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History, an educational tool in mathematics

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Outline

1 Introduction

Since the beginning of our school career, mathematics is presented as a finished product.
In primary school, pupils start with definitions and exercises. Then, they move onto secondary school, where mathematics become a collection of definitions, corollaries, and theorems. Similarly, at university the lecturer provides its students with a concluded theory, without explaining the processes a specific theory has to undergo in order to become a final and polished product.
In fact, textbooks or syllabi, which are used as pedagogical supports in mathematics courses are very accurate, detailing what should be taught and how. However, and even though, different mathematical subjects are given as precisely as they can, depending on the level of the students, the latter oftentimes have difficulties to get an overview of what is actually taught and why. Consequently, in the course of time, students increasingly start to question the necessity and usefulness of what they're seeing in class.
One of the most difficult tasks of mathematics teachers is to give an answer to these questions but, first and foremost, to show that mathematics makes, or can make, sense. A way to overcome these difficulties is to use history of mathematics as it provides with an inexhaustible source of real problems.
In addition, history shows that, in realty, mathematics developed through processes of trial and error, communication, criticism, rejection, etc. In fact, these are essential for accepting or rejecting a particular result and, subsequently might also serve as a basis for a new theory. Therefore, history is a key for providing new ways of thinking about mathematics.

This will be the subject of the first chapter of this thesis: the benefits of using history in mathematics classrooms. As is not a novel discovery, the first chapter starts with a brief review by teachers who acknowledge the importance for using history in mathematics lessons. This is followed by a non-exhaustive examination of why and how history can be a useful tool, as much for the student as for the teacher.
Of course, as it is the case with any other tool used in mathematics lessons to enrich the taught subject, the integration of history in mathematics must inevitably face challenges and difficulties, which shall also be shortly discussed within the first chapter.
The first chapter thus treats the purpose and necessary methodology of implementing the history of mathematics as a part of mathematics lessons. However, there is another important question that needs to be addressed, yet: what happens if one leaves out history of mathematics?
Nowadays, mathematics is simplified or changed in order to make it more accessible for students of different ages. Consequently, students mostly assume, for instance, that the definitions conveyed in class are those that have been immutably used for centuries and therefore represent the ultimate truth. What follows then, is that these students carry on teaching this modified or simplified definition to other pupils, who won't be questioning the accuracy of the definition, and so on. As a result, the core definition and how the latter was developed fall into neglect, leading to a further neglect of the richness, subtlety, and the power of a definition. The latter being, most importantly, the fruit of centuries of work and mathematical thinking, which could be neglected, as for instance the Riemann's definition of the integral.

The purpose of the second chapter of this thesis is therefore going to be focusing and returning to the core of the mathematical essence by using original sources.
Nota bene: Several texts used in this paper were originally published in other languages than English; in an effort to render as faithful a translation as possible, I was compelled to sacrifice considerations of style and flow at times.
To visualise the differences between how the concept of Riemann Integral is given to students and how it deviates from the original vision of Riemann, the second chapter begins with four high school books; from the UK, Italy, Germany and Belgium, containing a definition of the definite integral and the definition I was taught at university. The latter is then followed by almost the entire development that the definition, given by Riemann, needed to undergo.
The beginnings are made by Newton and Leibniz which were the first to consider infinitesimal calculus to obtain the area under a curve or a function. At that time, neither the concept of the function, nor the concept of integral was the same as it is today. In fact, the word "function" did not even exist yet. In addition, functions were given in a different form, meaning, as finite or infinite series. Series play therefore an important role in the development of function and the definition of the definite integral. However, there will be hardly any theory regarding this concept but only principal results, or ideas to ensure that there would not be too much of a deviation from the main aim, which is to understand the purpose of the Riemann's definition of the integral.

2 HPM – History and pedagogy of mathematics

2.1 History of HPM

As seen in the introduction, integrating the history of mathematics in the teaching of this subject is not a new pedagogical urge but has been amongst the considerations of theorists and mathematicians for a considerable time. In the second half of the 19th century already, there has been a surge of interest in the use of historical concepts of mathematics in the context of mathematics education, as famous mathematicians such as De Morgan, Glaisher, Poincaré and Klein became aware of the different benefits of bringing in history.

As a matter of fact, De Morgan claimed that “[t]he history of mathematics may be instructive as well as agreeable; it may not only remind us of what we have, but may also teach us to increase our store” and this idea of instruction by reminiscence is backed up by historians of mathematics like Cajori who also believes that "The early history of the mind of men with regards to mathematics leads us to point out our own errors; and in this respect it is well to pay attention to the history of mathematics." (Moritz, 1993, p. 90) Hence, they seem to claim that the study of the history of mathematics may lead to a better and deeper understanding of the subject destined to be taught.

However, the consideration of history is not only pertinent in the understanding of the innate flaws and errors of the human mind in dealing with these subjects, but can also enrich the very subject itself, as is highlighted by J.W.L. Glaisher who is “[…] sure that no subject loses more than mathematics by any attempt to dissociate it from its history.” (Moritz, 1993, p. 96) Moreover, taking into account the history of the subject must necessarily inform a better understanding of the future, as Poincaré affirmed when he said that “in order to anticipate the future of mathematics, the true method is to study their history and their present state.” (Poincaré, 1908, p. 20) It appears obvious that it is counterproductive and maybe even counterintuitive to separate one slice of knowledge and hail it as independently and invariably true. On the contrary, if one is to achieve true understanding in order to inscribe a concept and its transmission throughout the generations into a continuity, one must take into consideration the past to be able to project a future.

German mathematician F.C. Klein has extensively touched upon this idea in the context of his vision of textbooks that differed from the habitual presentation. As a matter of fact, he projected four key points that he considered to be especially important: the visualisation of abstract considerations by means of figures, the emphasis put on the relation of mathematics to neighbouring fields, the emphasis put on historical growth, and lastly, the exhibition of samples of popular literature to mark the deviation of the thus induced viewpoints in the public from those of the professional mathematician.
It becomes obvious that the contextualisation of mathematics plays a key role in Klein’s vision of the ideal instruction of the matter, and he does encourage the “prospective teacher” (Klein, 2016, pp. 256,257) to take into account those aspects, as he or she would, inevitably, be confronted with the “popular views” (Klein, 2016, pp. 256,257) which leads to what he calls an “insecure” footing if there is a lack of knowledge and awareness of “[the] intuitive elements of mathematics as well as the vital relations with neighbouring fields [and] above all, […] the historical development.” (Klein, 2016, pp. 256,257)

Consequently, this lack of caution and situational awareness must lead to one of two pedagogical failures, that he describes in his impassionate warning to future teachers as follows:

> "You will then either withdraw to the ground of the most modern pure mathematics, and fail to be understood in the school, or you will succumb to the assault, give up what you learned in higher education, and even in your teaching allow yourself to be buried in the traditional routine." (Klein, 2016, pp. 256,257)

Klein was thus fiercely aware of the "discontinuity between school and university" (Klein, 2016, pp. 256,257), so between what was learned and what was to be taught, between the input received and the output produced, which he judged to be most severe "[…] precisely in the field of infinitesimal calculus" (Klein, 2016, pp. 256,257). This claim shall be inspected further in this thesis, but it is notable that an acclaimed mathematician such as Klein has judged this awareness to be "useful armour" (Klein, 2016, pp. 256,257) in the profession of teaching, especially when it comes to mathematics, even though his cautions apply to any subject or field.

Despite these early ventures into the subject, it was mainly the New Math reform in the 1950's to 1970's that truly challenged the conservative vision of school according to which facts should be memorized and reproduced. Instead, it promoted the idea of mathematics serving as a logic for solving scientific and social issues in various creative ways (The TIME Vault, 1964, p. 36), initiating discussions that would largely be in the same vein than the contemplations that Klein had already elicited a century ago.
In 1969, the introduction of the history of mathematics as a teaching tool became of primary focus in the 31st Yearbook of the National Council of Teachers of Mathematics in the USA. Besides, the *First International Congress On Mathematical Education* (ICME-1) took place the same year, which helped establish regularly organized international meetings, devoted to discuss mathematics education. Then, at the ICME-2 of 1972, 38 distinct Working Groups (WG) were created to address diverse themes around mathematics education such as the "Relations between the History and Pedagogy of Mathematics" organized by P.S. Jones and L. Rogers.
In 1976, the Working Group continued at the ICME-3 (Clark, et al., 2019).The interest and acknowledgment of the importance of historical-pedagogical studies then grew to such an extent that a system was set up to ensure regular sessions at future ICMEs addressing this particular theme with the approval of the International Commission on Mathematical Instruction (ICMI). This led to the approval, in 1996, of a 4-year ICMI study assessing their work and reporting on major issues for further studies. Originally, the study group went by the name of *"International Study Group on Relations between History and Pedagogy of Mathematics, cooperating with the International commission on Mathematical Instruction"* and is nowadays abbreviated as the HPM Group. The HPM Group grandly supported and encouraged the interest and research into the field of mathematics education on an international level and has hence inspired ground-breaking pedagogical practices worldwide (Clark, et al., 2019).

Up to this day, the HPM group pursues diverse interests and fulfils several important duties in the name of progress in mathematical thinking and pedagogical transmission:

- The study and developments of theoretical and conceptual frameworks for integrating history in mathematics education.
- Empirical research based on actual classroom units.
- Development of various kinds of teaching aids.
- Design of specific teaching units.
- The development of various kinds of teaching aids.
- The investigation and understanding of students' response to the introduction of the history of mathematics in teaching.
- Designing, applying, and evaluating interdisciplinary teaching.
- The exploration of eventual parallels between the historical development and learning in modern classrooms.

…

(Clark, et al., 2019, p. 4)

2.2 Benefits and issues in the HPM domain

Over the past few decades, the HPM group has managed to collect an extensive amount of research data covering a broad spectrum of activities based on actual implementation in class. Data has for instance been collected to develop conceptual and theoretical frameworks to understand the benefits of including history in mathematics lessons through the design of concrete teaching units and the development of diverse teaching aids aiming to fully understand the student's and the teacher's response to this approach (Clark, et al., 2019).This most particularly led to the creation, application and evaluation of an interdisciplinary teaching approach by exploring the parallels between the historical development and the learning processes in current teaching spaces.

In his work *Using History in mathematics education*, John Fauvel goes into the details of "why" and "how" the integration of historical components into the teaching of mathematics can be beneficial.

Hence, he presents the following arguments as being leading to success from the point of view of the students (Fauvel, June 1991, pp. 4,5):

- **Why**
  - Helps to increase motivation for learning
  - Gives mathematics a more human touch
  - Helps to order the presentation of the topics of the curriculum
  - Changes pupils' perceptions of mathematics
  - Provides pupils with a sense of comfort; struggling to understand is part of the process
  - Encourages quick learners to look further
  - Bridges the gab between mathematics and other disciplines
  - Historical development versus polished mathematics
  - Helps understanding the nature of mathematics and mathematical activity
  - enriches the didactical repertoire
  - Increases the development of students' mathematical reasoning skills by example of the problems faced trough history
  - Shedds light on mathematicians' moral values that lead them troughout their career and can be a source of insipration for the students to work with honest, diligence or determinagion for instance
  - Provides matemathics teachers with a guidance when structuring their lessons

- **How**
  - Mention past mathematicians anecdotally
  - Provide historical introductions to concepts which are new to pupils
  - Encourage pupils to understand the historical problems to which the concepts they learning are answers
  - Give "history of mathematics" lessons
  - Devise classroom or homework exercises using mathematical texts from the past
  - Using critical examples from the past to illustrate techniques or methods
  - Explore past misconceptions/errors/alternative views to help in understanding and resolving difficulties for today's learners
  - Devise the pedagogical approach to a topic in sympathy with its historical development
  - Devise the ordering and structuring of topics within the syllabus on historically-informed grounds

As for the teachers, the including of history of mathematics in the classroom does not automatically lead to higher scores obtained by the students overnight, but it could make the process of learning mathematics a more meaningful and lively experience, which, one hopes, makes it come easier. Thus, “[t]he awareness of this evolutionary aspect of mathematics can make a teacher more patient, less dogmatic, more humane, less pedantic. (Siu, 1996, p. 12)”
Moreover, by creating a strong link between a historical aspect and the subject, it applies to also valorise the importance of history inevitably. As a matter of fact, by turning history into a completely hermetic, autonomous, and impermeable discipline, we risk rendering it obsolete. It seems vital to combine history and teaching, to date the invention of a concept, to explain the historical range of a concept, to get people to read ancient texts, but also to resolve “historical problems”. One should not seek to create a discipline or an academic “moment” that would be detached from the mathematical method, on the contrary, one should combine them to get a deeper understanding of a specific subject.

Indeed, the integration of the historical context in the teaching of mathematics goes against a "random" approach to the mathematical concepts taught, in the sense that historical knowledge and awareness relieves the teacher, when questioned about the usefulness of a notion, from giving answers such as "in mathematics, that's how it happens" or "you'll see that later". (Barbin, 2012)
The epistemological instruction of the teaching staff that is the study of ancient texts relating to the fundamental notions of a given subject, seems particularly relevant for a deeper understanding of a certain subject.
Consequently, those documents are not to be read in comparison or opposition with our own knowledge but rather within the context of their own genesis. The study becomes then a source of “epistemological astonishment” as it calls into question the knowledge and procedures that are generally considered “self-evident”. The question that arises is therefore no longer “Why don’t the pupils understand?” but rather “Why were the contemporaries of this document unable to process this novel idea?”.
Subsequently, the historical thought leads to the idea of the “rectification of notions” since any given notion may change in the course of history. Mostly, the taught notions, which are often too complex and opposed to the studied problems, which often turn out to be too simple, can be clarified by the study and consideration of history. Consequently, this could help students to give more sense to what they are learning and hopefully to keep the notion as long-term memory.

To achieve a successful education, one cannot exaggerate the importance of a skillset that allows the teacher to give a comprehensive presentation of any given topic: this requires, on the one hand, a mathematical concept formation, including definitions, corollaries, theorems, and calculations.
On the other hand, it is also essential to actually motivate the potential learner to take on a persevering, frustration tolerant, mental work in the aforementioned way (Nickel, 2016).

Nevertheless, G. Heppel also touches on the restrictions and limitations of history in the context of teaching mathematics in his paper “The Use of History in Teaching Mathematics” (Heppel, 1893). Of the many arguments to be found in this document, three seem particularly relevant though:

First, the history of mathematics should in no way develop into a separate subject of education, but should be strictly auxiliary and subordinate to the mathematical teaching, which corresponds with the point of view of the abovementioned French article.
Likewise, it is up to the teacher to only bring in historical elements if it is truly beneficial to the learner at hand; the integration of history should be an added value and not an added difficulty that would confuse and frustrate the learner rather than make him progress in his study.
Finally, the historical aspects are destined to an aid and an additional motivation to reach a deeper understanding of the topic at hand. They should never be the subject of any kind of examination within the field of mathematics.

Lest these conditions are respected, there is a risk that the introduction of history as a new matter for instruction would have the very opposite effect than was originally anticipated. That's to say that it would be harmful to the success and the sustained motivation of the learner, who, in general, has already sufficient subjects to study and who, no doubt, would be overwhelmed and overstrained by the addition of another one (Ho, 2008).

Furthermore, there is a considerable lack of pedagogical training when it comes to the history of mathematics, and it is obvious that one should be well familiar with the subject if one wants to provide these elements as tools for learners. Additionally, the schedule of the educational program often does not allow for such diversions as it would take time away from the preparation of the actual program. This might potentially lead to an underwhelming and unsatisfactory performance during the national examinations. Hence, neither the teachers nor the learners are usually willing to “waste” their precious time on elements that won't be subjected to examination.
Besides, insufficient studies have been conducted in order to ascertain the usefulness and beneficial effect of history as a component of mathematics teaching, so there remains a distinct lack of definitive results for want of more findings on the subject.

Nonetheless, Uffe Jankvist, for example, provides a list of empirical studies that recognises, for the most part, the use of history as either an affective and/or motivational tool or as a cognitive tool. Cecil C. McBride and James H. Rollins also report that the implication of original sources by famous scientists such as Descartes, Pythagoras, Kepler *et alii* leads to a positive change in attitude towards mathematics.
Most interestingly, Siew Yee Lim and Elaine Chapman describe an experimental study in which four classes of school in Singapore have participated. Two of them were assigned as control groups, so they had no history classes, whereas the other two acted as the active experimental classes and had history of mathematics classes. The study tried to measure variations in different areas such as attitude towards mathematics, anxiety levels, motivation levels and academic achievement.
Regarding their attitude towards mathematics, students in the experimental group saw a higher perceived value in mathematics than the control group, even though it seems that the effects observed only persisted for a short-term period.
As for anxiety, defined by subscale comprising four items and measures: "feelings of anxiety, dread, nervousness, and associated bodily symptoms related to doing mathematics", no differences between the experimental and the control groups were observed.

