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The classifying topos

A tool to understand mathematical theories

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The classifying topos : a tool to understand mathematical theories

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Introduction

In 1964, Wiliam Lawvere was working in giving a version of set theory based on functions and composition of functions, the language of category theory. Meanwhile, Alexander Grothendieck was working in algebraic geometry, attempting to prove the Weil conjectures. While doing so, he introduced the notion of topos, now known as Grothendieck topos. In 1971, William Lawvere and Myles Tierney realized Grothendieck topoi could be seen as abstract mathematical universe in which most usual set-theoretic construction can be carried. They generalized the notion of Grothendieck topos to the notion of (elementary) topos. A topos is a category where, similarly to **Set** the category of sets, the collection of all morphisms between two objects $\mathbf{Hom}(X,Y)$ is again an object in the topos and subobjects of an object are elements of the "power object" $\mathcal{P}(X)$, similar to the notion of power set. Ultimately, few years later, the theory of classifying topoi brought a fundamental result : any Grothendieck topos can be associated to a geometric theory, a theory whose axioms are expressed with a finite number of "and" $\wedge$ and a small number of "or" $\vee$. Grothendieck topoi are thus central when it comes to the study of mathematical theories in category theory.

We will focus our work in explaining what is a mathematical theory in the context of category theory and we will devise a classifying topos for such a theory. This topos classifies the theory in the sense that for any other topos with a model of the theory internal to it, this model corresponds to a specific kind of adjunction (the geometric morphism) between the two topoi.

The first chapter will be an introduction to the Grothendieck topoi. We expect the reader to be already familiar with basic categorical notions such as regular categories and their properties, the notion of adjunction, limits and colimits. We will see that by their very definition, the Grothendieck topoi are nice categories with a lot a useful properties that are directly inherited from the properties of **Set**. We will end this chapter by the definition of a classifying topos. At that moment in time giving examples of classifying topoi will not be an easy task.

In the second chapter we will sketch theories. We will associate to a theory a sketch, a collection of suitable diagrams such that a model of that sketch (for example a group in the theory of groups) in any category will be a collection of limit and colimit cones corresponding to the right diagrams. In this chapter we will provide a proof to the fact that any Grothendieck topos is the classifying topos of some theory defined by a sketch and conversely any theory determined by a geometric sketch admits a classifying topos.

The last chapter will be dedicated to operating a full translation of the set-theoretic language of a theory to any language internal to a topos. Even though sketches are a nice concept, drawing the right diagrams to get the right theory is not necessarily an easy task. Therefore instead of trying to translate the usual set-theoretic language into the language of category theory, we will adopt a similar language inside a topos. This chapter will bring great insights as to why the study of classifying topoi is relevant as we will prove the validity of a formula is preserved by geometric morphisms. We will end this chapter with a construction, for a given geometric theory, of a sketch such that its models in a topos are precisely the ones of the geometric theory. Hence, any geometric theory admits a classifying topos.

Through out this work, we will use a lot various properties of limits and colimits as well as two important categorical notions : the Yoneda embedding and the Kan extensions. There are appendices in the end of this document listing all the properties we will need with some proofs.

Our main source for chapter 1 and 2 will be [3], while our main source for chapter 3 will be [6].

Contents

Chapter 1

Grothendieck topoi

We start with an introduction to Grothendieck topoi. There is a lot to say about Grothendieck topoi, a full account can be found in [6, Chapters II, III]. We will focus our attention to the fact that Grothendieck topoi are complete, cocomplete, regular categories and that the suitable notion of morphisms between Grothendieck topoi is the one of geometric morphism.

1.1 Grothendieck topoi

Definition 1.1.1. A *Grothendieck topology* T on a small category $\mathscr{C}$ is the data for each object $C \in \mathscr{C}$ of a family $T(C)$ of subfunctors $r : R \hookrightarrow \mathbf{Hom}_{\mathscr{C}}(-, C)$ called the *covering sieves* such that :

1. $\forall C \in \mathscr{C}$, $\mathbf{Hom}_{\mathscr{C}}(-, C) \in T(C)$;
2. $\forall f : D \to C$ morphism in $\mathscr{C}$ and $\forall r \in T(C)$ if R_f is the pullback of $\mathbf{Hom}_{\mathscr{C}}(-, f)$ along r then $r_f \in T(D)$;

$$\begin{tikzcd} R_f \arrow[r] \arrow[d, hook, "r_f"'] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, hook, "r"] \\ \mathbf{Hom}_C(-, D) \arrow[r, "f\circ-"'] & \mathbf{Hom}_{\mathscr{C}}(-, C) \end{tikzcd}$$

3. $\forall C \in \mathscr{C}$, $S \in T(C)$ and $r : R \hookrightarrow \mathbf{Hom}_{\mathscr{C}}(-, C)$ an arbitrary subfunctor, if $\forall D \in \mathscr{C}$ and $f \in S(D)$ we get $r_f \in T(D)$ then $r \in T(C)$.

The pair $(\mathscr{C}, T)$ is called a *Grothendieck site.*

Definition 1.1.2. Let $(\mathscr{C}, T)$ a Grothendieck site.

- A *presheaf* on $\mathscr{C}$ is a functor $F : \mathscr{C}^{op} \to \mathbf{Set}$.

- A *sheaf* on a site $(\mathscr{C}, T)$ is a presheaf F on $\mathscr{C}$ such that for any object $C \in \mathscr{C}$ and covering sieve $R \in T(C)$, every natural transformation $\alpha : R \to F$ extends to a unique $\beta : \mathbf{Hom}_{\mathscr{C}}(-, C) \to F$.

$$\begin{tikzcd} R \arrow[r, hook, "r"] \arrow[dr, "\alpha"'] & \mathbf{Hom}_{\mathscr{C}}(-, C) \arrow[d, "\beta"] \\ & F \end{tikzcd}$$

In particular this defines a category $\mathbf{Pr}(\mathscr{C})$ (resp. $\mathbf{Sh}(\mathscr{C}, T)$) with objects the presheaves (resp. sheaves) on $\mathscr{C}$ and with morphisms the natural transformations. $\mathbf{Sh}(\mathscr{C}, T)$ is a full subcategory $i : \mathbf{Sh}(\mathscr{C}, T) \hookrightarrow \mathbf{Pr}(\mathscr{C})$. Moreover by [6, Theorem III.5.1], there is a specific adjunction between the category of sheaves and presheaves .

Theorem 1.1.3 (Sheafification)**.** *The inclusion i has a left adjoint $\#$ that preserves colimits and finite limits.*

Thus $\mathbf{Sh}(\mathscr{C}, T)$ is an exact reflective subcategory of $\mathbf{Pr}(\mathscr{C})$. If the reader is not familiar with reflective subcategories and their properties, see [2, Section 3.5]. A reflective subcategory with an exact reflection is called a *localization*. There is a strong bound between localization of a finitely complete category $\mathscr{B}$ and universal closure operation on $\mathscr{B}$. We shall recall to the reader the definitions and properties we will need to prove 3.2.11. The proofs can be found in [2, Sections 5.6, 5.7] and in [3, Section 3.5].

Definition 1.1.4. Let be a finitely complete category $\mathscr{B}$. A *universal closure operator* on $\mathscr{B}$ is the data for each subobject $S \hookrightarrow B$ in $\mathscr{B}$ of another subobject $\overline{S} \hookrightarrow B$ called the *closure* of S in B, such that for any subobjects S, T of B and any morphism $f : A \to B$ of $\mathscr{B}$:

1. $S \subseteq \overline{S}$,
2. $S \subseteq T \Rightarrow \overline{S} \subseteq \overline{T}$,
3. $\overline{\overline{S}} = \overline{S}$,
4. $f^{-1}(\overline{S}) = \overline{f^{-1}(S)}$.

where $f^{-1}(S)$ denotes the pullback of $S \hookrightarrow B$ along f.

We call a subobject $S \hookrightarrow B$:

- *dense* when $\overline{S} = B$.
- *closed* when $\overline{S} = S$.

Proposition 1.1.5. *Consider a localization* $\mathscr{A} \underset{i}{\overset{r}{\rightleftarrows}} \mathscr{B}$ *of a finitely complete category* $\mathscr{B}$*. Then the localization induces a universal closure operator on* $\mathscr{B}$*. The closure of* $S \hookrightarrow B$ *is the pullback of* $ir(S) \hookrightarrow ir(B)$ *along* $\eta(B) : B \to ir(B)$*. In particular any subobject in* $\mathscr{A}$ *is closed.*

Proposition 1.1.6. *Let* $\mathscr{C}$ *be a small category and* $\mathbf{Pr}(\mathscr{C})$ *the category of presheaves on* $\mathscr{C}$*. There is a bijection between*

1. *the universal closure operators on* $\mathbf{Pr}(\mathscr{C})$*.*
2. *the localizations of* $\mathbf{Pr}(\mathscr{C})$*.*

We admit without a proof (can be found in [3, Lemma 3.5.1]) this next result :

Proposition 1.1.7. *Let* $(\mathscr{C}, T)$ *be a Grothendieck site,* $C \in \mathscr{C}$ *an object and* $r : R \hookrightarrow \mathbf{Hom}_{\mathscr{C}}(-, C)$ *a subobject in* $\mathbf{Pr}(\mathscr{C})$*. The following are equivalent :*

1. $r \in T(C)$*,*
2. $\#r$ *is an isomorphism.*

Any category of sheaves on a site $(\mathscr{C}, T)$, $\mathbf{Sh}(\mathscr{C}, T)$, is uniquely determined by the closure operator 1.1.5 on $\mathrm{Pr}(\mathscr{C})$. In particular, any subobject in $\mathbf{Sh}(\mathscr{C}, T)$ is closed and any covering sieve is dense. This will come handy later on.

It is now time to define one of the main concept we are going to work with.

Definition 1.1.8. A *Grothendieck topos* is a category equivalent to a category of sheaves on a certain site, i.e $\mathscr{E}$ is a Grothendieck topos then there exists a small category $\mathscr{C}$ and a Grothendieck topology T on $\mathscr{C}$ such that $\mathscr{E} \cong \mathbf{Sh}(\mathscr{C}, T)$.

Examples 1.1.9.

1. The category of presheaves $\mathbf{Pr}(\mathscr{C})$ on a small category $\mathscr{C}$ is a Grothendieck topos. Indeed $\mathbf{Pr}(\mathscr{C})$ is precisely $\mathbf{Sh}(\mathscr{C}, T_{\mathrm{ind}})$ the category of sheaves on $\mathscr{C}$ with respect to the *indiscrete* topology T_{ind} defined for any $C \in \mathscr{C}$ as $T_{\mathrm{ind}}(C) = \{\mathrm{Id} : \mathbf{Hom}_{\mathscr{C}}(-, C) \to \mathbf{Hom}_{\mathscr{C}}(-, C)\}$. This is a well defined topology since the first axiom is trivially satisfied and the second too since any pullback along the identity is again an identity. The third axiom is also satisfied : for an object $C \in \mathscr{C}$, let $r : R \hookrightarrow \mathbf{Hom}_{\mathscr{C}}(-, C)$ be a sieve such that for any $f \in \mathbf{Hom}_{\mathscr{C}}(D, C)$ for any $D \in \mathscr{D}$, the pullback of $\mathbf{Hom}_{\mathscr{C}}(-, f)$ along r is $R_f = \mathbf{Hom}_{\mathscr{C}}(-, D)$. Then in particular take $\mathrm{Id}_C \in \mathbf{Hom}_{\mathscr{C}}(C, C)$, the pullback $R_{\mathrm{Id}_C} = \mathbf{Hom}_{\mathscr{C}}(-, C)$ is R. Thus $R \in T_{\mathrm{ind}}(C)$. Any presheaf on $\mathscr{C}$ is then a sheaf on the site $(\mathscr{C}, T_{\mathrm{ind}})$.
2. $\mathbf{Set}$ is a Grothendieck topos. Indeed, $\mathbf{Set}$ is equivalent to $\mathbf{Pr}(1)$ the category of presheaves on 1 the category with one single object and just the identity morphism. It is easy to see there is exactly one presheaf $F : 1 \to \mathbf{Set}$ for each set in $\mathbf{Set}$ and one natural transformation for each map between sets, hence the equivalence.

We will now prove some properties of Grothendieck topoi.

Proposition 1.1.10. *Grothendieck topoi are complete, cocomplete and the colimits are pullback stable.*

Proof. Let $\mathscr{C}$ be a small category. As a direct corollary of B.0.1, $\mathbf{Pr}(\mathscr{C})$ is complete, cocomplete and colimits are universal since all limits and colimits are computed pointwise and $\mathbf{Set}$ is complete, cocomplete and its colimits are pullback stable.

Now let be a site $(\mathscr{C}, T)$. Since $\mathbf{Sh}(\mathscr{C}, T)$ is a reflective subcategory of $\mathbf{Pr}(\mathscr{C})$, it is complete and cocomplete. Since # preserves colimits and pullback, it preserves universal colimits. Since the colimits on $\mathbf{Sh}(\mathscr{C}, T)$ are precisely obtained by applying # to the colimits of $\mathbf{Pr}(\mathscr{C})$, we get that the colimits of $\mathbf{Sh}(\mathscr{C}, T)$ are pullback stable. $\square$

Proposition 1.1.11. *In a Grothendieck topos, every monomorphism is regular and every epimorphism is regular.*

Proof. In $\mathbf{Set}$ monomorphisms are the injections and the epimorphisms are the surjections. Moreover a monomorphism $f : A \hookrightarrow B$ is the equalizer of its cokernel pair and a epimorphism $g : A \to B$ is the coequalizer of its kernel pair:

Let (X, i_1, i_2) be the cokernel pair of f, i.e the pushout of f along itself. The pushout X in $\mathbf{Set}$ is defined as $X := (B \sqcup B)/\sim = \{(b, 0), (0, b) | b \in B\}/\sim$ where $\sim$ is the equivalence relation generated by $(f(a), 0) \sim (0, f(a))$, $\forall a \in A$ and i_1, i_2 are the canonical inclusions. It is easy to see that for any two arrows $\alpha, \beta : B \to Y$ such that for all $a \in A$, $\alpha(f(a)) = \beta(f(a))$, they yield a unique arrow $\gamma : X \to Y$ defined as $\gamma([b, 0]) = \alpha(b)$ and $\gamma([0, b]) = \beta(b)$ satisfying that $\gamma \circ i_1 = \alpha$ and $\gamma \circ i_2 = \beta$. Thus (i_1, i_2) is indeed the cokernel pair of f. Now let be another morphism $m : C \to B$ such that $i_1(m(c)) = i_2(m(c))$ for any $c \in C$. Thus $(m(c), 0) \sim (0, m(c))$ and by definition of the relation there exists $a \in A$ such that $m(c) = f(a)$. Hence we get an arrow $\lambda : C \to A$ mapping any c to its associated a. λ is unique since f is injective. Therefore f is the equalizer of its cokernel pair, f is a regular monomorphism.

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[d, "i_2"] \arrow[ddr, bend left, "\beta"] & \\
B \arrow[r, "i_1"'] \arrow[drr, bend right, "\alpha"'] & X \arrow[dr, dashed, "\lambda"'] & \\
& & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f"'] & B \arrow[r, shift left, "i_1"] \arrow[r, shift right, "i_2"'] & X \\
C \arrow[u, dashed, "\lambda"'] \arrow[ur, "m"] & &
\end{tikzcd}
$$

Now let (R, p_1, p_2) the kernel pair of g, i.e the pullback of g along itself. One can show that the pullback is $R := \{(a, a') | g(a) = g(a')\}$ with the canonical projections p_1 and

p_2. Now assume we have a morphism $q : A \to C$ such that $q \circ p_1 = q \circ p_2$. Since g is surjective, for any $b \in B$ there exists a $a \in A$ such that $g(a) = b$. This yields a unique map $\lambda : B \to C$ such that $\lambda(b) = q(a)$, the unicity is given by the fact that if we have $a' \in A$ such that $g(a) = g(a')$ then $(a, a') \in R$ and thus $q(a) = q(a')$ too. Hence g is the coequalizer of its kernel pair, g is a regular epimorphism.

$$\begin{tikzcd} R \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & A \arrow[r, "g"'] \arrow[dr, "q"] & B \arrow[d, dashed, "\lambda"'] \\ & & C \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "p_2"] \arrow[d, "p_1"'] & A \arrow[d, "g"] \\ A \arrow[r, "g"'] & B \end{tikzcd}$$

Now, let $\mathscr{C}$ be a small category. Since in the category of presheaves $\mathbf{Pr}(\mathscr{C})$ all colimits and limits are computed pointwise, every monomorphism in $\mathbf{Pr}(\mathscr{C})$ is the equalizer of its cokernel pair and every epimorphism is the coequalizer of its kernel pair.

Finally, let $(\mathscr{C}, T)$ be a site and $f : F \to G$ a monomorphism in the category of sheaves $\mathbf{Sh}(\mathscr{C}, T)$. Then $i \circ f$ is monomorphism in the category of presheaves since i preserves limits. $i \circ f$ is an equalizer. Since $\#$ preserves equalizers, $\# \circ i \circ f = f$ is an equalizer, that is, f is a regular monomorphism.

Now assume $g : F \to G$ is an epimorphism in $\mathbf{Sh}(\mathscr{C}, T)$. The inclusion i does not preserves epimorphisms so we cannot assume $i \circ g$ is an epi. But for any object $C \in \mathscr{C}$, let $\mathrm{Im}(g(C))$ be the image of $g(C)$ in $\mathbf{Set}$ and denote $p(C) : F(C) \to \mathrm{Im}(g(C))$ the surjection and $m(C) : \mathrm{Im}(g(C)) \to G(C)$ the inclusion. In $\mathbf{Set}$ any epimorphism is regular and in particular is strong. We get, by this remark and the fact that g is natural transformation, for any arrow $h : C' \to C$ in $\mathscr{C}$ a unique arrow $\mathrm{Im}(h) : \mathrm{Im}(g(C)) \to \mathrm{Im}(g(C'))$ making the diagram commutative:

$$\begin{tikzcd} F(C) \arrow[r, two heads, "p(C)"] \arrow[d, "p(C') \circ F(h)"'] & \mathrm{Im}(g(C)) \arrow[d, "G(h) \circ m(C)"] \arrow[dl, dashed] \\ \mathrm{Im}(g(C')) \arrow[r, hook, "m(C')"'] & G(C') \end{tikzcd}$$

This construction defines $\mathrm{Im}(g(-))$ as a presheaf in $\mathbf{Pr}(\mathscr{C})$ and $p = \{p(C)\}_{C \in \mathscr{C}}$ and $m = \{m(C)\}_{C \in \mathscr{C}}$ as natural transformations in $\mathbf{Pr}(\mathscr{C})$. In particular p is an epimorphism, m is a monomorphism and $g = m \circ p$. Let (u, v) be the kernel pair of g in $\mathbf{Sh}(\mathscr{C}, T)$. Since i preserves limits, $(i \circ u, i \circ v)$ is the kernel pair of $i \circ g$ in $\mathbf{Pr}(\mathscr{C})$. Since m is a monomorphism, $(i \circ u, i \circ v)$ is also the kernel pair of p. Thus p is the coequalizer of $(i \circ u, i \circ v)$. Applying $\#$ we get that by preservation of colimits, $\#p$ is the coequalizer of (u, v). Since $g = \#m \circ \#p$ and g is an epi, so is $\#m$. But $\#m$ is also a regular monomorphism, as we proved earlier. Thus since $\#m$ is an epi and a regular mono, it is an isomorphism. Hence $g = \#m\#p$, is a coequalizer because $\#p$ is, thus a regular epimorphism. ☐

Corollary 1.1.12. *In a Grothendieck topos every morphism that is both a monomorphism and an epimorphism is an isomorphism.*

Proof. This directly follows from 1.1.11, since any morphism that is both a regular monomorphism and an epimorphism (respectively a monomorphism and a regular epimorphism) is an isomorphism. ☐

We assume our readers are already familiar with the notion of regular categories and the property of image factorization in such categories. Just to fix the definition, we shall recall what is a regular category:

Definition 1.1.13. A *regular* category is a category $\mathscr{C}$ where:

- Every arrow in $\mathscr{C}$ has a kernel pair.
- Every kernel pair has a coequalizer.
- The pullback of any regular epimorphism along any morphism exists and regular epimorphisms are pullback stable.

