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Louvain School of Management

# Inventory optimization under demand uncertainty: algorithmic approaches

Master thesis

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Submitted in fulfillment of the requirements for the
**Master 120 in Business Engineering, with a professional focus**

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Academic years 2017-2018

# ACKNOLEDGMENTS

This master thesis was conducted as part of a supply chain project at N-side, a Belgian company specialized in optimization and applied mathematics.

First and foremost, I would like to thank all the people from N-side for their hospitality. In particular, I would like to express my gratitude to my internship advisor, Jean-Baptiste Mairy, for the time devoted to me and his advice.

Furthermore, I would like to thank my thesis advisor, Philippe Chevalier, for his availability and guidance.

I would also like to express my greatest appreciation to all SME’s managers, who accepted to spend their time for an interview and share their experience.

Finally, I would like to express my gratitude to my family and friends for their encouragement.

# Contents

# Introduction

Inventory management is an active field of operational research. The goal of this master thesis is to suggest algorithmic methods that can be used to solve inventory management problems. The methods should be able to solve real-life problems. Moreover, the approaches should be scalable in order to be adapted to specific contexts and fit to many firms' needs. Finally, the methods should explicitly consider demand uncertainty to propose robust solutions.

This master thesis is structured as follows:

- The first part is dedicated to the presentation of inventory management problems and the definition of important concepts.
- Secondly, we briefly review common types of inventory management problems. The aim of this part is to provide a global framework of the common problems studied in the scientific literature.
- Thirdly, we discuss the theory about the dynamic lot-sizing problem, which refers to an inventory problem under a periodic review system. Special emphasis is put on the extensions developed to consider demand uncertainty.
- In the fourth part, we suggest methods that can solve inventory management problems. Each method is based on a specific replenishment strategy and a specific algorithmic approach. Then, what we call a method is the combination of a strategy and an algorithmic approach.
- In the fifth part, we illustrate the methods on a common test bed and we carry out a sensitivity analysis to identify the strengths and weaknesses of the methods.
- Part six is dedicated to the presentation of two case studies used to establish the extensions that we consider for the methods.
- In part seven, the methods suggested are extended to consider specific constraints. Then, we carry out another sensibility analysis to assess the scalability of the methods.
- In part eight, we discuss how the methods can be extended to consider multi-item settings and we propose a test bed. Again, the goal is to assess the scalability of the methods.
- In the last part, we summarize the work that has been done, present some recommendations about the methods studied and conclude by suggesting directions for next research.

# Part 1: Inventory management problem

## 1.1 Definition and reasons for inventory

Generally speaking, inventory can be defined as the stock of resources or items used by the company. It incorporates stocks related to the manufacturing process (i.e. raw materials, work-in-process, finished products, etc.), the distribution (i.e. in-transit inventory) and the administration of services (Chase & Jacobs, 2013). There is a paradox in the concept of inventory. We use inventory to be protected from environment's uncertainties. Indeed, it allows to be flexible and to react quickly. But inventory is also costly because there are opportunity costs for holding stocks. The paradox is that companies want to keep some stocks to ensure the production or the service level and at the same time reduce stocks for financial purposes (Vrat, 2014).

Concretely, there are many reasons for keeping inventory. But, four main functions can be identified (Cavinato & Kauffman, 2000).

1. Anticipation inventory: Inventory is used to prevent against future events such as high seasonal demands, low production rate or special promotions.
2. Fluctuation inventory: Inventory is considered as a safety stock to prevent uncertainty of demand, production, lead-time, etc.
3. Lot-size inventory: Inventory is built to benefit from economies of scale or because minimum order quantities (or batch sizes) are imposed.
4. Transportation inventory: Inventory is the result of multi-location supply chain. It is the inventory in transit from one site to another.

Despite all these reasons, inventory is one of the seven wastes according to the lean manufacturing theory. Indeed, inventory brings no added value to the customer and then it is a waste that should be eliminated (Wilson, 2010).

## 1.2 What is an inventory management problem?

Inventory management tries to answer to three general questions (Silver, 1981):

1. How often should the inventory position be reviewed?
2. When to send orders to the supplier (or start the production)?
3. How much to order (or how much to produce)?

Those three questions form the inventory management problem. Moreover, the set of strategies/policies used to monitor the inventory position, determine the level of safety stock, decide of the replenishment timing and fix the replenishment size is called the inventory system (Chase & Jacobs, 2013). One reason why inventory management problems can be difficult to solve is because it requires to balance different costs (see section 1.3). Another reason is that inventory is coupled with other functions. For instance, the production department may ask for high raw material availability. The sales department may ask for large finished good inventory to easily meet demand, while the financial department may need to reduce stocks to keep cash for other purposes. All these conflicting goals make inventory management a complex task (Axsäter, 2015).

## 1.3 Types of inventory costs

An inventory management problem may involve many different costs. The following costs are generally considered (Atanga, Dadzie, & Ghansah, 2016):

1. Acquisition costs: Direct costs for purchasing items.
2. Ordering costs (or setup costs): Costs associated to send an order (or launch production). It includes real or opportunity costs such as negotiation, order preparation, delivery, reception of goods, inspection, etc.
3. Holding costs: Costs associated to carry inventory over time. It includes costs proportional to the value of the inventory such as capital costs, insurance costs or risk costs (deterioration or obsolescence for example). It also includes costs associated to the physical characteristics of the inventory such as labour costs (e.g. for handling), storage costs (e.g. storage place or heat/light) and clerical costs (e.g. keep records).
4. Stockout costs: Costs incurred when running out of inventory. It includes opportunity costs for production losses (due to shortage of raw materials), idle times, actions to tackle stockouts, potential lost customers, etc.

Additionally, there are 'system control costs' which are often ignored in the literature. It is all the costs associated to the selected inventory system such as data collection, computational effort, implementation costs, ... (Silver, 1981).

## 1.4 Inventory monitoring systems & replenishment strategies

The inventory position can be monitored either by a periodic review system or by a continuous (or perpetual) review system. In the first case, the stock level is assessed periodically (e.g.

every day/week). By contrast, with a continuous review system, the inventory position is tracked in real time (Axsäter, 2015).

A replenishment strategy specifies how a replenishment order is triggered and what is the size of the replenishment (Tempelmeier, n.d.). Replenishment strategies are closely related to inventory monitoring systems. We can summarize the most common replenishment strategies used both in the literature and in practice, with the following table (Bijvank & Vis, 2011).

| With periodic review system | With continuous review system |
|---|---|
| $(r, s, Q)$ strategy<br>$(r, S)$ strategy<br>$(r, s, S)$ strategy | $(s, Q)$ strategy<br>$(s, S)$ strategy |

Where:

$r$: denotes the review interval.
$Q$: denotes the order quantity.
$s$: denotes the reorder point.
$S$: denotes the order-up-to level.

Obviously, if the review interval, $r$, tends to zero, the $(r, s, Q)$ and the $(r, s, S)$ strategies under a periodic review system respectively tend to the $(s, Q)$ and the $(s, S)$ strategies under a continuous review system. Continuous replenishment strategies allow to carry fewer stocks than their periodic forms. In the continuous cases, the reorder point, $s$, must be large enough to cover the demand during the leadtime. While, in the periodic cases, the reorder point must be sufficient to cover the demand during the review interval plus the demand during the leadtime. Since it is the reorder point that determines the level of safety stock, it should be as small as possible (Taylor, 2004). Thus, at first glance, replenishment strategies with periodic review systems might only look like restrictive forms of their equivalent under continuous review systems. However, periodic replenishment strategies have their advantages. Indeed, those strategies are often used in practice because it reduces transactional efforts and it eases planning and coordination (mainly with multi-item system) (Cunha, Oliveira, & Raupp, 2017).

Then, periodic replenishment strategies involve less 'system control costs'. Later, we will focus on inventory management problems for periodic review systems.

## 1.5 Stockout policy

In an inventory management problem under demand uncertainty, stockouts may occur. Then, a policy should be defined to specify how stockouts are managed. Two decisions must be taken: (1) How the unsatisfied demand is considered and (2) how stockouts are controlled. There are two options generally adopted in the literature to consider unsatisfied demand. The excess demand can be considered lost (i.e. lost sales) or backlogged. The stockouts can be controlled either by penalizing the quantity of lost sales/backlogs or by imposing a service level criterion. In the literature, various criteria are used to measure the service level (i.e. capacity to fill the demand). Three criteria are traditionally considered (Tempelmeier, 2013):

- $\alpha$-service level: It corresponds to the probability to fill entirely the demand during a replenishment cycle. In other words, it is the probability to not have a shortage during the replenishment cycle.
- $\beta$-service level: It corresponds to a fill rate criterion. It measures the unfilled demand compared to the total demand. Generally, the long-term average fill rate is used but the fill rate can also be computed for a single period.

$$\beta_t = 1 - \frac{unfilled\ demand\ in\ period\ t}{demand\ in\ period\ t}$$

- $\gamma$-service level: It corresponds to a time-weighted fill rate. Not only the ratio of unfilled demand matters, but the waiting time is also considered. However, this criterion may be meaningless in a poorly managed supply chain since the mean cumulative unfilled demand may be greater than the average demand and then $\gamma$ is negative.

$$\gamma = 1 - \frac{mean\ cumulative\ unfilled\ demand\ per\ period}{mean\ demand\ per\ period}$$

  Obviously, it does not make sense to use the $\gamma$-service level if unsatisfied demand is considered as lost (Schneider, 1981).

# Part 2: Framework for the inventory problem

In 1913, Ford W. Harris published a paper called 'how many parts to make at once'. This paper introduced for the first time the Economic Order Quantity (EOQ). Generally speaking, this paper formulates a new problem where the decision maker has to balance different inventory costs (i.e. ordering costs and holding costs) to find the cheapest lot-size to produce/order. Understandably, the lot-size directly influences the average inventory level. Indeed, in an ideal world where producing/ordering is free and instantaneous, the batch-size is equal to one (i.e. we produce/order unit by unit when needed) and no inventory is required. During the last century, this problem has been widely studied. Many extensions and models have been developed to cover a wide range of realistic situations (Andriolo, Battini, Grubbström, Persona, & Sgarbossa, 2014). The main goal of this part is to provide a framework of the different types of inventory management problems studied in the literature. From our research, the following problems seem to be the most common:

1. Lot-sizing problem
    a. Economic Lot Scheduling Problem (ELSP)
    b. Dynamic lot-sizing problem
2. Newsvendor problem
3. Joint Economic Lot Scheduling (JELS)
4. Joint Replenishment Problem (JRP)

## 2.1 Common inventory management problems

### 2.1.1 Lot-sizing problem

Lot-sizing problems consist in establishing a production or a replenishment planning. This means determining the optimal production/replenishment timing and the optimal quantities to satisfy the demand and minimize costs (production/procurement costs, setup/ordering costs and inventory holding costs) (Goren, Jans, & Tunali, 2008). Inventory lot-sizing models are maybe the most often used for inventory management problems in the literature and many extensions have been developed to cover a wide range of situations (e.g. capacity restriction, multi-item, multi-stage…). Many authors have tried to summarize research about inventory lot-sizing problems. To the best of our knowledge, no common framework has been developed to classify inventory lot-sizing models. However, many authors suggest their own classification scheme (Glock, Grosse, & Ries, 2014; Ullah & Parveen, 2010; Drexl & Kimms, 1997; Goren, Jans, & Tunali, 2008; Prasad, 1994; Bushuev, Guiffrida, Jaber, & Khan, 2014). These classifications differ mainly according to how broadly extensions are considered.

From a modelling point of view, an important distinction should be made regarding how time is modeled. As illustrated in figure 1, the time horizon can be either continuous or discrete. Thus, lot-sizing problems can be divided in two main categories. The first category concerns problems with static demand (i.e. constant over time, not seasonal) and infinite horizon. These problems correspond to Economic Lot Scheduling Problem (ELSP). The second category is about problems with dynamic demand (non-stationary or seasonal) and finite horizon. These problems are dynamic lot-sizing problems (Goren, Jans, & Tunali, 2008).

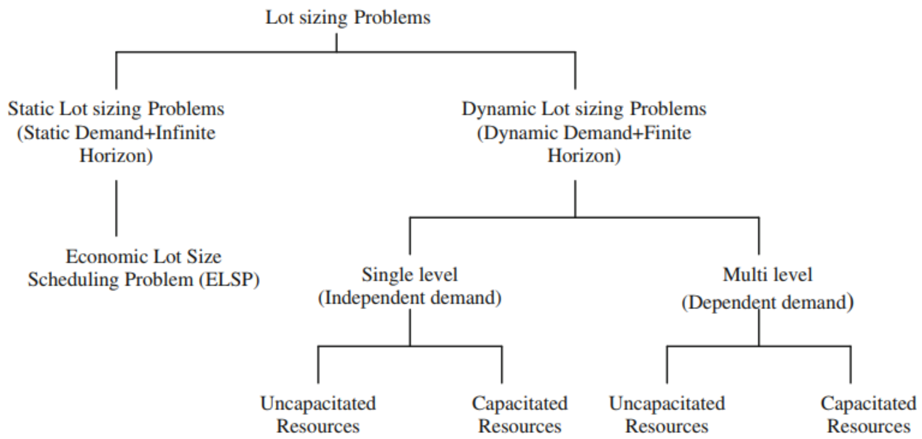


*Figure 1: Example of classification of lot sizing problems. Retrieved from (Goren, Jans, & Tunali, 2008).*

### 2.1.2 Newsvendor problem

Newsvendor problems are traditional problems in inventory management (Qin, Wang, Vakharia, Chen, & Seref, 2011). Essentially, the problem consists in finding the optimal stock level to satisfy a random demand and maximize profit. Initially, it is a single period problem where excess inventory after demand realization is salvaged. The goal is to find the optimal stock level that balances the expected lost sales costs and the expected costs of overstocking (Dada & Petruzzi, 1999). Many extensions have been studied to consider other objective functions, other pricing policies, multi-product/echelon/location systems (Khouja, 1997) and multi-period configurations (Kim, Huang, & Wu, 2015).

### 2.1.3 Joint economic lot scheduling problem (JELS)

Coordination is the key aspect of the joint economic lot scheduling problem. In JELS problems, the goal is to determine the optimal lot-size and shipment planning to minimize the total cost for the whole supply chain. In the basic setting, there is a manufacturer with a given production rate, setup cost per lot and holding cost. And, there is a buyer with a fix ordering cost and holding cost (Ben-Daya, Darwish, & Ertogral, 2008). Conceptually, the JELS problem can be seen as an extension of the 'inventory lot-sizing' problem described above. JELS problems are appropriate for companies involved in long-term partnerships with their customers or suppliers. Indeed, coordination allows to reduce the total cost incurred and may benefit to all members of the supply chain (Glock C. H., 2012).

### 2.1.4 Joint replenishment problem (JRP)

The joint replenishment problem is an important model with applications in inventory management and logistics (Cheung, Elmachtoub, Levi, & Shmoys, 2015). The goal is to coordinate the replenishment of several products to minimize the total cost. In the basic configuration, there are both a holding cost incurred for holding inventory and ordering costs. The ordering costs are composed of two parts: (1) a major ordering cost independent of the subset of item types ordered and (2) a minor ordering cost depending on the number of different items ordered (Goyal & Khouja , 2008). The JRP is generally NP-hard due to the combinatorial aspect of the problem. Then, several solution procedures exist to solve the problem either heuristically or optimally (Bastos, Carneiro, Melo, Mendes, & Nunes, 2017).

## 2.2 Inventory problem selection

We briefly presented common inventory management problems in sections 2.1. To detail each problem and review the state of the art is a complex task that is out of the scope of this work. As a reminder, the goal of this work is to suggest methods based on algorithmic approaches to solve generic inventory management problems. And, the solutions found should be adaptable to more specific settings. Therefore, we are particularly interested in 'lot-sizing problems'. We select this category because to the top of our knowledge, this is the category with the least specific characteristics. Then, solutions for this category of problems should be applicable for a large set of companies. As mentioned in the previous section, it is customary to consider that 'lot-sizing problems' can be divided into two sub-categories of problems: (1) The 'dynamic lot-sizing problem' and (2) the 'Economic Lot Scheduling Problem'. In the next section, we review the dynamic lot-sizing problem in greater detailed. Dynamic lot-sizing models are particularly suitable for inventory management problems under periodic review systems. We decide to focus our effort on this category because interesting extensions have been developed to tackle demand uncertainty.

# Part 3: Dynamic lot-sizing problems

This part is devoted to the dynamic lot-sizing problem. First the traditional deterministic lot-sizing is briefly discussed. Then, the extensions to consider demand uncertainty are studied.

## 3.1 Deterministic dynamic lot-sizing problem

In the dynamic lot-sizing problem, parameters vary over time (traditionally at least the demand). To tackle this problem, the time horizon is represented by time buckets representing the periods (e.g. days, week, etc.) The dynamic lot-sizing has been extensively studied and a large set of models and extensions have been developed. In 1958, Wagner and Within published their paper called 'Dynamic version of the economic lot size model'. In this seminal paper, the authors provide an efficient algorithm to find the optimal solution of the classic lot-sizing problem by dynamic programming. Since then, many extensions of the problem have been imagined, where unfortunately the solution of Wagner and Within does not hold anymore. For both classic formulations and extensions of the problem, the running time can be intractable with moderate to large instances. Consequently, heuristic procedures have been developed (Ullah & Parveen, 2010).

### 3.1.1 Formulation of the classic dynamic lot-sizing problem

Let us consider the classic lot-sizing problem studied by Wagner and Within (1958) with a time horizon composed of $T$ periods where $t \in \{1, 2, \dots, T\}$. Then we have:

$d_t$: The demand in period $t$.

$h_t$: Holding cost per unit stored from period $t$ to $t + 1$.

$s_t$: Ordering cost in period $t$.

$x_t$: Amount ordered in period $t$.

$d_t, h_t, s_t, x_t \geq 0 \;\; \forall t$

The total cost is given by: $\sum_{t=1}^{T}[h_t + \delta(x_t)s_t]$ where $\delta(x_t) = \begin{cases} 1, if\ x_t > 0 \\ 0, else \end{cases}$

Obviously, the optimal solution can be found by enumeration of all possible combinations or equivalently with a branch-and-bound algorithm. However, Wagner and Within (1958) showed that the optimal solution can be found more efficiently with a dynamic programming formulation. The reader can refer to their original paper.

## 3.2 Stochastic dynamic lot-sizing problem

In real life, information is not perfectly known, and decisions have to be made using forecasts or other uncertain information. As mentioned, the dynamic lot-sizing is a traditional model for inventory management problems and many extensions have been studied in the literature, including for stochastic environments. In this section, we will focus on the specific cases where demand is uncertain. In the stochastic equivalent of the classic lot-sizing problem, demand is uncertain for all periods. Uncertainty is represented by probability distributions. For each period, the probability distributions of the demands are assumed fully known and these random variables are supposed mutually independent (Tempelmeier, 2013). Stochastic dynamic lot-sizing studied in the literature vary in function of the replenishment strategy chosen and the stockout policy.

### 3.2.1 Replenishment strategies

In their seminal paper, Bookbinder and Tan (1988) discuss three replenishment strategies for the stochastic dynamic lot-sizing:

1. *Static uncertainty strategy*: The timing of the orders and the order quantities must be determined at the beginning of the planning horizon, and are decided once for all. This strategy is the most restrictive because it supposes that decision makers cannot react to demand realization. However, this strategy may be interesting from an organizational point of view because there is no planning nervousness (Tempelmeier, 2013). Indeed, production (or ordering) periods and production (or order) quantities are known. This may be particularly adapted to industrial environments with low level of flexibility. Traditionally, the timing of orders is denoted by $R$ and the quantity by $Q$. Then it is referred as the $(R_t, Q_t)$ strategy (Dural-Selcuk, Kilic, Tarim, & Rossi, 2016).
2. *Dynamic uncertainty*: In this strategy, a reorder point and an order-up-to level are determined for each period of the planning horizon. At the beginning of each period, the current stock level is measured. If the inventory position is below the reorder point, an order is placed to bring back the stock to the order-up-to level of the current period (Dural-Selcuk, Kilic, Tarim, & Rossi, 2016). It has been shown that this is the optimal policy under linear holding costs and shortage costs if unsatisfied demand is backlogged (Scarf, 1960). Unfortunately, this strategy has two important drawbacks. Firstly, finding the optimal reorder point and order-up-to level for all periods is

complex. Secondly, since neither the replenishment periods and the order quantities are known, it introduces a lot of nervousness in the planning[1]. This strategy is often referred as the $(s_t, S_t)$ strategy where $s_t$ refers to the reorder point and $S_t$ is the order-up-to level (Tempelmeier, 2013).

3. *Static-dynamic uncertainty*: This strategy is a hybrid between the two extremes described above. This strategy has been introduced by Bookbinder and Tan (1988), and since then this strategy has been studied in many other papers. With this policy, the timing of orders is determined at the beginning of the planning horizon and once for all. However, order quantities are decided only at the beginning of the ordering periods, after observing the demand realization of all previous periods. The quantities to order are determined by an order-up-to level system (i.e. the order quantity is the quantity that will bring back the stock to the order-up-to level). Traditionally, this policy is referred as the $(R_t, S_t)$ strategy. Since the timing of the orders is known at the beginning of the planning horizon, this strategy is appropriate for joint replenishments, shipment consolidation and material requirement planning (Dural-Selcuk, Kilic, Tarim, & Rossi, 2016).

### 3.2.2 Stockout policy

In stochastic dynamic lot-sizing problems, the demand is uncertain, and the possibility of stockouts cannot be ignored. As mentioned earlier, a policy must be defined to clarify how these stockouts are managed. More concretely, it should specify how the unsatisfied demand is considered and how the expected amount of unsatisfied demand is controlled or penalized. There are two common ways to consider unfilled demand. (1) Unsatisfied demand can be considered as backordered to the next period with a positive inventory position. Or, (2) unsatisfied demand can be considered as lost.

The way uncertainty is controlled or penalized may have a significant impact on the solution approach. Generally, the expected amount of unsatisfied demand is either controlled by a constraint based on a service level criterion (e.g. $\propto$ service level should be greater than 90%) or directly penalized in the objective function (Tempelmeier, 2013). The stockout policy can have an important impact on the complexity of the problem because it influences how we

[1] This nervousness is a good example of the "system control costs" (Silver, 1981). Of course, these costs are not taken into consideration when proving the optimality of the dynamic uncertainty strategy.

consider the expected inventory on-hand and the expected stockout quantity. Typically, these variables are given by non-linear functions.