However, there were significant differences ascertained between the two groups pertaining to the elements of motivation and achievement. In fact, the experimental group developed a higher interest in mathematics due to the use of history and therefore acknowledged more sense in what they were learning, which, in turn, had a positive impact on the motivation and, consequently, on the achievements of the experimental groups, that outperformed the control groups in every test following the split.

Generally speaking, it is also the apprehension of the teachers to find themselves unable to relate the history to the topic in an efficient and relevant manner, due to lack of time amongst other elements. Correspondingly, they seem intimidated by the possibility that the students may well struggle to familiarise themselves with the history because of the considerable cultural differences that exist between their present reality and the, sometimes distant, past. On top of these reservations, they do not want to use the title of "history", afraid of having it, falsely, associated with a humanities subject. It also may trigger their own biased memories of their school history lessons that is comprising a boring search for sources, for obscure ages and strange events of no visible, actual concern in their own reality (Führer, 1991).

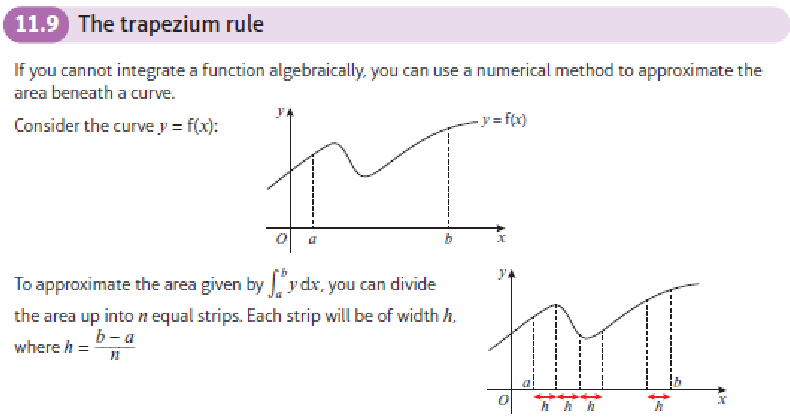
Summa summarum, the very nature and essence of history conjures up memories, apprehensions and trauma and the integration of history into another, often unloved, subject demands a good measure of susceptibility to the conciliations that have to be made to avoid falling prey to one of the many mistakes, too easily made, out of lack of preparation or commitment.

3 History of the *definite integral*

As presented in the previous chapter, history may be a beneficial tool but also allowed the real purpose of some notions to emerge. To illustrate the latter, the concept of Riemann's integral followed by Darboux's integral is going to be analysed.

Students begin the process of familiarising with the concept of integration in high school. They first learn that integrating a function represents an area under the graph of a given positive function, which in some textbooks is presented under the name of Riemann's integral, Darboux's integral or definite integral[1]. To get an idea on how the definition is taught in different high schools, I asked some friends of mine, who studied in the UK, Italy, Germany and Belgium to send me the references of the book they used in their final year of their mathematics class, in which the definition of definite integral is presented.

As shown below, in the UK, they are interested in calculating the area under a curve of a continuous and positive function. They start the section with "if you cannot integrate a function algebraically", which means that "if you do not know the antiderivative of the function", then the trapezium rule can be used (Attwood, et al., 2017, pp. 317,318).


11.9 The trapezium rule

If you cannot integrate a function algebraically, you can use a numerical method to approximate the area beneath a curve.

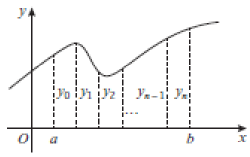
Consider the curve $y = f(x)$:

To approximate the area given by $\int_a^b y \, dx$, you can divide the area up into n equal strips. Each strip will be of width h, where $h = \frac{b-a}{n}$

[1] Definite integrals means that a function is integrated on a given interval $[a; b]$ with a and b two real numbers.

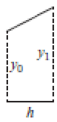
Next you calculate the value of y for each value of x that forms a boundary of one of the strips. So you find y for $x = a$, $x = a + h$, $x = a + 2h$, $x = a + 3h$ and so on up to $x = b$.
You can label these values $y_0, y_1, y_2, y_3, \dots, y_n$.

Hint Notice that for n strips there will be $n + 1$ values of x and $n + 1$ values of y.



Finally you join adjacent points to form n trapezia and approximate the original area by the sum of the areas of these n trapeziums.

You may recall from GCSE maths that the area of a trapezium like this:



is given by $\frac{1}{2}(y_0 + y_1)h$. The required area under the curve is therefore given by:

$$\int_a^b y\,dx \approx \tfrac{1}{2}h(y_0 + y_1) + \tfrac{1}{2}h(y_1 + y_2) + \dots + \tfrac{1}{2}h(y_{n-1} + y_n)$$

Factorising gives:

$$\int_a^b y\,dx \approx \tfrac{1}{2}h(y_0 + y_1 + y_1 + y_2 + y_2 \dots + y_{n-1} + y_{n-1} + y_n)$$

or $$\int_a^b y\,dx \approx \tfrac{1}{2}h(y_0 + 2(y_1 + y_2 \dots + y_{n-1}) + y_n)$$

This formula is given in the formula booklet but you will need to know how to use it.

- **The trapezium rule:**

  $$\int_a^b y\,dx \approx \tfrac{1}{2}h(y_0 + 2(y_1 + y_2 \dots + y_{n-1}) + y_n)$$

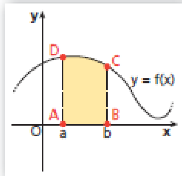
  where $h = \frac{b-a}{n}$ and $y_i = f(a + ih)$

Similarly, in an Italian textbook, the area under a curve of a continuous and positive function is calculated (la funzione è continua e positiva). Moreover, the term trapezium is also used (trapezoide). In this case however, "trapezium" refers to the entire area under the curve ("S") which looks like a trapezium, as shown in figure 6 (figura 6). The area of the trapezium is approximated by using rectangles. They first divide the interval $[a; b]$ in n equal parts of length "h". "m_i"and "M_i" represent the infimum and the supremum of the function $f(x)$ in the i-th interval. "s_n" and "S_n" are the upper and lower sums (somma integrale inferiore e superiore). The area of the trapezium "S" is between the lower and upper sum. Expressed in symbols, we get: "$s_n \leq S \leq S_n$". Besides, in this textbook, it is also mentioned that by using the continuity of the function, it can be shown that the limit of the two sums is equal, "$\lim_{n\to\infty} s_n = \lim_{n\to\infty} S_n$". This limit is called definite integral (integrale definito) (Bergamini, et al., 2012, pp. 1501,1502).

5. L'INTEGRALE DEFINITO

La funzione è continua e positiva

Dati una funzione $y = f(x)$ e un intervallo chiuso e limitato $[a; b]$ nel quale la funzione è *continua* e *positiva* (o nulla), si chiama **trapezoide** la figura piana delimitata dall'asse x, dalle rette $x = a$ e $x = b$ e dal grafico di $f(x)$ (figura 6). Il trapezoide viene chiamato così perché somiglia a un trapezio con le basi parallele all'asse y.
L'*area S di un trapezoide* non può essere calcolata in modo elementare, tuttavia possiamo approssimarla utilizzando il seguente procedimento:



▲ Figura 6 Il trapezoide.

● Poiché la funzione è continua, il teorema di Weierstrass garantisce che negli intervalli considerati esistono il minimo e il massimo assoluto.

- dividiamo l'intervallo $[a; b]$ in n parti uguali di ampiezza $h = \dfrac{b-a}{n}$ (per esempio, nella figura 7 abbiamo considerato $n = 6$);
- consideriamo gli n rettangoli aventi ciascuno per base un segmento di suddivisione e per altezza il segmento associato al minimo m_i che la funzione assume in tale intervallo;
- indichiamo con s_n la somma delle aree di tutti questi n rettangoli:

$$s_n = m_1h + m_2h + \ldots + m_nh = \sum_{i=1}^{n} m_ih.$$

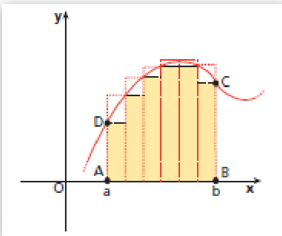
L'area del trapezoide viene così approssimata per difetto da s_n.

In maniera analoga, possiamo approssimare per eccesso l'area del trapezoide tramite la somma delle aree dei rettangoli associati a una scomposizione dell'intervallo $[a; b]$ in n parti uguali e aventi per altezza il segmento associato al massimo M_i della funzione nel corrispondente intervallo. Indichiamo la somma con S_n:

$$S_n = M_1h + M_2h + \ldots + M_nh = \sum_{i=1}^{n} M_ih.$$

s_n e S_n vengono chiamate rispettivamente **somma integrale inferiore** e **somma integrale superiore**. L'area S del trapezoide risulta compresa fra l'area per difetto e quella per eccesso, ossia possiamo scrivere:

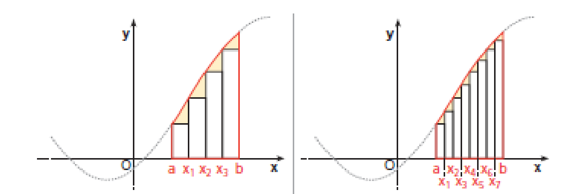
$$s_n \le S \le S_n.$$



► Figura 7 L'area del trapezoide *ABCD* è compresa tra il plurirettangolo inscritto s_6 e il plurirettangolo circoscritto S_6.

L'approssimazione delle due aree s_n e S_n risulta migliore man mano che si scelgono più piccoli gli intervalli di suddivisione di $[a; b]$.

► Figura 8



a. Per $n = 4$ l'intervallo è suddiviso in 4 parti. La somma delle aree dei 4 rettangoli approssima, per difetto, l'area del trapezoide.

b. Se $n = 8$, la somma delle aree degli 8 rettangoli approssima meglio della precedente, per difetto, l'area del trapezoide.

Per $n = 1, 2, 3, \ldots$ i valori di s_n e S_n formano le due successioni:

$$s_1, s_2, s_3, \ldots, s_n, \ldots$$

$$S_1, S_2, S_3, \ldots, S_n, \ldots$$

Si può dimostrare, utilizzando l'ipotesi della continuità della funzione $f(x)$ in $[a; b]$, che tali successioni convergono allo stesso limite ossia che:

$$\lim_{n \to +\infty} s_n = \lim_{n \to +\infty} S_n.$$

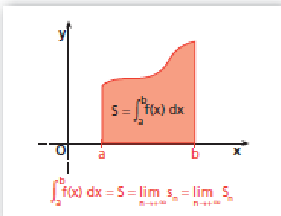
Il limite delle due successioni si chiama **integrale definito** e viene indicato con il simbolo

$$\int_a^b f(x)\,dx,$$

che si legge «integrale da a a b di $f(x)$ in dx».

Tale limite fornisce la misura dell'area S del trapezoide relativo a $f(x)$ e nell'intervallo $[a; b]$.

● Il simbolo $\int$ rappresenta una S allungata per ricordare che, nella rappresentazione grafica, a un integrale corrisponde una somma di aree di rettangoli aventi altezza $f(x)$ e base dx.



▲ Figura 9

In the German textbook, a positive and continuous function is still used but the approach is a little bit different. In fact, only the lower sum is calculated and denoted by "A_n". Furthermore, it is explained that, an alternative value can be chosen in order to calculate the height of the rectangle, which is not necessarily the infimum of the function. The alternative value is denoted as "z_i" and represents any real number of the i-th interval. Therefore, depending on the chosen value, the height of the rectangle changes. They conclude that $\lim_{n\to\infty} A_n$ represents the integral of $f(x)$ between the boundary value a and b (…dann heisst der Grenzwert $\lim_{n\to\infty} A_n$ Integral der function f zwischen den Grenzen a und b), and is written as: $\int_a^b f(x)dx$ (Lambacher Schweizer, 2012, pp. 139,140).

2 Das Integral

Der Inhalt der Fläche unter dem Graphen von f mit $f(x) = x^2$ soll über dem Intervall $[0; 1]$ bestimmt werden. Dazu füllt man den Inhalt zunächst näherungsweise mit gleich breiten Rechtecken (gelb in Fig. 1).

Einteilung in z. B. vier Teilintervalle

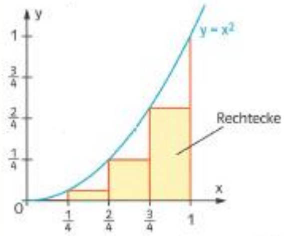


Fig. 1

Der Inhalt der vier Rechtecke beträgt

$$A_4 = \frac{1}{4}\cdot 0^2 + \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{2}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2 \approx 0{,}2188.$$

Eine solche Rechteckssumme nähert den gesuchten Flächeninhalt umso besser an, je kleiner die Teilintervalle sind. In der Tabelle sind einige Werte zusammengestellt.

Anzahl der Teilintervalle	10	100	1000
Rechteckssumme	$A_{10} \approx 0{,}2850$	$A_{100} \approx 0{,}3284$	$A_{1000} \approx 0{,}3328$

Die Rechtecke in Fig. 1 liegen alle *unter* dem Graphen. Man nennt diese Rechteckssumme **Untersumme** U_4.
Die **Obersumme** O_4 ist größer als der gesuchte Flächeninhalt:

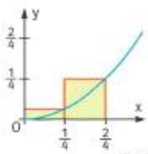


Fig. 2

Zur Untersuchung von A_n für $n \to \infty$ muss A_n in Abhängigkeit von der Anzahl n der Teilintervalle ausgedrückt werden: $A_n = \frac{1}{n}\cdot 0^2 + \frac{1}{n}\cdot\left(\frac{1}{n}\right)^2 + \frac{1}{n}\cdot\left(\frac{2}{n}\right)^2 + \ldots + \frac{1}{n}\cdot\left(\frac{n-1}{n}\right)^2$.

Ausklammern von $\left(\frac{1}{n}\right)^3$ ergibt: $A_n = \left(\frac{1}{n}\right)^3\cdot[0^2 + 1^2 + 2^2 + \ldots + (n-1)^2]$.

Einsetzen der auf dem Rand angegebenen Formel für die Summe von Quadratzahlen ergibt:

$$A_n = \frac{1}{n^3}\cdot\frac{1}{6}(n-1)\cdot n\cdot(2n-1) = \frac{1}{6}\cdot\frac{n-1}{n}\cdot\frac{n}{n}\cdot\frac{2n-1}{n} = \frac{1}{6}\cdot\left(1-\frac{1}{n}\right)\cdot 1\cdot\left(2-\frac{1}{n}\right).$$

Für $n \to \infty$ ergibt sich: $\lim_{n\to\infty} A_n = \frac{1}{6}\cdot 1\cdot 1\cdot 2 = \frac{1}{3}$.

Für den gesuchten Flächeninhalt ist es sinnvoll, den Wert $A = \lim_{n\to\infty} A_n = \frac{1}{3}$ festzusetzen.

Summenformel für die Summe der ersten $z-1$ Quadratzahlen:
$1^2 + 2^2 + 3^2 + \ldots + (z-1)^2 = \frac{1}{6}\cdot(z-1)\cdot z\cdot(2z-1)$.

Man kann für die Höhe der Rechtecke auch andere Funktionswerte nehmen (Fig. 3). Den Grenzwert einer Rechteckssumme A_n kann man dann allgemein so darstellen:

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty}[f(z_1)\cdot(x_2-x_1) + f(z_2)\cdot(x_3-x_2) + \ldots + f(z_n)\cdot(x_{n+1}-x_n)].$$

Kürzt man die gleichen Differenzen $x_1 - x_0$, $x_2 - x_1$ usw. mit Δx (lies: Delta x) ab, ergibt sich:

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty}[f(z_1)\cdot\Delta x + f(z_2)\cdot\Delta x + \ldots + f(z_n)\cdot\Delta x].$$

Bei stetigen Funktionen ergibt sich unabhängig von der Art der Rechteckssumme immer der gleiche Grenzwert.

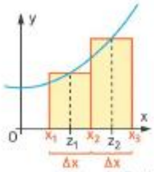


Fig. 3

In Fig. 1 verläuft der Graph der Funktion f teilweise unterhalb der x-Achse. Es wird jeweils beispielhaft der Flächeninhalt eines oberhalb und eines unterhalb der x-Achse liegenden Rechtecks berechnet.
Da die Inhalte von unterhalb der x-Achse liegenden Rechtecken dabei negativ gezählt werden, erhält man bei diesem Vorgehen orientierte Flächeninhalte.

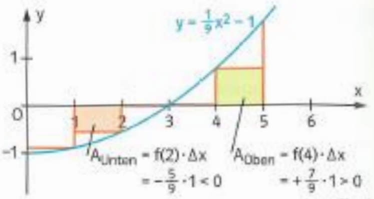


Fig. 1

Damit kann man mittels des Grenzwertes von Rechteckssummen auch bei nicht stückweise linearen Funktionen orientierte Flächeninhalte und Gesamtänderungen von Größen bestimmen.