Proposition 1.1.14. *Any Grothendieck topos is a regular category.*

Proof. By 1.1.10 we already know that any morphism has a kernel pair and that any kernel pair has a coequalizer. What remains is to prove that the regular epimorphisms are pullback stable. By 1.1.11, since any epimorphism is a regular epimorphism, what we really need to check is that every epimorphism is pullback stable. Let $\mathscr{C}$ be a small category. Since **Set** is regular and limits and colimits are computed pointwise in the category of presheaves $\mathbf{Pr}(\mathscr{C})$, $\mathbf{Pr}(\mathscr{C})$ is also regular.

Let be a site $(\mathscr{C}, T)$, an epimorphism $g : F \to G$ and a morphism $f : H \to G$ in the category of sheaves $\mathbf{Sh}(\mathscr{C}, T)$). Take the pullback of g along f in $\mathbf{Sh}(\mathscr{C}, T)$. Now in $\mathbf{Pr}(\mathscr{C})$, since i preserves limits, we get a pullback of $i \circ g$ along $i \circ f$. In the proof of 1.1.11 we had an image factorization $i \circ g = m \circ p$. Pulling back along m and p gives us the pullback of $i \circ g$ along $i \circ f$

$$\begin{tikzcd}
L \arrow[r, two heads, "q"'] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & I' \arrow[r, hook, "h"'] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & H \arrow[d, "f"'] \\
F \arrow[r, two heads, "p"] & I \arrow[r, hook, "m"] & G
\end{tikzcd}$$

Since $\mathbf{Pr}(\mathscr{C})$ is regular and p is an epimorphism, so it q. Since m is a monomorphism, so is h. Since we know from the proof of 1.1.11 that $\#m$ is an iso, and that $\#$ preserves pullback, we get that $\#h$ is an isomorphism too. By our construction, $h \circ q$ is a morphism in $\mathbf{Sh}(\mathscr{C})$, thus $h \circ q = \#(h \circ q) = \#h \circ \#q$. Since $\#$ preserves the epimorphisms, $\#q$ is an epi and thus so is $\#h \circ \#q$, and therefore so is $h \circ q$. $\square$

Finally we give the following characterisation of Grothendieck topologies due to Giraud, a reference of this fondamental result of Grothendieck topoi theory can be found in [3, Theorem 3.6.1]. We will not use this characterisation, but it felt important to mention it since it is a very important result.

Theorem 1.1.15 (Giraud)**.** *A category $\mathscr{E}$ is a Grothendieck topos if and only if the following conditions hold:*

1. *$\mathscr{E}$ has a set of generators.*
2. *$\mathscr{E}$ has all finite limits.*
3. *$\mathscr{E}$ has all coproducts and they are disjoint and universal.*
4. *Every relation in $\mathscr{E}$ has a universal coequalizer.*
5. *Every equivalence relation in $\mathscr{E}$ is effective.*
6. *Every epimorphism in $\mathscr{E}$ is regular.*

Definition 1.1.16. A *geometric morphism* between two finitely complete categories $f = (f_*, f^*) : \mathscr{A} \to \mathscr{B}$ is a pair of functors $f_* : \mathscr{A} \to \mathscr{B}$ and $f^* : \mathscr{B} \to \mathscr{A}$ such that f^* is left adjoint to f_* and f^* is furthermore left exact. The composition of two geometric morphisms $(f_*, f^*) : \mathscr{A} \to \mathscr{B}, (g_*, g^*) : \mathscr{B} \to \mathscr{C}$ is $(g_* f_*, f^* g^*) : \mathscr{A} \to \mathscr{C}$.

There is a notion of morphisms between two Grothendieck topoi, the geometric morphisms. We shall list some properties involving geometric morphisms. Proofs of those properties are available in [3, Section 4.1].

Example 1.1.17. The adjunction $(i, \#) : \mathbf{Sh}(\mathscr{C}, T) \to \mathbf{Pr}(\mathscr{C})$ is a geometric morphism.

Lemma 1.1.18. *Let $\mathscr{E}, \mathscr{F}$ be Grothendieck topoi. There is a bijection between isomorphism classes of*

1. *geometric morphisms form $f : \mathscr{E} \to \mathscr{F}$,*
2. *functors $f^* : \mathscr{F} \to \mathscr{E}$ preserving finite limits and small colimits.*

Lemma 1.1.19. *Let $\mathscr{E}, \mathscr{F}$ be two Grothendieck topoi, and two geometric morphisms $(f_*, f^*) : \mathscr{E} \to \mathscr{F}$ and $(g_*, g^*) : \mathscr{E} \to \mathscr{F}$. The collection of natural transformations $\alpha : f^* \Rightarrow g^*$ is a set.*

Hence by the last two lemmas, $\mathbf{Geom}(\mathscr{E},\mathscr{F})$ the category of geometric morphisms between two Grothendieck topoi $\mathscr{E}$ and $\mathscr{F}$ with morphisms $\alpha : (f_*, f^*) \to (g_*, g^*)$ the natural transformations $\alpha : f^* \Rightarrow g^*$, is a well defined category.

The next lemma is a small remark to show that the data of the natural transformations $\alpha : f^* \Rightarrow g^*$ is the same as the data of the natural transformations $\beta : g_* \Rightarrow f_*$.

Lemma 1.1.20. *For two Grothendieck topoi $\mathscr{E}$ and $\mathscr{F}$, $\mathbf{Geom}(\mathscr{E},\mathscr{F})$ is isomorphic to the category whose objects are the geometric morphisms from $\mathscr{E}$ to $\mathscr{F}$ and the morphisms $\beta : (f_*, f^*) \to (g_*, g^*)$ are the natural transformations $\beta : g_* \Rightarrow f_*$.*

Proposition 1.1.21. *For any Grothendieck topos $\mathscr{F}$ there exists a unique geometric morphism $\Gamma : \mathscr{F} \to \mathbf{Set}$.*

Definition 1.1.22. A *point* of a Grothendieck topos $\mathscr{F}$ is a geometric morphism $(f_*, f^*) : \mathbf{Set} \to \mathscr{F}$.

For the category of points we use the notation $\mathbf{Pt}(\mathscr{F}) = \mathbf{Geom}(\mathbf{Set}, \mathscr{F})$.

1.2 Classifying topos

Something remarkable about Grothendieck topoi is that a lot of mathematical theories can be studied inside a Grothendieck topos. We don't expect the reader to completely understand what is claimed here since we will only discuss this seriously in chapter 3. Although there is a need to be clearer with what we mean by "a lot of mathematical theories can be studied inside a Grothendieck topos", let us first give some explanations and intuitions on what is a theory inside a category :

What is a group ? We know a group is a set G together with an operation $+ : G\times G \to G$ and a neutral element $e \in G$ such that $+$ is associative and for any element $g \in G$, $g + e = g = e + g$ and there exists an inverse $-g$, $g - g = e = -g + g$. This definition uses a lot of the language of set theory, for instance the notion of element of a set $g \in G$. A categorical approach of the same definition would be a set G together with an operation $+ : G \times G \to G$, an inverse $- : G \to G$ and a neutral element $e : \{*\} \to G$ such that the following diagrams commute

$$\begin{tikzcd} G \arrow[r, "\Delta"] \arrow[d] & G \times G \arrow[r, "\mathrm{Id}\times -"] & G \times G \arrow[d, "+"] \\ \{*\} \arrow[rr, "e"] & & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\Delta"] \arrow[d] & G \times G \arrow[r, "-\times\mathrm{Id}"] & G \times G \arrow[d, "+"] \\ \{*\} \arrow[rr, "e"] & & G \end{tikzcd}$$

for the inverse

$$\begin{tikzcd} G\times G\times G \arrow[r, "+\times \mathrm{Id}"] \arrow[d, "\mathrm{Id}\times +"'] & G\times G \arrow[d, "+"] \\ G\times G \arrow[r, "+"'] & G \end{tikzcd}$$

for the associativity and

$$\begin{tikzcd} \{*\}\times G \arrow[r, "+\circ(e\times \mathrm{Id}_G)"] \arrow[d, "\pi_2"'] & G \arrow[dl, "\mathrm{Id}"] \\ G & \end{tikzcd} \qquad \begin{tikzcd} G\times \{*\} \arrow[r, "+\circ(\mathrm{Id}_G\times e)"] \arrow[d, "\pi_1"'] & G \arrow[dl, "\mathrm{Id}"] \\ G & \end{tikzcd}$$

for the neutral element. With this approach it is easy to understand that a group can be defined internally to any category with products and terminal object, where G and $+,-,e$ are object and morphisms of this category. A model of the theory of group in a Grothendieck topos $\mathscr{F}$ is then an object G in $\mathscr{F}$ with the right morphisms satisfying the diagrams above. Similarly, a homomorphism of groups in $\mathscr{F}$ is then just an arrow between two group objects $G \to G'$ in $\mathscr{F}$ such that the following diagrams commute

$$\begin{tikzcd} G\times G \arrow[r, "+"] \arrow[d, "(f,f)"'] & G \arrow[d, "f"'] \\ G'\times G' \arrow[r, "+'"] & G' \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "-"] \arrow[d, "f"'] & G \arrow[d, "f"'] \\ G' \arrow[r, "-'"] & G' \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "e"] \arrow[dr, "e'"'] & G \arrow[d, "f"'] \\ & G' \end{tikzcd}$$

with 1 the terminal object in $\mathscr{F}$. Thus from a theory $\mathcal{T}$ in a Grothendieck topos $\mathscr{F}$, one can extract $\mathbf{Mod}_{\mathscr{F}}(\mathcal{T})$ the category of models of $\mathcal{T}$ in $\mathscr{F}$. This is a full subcategory of $\mathscr{F}$. Sometimes, for a given theory, a Grothendieck topos allow for a better understanding of the models of the theory in any Grothendieck topos.

Definition 1.2.1. Let be a mathematical theory $\mathcal{T}$. A classifying topos for this theory is the data of :

1. a Grothendieck topos $\mathscr{E}[\mathcal{T}]$,
2. a model $M \in \mathbf{Mod}_{\mathscr{E}[\mathcal{T}]}(\mathcal{T})$ of the theory $\mathcal{T}$ in $\mathscr{E}[\mathcal{T}]$,

such that, for every Grothendieck topos $\mathscr{F}$, there is an equivalence of categories

$$\mathbf{Geom}(\mathscr{F}, \mathscr{E}[\mathcal{T}]) \to \mathbf{Mod}_{\mathscr{F}}(\mathcal{T})$$

obtained by sending the model M to $\mathscr{F}$ by applying the left adjoint part f^* of geometric morphisms, i.e the equivalence is $(f_*, f^*) \mapsto f^*(M)$

If the pair exists, $\mathscr{E}[\mathcal{T}]$ is called a classifying topos of the theory $\mathcal{T}$, and M the generic model of the theory.

Before giving examples, just note that by ring we mean commutative ring with unit element. We will explore some of the following examples, but they are detailed in [6, Chapter VIII].

Examples 1.2.2.

1. The classifying topos for the theory of object is the category of presheaves over $\mathbf{FinSet}^{op}$ the opposite category of finite sets. The universal model is $\mathbf{Hom}_{\mathbf{FinSet}^{op}}(-,\{*\})$ the Yoneda embedding evaluated on the finite set $\{*\}$.

2. The classifying topos for the theory of rings is the category of presheaves over $\mathbf{fp\text{-}rings}^{op}$ the opposite category of finitely presented rings. The universal model is $\mathbf{Hom}_{\mathbf{fp\text{-}rings}^{op}}(-,\mathbb{Z}[X])$ the Yoneda embedding evaluated on the polynomial ring over integers $\mathbb{Z}[X]$.

3. The classifying topos for the theory of local rings is the category of sheaves over the site $(\mathbf{fp\text{-}rings}^{op}, J)$ where J is the Zariski covering. The universal model is the structure sheaf $\mathcal{O}$.

4. The classifying topos for the theory of linear orders is the category of presheaves over Δ the category of simplicial sets. The universal model is $\mathbf{Hom}_{\Delta}(-,\{0,1\})$ the Yoneda embedding evaluated on the ordered set $\{0,1\}$.

We need some work before proving those examples. The next chapter will be consecrated to the construction of classifying topoi. In particular we will fully prove in 2.1.7 the first example and then mention the two next examples briefly. In the end of chapter 3, we will address some remarks concerning the last example of local rings in 3.5.2. We will not prove the example, but we will see how the proof in the literature can be related to certain tools yielded through out this thesis.

Chapter 2

Sketching a theory

In this chapter we will introduce the notion of sketches. Sketches are a nice way to deal with theories while working with a categorical language. As we have seen in the end of last chapter, a model of a theory such as group theory can be interpreted as suitable diagrams involving limits and colimits. A sketch will be the data of such diagrams with the data of the right elements in the diagrams that need to be associated to limits or colimits. A model of a sketch in a category will be a functor sending the elements associated to limits and colimits, to limit and colimit cones. We will first prove that if the models of a sketch in a Grothendieck topos $\mathscr{F}$ are precisely the left exact functors from a fixed finitely complete category $\mathscr{C}$, then $\mathbf{Pr}(\mathscr{C})$ the category of presheaves on $\mathscr{C}$ is a classifying topos for the sketch. This will gives us our first examples of classifying topos. We will then move on to proving the more broader theorems that any Grothendieck topos is the classifying topos of some theory defined by a sketch, and any geometric sketch (sketch whose data carries a finite amount of cones and a small amount of cocones) admits a classifying topos.

2.1 Sketch

Definition 2.1.1. A *sketch* is a triple $\mathcal{S} = (\mathscr{T}, \mathcal{P}, \mathcal{I})$ such that

- $\mathscr{T}$ is a small category.
- $\mathcal{P}$ is a set of cones in $\mathscr{T}$.
- $\mathcal{I}$ is a set of cocones in $\mathscr{T}$.

A *geometric* sketch is a sketch $(\mathscr{T}, \mathcal{P}, \mathcal{I})$ where $\mathcal{P}$ is a set of finite cones.

Definition 2.1.2. Let $\mathcal{S} = (\mathscr{T}, \mathcal{P}, \mathcal{I})$ be a sketch. A model of $\mathcal{S}$ on a category $\mathscr{C}$ is a functor $M : \mathscr{T} \to \mathscr{C}$ such that :

- if $(T, p_D)_{D \in \mathscr{D}} \in \mathcal{P}$ is a cone on a functor $F : \mathscr{D} \to \mathscr{T}$, then $(M(T), M(p_D))_{D \in \mathscr{D}}$ is the limit of $M \circ F$,

- if $(T, s_D)_{D \in \mathscr{D}} \in \mathcal{I}$ a cocone on a functor $F : \mathscr{D} \to \mathscr{T}$, then $(M(T), M(s_D))_{D \in \mathscr{D}}$ is the limit of $M \circ F$.

$\mathbf{Mod}_{\mathscr{C}}(\mathcal{S})$ is the category of models of $\mathcal{S}$ on $\mathscr{C}$, with natural transformation of models as morphisms.

Examples 2.1.3. 1. Let $\mathcal{S} = (1, \mathcal{P}, \mathcal{I})$ be the sketch of the theory of objects, where 1 is the category with one unique object and the identity as sole morphism, $\mathcal{P}$ and $\mathcal{I}$ are empty set of cones and cocones. Then there is exactly one model in any category $\mathscr{C}$ for each object of $\mathscr{C}$, and the morphisms of models are just the morphisms of objects in $\mathscr{C}$. Thus $\mathbf{Mod}_{\mathscr{C}}(\mathcal{S}) \cong \mathscr{C}$.

2. Let us construct the sketch for the theory of unital magmas. A unital magma is a set M with a binary operation $+$ and a left and right unit e. The sketch of the theory of unital magmas is the data of:

   - The category

$$\begin{tikzcd}
 & A \arrow[dl, "p_1"'] \arrow[d, "p_2"] \arrow[dr, "{e,-}"] & \\
C \arrow[r, "e"] & B & D \arrow[l, shift right=2, "\pi_1"'] \arrow[l, "+" description] \arrow[l, shift left=2, "\pi_2"] \\
 & E \arrow[ul, "q_2"] \arrow[u, "q_1"'] \arrow[ur, "{-,e}"'] &
\end{tikzcd}$$

   where everything commutes.

   - The set of cones is the collection of the four following cones

$$C \qquad \begin{tikzcd} & A \arrow[dl, "p_1"'] \arrow[d, "p_2"] \\ C & B \end{tikzcd} \qquad \begin{tikzcd} C & B \\ & E \arrow[ul, "q_2"] \arrow[u, "q_1"'] \end{tikzcd} \qquad \begin{tikzcd} B & D \arrow[l, shift right=1, "\pi_1"'] \arrow[l, shift left=1, "\pi_2"] \end{tikzcd}$$

   - The set of cocones is the empty set

   A model of this sketch is then a functor that maps B to a unital magma object M, where C is sent to the initial terminal object 1, A to the product $1 \times M$, D to the product $M \times M$ and E to the product $M \times 1$. Moreover $e, -$ is send to $e \times \mathrm{Id}_M$ and similarly $-, e$ to $\mathrm{Id}_M \times e$.

Lemma 2.1.4. *Let $\mathscr{C}$ be a finitely complete small category. The category of left exact functors from $\mathscr{C}$ to* **Set** *is equivalent to the category of points of the category of presheaves on $\mathscr{C}$. This equivalence is obtained by the following functor :*

$$\pi : \mathbf{Lex}(\mathscr{C}, \mathbf{Set}) \to \mathbf{Pt}(\mathbf{Pr}(\mathscr{C})),$$

mapping a left exact functor $F : \mathscr{C} \to \mathbf{Set}$ to the geometric morphism associated by 1.1.18 to the Kan extension of F along the Yoneda embedding $\pi^*(F) := \mathbf{Lan}_Y F : \mathbf{Pr}(\mathscr{C}) \to \mathbf{Set}$.

Proof. Let be $\pi^*(F) = \mathbf{Lan}_Y F$ for any $F \in \mathbf{Lex}(\mathscr{C}, \mathbf{Set})$. By C.0.8 π^* is left exact. Now let be a presheaf $G \in \mathbf{Pr}(\mathscr{C})$. By C.0.9 and C.0.5

$$\pi^*(F)(G) = \mathbf{Lan}_Y F(G) = \mathbf{Lan}_{Y^*} G(F) = \mathrm{colim}(G \circ \phi_F),$$

where $\phi_F : \mathbf{Elts}(F) \to \mathscr{C}$ is the forgetful functor.