### 3.2.3 What makes stochastic lot-sizing problems more complex

Stochastic lot-sizing problems are more complex than deterministic lot-sizing problems because they often involve non-linear terms. As an example, let us consider the single-item stochastic lot-sizing problem under the $(R_t, Q_t)$ strategy studied by Sox (1997). Demands are independent random variables with a known probability distribution. All unfilled demands are backlogged to the next period. The following notations are used:

$d_t$: Demand in period $t$ (random variable).
$D_t$: Cumulative demand from period 1 to $t$ (random variable).
$h$: Holding cost.
$b$: Penalty cost for backlog.
$s$: Setup cost.
$c$: Procurement cost.
$I_t^+$: Ending inventory in period $t$ (random variable).
$I_t^-$: Backorder quantity in period $t$ (random variable).
$I_t^n$: Ending net inventory position in period $t$ (random variable).
$x_t$: The order quantity in period $t$.
$X_t$: The cumulative order quantity from period one to $t$.
$Y_t$: Binary variable equal to 1 if an order is received in the beginning of period $t$.

The problem consists in minimizing the sum of the setup costs, procurement costs, holding costs and backlog costs. Sox (1997) proposed the following model:

$$Min: \sum_{t=1}^{T} [sY_t + cx_t + hE\{I_t^+\} + bE\{I_t^-\}]$$

Subject to:

$$E\{I_t^n\} = I_0^n + X_t - E\{D_t\} \qquad \forall t \qquad \text{Inventory balance constraint}$$

$$x_t \le MY_t \qquad \forall t \qquad \text{Big-M constraint}$$

$$X_t = \sum_{i=1}^{t} x_i \qquad \forall t$$

$$x_t \ge 0 \text{ and } Y_t \in \{1,0\} \qquad \forall t$$

Where: $E\{I_t^+\} = E\{Max(X_t - D_t, 0)\}$ & $E\{I_t^-\} = E\{Max(X_t - D_t, 0)\}$

We are confronted to a mixed integer non-linear program (MINLP). The non-linearity comes from $E\{I_t^+\}$ and $E\{I_t^+\}$. These values can be computed for a given $X_t$ with first order loss functions (see appendix 1). Sox (1997) provides an optimal solution procedure, similar to the Wagner-Whitin algorithm, using a dynamic programming formulation and a decomposition of the objective function into multi-period newsvendor problems.

### 3.2.4 $(\boldsymbol{R_t}, \boldsymbol{Q_t})$ strategy: A review

Under the $(R_t, Q_t)$ strategy (sometimes called static uncertainty strategy), the timing of the orders and the order quantities must be determined at the beginning of the planning horizon, and it is decided once for all. As mentioned previously, models differ according to the stockout policy selected.

Bookbinder and Tan (1988) studied the stochastic lot-sizing problem under the $(R_t, Q_t)$ strategy, backlog policy and $\alpha$-service level constraint (i.e. the probability of non-stockout should be greater than a threshold called $\alpha$ for all periods). When working with a $\alpha$-service level constraint, it is not necessary to compute the expected stockout quantities given by the non-linear function. Moreover, Bookbinder and Tan (1988) make a convenient assumption. The authors supposed that $\alpha$ (i.e. the desired service level) is relatively high. Typically, $\alpha$ is greater than 0.9. Then, the expected inventory on-hand can be approximated by the net inventory position[2]. Therefore, the problem becomes totally deterministic and linear. Actually, the authors even demonstrated that the remaining problem has the same mathematical structure than the original deterministic problem. A parallelism can be established. Then, any method developed in the literature to solve the deterministic problem can be used to solve the stochastic model under the $(R_t, Q_t)$ strategy, $\alpha$-service level constraint and backlog. This result only holds thanks to the assumption of a high service level that allows to approximate the inventory on-hand by a linear term.

In the previous section, a stochastic lot-sizing problem under the $(R_t, Q_t)$ strategy, backlog and penalty cost was described. This problem is much more complex to solve due to the two non-linear terms. Beside the dynamic programming formulation of Sox (1997), other solving procedures have been developed. Notably, there are a shortest path formulation (Vargas, 2009) and a MIP formulation with piecewise linear functions to approximate the inventory on-

[2] The net inventory position is the inventory on-hand minus the eventual backlog quantity.

hand function and the backlog function (Helber, Sahling, & Schimmelpfeng, 2013). The shortest past algorithm is the most efficient. However, the MIP formulation is more flexible and capacity constraints or multi-item extensions can be easily implemented (Tempelmeier, 2013).

The two configurations presented above (i.e. with penalty costs or with a $\alpha$-service level constraint) illustrate how the stockout policy might influence the complexity of the problem. With a $\beta$-service level constraint, the non-linear term corresponding to the expected stockout quantity should still be evaluated. Actually, the $\alpha$-service level constraint with the high service level assumption is particularly convenient. The problem becomes easier to solve and more scalable. Moreover, the assumption of a high service level seems realistic in many situations.

### 3.2.5 $(\boldsymbol{s_t}, \boldsymbol{S_t})$ strategy: A review

Under the $(s_t, S_t)$ strategy (sometimes called dynamic uncertainty strategy), neither the timing of the orders or the order quantities are determined at the beginning of the planning horizon. Instead, a reorder point, $s_t$, and an order-up-to level, $S_t$, must be determined for each period. At the beginning of period $t$, if the inventory position is below $s_t$, an order is placed to bring back to the stock to $S_t$. The literature on the stochastic lot-sizing under $(s_t, S_t)$ strategy is limited and divided between research on the stationary and non-stationary problem. For the stationary problem, the goal is to find a unique pair $(s, S)$. By contrast, for the non-stationary problem, $T$ pairs $(s_t, S_t)$ should be determined (i.e. one pair for each period) (Martin-Barragan, Rossi, Tarim, & Xiang, 2017). Efficient algorithms have been developed to solve the stationary case. We can for example mention the papers of Zheng and Federgrue (1991) or Feng and Xiao (2000). Both focus on computing the optimal $(s, S)$ policy in the case of fixed setup costs, linear holding costs and linear backlogging costs. For the non-stationary case, not much progress has been done since the seminal work of Scarf (1960). Scarf demonstrated the optimality of the $(s_t, S_t)$ policy under linear holding costs and penalty costs with backlogs. Moreover, Scarf also suggested a dynamic programming method to compute the optimal solution. However, the method is computationally intensive and not very scalable. Martin-Barragan et al. (2017) recently developed a heuristic procedure to solve the same problem and argued that before their work, there were only two other heuristics to tackle the same problem. These heuristics have been developed by Askin (1981) and Bollapragada and

Morton (1999). The optimality gap of the two heuristics is at 3,9% and 4,9% respectively (Dural-Selcuk, Kilic, Tarim, & Rossi, 2016).

- The heuristic of Askin is based on the same logic than the Silver-Meal algorithm but it takes into consideration demand uncertainty.
- The heuristic of Bollapragada and Morton approximates the optimal $(s_t, S_t)$ policy by computing the optimal $(s, S)$ policy for the stationary problem with the same average demand on a portion of the non-stationary problem. The portion is determined by a myopic estimation of the cycle length. Askin's heuristic performs better on average (Dural-Selcuk, Kilic, Tarim, & Rossi, 2016). But as explained by Bollapragada and Morton (1999), the more the demand pattern is close to the stationary setting, the better their heuristic performs.
- Recently, Martin-Barragan et al. (2017) suggested two heuristics based on MINLP mathematical models that can be solved using a commercial solver and piecewise linearization. The first heuristic derived a good approximation of the $(s_t, S_t)$ strategy using the $(R_t, S_t)$ strategy. The second uses a binary search approach combined with a model similar to the first heuristic. This allows to tackle larger instances. According to the authors, both heuristics have an optimality gap around 0.3% and the second heuristic can solve instances of 25 periods in approximately 14 minutes on average.

All those formulations have been developed for the problem introduced by Scarf (1960) (i.e. the stochastic dynamic lot-sizing problem under linear holding costs and penalty costs with backlogs). And, not much attention has been devoted to the same problem with different stockout policies (e.g. $\alpha$ or $\beta$ service level constraint).

It has been proved that the $(s_t, S_t)$ strategy is optimal for the stochastic dynamic lot-sizing problem under linear holding costs and penalty costs with backlogs. However, this strategy is not especially recommended because it introduces nervousness in the system (Tempelmeier, 2013). Commonly, two types of nervousness are considered for replenishment systems: the setup-oriented nervousness and the quantity-oriented nervousness. The setup-oriented nervousness refers to changes in the replenishment planning and the quantity-oriented nervousness concerns changes in the replenishment quantities planned (Eksioglu, Kilic, Tarim, & Tunc, 2013). By comparing the three replenishment strategies, it has been shown that the most critical nervousness is generally the setup nervousness. And, it can be removed at minor cost using the $(R_t, S_t)$ strategy (i.e. fixing the replenishment schedule, but not the replenishment quantities). Numerical experiences suggest an average gap between 2.5% and 3% for using the $(R_t, S_t)$ strategy instead of the $(s_t, S_t)$ strategy. By contrast, the quantity-oriented nervousness, which is considered as less critical, turns out to be costly to eliminate. Indeed, it has been shown that that the $(R_t, Q_t)$ strategy performs poorly compared to the two other strategies, especially with a long planning horizon (Eksioglu, Kilic, Tarim, & Tunc, 2013). Moreover, the efficiency of the $(R_t, S_t)$ strategy is illustrated by the fact that state-of-the-art heuristics for the $(R_t, S_t)$ strategy perform better than heuristics from Asking and Bollapragada and Morton for the $(s_t, S_t)$ strategy (Dural-Selcuk, Kilic, Tarim, & Rossi, 2016). Lastly, the effect of re-planning has also been studied (i.e. the optimal parameters found are recomputed at the beginning of each period and demand realization is taken into consideration). As expected, the $(R_t, S_t)$ strategy is even closer to the optimal $(s_t, S_t)$ strategy. More surprisingly, under re-planning, the $(R_t, Q_t)$ strategy turns out to be also near optimal. As a matter of fact, a numerical experience measures an optimality gap of only 0.5% for the $(R_t, Q_t)$ strategy and 0.3% for the $(R_t, S_t)$ strategy. It demonstrates that, under re-planning, the $(R_t, Q_t)$ strategy is a good alternative to the $(s_t, S_t)$ strategy (Dural-Selcuk, Kilic, Tarim, & Rossi, 2016).

Again, all the results discussed above only hold for the stochastic dynamic lot-sizing problem under linear holding costs and penalty costs with backlogs.

### 3.2.6 $(\boldsymbol{R_t}, \boldsymbol{S_t})$ strategy: A review

Under the $(R_t, S_t)$ strategy, the replenishment planning is fixed but not the replenishment quantities. In other words, the timing of orders is determined at the beginning of the planning horizon, but the order quantities are only determined when the order should be placed (after observing demand realization of all previous periods). The order quantities will be the quantities that will bring back the inventory to the order-up-to level, $S_t$.

Bookbinder and Tan (1988) introduce this strategy for the first time and suggest a two-step method to obtain a good approximation of the optimal solution. The authors suggest to determine the planning of orders with the $(R_t, Q_t)$ strategy. Then, the best order-up-to levels for this planning can be computed with a linear programming formulation. Since then, the $(R_t, S_t)$ strategy has been extensively studied compared to the $(s_t, S_t)$ strategy.

Tarim and Kingsman (2004) propose a MIP model to optimally solve the problem under backlog policy and a $\alpha$-service level constraint. In their paper, they approximate the expected inventory on-hand by the net inventory position[3], assuming a high service level target. For the same problem, a constraint programming formulation that performs a bit better than the original formulation has also been proposed (Smith & Tarim, 2008). After, cost-based filtering techniques have been developed to improve the efficiency of the CP approach. And because of that, the formulation can tackle realistic instances (Hnich, Prestwich, Rossi, & Tarim, 2009). Since then, other methods have also been developed to improve the tractability of the original formulation of Tarim and Kingsman (2004) (Dogru, Özen, Rossi, & Tarim, 2011; Eksioglu, Kilic, Tarim, & Tunc, 2014).

Researchers also focus on the same problem with other stockout policies (e.g. penalty costs or $\beta$-service level constraints). Typically, the difficulty is to compute the expected on-hand inventory and the expected stockout quantities. Tarim and Kingsman (2006) propose an approximation with a MIP model obtained by piecewise linearization of the non-linear functions (i.e. the expected on-hand inventory and the expected stockout quantity functions). Hnich et al. (2012) suggest an exact constraint programming model and propose cost-based filtering techniques to boost the efficiency of the model. Dogru et al. (2012) propose two heuristics based on a dynamic formulation of the problem where either penalty costs or $\alpha$-service level constraints are considered. The authors also propose extensions to take into consideration capacity and minimum order quantity constraints. Moreover, they claim that the proposed heuristics can provide building blocks for multi-item extensions. Kilic et al. (2015) propose a unified MIP approach to solve the problem under lost sale or backlog policy and penalty costs or $\alpha/\beta$-service level constraints. The main advantage of the suggested method is to allow seamless modelling for the different configurations of the problem. Recently, Hilger and Tempelmeier (2015) propose a MIP model obtained by piecewise linearization for the extended multi-item stochastic lot-sizing under the $(R_t, S_t)$ strategy.

[3] The net inventory position is the inventory on-hand minus the eventual backlog quantity.

# Part 4: Presentation of the methods

## 4.1 Introduction

The goal of this section is to propose methods to solve a standard inventory management problem under specific replenishment strategies. Each method uses an algorithmic approach that corresponds to the solving procedure, while the strategies refer to how decisions regarding the replenishments are taken. So, what we call a 'method' is the combination of a replenishment strategy and an algorithmic approach. For instance, a strategy solved with a mixed-integer programming formulation is another method than the same strategy solved with a constraint programming formulation.

We propose two new methods and compare them with two others from the literature:

1. The *BT method* from Bookbinder and Tan (1988) based on the $(R_t, Q_t)$ strategy (i.e. static uncertainty strategy) and a MIP formulation.
2. The *TK method* from Tarim and Kingsman (2004) based on the $(R_t, S_t)$ strategy (i.e. static-dynamic uncertainty strategy) and a MIP formulation.
3. The *DP method:* We propose this new method based on the $(s_t, S_t)$ strategy (i.e. dynamic uncertainty strategy) and a dynamic programming formulation inspired from Scarf (1960).
4. The *DP-EOQ method:* We propose this new method based on the $(r, s_t, S_t)$ strategy and a dynamic programming formulation coupled with an EOQ approach.

The three first methods use strategies that have already been discussed because it corresponds to the traditional strategies of the stochastic lot-sizing problem. The fourth strategy, close to the $(s_t, S_t)$ strategy, is a traditional strategy for periodic review system (Cunha, Oliveira, & Raupp, 2017). We have decided to consider this strategy because we believe that the behaviour will be similar to the $(s_t, S_t)$ strategy but with less nervousness. Under the $(r, s_t, S_t)$ strategy, a review interval, denoted by $r$, is determined. The notation $r$ (lower case) should not be confused with $R$ (upper case). The first one is used to denote the review interval (i.e. the time interval between two reviews) while the second is used to denote the replenishment periods (i.e. periods with a replenishment). Here, $r$ is an integer corresponding to the number of periods between two reviews of the inventory position. When reviewing the stock level, if the inventory position is below $s_t$, then an order is placed to bring back the stock to $S_t$. Else, we wait until the next review period. Naturally, the $(r, s_t, S_t)$ strategy with $r$ equal to 1 corresponds to the $(s_t, S_t)$ strategy.

As a reminder, the methods consider demand uncertainty to propose robust solutions. But under demand uncertainty, stockouts may occur. To ensure proper comparisons, we decide to keep a common framework regarding the stockout policy[4]. Firstly, we select the $\alpha$-service level criterion as a system to control demand uncertainty. This criterion has been selected because it seems that service level constraints are more popular than penalty costs in real inventory systems. One reason is that penalty costs are hard to estimate (Aktürk, Koca, & Yaman, 2014). Moreover, the $\alpha$-service level is easier to implement into mathematical models than the $\beta$-service level. Secondly, we consider that all the unsatisfied demand is backlogged to the next period.

## 4.2 Standard inventory management problem

In order to explain easily each method, a standard inventory management problem is proposed. In this way, a common context can be used to illustrate the algorithms. We consider a simple problem where the goal is to minimize the total cost over a finite time horizon.

The total cost is simply the sum of:

1. The holding cost incurred for carrying inventory over time (incurred for each unit carried from one period to the next).
2. The ordering cost incurred for each order placed whatever the size of the order.
3. The procurement cost for purchasing items.

The time horizon is composed of $T$ periods (e.g. days) with $t \in \{1, \dots, T\}$. Replenishments occur at the beginning of these periods. Replenishments are considered instantaneous and there is no leadtime. Demands of all periods are uncertain and follow known and independent probability distributions. The stockout probability should be less than $1 - \alpha$ for all periods. And, if a stockout occurs, the unsatisfied demand is backlogged to the next period.

[4] As a reminder, the stockout policy specifies how unsatisfied demand is considered (backlog vs lost sales) and controlled (with penalty costs or with an $\alpha/\beta$-service level constraint).

The following notations are used:

$d_t$: Expected demand in period $t$.

$h$: Unit cost for carrying an item from one period to the next.

$a$: Ordering cost for placing an order.

$c$: Unit procurement cost for purchasing an item.

For convenience, random variables are followed by empty brackets. For instance, $d_t(\,)$ is the demand realization in period $t$ while $d_t$ is the expected demand in period $t$.

## 4.3 Description of the methods

### 4.3.1 BT method: $(\boldsymbol{R_t}, \boldsymbol{Q_t})$ strategy & MIP approach

Under the $(R_t, Q_t)$ strategy, $\alpha$-service level constraint and backlog policy, Bookbinder and Tan (1988) propose to use the net inventory position as an approximation of the on-hand inventory, assuming a high $\alpha$-service level. As a reminder, the net inventory position is the actual inventory on-hand minus the eventual backlog quantity. The net inventory position can be negative, but it is physically impossible for the inventory on-hand. The authors also show that, under this configuration, the stochastic lot-sizing is equivalent to its deterministic version and the same methods can be applied to solve both models (heuristics, strong formulations, Wagner-Within algorithm…). Therefore, the stochastic lot-sizing under the $(R_t, Q_t)$ strategy can have a good tractability if tailor-made algorithms developed for the deterministic lot-sizing are used. In this section, we propose the MIP formulation of Bookbinder and Tan (1988). From now on, we denote this method by the *BT method*. This formulation is probably less tractable than other tailor-made algorithms. But, it should be tight enough to tackle medium-large size instances and it is very scalable. Indeed, the model can be easily extended to multi-item or multi-stage settings and specific constraints can be easily added into the MIP formulation. However, fundamentally, the $(R_t, Q_t)$ strategy is not flexible and the company cannot react to demand realization. Moreover, this approach requires to know the probability distribution of the demand for all time periods. It can be an issue if only rough estimations are known.

**MIP formulation**

We introduce the following variables:

$R_t \in \{0,1\}$: Variable equal to 1 if a replenishment occurs in period $t$. Else, equal to 0.

$Q_t \geq 0$: Quantity ordered in period $t$.

$I_t \geq 0$: Expected ending inventory in period $t$.

Objective function:

$$Min \sum_{t=1}^{T} hI_t + aR_t + cQ_t$$

Subject to:

| | |
|---|---|
| $I_t = I_0 + \sum_{k=1}^{t} Q_k - d_k$ | $\forall t$ |
| $\sum_{k=1}^{t} Q_k \geq G^{-1}_{d_1()+\cdots+d_t()}(\alpha) - I_0$ | $\forall t$ |
| $Q_t \leq M \times R_t$ | $\forall t$, where $M$ is a big number. |

$G^{-1}_{d_1()+\cdots+d_t()}(\alpha)$ is the inverse cumulative distribution function of the demand from period 1 to $t$ such that $G^{-1}_{d_1()+\cdots+d_t()}(\alpha) = u$ means that $Proba(d_1(\,) + \cdots + d_t(\,)\ \leq u\,) = \alpha$.

The constraint $\sum_{k=1}^{t} Q_t \geq G^{-1}_{d_1()+\cdots+d_t()}(\alpha) - I_0$ ensures that the $\alpha$-service level constraint is respected. Indeed, we want:

$$proba(I_t(\,) \geq 0) \geq \alpha$$

$$proba\left(I_0 + \sum_{k=1}^{t} Q_k - \sum_{k=1}^{t} d_k(\,) \geq 0\right) \geq \alpha$$

$$proba\left(\sum_{k=1}^{t} Q_k \geq \sum_{k=1}^{t} d_k(\,) - I_0\right) \geq \alpha$$

$$\sum_{k=1}^{t} Q_k \geq G^{-1}_{d_1()+\cdots+d_t()}(\alpha) - I_0$$

We assume that the probability distributions of the demands are known. Then, we can compute easily $G^{-1}_{d_1()+\cdots+d_t()}(\alpha)$ for all $t$ with a spreadsheet software. Finally, the model can be solved using a traditional solver software. With this model, the set of replenishment periods is given by the set of $t \in T$ such that $R_t = 1$. And, the corresponding quantities to order are given by $Q_t$.

### 4.3.2 TK method: $(\boldsymbol{R_t}, \boldsymbol{S_t})$ strategy & MIP approach

Under the $(R_t, S_t)$ strategy, the replenishment planning is fixed but not the replenishment quantities. Tarim and Kingsman (2004) propose an optimal MIP formulation for the case with an $\alpha$-service level constraint and a backlog policy. They also assume a high service level to approximate the inventory on-hand by the net inventory position. We denote this method by the *TK method*. The formulation is less tractable than the *BT method* suggested above because additional binary variables are required. It is possible to improve the tractability by using a simple heuristic. It consists to find good replenishment periods using the *BT method*. Then, the replenishment periods found can be used as inputs of the MIP model for the $(R_t, S_t)$ strategy given below (i.e. *TK method)*. When the replenishment periods are known, the remaining problem is linear, and the solution is straightforward. This heuristic was suggested by Bookbinder and Tan (1988) and generally provides a good solution[5]. Regarding the scalability, the *TK method* seems almost as scalable as the *BT method.* But this time, the $(R_t, S_t)$ strategy allows more responsiveness to demand realization. Again, this approach requires to know to the probability distribution of the demand for all time periods.