Definition: Die Funktion f sei auf dem Intervall $[a; b]$ stetig und $A_n = f(z_1) \cdot \Delta x + f(z_2) \cdot \Delta x + \ldots + f(z_n) \cdot \Delta x$ sei eine beliebige Rechteckssumme zu f über dem Intervall $[a; b]$.
Dann heißt der Grenzwert $\lim_{n \to \infty} A_n$ **Integral** der Funktion f zwischen den Grenzen a und b.
Man schreibt dafür: $\int_a^b \mathbf{f(x)\,dx}$ (lies: Integral von f(x) von a bis b).

The last selected textbook is Belgian. In this textbook, it is only mentioned that the function needs to be continuous. To calculate the integral, the lower and upper sum are used but it is also specified that they use the definition given by Darboux (somme inférieure et supérieure de Darboux). If the limit of the two sums is equal, "$\lim_{n\to\infty} \underline{S_n} = \lim_{n\to\infty} \overline{S_n}$", then this limit is the definite integral of f between the boundary value a and b, which is denoted by $\int_a^b f(x)dx$ (cette limite est l'intégrale définie de f entre les bornes a et b et est notée $\int_a^b f(x)dx$) (Adam & Lousberg, 2011, pp. 192,193).

INTEGRALE DEFINIE

SURFACE SOUS LA COURBE

VOCABULAIRE ET NOTATIONS

La fonction $f : \mathbb{R} \to \mathbb{R} : x \to f(x)$ est **continue sur $[a; b]$**.
L'intervalle $[a; b]$ est divisé en **n parties de même longueur** : Δx.

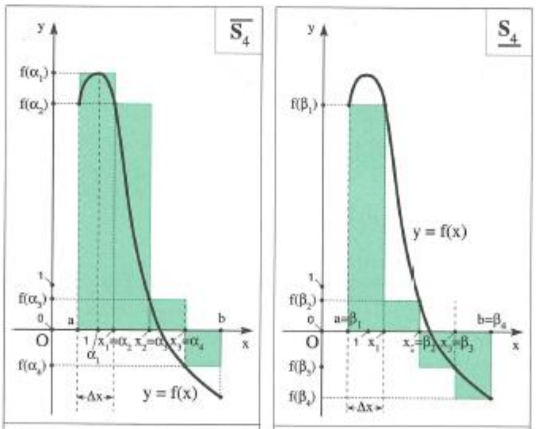
$$\text{Donc } \Delta x = \frac{b-a}{n}.$$

Sur chacune des parties de $[a; b]$, on note

- $\mathbf{f(\alpha_i)}$ la **plus grande valeur** prise par f sur la i^e partie;
- $\mathbf{f(\beta_i)}$ la **plus petite valeur** prise par f sur la i^e partie.

Enfin, on note

- $\overline{S_n}$, la somme $\displaystyle\sum_{i=1}^{n} f(\alpha_i)\Delta x$ nommée **somme supérieure de Darboux**;
- $\underline{S_n}$, la somme $\displaystyle\sum_{i=1}^{n} f(\beta_i)\Delta x$ nommée **somme inférieure de Darboux**.



$\overline{S_4} = f(\alpha_1)\Delta x + f(\alpha_2)\Delta x + f(\alpha_3)\Delta x + f(\alpha_4)\Delta x$

$\underline{S_4} = f(\beta_1)\Delta x + f(\beta_2)\Delta x + f(\beta_3)\Delta x + f(\beta_4)\Delta x$

DÉFINITION

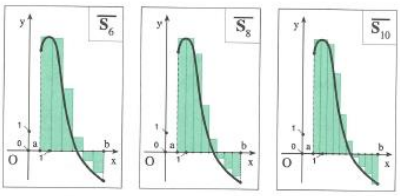
La fonction $f : \mathbb{R} \to \mathbb{R} : x \to f(x)$ étant continue sur $[a; b]$,

si $\lim\limits_{n\to+\infty} \underline{S_n} = \lim\limits_{n\to+\infty} \overline{S_n}$,

alors

- f est **intégrable** (au sens de Riemann) sur $[a; b]$;
- cette limite est l'**intégrale définie de f entre les bornes a et b** et est notée $\displaystyle\int_a^b f(x)\,dx$.

B. Riemann
(1826 - 1866)



At university (Bonheure, et al., 2014, pp. 132,133), the definition of Riemann integral, which is actually the definition given by Darboux, is revisited and the notations changes.

Definition

Let $P = \{x_0, x_2, \ldots, x_n\}$ be a partition of the interval $[a, b]$. Let $f(x): [a, b] \to \mathbb{R}$ be a bounded function, the *upper sum* of f is given by

$$U(f, P) = \sum_{i=1}^{n} M_i (x_i - x_{i-1})$$

where $M_i = sup\{f(x): x \in [x_{i-1}, x_i]\}$. The *lower sum* of f is given by

$$L(f, P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1})$$

where $m_i = inf\{f(x): x \in [x_{i-1}, x_i]\}$.

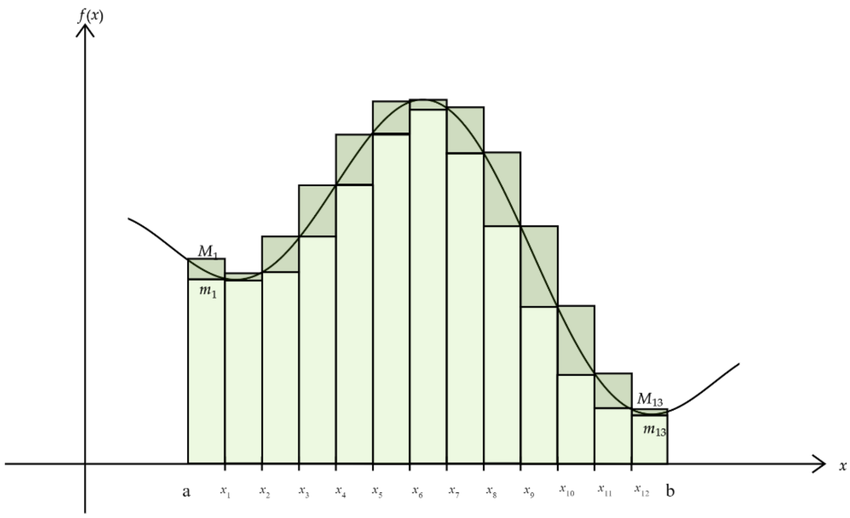


Figure: Upper and lower sum of f *with* $n = 13$

Definition

Let $f(x): [a, b] \to \mathbb{R}$ be a bounded function, the quantity

$$\mathcal{L}(f) = \sup\{L(f, P): P\ a\ partion\ of\ [a, b]\}$$

is called the *lower integral of* f.

The quantity

$$\mathcal{U}(f) = inf\ \{L(f,P): P\ a\ partion\ of\ [a,b]\}$$

is called the *upper integral of* f.

Definition

Let $f(x): [a,b] \to \mathbb{R}$ be a bounded function, is said to be *integrable* if $\mathcal{L}(f) = \mathcal{U}(f)$. In this case, this common value is noted

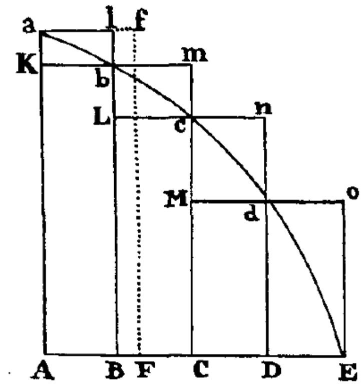
$$\int_a^b f(x)dx$$

3.1 Newton and Leibniz

In school, the definition of a *definite integral* is given as a finished product as can be observed in the previous section. However, the process that led to said definition is barely mentioned. Therefore, this thesis is going to start with Newton and Leibniz, who both used *infinitesimal calculus* around the same period to calculate the area under a curve.

Newton explains in section 1 of his first volume of his book *Mathematical Principles of Natural Philosophy*, written in 1680 and published in 1687. He writes:

> *"LEMMA I*
>
> *Quantities, and ratio of quantities, which in any finite time converge continually to equality, and before the end of that time approach nearer to each other than by any given difference, become ultimately equal.*
>
> If you deny it, suppose them to be ultimately unequal, and let D be their ultimate difference. Therefore, they cannot approach nearer to equality than that difference D; which is contrary to the supposition



LEMMA II

If in any figure $AacE$, terminated by the right lines Aa, AE, and the curve acE, there be inscribed any number of parallelograms Ab, Bc, Cd, &c., comprehended under equal bases AB, BC, CD, &c., and the sides,Bb, Cc, Dd, &c., parallel to one side Aa of the figure; and the parallelograms $aKbl, bLcm, cMdn$, &c., are completed : then if the breath of those parallelograms be supposed to be diminished, and their number to be augmented in infinitum, I say, that the ultimate ratios which the inscribed figure $AKbLcMdD$, the circumscribed figure $AalbmcndoE$, and curvilinear figure $AabcdE$, will have to one another, are ratios of equality.

For the difference of the inscribed and circumscribed figures is the sum of the parallelograms Kl, Lm, Mn, Do, that is (from the equality of all their bases), the rectangle under one of their bases Kb and the sum of their altitudes Aa, that is, the rectangle $ABla$. But this rectangle, because its breadth is supposed diminished *in infinitum,* becomes less than any given space. And therefore (by Lem. I) the figure inscribed and circumscribed become ultimately equal one to the other; and much more will the intermediate curvilinear figure be ultimately equal to either. Q.E.D.

LEMMA III

The same ultimate ratio are also ratios of equality, when the breaths AB, BC, DC, &c., of the parallelograms are unequal, and are all diminished in infinitum.

For suppose AF equal to the greatest breath, and complete the parallelogram $FAaf$. This parallelogram will be greater than the difference of the inscribed and circumscribed figures; but, because its breath AF is diminished *in infinitum*, it will become less than any given rectangle. Q.E.D." (Newton, 1687, pp. 29,30)

Newton does not use the term "function" but rather speaks of quantities. In fact, at that time, the notion of function, like it is used in current times, did not exist.

Today the Lemma could be given in the following way:

<u>Lemma 1</u>

Let $f(t)$ and $g(t)$ be a function of t so that the ratio is given by $\frac{f(t)}{g(t)}$. Let t be the time which varies in finite interval $[t_1, t_2]$ with t_1 the beginning of the time and t_2 the end of the time. When $t < t_2$ but near enough to t_2 then $| f(t) - g(t) | < D$. To put it in mathematical writing: $\lim_{t \to t_2} f(t) = g(t)$ and $\lim_{t \to t_2} \frac{f(t)}{g(t)}$ =1.

Proof

Let D be the ultimate difference so that $|\, f(t) - g(t) \,| = D$ then $f(t)$ cannot approach $g(t)$ since the difference $|\, f(t) - g(t) \,|$ does not become smaller than D, which goes against the assumption made.

Regarding Lemma II, there are some aspects that need to be specified. The curve acE represents a monotone decreasing function f with $f(E) = 0$ and the inscribed and circumscribed figures $AKbLcMdD$ *and* $AalbmcndoE$ correspond to the lower and upper sum. Furthermore, the last term that needs to be clarified is what Newton calls the *ultimate ratio*.

In the last *SCHOLIUM* of section I, he explains:

> "Those ultimate ratios with which quantities vanish are not truly the ratios of ultimate quantities, but limits towards which the ratios of quantities decreasing without limit do always converge; and to which they approach nearer than by any given difference, but never go beyond, nor in effect attain to, till the quantities are diminished *in infinitum*. This thing will appear more evidently in quantities indefinitely great. If two quantities, whose difference is given, is augmented *in infinitum*, the ultimate ratio of these quantities will be given, namely, the ratio of equality; but it does not form thence fallow, that the ultimate or greatest ratio themselves, whose ratio that is, will be given." (Newton, 1687, p. 39)

The term "ultimate" as used by Newton could be seen as the modern concept of limits.

By replacing the different terms in the statement of Lemma II, we get: When the breadth of the rectangles $Kalb, Lbmc, Mcnd$ and $DdoE$ converge to 0 then the limit of the ratio of the upper sum and of the area under the curve $AabcdE$ and of the ratio of the lower sum and of the area under the curve $AabcdE$ will converge to 1.

In Lemma III, Newton changes the breath of the fist rectangle. By doing this, the subintervals are not equal anymore but since the longest subinterval converge to 0, the limit of the ratio stays the same as before.

Finally, by comparing the way Newton calculates the area under the curve and the definition of Riemann's integral, a similarity can be observed.

Around 1680, Leibniz published a paper where he explains how to calculate the area under a curve by using infinitesimal techniques. However, before him, Cavalieri, like Newton, made use of rectangles to calculate the area under a certain curve. Cavalieri represented the summation by using the symbol "omn" (Jahnke, 2003, p. 89), which is the abbreviation of the Latin word *omnia* (Gaffiot, 2001, p. 508) meaning "all" or "the whole", mathematically: "the sum of all". But Leibniz was the first to introduce the symbol $\int$ as a replacement for symbol omn. In that context, $omn.\overline{omn.l}$ means "the sum of the sum of all l's" (Cajori, 1929, p. 210), in the manuscript of October 1675, he writes:

> "Let $BL = y, WL = l, BP = p, TB = t, GW = a$, then $y = omn.l$. [...] $\frac{l}{a} = \frac{p}{omn.l} = \frac{p}{y}$,therefore $p = \frac{\overline{omn.l}}{a} l$. Hence, $omn. y \frac{l}{a}$ does not mean the same thing as $omn. y$ into $omn. l$, nor yet y into $omn. l$; for, since $p = \frac{y}{a} l$ or $\frac{\overline{omn.l}}{a} l$, it means the same thing as $omn. l$ multiplied by that one l that corresponds with a certain p ; hence $omn. p = omn. \frac{\overline{omn.l}}{a} l$. Now I have otherwise proved $omn. p = \frac{y^2}{2}$, i.e. $= \frac{\overline{omn.l}^2}{2}$; therefore we have a theorem that to me seems admirable, and one that will be of great service to this new calculus, namely,
>
> $$\frac{\overline{omn.l}^2}{2} = omn. \overline{\overline{omn.l}\, \frac{l}{a}}, \text{ whatever } l \text{ may be;}$$
>
> that is, if all the l's are multiplied by their last, and so on as often as it can be done, the sum of all these products will be equal to half the sum of the squares, of which the sides are the sum of the l's or all the l's. This is a very fine theorem, and one that is not at all obvious." (Leibniz, 1675, p. 80)

In the modern notation, l means dy, and a is dx. (Leibniz, 1675, p. 73)So, $y = omn. l = \int dy$. By replacing the different notations in the equality $\frac{\overline{omn.l}^2}{2} = omn. \overline{\overline{omn.l}\, \frac{l}{a}}$ we obtain $\frac{(\int dy)^2}{2} = \int (\int dy) \frac{dy}{dx}$ and finally $\frac{y^2}{2} = \int y \frac{dy}{dx}$.

In the same paper, Leibniz writes:

> "I propose to return to former considerations.
>
> Given l, and its relation to x, to find l.This is to be obtained from the contrary calculus, that is to say, suppose that $\int l = ya$. Let $l = ya/d$; then just as will increase, so d will diminish the dimensions. But $\int$ means a sum, and d a difference. From the given y, we can always find y/d or l, that is, the difference of the y's." (Leibniz, 1675, p. 82)

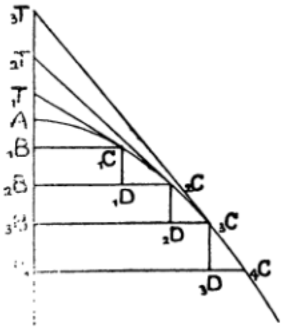
This shows that Leibniz began to interpret the integral as the inverse of the differential, which, at that time, he called "difference" instead of "differential". $\frac{x}{d}$ then means "the differential of x" (Cajori, 1929, p. 209). By putting the d on the numerator instead of the denominator, we would get $l = d(ya)$, which is a more familiar notation. In the paper *Examples of the inverse method of tangents*, he investigates how to find a curve while knowing its integral. We are not going to further explore this concept since we are interested in understanding how to find the area under a curve.

Now that the symbols have been explained, we can access Leibniz's paper published around 1680[2] under the name *The elements of the new calculus for differences and sums, tangents and quadratures, maxima and minima, dimensions of lines, surfaces, and solids, and for other things that transcend other means of calculation.*

---

[2] In *J. M. Child - The Early Mathematical Manuscripts Of Leibniz* is mentioned that there is not exact date of the publication but they guess, based on the dated published papers, that it should not be far from 1680

"Let CC be a line, of which the axis is AB, and let BC be ordinates perpendicular to this axis, these being called y, and let AB be the abscissae cut off along the axis, these being called x.



