,\vec{s}_2-\vec{e}_{j_2}}) \Bigg] \\ + H_1^{\vec{r}_1,\vec{s}_1} \otimes \Bigg[&\sum_{1\le i_2\le n_2} \epsilon_{H,x}^{\vec{r}+\vec{e}_{i_2},\vec{s}} \epsilon_{A_1\otimes A_2}^{\vec{r}_1,\vec{r}_2,i_2} (G_2^{\vec{r}_2+\vec{e}_{i_2},\vec{s}_2} \circ \alpha_{i_2}^{\vec{r}_2}) \\ &+ \sum_{1\le j_2\le m_2} (-1)^{\beta_{i_2}^{s_{j_2}-1} \cdot H_1^{\vec{r}_1,\vec{s}_1}} \epsilon_{B_1\otimes B_2}^{\vec{s}_1,\vec{s}_2-\vec{e}_{j_2},j_2} \epsilon_{H,x}^{\vec{r},\vec{s}-\vec{e}_{j_2}} (\beta_{j_2}^{\vec{s}_2-\vec{e}_{j_2}} \circ G_2^{\vec{r}_2,\vec{s}_2-\vec{e}_{j_2}}) \Bigg] \end{aligned}$$

This is similar to the previous case, and therefore we only give the calculations.

- Setting $s \equiv f_2^{\vec{r}_2+\vec{e}_{i_2},\vec{s}_2} + \alpha_{i_2}^{\vec{r}_2} \equiv \beta_{j_2}^{\vec{s}_2-\vec{e}_{j_2}} + f_2^{\vec{r}_2,\vec{s}_2-\vec{e}_{j_2}}$ and recalling that $|\vec{r}_1| = |\vec{s}_1| + 1$:

$$\begin{aligned} \sigma_{h,w}^{\vec{r}+\vec{e}_{i_2},\vec{s}} + \sigma_{A_1\otimes A_2}^{\vec{r}_1,\vec{r}_2,i_2} &\equiv |\beta_2|(\vec{s}_2) \cdot (h_1^{\vec{r}_1,\vec{s}_1} + |\alpha_1|(\vec{r}_1) + |\beta_1|(\vec{s}_1)) \\ &\quad + f_2^{\vec{r}_2+\vec{e}_{i_2},\vec{s}_2} \cdot |\alpha_1|(\vec{r}_1) + |\vec{r}_1| + \alpha_{i_2}^{r_{i_2}} \cdot |\alpha_1|(\vec{r}_1) \\ &\equiv |\beta_2|(\vec{s}_2) \cdot (h_1^{\vec{r}_1,\vec{s}_1} + |\alpha_1|(\vec{r}_1) + |\beta_1|(\vec{s}_1)) + |\vec{r}_1| + s \cdot |\alpha_1|(\vec{r}_1), \end{aligned}$$

$$\begin{aligned} &\beta_{i_2}^{s_{j_2}-1} \cdot h_1^{\vec{r}_1,\vec{s}_1} + \sigma_{B_1\otimes B_2}^{\vec{s}_1,\vec{s}_2-\vec{e}_{j_2},j_2} + \sigma_{h,w}^{\vec{r},\vec{s}-\vec{e}_{j_2}} \\ &\quad \equiv \beta_{i_2}^{s_{j_2}-1} \cdot h_1^{\vec{r}_1,\vec{s}_1} + |\vec{s}_1| + \beta_{j_2}^{s_{j_2}-1} |\beta|(\vec{s}_1) \\ &\quad\quad + |\beta_2|(\vec{s}_2) \cdot (h_1^{\vec{r}_1,\vec{s}_1} + |\alpha_1|(\vec{r}_1) + |\beta_1|(\vec{s}_1)) + f_2^{\vec{r}_2,\vec{s}_2} \cdot |\alpha_1|(\vec{r}_1) \\ &\quad \equiv |\vec{s}_1| + |\beta_2|(\vec{s}_2) \cdot (h_1^{\vec{r}_1,\vec{s}_1} + |\alpha_1|(\vec{r}_1) + |\beta_1|(\vec{s}_1)) + s \cdot |\alpha_1|(\vec{r}_1). \end{aligned}$$

- Setting $s \equiv g_2^{\vec{r}_2+\vec{e}_{i_2},\vec{s}_2} + \alpha_{i_2}^{\vec{r}_2} \equiv \beta_{j_2}^{\vec{s}_2-\vec{e}_{j_2}} \circ g_2^{\vec{r}_2,\vec{s}_2-\vec{e}_{j_2}}$ and recalling that $|\vec{r}_1| = |\vec{s}_1| + 1$:

$$\begin{aligned}\sigma_{h,x}^{\vec{r}+\vec{e}_{i_2},\vec{s}} + \sigma_{A_1\otimes A_2}^{\vec{r}_1,\vec{r}_2,i_2} &\equiv |\beta_2|(\vec{s}_2)\cdot(h_1^{\vec{r}_1,\vec{s}_1} + |\alpha_1|(\vec{r}_1) + |\beta_1|(\vec{s}_1)) \\ &\quad + g_2^{\vec{r}_2+\vec{e}_{i_2},\vec{s}_2}\cdot|\alpha_1|(\vec{r}_1) + |\vec{r}_1| + \alpha_{i_2}^{r_{i_2}}\cdot|\alpha_1|(\vec{r}_1) \\ &\equiv |\vec{r}_1| + |\beta_2|(\vec{s}_2)\cdot(h_1^{\vec{r}_1,\vec{s}_1} + |\alpha_1|(\vec{r}_1) + |\beta_1|(\vec{s}_1)) + s\cdot|\alpha_1|(\vec{r}_1),\end{aligned}$$