**MIP Formulation**

We introduce the following variables:

- $R_t \in \{0,1\}$: Variable equal to 1 if a replenishment occurs in period $t$. Else, equal to 0.
- $P_{tj} \in \{0,1\}$: Variable equal to 1 if the most recent replenishment prior to period $t$ was in period $t-j+1$ (i.e. if the last replenishment was $j-1$ periods ago). Else, equal to 0.
- $I_t \geq 0$: Expected ending inventory in period $t$.
- $S_t \geq 0$: Expected opening inventory in period $t$, taking into account the eventual replenishment at the beginning of the period.

Objective function[6]:

$$Min \sum_{t=1}^{T} hI_t + aR_t + (cS_t - cI_{t-1})$$

Subject to:

$$I_t = S_t - d_t \qquad \forall t$$

$$S_t \geq I_{t-1} \qquad \forall t$$

$$S_t - I_{t-1} \leq M \times R_t \qquad \forall t\text{, where } M \text{ is a big number}$$

$$I_t \geq \textstyle\sum_{j=1}^{t} (G^{-1}_{d_{t-j+1}()+d_{t-j+2}()+\cdots+d_t()}(\alpha) - \sum_{k=t-j+1}^{t} d_k) P_{tj} \qquad \forall t$$

$$\textstyle\sum_{j=1}^{t} P_{tj} = 1 \qquad \forall t$$

$$P_{tj} \geq R_{t-j+1} - \textstyle\sum_{k=t-j+2}^{t} R_k \qquad \forall t, \forall j \leq t$$

---

5 However, the solution is not optimal because the choice of the replenishment periods is made independently of the choice of the order-up-to levels (Armagan Tarim B. K., 2004).

6 The procurement costs correspond to $\sum_{t=1}^{T}(cS_t - cI_{t-1})$ because the expected quantity delivered in period $t$ is: $S_t - I_{t-1}$

The variable $P_{tj}$ uniquely identifies the last replenishment period prior to period $t$. Then, the minimum ending inventory to satisfy the $\alpha$-service level constraint can be computed using $G^{-1}_{d_{t-j+1}()+d_{t-j+2}()+\cdots+d_t()}(\alpha)$. As before, the probability distributions of demands are known, and we can compute $G^{-1}_{d_{t-j+1}()+d_{t-j+2}()+\cdots+d_t()}(\alpha)$ for all combinations of $t$ and $j$ such that $j \leq t$. After that, we can solve the problem with a solver software. For more explanations about this model, the reader can refer to the original paper of Tarim and Kinsman (2004). It is possible to formulate the model without the variable $S_t$ by substituting $S_t$ with $I_t + d_t$ (see the first constraint). However, we keep $S_t$ for the sake of clarity. As before, the set of replenishment periods is given by the set of $t \in T$ such that $R_t = 1$. And, the corresponding order-up-to levels are given by $S_t$.

### 4.3.3 DP method: $(\mathbf{s_t}, \mathbf{S_t})$ strategy & Dynamic programming approach

Under the $(s_t, S_t)$ strategy, the goal is to determine the best reorder point and order-up-to level for each period. At the beginning of each period, if the inventory position is below the reorder point, an order is placed to bring back the stock to the order-up-to level of the current period.

To the best of our knowledge, there is no formulation to solve the stochastic lot-sizing under the $(s_t, S_t)$ strategy, $\alpha$-service level constraints and backlog policy. We then propose a dynamic programming heuristic inspired from the formulation of Scarf (1960) for the penalty cost scheme. The approach of Scarf is based on the k-convexity property of the cost function (where ‘k’ denotes the setup cost). In this formulation, the program computes recursively the $(s_t, S_t)$ points by evaluating gradually the expected costs until the solution obtained is higher by k than the best solution found. Scarf proved the optimality of the $(s_t, S_t)$ strategy for the dynamic stochastic lot-sizing problem under penalty costs and backlog policy. This proof does not cover the same problem under other stockout policies. Then, we cannot guarantee that the $(s_t, S_t)$ strategy is optimal for the stochastic lot-sizing problem under $\alpha$-service level constraints and backlog policy. However, we suppose that the $(s_t, S_t)$ strategy provides good solutions because the problems are very similar. We denote our new method based on a dynamic programming approach by *the DP method*.

The scalability of our method is mitigated because the formulation can be easily extended to other stockout policies. But, the formulation does not allow to be easily extended to multi-item settings or specific constraints. The $(s_t, S_t)$ strategy benefits from a high responsiveness to demand realization. But, it increases the nervousness in the inventory system as explained in section 3.2.5. Finally, this formulation also requires to know the probability distribution of the demand for all time periods.

The *BT method* and the *TK method* are both MIP formulations. In these formulations, we made an important assumption to keep a linear setting. We assume that the $\alpha$-service level is high and thus, we can approximate the inventory on-hand by the net inventory position. By contrast, the dynamic programming formulation does not require linearity. Then, if needed we can compute the expected inventory on-hand without approximation. As mentioned before, it requires to use special mathematical functions called 'first order loss functions' (see appendix 1). However, for all instances tested in our research, the $\alpha$-service level is at least equal to 80% and we do not observe any change of the outputs. And this, no matter if we approximate the inventory on-hand by the net inventory position or not. We then describe the *DP method* using this approximation for the sake of clarity.

**How to compute $s_t$ and $S_t$**

We propose a dynamic programming approach based on the intuition that the cost structure is often such that there is no interest for early ordering[7]. In other words, it is cheaper to order as late as possible. It happens if holding costs are bigger than the potential benefit from early ordering. Mathematically, it happens if $h_t x' \geq (a_{t+1} + x' c_{t+1}) - (a_t + x' c_t)$ for all $t$, where $h_t$, $a_t$ , and $c_t$ denote the unit holding cost, the ordering cost, and the unit procurement cost in period $t$ respectively. And, where $x'$ is the order quantity. If there is no interest for early ordering, we order only when the current inventory position does not allow to wait until the next replenishment period, while ensuring that the $\alpha$-service level constraint is respected. In other cases, it is cheaper to order at the next period. It means that we can directly compute $s_t$ for all $t$ so that we have $Proba(s_t \geq d_t(\,)) = \alpha$.

Next, we must find the $S_t$. Theoretically, we believe that it is possible to find the optimal $S_t$ with a recursive approach. Starting from the end of the planning horizon, we select the $S_t$ such that the expected total cost is minimized. Unfortunately, computing the exact expected total

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[7] This is the case for the inventory management problem used as an example. Indeed, the costs are constant over time and it is better to order as late as possible.

cost is a complex task because, at each period, there is a probability that a replenishment occurs. And, if there is a replenishment, it affects both the inventory position and the next replenishments for the rest of the planning horizon. Then, to overcome this issue, we approximate the expected total cost by supposing that the demand will occur as expected. Starting from the end of the planning horizon, we find $S_t$ such that the costs are minimized if the demand is as expected (i.e. if the demand is deterministic). Due to this approximation, our method is not optimal. Concretely, $S_t$ is found by gradually evaluating the cost function, $J(t, x)$, and storing the lowest value found. $J(t, x)$ is the optimal cost from period $t$ to $T$, if the opening inventory position in period $t$ is $x$ and if the demand realization is exactly as expected (i.e. if the demand is deterministic). This function can only be computed if the optimal reorder points, $s_t$, and order-up to levels, $S_t$, are known for all $t \in \{t, \dots, T\}$.

$for\ all\ t \in \{1, \dots, T\}$,

$$J(t, x) = \begin{cases} h \times (S_t - d_t) + a + c(S_t - x) + J(t+1, S_t - d_t) & if\ \ x < s_t \\ h \times (x - d_t) + J(t+1, x - d_t) & if\ \ x \geq s_t \end{cases}$$

with $J(T+1, x) = 0$

Concretely, we select $S_t$ to minimize the estimated expected cost from period $t$ to $T$. Since we do not know the exact inventory position when a replenishment is needed, we approximate the procurement cost in period $t$ by the lower bound $c \times (S_t - s_t)$. Mathematically, we select $S_t$ to minimize $a + h \times (S_t - d_t) + J(t+1, S_t - d_t) + c \times (S_t - s_t)$.

**Algorithmic approach**

Step 1: - Compute $s_t$ for all $t$ such that $Proba\big(s_t \geq d_t(\,)\big) = \alpha$

- Define an upper bound for all $S_t$, we name it: $\overline{S_t}$

Step 2: Set $t = T$

Step 3: - If $(t \geq 1)$

$BSF$[8] $= a + h \times (\overline{S_t} - d_t) + J(t+1, \overline{S_t} - d_t) + c \times (\overline{S_t} - s_t)$

$IndexBSF$[9] $= \overline{S_t}$

Go to step 4

- Else, stop. All $(s_t, S_t)$ has been found!

Step 4: Set $i = s_t$ and go to step 5

Step 5: - Compute: $current = a + h \times (i - d_t) + J(t+1, i - d_t) + c \times (i - s_t)$

- If $(current \leq BSF)$

$BSF = current$

$IndexBSF = i$

- If $(i \geq \overline{S_t})$

$S_t = IndexBSF$

Go to step 3 with $t = t - 1$

- Increment $i$: $i = i + 1$ and repeat step 5

## 4.3.4 DP-EOQ method: $(\boldsymbol{r}, \boldsymbol{s_t}, \boldsymbol{S_t})$ strategy & DP + EOQ approach

Under the $(r, s_t, S_t)$ strategy, two types of information should be determined. Firstly, an appropriate review interval, $r$, must be selected. Secondly, a reorder point, $s_t$, and an order-up-to level, $S_t$, for each period must be computed. The idea behind this strategy is to reduce the nervousness of the $(s_t, S_t)$ strategy by reviewing the inventory position less often and use aggregated data rather than disaggregated data (e.g. using weekly forecasts rather than daily forecasts). We propose a two-step heuristic to find these parameters. The idea is to find a good review interval based on an Economic Order Quantity approach (see appendix 2 for a brief reminder about the EOQ approach). Therefore, the review interval is computed using the

[8] $BSF$ is a variable that stores the Best Solution Found so far (i.e. the minimal cost) for the current $t$.

[9] $IndexBSF$ is a variable that stores the index (i.e. the order-up-to level) of the Best Solution Found.

restrictive assumptions of the EOQ model (e.g. stationary demand pattern). As soon as we have found the review interval (e.g. $r = 3$), we aggregate the demands during all the review intervals and create a new planning horizon composed of $roundUP$[10]$(\frac{T}{r})$ periods, where $T$ is the initial number of periods. This is illustrated in figure 2.

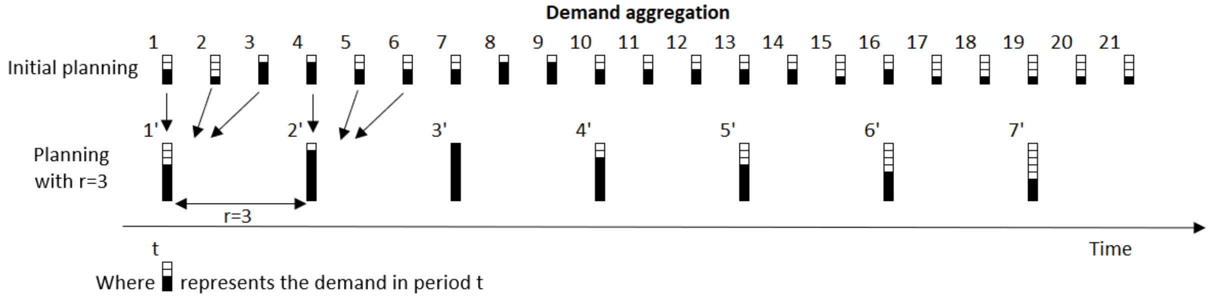


*Figure 2: Example of the demand aggregation process.*

Next, the $s_t$ and $S_t$ levels of all review periods (i.e. the aggregated period) are found using the dynamic programming algorithm described in the previous section. Therefore, we assume again that there is no interest for early ordering. Moreover, to conserve the idea of aggregation, we consider that the demand occurs linearly when computing the holding cost in the $J(t, x)$ function.

E.g. with $r = 3,\ d_1 = 20, d_2 = 10\ and\ d_3 = 30$, we find that the aggregated demand during the review interval is $d_{1'} = 60$. Knowing that the initial stock is 80 units, we estimate the expected holding cost by:

$$h \times \left(\left(80 - \frac{60}{3}\right) + \left(80 - \frac{2\times60}{3}\right) + \left(80 - \frac{3\times60}{3}\right)\right)$$

Mathematically, it implies that the cost function, $J(t, x)$, becomes:

$$for\ all\ t \in \{1, \dots, T\},\ J(t, x) =$$

$$\begin{cases} h \times \left((S_t - \frac{1}{r}d_t) + (S_t - \frac{2}{r}d_t) + \cdots + \ (S_t - \frac{r}{r}d_t)\right) + a + c(S_t - x) + J(t+1, S_t - d_t) & if\ x < s_t \\ h \times \left((x - \frac{1}{r}d_t) + (x - \frac{2}{r}d_t) + \cdots + (x - \frac{r}{r}d_t)\right)\ + J(t+1, x - d_t) & if\ x \geq s_t \end{cases}$$

with $J(T + 1, x) = 0$

We denote this method by the *DP-EOQ method*. This heuristic should be faster than the *DP method* because min/max levels must be computed only for the review periods and not for all periods. For instance, if $r$ is equal to 5 and the planning horizon is composed of 100 periods,

[10] We round up the ratio $\frac{T}{r}$.

only twenty min/max levels must be computed instead of one hundred. The *DP-EOQ method* keeps the same mitigated scalability than the *DP method.* Also, the strategy is quite responsive even if it allows less reactivity than the $(s_t, S_t)$ strategy, especially if $r$ is big. Compared to the other methods presented so far, it is not required to know the probability distribution of the demands for all time periods. Indeed, the review interval can be computed using only average data. And, we only need the aggregated probability distribution for the demand of each review interval.

**Algorithmic approach**

Step 1: Find a good review interval using an EOQ approach

$$r = \frac{EOQ^*}{Total\ demand} \times horizon\ length$$

Step 2: Aggregate the forecasted demands between all the review intervals.

Step 3: Find $s_t\ and\ S_t$ for all $t$ by dynamic programming (see: $(s_t, S_t)$ algorithmic approach)

## 4.7 Choice of the probability distribution

In inventory management problems with demand uncertainty, the choice of the probability distribution to model the demand is important. The normal distribution is an appropriate choice for fast-moving items. By contrast, for slow-moving products, a Poisson or Laplace distribution is more adapted (Janssens & Ramaekers, 2008). The *BT method* and the *TK method* require to compute the inverse cumulative function of the demand denoted by $G_{d()}^{-1}(\alpha)$. If the demands are independents and normally distributed, then $G_{d()}^{-1}(\alpha)$ can be obtained on Excel with the following command: $norm.inv(\alpha, \mu, \sigma)$. To the best of our knowledge, there is no direct function to compute $G_{d()}^{-1}(\alpha)$ on Excel, if the demands are independent Poisson variables. However, it is easy to create a VBA function that compute it (see appendix 3).

# Part 5: Test Bed and sensitivity analysis

The goal of this part is double. Firstly, we aim to illustrate the strategies and their corresponding algorithmic approaches on a common test bed. Secondly, we carry out a sensibility analysis to assess the performance of the methods under different settings.

## 5.1 Test Bed

### 5.1.1 Problem

To briefly illustrate all methods, we propose a small example inspired from Tarim and Kingsman (2004). We suppose a time horizon of 30 periods. The demands are independent Poisson random variables with known means. There are three costs in the inventory management problem.

1. Procurement cost: $c = 4$
2. Holding cost: $h = 0.2$
3. Setup cost: $a = 250$

Moreover, we assume that the desired service level is set to 95% and that the unsatisfied demand is backlogged to the next period. Table 1 summarizes the expected demands.

| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $d_t$ | 50 | 75 | 43 | 66 | 64 | 62 | 50 | 62 | 44 | 64 | 71 | 59 | 61 | 44 | 45 |

| $t$ | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $d_t$ | 66 | 66 | 57 | 52 | 67 | 53 | 73 | 71 | 71 | 79 | 48 | 40 | 77 | 78 | 42 |

*Table 1: Means of the Poisson distributed demands.*

### 5.1.2 Results

We apply the four methods discussed to the test bed problem stated above[11]. The results are summarized for each method in table 2. Under the $(R_t, Q_t)$ strategy, both the ordering cost and the procurement cost are independent of the demand realization. And, with the $(R_t, S_t)$ strategy only the ordering cost is independent of the demand realization. By contrast, with the two other strategies, all costs are dependent of the demand realization. This is particularly true for the $(s_t, S_t)$ strategy, where the ordering cost is hardly predictable. Table 3 summarizes the total cost computed for each method if the demand realization is exactly what is forecasted (i.e. if the demand is deterministic). We observe that the *DP method* provides the lowest cost under determinist setting. Next, it is the *TK method* that brings the best results

[11] We use the software Aimms to solve the MIP models of the *BT method* and the *TK method*. For the *DP method* and the *DP-EOQ method*, the recursive algorithms are written in Scala.

but the *DP-EOQ method* has very similar performances. Finally, as it could be expected, the *BT method* is the least efficient.

| Periods | 1. BT method | | 2. TK method | | 3. DP method | | 4. DP-EOQ method | | |
|---|---|---|---|---|---|---|---|---|---|
| | R | Q | R | S | s | S | R (r=6) | s | S |
| 1 | 1 | 444 | 1 | 444 | 62 | 422 | 1 | 391 | 391 |
| 2 | 0 | 0 | 0 | 394 | 90 | 477 | 0 | 0 | 0 |
| 3 | 0 | 0 | 0 | 319 | 54 | 402 | 0 | 0 | 0 |
| 4 | 0 | 0 | 0 | 276 | 80 | 359 | 0 | 0 | 0 |
| 5 | 0 | 0 | 0 | 210 | 77 | 293 | 0 | 0 | 0 |
| 6 | 0 | 0 | 0 | 146 | 75 | 295 | 0 | 0 | 0 |
| 7 | 0 | 0 | 0 | 84 | 62 | 511 | 1 | 381 | 381 |
| 8 | 1 | 465 | 1 | 485 | 75 | 461 | 0 | 0 | 0 |
| 9 | 0 | 0 | 0 | 423 | 55 | 399 | 0 | 0 | 0 |
| 10 | 0 | 0 | 0 | 379 | 77 | 355 | 0 | 0 | 0 |
| 11 | 0 | 0 | 0 | 315 | 85 | 360 | 0 | 0 | 0 |
| 12 | 0 | 0 | 0 | 244 | 72 | 355 | 0 | 0 | 0 |
| 13 | 0 | 0 | 0 | 185 | 74 | 403 | 1 | 370 | 370 |
| 14 | 0 | 0 | 0 | 124 | 55 | 342 | 0 | 0 | 0 |
| 15 | 0 | 0 | 0 | 80 | 56 | 418 | 0 | 0 | 0 |
| 16 | 1 | 444 | 1 | 393 | 80 | 448 | 0 | 0 | 0 |
| 17 | 0 | 0 | 0 | 327 | 80 | 453 | 0 | 0 | 0 |
| 18 | 0 | 0 | 0 | 261 | 70 | 387 | 0 | 0 | 0 |
| 19 | 0 | 0 | 0 | 204 | 64 | 401 | 1 | 420 | 420 |
| 20 | 0 | 0 | 0 | 152 | 81 | 349 | 0 | 0 | 0 |
| 21 | 0 | 0 | 0 | 85 | 65 | 282 | 0 | 0 | 0 |
| 22 | 0 | 0 | 1 | 414 | 87 | 590 | 0 | 0 | 0 |
| 23 | 1 | 517 | 0 | 341 | 85 | 517 | 0 | 0 | 0 |
| 24 | 0 | 0 | 0 | 270 | 85 | 446 | 0 | 0 | 0 |
| 25 | 0 | 0 | 0 | 199 | 94 | 375 | 1 | 396 | 396 |
| 26 | 0 | 0 | 0 | 120 | 60 | 296 | 0 | 0 | 0 |
| 27 | 0 | 0 | 0 | 72 | 51 | 248 | 0 | 0 | 0 |
| 28 | 0 | 0 | 1 | 220 | 92 | 208 | 0 | 0 | 0 |
| 29 | 0 | 0 | 0 | 143 | 93 | 131 | 0 | 0 | 0 |
| 30 | 0 | 0 | 0 | 65 | 53 | 53 | 0 | 0 | 0 |

Table 2: Results obtained for each method.

| Cost summary with deterministic demand | | | | |
|---|---|---|---|---|
| Criteria\Methods | *BT method* | *TK method* | *DP method* | *DP-EOQ method* |
| Ordering cost | 1000.0 | 1250.0 | 1000.0 | 1250.0 |
| Procurement cost | 7480.0 | 7292.0 | 7244.0 | 7328.0 |
| Holding cost | 1464.0 | 1114.8 | 1215.0 | 1117.4 |
| Total cost | 9944.0 | 9656.8 | 9459.0 | 9695.4 |

Table 3: Cost summary if demand is deterministic.

By looking at table 2, we see that the *DP-EOQ method*, which should return a $(r, s_t, S_t)$ strategy, returns a $(R_t, S_t)$ strategy as output because $s_t = S_t$ for all $t$. At first glance, this result may seem surprising. However, we remind that the review interval, $r$, is determined in such a way that a replenishment will probably be needed when reviewing the inventory position. Then, this result was predictable.