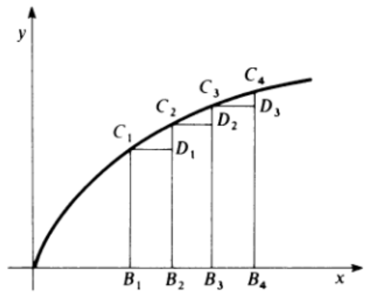


Figure: clarification of the original one

Then CD , the differences of the abscissae, will be called dx ; such are ${}_1C\,{}_1D, {}_2C\,{}_2D, {}_3C\,{}_3D$ [in modern notation: C_1D_1, C_2D_2, C_3D_3] etc. Moreover, the straight lines ${}_1D\,{}_2C, {}_2D\,{}_3C, {}_3D\,{}_4C$, the differences of the ordinates, will be called dy. If now these dx and dy are taken to be infinitely small, or the two points on the curve are understood to be at a distance apart that is less than any given length, i. e., if ${}_1D\,{}_2C, {}_2D\,{}_3C$, etc. are considered as the *momentaneous increments* of the line BC, increasing continuously as it descends along AB, then it is plain that the straight line joining these two points, ${}_2C\,{}_1C$ say, (which is an element of the curve or a side of the infinite-angled polygon that stands for the curve) [...] thus ${}_4B\,{}_4C$ is the sum of all the differences such as ${}_3D\,{}_4C, {}_2D\,{}_3D$ etc., as far as A, even if they are infinite in number. This fact I represents thus, $\int dy = y$. I also represent the area of a figure by the sum of all the rectangles contained by the ordinates and the differences of the abscissae, i. e., by the sum ${}_1B\,{}_1D + {}_2B\,{}_2D + {}_3B\,{}_3D +$ etc. For the narrow triangles ${}_1C\,{}_1D\,{}_2C$, ${}_2C\,{}_2D\,{}_3C$ etc., since they are infinitely small compared to the said rectangles, they may be omitted without risk; and thus I represents in my calculus the area of the figure by $\int y\,dx$, or the sum of the rectangles contained by each y and the dx that corresponds to it; here, if the dx's are taken equally to one another." (Leibniz, c. 1680, pp. 137,138)

Leibniz's notation for integral and differential are still used today. Nowadays, the distinction between the defined integral and the undefined integral is additionally made. Seemingly, Leibniz did not make it at that time; he juggled with the definition of integral and differentiation and, ostensibly, using the symbol of the undefined integral worked just fine for him.

Newton and Leibniz's techniques of finding the area under the curve are similar but there is a big difference in the way they approach the problem. Newton works with limit ratio in contrast to Leibniz, who dealt directly with infinitely small increments. Here again, Leibniz does not explicitly use the word "function" but writes instead "the line CC". The combination of the technique used by Newton and the notation used by Leibniz is still used today. However, they are rarely mentioned in textbooks regarding finding the area under the curve.

3.2 Concept of the function

The concept of function is another important part which contributes to understanding the need for Riemann's integral. In *La géométrie* (1637), Descartes introduces the idea of analytical function, in which he reduces the solution of all algebraic problem and equations to some standard procedure for constructing their roots which was possible thanks to intersections of appropriate plane curve of the lowest order (Youschkevitch, 1975, pp. 52,53). He then noticed, thanks to an equation, that all points of a curve can be linked to points on a line, for example on the x-axis. Here, the analytical expression was restricted to the algebraic one. Thereinafter, some mathematicians like Newton and Leibniz developed the theory of infinite series. Moreover, other infinite expressions of functions were added, like infinite products. Seeing the concept of function as the convergence of series, which is possible under certain conditions, became very popular and occupied a central role in the development of mathematical analysis. Indeed, finite and infinite series are easier to integrate or differentiate since they could do it term-wise (Edwards, Jr., 1979, p. 166).

The word "function" appeared for the first time in Leibniz's manuscript[3] from August 1673. It derives from the Latin word *fungŏr* which means "to perform, to execute" (Gaffiot, 2001, p. 508). He chose the term "function" in relation to everyday actions and to the everyday language. However, in the same manuscript, the meaning of the term function changes. The term function now refers to quantities that vary relatively to a low, abscissa or ordinate. (Gayon & Petit, 2018, p. 163)

Johann Bernoulli was the first to provide an explicit definition of function in his paper *Remarks on what we have given so far of solutions of the problems on isoperimetries, with a new short and easy method of solving them in the computation, which also extends to other problems which are related to those (Remarques sur ce qu'on a donné jusqu'ici de solutions des problèmes sur les isopérimètres, avec une nouvelle méthode courte et facile de les résoudre sans le calcul, laquelle s'étend aussi à d'autres des problèmes qui ont rapport à ceux-là)* published in 1718.

> "Definition. We call here Function of a variable quantity, a quantity composed in any way of this variable quantity and of constants." (Bernoulli, 1718, p. 106)

By function, Bernoulli meant an analytical expression of functions, since, as mentioned above, it was the concept of function used at that time. Euler, Bernoulli's student, further developed the concept of function. He was the first one to use the abbreviation f for function in his paper "*Addendum to the dissertation on infinite(ly many) curves of the same type"* published in 1740. He writes:

> "$f\left(\frac{x}{a}+c\right)$ denotes some function of $\frac{x}{a}+c$" (Euler, 1748, p. 3)

[3] *Methodus tangentium inversa seu de functionibus*, published in the Academy edition under the title *De functionibus plagulae quattuor*

In the first chapter *On Functions in General* of the first Volume of *Introduction to analysis of the infinite* (finished in 1744 and published 1748), Euler gives the following definitions:

> *"1. A constant quantity is a determined quantity which always keeps the same value.*
>
> Quantities of this type are numbers of any sort which keep the same constant value, once they have been assigned; if it is required to represent constant quantities by a symbol, the initial letters of the alphabet are used a, b, c, etc. […]
>
> *2. A variable quantity is one which is not determined or is universal, which can take on any value.*
>
> Since all determined values can be expressed as numbers, a variable quantity takes on all possible numbers (all numbers of all types). […] Variable quantities of this kind are usually represented by the final letters of the alphabet z, y, x, etc.
>
> *3. A variable quantity is determined when some definite value is assigned to it.*
>
> Hence a variable quantity can be determined in infinite ways, since absolutely all numbers can be substituted for it. Nor is the symbol of the variable quantity exhausted until all definite numbers have been assigned to it. Thus, a variable quantity encompasses within itself absolutely all numbers, both positive and negative, integers and rationals, irrationals and transcendentals. Even zero and complex numbers are not excluded from the signification of a variable quantity.
>
> *4. A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities.*
>
> Hence every analytical expression, in which all component quantities except the variable z are constants, will be a function of that z; thus $a + 3z; az - 4z^2; az + b\sqrt{a^2 - z^2}$; c^z; etc. are functions of z. " (Euler & translated by Blandon, 1988, pp. 2,3)

These definitions give a clearer meaning to the concept of function and what was meant by analytical expression. Euler goes further and gives some more definitions:

> "6*. The principal distinction between functions, as to the method of combining the variable quantity and the constant quantities is here set down*.
>
> In fact, it depends on the operations by which the quantities are arranged and mixed together. These operations are addition, subtraction, multiplication, division, raising to a power, and extraction of roots. Furthermore, the solution of equations has to be considered. Besides these operations, which are usually called algebraic, there are many others which are transcendental, such as exponentials, logarithms, and others which integral calculus supplies in abundance.
>
> *7.Functions are divided into algebraic and transcendental. The former are those made up from only algebraic operations, the latter are those which involve transcendental operations.*" (Euler & translated by Blandon, 1988, pp. 2,3)

Here, Euler makes a distinction between algebraic analytical expression (analytical function) and transcendental analytical expression (transcendental function). He specifies that analytical functions are those which can be expressed in finite terms of addition, subtraction, division, raising to the power and extraction of roots, so, those which satisfy a polynomial equation which is similar to what Descartes did. Furthermore, transcendental functions are those who do not satisfy any polynomial equation and cannot be written as a finite sum of algebraic operation. He gives as example the exponential, logarithm function and an infinite number of other functions which can be found through the calculation of integrals. Transcendental functions are therefore functions that can be expressed by infinite series and, as mentioned before, these series simplify the integration and differentiation of those functions. For example, by supposing that $|x| < 1$ and integrating the following infinite series termwise (or by transcendental operations),

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

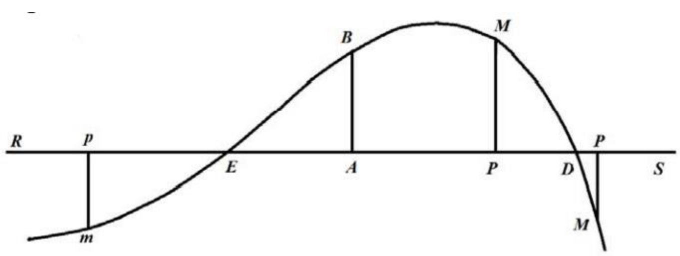
becomes

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

The term transcended is somehow related to the idea of infinity. To resume, for Euler $log(x)$ is a transcendent function but $log(25) + x$ is an algebraic function.

Euler also published a second volume, in which he focuses on the theory of plane curves. In this volume, he explains that other types of function exist which he called "*discontinuous or mixed curves*". He defines these types of analytical expressions in the following way:

"



Some function of x will supply the needs either of certain straight or curved lines, from which in turn the curved lines will be permitted to be called functions. Therefore, the nature of each curved line may be expressed by a function of x of this kind, which may always show the true length of this applied line MP, while the interval AP to which the perpendiculars MP may be sent from the individual points of the curve M towards the right line RS, are indicated by the variable x.

9. From this idea of curved lines the division of these follows at once into continuous and discontinuous or mixed curves. Clearly a continuous curved line has been prepared thus, so that its nature may be expressed by a single defined function of x. But if moreover the curved line may be prepared thus, so that the various parts of this BM, MD, DM etc. may be expressed by various functions of x, thus so that, after the part BM had been described from a single function, then from another function the part MD may be described, we will call curved lines of this kind discontinuous, or mixed and irregular, on account of which they may not be formed from a single constant law, and instead are composed from parts of various curves. " (Euler, 1748, pp. 3,4)

Euler's interpretation of continuity and discontinuity is different from the present understanding of the two words, as A.P. Youschkevitch explains:

> "In EULER'S sense continuity meant invariability, immutability of the law - of the equation determining the function over all the domain of values of the independent variable, while discontinuity of a function meant a change of the analytical law, an existence of different laws on two or more intervals of this domain. *Discontinuous* curves, explained EULER, are composed from *continuous* parts, being exactly for this reason called *mixed* or *irregular (irregulares);* also, he sometimes called such curves *mechanical (mechanices).* In geometry, according to EULER, mainly *continuous (i.e.* analytic) curves are studied. *Discontinuous,* or *mixed* functions and curves of Volume 2 of the *Introductio* correspond to our piecewise analytic functions thus their inclusion into mathematical analysis offered no essential extension of the concept of function." (Youschkevitch, 1975, p. 64)

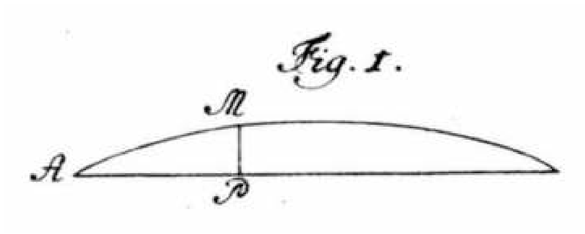
In other words, a *continuous* function could be expressed by a single analytical or transcendental expression. For instance, $y = ax^3 + bx^2 + cx + d$ with $a, b, c, d, x \epsilon \mathbb{R}$ or $y = \sin(x)$ with $x \epsilon \mathbb{R}$. On the other hand, *discontinuous* functions can be expressed by more than one single analytical expression. For example, $y = \begin{cases} x & for\ x \geq 0 \\ -x & for\ x < 0 \end{cases}$. Currently, this function is called the *absolute value function* which is a continuous function in modern sense. This shows that the meaning of continuity and discontinuity are not the same as nowadays. In fact, for Euler $f(x) = \frac{1}{x}$ is a continuous function which is not the case according to the current understanding. This function is continuous on the domain $\mathbb{R}\backslash\{0\}$, but is not continuous over the domain $\mathbb{R}$ because it is not defined at $x = 0$. Euler sees the meaning of continuity more as a relation between the values of the function than "being connected geometrically".

The interest in the problem of a vibrating string incurred new attention and, by extension, so did the concept of function as well. Taylor and J. Bernoulli were, for instance, both interested in determining the motion of a string and found that the curve needed to be sinusoidal, naming the phenomenon "Companion of the Cycloid" at that time. (Jan Struik, 2014, pp. 351,352)

In response to this, Jean d'Alembert published the paper *Research on the curve which forms a taut string set in vibration (Recherches sur la courbe qui forme une corde tendue mise en vibration)* in 1747.

He starts his paper by the following paragraph:

> "I propose myself to show in this thesis, that there are infinitely other curves than *the Companion of the elongated cycloid,* which does to the problem in question. I always suppose 1st, that the excursion or vibration of the string are very small, such that the arc AM of the curve which if forms, may always be supposed sensibly equal to correspondent abscissas AP. 2nd that the string is uniformly thick in the entire length. […] We appoint AP or AM , s , PM ,y." (D'Alembert, 1750, p. 214)



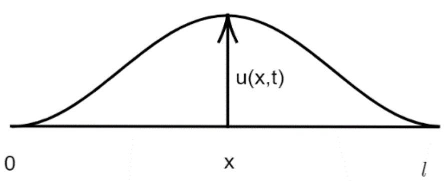
He also denotes the weight by m and the length of the string by l.

> “Let t be the time elapsed since the string started vibrating; it is certain that the ordinate PM can only be expressed by a function of time t and an abscissa or the corresponding arc s or AP.[…]
>
> V. Let α ne the space of a heavy body animated by the gravity p would travel in a given and constant time θ,we get [..] $\alpha = \beta \frac{2aml}{\theta^2}$.
>
> VI. We first not, that we can represent the time θ by a constant line of such magnitude as we like: it will only be necessary to take care to take, in order to express the variable and indetermined parts of the time, lines t that are to the line that we going to take to represent θ, in the ratio of those variable parts of the time to a given constant time, during which a heavy body traverse the space a. We can therefore suppose that θ to be such that $\theta^2 = 2aml$” (D'Alembert, 1750, pp. 214-216)

D’Alembert wrote $\alpha = \beta \frac{2aml}{\theta^2}$ instead of $\frac{\partial^2 y}{\partial^2 t} = \frac{2aml}{\theta^2}\frac{\partial^2 y}{\partial^2 s}$ and since the unit of time can be chosen, he takes θ such that $\theta^2 = 2aml$. In a modern notation, s would be written as x and y as $u(x,t)$. Finally, we get

$$\frac{\partial^2 u(x,t)}{\partial^2 t} = \frac{\partial^2 u(x,t)}{\partial^2 x}$$

To resume, $u(x,t)$ is the distance of the string to his resting position at a time t and at an abscissa x.



After some further thoughts and manipulations, D’Alembert introduced the general solution of the differential equation, given by $u(x,t) = \varphi(t+x) + \psi(t-x)$. The first function, φ, represents a right traveling wave and the second function, ψ, represents a left traveling wave. This general solution not only has to satisfy the differential equation i.e. need to be twice differentiable, but also had to satisfy some other initial conditions. The latter means, $u(x,t)$ must be 0 at the points where the string is attached, here at $x = 0$ and $x = l$ and at time $t = 0$.

Therefore, d'Alembert gets at the abscissa point $x = 0$, $0 = \varphi(t) + \psi(t)$ which gives $\varphi(t) = -\psi(t)$ and so $-\psi(t - x) = \varphi(t - x)$. Finally, he gets

$$u(x,t) = \varphi(t + x) - \varphi(t - x).$$

At time point $t = 0$, we also have $u(x, 0) = 0$ since the string is not set in vibration yet and is therefore still in his resting position. At $t = 0$, $u(x,t) = \varphi(t + x) - \varphi(t - x)$ becomes $\varphi(x) = \varphi(-x)$ hence φ is an even function.

He continues, since $u(l,t) = 0$ when $x = l$, $u(x,t) = \varphi(t + x) - \varphi(t - x)$ becomes $0 = \varphi(t + l) - \varphi(t - l)$.

So, he needs to find $\varphi(t + x)$ so that $\varphi(x) - \varphi(-x) = 0$ and $\varphi(t + l) - \varphi(t - l) = 0$.