Thus let $\mathrm{colim}_i G_i$ be a colimit in $\mathbf{Pr}(\mathscr{C})$, since colimits in $\mathbf{Pr}(\mathscr{C})$ are computed pointwise and by the interchange of colimits :

$$\begin{aligned} \pi^*(F)(\mathrm{colim}_i G_i) &= \mathrm{colim}\left((\mathrm{colim}_i G_i) \circ \phi_F\right) \\ &= \mathrm{colim}_i \left(\mathrm{colim}(G_i \circ \phi_F)\right) \\ &= \mathrm{colim}_i \left(\pi^*(F)(G_i)\right). \end{aligned}$$

Hence $\pi^*(F)$ preserves finite limits and small colimits. In virtue of 1.1.18, π maps a left exact functor F to the geometric morphism associated to $\pi^*(F)$. Now if $\beta : F \Rightarrow F'$ is a morphism in $\mathbf{Lex}(\mathscr{C}, \mathbf{Set})$ and we denote $\alpha : F \Rightarrow \mathbf{Lan}_Y F \circ Y$ and $\alpha' : F' \Rightarrow \mathbf{Lan}_Y F' \circ Y$ the two canonical natural transformations, then by the definition of Kan extension, since $\alpha' \circ \beta : F \Rightarrow \mathbf{Lan}_Y F' \circ Y$ is a natural transformation, we get a unique natural transformation $\gamma_\beta : \mathbf{Lan}_Y F \Rightarrow \mathbf{Lan}_Y F'$ such that $(\gamma_\beta \bullet \mathrm{Id}_Y) \circ \alpha = \alpha' \circ \beta$. We set $\pi(\beta) = \gamma_\beta$. It is easy to verify π is a well defined functor.

Now let us prove π induces an equivalence of categories. Since Y is fully faithful (A.0.3), the canonical natural transformation α and α' are isomorphism by C.0.6. Thus if given two natural transformations $\beta, \beta' : F \Rightarrow F'$ such that $\pi(\beta) = \pi(\beta')$, then by definition of the Kan extension,

$$\alpha' \circ \beta = (\pi(\beta) \circ \mathrm{Id}_Y) \circ \alpha = (\pi(\beta') \circ \mathrm{Id}_Y) \circ \alpha = \alpha' \circ \beta'.$$

Thus since α' is an isomorphism, $\pi(\beta) = \pi(\beta') \iff \beta = \beta'$. This shows that π is faithful.

Now assume we have a natural transformation $\gamma : \pi^*(F) \Rightarrow \pi^*(F')$ and let us construct a natural transformation $\beta : F \Rightarrow F'$. Define $\beta : F \overset{\alpha}{\Rightarrow} \pi^*(F) \circ Y \xRightarrow{\gamma \bullet \mathrm{Id}_Y} \pi^*(F') \circ Y \xRightarrow{\alpha'^{-1}} F'$. Then it is easy to check that $\pi(\beta) = \gamma$ since $(\gamma \bullet \mathrm{Id}_Y) \circ \alpha = \alpha' \circ \beta$. Thus π is full.

Finally let f be a geometric morphism in $\mathbf{Pt}(\mathbf{Pr}(\mathscr{C}))$. Since both f^* and Y are finite limit preserving, $f^* \circ Y$ is in $\mathbf{Lex}(\mathscr{C}, \mathbf{Set})$. Let $C \in \mathscr{C}$ then by C.0.5 and by previous considerations

$$\pi^*(f^* \circ Y)(\mathbf{Hom}_{\mathscr{C}}(-, C)) = f^* \circ Y(C) = f^*(\mathbf{Hom}_{\mathscr{C}}(-, C)).$$

Thus f^* and $\pi^*(f^* \circ Y)$ coincide on the representable functors. Since they both preserve small colimits and by B.0.3 any presheaf $G \in \mathbf{Pr}(\mathscr{C})$ is a small colimit of representable functors, f^* and $\pi^*(f^* \circ Y)$ coincide. Hence π is essentially surjective. Therefore π is an equivalence of categories.

□

Theorem 2.1.5. *Let $\mathscr{C}$ be a finitely complete small category. Then the theory whose category of models is equivalent to the category of left exact functors on $\mathscr{C}$ admits as classifying topos $\mathbf{Pr}(\mathscr{C})$ the topos of presheaves on $\mathscr{C}$, with the Yoneda embedding as the generic model.*

Proof. By 2.1.4 we have constructed an equivalence of categories

$$\pi : \mathbf{Lex}(\mathscr{C}, \mathbf{Set}) \to \mathbf{Geom}(\mathbf{Set}, \mathbf{Pr}(\mathscr{C}))$$

where a left exact functor $F : \mathscr{C} \to \mathbf{Set}$ is mapped to a geometric morphism $(\pi(F)_*, \pi(F)^*)$ with $\pi(F)^* = \mathbf{Lan}_Y F$ where $Y : \mathscr{C} \to \mathbf{Pr}(\mathscr{C})$, $C \mapsto \mathbf{Hom}(-, C)$ is the Yoneda embedding. The functor

$$\mathbf{Geom}(\mathbf{Set}, \mathbf{Pr}(\mathscr{C})) \to \mathbf{Lex}(\mathscr{C}, \mathbf{Set})$$

mapping a geometric morphism (f_*, f^*) to $f^* \circ Y$ is a quasi-inverse of π.

To prove $\mathbf{Pr}(\mathscr{C})$ is a classifying topos and Y is the generic model we need to prove that for any Grothendieck topos $\mathscr{F}$, $\mathbf{Geom}(\mathscr{F}, \mathbf{Pr}(\mathscr{C}))$ and $\mathbf{Mod}_{\mathscr{F}}(\mathscr{C})$ are equivalent and the models are obtained by precomposing the f^* part of the geometric morphisms with Y. So far it is proven in the specific case of $\mathscr{F}$ begin $\mathbf{Set}$. The next step is to reach the case $\mathscr{F}$ is of the form $\mathbf{Pr}(\mathscr{D})$ for any $\mathscr{D}$ small category.

Now assume $\mathscr{D}$ is a small category. Since Y is left exact and the left adjoint part of a geometric morphism is too, it is easy to see that

$$\mathbf{Geom}(\mathbf{Pr}(\mathscr{D}), \mathbf{Pr}(\mathscr{C})) \to \mathbf{Lex}(\mathscr{C}, \mathbf{Pr}(\mathscr{D})),\ (f_*, f^*) \mapsto f^* \circ Y$$

is well defined and uses Y as a generic model. We are going to show that this functor is an equivalence. Let $F : \mathscr{C} \to \mathbf{Pr}(\mathscr{D})$ be a left exact functor. Since $\mathbf{Pr}(\mathscr{D})$ is cocomplete, there exist a Kan extension $\mathbf{Lan}_Y F$. Let's prove $\mathbf{Lan}_Y F$ is again the left adjoint part of a geometric morphism in $\mathbf{Geom}(\mathbf{Pr}(\mathscr{D}), \mathbf{Pr}(\mathscr{C}))$. Since finite limits and small colimits are computed pointwise in $\mathbf{Pr}(\mathscr{D})$, then we need to check the preservation of finite limits and small colimits pointwise, i.e $\mathbf{Lan}_Y F(-)(D) = \mathbf{ev}_D \circ \mathbf{Lan}_Y F$ preserves them. If $\mathbf{ev}_D$

has a right adjoint, then $\mathbf{ev}_D \circ \mathbf{Lan}_Y F = \mathbf{Lan}_Y(\mathbf{ev}_D \circ F)$ by [insert] and we have proven $\mathbf{Lan}_Y$ is preserving finite limits and small colimits in $\mathbf{Set}$. Hence all we have to do is to prove $\mathbf{ev}_D$ has a right adjoint. Consider the functor

$$\mathbf{Set} \to \mathbf{Pr}(\mathscr{D}), \quad X \mapsto \mathbf{Hom}(\mathbf{Hom}_{\mathscr{D}}(D, -), X).$$

For any $G \in \mathbf{Pr}(\mathscr{D})$, G can be seen as a colimit of representable functors $\text{colim} G = \text{colim } \mathscr{D}(-, D_i)$ by B.0.3, moreover contravariant representable functors (presheaves) transform colimits into limits by B.0.2. Hence, with that in mind and with the help of the Yoneda lemma we get the following computation :

$$\begin{aligned}
\mathbf{Pr}(\mathscr{D})(G, \mathbf{Hom}(\mathscr{D}(D, -), X)) &\cong \mathbf{Pr}(\mathscr{D})(\text{colim } \mathscr{D}(-, D_i), \mathbf{Hom}(\mathscr{D}(D, -), X)) \\
&\cong \lim \mathbf{Pr}(\mathscr{D})(\mathscr{D}(-, D_i), \mathbf{Hom}(\mathscr{D}(D, -), X)) \\
&\cong \lim \mathbf{Hom}(\mathscr{D}(D, D_i), X) \cong \mathbf{Hom}(\text{colim } \mathscr{D}(D, D_i), X) \\
&\cong \mathbf{Hom}(G(D), X) \cong \mathbf{Hom}(\mathbf{ev}_D(G), X).
\end{aligned}$$

Yoneda lemma is used to go from the second line to the third. Therefore $\mathbf{ev}_D$ has a right adjoint, hence $\mathbf{Lan}_Y F$ preserves finite limits and small colimits. Hence $\mathbf{Lan}_Y F$ is the left part of a geometric morphism. The following functor is well defined.

$$\mathbf{Lex}(\mathscr{C}, \mathbf{Pr}(\mathscr{D})) \to \mathbf{Geom}(\mathbf{Pr}(\mathscr{D}), \mathbf{Pr}(\mathscr{C})), \quad F \mapsto \mathbf{Lan}_Y F.$$

To prove the equivalence, one way is easy. By A.0.3, since Y is fully faithful, then $(\mathbf{Lan}_Y F) \circ Y \cong F$. In the other way, we need to show $\mathbf{Lan}_Y(f^* \circ Y) \cong f^*$. Exactly by the previous argument, $\mathbf{Lan}_Y(f^* \circ Y) \circ Y \cong f^* \circ Y$, they coincide on representable functors. Since f^* and $\mathbf{Lan}_Y(f^* \circ Y)$ both preserves small colimits, and that any presheaf of $\mathbf{Pr}(\mathscr{C})$ can be seen as a small colimit of representable functors, then $(f^* \circ Y) \cong f^*$.

We have extended our equivalence for $\mathscr{F} = \mathbf{Set}$ to $\mathscr{F} = \mathbf{Pr}(\mathscr{D})$. Now let finally extend it to any Grothendieck topos. The study of Grothendieck topos reduces to the study of topos of sheaves $\mathbf{Sh}(\mathscr{D}, \mathcal{T})$ of a site $(\mathscr{D}, \mathcal{T})$. For the same argument as above, the following functor is well defined :

$$\mathbf{Geom}(\mathbf{Sh}(\mathscr{D}, \mathcal{T}), \mathbf{Pr}(\mathscr{C})) \to \mathbf{Lex}(\mathscr{C}, \mathbf{Sh}(\mathscr{D}, \mathcal{T})), \ (f_*, f^*) \mapsto f^* \circ Y,$$

and again the quasi-inverse of the equivalence is given by

$$\mathbf{Lex}(\mathscr{C}, \mathbf{Sh}(\mathscr{D}, \mathcal{T})) \to \mathbf{Geom}(\mathbf{Sh}(\mathscr{D}, \mathcal{T}), \mathbf{Pr}(\mathscr{C})), \quad F \mapsto \mathbf{Lan}_Y F.$$

First we prove $\mathbf{Lan}_Y F$ is the left part of a geometric morphism in $\mathbf{Geom}(\mathbf{Sh}(\mathscr{D}, \mathcal{T}), \mathbf{Pr}(\mathscr{C}))$. Let $F : \mathscr{C} \to \mathbf{Sh}(\mathscr{D}, \mathcal{T})$ a left exact functor, and denote $i : \mathbf{Sh}(\mathscr{D}, \mathscr{C}) \to \mathbf{Pr}(\mathscr{D})$ for the inclusion and $\#$ for the adjoint sheaf functor. Since $\#$ is left adjoint we have by C.0.7 that

$$\# \circ \mathbf{Lan}_Y(i \circ F) \cong \mathbf{Lan}_Y(\# \circ i \circ F) \cong \mathbf{Lan}_Y F.$$

i is left exact, hence $i \circ F$ is too and $\mathbf{Lan}_Y(i \circ F)$ preserves finite limits and small colimits by previous work. But $\#$ also preserves finite limits and small colimits, hence $\# \circ \mathbf{Lan}_Y(i \circ F) \cong \mathbf{Lan}_Y F$ too. That means $\mathbf{Lan}_Y F$ is the left part of a geometric morphism. In an identical way as for the topos of presheaves, one can prove those two functors to form an equivalence. $\square$

Lemma 2.1.6. *For any category with finite colimits $\mathscr{D}$, $\mathbf{Rex}(\mathbf{FinSet}, \mathscr{D})$ and $\mathscr{D}$ are equivalent categories.*

Proof. First, notice that in **FinSet** any set S can be written as the finite coproduct $\amalg_{s\in S}\{*\}$. Let $F : \mathbf{FinSet} \to \mathscr{D}$ be a functor preserving finite coproducts and let us a prove it preserves finite colimits. Consider in **FinSet** the coequalizer

$$S \rightrightarrows T \longrightarrow R$$

Since each set is a finite coproduct of $\{*\}$ and F preserves finite coproducts, by writting $F(\{*\}) = D$, we get the following diagram in $\mathscr{D}$:

$$\amalg_{s\in S} D \rightrightarrows \amalg_{t\in T} D \longrightarrow \amalg_{r\in R} D$$

This diagram is a coequalizer if and only if for any $X \in \mathscr{D}$

$$\mathbf{Hom}(R, \mathbf{Hom}(D, X)) \longrightarrow \mathbf{Hom}(T, \mathbf{Hom}(D, X)) \rightrightarrows \mathbf{Hom}(S, \mathbf{Hom}(D, X))$$

is an equalizer in **Set**. By B.0.2 it is an equalizer. Hence F preserves coequalizers and thus all finite colimits.

Now consider the functor

$$\mathrm{ev} : \mathbf{Rex}(\mathbf{FinSet}, \mathscr{D}) \to D,$$

mapping a right exact functor $F : \mathbf{FinSet} \to \mathscr{D}$ to $F(\{*\}) \in \mathscr{D}$ and mapping any natural transformation $\eta : F \Rightarrow G$ to the component $\eta(\{*\}) : F(\{*\}) \to G(\{*\})$. Consider a second functor

$$\phi : \mathscr{D} \to \mathbf{Rex}(\mathbf{FinSet}, \mathscr{D})$$

who maps any object $D \in \mathscr{D}$ to the functor sending any finite set S to the coproduct $\phi(D)(S) = \amalg_{s\in S} D$ and any map $f : S \to T$ in **FinSet** to the unique arrow $\phi(D)(f) : \amalg_{s\in S} D \to \amalg_{t\in T} D$ yielded by universal property of the coproduct. By definition $\phi(D)$ preserves finite coproducts and thus it is a right exact functor. For any morphism $D \to D'$ in $\mathscr{D}$, $\phi(d)$ is the natural transformation defined component wise by $\phi(d)(S) : \amalg_{s\in S} D \to \amalg_{s\in S} D'$ the unique arrow from the property of coproducts. The uniqueness makes sure this is a well defined functor.

$$
\begin{tikzcd}
\amalg_{s\in S} D \arrow[rr, dashed, "\phi(D)(f)"] & & \amalg_{s\in T} D \\
& D \arrow[ul, "i_s"] \arrow[ur, "i_{f(s)}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\amalg_{s\in S} D \arrow[r, dashed, "\phi(d)(S)"] & \amalg_{s\in S} D' \\
D \arrow[u, "i_s"] \arrow[r, "d"] & D' \arrow[u, "i_s"']
\end{tikzcd}
$$

By straight forward computation one can check that $\mathrm{ev} \circ \phi$ is isomorphic to the identity. Conversely let F be a right exact functor and S be a finite set, then

$$
\begin{aligned}
\phi \circ \mathrm{ev}(F)(S) &= \phi(F(\{*\})(S) \\
&= \phi \amalg_{s\in S} F(\{*\}) && \text{since } F \text{ preserves finite coproducts} \\
&= F(S).
\end{aligned}
$$

And similarly $\phi \circ \mathrm{ev}(\eta)(S) = \eta(S)$ for any natural transformation $\eta : F \Rightarrow G$. Hence $\phi \circ \mathrm{ev}$ is isomorphic to the identity. Thus we have an equivalence between $\mathbf{Rex}(\mathbf{FinSet}, \mathscr{D})$ and $\mathscr{D}$. □

Theorem 2.1.7. $\mathbf{Pr}(\mathbf{FinSet}^{op})$ *is a classifying topos of the theory of objects in a topos.*

Proof. By the dual of 2.1.6, for any finitely complete category $\mathscr{C}$ there is an equivalence $\mathbf{Lex}(\mathbf{FinSet}^{op}, \mathscr{C}) \to \mathscr{C}$. Then applying 2.1.5 we get for any Grothendieck topos $\mathscr{E}$,

$$\mathbf{Geom}(\mathscr{E}, \mathbf{Pr}(\mathbf{FinSet}^{op})) \cong \mathbf{Lex}(\mathbf{FinSet}^{op}, \mathscr{E}) \cong \mathscr{E}.$$

$\mathbf{Pr}(\mathbf{FinSet}^{op})$ is a classifying topos of the theory of objects in a topos. □

The examples of classifying topos for the theory of rings and the theory of linear order are similar to the one for the theory of objects. The point of the proofs are to find a suitable category such that the category of left exact functors from that category to a Grothendieck topos is equivalent to the category of models of the theory of rings or linear order. Then applying 2.1.5, one immediately get a classifying topos for those theories. Those claims are done with details in [6, Chapter VIII].

We now focus on the relationship between sketches and Grothendieck topoi. We shall show in the next theorem that any Grothendieck topos is the classifying topos of some theory determined by a sketch

Theorem 2.1.8. *Any Grothendieck topos is the classifying topos of some theory defined by a sketch.*

Proof. A Grothendieck topos $\mathscr{E}$ is equivalent to a topos of sheaves $\mathbf{Sh}(\mathscr{C}, T)$ on a site $(\mathscr{C}, T)$ with $\mathscr{C}$ finitely complete (the fact that $\mathscr{C}$ can be chosen finitely complete is a consequence of Giraud's Theorem (1.1.15), see [3, Corollary 3.6.2]. From the site $(\mathscr{C}, T)$ we create the sketch $\mathcal{S} = (\mathscr{C}, \mathcal{P}, \mathcal{I})$ where :

1. $\mathscr{C}$ is the underlying category of the site,
2. $\mathcal{P}$ is the set of all finite limit cones in $\mathscr{C}$,
3. $\mathcal{I}$ is the set of all cocones corresponding to the elements of the Grothendieck topology T, i.e for any object $C \in \mathscr{C}$ and sieve $R \in T(C)$, $R : \mathscr{C}^{op} \to \mathbf{Set}$, one can consider the diagram $\phi_R : \mathbf{Elts}(R) \to \mathscr{C}$. By definition of $\mathbf{Elts}(R)$

$$\left(C, \left\{s_{(D,d)} : \phi_R(D,d) = D \xrightarrow{d} C)\right\}_{(D,d)\in\mathbf{Elts}(R)}\right)$$

is a cocone. $\mathcal{I}$ is the set of all such cocones for all $C \in \mathscr{C}$ and $R \in T(C)$.

The goal of the proof is to show that $\mathscr{E}$ is the classifying topos of the theory modeled by the sketch $\mathcal{S}$ with generic model the functor $\# \circ Y : \mathscr{C} \to \mathbf{Sh}(\mathscr{C}, T)$ where $Y : \mathscr{C} \to \mathbf{Pr}(\mathscr{C})$ is a associated Yoneda embedding and $\# : \mathbf{Pr}(\mathscr{C}) \to \mathbf{Sh}(\mathscr{C}, T)$ is the associated sheafification functor.