To assess the real performance of the methods when demand is uncertain (i.e. under stochastic demand), we carry out a small experiment based on simulations. Concretely, we estimate the expected costs for each method by running 1000 simulations[12]. During the simulation process, demands are randomly generated according to a Poisson distribution with the means from table 1. The results of the experiment are summarized in table 4.

| Average results obtained by simulation | | | | |
|---|---|---|---|---|
| **Criteria\Methods** | ***BT method*** | ***TK method*** | ***DP method*** | ***DP-EOQ method*** |
| **Costs** | | | | |
| Ordering cost | 1000.0 | 1250.0 | 1190.8 | 1250.0 |
| Procurement cost | 7480.0 | 7293.1 | 7283.0 | 7331.0 |
| Holding cost | 1459.2 | 1114.2 | 1297.3 | 1118.0 |
| Total cost | 9939.2 | 9657.2 | 9771.1 | 9698.9 |
| **Service level** | | | | |
| Cumulative backlog quantity | 3.233 | 2.153 | 0.134 | 2.186 |
| Number of stockouts | 0.213 | 0.248 | 0.038 | 0.251 |

*Table 4: Results obtained by simulation for each method.*

First, we notice that under demand uncertainty (i.e. stochastic setting), the *TK method* becomes the most efficient. Also, we observe that the average costs obtained by simulation are close to the estimated expected costs from the MIP formulation. This is because, the only approximation made is the estimation of the inventory on-hand (which is non-negative) by the net inventory position (which can be negative due to backlogs). But, since the service level is high, it does not affect the results. The same reasoning holds for the *BT method* but this time, this is the least efficient method. By contrast, the *DP method* gives disappointing results from a cost point of view as it highly underestimates the costs. One reason is that the strategy introduces a lot of nervousness regarding setups. This is costly because the $(s_t, S_t)$ strategy may require many replenishments, which is not desirable if the ratio $A/h$ is high (Kingsman & Tarim, 2004). Another reason comes from the algorithmic approach itself. The $s_t$ are determined in such a way to ensure the service level constraint. And the $S_t$ are determined to minimize our estimation of the expected total cost. As a reminder, we approximate the expected total cost by computing the total cost if the demand realization happens as expected (i.e. if the demand is deterministic). However, the demand realization will differ from the expected values because the demands are given by random variables. These deviations are costly. Often, an order is placed because the opening inventory position is slightly below $s_t$.

[12] We use the software AnyLogic.

However, the inventory position should ideally be equal to zero before ordering because the delivery is instantaneous. This reasoning is also true with non-zero leadtime but then, before ordering, the inventory position should ideally be equal to the demand during the lead-time. This phenomenon occurs also with the *DP-EOQ method.* However, the effects are reduced because the review interval is fixed in such a way that we will probably make a replenishment. Consequently, it reduces the setup nervousness and the expected costs are closer to the estimated costs of the method.

As already explained, the $(s_t, S_t)$ strategy of the *DP method* implies uncertainty about the replenishment periods. Therefore, the expected total cost is more variable than with the other strategies. This is clearly illustrated by the box plots in figure 3 that represent the distribution of the total cost for each method over the 1000 simulations.

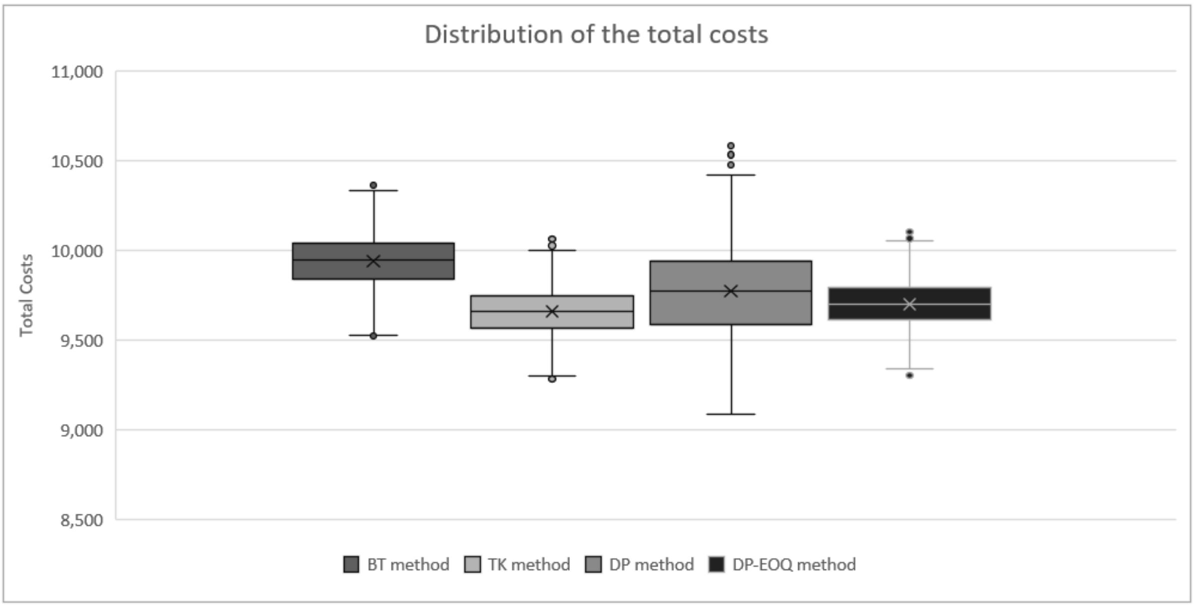


*Figure 3: Distribution of the total costs for each method.*

Some information about the observed service level is also summarized in table 4. With the *BT method,* we see that the average cumulative quantity backlogged is slightly over 3 units. It means that, on average, 3 units are not served on time during the planning horizon. With the *DP-EOQ method* and *TK method*, the average cumulative quantity backlogged is approximately 2 units. However, for these three methods, the average number of shortages over the planning horizon is between 0.2 and 0.3, which means that most of the time, no shortage happens. But, it also means that when backlogging happens, the quantity backlogged is substantially greater than the average cumulative backlog quantity during the planning horizon. This is illustrated in figure 4 with the distribution of the backlog quantity when a stockout occurs. Lastly, the *DP method* shows an impressive service level. Indeed, the average number of shortages over the planning horizon is close to zero and the average backlog quantity is very small.

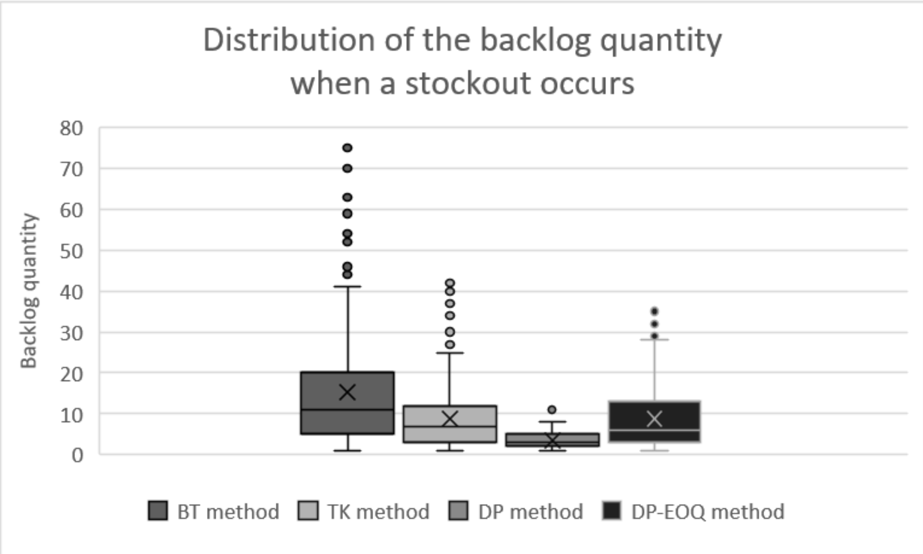


*Figure 4: Distribution of the backlog quantity when shortages occur.*

In the test bed, we imposed a maximum stockout probability of 5%. In theory, all methods should satisfy this constraint. We can estimate the stockout probability for each period by looking at the experiment outputs. All non-zero observed probabilities are summarized in table 5.

| **Observed stockout probabilities per period (%)** | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Methods\Periods** | **6** | **7** | **12** | **14** | **15** | **18** | **20** | **21** | **22** | **24** | **27** | **29** | **30** |
| ***BT method*** | 0.0 | 5.3 | 0.0 | 0.0 | 4.9 | 0.0 | 0.0 | 0.0 | 5.8 | 0.0 | 0.0 | 0.5 | 4.8 |
| ***TK method*** | 0.0 | 5.3 | 0.0 | 0.0 | 5.6 | 0.0 | 0.0 | 4.5 | 0.0 | 0.0 | 4.9 | 0.0 | 4.5 |
| ***DP method*** | 0.0 | 0.3 | 0.0 | 0.1 | 0.3 | 0.0 | 0.1 | 0.3 | 0.2 | 0.1 | 0.0 | 0.1 | 2.3 |
| ***DP-EOQ method*** | 5.7 | 0.0 | 4.8 | 0.0 | 0.0 | 4.7 | 0.0 | 0.0 | 0.0 | 4.7 | 0.0 | 0.0 | 5.2 |

*Table 5: Observed stockout probabilities per period.*

Globally, we observe that all the methods satisfy the $\alpha$-service level constraint. As already mentioned, the *DP method* shows a particularly high service level. The reason is that the algorithm fixes the $s_t$ to guarantee at least a non-stockout probability of 95% for each period but, most of the time, it will be much larger. If we start a period with exactly the minimum stock required, the shortage probability is equal to 5%. But, in practice, the inventory position is rarely equal to $s_t$. The inventory position is often either greater or lower than $s_t$. When the inventory position is greater than $s_t$, the probability of shortage is lower than 5%. And, when it is lower than $s_t$, a replenishment occurs, which means that the probability of shortage will be much lower than 5%.

## 5.2 Sensitivity analysis

### 5.2.1 Methodology

In this section, we aim to assess the performance of the strategies with their corresponding algorithmic approaches. We use the same problem structure as before with the three costs unchanged (i.e. $c = 4, h = 0.2, a = 250$) and a time horizon of 30 periods. We want to assess the performance under different seasonality levels, service levels and uncertainty levels.

To evaluate the effectiveness of the methods with time varying demands, we consider several demand patterns. Altogether, we consider six demand patterns: constant (*Const*), random (*Rand*), low life cycle (*Lcy*), high life cycle (*Hcy*), low sinusoidal (*Lsin*) and high sinusoidal (*Hsin*). The demand patterns are adapted[13] from the research of Dural-Selcuk et al. (2016). We decide to use these patterns because, as claimed by the authors, these patterns have been largely used in other studies. We keep a constant average demand per period equal to 60 for all instances. The demand patterns are illustrated in figure 5 (see appendix 4 for exact data). Moreover, we consider three different service levels, $\alpha \in \{95\%, 90\%, 80\%\}$, corresponding to high, medium and low service levels respectively. Finally, we suppose again that the demands are independent Poisson random variables. But, to assess the robustness against uncertainty, we do not always assume that the means are perfectly known. Instead, we introduce a new parameter in the simulation process, $\rho \in \{0, 0.05, 0.1, 0.2\}$, that represents the coefficient of variation of the forecast error[14] for the expected mean of the probability distribution. More concretely, it means that all methods assume that the demand in period $t$ is given by $y \sim Poisson(d_t)$, where $d_t$ is the expected demand in period $t$. But, in reality, the demand in period $t$ is given by $y' \sim Poisson\big(d'_t \sim Normal(d_t, \rho \times d_t)\big)$. For instance, we may forecast that the demand in period $t$ is a Poisson random variable with a mean ($d_t$) equal to 60, while the true demand realization has a mean ($d'_t$) of 70. Also, if $\rho$ tends to zero, it means that there is no forecast error about the expected mean.

Then, we perform a sensitivity analysis by varying three parameters: (1) The demand pattern, (2) the $\alpha$-service level, and (3) the coefficient of variation of the forecast error, $\rho$. None of the methods can consider $\rho$ because this parameter is unknown and only introduced in the

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[13] We do not use the same absolute values, but we use the same trends.

[14] We assume that the forecast error is Normally distributed.

simulation process. Since we consider 6 demand patterns and 3 $\alpha$-service levels, it leads to 18 instances to solve with the 4 methods. We evaluate by simulation the costs associated to the 72 configurations with the 4 possible values of $\rho$. Then, in total, we carry out 288 experiments[15] and each experiment is composed of 1000 simulations using randomly generated data.

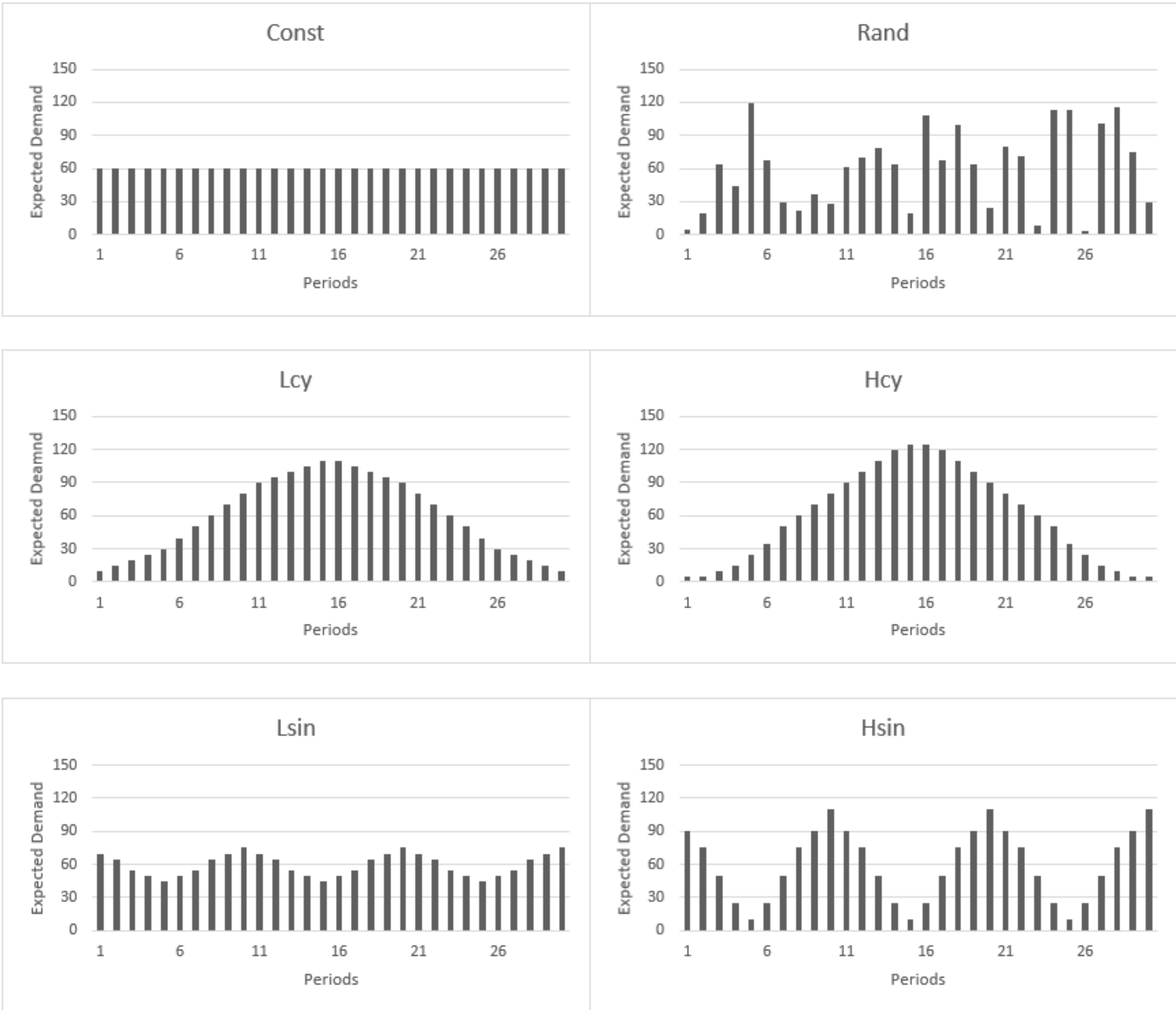


*Figure 5: Demand patterns.*

15 We use the software AnyLogic.

### 5.2.2 Results

We carry out the 288 experiments and we summarize the global average results (i.e. averages of all configurations tested) obtained for each method in table 6:

| Global results | | | | |
|---|---|---|---|---|
| **Criteria\Methods** | ***BT method*** | ***TK method*** | ***DP method*** | ***DP-EOQ method*** |
| **Average Costs** | | | | |
| Ordering cost | 1111.11 | 1236.11 | 1292.32 | 1207.77 |
| Procurement cost | 7413.33 | 7292.73 | 7277.72 | 7301.01 |
| Holding cost | 1186.36 | 979.34 | 1136.59 | 1108.53 |
| Total cost | 9710.81 | 9508.18 | 9706.64 | 9617.31 |
| **Service Level** | | | | |
| Cumulative backlog quantity | 29.80 | 16.08 | 1.16 | 12.83 |
| Number of stockouts | 1.02 | 0.95 | 0.16 | 0.79 |

*Table 6: Average results of the 72 configurations for each method.*

To perform our sensitivity analysis, we fix one parameter and look at the average total cost obtained by fixing this parameter. The results are in table 7. Moreover, we also look at the average cumulative backlog quantity during the planning horizon and we summarize the information in table 8.

| Sensitivity analysis: average total cost | | | | |
|---|---|---|---|---|
| | ***BT method*** | ***TK method*** | ***DP method*** | ***DP-EOQ method*** |
| **Demand patterns** | | | | |
| Const | 9809.72 | 9589.93 | 9758.58 | 9589.93 |
| Rand | 9716.54 | 9536.65 | 9762.54 | 9712.09 |
| Lcy | 9688.05 | 9486.06 | 9684.70 | 9639.50 |
| Hcy | 9628.99 | 9429.79 | 9654.71 | 9569.75 |
| Lsin | 9796.51 | 9578.76 | 9754.06 | 9595.28 |
| Hsin | 9625.05 | 9427.91 | 9625.23 | 9597.32 |
| **Service levels** | | | | |
| 95% | 9848.74 | 9580.37 | 9735.19 | 9684.80 |
| 90% | 9716.72 | 9512.16 | 9709.34 | 9620.61 |
| 80% | 9566.97 | 9432.01 | 9675.38 | 9546.52 |
| **Forecast errors** | | | | |
| 0% | 9706.48 | 9507.81 | 9682.65 | 9611.09 |
| 5% | 9711.41 | 9502.31 | 9685.67 | 9607.27 |
| 10% | 9705.66 | 9510.17 | 9711.53 | 9620.35 |
| 20% | 9719.69 | 9512.43 | 9746.69 | 9630.53 |

*Table 7: Average total cost for different parameters.*

For example, according to our results, the average total cost of the *BT method* under 90% service level is equal to 9716.72. It corresponds to the average result of the 24 configurations computed with the *BT method* and a service level equal to 90%.

| Sensitivity analysis: average cumulative backlog quantity | | | | |
|---|---|---|---|---|
| | ***BT method*** | ***TK method*** | ***DP method*** | ***DP-EOQ method*** |
| **Demand patterns** | | | | |
| Const | 19.83 | 11.62 | 1.07 | 11.62 |
| Rand | 24.25 | 16.57 | 0.85 | 13.41 |
| Lcy | 26.56 | 13.71 | 0.73 | 9.20 |
| Hcy | 34.69 | 16.12 | 0.69 | 9.93 |
| Lsin | 21.90 | 12.17 | 1.21 | 10.28 |
| Hsin | 22.76 | 12.66 | 1.39 | 13.40 |
| **Service levels** | | | | |
| 95% | 13.78 | 7.72 | 0.51 | 6.41 |
| 90% | 22.89 | 12.82 | 0.86 | 10.64 |
| 80% | 39.97 | 22.19 | 1.64 | 17.54 |
| **Forecast errors** | | | | |
| 0% | 10.02 | 5.99 | 0.39 | 5.04 |
| 5% | 12.95 | 7.76 | 0.53 | 6.58 |
| 10% | 23.74 | 13.61 | 0.99 | 11.70 |
| 20% | 68.73 | 35.60 | 2.58 | 27.55 |

*Table 8: Average cumulative backlog quantity over the planning horizon for different parameters.*

For instance, according to our results, the average cumulative backlog quantity (i.e. the demand not directly satisfied over the planning horizon), with the *TK method* and no forecast error (i.e. $\rho = 0\%$), is 5.99. It corresponds to the average result of the 18 configurations tested with these settings.

Firstly, we examine how the methods perform compared to each other and we observe that:

1. The *TK method* is the most efficient under all configurations tested.
2. The *DP method* performs poorly.
3. Surprisingly, the *DP-EOQ method* is the second top performing method.
4. Under constant demand patterns, the *DP-EOQ method* gives the same results than the *TK method*.

It is important to notice that we cannot conclude that under $\alpha$-service level constraint and backlog policy, the $(R_t, S_t)$ strategy performs better than the $(s_t, S_t)$ strategy because it also depends of the algorithmic approach used. We can only conclude that the state of the art MIP method of Tarim and Kingsman (2004) (i.e. the *TK method*) is more efficient than our *DP method* for the $(s_t, S_t)$ strategy. As a reminder, to the best of our knowledge, there is no proof that the $(s_t, S_t)$ strategy is optimal under $\alpha$-service level constraint and backlog policy. Then, it may be possible than even the best formulation for the $(s_t, S_t)$ strategy cannot outperform a strong formulation for the $(R_t, S_t)$ strategy.

Undoubtedly, the poor performance of the *DP method* comes at least partially from the algorithmic approach. As explained section 5.1.2, the method finds min/max levels with which an order is often placed because the inventory position is slightly below $s_t$. It means that inventory has been carried unnecessarily. The source of the problem is the recursive computation of $S_t$ that assumes a deterministic demand. As demands are given by random variables, ignoring the deviations from the expected values are costly. However, the poor score of the *DP method* can be balanced by looking at table 8. We see that the average cumulative backlog quantity is much lower with the *DP method* than with other methods. The reason behind that has already been discussed earlier. Also, table 7 illustrates that reducing the required service level reduces the costs. Then, one way to improve the performance of the *DP method* would be by fixing a more appropriate (i.e. lower) service level in the algorithmic approach, knowing that the observed service level will be higher than the constrained service level in the algorithmic approach.

The effectiveness of the *DP-EOQ method* is not intuitive because the $(r, s_t, S_t)$ strategy is a kind of constrained $(s_t, S_t)$ strategy and both methods use the same dynamic programming approach for the determination of the min/max. In fact, the *DP-EOQ method* is more efficient because $r$ is determined in such a way that a replenishment will probably be required every review period. Consequently, we observe often that $s_t = S_t$ and that a replenishment will occur almost each time the inventory position is reviewed. It means that we order only if the inventory level does not allow to cover the demand during the review interval. And, if we make a replenishment, we order just enough to cover the demand until the next review. Therefore, the inventory position before making a replenishment is closer to zero, which means that less inventory is carried unnecessarily. With the cost configuration used in the sensitivity analysis, we find $r = 6$. Then, the inventory position is reviewed five times over the planning horizon. For all configurations tested, we observe that $s_t = S_t$, except for the penultimate review periods with cycling demand patterns. In these cases, $S_t \geq s_t$ because there is an incentive to make only one replenishment due to the low demand at the end of the planning horizon.