To finalise this, he takes the curve tOT and discovers that tO and OT are of equal portions. Then, he constructs TK which is similar and equal to tO and KX is similar and equal to OT. Since the pair tO and OT are similar and equal, TK and KX are also similar and equal. This procedure can be repeated infinitely on both sides of the point O and so the curve must be periodic (like the cycloid found by Taylor and Bernoulli).

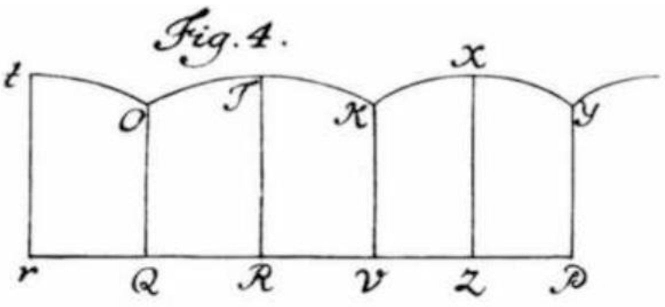


Figure: (D'Alembert, 1750)

The same year, D'Alembert published a sequel of his theses where he analyses different types and times of vibration. (D'Alembert, 1750) In thesis 1 of *Mathematical booklets (Opuscules Mathématiques)* (D'Alembert, 1747, pp. 6,7) from 1747, he comments on two previously published papers. He specifies that if we take the general solution $u(x,t) = \varphi(t + x) + \psi(t - x)$, at $x = 0$ and $x = l$ we know that $u(x,t) = 0$ and we get $\varphi(t) = -\psi(t)$ and $\varphi(t + l) = -\psi(t - l)$. So, $\varphi(t) = -\psi(t) = -\psi(t + l - l) = -\psi\big((t + l) - l\big) = \varphi\big((t + l) + l\big) = \varphi(t + 2l)$. He concludes that φ would then be an odd arbitrary function which is periodic and needs to satisfy one and the same equation. If the latter is not satisfied, then "the problem cannot be solved and will not be analysed" (D'Alembert, 1747, p. 7). So, the function, in Euler's sense, is continuous, since the function needs to satisfy only one and the same equation or, in other words, the law of the equation determining the function must be invariable.

Hereupon, Euler in 1750 publishes the paper *About the vibrating string* and starts the following way:

"Although all that Messrs. Taylor, Bernoulli, and some others, have said and discovered so far on the subject of the vibratory motion of strings, seems to have exhausted the matter, however there are still a double limitation, it restricts it so much, that there are hardly any cases, where the true motion of a vibrating string can be determined. Because first, they have supposed that the stretched strings make only almost infinitesimally small vibrations, so that in this motion, of the string, whether it is a straight or a curve, can always be considered of the same length. The other limitation consists, in that they assume that all vibrations are regular, claiming that in every vibration the entire string, and all at once, is stretched directly, and, looking for its curved figure starting from this situation, they have found it to be a trochoid[4] prolonged to infinity." (Euler, 1750, p. 69)

Euler rejects the first limitation since, no matter what curvature of the curve is chosen, the conclusion does not change significantly. Regarding the second limitation, Euler explains that, if we do not consider a regular vibration, then after a short uniform elapse of time, we will not have the trochoid. As a matter of fact, every vibration depends on the previous vibration i.e. if we start with an irregular vibration then the vibration will never reach a perfect regular movement.

"Now the first vibration depends of our good pleasure, since, before we drop the string, we give him an arbitrary shape; which makes the arbitrary movement of the same string vary at infinity, depending on the shape that we give to the string at the beginning of the motion.

IV. From this there arises therefore the following question, in which the whole research is comprised.

If a string of given length and mass is stretched by a given force or weight; and if instead of the straight situation we give it an arbitrary shape, which however differs by only an infinitesimal amount from a straight line, and if it then released suddenly; determine the total vibratory motion, with which it will be agitated." (Euler, 1750, p. 70)

The straight situation has been covered by D'Alembert and Euler underlines that the general solution given by D'Alembert is "very beautiful" and that it does not differ much from his own findings, but given the fact that he has considered an arbitrary shape for the string i.e. "eellike curve, either regular, contained in a certain equation, or irregular or mechanical" (Euler, 1750, p. 80), Euler is convinced that he gives some interesting observations in the use of the general solution.

Euler's general solution is $u(x,t) = f\left(x + t\sqrt{b}\right) + \varphi\left(x - t\sqrt{b}\right)$ with $b = \frac{F\,l}{2\,M}$. l is the length of the string, M the mass, F the force by which the string is stretched and φ, ψ are two arbitrary functions. By taking the same initial conditions as D'Alembert took, he gets $u(x,t) = \frac{1}{2}f\left(x + t\sqrt{b}\right) + \frac{1}{2}f\left(x - t\sqrt{b}\right)$. (Euler, 1750, p. 81)

At the end of the paper, he writes:

[4] denoting a curve traced by a point on a radius of a circle rotating along a straight line or another circle (a cycloid, epicycloid, or hypocycloid)

"Thus having given the general solution, let us now consider some special cases of it, in which the eellike curve is a continuous curve, of which the parts are related by virtue of the law of continuity, in such

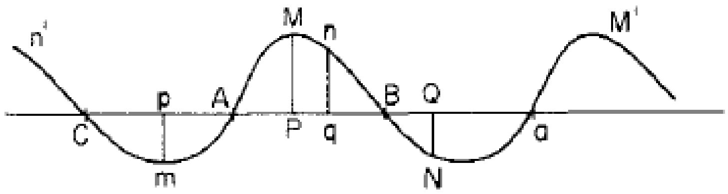


Figure: eellike curve

a way that its nature can be expressed by an equation. And first: these curves always will be transcendental, since they are out by the axis in an infinity of points. If the length of the string $AB = a$, and an arbitrary abscissa $AP = u$, and if $1 : \pi$ as the diameter of the circle to the circumference, then it is clear that the following equation, expressed by sines, provides a curve of the required form:

$$PM = \alpha \sin\left(\frac{\pi u}{a}\right) + \beta \sin\left(\frac{2\pi u}{a}\right) + \gamma \sin\left(\frac{3\pi u}{a}\right) + \cdots$$

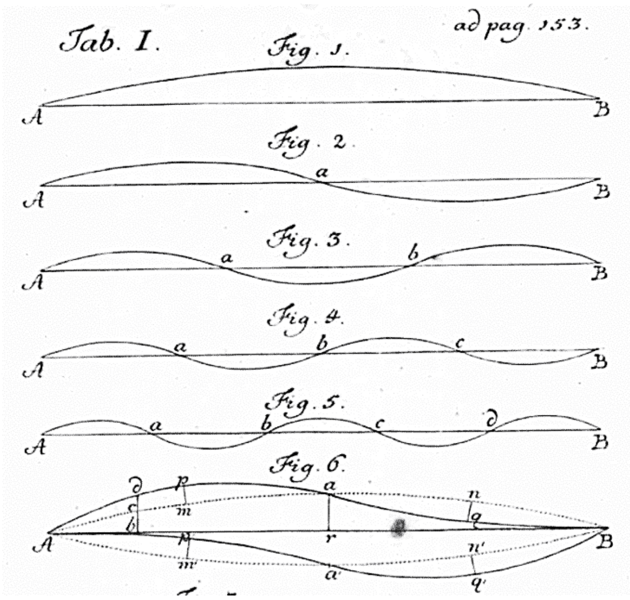
"with $\alpha, \beta, \gamma, \ldots \epsilon\ \mathbb{R}$. (Euler, 1750, p. 84)

D'Alembert did not agree with Euler since Euler considers an eellike curve which can be either irregular or mechanical. A long controversy began. The main disagreement between these two mathematicians was the nature of the arbitrary function. For D'Alembert, the function needs to be expressed by one and the same equation, which could be algebraic or transcendental, as mentioned before. The main one of D'Alembert's arguments, to explain why the use of Euler's change of the analytical expression is not appropriate, is based on his beliefs concerning the basis of analysis, as he testifies in his first thesis of the *Opuscules*:

"I am adding, that is against all rules of analysis, to change the shape, according to the need we believe, to the integral of a differential equation." (D'Alembert, 1747, p. 32)

In 1753, Daniel Bernoulli introduced a new way to approach the problem thanks to his paper *Reflections and clarifications on the new vibrations chords exhibited in the memories of the Academy of 1747 & 1748 (Réflexions et éclaircissements sur les nouvelles vibrations des cordes exposées dans les mémoires de l'Académie de 1747 & 1748).* He writes:

"



I will use the same notation as Mr. Euler. Let $AB = a$ be the length of the curve; $\pi =$ to the semicircumference of the circle whose radius is expressed by the unit, the largest ordinate at the middle of every loop for the first figure be $= \alpha$, for the third $= \gamma$, for the fourth $= \delta$, and finally let x be an arbitrary abscissa, and y the ordinate for the abscissa; then we shall have according to MR. Taylor:

for the first figure $y = \alpha \sin\left(\frac{\pi x}{a}\right)$,

for the second figure $y = \beta \sin\left(\frac{2\pi x}{a}\right)$,

for the third figure $y = \gamma \sin\left(\frac{3\pi x}{a}\right)$,

for the third figure $y = \delta \sin\left(\frac{4\pi x}{a}\right)$ and so forth.

By combing all these curves, according to Fig. 6, for which we have combined only the first two figures, we shall have in the general case the following equation for the same abscissa x :

$$y = \alpha \sin\left(\frac{\pi x}{a}\right) + \beta \sin\left(\frac{2\pi x}{a}\right) + \gamma \sin\left(\frac{3\pi x}{a}\right) + \delta \sin\left(\frac{4\pi x}{a}\right) + \cdots$$

in which the quantities $\alpha, \beta, \gamma, \delta$, etc. are positive or negative quantities.

XIII. Here we have therefore found this infinity of curves without any computation, and our equation is the same as that of Mr. Euler." (Bernoulli, 1747,1748, pp. 156,157)

Daniel Bernoulli finds the same solution as Euler, but he claims that this is the most general one, as opposed to Euler, to whom this solution is only a special case. Euler replies by writing the paper *Remark on the previous thesis of M.Bernoulli (*Remarque sur les mémoires précédents de M.Bernoulli) published in 1755. In this paper, he undertakes to prove that all the curves representing the vibrations of the string are not necessarily included in D. Bernoulli's solution and therefore, he rejects the superpositions of infinitely many oscillations.

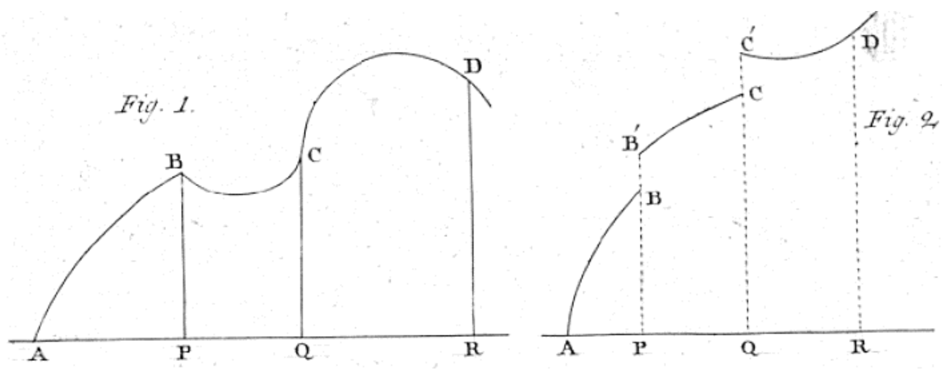
D'Alembert rejected D. Bernoulli solution too. Time went by and the controversy did not end. In 1755, Lagrange entered the debate. His solutions agreed with those of Euler and highlighted that it is necessary to consider the functions called *irregular* or *discontinuous* (Lagrange, 1760-1761, p. 158).

The consequence of all the debates and discussions was that Euler could improve upon his definition of a function which is very similar to the modern definition still in use. The definition was published 1755, in his paper *Lectures on differential calculus (Institutiones calculi differentialis)*:

> "If some quantities so depend on other quantities that if the latter are changed the former undergo change, then the former quantities are called functions of the latter. This denomination is of broadest nature and comprises every method by means of which one quantity could be determined by others. If, therefore, x denotes a variable quantity, then all quantities which depend upon x in any way or are determined by it are called functions of it" (Edwards, Jr., 1979, p. 271)

With regards to the continuous or discontinuous functions, the definition of continuous function has not changed to a great extent, however, a clear definition of the discontinuous function has not been given. The definition closest to what we use nowadays is that given by Louis Arbogast in his paper *Thesis on the nature of the arbitrary functions which enter the integrals of the partial differential equations (Mémoire sur la nature des fonctions arbitraires qui entrent dans les intégrales des équations aux différentielles partielles)* published in 1791, even though it is not given under the name of discontinuity, but discontiguity instead. He wrote:

"



The law of continuity consists in that a quantity cannot pass from one state to another without passing through all the intermediate states which are subject to the same law. Algebraic functions are regarded as continuous because the different values of these functions depend in the same manner on those of the variable; and, supposing that the variable increases continually, the function will receive corresponding variations; but it will not pass from one value to another without also passing through all the intermediate values. Thus the ordinate y of an algebraic curve, when the abscissa x varies, cannot pass brusquely from one value to another; there cannot be a saltus from one ordinate to another which differs from it by an assignable quantity; but all the successive values of y must be linked together by one and the same law which makes the extremities of these ordinates make up a regular and continuous curve.

This continuity may be destroyed in two manners.

1. The function may change its form, that is to say, the law by which the function depends on the variable may change all at once. A curve formed by the assemblage of many portions of different curves is of this kind. Let (Fig 1.) the curve $ABCD$ be a piece of the parabola from A to B, from B to C a piece of an ellipse, from C to D a piece of a circle. For this curve, the ordinate y, which is in general a function of x, cannot be determined for every point by one and the same equation, but by different equations, where one starts, where the other ceases to take place. [...] It is not even necessary that the function y should be expressed by an equation for a certain interval PQ of the variable; it may continually change its form, and the line $ABCD$ representing it, instead of being an assemblage of regular curves, may be such that at each of its points it becomes a different curve; that is to say, it may be entirely irregular and not follow any law for any interval however small. Such would be a curve traced at hazard by the free movement of the hand. These kinds of curves can neither be represented by one nor by many algebraic or transcendental equations.

We will call curves of this kind discontiguous curves […].

2. The law of continuity is again broken when the different parts of a curve do not join to one another. Such as in fig. 2, composed by the parts $AB, B'C, C'D$. Here the second part $B'C$ does not start at point B where the fist finish, the part $B'C$ is more elevated, such that at the same abscissa AP the are to ordinates PB & PB' how corresponds, whereas the first belongs to the part AB, and the other one to the part $B'C$.

We will call curves of this kind discontiguous curves, because all their parts are not contiguous (continuous), and similarly for discontiguous functions" (Arbogast, 1791, pp. 9-11) (Arbogast, 1791, pp. 9-11)

No important developments were made regarding the concept of "function" and its representation by trigonometric series until 1822, when Joseph Fourier published a paper named *The analytical theory of heat (Théorie analytique de la chaleur).* In section VI of his publication, titled *Development of an arbitrary function by trigonometric series (Développement d'une fonction arbitraire en séries trigonométriques)* (Fourier, 1822, pp. 210-261)*,* he explains that an arbitrary function could be written in the following way:

"

$$F(x) = A + \begin{cases} a_1 \, \cos x + a_2 \cos 2x + a_3 cos 3x + a_4 \cos 4x + \cdots \\ b_1 \, \sin \mathrm{x} + b_2 \sin 2x + b_3 \sin 3x + b_4 \sin 4x + \cdots \end{cases}$$

Every coefficient is a definite integral. Generally we have

$$2\pi A = \int_{-\pi}^{\pi} dx\ Fx$$

$$\pi a_i = \int_{-\pi}^{\pi} dx\ Fx \cos ix$$

and

$$\pi b_i = \int_{-\pi}^{\pi} dx\ Fx \sin ix\,."$$

(Fourier, 1822, p. 619)

Today's notations are slightly different. First a_0 and b_0 need to be found.
So since (and by using the modern notation),

$$2\pi A = \int_{-\pi}^{\pi} f(x)\, dx \qquad (1)$$

it results in

$$\pi a_0 = \int_{-\pi}^{\pi} f(x)\cos(0\cdot x)dx \Leftrightarrow a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \overset{(1)}{\Leftrightarrow} a_0 = \frac{1}{\pi}\cdot 2\pi A \Leftrightarrow A = \frac{a_0}{2}$$

and

$$\pi b_0 = \int_{-\pi}^{\pi} f(x)\sin(0\cdot x)dx \Leftrightarrow \pi b_0 = 0 \Leftrightarrow b_0 = 0.$$

Consequently, an arbitrary function $f(x)$ can be written as

$$f(x) = \begin{cases} \frac{1}{2}a_0 + a_1\ \cos(x) + a_2\cos(2x) + a_3\cos(3x) + a_4\cos(4x) + \cdots \\ \qquad\qquad b_1\ \sin(x) + b_2\sin(2x) + b_3\sin(3x) + b_4\sin(4x) + \cdots \end{cases}$$

with the coefficients given by the following formulas;

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\, dx$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx\ .$$

Fourier was probably not the first person using the name *definite integral.* However, in his book, he explains the meaning of $\int_a^b$, which is supposed to symbolise the definite integral. In fact, he writes:

"Generally, we are going to name, by the sign, $\int_a^b$ the integral which begins when the variable (x) is equal to a, and which will be completed when the variable is equal to b." (Fourier, 1822, p. 252)

In the final chapter of his book, he also gives the definition of an arbitrary function:

"In general, the function $f(x)$ represents a succession of values or ordinates, each of which is arbitrary. The abscissa x which can receive an infinite number of values, there is an equal number of ordinates $f(x)$. [...] These ordinates are not supposed to be subject to a common law; they succeed one another in any way possible, and each of them is given as if it was a single quantity. It may follow from the very nature of the problem, and from the analysis which is applicable to it, so that the passage from one ordinate to the following is affected in a continuous manner. However, there are special conditions that must be taken into consideration. A general equation, considered by itself is independent of these conditions, which is rigorously applicable to discontinuous functions." (Fourier, 1822, p. 552)

When Fourier refers to *discontinuous functions*, he means a piecewise-smooth function, i.e. discontinuity in Euler's sense.