$$\begin{aligned}&\beta_{i_2}^{s_{j_2}-1}\cdot h_1^{\vec{r}_1,\vec{s}_1} + \sigma_{B_1\otimes B_2}^{\vec{s}_1,\vec{s}_2-\vec{e}_{j_2},j_2} + \sigma_{h,x}^{\vec{r},\vec{s}-\vec{e}_{j_2}} \\ &\quad\equiv \beta_{i_2}^{s_{j_2}-1}\cdot h_1^{\vec{r}_1,\vec{s}_1} + |\vec{s}_1| + \beta_{j_2}^{s_{j_2}-1}\cdot|\beta_1|(\vec{s}_1) \\ &\quad\quad + |\beta_2|(\vec{s}_2-\vec{e}_{j_2})\cdot(h_1^{\vec{r}_1,\vec{s}_1} + |\alpha_1|(\vec{r}_1) + |\beta_1|(\vec{s}_1)) + g_2^{\vec{r}_2,\vec{s}_2-\vec{e}_{j_2}}\cdot|\alpha_1|(\vec{r}_1) \\ &\quad\equiv |\vec{s}_1| + |\beta_2|(\vec{s}_2)\cdot(h_1^{\vec{r}_1,\vec{s}_1} + |\alpha_1|(\vec{r}_1) + |\beta_1|(\vec{s}_1)) + s\cdot|\alpha_1|(\vec{r}_1).\end{aligned}$$

□

Bibliography

[1] D. Bar-Natan. “Fast Khovanov homology computations”. In: *J. Knot Theory Ramifications* 16.3 (2007), pp. 243–255. ISSN: 0218-2165. DOI: 10.1142/S0218216507005294.

[2] D. Bar-Natan. “Khovanov’s homology for tangles and cobordisms”. In: *Geom. Topol.* 9 (2005), pp. 1443–1499. ISSN: 1465-3060. DOI: 10.2140/gt.2005.9.1443.

[3] F. Borceux. *Handbook of categorical algebra. 2.* Vol. 51. Encyclopedia of Mathematics and its Applications. Categories and structures. Cambridge University Press, Cambridge, 1994, pp. xviii+443. ISBN: 0-521-44179-X.

[4] J. Brundan and A. P. Ellis. “Monoidal supercategories”. In: *Comm. Math. Phys.* 351.3 (2017), pp. 1045–1089. ISSN: 0010-3616. DOI: 10.1007/s00220-017-2850-9.

[5] J. Brundan and A. P. Ellis. “Super Kac-Moody 2-categories”. In: *Proc. Lond. Math. Soc. (3)* 115.5 (2017), pp. 925–973. ISSN: 0024-6115. DOI: 10.1112/plms.12055.

[6] M. Khovanov. “A categorification of the Jones polynomial”. In: *Duke Math. J.* 101.3 (2000), pp. 359–426. ISSN: 0012-7094. DOI: 10.1215/S0012-7094-00-10131-7.

[7] A. D. Lauda. “An introduction to diagrammatic algebra and categorified quantum $\mathfrak{sl}_2$”. In: *Bull. Inst. Math. Acad. Sin. (N.S.)* 7.2 (2012), pp. 165–270. ISSN: 2304-7909.

[8] A. D. Lauda, H. Queffelec, and D. E. V. Rose. “Khovanov homology is a skew Howe 2-representation of categorified quantum $\mathfrak{sl}_m$”. In: *Algebr. Geom. Topol.* 15.5 (2015), pp. 2517–2608. ISSN: 1472-2747. DOI: 10.2140/agt.2015.15.2517.

[9] M. Mackaay, M. Stošić, and P. Vaz. “A diagrammatic categorification of the q-Schur algebra”. In: *Quantum Topol.* 4.1 (2013), pp. 1–75. ISSN: 1663-487X. DOI: 10.4171/QT/34.

[10] S. MacLane. *Categories for the working mathematician.* Graduate Texts in Mathematics, Vol. 5. Springer-Verlag, New York-Berlin, 1971, pp. ix+262.

[11] G. Naisse and K. Putyra. “Odd Khovanov homology for tangles” (2020). arXiv: 2003.14290 [math.QA].

[12] nLab. URL: https://ncatlab.org/nlab/show/HomePage (visited on 05/11/2020).

[13] nLab. *monoidal category.* URL: https://ncatlab.org/nlab/show/monoidal+category (visited on 05/11/2020).

[14] T. Ohtsuki. *Quantum invariants*. Vol. 29. Series on Knots and Everything. A study of knots, 3-manifolds, and their sets. World Scientific Publishing Co., Inc., River Edge, NJ, 2002, pp. xiv+489. ISBN: 981-02-4675-7.

[15] P. S. Ozsváth, J. Rasmussen, and Z. Szabó. "Odd Khovanov homology". In: *Algebr. Geom. Topol.* 13.3 (2013), pp. 1465–1488. ISSN: 1472-2747. DOI: 10.2140/agt.2013.13.1465.

[16] L. Schelstraete and P. Vaz. "Supercategorification and $\mathfrak{sl}(2)$-tangle invariants" (in preparation).

[17] A. N. Shumakovitch. "Patterns in odd Khovanov homology". In: *J. Knot Theory Ramifications* 20.1 (2011), pp. 203–222. ISSN: 0218-2165. DOI: 10.1142/S0218216511008802.

[18] P. Vaz. "Not even Khovanov homology". In: *Pac. J. Math.* (to appear). arXiv: 1909.03913 [math.QA].

[19] B. Webster. "Knot invariants and higher representation theory". In: *Mem. Amer. Math. Soc.* 250.1191 (2017), pp. v+141. ISSN: 0065-9266. DOI: 10.1090/memo/1191.

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