When the demand pattern is constant, we observe that the *DP-EOQ method* gives the same results than the *TK method.* This is because the review interval, $r$, found with the EOQ approach gives the optimal replenishment planning obtained with the *TK method*. It happens because the stationarity assumption of the EOQ model is respected. Moreover, the *DP-EOQ* *method* gives output values such that $s_t = S_t$. Then, the $(r, s_t, S_t)$ strategy of the *DP-EOQ method* is equivalent to the $(R_t, S_t)$ strategy of the *TK method.*

Next, we examine how the methods perform under the different settings and we observe that:

1. All methods manage efficiently demand seasonality.
2. The *BT method* is particularly sensitive to a service level reduction.
3. All methods are robust to forecast errors from a cost point of view but are significantly affected from a service level perspective.

All methods manage efficiently demand seasonality from a cost point of view. This is also true for the *DP-EOQ method,* even though the review interval, $r$, is computed thanks to an EOQ approach that supposes no seasonality. Furthermore, even the distribution of the total cost does not change significantly with the different demand patterns as illustrated in figure 6. From a service level point of view, seasonality is also well managed by all the methods, but it should be noticed that an increase of the seasonality increases the average cumulative backlog quantity.

Naturally, decreasing the required service level reduces the average total cost and increases the average cumulative backlog quantity. As illustrated in figure 7, the distribution of the total cost around the mean is roughly the same for all service levels considered. Regarding the costs, the *BT method* benefits the most from a service level reduction. But, the average cumulative backlog quantity also increases significantly. This result was predictable. In the $(R_t, Q_t)$ strategy, both the replenishment periods and the order quantities must be determined at the beginning of the planning horizon. This strategy has no freedom to react to demand realization. So, if the service level constraint is reduced, it is highly beneficial.

When forecast errors are introduced, the average cumulative backlog quantities generally explode, except for the *DP method*. By contrast, the *DP method* is the only method significantly impacted from a cost point a view. This is because the higher responsiveness of the $(s_t, S_t)$ strategy allows to react efficiently to demand realization. These deviations from the 'usual' planning are costly, but it allows to keep the average cumulative backlog quantity low. The other methods fail to react to the unexpected demand realization. Then, there is no deviation from the 'usual' planning. It leads to almost unchanged costs and a significant increase of the average cumulative backlog quantity. And, while the average total cost stays roughly the same, the variability around the mean increases significantly with the forecast error, as illustrated in figure 8.

Finally, we observe that all the methods are quite tractable[16]. The *BT method* solves each instance in approximately 0.5 second while the *TK method* requires about 5 seconds. These are acceptable results knowing that these methods give the optimal solution for their corresponding strategies. Comparatively, the *DP method* and the *DP-EOQ method* take about 1 second to solve an instance. However, these methods are only heuristics and do not provide the optimal solution.

[16] We use the software Aimms with IBM ILOG CPLEX 12.6.3 to solve the MIP models of the *BT method* and the *TK method*. For the *DP method* and the DP-EOQ method, the recursive algorithms are written in Scala. Moreover, the computer is equipped with a 2.0Ghz AMD A8-6410 and 12GB of RAM.

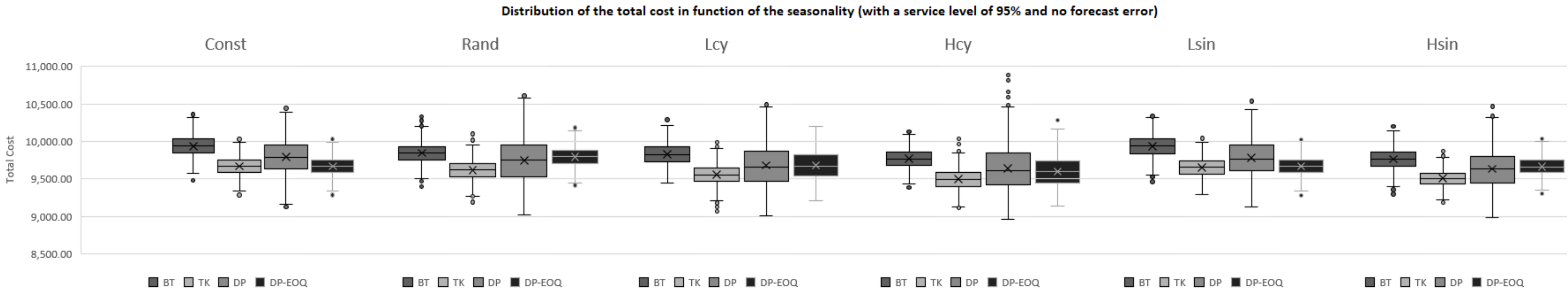


*Figure 6: Distribution of the total cost in function of the seasonality with $\alpha = 0.95$ and $\rho = 0.0$.*

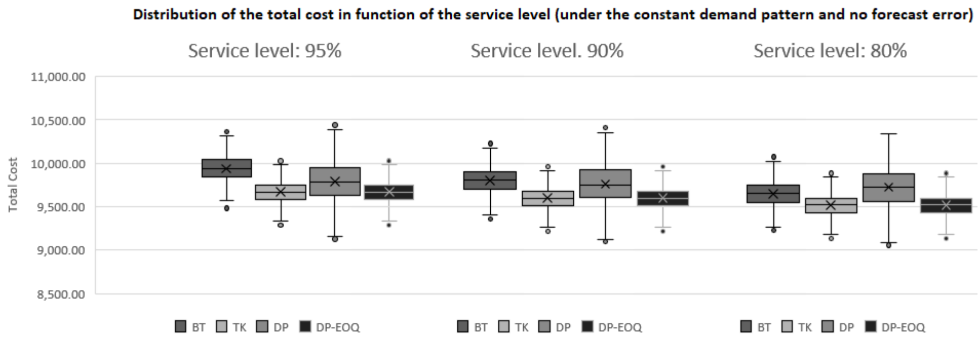


*Figure 7: Distribution of the total cost in function of the service level under constant demand pattern and with $\rho = 0.0$.*

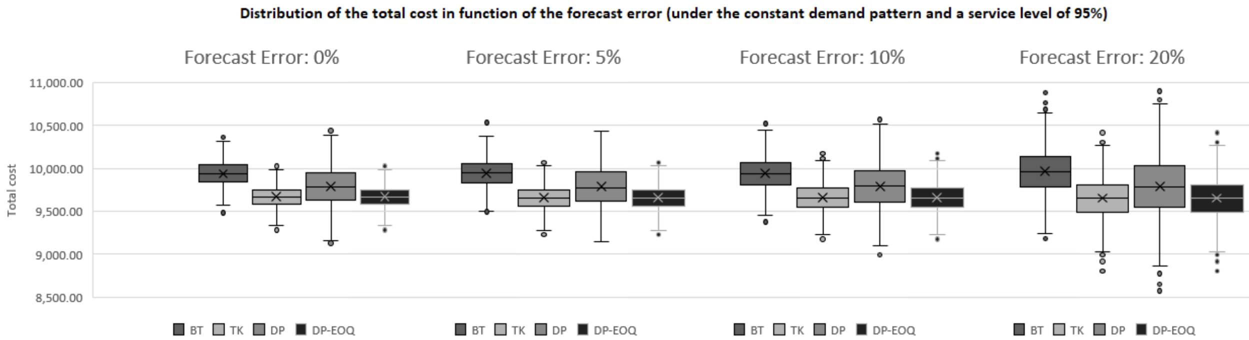


*Figure 8: Distribution of the total cost in function of the forecast error under constant demand pattern and with $\alpha = 0.95$.*

## 5.3 Key findings

The goal of this master thesis is to discuss algorithmic methods to solve inventory problems. The methods should be applicable to real-life problems, scalable and robust to demand uncertainty. Previously, we have discussed some methods to solve inventory problems under periodic review systems. Two methods come from the scientific literature and we proposed two other heuristics based on dynamic programming. In this part, we have illustrated the methods on a common test bed and we have assessed their performances under different settings.

First, we observe that all methods have reasonable running times. In addition, it seems that they are all quite robust to demand uncertainty in the sense that the expected costs computed by the methods are close to the real expected costs observed by simulation. However, we observe that the *DP method* is less efficient under demand uncertainty than expected. The problem is that the expected costs are estimated by the costs occurring if the demand is deterministic. This issue could theoretically be solved by computing the real expected costs, but this task would probably make the method intractable in a reasonable time. By contrast, the *DP method* is the most effective method in terms of service levels. This is why we believe that the *DP method* is appropriate for items needing a high service level.

Moreover, we observe that the *TK method*, which provides optimal solutions for the $(R_t, S_t)$ strategy, is the most efficient under all settings tested. We also notice that the *DP-EOQ method* is the second top performing method and it can reach the same performances as the *TK method* under constant demand patterns.

Furthermore, we notice that all methods efficiently tackle seasonality. We also observe that the *BT method* is the most sensitive to a service level reduction. And finally, we see that introducing forecast errors mainly affects the service level performances and lets the costs roughly unchanged (except for the *DP method*).

To sum up, we believe that the *TK method* is the most promising method, because it is the most cost-effective option while satisfying the service level constraint. Besides, this method is tractable although it is the method with the highest running time.

# Part 6: Extensions

## 6.1 Methodology

The final goal of this report is to provide insights about some methods to solve concrete inventory management problems. Then, to assess the scalability of the methods to real-life settings, we confront them to two case studies with new constraints inspired from real life situations. To undertake these case studies, we interviewed two SMEs' managers[17] (i.e. small and medium-sized enterprises). In this part, we will simply describe the respective inventory management problems inspired from the two companies. Concretely, we use these cases to select the extensions that we want to implement in the methods discussed before. Later, the extended methods are assessed by simulation to measure how they can address these real-life constraints.

## 6.2 Case study 1: Single-item replenishment problem with constraints

A manufacturer, working under a Make-To-Order policy, wants to establish his weekly replenishment planning of raw materials. The uncertainty about the raw material requirements comes from the last-minute orders. The company want to know the optimal stock level of each raw material based on the forecasted needs.

The optimal stock level should minimize the following costs:

- *Procurement cost:* Variable cost associated with the procurement of the raw material.
- *Ordering cost:* Fixed cost occurring each time an order is placed.
- *Holding cost:* Variable cost related to storing items from one period (e.g. a week) to the next.

Moreover, the solution should ideally consider that:

- Raw materials generally have a 'MOQ' (Minimum Order Quantity).
- Raw materials generally have a maximum availability (maximum order quantity).
- Storage capacity is limited (maximum level of inventory).
- Deliveries generally take some time (non-zero leadtime).
- A maximum probability of shortage is tolerated (service level constraint).

The goal is to find the replenishment planning that would minimize the costs while satisfying the raw material needs and the constraints.

[17] We do not mention the name of the companies and managers to ensure anonymity.

## 6.3 Case study 2: Multi-item joint replenishment problem

A Belgian retailer offers a wide range of products from many suppliers. Consolidation of orders intended to the same supplier is essential because there is a fix delivery cost, no matter the number of SKUs (Stock Keeping Units) ordered. The goal is to establish a replenishment planning based on the forecasted sales.

The goal is to minimize the sum of the following costs while satisfying the demand (we suppose an $\alpha$-service level constraint):

- *Procurement cost:* Variable cost associated to the procurement of the items.
- *Delivery cost:* Fixed cost for the delivery from the supplier to the retailer (independent of the number of SKUs).
- *Ordering cost:* Fixed cost that occurs each time a different item is ordered.
- *Holding cost:* Variable cost related to storing items from one period to the next.

To sum up, the problem consists in determining the multi-item replenishment planning to minimize the total cost and satisfy the demand.

# Part 7: Case study 1 – Extended methods

In the previous part, we introduced a single-item replenishment problem with specific constraints. This part gives a mathematical problem statement of this case study. Next, to solve this problem, we suggest extensions for the methods discussed in part 4. Finally, we assess the effectiveness of the extended methods by conducting experiments.

## 7.1 Mathematical problem statement

We consider a planning horizon of $T$ periods denoted by $t \in \{1, \dots, T\}$. The probability to have enough inventory of the raw material should be at least greater than $\alpha$. If there is not enough inventory, the excess requirements are backlogged to the next period. Each time a replenishment order is sent, an order cost, $a$, occurs whatever the size of the order. A linear holding cost, $h$, occurs for each unit stored from one period to the next. Finally, the unit procurement cost, $c$, should also be considered. The goal is to minimize the total cost incurred over the planning horizon. For convenience, random variables are always followed by brackets. For instance, $d_t(\,)$ denotes the requirement of the raw material in period $t$.

The following notations are used:

- $d_t$: Expected demand in period $t$.
- $a$: Ordering cost.
- $h$: Holding cost.
- $c$: Procurement cost.
- $LT$: Leadtime duration.
- $CapRM$: Maximum order quantity.
- $CapWH$: Capacity available for the raw material in the warehouse.
- $MOQ$: Minimum order quantity.

The solution found should ideally respect the following constraints:

- The MOQ constraint: The raw material has a minimum order quantity.
- The CapRM constraint: The raw material has a maximum availability.
- The CapWH constraint: The storage capacity is limited.
- The service level constraint: A maximum probability of shortage is tolerated.
- The leadtime constraint: There is a non-zero leadtime for the replenishments (we assume that the leadtime is an integer multiple of the period length).

To make the problem feasible, we allow a replenishment in the first period without a leadtime to start the planning horizon with an initial stock. In the next four sections, we explain how the methods can be extended to solve this problem.

## 7.2 Extended methods

### 7.2.1 Extended *BT method*: $(\boldsymbol{R_t}, \boldsymbol{Q_t})$ strategy and MIP approach

The *BT method* can be extended effortlessly. We just have to introduce new constraints to fit with the new problem requirements. Again, the MIP can be solved with a traditional solver software. We propose the following MIP with the same variables as before:

- $I_t$: Expected ending inventory in period $t$. $I_0$ is the initial inventory position.
- $R_t$: Variable equal to 1 if the raw material is received in period $t$. Else, equal to 0.
- $Q_t$: Quantity of the raw material received in period $t$.

**MIP formulation:**

$$Min \sum_{t=1}^{T} [\, (R_t \times a) + (I_t \times h) + (Q_t \times c)\, ]$$

Subject to:

| | | |
|---|---|---|
| Flow constraint: | $I_t = I_0 + \sum_{k=1}^{t} \{\, Q_k - d_k \}$ | $\forall t$ |
| Setup constraint: | $Q_t \leq M \times R_t$ | $\forall t$ |
| Service level constraint: | $\sum_{k=1}^{k=t} Q_k \geq G^{-1}_{d_1()+\cdots+d_t()}(\alpha) - I_0$ | $\forall t$ |
| CapRM constraint: | $Q_t \leq CapRM$ | $\forall t$ |
| CapWH constraint: | $I_{t-1} + Q_t \leq CapWH$ | $\forall t$ |
| MOQ constraint: | $Q_t \geq MOQ \times R_t$ | $\forall t$ |
| Leadtime constraint: | $R_t = 0$ | $\forall 1 < t \leq LT$ |

**Comments**

1. There are little changes with the leadtime constraint because, under the $(R_t, Q_t)$ strategy, we commit about replenishment periods and order quantities. We simply exclude other replenishments than the first one during the beginning of the planning horizon (i.e. the first $LT$ periods) because it is impossible due to the leadtime.

2. Because we commit about the order quantities, constraints on these quantities are easy to implement. Here, the CapRM constraint ensures that the order quantity respects the maximum capacity of the supplier. Moreover, the MOQ constraint forces the order quantity to be at least equal to the minimum order size of the supplier.
3. In the MIP formulation above, the efficiency of the CapWH constraint can be debated. Currently, it simply forces the expected opening inventory to be lower than the storage capacity. The expected opening inventory in period $t$ is the expected ending inventory in period $t-1$ plus the eventual quantity delivered in period $t$. However, the effective inventory level could be higher/lower than expected because it is a random variable linked to the demand realization.
   - To better ensure that the expected opening inventory do not exceed the storage capacity, it is possible to force this expected opening inventory to be lower than a given percentage of the storage capacity:
   $$I_{t-1} + Q_t \leq CapWH \times \beta$$
   - Another option is to use a chance constraint method[18] to enforce that the constraint is respected with a probability equal or greater than $\beta$. The chance constraint can be formulated in the MIP after few transformations:
   $$Proba(I_{t-1}() + Q_t \leq CapWH) \geq \beta$$
   $$Proba\left(I_0 + \sum_{k=1}^{t-1}\{Q_k - d_k()\} + Q_t \leq CapWH\right) \geq \beta$$
   $$Proba\left(\sum_{k=1}^{t} Q_k \leq CapWH - I_0 + \sum_{k=1}^{t-1} d_k()\right) \geq \beta$$
   $$\sum_{k=1}^{t} Q_k \leq G'^{-1}_{\sum_{k=1}^{t-1} d_k()}(\beta) - I_0 + CapWH$$
   Where $G'^{-1}_{d_t()}(\beta)$ is the right-tailed inverse cumulative distribution function such that $G'^{-1}_{d_t()}(\beta) = u$ means that $Proba(d_t() \geq u) = \beta$. And, $G'^{-1}_{d_t()}(\beta)$ can be computed with a spreadsheet software for all $t$ before solving the

[18] A chance constraint takes the form $proba\{C\} \geq \beta$, where $C$ is a relationship including random and decision variables. And $proba\{C\}$ is the probability that the constraint is respected (Hnich, Zghidi, & Rebai, 2018)

problem[19]. It is easy to see that if $\beta = 0.5$, the chance constraint is similar to the initial constraint suggested.

we assume that imposing the expected opening inventory to be below the capacity is sufficient because we do not know what would be the appropriate $\beta$ coefficient for this problem.

### 7.2.2 Extended *TK method*: $(\boldsymbol{R_t}, \boldsymbol{S_t})$ strategy and MIP approach

The *TK method* can be extended to consider the leadtime and the specific constraints. The problem is still formulated as a MIP and solved using a traditional solver.

We consider the same variables as before:

$R_t \in \{0,1\}$: Variable equal to 1 if a replenishment occurs in period $t$. Else, equal to 0.

$P_{tj} \in \{0,1\}$: Variable equal to 1 if the most recent replenishment prior to period $t$ was in period $t-j+1$ (i.e. if the last replenishment was $j-1$ periods ago). Else, equal to 0.

$I_t \geq 0$: Expected ending inventory in period $t$.

$S_t \geq 0$: Expected opening inventory in period $t$ taking into account the eventual replenishment at the beginning of the period.

Without the leadtime, the replenishment periods and the order-up-to levels are directly derived from the variables $R_t$ and $S_t$ respectively. This is still the case when introducing a leadtime, but it is important to notice that the replenishment decision must be taken $LT$ periods before the actual replenishment. Also, the order quantity in period $t - LT$ is the quantity that brings back the expected opening inventory in period $t$ to $S_t$ (considering the demand during the leadtime and the ongoing replenishments). This is because $S_t$ denotes the expected opening inventory in period $t$ considering the eventual replenishment. When the leadtime is zero and only in this case, $S_t$ also corresponds to the optimal order-up-to level.

**MIP formulation**:

$$Min\ TC\ with\ TC = \sum_{t=1}^{T} [\ aR_t + hI_t + c(S_t - I_{t-1})]$$

[19] $G'^{-1}_{d_t()}(\beta) = G^{-1}{}_{d_t()}(1-\beta)$, if the demand is normally distributed because the Normal distribution is symmetric.

Subject to:

| | | |
|---|---|---|
| Flow constraint: | $I_t = S_t - d_t$ | $\forall t$ |
| No return constraint: | $S_t \geq I_{t-1}$ | $\forall t$ |
| Setup constraint: | $S_t - I_{t-1} \leq M \times R_t$ | $\forall t$ |
| Service level constraint: | $I_t \geq \sum_{j=1}^{t} (G^{-1}_{d_{t-j-LT+1}()+\cdots+d_t()}(\alpha) - \sum_{k=t-j-LT+1}^{t} d_k) \times P_{tj}$ | $\forall t$ |
| Replenishment cycle constraint: | $\sum_{j=1}^{t} P_{tj} = 1$ | $\forall t$ |
| Replenishment cycle constraint: | $P_{tj} \geq R_{t-j+1} - \sum_{k=t-j+2}^{t} R_k$ | $\forall t$ $\forall j \leq t$ |
| CapRM constraint: | $S_t - I_{t-1} \leq CapRM$ | $\forall t$ |
| CapWH constraint: | $S_t \leq CapWH$ | $\forall t$ |
| MOQ constraint: | $S_t - I_{t-1} \geq MOQ \times R_t$ | $\forall t$ |
| Leadtime constraint: | $R_t = 0$ | $\forall t \in \{2, \dots, LT\}$ |

## Comments

1. In the original paper from Tarim and Kingsman (2004), the authors mentioned that their model can be extended to non-zero leadtime but they did not provide further explanation. Later, Tarim and Smith (2008) revealed that the extention for the leadtime is simply obtained by replacing the initial service level constraint by:

   $$I_t \geq \sum_{j=1}^{t} (G^{-1}_{d_{t-j-LT+1}()+d_{t-j-LT+2}()+\cdots+d_t()}(\alpha) - \sum_{k=t-j-LT+1}^{t} d_k) \times P_{tj} \qquad \forall t$$

   However, the authors did not provide any justification. Then, we propose a demonstration that justifies the use of this constraint in appendix 5. Moreover, we exclude other replenishments than the first one during the beginning of the planning horizon (i.e. the first $LT$ periods) because it is impossible due to the leadtime.
2. As before, the CapWH constraint simply forces the expected opening inventory (considering the eventual replenishment) to be lower than the storage capacity. Again, it is possible to use a chance constraint method to enforce that the constraint is

respected with a probability equal or greater than $\beta$. The chance constraint can be formulated in the MIP after few transformations:

$$proba\{S_t(\,) \leq CapWH\} \geq \beta \quad \forall t$$

[...] (see appendix 6 for the demonstration)

$$S_t \leq G'^{-1}_{d_{t-LT}+\cdots+d_{t-1}}(\beta) - \sum_{k=t-LT}^{t-1} d_k + CapWH \quad \forall t \; such\; that\; R_t = 1$$

Where $G'^{-1}_{d_t(\,)}(\beta)$ is the right-tailed inverse cumulative distribution function such that $G'^{-1}_{d_t(\,)}(\beta) = u$ means that $Proba(d_t(\,) \geq u) = \beta$. Again, $G'^{-1}_{d_t(\,)}(\beta)$ can be computed with a spreadsheet software for all $t$ before solving the problem. As before, we assume that imposing the expected opening inventory to be below the capacity is sufficient because we do not know what would be the appropriate $\beta$ coefficient for this problem.