3.3 Cauchy's definition of definite integral

Fourier however does not provide a clear definition of the integral of these arbitrary functions and how to calculate it in his book. Therefore, one year later, Cauchy follows with the publication of his paper *Summary of the lessons on infinitesimal calculus given at the Poltytechnic School (Résumé des leçons sur le calcul infinitésimal donné à l'école polytechnique)* in which he gives a rigorous definition of a definite integral of a continuous function.

"INTEGRAL CALCULATION

TWENTYFIRST LESSON

DEFINITE INTEGRAL

Let us assume that the function $y = f(x)$ is continuous [in the modern sense] with respect to the variable x between the finite limits $x = x_0, x = X$, we designate by $x_1, x_2, \dots, x_{n-1}$ the new values of x , laying between these two limits. The letters are always increasing or decreasing from the first limit to the second one. We can use the values to divide the difference $X - x_0$ into elements

$$(1) \qquad x_1 - x_0, x_2 - x_1, x_3 - x_2, \dots, X - x_{n-1}$$

all of which have the same sign. That set, let us multiply every element [subinterval] with the value $f(x)$, that corresponds to the *origin* of the same element, namely the element $x_1 - x_0$ with $f(x)$, the element $x_2 - x_1$ with $f(x_1)$,..., and finally the element $X - x_{n-1}$ with $f(x_{n-1})$ and let

(2) $$S = (x_1 - x_0)f(x_0) + (x_2 - x_1)f(x_1) + \cdots + (X - x_{n-1})f(x_{n-1})$$

be the sum of the products that we obtain. The quantity S will clearly depend upon:

1°, the chosen number n of the elements by which we have divided the difference $X - x_0$ and

2°, the values of these elements and consequently on the mode of division [partition]. It is important to note that if the numeric values [lengths] of these elements become very small and the number n very large, the mode of subdivision will only have an imperceptible influence on the value of S. This will be demonstrated as follows.

On the first hand, if we suppose that all the elements of the difference $X - x_0$ are reduced to only one difference, which would be this difference itself, we would simply have

(3) $$S = (X - x_0)f(x_0).$$

On the other hand, when we take the expression (1) as the elements of the difference $X - x_0$, the value of S, defined in this case by the equation (2), is equal to the sum of the elements multiplied by the average of the coefficients

$$f(x_0), f(x_1), f(x_2), \ldots, f(x_{n-1}).$$

(see, in the preliminary *analysis courses*, the corollary of the theorem III)." (Cauchy, 1823, pp. 122,123)

The corollary of the theorem III of his *analysis courses (Cours d'analyse)* says: If $\alpha_1, \ldots \ldots, \alpha_n$ are positive numbers, and $a_1, \ldots \ldots, a_n$ are arbitrary numbers, then

$$\sum_{i=1}^{n} \alpha_i a_i = M(a_1, \ldots \ldots, a_n)$$

with $M(a_1, \ldots \ldots, a_n)$ representing the average of the quantities $a_1, \ldots \ldots, a_n$. (Cauchy, 1821, pp. 28,29)

"Moreover, these coefficients are special values of the expression

$$f[x_0 + \theta(X - x_0)]$$

with θ lying between zero and the unit $[\theta \in [0,1]]$. By similar reasonings to those of the seventh lesson; the average in question is of another value of the same expression with θ lying between the same limits [the intermediate value theorem]. We can therefore substitute the equation (2) by

(4) $$S = (X - x_0)f[x_0 + \theta(X - x_0)]$$

with θ being a number smaller than the unit $[\theta \in [0,1]]$.

In order to switch from one mode of division to another, where the numeric values of the elements $X - x_0$ are even smaller, it suffices to divide the expression (1) in new elements [new partition that lies in some subintervals of (1)]. Then, we should replace the product of $(x_1 - x_0)f(x_0)$ by a sum of similar products in the second member of the equation (2) and which we can substituted by an expression in the following form

(5) $$(x_1 - x_0)f[x_0 + \theta_0(x_1 - x_0)]$$

with θ_0 a number smaller than the unit $[\theta_0 \epsilon [0,1]]$, expecting that, between this sum and the product $(x_1 - x_0)f(x_0)$, there exists a similar relationship to the one that exists between the values of S , provided by the equations (4) and (3) [Cauchy applies the intermediate value theorem of the interval $[x_0, x_1]$]. By the same reasoning, we need to substitute the product of $(x_2 - x_1)f(x_1)$ by a sum of terms, which can be presented in the form of

$$(x_2 - x_1)f[x_1 + \theta_1(x_2 - x_1)]$$

with θ_1 , a number smaller than the unit. By proceeding in the same way, we will conclude that by using the new mode of division, the value of S will be

$$\begin{aligned} S = (x_1 - x_0)&f[x_0 + \theta_0(x_1 - x_0)] \\ &+(x_2 - x_1)f[x_1 + \theta_1(x_2 - x_1)] + \cdots \\ &+(X - x_{n-1})f[x_{n-1} + \theta_{n-1}(X - x_{n-1})]. \end{aligned}$$

If we make the latter equation

$$\begin{aligned} &f[x_0 + \theta_0(x_1 - x_0)] = f(x_0) \pm \varepsilon_0, \\ &f[x_1 + \theta_1(x_2 - x_1)] = f(x_1) \pm \varepsilon_1, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots., \\ &f[x_{n-1} + \theta_{n-1}(X - x_{n-1})] = f(x_{n-1}) \pm \varepsilon_{n-1.} \end{aligned}$$

We will obtain

$$(6) \qquad \begin{aligned} S = (x_1 - x_0)[f(x_0) \pm \varepsilon_0] + (x_2 - x_1)[f(x_1) \pm \varepsilon_1] + \cdots \\ +(X - x_{n-1})[f(x_{n-1}) \pm \varepsilon_{n-1}], \end{aligned}$$

then, by developing the products

$$(7) \qquad \begin{aligned} S = (x_1 - x_0)f(x_0) + (x_2 - x_1)f(x_1) + \cdots (X - x_{n-1})f(x_{n-1}) \\ \pm \varepsilon_0(x_1 - x_0) \pm \varepsilon_1(x_2 - x_1) \pm \cdots \pm \varepsilon_{n-1}(X - x_{n-1}). \end{aligned}$$

It should be added that, if the elements $x_1 - x_0, x_2 - x_1, \dots, X - x_{n-1}$have very small numerical values, each and one of the quantities $\pm \varepsilon_0, \pm \varepsilon_1, \dots, \pm \varepsilon_{n-1}$ will differ very little from zero, and consequently the same will apply to the sum

$$\pm \varepsilon_0(x_1 - x_0) \pm \varepsilon_1(x_2 - x_1) \pm \cdots \pm \varepsilon_{n-1}(X - x_{n-1}),$$

which is equivalent to the product of $X - x_0$ by an average between these various quantities. This set, and following from the equations (2) and (7), compared with each other, one will not alter perceptibly the value S. The latter is calculated by the mode of division in which the elements of the difference $X - x_0$ have very small numerical values, if one passes to a second mode in which each of these elements is subdivided into many others.

Conceive for the present, that one considers at the same time two modes of division of the difference $X - x_0$, in each of which the elements of the difference have very small numerical values. One will be able to compare these two modes with a third in such a way that each element, be it from the first, or from the second mode is formed by the union of several elements of the third. For this condition to be satisfied, it will suffice that each of the values of x interplaced in the first two modes between the limits x_0 and X be employed in the third, and one will prove that one alters the value of S very little in passing from the first or the second mode to the third, and consequently, in passing from the first to the second. Therefore, when the elements of the difference $X - x_0$ become infinitely small, the division mode has only an insensitive influence on the value of S; and, if we decrease the numerical values of these elements indefinitely, by increasing their number, the value of S will end up being approximately constant or, in other words, it will end up reaching a certain limit which will depend only on the form of the function $f(x)$ and the extreme values x_0, X attributed to the value x. This limit is what we call a *definite integral.*

Let us now observe that, if we refer to $\Delta x = h = dx$ as a finite increment of the variable x, the different terms of which the value S is composed, such that the products,

$$(x_1 - x_0)f(x_0), \quad (x_2 - x_1)f(x_1), \qquad \dots$$

will all be included in the general form

$$hf(x) = f(x)dx, \tag{8}$$

from which they will be deduced one after the other, by first posing

$$x = x_0 \qquad \text{and} \qquad h = x_1 - x_0$$

then

$$x = x_1 \qquad \text{and} \qquad h = x_2 - x_1, \ \dots\dots$$

We can therefore state that the quantity S is a sum of products, similar to the expression (8), which is sometimes expressed by the characteristic $\sum$, by writing

$$S = \sum hf(x) = \sum f(x)\Delta x. \tag{9}$$

The quantity S converges to the definite integral, while the elements of difference $X - x_0$ become infinitely small, which it is agreed to be represented by the notation $\int$, substitute for $\sum$, indicating, no longer a sum of products similar to the expression (8), but a limit of a sum of this kind. Moreover, as the value of the definite integral we are considering, depends on the extreme values x_0, X attributed to the value x, and we place these two values, the first above, the second below the letter $\int$, or to write them next to the integral, which we consequently designate by one of the notations

$$\int_{x_0}^{X} f(x)dx, \qquad \int f(x)dx \begin{bmatrix} x_0 \\ X \end{bmatrix}, \qquad \int f(x)dx \begin{bmatrix} x = x_0 \\ x = X \end{bmatrix}, \tag{10}$$

The first of these notations, devised by M. Fourier, is the simplest. In the particular case, where the function $f(x)$ is replaced by a constant quantity a , regardless of how the difference $X - x_0$ is divided,

$$S = a(X - x_0)$$

And we conclude

$$\int_{x_0}^{X} a\, dx = a(X - x_0). \tag{11}$$

If, in this last formula, we put $a = 1$, we will get

$$\int_{x_0}^{X} dx = (X - x_0)." \tag{12}$$

(Cauchy, 1823, pp. 122-127)

To resume, the limit of the sum $S = \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1})$ over every division of the interval $[a, b]$, $a = x_0 < x_1 < x_2 < \cdots x_{n-1} < x_n = b$, is $\int_a^b f(x)dx$, when the maximum length of all the subintervals of the interval $[a, b]$, tends to 0 when n tends to infinity. It can be observed that Cauchy does not divide the partition in n equal parts. However, in one of his lessons, he explains how to find the integrals of certain continuous functions and divides the intervals in equal parts for simplicity. In other lessons, he derives the linearity laws of the integral, as well as other properties of the integral which are still used nowadays.

The research on how to integrate an arbitrary function was still incomplete since Cauchy only took into account continuous function or piecewise continuous function. At that time in fact, a rigorous definition of discontinuous function in the modern sense still did not exist yet. Mathematicians still thought that all functions could be classified in two categories, as Euler defined them, i.e. continuous function and mixed function. However, the idea of a mixed function was more and more criticised by the scientific community. Therefore, Cauchy himself felt the need to reanalyse Euler's definition of mixed function. In his paper *Thesis about continuous function (Mémoire sur les fonctions continues)* published in 1844, Cauchy explains, as mentioned before, that Euler classified the function $y = \begin{cases} x & for\ x \geq 0 \\ -x & for\ x < 0 \end{cases}$ as a discontinuous function for instance. However, since it can be represented by the radical $\sqrt{x^2}$ and consequently by a single equation, it could also be classified as continuous. This proved that the distinction that Euler made, between mixed function and a continuous function, could not be sustained anymore. (Cauchy, 1844)

Returning however pertaining to the question how to represent a function by a trigonometric series, further developments were made by Cauchy and Dirichlet. This is due to the fact that both were, *a priori,* interested in the convergence of trigonometric series. Cauchy did not manage to give a rigorous proof, but Dirichlet did, in his paper *On the convergence of trigonometric series which are used to represent an arbitrary function between limits (Sur la convergence des séries trigonométriques qui servent à représenter une fonction arbitraire entre des limites)* published in 1829. In the latter for instance, he proves that a 2π periodic function needs to be bounded, piecewise continuous and does not have to have too many minima and maxima in the interval $[-\pi, \pi]$ for the trigonometric series to converge. So, he explains and adds:

> "It is necessary that the function $f(x)$ is such that, if we nominate by a and b two arbitrary quantities lying between $-\pi$ and π, we can always place between a and b other quantities r and s relatively close together such that the function stays continuous in the interval from r to s." (Lejeune Dirichlet, 1829, p. 169)

At the end of his paper, he then provides us with the famous example, known as the *Dirichlet's function,* which does not fulfil the above-mentioned condition. In fact, *Dirichlet's function*

$$f(x) = \begin{cases} c, & when\ x\ is\ rational \\ d, & when\ x\ is\ irractional \end{cases}, \quad c \neq d, \quad c, d \in \mathbb{R}$$

is discontinuous at every $x \in \mathbb{R}$ i.e. has too many minima and maxima.

Proof:

Let $\varepsilon = \frac{|c-d|}{2}$ and let δ be a small real number.

1. Let $x, y \in \mathbb{Q}$ such that $|x-y| < \delta$. Suppose that $y > x$, since between two rational numbers there exists an irrational number, it exists an $z \in \mathbb{R}\backslash\mathbb{Q}$ such that $x < z < y$ and so $|f(x) - f(z)| = |c-d| > \varepsilon$. Similarly, if $y < x$ it exists an $z \in \mathbb{R}\backslash\mathbb{Q}$ such that $y < z < x$ and so $|f(x) - f(z)| = |c-d| > \varepsilon$.
2. Let now, $x, y \in \mathbb{R}\backslash\mathbb{Q}$ be such that $|x-y| < \delta$. Suppose that $y > x$, since between two real numbers there exists a rational number, it exists an $z \in \mathbb{Q}$ such that $x < z < y$ and so $|f(x) - f(z)| = |c-d| > \varepsilon$. Similarly, if $y < x$ it exists an $z \in \mathbb{Q}$ such that $y < z < x$ and so $|f(x) - f(z)| = |c-d| > \varepsilon$.

□

Dirichlet concludes:

> "The restriction that I have just specified and, in addition, the condition of not becoming infinite, are the only ones to which the function $f(x)$ is subjected. […] However, in order to reach as much clarity as one desires, requires some details linked to the main fundamentals of the analysis, which will be exposed in another note." (Lejeune Dirichlet, 1829, p. 169)

Unfortunately, he never kept this promise. Nevertheless, he published another paper in 1837 titled *On the representation of completely arbitrary functions by means of sinusoidal and consinusoidal series (Über die Darstellung ganz willkürlicher Funktionen durch Sinus- und Consinusreihen)* in which he gives the definition of a function and of the definite integral.