We check that $\# \circ Y$ is a model of $\mathcal{S}$. Y preserves finite limits (by A.0.3) and $\#$ too (by 1.1.3). Hence, $\# \circ Y$ preserves finite limits and thus each cone of $\mathcal{P}$ is mapped to a limit cone by $\# \circ Y$. Now let us fix an object $C \in \mathscr{C}$ and a sieve $R \in T(C)$. By A.0.3 we associate to any $d \in R(D)$ for any $D \in \mathscr{C}$ a natural transformation

$$t_{(D,d)} : \mathbf{Hom}_{\mathscr{C}}(-, D) \Rightarrow R$$

defined for any $B \in \mathscr{C}$ and $f \in \mathbf{Hom}_{\mathscr{C}}(B, D)$ by $t_{(D,d)}(B)(f) = R(f)(d)$. By B.0.3 the cocone $\left(R, \left\{t_{(D,d)}\right\}_{(D,d)\in\mathbf{Elts}(R)}\right)$ is the colimit of $Y \circ \phi_R$ in $\mathbf{Pr}(\mathscr{C})$. Since by 1.1.3, $\#$ preserves colimits, the cocone $\left(\#R, \left\{\#t_{(D,d)}\right\}_{(D,d)\in\mathbf{Elts}(R)}\right)$ is a colimit in $\mathbf{Sh}(\mathscr{C}, T)$. By 1.1.7, the monomorphism $r : R \hookrightarrow \mathbf{Hom}_{\mathscr{C}}(-, C)$ is mapped to an isomorphism $\#r : \#R \cong \#\mathbf{Hom}_{\mathscr{C}}(-, C)$. Thus the cone

$$\left(\#\mathbf{Hom}_{\mathscr{C}}(-, C), \left\{\#r \circ \#t_{(D,d)}\right\}_{(D,d)\in\mathbf{Elts}(R)}\right)$$

is also a colimit in $\mathbf{Sh}(\mathscr{C}, T)$. Now let us compute for any $B \in \mathscr{C}$ and $f : B \to D$:

$$\begin{aligned}
(r \circ t_{(D,d)})(B)(f) &= r(B)(R(f)(d)) \\
&= (r(B) \circ R(f))(d) \\
&= (\mathbf{Hom}_{\mathscr{C}}(f, C) \circ r(D))(d) && \text{by naturality of } r \\
&= \mathbf{Hom}_{\mathscr{C}}(f, C)(d) && r(D) \text{ is the inclusion in } \mathbf{Hom}_{\mathscr{C}}(D, C) \\
&= d \circ f = Y(d)(B)(f) = Y(s_{(D,d)})(B)(f).
\end{aligned}$$

thus $r \circ t_{(D,d)} = Y(d) = Y(s_{(D,d)})$. Hence

$$\left(\# \circ Y(C), \left\{\# \circ Y(s_{(D,d)})\right\}_{(D,d)\in\mathbf{Elts}(R)}\right)$$

is a colimit in $\mathbf{Sh}(\mathscr{C}, T)$. $\# \circ Y$ maps the cocones of $\mathcal{I}$ to colimits. Therefore $\# \circ Y$ is a model of $\mathcal{S}$.

Let $\mathscr{F}$ be a Grothendieck topos and $(g_*, g^*) \in \mathbf{Geom}(\mathscr{F}, \mathbf{Sh}(\mathscr{C}, T))$. Since g^* preserves finite limits and small colimits, $g^* \circ \# \circ Y$ is a model of $\mathcal{S}$ in $\mathscr{F}$. Moreover since a morphism of geometric morphisms $(h_*, h^*) \to (g_*, g^*)$ is by definition a natural transformation $h^* \Rightarrow g^*$, this defines automatically a natural transformation on the models $h^* \circ \# \circ Y \Rightarrow g^* \circ \# \circ Y$. Thus composing any left adjoint part g^* of such geometric morphism with $\# \circ Y$ yields a functor

$$\sigma : \mathbf{Geom}(\mathscr{F}, \mathbf{Sh}(\mathscr{C}, T)) \to \mathbf{Mod}_F(\mathcal{S})$$

We now prove σ is an equivalence of categories. First consider the following diagram

$$\begin{tikzcd}
\mathbf{Geom}(\mathscr{F}, \mathbf{Pr}(\mathscr{C})) \arrow[r, "\theta"] & \mathbf{Lex}(\mathscr{C}, \mathscr{F}) \\
\mathbf{Geom}(\mathscr{F}, \mathbf{Sh}(\mathscr{C}, T)) \arrow[u, "\alpha"] \arrow[r, "\sigma"'] & \mathbf{Mod}_F(\mathcal{S}) \arrow[u, "\beta"']
\end{tikzcd}$$

where α is the composition with the geometric morphism $(i, \#) : \mathbf{Sh}(\mathscr{C}, T) \to \mathbf{Pr}(\mathscr{C})$ (the evaluation of α on morphisms is defined in a similar fashion as σ), β is the canonical inclusion since any model of $\mathcal{S}$ is left exact and θ is the equivalence given by 2.1.5. By definition the diagram commutes. Moreover, β is fully faithful since it is just the inclusion of a full subcategory, α is fully faithful since any morphism of geometric morphisms $(h_*, h^*) \to (g_*, g^*)$ is equivalent to a natural transformation $g_* \Rightarrow h_*$ by 1.1.20 and $i : \mathbf{Sh}(\mathscr{C}, T) \to \mathbf{Pr}(\mathscr{C})$ is fully faithful. Hence, since α, β and θ are fully faithful, so is σ.

All that remains now is to prove σ is essentially surjective, i.e, for any model $F : \mathscr{C} \to \mathscr{F}$ of $\mathcal{S}$, there exists $(g_*, g^*) \in \mathbf{Geom}(\mathscr{F}, \mathbf{Sh}(\mathscr{C}, T))$ such that $F \cong \sigma(g_*, g^*)$. Since F is a left exact functor and θ is an equivalence, then by 2.1.5 there exists $(f_*, f^*) \in \mathbf{Geom}(\mathscr{F}, \mathbf{Pr}(\mathscr{C}))$ such that $F \cong f^* \circ Y$. Hence it suffices to show that (f_*, f^*) factors through $\mathbf{Sh}(\mathscr{C}, T)$. We compute for any objects $X \in \mathscr{F}$, $C \in \mathscr{C}$:

$$\begin{aligned}
f_*(X)(C) &\cong \mathbf{Nat}(\mathbf{Hom}_{\mathscr{C}}(-, C), f_*(X)) && \text{by Yoneda lemma} \\
&\cong \mathbf{Hom}_{\mathscr{F}}(f^*(\mathbf{Hom}_{\mathscr{C}}(-, C)), X) && \text{by adjunction} \\
&\cong \mathbf{Hom}_{\mathscr{F}}((f^* \circ Y)(C), X) \\
&\cong \mathbf{Hom}_{\mathscr{F}}(F(C), X).
\end{aligned}$$

We want to show that $f_*(X) = \mathbf{Hom}_{\mathscr{F}}(F(-), X)$ is a sheaf. Let $C \in \mathscr{C}$ be an object, $R \in T(C)$ a sieve and $\gamma : F \Rightarrow \mathbf{Hom}_{\mathscr{F}}(F(-), X)$ a natural transformation. For any $D \in \mathscr{C}$ and $d \in R(D)$, $\gamma(D)(d) : F(D) \to X$ and for any morphism of elements $f : (D', d') \to (D, d)$, i.e a morphism $f : D' \to D$ and $d' = R(f)(d)$, we have by naturality of γ that $\gamma(D')(d') = \gamma(D)(d) \circ F(f)$. Thus

$$\Big(X, \{\gamma(D)(d)\}_{(D,d)\in\mathbf{Elts}(R)}\Big)$$

is a cocone on the diagram $F \circ \phi_R$. But since F is a model of $\mathcal{S}$ in $\mathscr{F}$,

$$\Big(F(C), \{F(d) : F(D) \to F(C)\}_{(D,d)\in\mathbf{Elts}(R)}\Big)$$

is the colimit of $F \circ \phi_R$. Hence there exists a unique morphism $k \in \mathbf{Hom}_{\mathscr{F}}(F(C), X)$ such that $k \circ F(d) = \gamma(D)(d)$ for all $(D, d) \in \mathbf{Elts}(R)$. By the Yoneda lemma (A.0.1), k corresponds to a natural transformation $\xi : \mathbf{Hom}_{\mathscr{C}}(-, C) \Rightarrow \mathbf{Hom}_{\mathscr{F}}(F(-), X)$ such that for any $D \in \mathscr{C}$ and $d : D \to C$, $\xi(D)(d) = \mathbf{Hom}_{\mathscr{F}}(F(d), X)(k) = k \circ F(d)$. Hence if $(D, d) \in \mathbf{Elts}(R)$ then

$$\xi(D)(d) = \gamma(D)(d).$$

ξ is the unique extension of γ to $\mathbf{Hom}_{\mathscr{C}}(-, C)$, thus $\mathbf{Hom}_{\mathscr{F}}(F(-), X)$ is indeed a sheaf. Since $\mathbf{Hom}_{\mathscr{F}}(F(-), X)$ is a sheaf so is $f_*(X)$. Thus by putting $g_* = f_*$ and taking as g^* the restriction of f^* to $\mathbf{Sh}(\mathscr{C}, T)$ we get that

$$F \cong f^* \circ Y = g^* \circ \# \circ Y = \sigma(g_*, g^*).$$

□

2.2 Classifying topos for a geometric sketch

In this section we will construct for a given geometric sketch, a Grothendieck topos that is the classifying topos for the models of this sketch. This is a big proof that will be split into a series of lemmas, the main trick in the observation that a functor sending a cocone to a cocone limit is equivalent to sending a certain family of morphisms to an epimorphic family. Our starting point will be that for a given sketch with an empty set of cocones, there exists a finitely complete small category $\mathscr{C}$ such that the models of the sketch in a Grothendieck topos $\mathscr{E}$ correspond to left excat functors from $\mathscr{C}$ to $\mathscr{E}$.

The key argument in the proof is proving that the category of models of a sketch without assigned cocones in the category **Set** is a locally finitely presented category. This fact will prove the claim for the case $\mathscr{E} = \mathbf{Set}$. Based on this case one can then prove for the $\mathscr{E} = \mathbf{Pr}(\mathscr{C})$ and then for any Grothendieck topos $\mathbf{Sh}(\mathscr{C}, T)$. Since the theory of locally presented categoies falls out of the scope of this thesis, we do not provide a proper proof for this next lemma. A proof of this statement can be found in [3, Lemma 4.2.2].

Lemma 2.2.1. *Let $\mathcal{S} = (\mathcal{T}, \mathcal{P}, \mathcal{I})$ be a sketch with $\mathcal{T}$ a small category and $\mathcal{P}$ a set of finite cones of $\mathcal{T}$ and $\mathcal{I}$ the empty set of cocones. Then there exists a functor $\tau : \mathcal{T} \to \mathscr{C}$ with $\mathscr{C}$ a small finitely complete category such that :*

1. *τ maps any cone of $\mathcal{P}$ to a limit in $\mathscr{C}$.*
2. *For every Grothendieck topos $\mathscr{E}$, precomposing τ yields an equivalence between $\mathbf{Lex}(\mathscr{C}, \mathscr{E})$ and $\mathbf{Mod}_{\mathscr{E}}(\mathcal{S})$.*

Lemma 2.2.2. *Let be $\mathscr{E}$ a Grothendieck topos and a functor $F : \mathscr{D} \to \mathscr{E}$ with $\mathscr{D}$ a small category. Let $(M, \{s_D : F(D) \to M\}_{D \in \mathscr{D}})$ be a cocone on F. Assume that for each two objects $D, D' \in \mathscr{D}$ there exists $(p_{DD'}, t_D : p_{DD'} \to D, t_{D'} : p_{DD'} \to D')$ such that $(F(p_{DD'}), F(t_D), F(t_{D'}))$ is the pullback of s_D along $s_{D'}$. Then the following conditions are equivalent:*

1. *$(M, \{s_D : F(D) \to M\}_{D \in \mathscr{D}})$ is a colimit.*
2. *$\{s_D : F(D) \to M\}_{D \in \mathscr{D}}$ is a jointly epimorphic family.*

Proof. The implication 1. $\Rightarrow$ 2. is immediate. Let be $f, g : M \to L$ two morphisms such that for any $D \in \mathscr{D}$, $f \circ s_D = g \circ s_D$. Then $(L, \{f \circ s_D\}_{D \in \mathscr{D}})$ is a cocone and thus by universal property of the colimit $f = g$.

Conversely, consider $(L, \{\lambda_D : F(D) \to L\}_{D \in \mathscr{D}})$ the colimit of F and the unique arrow $f : L \to M$ such that $f \circ \lambda_D = s_D$ for all $D \in \mathscr{D}$. We shall prove f is an isomorphism, i.e f is an epimorphism and a monomorphism (see 1.1.12). f is an epimorphism since the s_D are jointly epimorphic. Indeed, assume v, u such that $v \circ f = u \circ f$. Then

$$(v \circ f = u \circ f) \Rightarrow (\forall D \in \mathscr{D},\ v \circ s_D = v \circ f \circ \lambda_D = u \circ f \circ \lambda_D = u \circ s_D) \Rightarrow (v = u).$$

The fact that f is monomorphic needs a little bit of work. Assume $\mathscr{E}$ is **Set** and let be $l, l' \in L$ such that $f(l) = f(l')$. Since $\{\lambda_D : F(D) \to L\}_{D \in \mathscr{D}}$ constitute a colimit, it is a jointly epimorphic family which is a jointly surjective family in **Set**. Thus there exist $D, D' \in \mathscr{D}$ and $x \in F(D), x' \in F(D')$ such that $l = \lambda_D(x)$ and $l' = \lambda_{D'}(x')$. We get that

$$s_D(x) = f \circ \lambda_D(x) = f(l) = f(l') = f \circ \lambda_{D'}(x') = s_{D'}(x').$$

This means that $(x, x') \in F(p_{DD'})$ with $F(t_D)(x, x') = x$ and $F(t_{D'})(x, x') = x'$. But since $p_{DD'}, t_D, t_{D'}$ are respectively an object and morphisms of $\mathscr{D}$ and the λ_D's constitute a cocone, we get that $\lambda_D \circ F(t_D) = \lambda_{D'} \circ F(t_{D'})$. Hence we compute

$$l = \lambda_D(x) = (\lambda_D \circ F(t_D))(x, x') = (\lambda_{D'} \circ F(t_{D'}))(x, x') = \lambda_{D'}(x') = l'.$$

Therefore in **Set**, f is a monomorphism.

Now assume $\mathscr{E}$ is the topos of presheaves $\mathbf{Pr}(\mathscr{C})$ for a certain small category $\mathscr{C}$. For each object $C \in \mathscr{C}$, $(L(C), \{\lambda_D(C) : F(D)(C) \to L(C)\}_{D \in \mathscr{D}})$ is a colimit cone of $F(-)(C)$:

$\mathscr{C} \to \mathbf{Set}$ and $(M(C), \{s_D(C) : F(D)(C) \to M(C)\}_{D \in \mathscr{D}})$ is a cocone of $F(-)(C) : \mathscr{C} \to \mathbf{Set}$. Thus there is for each $C \in \mathscr{C}$ a canonical morphism $f(C) : L(C) \to M(C)$. By the previous argument, $f(C)$ is a monomorphism. Let us denote by $Pb(f, f)$ the pullback of f along itself. Since limits are computed pointwise in $\mathbf{Pr}(\mathscr{C})$ and for any $C \in \mathscr{C}$ $f(C)$ is a monomorphism, $Pb(f, f)(C) = Pb(f(C), f(C)) = L(C)$ and the projections are pointwise the identities on $L(C)$. Hence $(L, \mathrm{Id}_L, \mathrm{Id}_L)$ is the pullback of f along itself, thus f is a monomorphism.

Finally if $\mathscr{E}$ is a Grothendieck topos, there exists a site $(\mathscr{C}, T)$ such that $\mathscr{E} \cong \mathbf{Sh}(\mathscr{C}, T)$. Since $\mathbf{Sh}(\mathscr{C}, T)$ is a reflective subcategory of $\mathbf{Pr}(\mathscr{C})$, its colimits are computed by applying the sheaf functor $\#$ on the colimits in $\mathbf{Pr}(\mathscr{C})$. But since the unique arrow $f : L \to M$ is a monomorphism in $\mathbf{Pr}(\mathscr{C})$ and $\#$ is left exact, then $\#f$ is a monomorphism. $\square$

Corollary 2.2.3. *Let $\mathscr{C}$ be a finitely complete small category, $C \in \mathscr{C}$ an object and a subfunctor $R \hookrightarrow \mathbf{Hom}_{\mathscr{C}}(-, C)$. If $F : \mathscr{C} \to \mathscr{E}$ is a left exact functor to any Grothendieck topos $\mathscr{E}$, then the following conditions are equivalent:*

1. *F maps R, viewed as the cocone $\left(C, \left\{s_{(D,d)} : \phi_R(D, d) = D \to C\right\}_{(D,d) \in \mathbf{Elts}(R)}\right)$ on a colimit cone.*
2. *F maps R, viewed as a family $\{d : D \to C\}_{(D,d) \in \mathbf{Elts}(R)}$, to a jointly epimorphic family.*

Proof. Let be two objects (D, d) and (D', d') in $\mathbf{Elts}(R)$. Since $\mathscr{C}$ is finitely complete, let $(p_{DD'}, t_D : p_{DD'} \to D, t_{D'} : p_{DD'} \to D')$ be the pullback of d along d' in $\mathscr{C}$ and denote $d'' = d \circ t_D = d' \circ t_{D'}$. This yields two morphisms $t_D : (p_{DD'}, d'') \to (D, d)$ and $t_{D'} : (p_{DD'}, d'') \to (D', d')$ in $\mathbf{Elts}(R)$. Since F preserves pullbacks, it suffices to apply 2.2.2 to the functor $F \circ \phi_R$.

$\square$

Lemma 2.2.4. *Let $\mathcal{S} = (\mathscr{C}, \mathcal{P}, \mathcal{I})$ be a sketch with:*

1. *$\mathscr{C}$ a small finitely complete category.*
2. *$\mathcal{P}$ the set of all finite limit cones of $\mathscr{C}$.*
3. *$\mathcal{I}$ a set of cocones $\left(C, \left\{s_{(D,d)} : \phi_R(D, d) = D \to C\right\}_{(D,d) \in \mathbf{Elts}(R)}\right)$ for some subfunctors $R \hookrightarrow \mathbf{Hom}_{\mathscr{C}}(-, C)$ with some objects $C \in \mathscr{C}$.*

If T is the Grothendieck topology on $\mathscr{C}$ generated by $\mathcal{I}$, then the sketches $\mathcal{S}$ and $\overline{\mathcal{S}} = (\mathscr{C}, \mathcal{P}, T)$ have the same models in all Grothendieck topoi.

Proof. Since $\mathcal{I} \subseteq T$, it is obvious that every model of $\overline{\mathcal{S}}$ in $\mathscr{E}$ is a model of $\mathcal{S}$. Conversely, T generated by $\mathcal{I}$ is the set obtained inductively by applying the three axioms of 1.1.1 to $\mathcal{I}$. We then just need to verify that every left exact functor sending $R \in \mathcal{I}$ to a colimit cone maps (see 2.2.3) any R obtained by the three axioms from $\mathcal{I}$ to a jointly epimorphic family. We use the notation of 1.1.1.

For all $C \in \mathscr{C}$, F maps $\mathbf{Hom}_{\mathscr{C}}(-, C)$ to a jointly epimorphic family since that family contains the epimorphism $F(\mathrm{Id}_C)$.