3. Under the $(R_t, S_t)$ strategy, the order quantities are also random variables. In the proposed MIP, the CapRM constraint ensures that the excepted order quantity is lower than the maximum capacity. Meanwhile, the MOQ constraint ensures that the excepted order quantity is greater than the minimum order quantity. Unfortunately, this time we did not succeed to find a chance constraint method to express the constraint alternatively[20].

### 7.2.3 Extended *DP method*: $(\boldsymbol{s_t}, \boldsymbol{S_t})$ strategy and DP approach

In this section, we propose an extension for the *DP method*. $s_t$ and $S_t$ still denote a reorder point and an order-up-to level respectively. It means that, at the beginning of period $t$, if the inventory position is lower than $s_t$, an order is placed to bring back the stock to $S_t$. In other words, if the stock is below $s_t$ at the beginning of period $t$, an order of a size equal to $S_t - inventory\ position$ is placed.

It is not straightforward to extend the initial dynamic programming approach to fit with the new problem requirements.

First, the dynamic programming approach cannot be easily adapted to consider multi-period leadtime. If the cost structure is such that there is no interest for early ordering, we can still compute $s_t$ by imposing: $Proba(s_t \geq d_t + d_{leadtime}) = \alpha$. We then order when the total

[20] We can still increase the probability that the constraints are respected by imposing:

$S_t - I_{t-1} \leq CapRM \times \beta \quad or \quad S_t - I_{t-1} \geq MOQ \times (1 + \beta) \quad where\ 0 < \beta \leq 1$

But it is not accurate.

inventory position (i.e. considering the ongoing replenishments) is lower than $s_t$. However, with multi-period leadtime, we face circularity issues to find $S_t$. Indeed, starting from the end of the planning horizon, the $S_t$ are computed recursively by evaluating the cost function for a given order-up-to level. The problem is that the cost function cannot be computed without knowing the ongoing replenishments not yet delivered. Unfortunately, these values depend on some $S_{k<t}$ that cannot be computed without knowing $S_t$. To overcome this issue, we propose to fix a review interval equal to the leadtime duration. If the leadtime is equal to a period length, we still have an $(s_t, S_t)$ strategy where the inventory position is checked every period. By contrast, if the leadtime is equal to several period lengths, we have a kind of $(r = LT, s_t, S_t)$ strategy because we review the inventory position only every $LT$ periods. By imposing the review interval to be equal to the delivery time, we can compute $s_t$ such that $Proba(s_t \geq d_t + d_{t+1}) = \alpha$ for all $t \in \{1,2,\dots,T-1\}$ (where $d_t$ denotes the demand during the review interval $t$). Because the review interval is equal to the leadtime, there is no ongoing replenishment. Then, we can recursively compute $S_t$ by evaluating the cost function. We just have to slightly adapt the cost function[21] $J(t,x)$ to reflect that the order quantities are not instantaneously available.

$for\ all\ t \in \{t, \dots, T\}$,

$$J(t,x) = \begin{cases} h \times (x - d_t) + a + c(S_t - x) + J(t+1, S_t - d_t) & if\ \ x < s_t \\ h \times (x - d_t) + J(t+1, x - d_t) & if\ \ x \geq s_t \end{cases}$$

with $J(T+1, x) = 0$.

Next, regarding the constraints, the *DP method* offers less flexibility than the *TK method* or the *BT method*. This is because the $(s_t, S_t)$ strategy does not allow to consider explicitly capacity constraints on the order quantities due to the uncertain replenishment timing and the variability of the order quantities (Tempelmeier, 2013). Our solution is to constrain the order quantities by imposing an upper bound on $S_t$. Since the $s_t$ are determined to ensure a stockout probability of $1 - \alpha$, we are sure that the inventory position is greater than zero with a probability of at least $\alpha$. Then, by imposing $S_t \leq CapRM$, we ensure that the order quantity is lower than *CapRM* with a probability of at least $\alpha$. Unfortunately, this constraint might be

[21] $J(t,x)$ is the cost from period $t$ to $T$ if the inventory position at the beginning of period $t$ is $x$ and if the demand is deterministic.

unnecessarily restrictive because the inventory position is often slightly below $s_t$ and significantly above zero.

Similarly, our solution to implement the MOQ constraint is to impose a lower bound on $S_t$. If we specify that $S_t$ must be greater than $s_t + MOQ$ for all $t$, we are sure to respect the MOQ constraint in all cases. Again, this constraint might be too restrictive because the inventory position is often significantly below $s_t$.

Finally, it is easier to implement the capacity constraint on the inventory on-hand. A possible solution is to enforce $S_t \leq CapWH$. But it is unnecessarily restrictive if the leadtime is long because the inventory position will not reach $S_t$ due to the demand during the leadtime. A more accurate solution is to impose: $S_t \leq CapWH + G'^{-1}_{d_{TL}()}(\beta)$, where $G'^{-1}_{d_t()}(\beta)$ is the right-tailed inverse cumulative distribution function such that $G'^{-1}_{d_t()}(\beta) = u$ means that $Proba(d_t() \geq u) = \beta$. In this way, we enforce the CapWH constraint for each period with a probability of $\beta$. To be consistent with the other methods, we assume that imposing the expected inventory to be below the capacity is sufficient (i.e. $\beta = 0.5$). Then, the CapWH constraint is: $S_t \leq CapWH + d_{LT}$.

Ultimately, we consider the problem as infeasible with the extended *DP method* if there is no $S_t$ that respects all three constraints.

**Algorithmic approach**

Step 1: - if $(LT > 1)$

Fix the review interval to $LT$.

Aggregate the forecasted demands during the review intervals to obtain the planning horizon corresponding to a $(r = LT, s_t, S_t)$ strategy (see figure 2, p.26).

- Else, go to step 2

Step 2: - Compute $s_t$ for all $t$ such that $Proba(s_t \geq d_t + d_{t+1}) = \alpha$

Step 3: - Set $t = T$

Step 4: - if $(t \geq 1)$

$lowerBound = MOQ + s_t$

$upperBound = \min(CapRM, CapWH + d_t)$

$BSF$[22] $= J(t+1, upperBound - d_t) + C \times (upperBound - s_t)$

$indexBSF$[23] $= upperBound$

Go to step 5

- else, stop. All $(s_t, S_t)$ has been found!

Step 5: - Set $i = lowerBound$ and go to step 6

Step 6: - $current = J(t+1, i - d_t) + C \times (i - s_t)$

if ($current \leq BSF$)

$BSF = current$

$IndexBSF = i$

if ($i \geq upperBound$)

$S_t = IndexBSF$

Go to step 4 with $t = t - 1$

Increment $i$: $i = i + 1$ and repeat step 6

### 7.2.4 Extended *DP-EOQ method*: $(\boldsymbol{r}, \boldsymbol{s_t}, \boldsymbol{S_t})$ Strategy and DP + EOQ approach

The algorithmic approach used for the *DP-EOQ method* is heavily based on the same dynamic programming formulation than the *DP method*. Therefore, the method shares the same limitations. Concretely, the constraints are implemented in exactly the way as with the *DP method*. Actually, only the way to tackle the leadtime constraint differs slightly from the *DP method*.

Under the $(r, s_t, S_t)$ strategy, $s_t$ and $S_t$ are computed only for the periods reviewing the inventory position (i.e. every $r$ periods). Then, the approach can be extended to consider multi-period leadtime not bigger than the review interval. As before, a good review interval is determined by using the EOQ approach.

- If the leadtime is longer than the review interval found, we set the review interval at the length of leadtime. We get a $(r = LT, s_t, S_t)$ strategy and we apply the extended *DP method* described in the previous section.
- If the review interval found is longer than the leadtime, we can extend the *DP-EOQ method.* This is possible because there is no ongoing replenishment when reviewing the inventory position. However, the leadtime may represent a large part of the review

[22] $BSF$ is a variable that stores the Best Solution Found (i.e. the minimal cost) so far for the current $t$.

[23] $IndexBSF$ is a variable that stores the index (i.e. the order-up-to level) of the Best Solution Found.

interval. Then, it is not accurate to penalize the excess stock at the end of the review interval because it was not carried during the full review interval. Consequently, we adapt the cost function $J(t,x)$ to make a distinction between the holding cost prior and after the replenishment.

$For\ all\ t \in \{1,\dots,T\},$

$$J(t,x) = \begin{cases} h \times \left( (LT \times x) + \left((r - LT) \times S_t\right) - \frac{r + \cdots + 1}{r} d_t \right) + a + c(S_t - x) + J(t+1, S_t - d_t) & if\ x < s_t \\ h \times \left( (x - \frac{1}{r} d_t) + (x - \frac{2}{r} d_t) + \cdots + (x - \frac{r}{r} d_t) \right) + J(t+1, x - d_t) & if\ x \geq s_t \end{cases}$$

with $J(T+1,x) = 0$.

The constraints are implemented in exactly the same way as in the extended *DP method* described above. The idea is to bound $S_t$:

The MOQ constraint: $S_t \geq s_t + MOQ$

The CapRM constraint: $S_t \leq CapRM$

The CapWH constraint: $S_t \leq CapWH + d_{LT}$

If there is no $S_t$ that respects all three constraints for some $t$, it means that the review interval is not appropriate. We then gradually decrease the review interval until either we find a feasible solution, or we reach a review interval equal to the leadtime. If there is no feasible solution with the review interval equal to the leadtime, we consider that the problem is infeasible with the *DP-EOQ method*.

**DP-EOQ algorithmic approach**

Step 1: Find a good review interval by using an EOQ approach (see appendix 2)

$$r = \max\left(\frac{EOQ^*}{Total\ demand} \times horizon\ length\ , LT\right)$$

Step 2: Aggregate the forecasted demands between all the review intervals.

Step 3: Find $s_t\ and\ S_t$ for all $t$ with the same algorithm than for the extended *DP method* but with the new cost function (i.e. $J(t,x)$) given in this section.

## 7.3 Simulation

### 7.3.1 Methodology

In this section, we aim at assessing the performance of the extended methods described above. To reach this goal, we carry out another sensibility analysis. We want to know how the extended methods tackle the introduction of a leadtime constraint, a minimum/maximum order quantity constraint and a maximum inventory constraint.

We decide to use the same problem structure as before with the three costs unchanged (i.e. $c = 4, h = 0.2, a = 250$) and the time horizon of 30 periods. We consider the constant demand pattern from the previous sensibility analysis (see figure 5), an $\alpha$-service level criterion equal to 95% and no forecast error.

Regarding the sensitivity analysis, we decide to consider four intensity levels for each constraint:

$$LT \in \{2,4,6,8\}$$
$$MOQ \in \{0,50,100,150\}$$
$$CapRM \in \{350,400,500,\infty\}$$
$$CapWH \in \{300,450,600,\infty\}$$

We do not consider the 256 possible configurations because, with some of them, the problem is infeasible. Instead, we gradually tighten one constraint while leaving the others to their less binding value. As before, each experiment is composed of 1000 simulations based on randomly generated data.

As a reminder, with most methods, we cannot ensure that the constraints will always be respected. For instance, all methods ensure that the expected inventory is lower than the warehouse capacity denoted by *CapWH.* But, in practice, the real inventory position is a random variable that depends on the demand realization. Thus, the constraint might not always be respected. However, during the simulation process, we enforce that all constraints are respected. Indeed, if an order must be placed, the actual order size will be the maximum between the *MOQ* and the order quantity suggested by the method. The same methodology is used to enforce the *CapRM* constraint. Finally, regarding the *CapWH constraint,* we consider that any excess inventory over the warehouse capacity is discarded.

### 7.3.2 Results

We carry out all experiments and summarize the main results about the average total cost for each method in figure 9 (see appendix 7 for additional results about the total costs and the cumulative backlog quantities).

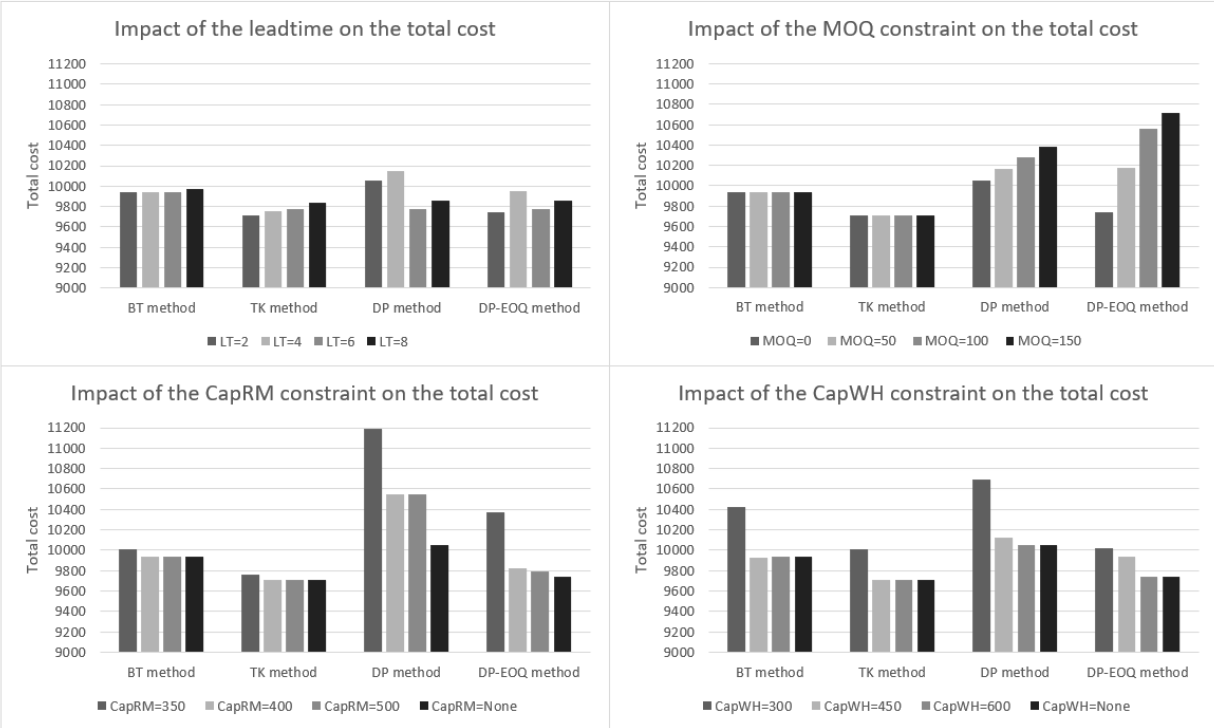


*Figure 9: Impact of the constraints on the average total cost.*

**Global overview**

As illustrated in figure 9, we observe that the *BT method* and the *TK method* manage the introduction of the constraints more efficiently than the *DP method* and *DP-EOQ method.* One reason is that the methods based on the dynamic programming formulation are less scalable because it uses the specific structure of the initial problem. Indeed, the problem is solved by recursion and $s_t/S_t$ must be determined without knowing the parameters of the previous periods. Then, it is difficult to add new constraints. By contrast, the MIP formulations are more scalable because new parameters and new constraints can be introduced without affecting the solving procedure. Another reason is the different degrees of freedom of the strategies. With the $(s_t, S_t)$ and the $(r, s_t, S_t)$ strategies, both the order quantities and the timing of orders are unknown. The additional uncertainty about timing of orders makes the methods harder to extend because information such as the expected inventory position is hardly predictable. This is because the inventory position depends on the timing of the last replenishment which is unknown.

From a service level point of view, we observe that the introduction of the new constraints is managed efficiently with all methods. Indeed, the backlog quantity stays under control under all configurations (see appendix 7). Regarding the *DP method* and the *DP-EOQ method*, this is because the constraints are implemented by bounding $S_t$ without affecting the computation of $s_t$ that ensures the service level constraint. Next, we review the effects of the introduction of each constraint individually.

**Leadtime constraint**

As expected, introducing a leadtime constraint has almost no effect on the *BT method* because the order quantities are determined at the beginning of the planning horizon. With this method, the total cost only slightly increases with long leadtimes because replenishments are not allowed during the first $LT$ periods (except for the first period). Then, it changes the optimal replenishment planning.

With the *TK method,* the total cost linearly increases with the leadtime length. This is because the leadtime duration affects the order quantities. Indeed, larger quantities are required to ensure the service level taking into account the additional uncertainty from the demand during the leadtime. In this way, it leads to higher procurement costs and holding costs.

With the extended *DP method,* the leadtime has an ambiguous effect. First, when there is a leadtime constraint, it requires more safety stocks, which increases the holding cost. However, our solution to tackle this leadtime constraint with the *DP method* is to review the inventory position less often. In some cases, it may be beneficial because it reduces the ordering cost nervousness.

The leadtime constraint has a curious impact on the *DP-EOQ method.* Overall, it increases the total cost because more safety stock is required to cover the uncertainty related to the demand during the leadtime. However, we also observe that a leadtime duration is particularly costly (i.e. when $LT = 4$, see figure 9). We believe that it comes from the algorithmic approach that estimates the costs by supposing a deterministic demand. Then, the method underestimates the possibility that a costly replenishment might be required at the end of the planning (i.e. in period 29).

**MOQ constraint**

The MOQ constraint is never binding with the *BT method*. The quantities are determined at the beginning of the planning horizon and the MOQ constraint is always respected.

With regards to the *TK method,* the MOQ constraint is also never binding. But this time, the model only enforces the expected order quantity to be lower than the $MOQ$. In practice, whatever the value of the $MOQ$, we never observe (over the 1000 simulations) that the constraint is not spontaneously respected.

For the *DP method*, the introduction of the MOQ constraint leads to a linear increase of the total cost. This is due to the algorithmic approach of the extended *DP method*. Indeed, the MOQ constraint is enforced by imposing $S_t - s_t \geq MOQ$. In this way, we ensure that the constraint is always respected. However, this formulation is very restrictive because the inventory position is often significantly lower than $s_t$. This means that the MOQ constraint would be respected without imposing a greater order-up-to level. Moreover, the constraint induces a negative end-of-horizon effect. For the last review, we should have $s_t = S_t$ because there is no interest to order more than the smallest quantity that respects the service level criterion. As stated, the MOQ constraint imposes to fix a higher order-up-to level for the last review period, which is costly. Then, the eventual order quantity is bigger, even if it is not needed for the service level and not required to respect the MOQ constraint.

Lastly, we observe that introducing the MOQ constraint leads to a significant increase of the total cost with the *DP-EOQ method*. This is because the extended *DP-EOQ method* also enforces the MOQ constraint by imposing $S_t - s_t \geq MOQ$. However, the review interval is determined in such a way that a replenishment will probably be required. Therefore, we often observe that $s_t = S_t$ (i.e. we order to have enough until the next review period). However, this is not possible anymore with the restrictive MOQ constraint. This leads to unnecessarily high order-up-to levels and a significant increase of the procurement costs and holding costs.

**CapRM constraint**

With the *BT method,* the CapRM constraint is only binding with the lowest value of $CapRM$ and it does not affect significantly the total cost.

The same observation holds for the *TK method*. However, the quantities are not determined at the beginning of the planning and the CapRM constraint enforces only the expected order quantity to be below $CapRM$. The method might then suggest an order quantity that does not respect the constraint. In the worst case, we observe that the *TK method* suggests infeasible order quantities for only 1.4% of the replenishments. This happens when $CapRM =$

$400$ but there is no significant consequence to order $CapRM$ instead of the infeasible quantity suggested by the *TK method.*

The extended *DP method* does not tackle efficiently the introduction of the CapRM constraint. Again, it comes from the algorithmic approach. To enforce the CapRM constraint, we impose: $S_t \leq CapRM$. In this way, the constraint is respected with a probability at least equal to the service level. However, this is unnecessarily restrictive because the inventory position is often greater than zero when ordering. This is why the total cost increases that much.

The extended *DP-EOQ method* does not manage efficiently the CapRM constraint. Firstly, with the initial review interval found with the EOQ approach, the problem is often infeasible. Then, we must gradually decrease the review interval length until the problem becomes feasible, but this solution increases the total cost. Secondly, the CapRM constraint is also imposed by $S_t \leq CapRM$ and, as before, it is often unnecessarily restrictive.

**CapWH constraint**

All methods enforce the CapWH constraint by imposing the expected inventory position to be lower than the warehouse capacity. For both the *BT method* and the *TK method,* the CapWH constraint is only binding at the most restrictive value of $CapWH$. With this setting, more replenishments must be planned which increases the total cost. Moreover, we observe that when the warehouse capacity is very limited, the expected inventory is close to the warehouse capacity, especially with the *BT method*. Therefore, it happens that some items must be discarded because there is no remaining space in the warehouse. More precisely, in the worst case (i.e. when the warehouse capacity is limited at 300 units), the storage limit is reached for 8.5% of the replenishments with the *BT method* and only for 0.2% of the replenishments with the *TK method.* We observe that the *DP method* and the *DP-EOQ method* also manage quite efficiently the CapWH constraint (except for very restrictive storage capacities). Moreover, in our experiment, we do not observe that the storage capacity is insufficient and that some items must be discarded.

## 7.4 Key findings

The goal of this part is to assess the scalability of the methods. We have done our best to extend them and we have assessed by simulation the effectiveness of the extended methods. We are concerned by the ability of the methods to be extended to new constraints because it shows how much they are scalable and how easily they would fit to different problems.

Firstly, we realize that the *BT method* and the *TK method* are easy to extend. This is because both the strategies and the algorithmic approaches are more convenient. With the $(R_t, Q_t)$ strategy of the *BT method*, both the timing of orders and the quantities are determined at the beginning of the planning horizon. Then, introducing leadtime constraints or capacity constraints on the order quantity is straightforward. It is also possible to add accurate chance constraints on the inventory position if needed. With the *TK method,* the timing of orders is known but not the order quantities. Consequently, it is harder to impose accurate constraints (i.e. chance constraints) on the order quantities. But we can easily constrain the expected order quantities. Moreover, it is still feasible to introduce leadtime constraints and precise chance constraints on the inventory position. In addition, for both the *BT method* and the *TK method,* the MIP formulation allows to introduce easily new parameters and express the new constraints without changing the solving procedure.