> "Let A and B be the end-points of a and b, and $\alpha\gamma\beta$ the curve corresponding to the function $f(x)$, so that it is clear, that this function can also be determined by the area $A\alpha\gamma\beta B$, which is delimitated by the ordinates $A\alpha$, $B\beta$ as by the piece AB of the abscissa and by the curve $\alpha\gamma\beta$, in case it cannot always be specified precisely. This area is known as the definite integral of the function $f(x)$ from a to b and is designated as
>
> $$\int_a^b f(x)\, dx.$$
>
> The origin of this symbol depends on how the infinitesimal calculus of an area is looked at. If the line $AB = b - a$ is divided in n equal parts, their common value is $= \frac{b-a}{n} = \delta$, and they will draw parallels to the abscissa passing through the ordinate α and the endpoints, which corresponds to the division points 1,2, … and so rectangles will arise, whose sum is
>
> $$\delta f(a) + \delta f(a+\delta) + \delta f(a+2\delta) \dots + \delta f(a+(n-1)\delta), \tag{1}$$
>
> will pass over to the area $A\alpha\gamma\beta B$ when the number n arises unstoppable i.e. one can always let n get so big that the sum (1) differs from the area less than a predetermined size, no matter

how small. If $b - a$ and δ are taken positively, so seems that rectangles of (1) appears positive or negative, depending if they are lying on the positive side or negative side of y. Conversely, it behaves the other way round when $b - a$ is negative. One can conclude from this, that the definite integral

$$\int_a^b f(x)\,dx$$

(if one sees this as the limit value of (1) when one supposes n to be infinite) can be seen as an area." (Dirichlet, 1900, pp. 4,5)

It can be observed that, unlike Cauchy, Dirichlet subdivides the interval $[a, b]$ into n equal parts. In doing so, he does not require to take first the maximum length of all the subintervals and thereafter let n tend to infinity. This way of partitioning an interval is the most used when it comes to giving the definition of the definite integral, especially in high school as it can be observed in section 2.1 of this thesis.

3.4 Reformulation of the Riemann Integral

Dirichlet's question on if and when a function $f(x)$ can be represented by a trigonometric series when both of the conditions

1. $f(x)$ is integrable
2. $f(x)$ has finitely many maxima and minima

are not fulfilled, was still not answered. Riemann, which was Dirichlet's pupil, set himself the goal to continue Dirichlet's work and to find the answer to this question. It is important to notice that "Cauchy's definition (which was also adopted in his work of 1829) is applicable when the integrand $f(x)$ is continuous in the interval of integration or presents at most finite number of points of discontinuity. However, it ceases to be valid if these points are infinite in number. But it was precisely this case that interested Riemann." (Sachse , 1880, pp. 229-276)

Accordingly, Riemann finished his thesis entitled *On the possibility of representation of a function by trigonometric series (Über die Darstellbarkeit einer Funktion durch eine trigonometrische Reihe)* in 1854, but it was published posthumously in 1868. The fourth section of his thesis starts as follows:

"The uncertainty that still dominates several fundamental points of the theory of definite integrals necessitates establishing a few things about the concept of the definite integral and the extent of its validity." (Riemann, 1854, published in 1868, p. 12)

Regarding the concept of the definite integral, he writes:

"First of all: What do we understand under $\int_a^b f(x)\,dx$?

In order to establish this, we take between a and b following the order of size, a sequence of values $x_1, x_2, \ldots\ldots, x_{n-1}$ and denominate for simplicity $x_1 - a$ by δ_1, $x_2 - x_1$ by $\delta_2, \ldots\ldots, b - x_{n-1}$ by δ_n and by ε $[\varepsilon_i]$a positive real fraction $[\varepsilon_i \in [0,1]]$. The value of the sum

$$S = \delta_1 f(a + \varepsilon_1 \delta_1) + \delta_2 f(x_1 + \varepsilon_2 \delta_2) + \delta_3 f(x_2 + \varepsilon_3 \delta_3) + \cdots + \delta_n f(x_{n-1} + \varepsilon_n \delta_n)$$

will depend on the value of δ $[\delta_i]$ and ε $[\varepsilon_i]$. If it has the property, however δ $[\delta_i]$ and ε $[\varepsilon_i]$ may be chosen, it tends towards a fixed limit A as soon as all the δ's become infinitely small, then this value is called $\int_a^b f(x)\,dx$.

If it does not have this property, then $\int_a^b f(x)\,dx$ has no meaning." (Riemann, 1854, published in 1868, pp. 12,13)

In modern notation, it could be reformulated as follows:

Definition

A *partition* of an interval $[a, b]$ with $a, b \in \mathbb{R}$, is a finite sequence of real numbers denoted as $x_0, x_1, x_2, \ldots, x_n$ with $n \in \mathbb{N}$ such that $a = x_0 < x_1 < x_2 < \ldots < x_n = b$.

Definition

Let f be a function over an interval $[a, b]$ with $a, b \in \mathbb{R}$. Let A be the area under the curve over the same interval. If $\forall\ \varepsilon' > 0$, $\exists \delta > 0$ such that for any partition, we have that the subintervals $|x_i - x_{i-1}| < \delta, \forall i$ and if for any set of tags x'_i such that $x'_i \in [x_{i-1}, x_i]$ then the *definite integral* of f is given by

$$\left| \sum_{i=1}^{n} f(x'_i)(x_i - x_{i-1}) - A \right| < \varepsilon'$$

when ε' becomes arbitrary small. We will denote it by

$$A = \int_a^b f(x)\ dx.$$

There are some clarifications that need to be made. First, the link between Riemann's definition and the modern notation of it demands explication, with the chosen δ corresponding to the maximum length of the δ_i, x'_i corresponding to $x_i + \varepsilon_i \delta_i$ with $i = 1, \ldots, n$ and $\varepsilon_i \in [0,1]$ and ε' corresponding to a small quantity, which can be arbitrarily small. By comparing this definition to Cauchy's, it can be observed that Riemann generalised Cauchy's definition by replacing the initial point x_i with an arbitrary point $x'_i \in [x_{i-1}, x_i]$. Secondly, one has to delve into how the definition of the *definite integral* is obtained. Mostly, necessary and sufficient conditions are turned into a definition. In fact, initially, one supposes that A exists, and then some necessary conditions are described which the function needs to satisfy, in this precise case that δ and ε can be made arbitrary small. Once the necessary conditions that the function needs to satisfy are clarified, so that the area A exists, one can treat these said necessary conditions as sufficient by adopting them as definition. In other words, if A exits then the function needs to satisfy some properties. If the function satisfies these properties, then one can say that the integral exists. Consequently, one can write "let A be the area". The question that arises naturally is: Which functions satisfy this definition? Or when does A exist?

Therefore, Riemann starts his fifth section as follows:

> "Secondly, let us now examine the range of validity of this notion, or the question, in which cases does a function allow for being integrable and in which cases does it not?
>
> We first consider the concept of the integral in the strict sense, i.e. we assume that the sum S converges, when all δ become infinitely small. Thus, let us denote the biggest variation of the function between a and x_1, i.e. the difference of its largest and its smallest value in this interval, by D_1, between x_1 and x_2 by D_2,..., between x_{n-1} and b by D_n, so must
>
> $$\delta_1 D_1 + \delta_2 D_2 + \cdots + \delta_n D_n$$
>
> become infinitely smaller with the quantities δ. We further assume, that, as long as all the δ remain smaller than d, the largest value that this sum can assume is Δ; Δ will then be a function of d, which will always decrease with d and which with this quantity becomes infinitely small. Is now the total size of the intervals, in which the variations are larger than σ, $= s$, so will the contribution of those intervals to the sum $\delta_1 D_1 + \delta_2 D_2 + \cdots + \delta_n D_n$ be obviously $\geqq \sigma s$. One therefore has
>
> $$\sigma s \overset{=}{<} \delta_1 D_1 + \delta_2 D_2 + \cdots + \delta_n D_n \overset{=}{<} \Delta, \text{ hence } s \overset{=}{<} \frac{\Delta}{\sigma}.$$
>
> $\frac{\Delta}{\sigma}$ can now, if σ is given, always be made arbitrarily small by a suitable choice of d; the same applies to s, and therefore it results:
>
> [Necessary condition] In order that the sum S, when all δ $[\delta_i]$ become infinitely small, converges, it is necessary, besides the finiteness of the function $f(x)$, that the total size of the intervals in which the variation $> \sigma$, no matter what σ is given, can be made arbitrary small by a suitable choice of d.
>
> The contrary is also true:
>
> [Sufficient condition] If the function $f(x)$ is always finite [bounded], and with the infinite decreasing of all quantities δ the total size s of the intervals, in which the variations of the function $f(x)$ exceed a given quantity σ, becomes infinitely small, then the sum S converges when all the δ become infinity small.
>
> For those intervals where the variation $> \sigma$ contributes an amount to the sum $\delta_1 D_1 + \delta_2 D_2 + \cdots + \delta_n D_n$ which is smaller than the product of s and the largest variation of the function between a and b, which is finite (by hypothesis); the remaining intervals contribute to an amount $< \sigma(b-a)$. Evidently one can now first take σ arbitrary small, so that s becomes arbitrary small whereby the sum $\delta_1 D_1 + \delta_2 D_2 + \cdots + \delta_n D_n$ also becomes arbitrary small, and consequently the value of the sum S can be enclosed in an arbitrary small narrow limit.
>
> We have thus found conditions which are necessary and sufficient so that the sum S converges when the $\delta's$ become arbitrary small, so that we can speak of an integral of the function $f(x)$ in the narrow sense.

> If we extend, as we have indicated above, the notion of integral in case where the function becomes infinite, for the integration to be possible, it will still be necessary that the second of the conditions found above must still be satisfied; but instead of the first, namely that the function always remains finite, the following must be true: that the function becomes infinite only when its argument approaches certain particular values, and, that we obtain a perfectly defined limit-value, when the limits of the integrations approach indefinitely those values for which the function becomes infinite. (Riemann, 1854, published in 1868, pp. 13-15)"

This can be rewritten using modern notations, which are in fact the notations that Darboux used (see below). First, let f be a discontinuous function and let $m_i = \inf_{[x_{i-1},x_i]} f(x)$ and $M_i = \sup_{[x_{i-1},x_i]} f(x)\ \ \forall i = 1, \dots n$. Riemann explained that if the function is integrable, then there will be some intervals containing the points of discontinuity of the function i.e. $D > \sigma$ (oscillation of the function), whose sum will be s, which will be smaller than $\frac{\varepsilon}{\sigma}$. In fact, by summing over the intervals, whose oscillation is larger than σ, it results that

$$\varepsilon > \sum_i (M_i - m_i)(x_i - x_{i-1}) \geq \sum_i \sigma(x_i - x_{i-1})$$

$$\Rightarrow \sum_i (x_i - x_{i-1}) < \frac{\varepsilon}{\sigma}$$

$$\Rightarrow \qquad s < \frac{\varepsilon}{\sigma}$$

Since ε can be made arbitrary small, the sum of the intervals of those on which the oscillations exceed σ can also be made arbitrary small. (Bressoud, 2009, p. 27)

Riemann proceeds and claims that the contrary is also true and gives the following integrability condition:

Condition for Riemann's Integrability (Bressoud, 2009, p. 27)

Let f be a bounded function on $[a, b]$. The function is integrable over $[a, b]$ if and only if for any $\sigma > 0$, a bound on the oscillation, and for any $s > 0$, a bound on the sum of the length of the intervals where the oscillation exceeds σ, we can find a response δ so that for any partition $[a, b]$ with subintervals of length smaller than δ, the subintervals, on which the oscillations are at least σ, have a combined length that is strictly less than s.

Proof:

Let $m_i = \inf_{[x_{i-1},x_i]} f(x)$ and $M_i = \sup_{[x_{i-1},x_i]} f(x)$ and let D be the difference between the least upper bound and the greatest lower bound of $\{f(x) | a \leq x \leq b\}$, so that $M_i - m_i < D\ \forall i$. Let $\sigma = \frac{\varepsilon}{2(b-a)}$ and choose a limit on the partition intervals so that those on which the oscillation is bigger than σ have a total length less than $\frac{\varepsilon}{2D}$ i.e $s < \frac{\varepsilon}{2D}$. By splitting the sum over the interval $[a, b]$ into the sum over those intervals where the oscillation exceeds

σ, denoted by $\sum_1$, and into the sum over the interval where the oscillations are strictly less than σ, denoted by $\sum_2$, it follows:

$$\begin{aligned}
&\sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1})\\
&= \sum_{1}(M_i - m_i)(x_i - x_{i-1}) + \sum_{2}(M_i - m_i)(x_i - x_{i-1})\\
&< \sum_{1} D(x_i - x_{i-1}) + \sum_{2} \sigma(x_i - x_{i-1})\\
&< D\sum_{1}(x_i - x_{i-1}) + \sigma\sum_{2}(x_i - x_{i-1})\\
&< Ds + \sigma(b-a)\\
&< D\frac{\varepsilon}{2D} + \frac{\varepsilon}{2(b-a)}(b-a)\\
&= \varepsilon
\end{aligned}$$

□

Riemann's definition is considered the most general one, but several mathematicians reformulated Riemann's definition of definite integral. This might be due to the difficulty handling the variability of the $x'_i \in [x_{i-1}, x_i]$. Darboux was one those who reformulated Riemann's definition. In fact, in his paper *Thesis about discontinuous function (Mémoire sur les fonctions discontinues)* published in 1875, he rewrites Riemann's integrability condition in the following way:

"Let us insert, between a and b, $n-1$ values $x_1, x_2, \dots, x_{n-1}$ and let us pose for brevity

$$x_1 - a = \delta_1,\ x_2 - x_1 = \delta_2, \dots,\ b - x_{n-1} = \delta_n.$$

In this way, we will form n intervals, and we will designate by M_i, m_i, Δ_i the maximum limit [supremum], the minimum limit [infimum] and the oscillation in the i-th interval. Let us form the three sums

$$M = M_1\delta_1 + M_2\delta_2 + \cdots + M_n\delta_n$$

$$m = m_1\delta_1 + m_2\delta_2 + \cdots + m_n\delta_n$$

$$\Delta = \Delta_1\delta_1 + \Delta_2\delta_2 + \cdots + \Delta_n\delta_n$$

between which the identical relationship exists

$$\Delta = M - m.$$

I say that, when n is sufficiently big, and that all intervals, δ $[\delta_i]$, converge to zero, the three previous sums, independently of the chosen function; continuous or discontinuous, will each converge to a finite and determined sum, only depending on the nature of the function and on the extreme values a, b which will mark the considered interval (Darboux, 1875, p. 65).

He specifies and explains also that the discontinuous functions to which Riemann refers, are continuous and cannot be derived. Moreover, he gives an explicit definition of the definite integral for a continuous function:

"Let $f(x)$ be a continuous function of x , which stays, when x varies from a to b, between two fixed numbers A, B, and let us take a series of values $x_1, x_2, \ldots, x_{n-1}$ between a and b, ranged by order of magnitude. Let us still designate $x_1 - a$ by δ_1; $x_2 - x_1$by $\delta_2, \ldots$, and in addition let $\theta_1, \theta_2, \ldots, \theta_n$ be positive numbers, smaller than or equal to the unity $\left[\theta_i \epsilon [0,1]\right]$. Let us therefore consider the sum

$$\Sigma = \delta_1 f(a + \theta_1 \delta_1) + \delta_2 f(x_1 + \theta_2 \delta_2) + \cdots + \delta_n f(x_{n-1} + \theta_n \delta_n).$$

It then becomes clear that the sum is simultaneously depended on the choice of the intervals δ $[\delta_i]$ and on the quantities θ $[\theta_i]$. Let us examine first how they vary when we provide θ with a system of possible values.

Let us denote by M_i, m_i the maxima and minima limit of the function of the i-th interval. The term $\delta_i f(a + \theta_i \delta_i)$ remains between $\delta_i M_i$ and $\delta_i m_i$ when θ_{i+1} varies and gets as close as one desires to either of these quantities. So, the sum Σ will remain between the two sums

$$M = M_1 \delta_1 + M_2 \delta_2 + \cdots + M_n \delta_n$$
$$m = m_1 \delta_1 + m_2 \delta_2 + \cdots + m_n \delta_n,$$

to which it can get as close as we want.

So, if we assume that the intervals δ $[\delta_i]$ tend to zero, their number will increase infinitely, so that the sum Σ has a limit, no matter what θ $[\theta_i]$ is. It is necessary and sufficient that the limits of M and of m are equal, and consequently we have

$$M_{ab} = m_{ab}, \qquad \Delta_{ab} = 0$$

So, *the necessary and sufficient condition for the sum Σ to have a limit, is that the total size of intervals, in which the oscillations are bigger than σ, converges to zero, when all the intervals tend to zero, with σ being fixed, but as small as we require it to be*.

If this condition is fulfilled, the limit of Σ is said to be the integral of $f(x)$ between the limits a, b. We therefore have

$$\lim \Sigma = \int_a^b f(x)\, dx = -\int_b^a f(x)\, dx.$$

Remark. – If the sum Δ_{ab} is not zero, the limit of Σ will depend on the choice of θ $[\theta_i]$ and can take all values lying between M_{ab} and m_{ab}. *Σ can, in any case, just converge to a limit or being indefinite.*" (Darboux, 1875, pp. 72,73)

A version combining Darboux's definition and the integrability condition is mostly taught at university, as illustrated in section 2.1 of this thesis.