Fo any $f : D \to C$ morphism in $\mathscr{C}$ and $R \in \mathcal{I}$, consider the pullback R_f (left diagram). For any object $X \in \mathscr{C}$ and $x \in R(X)$, consider the pullback (right diagram):

$$\begin{tikzcd} R_f \arrow[r, "f_R"] \arrow[d, hook, "r_f"'] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, hook, "r"] \\ \mathbf{Hom}_{\mathscr{C}}(-, D) \arrow[r, "\mathbf{Hom}_{\mathscr{C}}(-,f)"'] & \mathbf{Hom}_{\mathscr{C}}(-, C) \end{tikzcd} \qquad \begin{tikzcd} X_f \arrow[r, "f_x"] \arrow[d, "x_f"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "x"] \\ D \arrow[r, "f"'] & C \end{tikzcd}$$

Since limits are computed pointwise, $R_f(X_f)$ is the pullback in **Set** of $\mathbf{Hom}_{\mathscr{C}}(X_f, f)$ along $r(X_f) : R(X_f) \hookrightarrow \mathbf{Hom}_{\mathscr{C}}(X_f, C)$. Since

$$\mathbf{Hom}_{\mathscr{C}}(X_f, f)(x_f) = f \circ x_f = x \circ f_x = R(f_x)(x) \in R(X_f),$$

we get that $x_f \in R_f(X_f)$. Since F preserves pullbacks, since coproducts (see 1.1.10) and epimorphisms (see 1.1.11,1.1.14) are pullback stable and since by assumption the family $F(x)$ for all $X \in \mathscr{C}$, $x \in R(X)$ is mapped to a jointly epimorphic family, so is the family $F(x_f)$. Since the $F(x_f)$ are contained in the family $\{F(y)\}_{(Y,y)\in \mathbf{Elts}(R_f)}$, F maps R_f to a jointly epimorphic family.

Now let be $S \hookrightarrow \mathbf{Hom}_{\mathscr{C}}(-, C)$ in $\mathcal{I}$ and a subobject $R \hookrightarrow \mathbf{Hom}_{\mathscr{C}}(-, C)$ such that for any $D \in \mathscr{C}$ and any $f \in S(D)$, F maps R_f to a jointly epimorphic family. Consider two morphisms in $u, v : F(C) \to M$ in $\mathscr{E}$ such that $u \circ F(x) = v \circ F(x)$ for all $(X, x) \in \mathbf{Elts}(R)$. If $y \in R_f(X)$, then $f \circ y \in R(X)$ and thus

$$(u \circ F(f)) \circ F(y) = u \circ F(f \circ y) = v \circ F(f \circ y) = (v \circ F(f)) \circ F(y).$$

By assumption the $F(y)$ constitute a jointly epimorphic family, thus $u \circ F(f) = v \circ F(f)$. But again, the family $F(f)$ constitute a jointly epimorphic family. Therefore $u = v$.

Thus by induction if F maps all the $R \in \mathcal{I}$ to jointly epimorphic families, so does F for all the covering sieves of the topology T. □

Lemma 2.2.5. *Let $\mathscr{C}$ be a small category and $\{f_i : D_i \to C\}_{i\in I}$ a family of morphisms in $\mathscr{C}$. Let $R \hookrightarrow \mathbf{Hom}_{\mathscr{C}}(-, C)$ be the subfunctor generated by the f_i, i.e R is defined by $R(X) = \{f_i \circ x | x : X \to D_i,\ i \in I\}$ and the canonical composition $R(g)(-) = (-) \circ g$.*

For every functor $F : \mathscr{C} \to \mathscr{E}$ with $\mathscr{E}$ a Grothendieck topos, the following conditions are equivalent:

1. *$\{F(f_i)\}_{i\in I}$ is a jointly epimorphic family.*
2. *F maps R to a jointly epimorphic family.*

Proof. 1. $\Rightarrow$ 2. is trivial since $\{f_i\}_{i\in I}$ is part of the family $\{d : D \to C\}_{(D,d)\in\mathbf{Elts}(R)}$.

For 2. $\Rightarrow$ 1., consider $u, v : F(C) \to M$ in $\mathscr{E}$ such that $u \circ F(f_i) = v \circ F(f_i)$ for all $i \in I$. Since for any $X \in \mathscr{C}$, $y \in R(X)$ is of the form $f_i \circ x$ for $x : X \to D_i$ and $i \in \mathscr{C}$, we get that $u \circ F(y) = v \circ F(y)$ for any $y \in R(X)$ for any $X \in \mathscr{C}$. Thus by assumption $u = v$. □

Lemma 2.2.6. *Let $\mathcal{S} = (\mathscr{T}, \mathcal{P}, \mathcal{F})$ a triple such that*

- *$\mathscr{T}$ is a small category,*
- *$\mathcal{P}$ is a set of cones in $\mathscr{T}$,*
- *$\mathcal{F}$ is a set of families of morphisms $(f_i : T_i \to T)_{i\in I}$ in $\mathscr{T}$.*

A model of $\mathcal{S}$ in any Grothendieck topos $\mathscr{E}$ is a functor $F : \mathscr{T} \to \mathscr{E}$ mapping the cones of $\mathcal{P}$ to limit cones and the families of $\mathcal{F}$ to families of epimorphisms. Then the theory described by those models admits a classifying topos.

Proof. By 2.2.1, there exists a functor $\tau : \mathscr{T} \to \overline{\mathscr{T}}$ with $\overline{\mathscr{T}}$ a small finitely complete category, such that for every Grothendieck topos $\mathscr{E}$, $\mathbf{Lex}(\overline{\mathscr{T}}, \mathscr{E})$ is equivalent to $\mathbf{Mod}_{\mathscr{E}}(\mathscr{T}, \mathcal{P})$ the category of models of $\mathcal{S}$ without set of cocones. Let $\overline{\mathcal{S}} = (\overline{\mathscr{T}}, \overline{\mathcal{P}}, \overline{\mathcal{F}})$ be the triple with

- $\overline{\mathscr{T}}$, the small category obtained by 2.2.1,
- $\overline{\mathcal{P}}$ is the set of finite limit cones in $\overline{\mathscr{T}}$,
- $\overline{\mathcal{F}}$ is the set of all families $\{\tau(f_i)\}_{i\in I}$, for each family $\{f_i\}_{i\in I}$ in $\mathcal{F}$.

Composing any model of $\overline{\mathcal{S}}$ by τ yields an equivalence between $\mathbf{Mod}_{\mathscr{E}}\overline{\mathcal{S}}$ and $\mathbf{Mod}_{\mathscr{E}}\mathcal{S}$. Thus all we need is to prove the theory determined by $\overline{\mathcal{S}}$ admits a classifying topos.

By 2.2.5 we know the models stay the same if we replace the families $\{\tau(f_i)\}_{i\in I}$ of $\overline{\mathcal{F}}$ by the subfunctors $R \hookrightarrow \mathbf{Hom}_{\overline{\mathscr{T}}}(-, \tau(T))$ generated by each family. By 2.2.3 the sieves of L can be seen as cocones, L is the set of all those cocones. By 2.2.4 the models of $\overline{\mathcal{S}}$ don't change if $\overline{\mathcal{F}}$ is extended to the whole generated Grothendieck topology L. Then by 2.1.8 the theory sketched by $\overline{\mathcal{S}}$ admits the topos of sheaves on the site $(\overline{\mathscr{T}}, L)$ as a classifying topos. □

Let $H : \mathscr{D} \to \mathscr{E}$ be a functor with $\mathscr{D}$ a small category and $\mathscr{E}$ a Grothendieck topos. Since $\mathscr{E}$ is cocomplete it admits at least one cocone on H. Let $(C, \{s_D : H(D) \to C\}_{D\in\mathscr{D}})$ be a cocone on H. For each zigzag Σ in $\mathscr{D}$, since $\mathscr{E}$ is complete, consider the limit $(L_\Sigma, \{p_i : L_\Sigma \to H(D_i)\}_{D_i\in\Sigma})$ on H restricted to Σ with Σ denoted as

$$\begin{tikzcd}
& D_1 \arrow[dl, "d_1"'] \arrow[dr, "d_2"] & & D_3 \arrow[dl, "d_3"'] & \dots & D_{2n-1} \arrow[dr, "d_{2n}"] & \\
D = D_0 & & D_2 & & \dots & & D_{2n} = D'
\end{tikzcd}$$

Notice that since the p_i's and the s_D's form respectively a cone and a cocone,

$$\begin{aligned}
s_D \circ p_0 = s_{D_0} \circ p_0 &= s_{D_0} \circ H(d_1) \circ p_1 \\
&= s_{D_1} \circ p_1 = s_{D_2} \circ H(d_2) \circ p_1 \\
&= s_{D_2} \circ p_2 \\
&= \dots \\
&= s_{D_{2n}} \circ p_{2n} = s_{D'} \circ p_{2n}.
\end{aligned}$$

Thus by setting $(P_{DD'}, t_D, t_{D'})$ as the pullback of s_D along $s_{D'}$, we get a unique factorization $f_\Sigma : L_\Sigma \to p_{DD'}$.

$$\begin{tikzcd}
L_\Sigma \arrow[dr, dashed, "f_\Sigma"] \arrow[drr, bend left, "p_0"] \arrow[ddr, bend right, "p_{2n}"'] & & \\
& P_{DD'} \arrow[r, "t_D"] \arrow[d, "t_{D'}"'] & H(D) \arrow[d, "s_D"] \\
& H(D') \arrow[r, "s_{D'}"'] & C
\end{tikzcd}$$

Lemma 2.2.7. *Consider a functor $H : \mathscr{D} \to \mathscr{E}$ and a cocone $(C, \{s_D : H(D) \to C\}_{D\in\mathscr{D}})$ as in the previous paragraph. Then the following are equivalent:*

1. *$(C, \{s_D : H(D) \to C\}_{D\in\mathscr{D}})$ is a colimit cone of H.*
2. A. *$\{s_D : H(D) \to C\}_{D\in\mathscr{D}}$ is a jointly epimorphic family*

   B. *for each pair of objects $D, D' \in \mathscr{D}$, the family of morphisms $\{f_\Sigma : L_\Sigma \to P_{DD'}\}_\Sigma$ of all possible zigzags Σ between D and D' is jointly epimorphic family.*

Proof. Assume the s_D's and f_Σ are jointly epimorphic families. Let $(L, \{\lambda_D : H(D) \to L\}_{D\in\mathscr{D}})$ be the colimit of H. This yields a unique factorization $f : L \to C$ such that $f \circ \lambda_D = s_D$ for each $D \in \mathscr{D}$. Let be D, D' two objects of $\mathscr{D}$ and a zigzag Σ between D and D'. By the same argument as for s_D and s'_D, $\lambda_D \circ p_0 = \lambda_{D'} \circ p_{2n}$. Thus

$$\lambda_D \circ t_D \circ f_\Sigma = \lambda_D \circ p_0 = \lambda_{D'} \circ p_{2n} = \lambda_{D'} \circ t_{D'} \circ f_\Sigma.$$

Now let be the following commutative diagram where every square (yes, the circle is a square) is a pullback:

$$\begin{tikzcd}
P_{DD'} \arrow[r, "v_{DD'}"] \arrow[d, "u_{DD'}"] \arrow[dr, phantom, very near start, "\lrcorner"] \arrow[rr, bend left, "t_{D'}"] \arrow[dd, bend right, "t_D"'] & V(D') \arrow[r, "v'_D"] \arrow[d, "v_{D'}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & H(D') \arrow[d, "\lambda_{D'}"'] \arrow[dd, bend left, "s_{D'}"] \\
U(D) \arrow[r, "u_D"'] \arrow[d, "u'_D"] \arrow[dr, phantom, very near start, "\lrcorner"] & P \arrow[r, "v"] \arrow[d, "u"'] \arrow[dr, phantom, very near start, "\lrcorner"] & L \arrow[d, "f"'] \\
H(D) \arrow[r, "\lambda_D"] \arrow[rr, bend right, "s_D"'] & L \arrow[r, "f"] & C
\end{tikzcd}$$

Since by 1.1.10 colimits are pullback stable, we get that $(P, \{v_{D'} : V(D') \to P\}_{D'\in\mathscr{D}})$, $(P, \{u_D : U(D) \to P\}_{D\in\mathscr{D}})$, $(U(D), \{u_{DD'} : P_{DD'} \to U(D)\}_{D'\in\mathscr{D}})$ and $(V(D'), \{v_{DD'} : P_{DD'} \to V(D')\}_{D\in}$ are all colimits. And thus the families $\{u_D\}_{D\in\mathscr{D}}$, $\{v_{D'}\}_{D'\in\mathscr{D}}$, $\{v_{DD'}\}_{D\in\mathscr{D}}$ and $\{u_{DD'}\}_{D'\in\mathscr{D}}$ are all jointly epimorphic families.

We compute for any $D, D' \in \mathscr{D}$ and any zigzag Σ between D and D', that

$$\begin{aligned}
u \circ u_D \circ u_{DD'} \circ f_\Sigma &= \lambda_D \circ u'_D \circ u_{DD'} \circ f_\Sigma \\
&= \lambda_D \circ t_D \circ f_\Sigma \\
&= \lambda_{D'} \circ t_{D'} \circ f_\Sigma \\
&= \lambda_{D'} \circ v'_D \circ v_{DD'} \circ f_\Sigma \\
&= v \circ v_{D'} \circ v_{DD'} \circ f_\Sigma \\
&= v \circ u_D \circ u_{DD'} \circ f_\Sigma.
\end{aligned}$$

Since the u_D's, $u_{DD'}$'s and f_Σ are jointly epimorphic, so is their composition, thus $u = v$. This equality implies that f is a monomorphism.

Assume we have two morphisms x, y such that $x \circ f = y \circ f$. For any $D \in \mathscr{D}$

$$x \circ s_D = x \circ f \circ \lambda_D = y \circ f \circ \lambda_D = y \circ s_D.$$

Since by assumption $\{s_D\}_{D\in\mathscr{D}}$ is a jointly epimorphic family this implies $x = y$. Thus

f is an epimorphism. Hence f is an isomorphism by 1.1.12, this concludes the proof 2. $\Rightarrow_1$.

Conversely assume that $(C, \{s_D : H(D) \to C\}_{D\in\mathscr{D}})$ is a colimit. Thus the s_D's are jointly epimorphic. What remains to prove is that given $D, D' \in \mathscr{D}$ the f_Σ over all the zigzags Σ between D and D' are jointly epimorphic. To prove this, we are using the familiar trick to first prove it for the specific case $\mathscr{E} = \mathbf{Set}$, then prove it in the case $\mathscr{E}$ is a topos of presheaves, and then finally when $\mathscr{E}$ is any Grothendieck topos.

Assume $\mathscr{E} = \mathbf{Set}$, we know any colimit can be obtained from coproducts and coequalizers (see [2, 2.8.1]). C is the coequalizer of α and β.

$$\amalg_{d\in\mathscr{D}} H(\mathrm{dom}(d)) \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \amalg_{D\in\mathscr{D}} H(D) \xrightarrow{\quad q \quad} C$$

In the case of $\mathbf{Set}$, α is just the inclusion, $\beta(x) = H(d)(x)$ for $x \in H(\mathrm{dom(d)})$ and the coequalizer $q : \amalg_{D\in\mathscr{D}} H(D) \to C$ is the quotient of $\amalg_{D\in\mathscr{D}} H(D)$ by the equivalence relation generated by $(\alpha(x), \beta(x))$ for all $x \in \amalg_{d\in\mathscr{D}} H(\mathrm{dom}(d))$. Thus C is the quotient of $\amalg_{D\in\mathscr{D}} H(D)$ by the equivalence relation generated by $(x, H(d)(x))$ for any $d : D \to D'$ and any $x \in H(D)$. Hence for any $x \in H(D)$ and $x' \in H(D')$, they are equivalent if and only if there exists a sequence of elements $x = x_0, x_1, \ldots, x_k = x'$ such that for any two consecutive $x_i \in H(D_i)$ and $x_{i+1} \in H(D_{i+1})$ there either exists an arrow $d_{i+1} : D_i \to D_{i+1}$ such that $H(d_i)(x_i) = x_{i+1}$ or there exists an arrow $d_i : D_{i+1} \to D_i$ such that $x_i = H(d_i)(x_{i+1})$. After composing the consecutive d_i's that are composable and introducing identities at the beginning or at the end if necessary, we can express the equivalence between x and x' via a zigzag Σ as above. Thus an equivalence between x and x' is a zigzag Σ between D and D' such that for each $i \in \{0, \ldots, m\}$, $H(d_i)(x_i) = x_{i-1}$ or $x_{i+1} = H(d_{i+1})(x_i)$ for i odd.

But the limit L_Σ is precisely the equalizer e

$$L_\Sigma \xrightarrow{\quad e \quad} \Pi_{D\in\Sigma} H(D) \underset{\beta'}{\overset{\alpha'}{\rightrightarrows}} \Pi_{d\in\Sigma} H(\mathrm{codom}(d))$$

where $\alpha'((x_D)_{D\in\Sigma}) = (x_{\mathrm{codom}(d)})_{d\in\Sigma}$, $\beta'((x_D)_{D\in\Sigma}) = (H(d)(x_{\mathrm{dom}(d)}))_{d\in\Sigma}$ and the equalizer $L_\Sigma = \{(x_D)_{D\in\Sigma} \in \Pi_{D\in\Sigma} H(D) | \alpha((x_D)_{D\in\Sigma}) = \beta((x_D)_{D\in\Sigma})\}$. In particular the family $(x, \ldots, x')$ is in L_Σ. In the case of $\mathbf{Set}$, f_Σ is defined as $f_\Sigma((x_D)_{D\in\Sigma}) = (x_0, x_m)$. By definition for any given pair $(x, x') \in P_{DD'}$, x and x' are equivalent in $\amalg_{D\in\Sigma} H(C)$. Therefore there exists a zigzag Σ and a family $(x_D)_{D\in\Sigma} \in L_\Sigma$ such that $f_\Sigma((x_D)_{D\in\Sigma}) = (x, x')$. Hence the f_Σ's are jointly surjective.

Now if $\mathscr{E}$ is a topos of presheaves $\mathbf{Pr}(\mathscr{C})$ on a small category $\mathscr{C}$, since colimits and limits are computed pointwise, it directly follows form the computation in $\mathbf{Set}$ that the f_Σ's are jointly epimorphic.

Now assume $\mathscr{E}$ is any Grothendieck topos, that means there exists a site $(\mathscr{C}, T)$ such that $\mathscr{E} = \mathbf{Sh}(\mathscr{C}, T)$. Let be $(L', \{\lambda'_D : i \circ H(D) \to L'\}_{D\in\mathscr{D}})$ the colimit of $i \circ H$ in $\mathbf{Pr}(\mathscr{C})$

where i is the canonical inclusion. Let be $\{g_\Sigma\}_\Sigma$ the jointly epimorphic family obtained applying this theorem to the, already proven, case of presheaves. Since $\#$ is finite limit and colimit preserving, it maps the colimit $(L', \{\lambda'_D : i \circ H(D) \to L'\}_{D \in \mathscr{D}})$ to the colimit $(C, \{s_D : H(D) \to C\}_{D \in \mathscr{D}})$ and maps the whole construction (see the paragraph before the lemma) on $(L', \{\lambda'_D : i \circ H(D) \to L'\}_{D \in \mathscr{D}})$ to the construction on $(C, \{s_D : H(D) \to C\}_{D \in \mathscr{D}})$. Thus $\{g_\Sigma\}_\Sigma$ is mapped to the family $\{f_\Sigma\}_\Sigma$. Since $\#$ preserves in particular jointly epimorphic (since $\#$ preserves coproducts and epimorphisms), $\{f_\Sigma\}_\Sigma$ is a jointly epimorphic family. □

Theorem 2.2.8. *The theory determined by a geometric sketch admits a classifying topos.*

Proof. Let $\mathcal{S} = (\mathscr{T}, \mathcal{P}, \mathcal{I})$ a geometric sketch. By 2.2.1 the models of the sketch $\mathcal{S}' = (\mathscr{T}, \mathcal{P})$ ($\mathcal{S}$ without $\mathcal{I}$) are equivalent to the models of $\overline{\mathcal{S}}' = (\mathscr{C}, \overline{\mathcal{P}})$ where $\mathscr{C}$ is finitely complete and $\overline{\mathcal{P}}$ is the set of finite cones on $\mathscr{C}$ and this equivalence is yielded by the functor $\tau : \mathscr{T} \to \mathscr{C}$. Now for any cocone $\Gamma \in \mathcal{I}$, it is mapped by a model of $\mathcal{S}'$ to a colimit cone if and only if the cocone $\tau(\Gamma)$ is mapped by a model of $\overline{\mathcal{S}}'$ to a colimit cone. Thus by putting $\overline{\mathcal{I}}$ as the set of all cocones $\tau(\Gamma)$, we get an equivalence of models between $\mathcal{S}$ and $\overline{\mathcal{S}} = (\mathscr{C}, \overline{\mathcal{P}}, \overline{\mathcal{I}})$. Hence without lost of generality, we can consider $\mathscr{T}$ to be finitely complete and $\mathcal{P}$ to be the set of all limit cones of $\mathscr{T}$.