Secondly, we notice that the *DP method* and the *DP-EOQ method* are harder to extend because both the strategies and the algorithmic approaches are not convenient. From a strategic point of view, the problem is that both the timing of orders and the order quantities are unknown. It implies that the inventory position is hardly predictable and that constraints are hardly enforceable. Furthermore, because the dynamic programming formulation exploits the specific structure of the problem, a multi-period leadtime cannot be implemented.

Finally, from a cost point of view, our experiment shows that the *TK method* is the most efficient under constrained settings. The second top performing method is the *DP-EOQ method,* except when introducing minimum order quantity constraints (and very restrictive maximum order quantity constraints). Surprisingly, the extended *DP method* delivers a poor performance under constrained settings. Then, the *BT method* is often more efficient, even though its strategy is less flexible. Also, regarding the service level, we observe that all extended methods respect the service level criterion and that the average backlog quantity stays under control.

To sum up, we believe that the *TK method* is particularly interesting because it offers a good compromise between the cost efficiency and the scalability. Indeed, it is possible to extend the method with reasonable efforts to fit with specific problem requirements and our experiments show that it is the most efficient method from a cost perspective.

# Part 8: Case study 2 – Extended methods

In part 6, we introduced a multi-item joint replenishment problem concerning a retailer. This part gives a mathematical problem statement of this case study. Next, to solve this problem, we suggest extensions for the methods discussed in part 4. Finally, we assess the effectiveness of the extended methods on a test bed.

## 8.1 Mathematical problem statement

There are $I$ products individually identified by the index $i \in \{1, \dots, I\}$ and we want to consider a planning horizon of $T$ periods denoted by $t \in \{1, \dots, T\}$. The forecasted sales by products are translated into probability distributions. The probability to have enough inventory of product $i$ should be at least greater than $\alpha$. If there is not enough inventory, the unsatisfied demand is backlogged to the next period. Each time the product $i$ is ordered, an ordering cost, $a_i$, occurs whatever the order size. For each order placed, a delivery cost, $A$, should be paid. A unit holding cost, $h_i$, occurs for each unit of product $i$ stored from one period to the next. Finally, the procurement cost of each product, $c_i$, should also be considered. Obviously, the goal is to minimize the total cost incurred over the planning horizon.

## 8.2 Extended methods

### 8.2.1 Extended *BT method* and extended *TK method*

Both the *BT method* and the *TK method* are based on MIP formulations. It is surprisingly easy to extend these formulations to multi-item settings. The formulations are given in appendix 8 and 9 for both methods respectively. To extend the methods, we simply add a new index to identify the items individually. Consequently, the initial number of variables is multiplied by the number of items. We also introduce a new binary variable to identify the periods where at least one item is ordered. Unfortunately, the increasing number of variables makes the solving procedure computationally intensive. The same methodology can be applied for other multi-item problems. For example, it is easy to consider a joint capacity constraint on the inventory level or on the order quantities.

### 8.2.2 Extended *DP method*

The *DP method* is based on the $(s_t, S_t)$ replenishment strategy. Fundamentally, with this strategy, the ordering periods are unknown. Then, we cannot provide a direct extension to consider multi-item settings such as joint replenishment problems. A solution would be to transform the $(s_t, S_t)$ strategy into a "can-order" replenishment strategy. The "can-order" strategy is characterized by three parameters: $s_i$, $c_i$ and $S_i$. With this strategy, if the inventory position of item $i$ drops below $s_i$, a replenishment is made for all items with an inventory position below $c_i$. The quantity of item $i$ to order is the quantity that brings back the inventory position at $S_i$ (Melchiors, 2002). Initially, this strategy is used with stationary demand patterns, but it would be interesting to consider a $(s_{it}, c_{it}, S_{it})$ strategy for periodic review systems with seasonal demand patterns. However, this is out of the scope of this report. Consequently, we do not consider any true extension for the *DP method.* We simply compute the $s_{it}$ and $S_{it}$ with the standard *DP method,* by assuming that the ordering cost of item $i$ is $a_i + A$.

More generally speaking, even other multi-item settings cannot be considered due to the algorithmic approach. Indeed, as mentioned before, the dynamic programming approach suggested is based on the specific structure of the initial problem. Besides, we do not think that it is possible to consider multi-item constraints without changing radically the algorithmic approach.

### 8.2.3 Extended *DP-EOQ method*

The *DP-EOQ method* shares the same limits than the *DP method* in terms of adaptability to inventory management problems with multi-item constraints. However, for the specific case of the joint replenishment problem, it is possible to propose an extension without drastically changing the algorithmic approach. The *DP-EOQ method* is a method divided into two steps. First, a good review interval (i.e. $r$) is computed with an EOQ approach. Next, min/max levels (i.e. $s_t$ and $S_t$) are calculated for all review periods using a dynamic programming approach. A possible extension is to find review intervals that coordinate the replenishments. For this kind of problem, there are two main grouping strategies (Bastos, Carneiro, Melo, Mendes, & Nunes, 2017):

- Indirect Grouping Strategy (IGS): each item has its specific replenishment cycle, which is a multiple integer of a basic replenishment cycle.
- Direct Grouping Strategy (DGS): each product is grouped into a set where all items share the same replenishment cycle.

We suggest the heuristic of Chopra and Meindl (2013) to find good review intervals, $r_i$, under IGS strategy. Originally, this heuristic is used to determine replenishment cycles in multi-item

EOQ models. The idea is to review the inventory position every $r$ periods but the review will only concern a determined subset of items. Concretely, the inventory position of the product $i$ is reviewed every $r_i$ periods, where $r_i = m_i \times r$, and $m_i$ is an integer greater or equal to 1. The heuristic does not guarantee to find the optimal solution, but the authors claim that results are near-optimal. Since the heuristic was developed for EOQ models, the same restrictive assumptions must be made. For instance, only the total demand over the planning horizon is considered (i.e. the demand is assumed constant over time). The heuristic procedure has five steps (Chopra & Meindl, 2013):

**Step 1:** Identification of the most ordered product, $i^*$, considering that the delivery cost is entirely allocated to the item:

$$ordering\ frequency\ of\ item\ i: \overline{n_i} = \sqrt{\frac{D_i H_i}{2 \times (A + a_i)}}$$

Then, $i^*$ has an ordering frequency equal to $\overline{n} = \max(\overline{n_i}, i = 1, \ldots, I)$

**Step 2**: Compute the ordering frequency without the delivery cost for all $i \neq i^*$

$$\overline{\overline{n_i}} = \sqrt{\frac{D_i H_i}{2a_i}}$$

**Step 3**: Compute how often items are ordered in comparison to the most ordered product.

$$m_i = roundUP\left(\frac{\overline{n}}{\overline{\overline{n_i}}}\right) \forall i \neq i^*$$

$$m_{i^*} = 1$$

**Step 4:** Taking into consideration the ordering frequency of other items, re-compute the ordering frequency of $i^*$ given by $n$ such that:

$$n = \sqrt{\frac{\sum_{i=1}^{I} D_i H_i m_i}{2 \times \left(A + \sum_{i=1}^{I} \frac{a_i}{m_i}\right)}}$$

**Step 5:** For each product, a close to optimal ordering frequency is given by:

$$n_i = \frac{n}{m_i}$$

As soon as the review interval of each product is computed, the min/max levels for all review periods can be found for each product individually using the same dynamic programming approach than before.

## 8.3 Test bed

### 8.3.1 Methodology

We consider a joint replenishment problem with three items over a planning horizon of 30 periods. We suppose that the forecasted demand for each item is constant over time and follows a Poisson distribution. The products have different demand levels. We consider a product with a low demand ($d_{1t} = 30\ \forall t$), a product with a standard demand ($d_{2t} = 60\ \forall t$) and a product with a high demand ($d_{3t} = 120\ \forall t$). We use the following cost structure:

$$c_i = 4 \quad \forall i$$
$$h_i = 0.2 \quad \forall i$$
$$a_i = 250 \quad \forall i$$
$$A = 100$$

As before, we assume that unsatisfied demand is backlogged, and we impose a non-stockout probability of 95% for each period (i.e. $\alpha = 0.95$). To assess the performance of the new extended methods, we carry out a small experiment based on simulations. Concretely, we estimate the expected total cost for each method by running 1000 simulations.

### 8.3.2 Results

We apply the four extended methods discussed in section 8.2 to the inventory management problem stated above[24]. The *DP method* and the *DP-EOQ method* have running times about 4 and 3 seconds respectively, which are linearly proportional with the number of items. By contrast, we observe that the running times of the *BT method* and the *TK method* skyrocket with the number of items. Indeed, with the tested configuration, the *BT method* takes approximately 19 seconds to find the optimal solution. With the *TK method*, we stop after 540 seconds, and we obtain an optimality gap[25] of 1.78%. This drastic increase is due to the explosion of binary variables in the MIP formulations of the extended methods. The results found with the four methods are illustrated in table 9 and figure 10.

[24] We use the software Aimms with IBM ILOG CPLEX 12.6.3 to solve the MIP models of the *BT method* and the *TK method*. For the *DP method* and the DP-EOQ method, the recursive algorithms are written in Scala. Moreover, the computer is equipped with a 2.0Ghz AMD A8-6410 and 12GB of RAM.

[25] $Optimality\ gap = \frac{Best\ solution\ found - Best\ LP\ bound}{Best\ solution\ found}$

| Average cumulative backlog quantity | | | | |
|---|---|---|---|---|
| Items\Methods | BT method | TK method | DP method | DP-EOQ method |
| Low demand item | 1.370 | 1.038 | 0.101 | 1.174 |
| Medium demand item | 4.302 | 2.056 | 0.142 | 2.035 |
| High demand item | 5.804 | 2.902 | 0.304 | 2.883 |

*Table 9 : Average cumulative backlog quantity over the planning horizon for each method.*

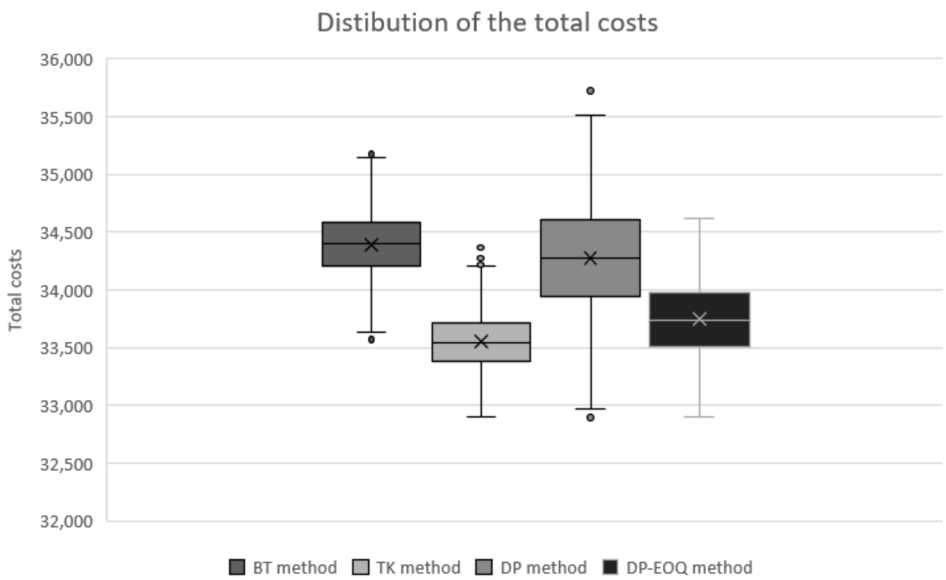


*Figure 10: Distribution of the total costs for each method.*

We observe that the extended *TK method* is again the most efficient from a cost perspective. The *DP-EOQ method* is the second top performing method in terms of costs and shows a service level very similar to the *TK method.* With the first sensitivity analysis given in part 5, we have shown that the *DP-EOQ method* is generally equivalent to the *TK method* under constant demand patterns. Here the *TK method* is probably more efficient because it determines better order timings. But it is precisely the reason why the *TK method* is time consuming. Indeed, finding the optimal replenishment planning is computationally intensive due to the combinatorial aspect of the task. In practice, heuristics can be used to improve the tractability of the *TK method.* For instance, we can use the same heuristic as for the *DP-EOQ method* if there are many items (i.e. the heuristic of Chopra and Meindl (2013)). In this way, a good planning of orders can be found in a trivial time and the remaining problem (i.e. determination of the order-up-to levels, $S_t$) is straightforward. Alternatively, if the number of items is limited, we can use the *BT method,* which performs faster, to determine a good replenishment planning (this heuristic was suggested by Bookbinder and Tan (1988) for the single-item setting).

The *DP method* performs slightly better than the extended *BT method,* even though it ignores the replenishment coordination due to the $(s_t, S_t)$ strategy. However, this may not be the case for larger ratios $\frac{A}{a_i}$. Moreover, we observe that the distribution of the total costs is highly variable with the *DP method.* We believe that this is because the setup nervousness is amplified when there are several items. Again, we see that the *DP method* is largely more efficient than the other methods from a service level point of view. This is due to the exact same reason than for the standard problem developed in part 5.

## 8.4 Key findings

The goal of this part is to assess the scalability of the suggested methods. Here, we are concerned about how efficiently they can tackle multi-item problems. Then, we have done our best to extend the methods to a common test bed problem and we have observed how the extended methods perform.

Firstly, we realize that the *BT method* and the *TK method* can be easily extended to multi-item settings because the MIP formulations allow to introduce new sets and identify items individually. Moreover, both methods are based on strategies that determine the order timing once for all, which is ideal for order consolidation. The only problem is that the number of binary variables increases with the number of items, which makes the methods computationally intensive.

Secondly, we notice that, generally speaking, the dynamic programming approach used for the *DP method* and the *DP-EOQ method* is not adaptable to multi-item settings. Nevertheless, for the specific case of joint replenishment problems, we suggest to coordinate the review intervals of the *DP-EOQ method* using a heuristic from the literature. We observe that it gives satisfying results.

Concretely, with the joint replenishment problem used as a test bed, we observe that the *TK method* is again the most efficient but the *DP-EOQ* method gives almost similar results. The *DP method* is still more efficient that the *BT method* with the configuration tested but it may not be the case for all cost configurations because the *DP method* ignores replenishment coordination due to the $(s_t, S_t)$ strategy.

To sum up, the *BT method* and *TK method* are easily scalable to multi-item settings, but the extended models are computationally intensive to solve if many items are considered. Then, heuristics may be required to improve the tractability. By contrast, the *DP method* and the *DP-EOQ method* cannot be easily extended to multi-item settings due to the strategies and the rigid algorithmic approach used. Therefore, we only manage to suggest an extension for the *DP-EOQ method* that can coordinate replenishments by determining appropriate review intervals with a heuristic from the literature.

# Conclusion

The purpose of this master thesis is to suggest algorithmic methods that can be used to solve real-life inventory management problems. More precisely, the methods should be applicable to real-life problems, scalable to fit to many problem settings and robust to demand uncertainty.

Our methodology to fulfil this threefold objective was the following:

First, we focus our efforts on the scientific literature related to inventory management problems. We identify the key characteristics of these problems and we review common types of inventory problems studied in the literature. We then focus on the dynamic lot-sizing problem, which concerns inventory problems with periodic review systems. We find this problem particularly attractive as it does not concern a specific problem configuration and then it should be more scalable. Also, interesting extensions have already been studied in the literature to consider demand uncertainty.

Next, we review the specific literature on the dynamic lot-sizing problem with stochastic demand. We observe that different algorithmic approaches exist to solve this problem under different replenishment strategies. There are three replenishment strategies commonly used in the literature:

- The $(R_t, Q_t)$ strategy: fix order timing and fix order quantities.
- The $(R_t, S_t)$ strategy: fix order timing and variable order quantities.
- The $(s_t, S_t)$ strategy: variable order timing and variable order quantities.

Thereafter, we decide to develop in detail two optimal methods from the literature. We select the MIP formulation of Bookbinder and Tan (1988) for the $(R_t, Q_t)$ strategy and we denote it by the *BT method*. Also, we select the MIP formulation of Tarim and Kingsman (2004) for the $(R_t, S_t)$ strategy and we denote it by the *TK method*. We also propose two other heuristic methods based on a dynamic programming formulation. The first one is used for the $(s_t, S_t)$ strategy and we name it the *DP method*. The second is combined with an EOQ approach to be used for the $(r, s_t, S_t)$[26] strategy and we name it the *DP-EOQ method.*

Subsequently, we confront the four methods to a test bed and we carry out a sensitivity analysis. This experiment is made to assess the performance of the methods under different

[26] The $(r, s_t, S_t)$ strategy is a $(s_t, S_t)$ apply every $r$ periods.

settings. Notably, we test the efficiency of the methods under seasonal demand patterns, with forecasting errors and under different service level requirements. And, in all cases, we are concerned about the efficiency under demand uncertainty. Thus, in all configurations tested, the demands are given by random variables.

Lastly, we want to assess the scalability of the methods. By scalability, we mean the capacity of the methods to be adapted for solving different inventory management problems. Then, we have met SME's managers and identified two cases inspired from real-life settings. The first one is an inventory management problem under specific constraints and the second is a multi-item joint replenishment problem. Then, to assess the scalability of the methods, we attempt to extend them, and we compare the performances of the extended methods on common test beds.

More concretely, our contribution can be summarized as follows:

- We develop two new formulations for the dynamic and stochastic lot-sizing problem under an $\alpha$-service level constraint.
- We compare the efficiency of these new methods with two other methods from the literature.
- We assess the impact of forecasting errors, service levels and seasonality.
- We propose extensions to adapt the methods to specific constraints and multi-item settings where it was feasible.

The key findings from our analysis are as follows:

- The *TK method* is surprisingly effective and even our *DP method* based on the $(s_t, S_t)$ strategy, which is more flexible, fails to out-perform it.
- The *DP method* gives a service level substantially higher than other methods, which may be interesting for critical items that have huge opportunity costs in case of stockouts (e.g. customer losses).
- With most methods, the introduction of forecasting errors mainly leads to a poorer service level but does not imply a significant increase of the expected total cost (except for the *DP method*).
- The *BT method* and the *TK method* are easy to extend to constrained settings and multi-item settings. This is because both methods are based on MIP formulations,

which are scalable, and because fixing the ordering period is convenient to formulate constraints.

- We realize that the *DP method* and the *DP-EOQ* method are much harder to extend to constrained settings and multi-item settings. This is because the dynamic programming formulation used is more rigid and because the $(s_t, S_t)$ strategy is not appropriate for constrained/multi-item formulations.

All in all, we observe that the *TK method* is the most efficient method that we have considered. We believe that it fits well with our primary objectives. The tractability may become an issue if the method is used for multi-item problems with many products. However, heuristics or tighter formulations can be developed to enhance the running time. Moreover, the *TK method* is based on the $(R_t, S_t)$ strategy which is a good tradeoff between the ability to react to demand realization and the stability from an organizational point of view. Indeed, by fixing the order timing at the beginning of the planning horizon, this strategy removes the setup nervousness from the inventory system. From a modelling point of view, the degree of freedom of the $(R_t, S_t)$ strategy is also very convenient because, by fixing $R_t$, only the order quantities are uncertain. Then, the expected inventory and the expected order quantities are easier to estimate. Moreover, constraints can be implemented effortlessly to fit with specific problem requirements.

In this master thesis, we test the *TK method* in different contexts and we propose modest extensions. We deeply regret not being able to complete the proof of concept about the effectiveness of the *TK method* by assessing the performances on a real situation. This is due to the lack of historical data and the short time frame to find a voluntary company. This would probably be a relevant next step for future research.

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# Appendix

## Appendix 1: First order loss functions

The expected inventory remaining at the end of the period and the expected backlog quantity or lost sales can be computed using the complementary first order loss function ($CFOLF$) and the first order function ($FOLF$) respectively (Hilger & Tempelmeier, 2015). The reader can refer to Hnich et al. (2013) for more explanation about these functions. Here, we just provide a short overview.

Let $x$ be the opening inventory and let $w$ be a continuous random variable that represent the demand. Moreover, let $g_w(x)$ and $G_w(x)$ be the density function and the cumulative distribution function of the random variable $w$ respectively. Then, $FOLF_w(x)$ and $CFOLF_w(x)$ are defined as:

- $FOLF_w(x) = E\{\max(w - x, 0)\} \ = \int_{-\infty}^{\infty} \max(t - x, 0)\, g_w(t)dt$
  $= \int_{x}^{\infty} (t - x) g_w(t)dt$
- $CFOLF_w(x) = E\{\max(x - w, 0)\} \ = \int_{-\infty}^{\infty} \max(x - t, 0)\, g_w(t)dt$
  $= \int_{x}^{\infty} (x - t) g_w(t)dt$

Hnich et al. (2013) also showed that for the case of a normally distributed random variable, we have:

- $FOLF_w(x) = \sigma_w \times [\phi(\beta) - ((1 - \Phi(\beta)) \times \beta)]$
- $CFOLF_w(x) = \sigma_w \times [\phi(\beta) + (\Phi(\beta) \times \beta)]$

where:

$\beta = \frac{x - \mu_w}{\sigma_w}$

$\phi(x)$ =The standard Normal probability density function.

$\Phi(x)$ = The standard Normal cumulative distribution function.

It is easy to see that the expected inventory at the end of a period is given by the complementary first order loss function while the expected lost sales/backlogs are given by the first order loss function:

- $Expected\ ending\ inventory(x) = E\{\max(x - w, 0)\} = CFOLF(x)$
- $Expected\ lost\ sales(x) = E\{\max(w - x, 0)\} = FOLF(x)$

## Appendix 2: Reminder about the EOQ approach

The standard EOQ model is based on the following assumptions (Reid & Sanders, 2010):

1. The demand of the product is perfectly known and constant over time.
2. The lead-time is constant.
3. The procurement cost is linear (i.e. no economy of scale).
4. The ordering cost is constant (i.e. whatever the size of the order, the cost of placing an order is the same).
5. The demand is perfectly known and stockouts are not allowed.
6. After the leadtime, the replenishment is immediate.