Riemann and Darboux were both interested in new kinds of functions, which were not considered at that time. Riemann was the first person taking into consideration an integrable function which is discontinuous at a dense set of points and therefore proposes the following example:

$$f(x) = \sum_{i=1}^{n} \frac{(nx)}{n^2} \qquad with \qquad (x) = \begin{cases} x - \lfloor x \rfloor, & \lfloor x \rfloor \le x < \lfloor x \rfloor + \frac{1}{2} \\ 0\,, & x = \lfloor x \rfloor + \frac{1}{2} \\ x - \lfloor x \rfloor - 1, & \lfloor x \rfloor + \frac{1}{2} < x < \lfloor x \rfloor + 1 \end{cases}$$

This example will not be analysed in detail in this thesis, since there are other examples that are easier to grasp, and which fulfil the same purpose i.e. to show how powerful Riemann's integrability condition and the definition are. As illustrated in the following graph, Riemann was however interested in different kinds of functions than those generally proposed to students in school

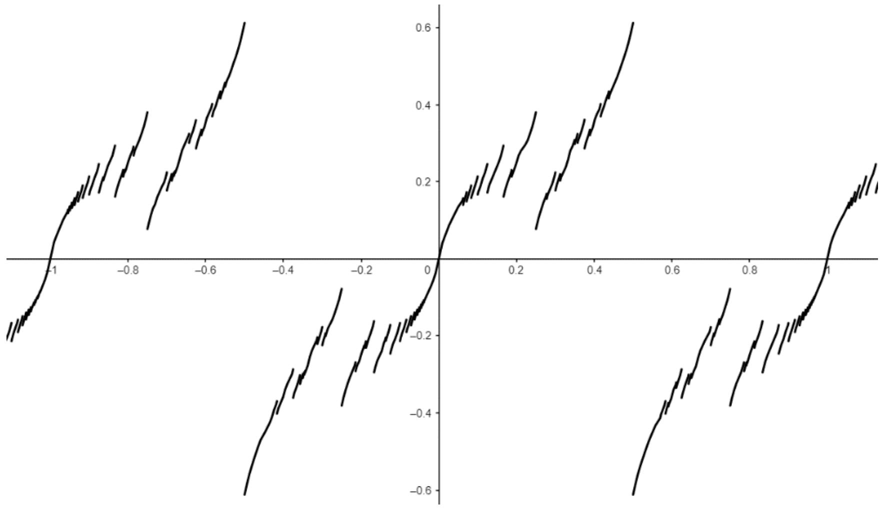


Figure: Riemann example with n=50 (used tool: GeoGebra)

The same year as Darboux, J.K. Thomae publishes the paper *Introduction to the theory of definite integrals (Einleitung in die Theorie der bestimmten Integrale)* in which, besides the example given by Riemann, he presents another example, known as the *Thomae function*:

$$f(x) = \begin{cases} \frac{1}{q} & if\ x = \frac{p}{q} \in \mathbb{Q}\ and\ (p,q) = 1 \\ 0 & if\ x \in \mathbb{R} \setminus \mathbb{Q} \\ 1 & if\ x = 0 \end{cases}$$

This function is Riemann integrable on the interval $[0,1]$.

Proof:

Let $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$ be a partition.

Let $m_i = \inf_{[x_{i-1},x_i]} f(x)$, $M_i = \sup_{[x_{i-1},x_i]} f(x)$, $d = \max_i \Delta x_i$ and $\sigma = \frac{1}{q_0}$.

m_i will be 0 since the set of irrational numbers are dense, so that there will always be an irrational number in every interval $[x_{i-1}, x_i]$.

Given a $q_0 \in \mathbb{N}$, the number of points in $A_{q_0} = \{x \in [0,1]: f(x) > \sigma\}$ is a finite set depending on q_0. In fact, $f(x) > \sigma > 0$ means that x must be a rational number. Assuming that it is of the form $\frac{p}{q}$ where $(p, q) = 1, 0 < p \leq q$, then the number of points in A_{q_0} are at most $q_0^2 = a$. So $|A_{q_0}| < a$.

Given $\varepsilon > 0$ such that $\sigma = \frac{1}{q_0} < \frac{\varepsilon}{2}$. We choose a partition such that each point of A_{q_0} is in an interval $[x_{i-1}, x_i]$ where $\Delta x_i = x_i - x_{i-1} < \delta_i < d$.

Let $B = \{x \in [0,1]: A_{q_0} \cap [x_{i-1}, x_i] = \emptyset\}$. If $x \in A_{q_0}$ then $M_i \leq 1$, if $x \in B$ then $f(x) \leq M_i < \sigma$. Thus,

$$\begin{aligned}
&\sum_{i=1}^{n} (M_i - m_i)\Delta x_i \\
&= \sum_{i=1}^{n} M_i \Delta x_i \\
&= \sum_{x \epsilon A_{q_0}, i=1}^{n} M_i \Delta x_i + \sum_{x \epsilon B, i=1}^{n} M_i \Delta x_i \\
&< 1 * \; ad + \sigma(1-0) \\
&< 1 * \; s + \sigma(1-0)
\end{aligned}$$

We chose $s < \frac{\varepsilon}{2}$, which implies that

$$\sum_{i=1}^{n} (M_i - m_i)\Delta x_i < \; s + \sigma < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$$

□

This proves that Thomae's function is integrable since Riemann's integrability condition is fulfilled. Consequently,

$$0 = \sum_{i=1}^{n} m_i \Delta x_i \leq \int_0^1 f(x)\, dx \leq \sum_{i=1}^{n} M_i \Delta x_i < \varepsilon$$

for all ε, so $\int_0^1 f(x)\, dx = 0$.

3.5 Monotonic functions

Another class of function, that is often considered during the school curriculum, are monotonic functions. This class of functions are interesting since every monotonic bounded function on a compact interval is Riemann integrable and some of them can have an infinite number of discontinuities.

Therefore, the next example that this thesis considers is the following monotonic function:

Let $\mathbb{Q} = \{\, q_n : n \in \mathbb{N}\}$ be an enumeration of the rational numbers. Let $Q = \mathbb{Q} \cap [0,1]$ and

$$f(x): Q \longrightarrow \mathbb{R} : x \longrightarrow \sum_{q_n \leq x} 2^{-n}.$$

This function is

1. monotonic (increasing).
   In fact, if $0 \leq x < y \leq 1$ then
   $$f(y) - f(x) = \sum_{q_n \leq y} 2^{-n} - \sum_{q_n \leq x} 2^{-n} = \sum_{x < q_n \leq y} 2^{-n} > 0$$
2. discontinuous at $x \in Q$.
   In fact, assume that $r = q_j$ is rational then
   $$\lim_{x \to r^-} f(x) = \sum_{q_n < r} 2^{-n}$$
   and
   $$\lim_{x \to r^+} f(x) = \sum_{q_n \leq r} 2^{-n} = \sum_{q_n < r} 2^{-n} + 2^{-j}$$
   Hence, $f(x)$ has a jump discontinuity at r of a height
   $$\lim_{x \to r^+} f(x) - \lim_{x \to r^-} f(x) = \sum_{q_n < r} 2^{-n} + 2^{-j} - \sum_{q_n < r} 2^{-n} = 2^{-j}$$
3. continuous at $x \in [0,1] \backslash Q$.
   In fact, if x is irrational, then for any $\varepsilon > 0$ it exists $N \in \mathbb{N}$ such that $\sum_{N<n} 2^{-n} < \varepsilon$. Let $\delta > 0$ be a sufficiently small value, then the interval $(x - \delta, x + \delta)$ will contain only rationals with indices lager than N. If $y \in (x - \delta, x + \delta)$, then $|f(x) - f(y)| < \varepsilon$. In other words, if $r = q_j$ is irrational, then
   $$\lim_{x \to r^+} f(x) = \lim_{x \to r^-} f(x) = \sum_{q_n < r} 2^{-n}.$$
4. Riemann Integrable thanks to theorem that follows and by replacing a by 0 and b by 1.

Theorem

A monotonic function $f(x)\colon [a,b] \to \mathbb{R}$ on a compact interval is Riemann integrable.

Proof:

Suppose that $f(x)$ is monotonic increasing ($f(x) \le f(y)$ for $x \le y$). Let $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ be a partition of $[a,b]$ into n intervals $[x_{i-1}, x_i]$ where $\Delta x_i = x_i - x_{i-1} < d$ with $d = \max_i \Delta x_i$.

Since $f(x)$ is increasing

$$m_i = \inf_{[x_{i-1},x_i]} f(x) = f(x_{i-1}), \qquad M_i = \sup_{[x_{i-1},x_i]} f(x) = f(x_i).$$

Hence,

$$\sum_{i=1}^{n} (M_i - m_i)\Delta x_i$$
$$< d \sum_{i=1}^{n} \big(f(x_i) - f(x_{i-1})\big)$$
$$= d\,\big(f(b) - f(a)\big)$$

It follows that $\sum_{i=1}^{n} (M_i - m_i)\Delta x_i \to 0$ when $d \to 0$. By the *Condition of Riemann's integrability* it implies that $f(x)$ is integrable.

The proof for a monotonic decreasing function $f(x)$ is similar, with

$$m_i = \inf_{[x_{i-1},x_i]} f(x) = f(x_i), \qquad M_i = \sup_{[x_{i-1},x_i]} f(x) = f(x_{i-1}).$$

□

The remaining part would be to calculate the integral of the function in Riemann's sense, on the interval [0,1].

Let $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$ be a partition of $[0,1]$ into 2^N intervals $[x_{i-1}, x_i] = \left[\frac{i-1}{2^N}, \frac{i}{2^N}\right]$ with $1 \le i \le 2^N$, where $\Delta x_i = x_i - x_{i-1} = \frac{1}{2^N}$ and let $x_i' = \frac{i}{2^N} \in \left[\frac{i-1}{2^N}, \frac{i}{2^N}\right]$.

$$\sum_{i=1}^{2^N} f(x'_i)(x_i - x_{i-1})$$
$$= \sum_{i=1}^{2^N} f\left(\frac{i}{2^N}\right)\frac{1}{2^N}$$

$$= \frac{1}{2^N} \sum_{i=1}^{2^N} \sum_{\left\{n : q_n \leq \frac{i}{2^N}\right\}} \frac{1}{2^n}$$

Since this function depends on the enumeration, an exact value of the limit is difficult to compute. Consequently, it follows that

$$\int_0^1 f(x) dx = \lim_{N \to \infty} \frac{1}{2^N} \sum_{i=1}^{2^N} \sum_{\left\{n : q_n \leq \frac{i}{2^N}\right\}} \frac{1}{2^n}.$$

One could also consider as sequence of partition of $[0,1]$ the set $P_k = \{0,1\} \cup \{q_1, \dots, q_k\}$ but the difficulty always lies in computing $f(q_k) = \sum_{\{n : q_n \leq q_k\}} \frac{1}{2^n}$.

4 Conclusion

Ultimately, this thesis has exposed the work and successes of the HPM group and the necessity to have an open dialogue about the implementation of such projects on every level of the academic career of students. As a matter of fact, the studies referred to in the first part of this thesis, show how fruitful this concept can be.
This dissertation has also revealed the challenges of the integration of the history of mathematics for teaching purposes, despite the need to conduct further analysis in the field to produce additional insights. In fact, one should not neglect the potential drawbacks and challenges that would have to be overcome in order to revolutionise the field of mathematics in their pedagogical application. For instance, many of the historical texts have been written in a language that is not easily accessible to students; because of it being a foreign language or because the mathematical notions and vocabulary have evolved and changed over the years. Undoubtedly every good theory has to be put to the test, criticised and expanded upon by other people in the sense that mathematics teaching has to start at some point. Moreover, it needs to undergo the same fluctuations, intermediate setbacks and adaptations than any other part of pedagogy.

The second part of this thesis covers almost the entire developmental steps of Riemann's definition of integral in a historical optic. The teacher could use this historical part in different ways. For example, he could pick some interesting facts or parts to feed the students' curiosity in order to make the topic more accessible to them. Interestingly, students usually start by seeing the definition of a function and, only afterwards, are introduced to the notion of series. However, historically, the mathematicians first showed interest in series, which then lead to the concept of function that we use today. The teacher could also use it to show that mathematics needs to undergo a whole process before it becomes the finished product presented in school. Moreover, the teacher could ask his/her students to compare the given definitions to those given by mathematicians in their original texts, as for example the definition of an arbitrary function given by Fourier[5] .
In addition, as seen in the second chapter, high school students discover continuous functions and learn how to analyse them, which is consistent with the historical development. It would probably be informative and encouraging to show the students the struggles that mathematicians had to overcome when they were about to discover and define what the students consider to be self-evident. In fact, when it comes to making the transition from continuous to discontinuous functions, it could be beneficial to share with the students the various doubts and difficulties people such as D'Alembert have encountered. Moreover, discovering the arguments and discussions between D'Alembert, Euler, Bernoulli and Lagrange on the subject may encourage them to question and discuss their thoughts with their teachers and peers. Maybe, their vison of discontinuous function is similar to Euler's vision of the "mixed function" and it would boost their motivation to discover that their own instinct might be "wrong" but has indeed been part of the thought process.
Furthermore, the aspect of discussion, exchange and even conflict might confer a livelier and a more dynamic appearance to what might otherwise be considered a dry, static definition to be learned by heart, rather than understood.

[5] p.34 of this thesis

Actually, as it has been seen, it was only due to the writing of Fourier of the arbitrary function, that it had become necessary to write down a formal definition of the integral. As it has been seen, Cauchy focused his research on the integration of continuous function and functions which have finite discontinuities.

Without the historical context of the evolution of these concepts, it is hard to fully understand the underlying differences in vision and aim. In fact, the integration of a continuous function is seen across the divide between high school and university, in the sense that Cauchy's continuous function is often seen in high school under the guise and name of "Riemann", whereas Riemann's or Darboux's integral and the function with finite discontinuities are only introduced at university level. Students tend to think that the transition between the continuous function and the function with finite discontinuities is an added difficulty, motivated and justified by the passage from high school to university. Therefore, they also believe that the definition given at university is only a more precise and more complicated version of what has been seen in high school, as both elements have been taught under the same name. Since, Riemann's definition is mainly applied to functions that would also have fallen into the area of interest and study of Cauchy, it becomes hard to distinguish the different authors and theories and to understand why this new version of Riemann's integral definition is needed. Indeed, when it comes to practice Riemann's definition, two types of functions are mostly proposed; functions with few discontinuities and functions with infinite discontinuities but which are not integrable, as Dirichlet's function. Consequently, students could conclude that all functions which have infinite discontinuities are not integrable, which is not the case. In fact, Riemann was interested in functions with infinite discontinuities and even proposed one example showing that there are such kinds of functions that are integrable.
In addition to that, since teachers mainly aim to teach the fundamental theorem of calculus in high school as well as at university, they do not need to expand on the concept of arbitrary functions that Riemann was interested in.
Another reason why the belief that the definition given at university is a more complex version of what students have previously seen, is enforced by the fact that teachers principally use graphs that illustrate continuous functions as a support to explain Riemann's definition of definite integral. In fact, the graphs used in high school and those used at university are very similar, if not even the same. Therefore, students tend to forget that Riemann's definition can also be used for discontinuous functions, since they have a vague memory of the definition and mostly remember the explanation given by the teacher with the graph of a continuous function.

Finally, due to all these factors, it seems clear why students do not understand the need for seeing and studying the "complicated" version of Riemann's integral definition. Aside from that, it also diminishes the importance and scope of Riemann's research and definition.

Summa summarum, the two big categories of functions which can be integrated are the continuous and the monotonic ones, with the latter having possibly infinite discontinuities and being barely ever seen in school. So, if functions with infinite discontinuities are to be set aside, why bring in Riemann's or Darboux's integral when their names might only cause confusion and mistaken conclusions to the actual complexity of their writings?

In that particular case, one could limit oneself to Cauchy's integral as it covers the applications taught in class. Additionally, his original texts might be given as supplementary reading that might make their understanding of the, brief and concise, definitions seen in class substantially easier.

It is often harder to follow a reasoning in reverse, to counteract the loss of relevant connection to contents and methods that have informed our present "knowledge". One could and should encourage students to follow the thought processes started, the problems encountered and overcome and the conclusions drawn by famous mathematicians of the past. So, they might gain a stronger connection and a deeper understanding for the actual definition that has resulted from those intellectual works.
Personally, one must agree with Gooday when he expertly states that

> "[t]he key role of history here is characterizing the complexities of how science ***changes***. So many science textbooks unhelpfully—and above all inaccurately—cultivate a rather static image of scientific disciplines, as if they were completed with comprehensive certainty. It is perhaps not difficult to understand how this gross simplification might arise as the result of a pedagogical need to 'tidy up' the presentation of science to meet the needs and capacities of students. But faced with the textbook spectacle of such an apparently unalterable monolith, is it any wonder that students can have difficulty conceiving how they might ever contribute to science?" (Gooday, et al., 2008, p. 326)

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