Let $(C, \{s_D : H(D) \to C\}_{D \in \mathscr{D}})$ be a cocone of $\mathcal{I}$ on a functor $H : \mathscr{D} \to \mathscr{T}$. By 2.2.7, a left exact functor $F : \mathscr{T} \to \mathscr{E}$ with $\mathscr{E}$ a Grothendieck topos, maps the cocone $(C, \{s_D : H(D) \to C\}_{D \in \mathscr{D}})$ to a colimit cone if and only if it maps the s_D's to a jointly epimorphic family as well as the f_Σ's. Let $\mathcal{F}$ be the set of those families deduced from the cocones in $\mathcal{I}$. Then the models of $\mathcal{S}$ coincide with the models of $(\mathscr{T}, \mathcal{P}, \mathcal{F})$ determined by 2.2.6, and this theory admits a classifying topos by 2.2.6. □

This concludes our study of classifying topoi. In the next chapter we will focus on giving another way to study mathematical theories in Grothendieck topoi that is more convenient. In the end of the chapter we will then go back to the last results of this chapter to prove our new approach for theories will still allow use to assign them a classifying topos.

Chapter 3

Interpreting a theory

Although a sketch of a theory is nice to manipulate, it is not necessary easy to compute a sketch for a given theory. The number of diagrams one needs to sketch a theory can be quite big even for simple theories such the theory of magmas (see example 2). Thus sketch theory is not that convenient in practice. In this chapter we will see that the set theoretic language, used to describe any mathematical theory, can be usually carried over certain categories called the topoi of which Grothendieck topoi are a subclass. Instead of translating this language into diagrams we can compute in categories, we will directly transpose that very language inside the topoi. Here is a little summary on how this will work.

First we of course need to have a good understanding on what is a mathematical theory. The first section will be dedicated to give a definition of mathematical theory and to explain what is a mathematical statement, a formula, in a theory. We will then observe how those formulas are deeply connected to the notion of subset and how logical operators such as "and" and "or" are merely the intersection and union of those subsets.

In the second section we will give the definition of a topos, a category with enough properties for its objects to behave like sets. In a topos the collection of morphisms between two objects gives rise to an object that we can manipulate inside the topos. In particular currying is possible in a topos. Currying is the process of seeing a map of several arguments $f : X \times Y \to Z$ as a family of maps $f_x : Y \to Z$. A topos also allows for a good understanding of the subobjects of an object, we can classify them in a topos. We will prove that the Grothendieck topoi are, as their name indicates, topoi and we will show how classifying subobjects and currying allows us to have an object corresponding to the collection of epimorphisms between two objects.

In the third section, we will study the behaviour of the subobjects in a topos. We will show that in a topos the intersection or union of subobjects can be defined and we will thus recover logical operators as "and" or "or" inside the topos.

Eventually, in the last section, we will show how from those logical operators inside a topos, one can define a whole language and thus interpret a mathematical theory in a topos. We will prove major results such as the fact that for any given geometric theory, if a statement is true in a topos, then it is true in **Set**, providing a strong motivation for the study of theories in topoi. We will end this chapter by proving that any geometric theory admits a classifying topos.

3.1 Logic

Definition 3.1.1. A *signature* Σ is the data $(S,\Sigma = \Sigma_{\text{op}} \amalg \Sigma_{\text{rel}})$ where $S, \Sigma_{\text{op}}, \Sigma_{\text{rel}}$ are disjoint sets.

1. S is the set of types (also called sorts).
2. Σ_{op} is the set of formal operation symbols and for each operation symbol $\sigma \in \Sigma_{\text{op}}$ is associated an n-arity $s_1 \times \cdots \times s_n \to s$ with $s_1, \ldots, s_n, s \in S$; noted $\sigma : s_1 \times \cdots \times s_n \to s$. An operation symbol of 0-arity, $\sigma : \to s$ is a *constant.*
3. Σ_{rel} is the set of formal relation symbols and for each relation symbol $\sigma \in \Sigma_{\text{rel}}$ is associated an n-arity $s_1 \times \cdots \times s_n$ with $s_1, \ldots, s_n \in S$; noted $\sigma \subseteq s_1 \times \cdots \times s_n$.

Definition 3.1.2. A Σ-*structure* A is an S-sorted set $|A| = (A_s)_{s\in S}$ together with operations $\sigma_A : A_{s_1} \times \cdots \times A_{s_n} \to A_s$ for each $\sigma : s_1 \times \cdots \times s_n \to s$ in Σ_{op} and relations $\sigma_A \subseteq A_{s_1} \times \cdots \times A_{s_n}$ for each $\sigma \subseteq s_1 \times \cdots \times s_n$ in Σ_{rel}. The constants in A of type A_s are the elements of the set A_s, that is the points $\{*\} \to A_s$.

A *homomorphism* of Σ-structures from A to B is a S-sorted map $f : |A| \to |B|$ such that $f_s(\sigma_A(a_1, \ldots, a_n)) = \sigma_B(f_{s_1}(a_1), \ldots, f_{s_n}(a_n))$ for each operation $\sigma : s_1 \times \cdots \times s_n \to s$ and $(f_{s_1} \times \cdots \times f_{s_n})(\sigma_A) \subseteq \sigma_B$ for each relation $\sigma \subseteq s_1 \times \cdots \times s_n$.

In the specific case with no relations, a structure is called an *algebra.* The category of Σ-structures and their homomorphisms is denoted $\mathbf{Str}\,\Sigma$, $\mathbf{Alg}\,\Sigma_{\text{op}}$ in the case of algebras.

Example 3.1.3. The signature of ordered groups is one sorted with $\Sigma_{\text{op}} = \{+, -, e\}$ and $\Sigma_{\text{rel}} = \{\leq\}$, with arities $+ : s \times s \to s$, $- : s \to s$, $e : \to s$ and $\leq \subseteq s \times s$.

It is important to note that a structure on this signature **is not** an ordered group, only the "backbones". The axioms of group theory are yet to be defined. One can also remark the signature for ordered groups and abelian ordered groups is the same.

For each type s of a signature Σ we assume we have a set (potentially countable infinite) of *variables* of type s.

Definition 3.1.4. The set of *terms* over an S-sorted set X of variables with respect to a signature Σ is defined inductively :

1. each variable of type s is a term of type s,

2. for each operation $\sigma : s_1 \times \cdots \times s_n \to s$ and each n-tuple of terms $(\tau_1, \ldots, \tau_n)$ of types $s_1, \ldots, s_n$, $\sigma(\tau_1, \ldots, \tau_n)$ is a term of type s.

Example 3.1.5. Let X be a set of variable and x, y, z variables in X. The following are examples of terms over X with the signature of ordered groups :

$$x, \quad e, \quad x+y, \quad -(z), \quad y+-(x+e), \quad \ldots$$

Definition 3.1.6. The set $\mathrm{Var}(\tau)$ of *free variables* of a term τ is inductively defined by :

1. $\mathrm{Var}(x) = x$ for any variable x,
2. $\mathrm{Var}(\sigma(\tau_1, \ldots, \tau_n)) = \mathrm{Var}(\tau_1) \cup \cdots \cup \mathrm{Var}(\tau_n)$.

An *atomic formula* is either an equation $\tau_1 = \tau_2$ of terms of same type or an expression $\sigma(\tau_1, \ldots, \tau_n)$ where σ is a relation of type $s_1 \times \cdots \times s_n$ and τ_i are terms of type s_i, or $\top$ (truth), or $\perp$ (falsity).

A *formula* is built from atomic formulas $\varphi, \psi, \ldots$ and the usual logical operators :

$$\begin{array}{ll}
\neg\varphi & \text{(negation)} \\
\varphi \Rightarrow \psi & \text{(implication)} \\
\varphi \wedge \psi,\ \varphi \vee \psi & \text{(conjunction and disjunction)} \\
(\forall x)\varphi,\ (\exists x)\varphi & \text{(universal and existential quantifiers)}
\end{array}$$

The free variables of a formula are the non-quantified variables of the formula. They are defined in the following way :

$$\begin{array}{rcl}
\mathrm{Var}(\tau_1 = \tau_2) & = & \mathrm{Var}(\tau_1) \cup \mathrm{Var}(\tau_2) \\
\mathrm{Var}(\sigma(\tau_1, \ldots, \tau_n)) & = & \mathrm{Var}(\tau_1) \cup \cdots \cup \mathrm{Var}(\tau_n) \\
\mathrm{Var}(\neg\varphi) & = & \mathrm{Var}(\varphi) \\
\mathrm{Var}(\varphi \Rightarrow \psi) & = & \mathrm{Var}(\varphi \wedge \psi) = \mathrm{Var}(\varphi \vee \psi) = \mathrm{Var}(\varphi) \cup \mathrm{Var}(\psi) \\
\mathrm{Var}((\forall x)\varphi) & = & \mathrm{Var}((\exists x)\varphi) = \mathrm{Var}(\varphi) \setminus \{x\} \\
\mathrm{Var}(\top) & = & \mathrm{Var}(\perp) = \emptyset.
\end{array}$$

Moreover in the case of infinitary disjunctions and conjunctions, for a given set I if one has a family of formulas $\{\phi_i\}_{i \in I}$ such that $\bigcup_{i \in I} \mathrm{Var}(\phi_i)$ is finite then

$$\begin{array}{rcl}
\mathrm{Var}(\bigwedge_{i \in I} \phi_i) & = & \bigcup_{i \in I} \mathrm{Var}(\phi_i) \\
\mathrm{Var}(\bigvee_{i \in I} \phi_i) & = & \bigcup_{i \in I} \mathrm{Var}(\phi_i)
\end{array}$$

$\varphi(x_1, \ldots, x_n)$ indicates that $\mathrm{Var}(\varphi) \subseteq \{x_1, \ldots, x_n\}$.

A *sentence* is a formula that has no free variables.

Example 3.1.7. Going back to our example with the signature of ordered groups. $x \leq y + z$ and $(\forall x)(\exists y)(x + y) = e$ are formulas and $\mathrm{Var}(x \leq y + z) = \{x, y, z\}$, $\mathrm{Var}((\forall x)(\exists y)(x + y) = e) = \emptyset$. $(\forall x)(\exists y)(x + y) = e$ is a sentence.

Definition 3.1.8. Let A be a Σ-structure and a formula $\varphi(x_1, \ldots, x_n)$. A *satisfies the formula* $\varphi(x_1, \ldots, x_n)$ under the assignment $x_i \mapsto a_i$, noted $A \models \varphi[a_1, \ldots, a_n]$, is defined inductively from :

$A \models (\tau_1 = \tau_2)[a_1, \ldots, a_n]$	iff $\tau_1(a_1, \ldots, a_n) = \tau_2(a_1, \ldots, a_n)$ in A
$A \models \sigma(\tau_1, \ldots, \tau_m)[a_1, \ldots, a_n]$	iff $(\tau_1(a_1, \ldots, a_n), \ldots, \tau_m(a_1, \ldots, a_n)) \in \sigma_A$
$A \models \neg\varphi[a_1, \ldots, a_n]$	iff it is not true that $A \models \varphi[a_1, \ldots, a_n]$
$A \models (\varphi \Rightarrow \psi)[a_1, \ldots, a_n]$	iff $A \models \neg\varphi[a_1, \ldots, a_n]$ or $A \models \psi[a_1, \ldots, a_n]$
$A \models (\varphi \wedge \psi)[a_1, \ldots, a_n]$	iff $A \models \varphi[a_1, \ldots, a_n]$ and $A \models \psi[a_1, \ldots, a_n]$
$A \models (\varphi \vee \psi)[a_1, \ldots, a_n]$	iff $A \models \varphi[a_1, \ldots, a_n]$ or $A \models \psi[a_1, \ldots, a_n]$
$A \models ((\forall x)\varphi)[a_1, \ldots, a_n]$	iff $\mathrm{Var}\varphi \subseteq \{x, x_1, \ldots, x_n\}$ and $A \models \varphi[a, a_1, \ldots, a_n]$ for any a in X of the same type of x
$A \models ((\exists x)\varphi)[a_1, \ldots, a_n]$	iff $\mathrm{Var}\varphi \subseteq \{x, x_1, \ldots, x_n\}$ and $A \models \varphi[a, a_1, \ldots, a_n]$ for some a in X of the same type of x.

Definition 3.1.9.
1. A *regular* formula is a formula built from terms, $=$, relations, $\top$, $\exists$ and finite $\wedge$.
2. A *coherent* formula is a formula built from terms, $=$, relations, $\top$, $\bot$, $\exists$, finite $\wedge$ and finite $\vee$.
3. A *geometric* formula is a formula built from terms, $=$, relations, $\top$, $\bot$, $\exists$ and finite $\wedge$ and small $\vee$.

Definition 3.1.10. A *theory* is a set T of sentences, called *axioms*, of the form

$$(\forall x_1, \ldots, x_n)(\varphi \Rightarrow \psi)$$

with formulas φ and ψ on the same free variables ($\mathrm{Var}(\varphi), \mathrm{Var}(\psi) \subseteq \{x_1, \ldots, x_n\}$). A *regular (coherent, ...)* theory is a theory whose axioms are all of the form $(\forall x_1, \ldots, x_n)(\varphi \Rightarrow \psi)$ with φ, ψ regular (coherent, ...) formulas.
A *essentially algebraic* theory is a theory whose axioms are of the form $(\forall x_1, \ldots, x_n)(\top \Rightarrow \psi)$ with ψ a formula built from equality of terms.

Examples 3.1.11.
1. The theory of groups (rings, modules over a ring, ...) is a essentially algebraic theory.
2. The theory of torsion-free abelian groups is not a limit theory because the theory has a countable number of axioms of the form

$$(\forall x)(x^n = 0 \Rightarrow x = 0)$$

where x^n is $((((x.x).x)\ldots).x)$ for every $n > 1$, making it a regular theory.

3. The theory of totally ordered groups is conformed by the following set of sentences with the structure $\Sigma = \{+, -, e\} \amalg \{\leq\}$:

$$\begin{aligned}
&(\forall x)(\top \Rightarrow x \leq x)\\
&(\forall x, y)(x \leq y \wedge y \leq x \Rightarrow x = y)\\
&(\forall x, y, z)(x \leq y \wedge y \leq z \Rightarrow x \leq z)\\
&(\forall x, y)(\top \Rightarrow_x \leq y \vee y \leq x)\\
&(\forall x, y, z)(x \leq y \Rightarrow x + z \leq y + z)\\
&(\forall x, y, z)(x \leq y \Rightarrow z + x \leq z + y)\\
&(\forall x, y, z)(\top \Rightarrow (x + y) + z = x + (y + z))\\
&(\forall x)(\top \Rightarrow x - x = e)\\
&(\forall x)(\top_{\Rightarrow} x + e = x = e + x),
\end{aligned}$$

it is in particular not a regular theory because of the sentence

$$(\forall x, y)(\top \Rightarrow_x \leq y \vee y \leq x),$$

but is a coherent theory.

4. The theory of local rings is a coherent theory.

5. The theory of torsion abelian groups is not a coherent theory because of the axiom

$$(\forall x)\left(\top \Rightarrow \bigvee_{n>1} ()x^n = 0)\right),$$

but is a geometric theory.

Definition 3.1.12. A *model* of a theory is a Σ-structure M such that M satisfies all the axioms of the theory.

The truth value and logical operators on formulas are deeply connected with operations on subsets such as the intersection or the union. Say we have the formula $\varphi(x, y)$: $x^2 + y = 1$ for x, y real numbers. It is clear this formula is true for the pair $(0, 1)$ and false for the pair $(1, 2)$. One can see such formula as a map $\varphi : \mathbb{R} \times \mathbb{R} \to \{0, 1\}$ where $\varphi(0, 1) = 1$ and $\varphi(1, 2) = 0$. Then the validity of this formula is determined by the subset $\{(x, y)|\varphi(x, y)\} \subseteq \mathbb{R} \times \mathbb{R}$. Now assume we have two formulas on the same free variables $\phi(x, y)$ and $\psi(x, y)$. $\phi(x, y) \wedge \psi(x, y)$ is true precisely when both formulas are true. If $\{(x, y)|\phi(x, y)\}$ and $\{(x, y)|\psi(x, y)\}$ are the biggest subsets making those formulas true, then it is clear their intersection is the biggest subset making the whole formula true. Therefore we shall focus our attention on the subsets and their operations.

Definition 3.1.13. Let $(L, \leq)$ be a poset (partially ordered set). For any two elements $a, b \in L$, $m \in L$ is the *meet* $a \wedge b$ if

1. $m \leq a$ and $m \leq b$.
2. For any $m' \in L$ such that $m' \leq a$ and $m' \leq b$ then $m' \leq m$.

Similarly $j \in L$ is the *join* $a \vee b$ if

1. $a \leq j$ and $b \leq j$.
2. For any $j' \in L$ such that $a \leq j'$ and $b \leq j'$ then $j \leq j'$.

In particular $a \wedge a = a = a \vee a$. Meets and joins don't necessarily exist but if they do they are unique. When they exist for any pair of elements in L, they then define binary operations $\wedge$ and $\vee$ that are commutative and associative.

Definition 3.1.14. A *lattice* is a poset $(L, \leq)$ where the two binary operations $\wedge$ and $\vee$ exist.

As a consequence of the definition, the following property holds : for any $a_1, a_2, b_1, b_2 \in L$ such that $a_1 \leq b_1$ and $a_2 \leq b_2$ then

$$a_1 \wedge a_2 \leq b_1 \wedge b_2, \quad a_1 \vee a_2 \leq b_1 \vee b_2.$$

A *bounded* lattice is a lattice with a top 1 and a bottom 0, i.e for any element $a \in L$

$$0 \leq a \leq 1.$$

One can notice that if we see $(L, \leq)$ as a category, then it is a bounded lattice when it has all binary products $\wedge$, all binary coproducts $\vee$, a terminal object 1 and an initial one 0.

Alternatively, a lattice can be defined as a set L with two binary operations $\wedge$ and $\vee$ such that

$$a \wedge (a \vee b) = a, \quad a \vee (a \wedge b) = a.$$

Then one can derive a partial order by defining $a \leq b \iff a = a \wedge b$ and $\wedge$ and $\vee$ are precisely the meet and join for this partial order.

Example 3.1.15. Let X be a set. Then the powerset $\mathcal{P}(X)$, the set of all subsets of X, equipped with the inclusion $\subseteq$, the intersection $\cap$, the union $\cup$ is a bounded lattice. Moreover, $\emptyset$ is the bottom element and X the top element.