The following notations are used:

$D$: Demand rate (e.g. annual demand rate).
$\lambda$: Production rate (e.g. annual production rate if annual demand rate is considered).
$Q$: Order quantity.
$Q^*$: Optimal order quantity.
$r^*$: Optimal order interval (expressed in the same time unit that the demand).
$H$: Holding cost (e.g. annual holding cost if annual demand rate is considered).
$S$: Ordering cost.
$C$: Unit procurement cost.
$TC$: Total cost (e.g. annual total cost if annual demand rate is considered).

Solving the EOQ model is straightforward and the optimal order quantity can simply be derived from the first order condition (Reid & Sanders, 2010).

| EOQ |
|---|
| $TC = \frac{D}{Q} \times S + \frac{Q}{2} \times H + D \times C$<br>First order condition:<br>$-DS \times \frac{1}{Q^2} + \frac{H}{2} = 0$<br>$Q^* = \sqrt{\frac{2DS}{H}}$<br>$r^* = \frac{Q^*}{D}$ |

# Appendix 3: VBA function to compute $\boldsymbol{G_{d()}^{-1}(\alpha)}$

If the demand follows a Poisson distribution. We then use the following VBA function to compute $G_{d()}^{-1}(\alpha)$:

```
Function ReversePoisson(mean, alpha)
    Dim i As Integer
    Dim searching As Boolean
    i = -1
    searching = True
    Do While (searching)
        i = i + 1
        If WorksheetFunction.Poisson_Dist(i, mean, True) >= alpha Then
            ReversePoisson = i
            searching = False
        End If
        If i >= 10000 Then
            ReversePoisson = "out of bound"
            searching = False
         End If
    Loop
End Function
```

# Appendix 4: Demand patterns

**Demand patterns**

| Periods | *Const* | *Rand* | *Lcy* | *Hcy* | *Lsin* | *Hsin* |
|---|---|---|---|---|---|---|
| 1 | 60 | 5 | 10 | 5 | 70 | 90 |
| 2 | 60 | 20 | 15 | 5 | 65 | 75 |
| 3 | 60 | 64 | 20 | 10 | 55 | 50 |
| 4 | 60 | 44 | 25 | 15 | 50 | 25 |
| 5 | 60 | 119 | 30 | 25 | 45 | 10 |
| 6 | 60 | 67 | 40 | 35 | 50 | 25 |
| 7 | 60 | 29 | 50 | 50 | 55 | 50 |
| 8 | 60 | 22 | 60 | 60 | 65 | 75 |
| 9 | 60 | 37 | 70 | 70 | 70 | 90 |
| 10 | 60 | 28 | 80 | 80 | 75 | 110 |
| 11 | 60 | 61 | 90 | 90 | 70 | 90 |
| 12 | 60 | 70 | 95 | 100 | 65 | 75 |
| 13 | 60 | 79 | 100 | 110 | 55 | 50 |
| 14 | 60 | 64 | 105 | 120 | 50 | 25 |
| 15 | 60 | 20 | 110 | 125 | 45 | 10 |
| 16 | 60 | 108 | 110 | 125 | 50 | 25 |
| 17 | 60 | 67 | 105 | 120 | 55 | 50 |
| 18 | 60 | 99 | 100 | 110 | 65 | 75 |
| 19 | 60 | 64 | 95 | 100 | 70 | 90 |
| 20 | 60 | 24 | 90 | 90 | 75 | 110 |
| 21 | 60 | 80 | 80 | 80 | 70 | 90 |
| 22 | 60 | 71 | 70 | 70 | 65 | 75 |
| 23 | 60 | 8 | 60 | 60 | 55 | 50 |
| 24 | 60 | 113 | 50 | 50 | 50 | 25 |
| 25 | 60 | 113 | 40 | 35 | 45 | 10 |
| 26 | 60 | 4 | 30 | 25 | 50 | 25 |
| 27 | 60 | 101 | 25 | 15 | 55 | 50 |
| 28 | 60 | 115 | 20 | 10 | 65 | 75 |
| 29 | 60 | 75 | 15 | 5 | 70 | 90 |
| 30 | 60 | 29 | 10 | 5 | 75 | 110 |

# Appendix 5: Demonstration of the $\boldsymbol{\alpha}$-service level constraint for the *TK method*

To demonstrate the $\alpha$-service level constraint with leadtime, we first justify the $\alpha$-service level constraint without leadtime because the same logic will be applied. For the zero leadtime case, the reader can refer to (Kingsman & Tarim, 2004) for the complete demonstration.

As a reminder, random variables are always followed by brackets. For instance, $d_t(\,)$ denotes the actual demand realization in period $t$ while $d_t$ is the expected demand in period $t$. Moreover, we introduce the new index $T_l \;\; \forall l \in \{1, \dots, m\}$ which corresponds to the period number of the $l^{th}$ replenishment.

**Without lead-time**

We want to ensure: $Proba\{I_t(\,) \geq 0\} \geq \alpha \quad \forall t$

We know that the actual ending inventory and the expected ending inventory are given by:

$$I_t(\,) = S_{T_l} - \textstyle\sum_{k=T_l}^{t} d_k(\,) \qquad \forall t, l \;\; where\; T_l \leq t < T_{l+1}$$

$$I_t = S_{T_l} - \textstyle\sum_{k=T_l}^{t} d_k \qquad \forall t, l \;\; where\; T_l \leq t < T_{l+1}$$

Then we have:

$$Proba\left\{S_{T_l} - \sum_{k=T_l}^{t} d_k(\,) \geq 0\right\} \geq \alpha \qquad \forall t, l \;\; where\; T_l \leq t < T_{l+1}$$

$$Proba\left\{S_{T_l} \geq \sum_{k=T_{T_l}}^{t} d_k(\,)\right\} \geq \alpha \qquad \forall t, l \;\; where\; T_l \leq t < T_{l+1}$$

$$Proba\left\{I_t + \sum_{k=T_l}^{t} d_k \geq \sum_{k=T_l}^{t} d_k(\,)\right\} \geq \alpha \qquad \forall t, l \;\; where\; T_l \leq t < T_{l+1}$$

$$I_t + \sum_{k=T_l}^{t} d_k \geq G^{-1}_{d_{T_l}(\,)+d_{T_l+1}(\,)+\cdots+d_t(\,)}(\alpha) \qquad \forall t, l \;\; where\; T_l \leq t < T_{l+1}$$

$$I_t \geq G^{-1}_{d_{T_l}(\,)+d_{T_l+1}(\,)+\cdots+d_t(\,)}(\alpha) - \sum_{k=T_l}^{t} d_k \qquad \forall t, l \;\; where\; T_l \leq t < T_{l+1}$$

Where $G^{-1}_{d_1(\,)+\cdots+d_t(\,)}(\alpha)$ is the inverse cumulative distribution function such that $G^{-1}_{d_1(\,)+\cdots+d_t(\,)}(\alpha) = u$ means that $Proba(d_1(\,) + \cdots + d_t(\,) \leq u) = \alpha$.

In the MIP model, Tarim and Kingsman (2004) do not use the index $T_l$. Instead, they use the binary variable $P_{tj}$ which is equal to 1 if the last replenishment prior to period $t$ was in period $t-j+1$ and zero otherwise. Then, the last equation obtained can be expressed as follows in the MIP model:

$$I_t \geq \sum_{j=1}^{t} (G^{-1}_{d_{t-j+1}()+d_{t-j+2}()+\cdots+d_t()}(\alpha) - \sum_{k=t-j+1}^{t} d_i) \times P_{tj} \qquad \forall t$$

**With lead-time**

The demonstration of the $\alpha$-service level constraint with a positive leadtime is similar to the demonstration for the zero leadtime case.

We still want to ensure: $Proba\{I_t(\,) \geq 0\} \geq \alpha \quad \forall t$

But now, the actual ending inventory and the expected ending inventory are given by:

$$I_t(\,) = S_{T_l}(\,) - \sum_{k=T_l}^{t} d_k(\,) \qquad \forall t, l \;\; where\; T_l \leq t < T_{l+1}$$

$$I_t = S_{T_l} - \sum_{k=T_l}^{t} d_k \qquad \forall t, l \;\; where\; T_l \leq t < T_{l+1}$$

In the zero leadtime case, the actual opening inventory on a replenishment day is not a random variable. But, with a positive leadtime, the actual opening inventory depends on the demand realization and the ongoing replenishments. We denote the replenishment quantity in the beginning of period $t$ by $Q_t$ . Mathematically, we have:

$$S_{T_l}(\,) = I_{T_l-LT-1}(\,) + Q_{T_l}(\,) - \sum_{k=T_l-LT}^{T_l-1} d_k(\,) + \sum_{k=T_l-LT}^{T_l-1} Q_k(\,) \;\; \forall t, l\; where\; T_l \leq t < T_{l+1}$$

$$Q_{T_l}(\,) = S_{T_l} - I_{T_l-LT-1}(\,) + \sum_{k=T_l-LT}^{T_l-1} d_k - \sum_{k=T_l-LT}^{T_l-1} Q_k(\,) \qquad \forall t, l\; where\; T_l \leq t < T_{l+1}$$

By substituting $Q_{T_l}(\,)$ in the equation of $S_{T_l}(\,)$, we obtain:

$$S_{T_l}(\,) = S_{T_l} - \sum_{k=T_l-LT}^{T_l-1} d_k(\,) + \sum_{k=T_l-LT}^{T_l-1} d_k \qquad \forall t, l\; where\; T_l \leq t < T_{l+1}$$

Then we have:

$$Proba\left\{S_{T_l}(\,) - \sum_{k=T_l}^{t} d_k(\,) \geq 0\right\} \geq \alpha \qquad \forall t, l\; where\; T_l \leq t < T_{l+1}$$

$$Proba\left\{S_{T_l} - \sum_{k=T_l-LT}^{T_l-1} d_k\,() + \sum_{k=T_l-LT}^{T_l-1} d_k - \sum_{k=T_l}^{t} d_k() \geq 0\right\} \geq \alpha \;\; \forall t, l \; where \; T_l \leq t < T_{l+1}$$

$$Proba\left\{I_t + \sum_{k=T_l}^{t} d_k + \sum_{k=T_l-LT}^{T_l-1} d_k - \sum_{k=T_l-LT}^{t} d_k() \geq 0\right\} \geq \alpha \;\; \forall t, l \; where \; T_l \leq t < T_{l+1}$$

$$Proba\left\{I_t + \sum_{k=T_l-LT}^{t} d_k \geq \sum_{k=T_l-LT}^{t} d_k()\right\} \geq \alpha \qquad \forall t, l \; where \; T_l \leq t < T_{l+1}$$

$$I_t + \sum_{k=T_l-LT}^{t} d_k \geq G^{-1}_{d_{T_l-LT}()+d_{T_l-LT+1}()+\cdots+d_t()}(\alpha) \qquad \forall t, l \; where \; T_l \leq t < T_{l+1}$$

$$I_t \geq G^{-1}_{d_{T_l-LT}()+d_{T_l-LT+1}()+\cdots+d_t()}(\alpha) - \sum_{k=T_l-LT}^{t} d_k \qquad \forall t, l \; where \; T_l \leq t < T_{l+1}$$

Where $G^{-1}_{d_1()+\cdots+d_t()}(\alpha)$ is the inverse cumulative distribution function such that $G^{-1}_{d_1()+\cdots+d_t()}(\alpha) = u$ means that $Proba(d_1() + \cdots + d_t() \leq u) = \alpha$.

In the suggested MIP, we do not use the index $T_l$. Instead, we use the same binary variable $P_{tj}$ than in the zero leadtime case. With this variable, the last equation is expressed as follows in the MIP model:

$$I_t \geq \left(G^{-1}_{d_{t-j-LT+1}()+d_{t-j-LT+2}()+\cdots+d_t()}(\alpha) - \sum_{k=t-j-LT+1}^{t} d_k\right) \times P_{tj} \quad \forall t$$

## Appendix 6: CapWH constraint for the *TK method*

We want: $proba\{S_t(\,) \le CapWH\} \ge \beta \quad \forall t$

We know that $d_t(\,) \ge 0\ \forall t$. So, the inventory peaks occur when there is a replenishment. Then, we introduce the new index $T_l \ \ \forall l \in \{1, \dots, m\}$ which corresponds to the period number of the $l^{th}$ replenishment. We have:

$$proba\{S_{T_l}(\,) \le CapWH\} \ge \beta \quad \forall l$$

$$S_{T_l}(\,) = I_{T_l-LT-1}(\,) + Q_{T_l}(\,) \ - \sum_{k=T_l-LT}^{T_l-1} d_k(\,) + \sum_{k=T_l-LT}^{T_l-1} Q_k(\,) \ \ \forall l$$

$$Q_{T_l}(\,) = S_{T_l} - \ I_{T_l-LT-1}(\,) + \sum_{k=T_l-LT}^{T_l-1} d_k - \sum_{k=T_l-LT}^{T_l-1} Q_k(\,) \quad \forall l$$

By substituting $Q_{T_l}(\,)$ in the equation of $S_{T_l}(\,)$, we obtain:

$$S_{T_l}(\,) = S_{T_l} - \sum_{k=T_l-LT}^{T_l-1} d_k\,(\,) + \sum_{k=T_l-LT}^{T_l-1} d_k \qquad \forall l$$

$$proba\left\{S_{T_l} - \sum_{k=T_l-LT}^{T_l-1} d_k\,(\,) + \sum_{k=T_l-LT}^{T_l-1} d_k \le CapWH\right\} \ge \beta \quad \forall l$$

$$proba\left\{S_{T_l} \le \sum_{k=T_l-LT}^{T_l-1} d_k\,(\,) - \sum_{k=T_l-LT}^{T_l-1} d_k + CapWH\right\} \ge \beta \quad \forall l$$

$$S_{T_l} \le G'^{-1}_{d_{T_l-LT}(\,)+\cdots+d_{T_l-1}(\,)}(\beta) - \sum_{k=T_l-LT}^{T_l-1} d_k + CapWH \quad \forall l$$

$$S_t \le G'^{-1}_{d_{t-LT}\,(\,)+\cdots+d_{t-1}(\,)}(\beta) - \sum_{k=t-LT}^{t-1} d_k + CapWH \quad \forall t \ such\ that\ R_t = 1$$

Where $G'^{-1}_{d_t(\,)}(\beta)$ is the right-tailed inverse cumulative distribution function such that $G'^{-1}_{d_t(\,)}(\beta) = u$ means that $Proba(d_t(\,) \ge u) = \beta$. Again, $G'^{-1}_{d_t(\,)}(\beta)$ can be computed with a spreadsheet software for all $t$ before solving the problem.

# Appendix 7: Additional results for case 1

| Summuray of the total cost | | | | |
|---|---|---|---|---|
| **Criteria\methods** | **BT method** | **TK method** | **DP method** | **DP-EOQ method** |
| **Leadtime** | | | | |
| LT=2 | 9938.44 | 9708.03 | 10055.58 | 9741.74 |
| LT=4 | 9938.44 | 9749.95 | 10148.76 | 9955.69 |
| LT=6 | 9938.44 | 9774.74 | 9774.74 | 9774.74 |
| LT=8 | 9972.56 | 9837.33 | 9860.65 | 9860.65 |
| **MOQ** | | | | |
| MOQ=0 | 9938.44 | 9708.03 | 10055.58 | 9741.74 |
| MOQ=50 | 9938.44 | 9708.03 | 10165.32 | 10178.95 |
| MOQ=100 | 9938.44 | 9708.03 | 10275.10 | 10561.35 |
| MOQ=150 | 9938.44 | 9708.03 | 10384.88 | 10715.73 |
| **CapRM** | | | | |
| CapRM=350 | 10005.16 | 9760.53 | 11195.07 | 10375.16 |
| CapRM=400 | 9938.44 | 9706.64 | 10546.67 | 9818.37 |
| CapRM=500 | 9938.44 | 9708.03 | 10546.67 | 9792.58 |
| CapRM=None | 9938.44 | 9708.03 | 10055.58 | 9741.74 |
| **CapWH** | | | | |
| CapWH=300 | 10424.51 | 10007.02 | 10697.51 | 10018.52 |
| CapWH=450 | 9924.71 | 9708.03 | 10126.35 | 9941.74 |
| CapWH=600 | 9938.44 | 9708.03 | 10055.58 | 9741.74 |
| CapWH=None | 9938.44 | 9708.03 | 10055.58 | 9741.74 |

| Summuray of the average cumulative backlog quantity | | | | |
|---|---|---|---|---|
| **Criteria\methods** | **BT method** | **TK method** | **DP method** | **DP-EOQ method** |
| **Leadtime** | | | | |
| LT=2 | 3.72 | 2.43 | 0.50 | 2.40 |
| LT=4 | 3.72 | 2.54 | 1.44 | 1.81 |
| LT=6 | 3.72 | 2.79 | 2.79 | 2.79 |
| LT=8 | 3.32 | 3.00 | 2.51 | 2.51 |
| **MOQ** | | | | |
| MOQ=0 | 3.72 | 2.43 | 0.50 | 2.40 |
| MOQ=50 | 3.72 | 2.43 | 0.29 | 0.63 |
| MOQ=100 | 3.72 | 2.43 | 0.29 | 0.69 |
| MOQ=150 | 3.72 | 2.43 | 0.29 | 0.69 |
| **CapRM** | | | | |
| CapRM=350 | 4.30 | 2.62 | 4.69 | 3.57 |
| CapRM=400 | 3.72 | 2.52 | 0.88 | 4.24 |
| CapRM=500 | 3.72 | 2.43 | 0.88 | 2.72 |
| CapRM=None | 3.72 | 2.43 | 0.50 | 2.40 |
| **CapWH** | | | | |
| CapWH=300 | 6.44 | 3.29 | 1.04 | 4.24 |
| CapWH=450 | 3.72 | 2.43 | 0.67 | 2.40 |
| CapWH=600 | 3.72 | 2.43 | 0.50 | 2.40 |
| CapWH=None | 3.72 | 2.43 | 0.50 | 2.40 |

## Appendix 8: Case study 2 – Extended *BT method*

To extend the *BT method,* we simply add a new index to identify the items individually. Concretely, there are $I$ products individually identify by the index $i \in \{1, \dots, I\}$ and we want to consider a planning horizon of $T$ periods denoted by $t \in \{1, \dots, T\}$.

We use the following notations:

$d_{it}$: Expected demand of item $i$ in period $t$.

$h_i$: Unit cost for carrying item $i$ from one period to the next.

$a_{it}$: Setup cost for ordering item $i$.

$A$: Delivery cost.

$c_i$: Unit procurement cost for purchasing item $i$.

We introduce the following variables:

$R_{it} \in \{0,1\}$: Variable equal to 1 if the item $i$ is ordered in period $t$. Else, equal to 0.

$\delta_t \in \{0,1\}$: Variable equal to 1 if an order $i$ is placed in period $t$.

$Q_{it} \geq 0$: Quantity of item $i$ ordered in period $t$.

$I_{it} \geq 0$: Expected ending inventory of item $i$ in period $t$.

$I_{i0}$ is the initial stock of item $i$ .

Objective function: $Min \ \sum_{t=1}^{T} \sum_{i=1}^{I} hI_{it} + aR_{it} + cQ_{it} \ + A\delta_t$

Subject to:

| | |
|---|---|
| $I_{it} = I_{i0} + \sum_{k=1}^{t} Q_{ik} - d_{ik}$ | $\forall t, i$ |
| $\sum_{k=1}^{t} Q_{ik} \geq G^{-1}_{d_{i1}()+\cdots+d_{it}()}(\alpha) - I_{i0}$ | $\forall t, i$ |
| $Q_{it} \leq M \times R_{it}$ | $\forall t, i$ where $M$ is a big number |
| $\sum_{i=1}^{I} R_{it} \leq M \times \delta_t$ | $\forall t$ |

# Appendix 9: Case study 2 – Extended *TK method*

To extend the *TK method,* we simply add a new index to identify the items individually. Concretely, there are $I$ products individually identify by the index $i \in \{1, \dots, I\}$ and we want to consider a planning horizon of $T$ periods denoted by $t \in \{1, \dots, T\}$.

We use the following notations:

$d_{it}$: Expected demand of item $i$ in period $t$.

$h_i$: Unit cost for carrying item $i$ from one period to the next.

$a_{it}$: Setup cost for ordering item $i$.

$A$: Delivery cost.

$c_i$: Unit procurement cost for purchasing item $i$.

We introduce the following variables:

$R_{it} \in \{0,1\}$: Variable equal to 1 if the item $i$ is ordered in period $t$. Else, equal to 0.

$P_{itj} \in \{0,1\}$: Variable equal to 1 if the most recent replenishment of item $i$ prior to period $t$ was in period $t - j + 1$ (i.e. if the last replenishment was $j - 1$ periods ago). Else, equal to 0.

$\delta_t \in \{0,1\}$: Variable equal to 1 if an order $i$ is placed in period $t$.

$I_{it} \geq 0$: Expected ending inventory of item $i$ in period $t$.

$S_{it} \geq 0$: Expected opening inventory of item $i$ in period $t$, taking into account the eventual replenishment at the beginning of the period.

Objective function[27]: $Min \sum_{t=1}^{T} \sum_{i=1}^{I} hI_{it} + aR_{it} + (cS_{it} - cI_{it-1}) \ + A\delta_t$

Subject to:

$I_{it} = S_{it} - d_{it}$ $\quad \forall t, i$

$S_{it} \geq I_{it-1}$ $\quad \forall t, i$

$S_{it} - I_{it-1} \leq M \times R_{it}$ $\quad \forall t, i$

and $M$ is a big number

$I_{it} \geq \sum_{j=1}^{t} (G^{-1}_{d_{i,t-j+1}()+d_{i,t-j+2}()+\dots+d_{it}()}(\alpha) - \sum_{k=t-j+1}^{t} d_{ik}) P_{itj}$ $\quad \forall t, i$

$\sum_{j=1}^{t} P_{itj} = 1$ $\quad \forall t, i$

$P_{itj} \geq R_{it-j+1} - \sum_{k=t-j+2}^{t} R_{ik}$ $\quad \forall t, i \quad \forall j \leq t$

[27] The procurement costs of item $i$ correspond to $\sum_{t=1}^{T} (cS_{it} - cI_{it-1})$ because the expected quantity delivered in period $t$ is: $S_{it} - I_{it-1}$

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