Definition 3.1.16. A *Heyting* algebra is a bounded lattice H equipped with a binary operation *implication* $\Rightarrow$, such that for any $a, b, c \in L$,

$$a \wedge b \leq c \iff a \leq b \Rightarrow c.$$

Again, if we see L as a category with finite products and finite coproducts, then L is a Heyting algebra precisely when for any object $b \in L$, the functor $- \wedge b$ has a right adjoint $b \Rightarrow -$. It is with no surprise that we define a *homomorphism* of Heyting algebras to be a morphism of sets preserving the Heyting algebra structure.

Example 3.1.17. Let X be a set and let us define an operation $\Rightarrow$ in $\mathcal{P}(X)$ by

$$(A \Rightarrow B) := \{x \in X | x \in A \Rightarrow x \in B\} \subseteq X.$$

It is easy to check that for any $A, B, C \in \mathcal{P}(X)$, $A \cap B \subseteq C \iff A \subseteq B \Rightarrow C$. Thus the powerset of a set is a Heyting algebra.

Proposition 3.1.18. *In any Heyting algebra H meets and joins are distributive, i.e for any $a, b, c \in H$,*

$$(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c),$$
$$(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c).$$

Proof. As a left adjoint $- \wedge c$ preserves coproducts, the first equality follows. The second equality can be obtained from the first one:

$$\begin{aligned}(a \vee c) \wedge (b \vee c) &= (a \wedge (b \vee c)) \vee (c \wedge (b \vee c)) \\ &= (a \wedge b) \vee (a \wedge c) \vee c \\ &= (a \wedge b) \vee c.\end{aligned}$$

□

Proposition 3.1.19. *In any Heyting algebra H and for any triple of elements $a, b, c \in H$:*

1. $a \leq b \iff (a \Rightarrow b) = 1$,
2. $a = (1 \Rightarrow a)$,
3. $a \Rightarrow (b \wedge c) = (a \Rightarrow b) \wedge (a \Rightarrow c)$.

Proof. $(a \Rightarrow b) = 1$ if and only if $1 \leq (a \Rightarrow b)$ which by adjunction is equivalent to $1 \wedge a \leq b$ i.e $a \leq b$.

$a \leq (1 \Rightarrow a)$ is always true since by adjunction this is equivalent to $a \wedge 1 \leq a$ i.e $a \leq a$. The other inequality is also true since $(1 \Rightarrow a) \wedge 1 \leq a$ is equivalent by adjunction to $(1 \Rightarrow a) \leq (1 \Rightarrow a)$.

Finally the last statement comes from the fact that $a \Rightarrow -$ has a left adjoint and thus preserves limits. □

Definition 3.1.20. Let H be a Heyting algebra. For any element $a \in H$, the *negation* is $\neg a := (a \Rightarrow 0)$.

Heyting algebras play a key role in intuitionistic logic.

Now, having a good look at the previous section, one can remark that every definition, especially the definition of structure 3.1.2, use only sets (the objects of **Set**) and arrows (the morphisms of **Set**). Hence it feels natural to forget about sets, and to generalise to any category with products and terminal object. A Σ-structure A is defined as a collection of sets, each associated to a different type, with operations defined as morphisms between sets of each type and relation defined as subsets.

Definition 3.1.21. Let $\mathscr{C}$ be a category with finite products and terminal object. Let $\Sigma = (S, \Sigma_{op} \amalg \Sigma_{rel})$ be a signature. A Σ-structure A in $\mathscr{C}$ is a set of objects $\{A_{s_1}, \ldots, A_{s_n}\}$ associated to each sort of S together with morphisms $\sigma_A : A_{s_1} \times \cdots \times A_{s_n} \to A_s$ for each $\sigma : s_1 \times \cdots \times s_n \to s$ in Σ_{op} and subobjects $\sigma_A \subseteq A_{s_1} \times \cdots \times A_{s_n}$ for each $\sigma \subseteq s_1 \times \cdots \times s_n$ in Σ_{rel}. The constants $e : \to s$ are associated to the points $e_A : 1 \to A_s$.

3.2 Topos

So far, when talking about models of a theory in a category $\mathscr{C}$, we have being sketching them or attributing them a number of diagrams in $\mathscr{C}$. A significant point to underline here is that there is a price to pay in exchange of a more generalised definition of what is a theory. When dealing with theories in our mathematician daily life, we do not draw such diagrams. It is more convenient to say, for example in the case of groups, $(\forall g, h, k \in G)(g + (h + k) = (g + h) + k)$, than to work with the diagram representing associativity. The usual set theoretic mathematical language we use allows us to manipulate things in **Set** without having to look at the actual category. Unfortunately this language does not hold in other categories. What does $g \in G$ means if G is only an object in any category and not a set ?

Another great thing about sets is that small collections of sets or maps, i.e objects and morphisms in **Set**, are again sets belonging to the category **Set**, while sets of objects or morphisms in another category $\mathscr{C}$ live outside the category.

Our next step is to find categories with enough structure, the topoi, to make sense of a language internal to them. First, we want to have an object that computes the morphisms between two objects X and Y, as $\mathbf{Hom}(X, Y)$ does for **Set**. Actually we want more than that, we want to evaluate our morphisms. If we have a morphism $f : X \times Y \to Z$, it is possible in **Set** to see this function as $f : X \to [Y \to Z]$ and for a fixed $x \in X$ one can evaluate this function and get a family of functions $f_x : Y \to Z$. This kind of manipulations are used a lot in mathematics. Notice $f : X \to [Y \to Z]$ is well defined as a morphism between two objects in **Set** because $\mathbf{Hom}(Y, Z)$ is a set. This leads us to the next definition

Definition 3.2.1. Let $\mathscr{C}$ be a category with finite products and terminal object. $\mathscr{C}$ is *cartesian closed* if for any object $Y, Z \in \mathscr{C}$, there exists an object $Z^Y \in \mathscr{C}$ such that there is a bijection

$$\mathbf{Hom}(X \times Y, Z) \cong \mathbf{Hom}(X, Z^Y)$$

natural in X, Y and Z. Z^Y is called the *object of morphisms (or internal* **Hom**) *from* Y *to* Z. For $f : X \to Z^Y$, we denote its *transpose* $\widehat{f} : X \times Y \to Z$.

In other terms, $\mathscr{C}$ is cartesian closed if all $Y \in \mathscr{C}$ the functor $- \times Y$ has a right adjoint $(-)^Y$, the *exponential map*. The counit of this adjonction is the *evaluation map* $\mathrm{ev}_{Y,Z} : Z^Y \times Y \to Z$.

Example 3.2.2. In **Set** Z^Y is $\mathbf{Hom}(Y, Z)$, the exponential map is the representable functor $\mathbf{Hom}(Y, -)$ and the evaluation map is the usual one $\mathrm{ev}(f, x) = f(x)$.

Proposition 3.2.3. *Any category of presheaves is cartesian closed.*

Proof. Consider the category of presheaves $\mathbf{Pr}(\mathscr{C})$ with $\mathscr{C}$ a small category.

Let be two presheaves $G, H : \mathscr{C}^{op} \to \mathbf{Set}$ in $\mathbf{Pr}(\mathscr{C})$. Define

$$H^G : \mathscr{C}^{op} \to \mathbf{Set},\ C \mapsto \mathbf{Nat}(\mathbf{Hom}_{\mathscr{C}}(-, C) \times G, H),$$

and defined obviously on the morphisms $f : C \to C'$ by

$$H^G(f)(\eta) := \eta \circ (\mathbf{Hom}(-, f) \times \mathrm{Id}_{G(-)}),$$

with $\eta \in \mathbf{Nat}(\mathbf{Hom}_{\mathscr{C}}(-, C') \times G, H)$. Let be another presheaf $F : \mathscr{C} \to \mathbf{Set}$, we compute

$$\begin{aligned}
\mathbf{Nat}(F \times G, H) &\cong \mathbf{Nat}\left((\mathrm{colim}_i \mathbf{Hom}_{\mathscr{C}}(-, C_i)) \times G, H\right) && \text{(by B.0.3)} \\
&\cong \mathbf{Nat}\left(\mathrm{colim}_i (\mathbf{Hom}_{\mathscr{C}}(-, C_i) \times G), H\right) && \text{($- \times G$ preserves colimits by B.0.1)} \\
&\cong \lim_i \mathbf{Nat}\left(\mathbf{Hom}_{\mathscr{C}}(-, C_i) \times G, H\right) && \text{(by B.0.2)} \\
&\cong \lim_i H^G(C_i) \\
&\cong \lim_i \mathbf{Nat}\left(\mathbf{Hom}_{\mathscr{C}}(-, C_i), H^G\right) && \text{(by Yoneda A.0.1)} \\
&\cong \mathbf{Nat}\left(\mathrm{colim}_i \mathbf{Hom}_{\mathscr{C}}(-, C_i), H^G\right) && \text{(by B.0.2)} \\
&\cong \mathbf{Nat}\left(F, H^G\right).
\end{aligned}$$

Thus $\mathbf{Pr}(\mathscr{C})$ is cartesian closed. □

We won't be able to prove that any Grothendieck topos is cartesian closed. The proof requires some tools useful when working with reflective subcategories that we have not introduced. However we can still give an intuition of the argument:

Let be a site $(\mathscr{C}, T)$. The only thing that is needed is to prove the functor H^G defined in $\mathbf{Pr}(\mathscr{C})$ is a sheaf in $\mathbf{Sh}(\mathscr{C}, T)$ if H is too. This would implies exponentiations in $\mathbf{Sh}(\mathscr{C}, T)$ are computed as in $\mathbf{Pr}(\mathscr{C})$ and since binary products of sheaves are also sheaves (by preservation of binary product by i and $\#$), the adjunction between $- \times G$ and $(-)^G$ in $\mathbf{Pr}(\mathscr{C})$ is also an adjunction in $\mathbf{Sh}(\mathscr{C}, T)$. Thus $\mathbf{Sh}(\mathscr{C}, T)$ is cartesian closed since $\mathbf{Pr}(\mathscr{C})$ is. But the proof that H^G is indeed a sheaf requires some reflective subcategories and orthogonality properties. A complete proof can be found in [3, Proposition 3.4.17].

Proposition 3.2.4. *Any Grothendieck topos is cartesian closed.*

Remark 3.2.5. *For all objects $Y, Z \in \mathscr{C}$ the points of the object of morphisms, $1 \to Z^Y$, are in bijection with the arrows $Y \to Z$.*

This last remark is the key behind this idea of "object of morphisms". Each element, each point of $1 \to Z^Y$ amounts for a morphism between Y and Z and vice-versa.

Now let $\mathrm{Epi}(X, Y)$ be the object computing the epimorphisms. For now we only have Y^X the object of morphisms. It is clear in the case of **Set** we expect $\mathrm{Epi}(X, Y)$ to be a subset of $\mathbf{Hom}(X, Y)$, precisely the subset of surjective maps. In our categorical language we already have a concept of subobject$S \hookrightarrow X$. But this is not sufficient. If we pick a subset of $\mathbf{Hom}(X, Y)$, how can we know it is the subset of surjective maps, without looking inside it ? We need to be able to classify the subobjects of an object.

In **Set** there are actually two ways to deal with subsets. The first one is obviously to consider them as sets themselves $S \subseteq X$. The second one is to consider them as maps that characterise them, the maps $\chi_S : X \to \{0, 1\}$ assigning 1 to $\chi_S(x)$ if $x \in S$ and assigning 0 otherwise. Here 1 and 0 act as respectively truth and false values. S is then characterised as the biggest subset of X such that $\chi_S(S) = \{1\}$. Biggest means that for any other subset of X, S' such that $\chi_S(S') = \{1\}$, then S' is a subset of S. Let us draw the diagram for the situation we have just described :

$$\begin{tikzcd}
S' \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, hook, bend right] & & \\
& S \arrow[r] \arrow[d, hook, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & \{1\} \arrow[d, hook, "\top"] \\
& X \arrow[r, "\chi_S"'] & \{0,1\}
\end{tikzcd}$$

We see that given the set $\{0, 1\}$ and the morphism $\top : \{1\} \to \{0, 1\} : 1 \mapsto 1$, one can associate to any subset $S \subseteq X$ a morphism χ_S such that the diagram is a pullback.

Conversely, any morphism with values in $\{0,1\}$ can be seen as the characteristic map of a subset. It is time to set up properly the definition :

Definition 3.2.6. Let be $\mathscr{E}$ a topos and an object $X \in \mathscr{E}$. A *subobject* of X is an equivalence class of monomorphism $s : S \hookrightarrow X$ with codomain X in $\mathscr{E}$ such that $s : s \hookrightarrow X$ and $s' : S' \hookrightarrow X$ are related if and only if there exists an arrow $e : S \to S'$ such that $s' \circ e = s$.

Definition 3.2.7. Let $\mathscr{C}$ be a category with finite limits. A *subobject classifier* is an object Ω in $\mathscr{C}$ with a monomorphism $\top : 1 \hookrightarrow \Omega$ such that for any monomorphism $S \hookrightarrow X$ there exists a unique morphism $\chi_S : X \to \Omega$ such that the following diagram is a pullback

$$\begin{tikzcd} S \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, hook, "\top"] \\ X \arrow[r, "\chi_S"'] & \Omega \end{tikzcd}$$

The morphism $\top : 1 \to \Omega$ is called the *true morphism* and we usually refer to Ω as the *subobject classifier* without specifying the true morphism. χ_S is called the *characteristic morphism* of S.

The name "true" morphism will appear clear later. For now all we can say about this name is the following : in **Set** a formula, say $x \in \mathbb{R}; x^2 = 0$, is either true or false. For some values of $x \in \mathbb{R}$ it is true and for the others it is false. Hence we can see the truth value of this formula as the subset of $\mathbb{R}$ that contains all the x making this formula true, here $\{0\}$. Therefore, the subobject classifier and the true morphism play a role when computing the truth value of a formula.

Example 3.2.8. In **Set** the subobject classifier is $\Omega = \{0,1\}$ and the true morphism is $\top : \{*\} \to \{0,1\}, * \mapsto 1$.

The next property is our main reason behind the name subobject classifier.

Proposition 3.2.9. *If $\mathscr{C}$ has a subobject classifier, then for any object X there is a natural bijection between the subobjects of X and the morphisms $X \to \Omega$.*

$$\mathrm{Sub}(X) \cong \mathbf{Hom}(X, \Omega)$$

Proof. Assume we have a subobject $S \subseteq X$, then there exists a morphism $\chi_S : X \to \Omega$. Taking the pullback of χ_S along t gives us back S by definition. Conversely let be $f : X \to \Omega$ and let S be the pullback of f along t. Since monomorphisms are pullback stable and t is a mono, the projection $S \to X$ is a mono, hence S is a subobject of X. Now for the naturality, assume we have a morphism $f : X' \to X$. For any subobject $B \subseteq X$, taking its pullback along f yields a subobject $f^{-1}(B) \subseteq X'$. The bijection is natural in X if and only if $\chi_{f^{-1}(B)} = \chi_B \circ f$. We only need to prove $\chi_B \circ f$ is the characteristic morphism of $f^{-1}(B)$. Consider the following diagram :

$$\begin{tikzcd}
f^{-1}(B) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, hook, "\top"] \\
X' \arrow[r, "f"'] & X \arrow[r, "\chi_B"'] & \Omega
\end{tikzcd}$$

Since the left and right square are pullbacks, the whole rectangle is a pullback. By definition this implies that $\chi_B \circ f$ is the characteristic morphism of $f^{-1}(B)$. □

Proposition 3.2.10. *Any category of presheaves has a subobject classifier.*

Proof. Let $\mathscr{C}$ be a small category. We want to construct a presheaf Ω in $\mathscr{C}$ such that for any presheaf $X \in \mathbf{Pr}(\mathscr{C})$, $\mathrm{Sub}(X) \cong \mathbf{Nat}(X, \Omega)$. But by the Yoneda lemma (A.0.1), for any object $C \in \mathscr{C}$, $\Omega(C) \cong \mathbf{Nat}(\mathbf{Hom}_{\mathscr{C}}(-, C), \Omega)$. Thus let us define

$$\Omega(C) = \mathrm{Sub}(\mathbf{Hom}_{\mathscr{C}}(-, C)).$$

Let $f : C' \to C$ be a morphism in $\mathscr{C}$. For any subobject $S \in \mathrm{Sub}(\mathbf{Hom}_{\mathscr{C}}(-, C))$, pulling back along $\mathbf{Hom}_{\mathscr{C}}(-, f)$ yields a subobject $\Omega(f)(S) \in \mathrm{Sub}(\mathbf{Hom}_{\mathscr{C}}(-, C'))$ since pullbacks preserve monomorphisms. By the next diagram, Ω is a well defined presheaf.

$$\begin{tikzcd}
\Omega(g) \circ \Omega(f)(S) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega(f)(S) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, hook] \\
\mathbf{Hom}_{\mathscr{C}}(-, C'') \arrow[r, "\mathbf{Hom}_{\mathscr{C}}(-, g)"'] & \mathbf{Hom}_{\mathscr{C}}(-, C') \arrow[r, "\mathbf{Hom}_{\mathscr{C}}(-, f)"'] & \mathbf{Hom}_{\mathscr{C}}(-, C)
\end{tikzcd}$$

Since limits are computed pointwise in $\mathbf{Pr}(\mathscr{C})$, see B.0.1, the terminal presheaf 1 is defined by $1(C) = \{*\}$ for any object $C \in \mathscr{C}$. Define the true morphism $\top : 1 \to \Omega$ component wise as $\top(C)(*) = \mathbf{Hom}_{\mathscr{C}}(-, C)$ for every $C \in \mathscr{C}$.

Now let $s : S \hookrightarrow X$ be a subobject (this is an equivalence class) in $\mathbf{Pr}(\mathscr{C})$ and let us construct its characteristic morphism $\chi_S : X \to \Omega$. In term of components, we want for any $C \in \mathscr{C}$ a map $\chi_S(C) : X(C) \to \mathrm{Sub}(\mathbf{Hom}_{\mathscr{C}}(-, C))$. By the Yoneda lemma (A.0.1), any $x \in X(C)$ corresponds to a natural transformation $\ulcorner x \urcorner : \mathbf{Hom}_{\mathscr{C}}(-, C) \to X$, defined as $\ulcorner x \urcorner(D)(f) = X(f)(x)$ for any morphism $f : D \to C$. Take the pullback $\chi_S(C)(x)$ of the mono s along $\ulcorner x \urcorner$. Since pullbacks preserve monomorphisms, $\chi_S(C)(x) \in \mathrm{Sub}(\mathbf{Hom}_{\mathscr{C}}(-, C))$.

Remark that for any morphisms $g : C' \to C$, $f : D \to C'$ both in $\mathscr{C}$ and for any $x \in X(C)$, by Yoneda lemma

$$\ulcorner Xg(x) \urcorner(D)(f) = X(f)(X(g)(x)) = X(g \circ f)(x) = \ulcorner x \urcorner(D)(g \circ f).$$

Thus $\ulcorner Xg(x) \urcorner = \ulcorner x \urcorner \circ \mathbf{Hom}_{\mathscr{C}}(-, g)$. Then

$$\begin{tikzcd}
\Omega(g)(\chi_S(C)(x)) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \chi_S(C)(x) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, hook, "s"] \\
\mathbf{Hom}_{\mathscr{C}}(-, C') \arrow[r, "\mathbf{Hom}_{\mathscr{C}}(-,g)"'] & \mathbf{Hom}_{\mathscr{C}}(-, C) \arrow[r, "\ulcorner x \urcorner"'] & X
\end{tikzcd}$$

since both squares are pullbacks by definition, so is the whole square. Hence $\Omega(g)(\chi_S(C)(x))$ is the pullback of s along $\ulcorner Xg(x) \urcorner$. Thus for any $x \in X(C)$ and any morphism $g : C' \to C$:

$$\Omega(g)(\chi_S(C)(x)) = \chi_S(C')(Xg(x)).$$

$\chi_S : X \to \Omega$ given by $\{\chi_S(C)\}_{C \in \mathscr{C}}$ is a natural transformation.

Now since pullbacks are computed pointwise in $\mathbf{Pr}(\mathscr{C})$, for any $D \in \mathscr{C}$, $\chi_S(C)(x)(D)$ is a pullback in **Set** given by

$$\chi_S(C)(x)(D) = \{f : D \to C | X(f)(x) \in S(D)\}.$$

If $x \in S(C)$, then $\ulcorner x \urcorner$ factors through S and thus $\chi_S(C)(x) = \mathbf{Hom}_{\mathscr{C}}(-, C)$. If $\chi_S(C)(x) = \mathbf{Hom}_{\mathscr{C}}(-, C)$, then $\mathrm{Id}_C \in \chi_S(C)(x)(C)$ and then $x = X(\mathrm{Id}_C)(x) \in S(C)$. Therefore

$$\chi_S(C)(x) = \mathbf{Hom}_{\mathscr{C}}(-, C) \iff x \in S(C).$$

This makes the following diagram into a pullback, since limits are computed pointwise and this diagram evaluated pointwise is a pullback by our computation.

$$\begin{tikzcd}
S \arrow[r] \arrow[d, hook] & 1 \arrow[d, hook, "\top"] \\
X \arrow[r, "\chi_S"'] & \Omega
\end{tikzcd}$$

What remains is to prove χ_S is unique. Let us remark in the specific case where $X = \mathbf{Hom}_{\mathscr{C}}(-, A)$ for some $A \in \mathscr{C}$, $\chi_S(C)(x) = \Omega(x)(S)$ for all $x \in \mathbf{Hom}_{\mathscr{C}}(C, A)$ and all $S \hookrightarrow \mathbf{Hom}_{\mathscr{C}}(-, A)$. Thus the characteristic morphism of a subobject of $\mathbf{Hom}_{\mathscr{C}}(-, A)$ for some $A \in \mathscr{C}$ is uniquely determined. Now for any subobject $s : S \hookrightarrow X$, let $\psi : X \to \Omega$ be another characteristic morphism candidate. For every C object in $\mathscr{C}$ and x element of $X(C)$, by the Yoneda lemma, the natural transformation $\psi \circ \ulcorner x \urcorner : \mathbf{Hom}_{\mathscr{C}}(-, C) \to \Omega$ is in correspondence with $\psi(C)(\ulcorner x \urcorner(C)(\mathrm{Id}_C)) = \psi(C)(X(\mathrm{Id}_C)(x)) = \psi(C)(x)$. Similarly $\chi_S \circ \ulcorner x \urcorner$ is in correspondence with $\chi_S(C)(x)$. Thus

$$\chi_S = \psi \iff (\chi_S \circ \ulcorner x \urcorner = \psi \circ \ulcorner x \urcorner \quad \forall C \in \mathscr{C}, x \in X(C)).$$

But $\chi_S \circ \ulcorner x \urcorner$ and $\psi \circ \ulcorner x \urcorner$ are now two potential characteristic morphisms for a subobject of $\mathbf{Hom}(-, A)$ (see diagram below). We know such characteristic morphism is uniquely defined. Thus $\chi_S \circ \ulcorner x \urcorner = \psi \circ \ulcorner x \urcorner \quad \forall C \in \mathscr{C}, x \in X(C)$, therefore $\chi_S = \psi$.

$$\begin{tikzcd}
\chi_S(C)(x) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[r] \arrow[d, hook, "r"] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, hook, "\top"] \\
\mathbf{Hom}_{\mathscr{C}}(-, C) \arrow[r, "\ulcorner x \urcorner"'] & X \arrow[r, "\chi_S"'] & \Omega
\end{tikzcd}$$

□

Proposition 3.2.11. *Any Grothendieck topos has a subobject classifier.*

Proof. Let $(\mathscr{C}, T)$ be a site and $\mathbf{Sh}(\mathscr{C}, T)$ the category of sheaves on this site. Remember from 1.1.5 and 1.1.6 that $\mathbf{Sh}(\mathscr{C}, T)$ is uniquely determined by its associated closure operator on $\mathbf{Pr}(\mathscr{C})$ defined for any subobject $S \hookrightarrow X$ in $\mathbf{Pr}(\mathscr{C})$ by the pullback of $i\#(S) \hookrightarrow i\#(X)$ along the unit $\eta(B) : B \to i\#(B)$. Since the counit $\epsilon(R) : R \to \#i(R)$ is an isomorphism, every subsheaf of a sheaf is closed.

Recall that in $\mathbf{Pr}(\mathscr{C})$, by 3.2.10, the subobject classifier is given by $\Omega(C) = \{S | \, S \in \mathrm{Sub}(\mathbf{Hom}_{\mathscr{C}}(-, C))\}$ with the true morphism $\top : 1 \to \Omega$, $\top(C)(*) = \mathbf{Hom}_{\mathscr{C}}(-, C)$. Let us take the subpresheaf of Ω

$$\Omega_{\mathrm{Cl}}(C) = \{S | \, S \text{ closed subpresheaf of } \mathbf{Hom}_{\mathscr{C}}(-, C)\}$$

with for any morphism $f : C' \to C$ in $\mathscr{C}$, $\Omega_{\mathrm{Cl}}(f) := \Omega(f)$ acting on the subpresheaves by pullback along $\mathbf{Hom}_{\mathscr{C}}(-, f)$. The restriction on the closed subpresheaves is well defined since the closure operator is pullback stable by $\alpha^{-1}(\overline{S}) = \overline{\alpha^{-1}(S)}$, for any morphism of presheaves $\alpha : Y \to X$. Remark that $\mathbf{Hom}_{\mathscr{C}}(-, C)$ for any object $C \in \mathscr{C}$ is a subpresheaf of $\mathbf{Hom}_{\mathscr{C}}(-, C)$ such that it must be contained on its closure that is again a subpresheaf of $\mathbf{Hom}_{\mathscr{C}}(-, C)$. Thus $\mathbf{Hom}_{\mathscr{C}}(-, C)$ is its own closure. Then $\mathbf{Hom}_{\mathscr{C}}(-, C) \in \Omega_{\mathrm{Cl}}(C)$. Hence the true morphism $\top : 1 \to \Omega$ factors through $\top : 1 \to \Omega_{\mathrm{Cl}}$.

Let us see how this subobject classifier candidate Ω_{Cl} associates a subsheaf to a characteristic morphism. Assume $S \hookrightarrow X$ is a subsheaf, i.e a subobject in $\mathbf{Sh}(\mathscr{C}, T)$. We already know this subsheaf can be seen as a closed subpreasheaf $iS \hookrightarrow iX$. Its characteristic morphism in $\mathbf{Pr}(\mathscr{C})$ is $\chi_{iS} : iX \to \Omega$ given by $\chi_{iS}(C)(x) = (\ulcorner x \urcorner)^{-1}(iS)$, where $x \in iX(C)$ and $\ulcorner x \urcorner$ is the correspondence of x by Yoneda (see 3.2.10). Since iS is closed,

$$(\ulcorner x \urcorner)^{-1}(iS) = (\ulcorner x \urcorner)^{-1}(\overline{iS}) = \overline{(\ulcorner x \urcorner)^{-1}(iS)}.$$

Thus $(\ulcorner x \urcorner)^{-1}(iS) \in \Omega_{\mathrm{Cl}}(C)$. The characteristic morphism factors through Ω_{Cl}. Since both χ_{iS} and $\top$ factor through Ω_{Cl}, and the inclusion $j : \Omega_{\mathrm{Cl}} \to \Omega$ is a monomorphism, iS is the pullback of χ_{iS} along $\top$ (as pullback over Ω or Ω_{Cl}). Now assume we have another characteristic morphism candidate $\psi : iX \to \Omega_{\mathrm{Cl}}$. Thus $j \circ \psi : iX \to \Omega$ is the characteristic morphism of iS in $\mathbf{Pr}(\mathscr{C})$. Thus $j \circ \psi = j \circ \chi_{iS}$, and since j is a monomorphism, we get $\psi = \chi_{iS}$.

$$\begin{tikzcd}
i(S) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\top"'] & \\
i(X) \arrow[r, "\psi"] & \Omega_{\mathrm{Cl}} \arrow[r, hook, "j"] & \Omega
\end{tikzcd}$$

Our last step is to show that Ω_{Cl} is a sheaf in $\mathbf{Sh}(\mathscr{C}, T)$. If Ω_{Cl} is a sheaf, then $\top : 1 \to \Omega_{\mathrm{Cl}}$ is in $\mathbf{Sh}(\mathscr{C}, T)$ since 1 is a sheaf because limits in $\mathbf{Sh}(\mathscr{C}, T)$ are precisely the ones in $\mathbf{Pr}(\mathscr{C})$ (see 1.1.10). Let be an object $C \in \mathscr{C}$, a covering sieve $r : R \to \mathbf{Hom}_{\mathscr{C}}(-, C)$, and a morphism $\alpha : R \to \Omega_{\mathrm{Cl}}$ in $\mathbf{Pr}(\mathscr{C})$. We have to show there exists a unique morphism $\beta : \mathbf{Hom}_{\mathscr{C}}(-, C) \to \Omega_{\mathrm{Cl}}$ such that $\beta \circ r = \alpha$. Take the pullback S of α along $\top : 1 \to \Omega_{\mathrm{Cl}}$. Since $\top$ is a monomorphism, we get a monomorphism $s : S \to R$. Hence S is a subobject of $\mathbf{Hom}_{\mathscr{C}}(-, C)$. Let χ_S, χ_R, ϕ_S be the characteristic morphisms of $r \circ s$, r and s in $\mathbf{Pr}(\mathscr{C})$. Recall that for any $D \in \mathscr{C}$ and $x \in \mathbf{Hom}_{\mathscr{C}}(D, C)$ in $\mathscr{C}$, χ_S, χ_R are defined as $\chi_S(D)(x)$,$\chi_R(D)(x)$ the pullbacks of $r \circ s$, r along $\ulcorner x \urcorner : \mathbf{Hom}_{\mathscr{C}}(-, D) \to \mathbf{Hom}_{\mathscr{C}}(-, C)$, where $\ulcorner x \urcorner = \mathbf{Hom}_{\mathscr{C}}(-, x)$. For any $D \in \mathscr{C}$ and $y \in R(D)$, $\phi_S(D)(y)$ is the pullback of s along $\ulcorner y \urcorner : \mathbf{Hom}_{\mathscr{C}}(-, D) \to R$.

$$\begin{tikzcd}
R \arrow[r] \arrow[d, hook, "r"] & 1 \arrow[d] \\
\mathbf{Hom}_{\mathscr{C}}(-, C) \arrow[r, "\chi_R"] & \Omega
\end{tikzcd}
\qquad
\begin{tikzcd}
S \arrow[r] \arrow[d, hook, "r \circ s"] & 1 \arrow[d] \\
\mathbf{Hom}_{\mathscr{C}}(-, C) \arrow[r, "\chi_S"] & \Omega
\end{tikzcd}
\qquad
\begin{tikzcd}
S \arrow[r] \arrow[d, hook, "s"] & 1 \arrow[d] \\
R \arrow[r, "\phi_S"] & \Omega
\end{tikzcd}$$

$$\begin{tikzcd}
\chi_R(D)(x) \arrow[r] \arrow[d] & R \arrow[d, hook, "r"] \\
\mathbf{Hom}_{\mathscr{C}}(-, D) \arrow[r, "\ulcorner x \urcorner"'] & \mathbf{Hom}_{\mathscr{C}}(-, C)
\end{tikzcd}
\qquad
\begin{tikzcd}
\chi_S(D)(x) \arrow[r] \arrow[d] & S \arrow[d, hook, "r \circ s"] \\
\mathbf{Hom}_{\mathscr{C}}(-, D) \arrow[r, "\ulcorner x \urcorner"'] & \mathbf{Hom}_{\mathscr{C}}(-, C)
\end{tikzcd}
\qquad
\begin{tikzcd}
\phi_S(D)(y) \arrow[r] \arrow[d] & S \arrow[d, hook, "s"] \\
\mathbf{Hom}_{\mathscr{C}}(-, D) \arrow[r, "\ulcorner y \urcorner"'] & R
\end{tikzcd}$$

Now in 3.2.10 we showed that $x \in R(D) \iff \chi_R(D)(x) = \mathbf{Hom}_{\mathscr{C}}(-, D)$. Thus if $x \in R(D)$ then the morphism p in the diagram of the pullback $\chi_R(D)(x)$ is of the form $p : \mathbf{Hom}_{\mathscr{C}}(-, D) \to R(D)$. In particular $p(D)(\mathrm{Id}_D) = x$, this can be computed knowing that pullback are pointwise, and thus p is by Yoneda (A.0.1) the natural transformation $\ulcorner x \urcorner$ associated to $x \in R(D)$. Hence the pullback of p along s is precisely $\phi_S(D)(x)$.

$$\begin{tikzcd}
\phi_S(D)(x) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \chi_R(D)(x) \arrow[r, hook] \arrow[d, "p = \ulcorner x \urcorner"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Hom}_{\mathscr{C}}(-, D) \arrow[d, "\ulcorner x \urcorner"'] \\
S \arrow[r, hook, "s"] & R \arrow[r, hook] & \mathbf{Hom}_{\mathscr{C}}(-, C)
\end{tikzcd}$$

If $x \in R(D)$, then both left and right squares are pullbacks and thus, so is the whole square. But this is precisely the pullback of $r \circ s$ along $\ulcorner x \urcorner$. Thus if $x \in R(D)$, $\chi_S(D)(x) = \phi_S(D)(x)$. S was defined as the pullback of $\alpha : R \to \Omega_{\mathrm{Cl}}$ along $\top : 1 \to \Omega_{\mathrm{Cl}}$. Since the inclusion $j : \Omega_{\mathrm{Cl}} \to \Omega$ is a monomorphism, then S is also the pullback of $j \circ \alpha$ along $\top : 1 \to \Omega$. But by uniqueness of the characteristic morphism, $j \circ \alpha = \chi_S$.

Hence for any $x \in R(D)$, $j(D)(\alpha(D)(x)) = \chi_S(D)(x)$. Now consider the morphism of presheaves $c : \Omega \to \Omega_{\mathrm{Cl}}$ defined on the objects as the closure operator, i.e for any object $C \in \mathscr{C}$ and subfunctor $S \hookrightarrow \mathbf{Hom}(-, A)$, $c(A)(S) = \overline{S}$. This is a well defined natural transformation since for any morphism $f : D \to C$ in $\mathscr{C}$, $f^{-1}\overline{S} = \overline{f^{-1}S}$ and thus $f^{-1}(c(C)(S)) = c(D)(f^{-1}S)$. Remark that $c \circ j = \mathrm{Id}_{\Omega_{\mathrm{Cl}}}$ since by definition of the universal closure operator $\overline{\overline{S}} = \overline{S}$. Thus for any $x \in R(D)$, $j(D)(\alpha(D)(x)) = \chi_S(D)(x)$ implies $c(D) \circ j(D)(\alpha(D)(x)) = c(D) \circ \chi_S(D)(x)$, so $\alpha(D)(x) = c(D) \circ \chi_S(D)(x)$. Therefore $\alpha = c \circ \chi_S \circ r$. $c \circ \chi_S$ is an extension of α.

What remains is to prove this extension is unique. Assume we have a natural transformation $\psi : \mathbf{Hom}_{\mathscr{C}}(-, C) \to \Omega_{\mathrm{Cl}}$ such that $\psi \circ r = \alpha$. Consider the following diagram where both squares are pullbacks

$$
\begin{tikzcd}
S \arrow[r, hook, "v"'] \arrow[d, hook, "s"] \arrow[dr, phantom, "\lrcorner", very near start] & U \arrow[r] \arrow[d, hook, "u"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, hook, "\top"'] \\
R \arrow[r, hook, "r"] & \mathbf{Hom}_{\mathscr{C}}(-, C) \arrow[r, "\psi"] & \Omega_{\mathrm{Cl}}.
\end{tikzcd}
$$

s is the subobject corresponding to α, it is in particular a closed subobject of $\mathbf{Hom}_{\mathscr{C}}(-, C)$. r is a covering sieve, thus by 1.1.7 it is dense. u is the subobject corresponding to ψ and thus is closed. Since s is dense so is v (pullback of an isomorphism is again an isomorphism). Hence U is the closure of S in $\mathbf{Hom}_{\mathscr{C}}(-, C)$. Therefore for any $x \in R(D)$, $\psi(D)(x) = \overline{\chi_S(D)(x)} = c(D)(\chi_S(D)(x))$. Thus $c \circ \chi_S = \psi$. $\square$

Definition 3.2.12. A *topos*, also called *elementary topos*, is a category $\mathscr{C}$ that

1. has all finite limits,
2. is cartesian closed,
3. has a subobject classifier.

Examples 3.2.13.
- **Set** is a topos.
- Any Grothendieck topos is a topos (by 1.1.10, 3.2.4, 3.2.11).
- **FinSet** is a topos that is not a Grothendieck topos. **FinSet** inherits its subobject classifier and its exponential map from **Set**. It has all finite limits, thus **FinSet** is a topos. But if **FinSet** were to be a Grothendieck topos, it would have all small limits by 1.1.10, which we know to be false since it doesn't have non finite products.

A subobject classifier allows us to understand subobjects of X as morphism $X \to \Omega$ and the cartesian closed property allows us to see this morphism as a a point of Ω^X. Hence, Ω^X is effectively the object computing the subobjects of X. In other terms one can see Ω^X as the power of X. In fact, it is even true that in the case of **Set**, $\Omega^X \cong \mathcal{P}(X)$.

Since we have objects of morphisms and objects of subobjects, we can compute in a topos the notion of domain and image of a morphism as we are used to in the case of **Set**. The next example is the construction of the object of epimorphisms between X and Y, i.e morphisms $X \to Y$ with image Y. For the sake of the construction we shall do, we will admit that a topos is a regular category. A proof will come later on, see 3.3.7.

Let $\mathscr{E}$ be a topos. We want to construct a subobject of the object of morphisms from X to Y, for any two objects of $\mathscr{E}$, computing the epimorphisms. We know intuitively from **Set** that epimorphisms are the morphisms whose image is the whole codomain. We shall then construct such object with the help of image factorization. The idea would be for any morphism $f : X \to Y$ to take the characteristic morphism of its image and then generalize to a map $\mathrm{im} : \mathbf{Hom}(X, Y) \to \mathbf{Hom}(Y, \Omega)$ that associates to each morphism the constructed characteristic map. Thus we could look for the subobject of $\mathbf{Hom}(X, Y)$ sent by im to $Y \to 1 \xrightarrow{\top} \Omega$. But there a slight problem to this plan. As one may have noticed, we started by saying thing about morphisms $X \to Y$ in $\mathscr{E}$, and ended up externally to $\mathscr{E}$ by considering a map $\mathbf{Hom}(X, Y) \to \mathbf{Hom}(Y, \Omega)$. Therefore we need a technical trick to make sure we stay in our category $\mathscr{E}$.

Let us introduce $C \in \mathscr{E}$ a parameter object. Let be $f : C \to Y^X$ and its transpose $\widehat{f} : C \times X \to Y$. Let $\mathrm{Im}_C(f)$ be the image of $(\pi_1^{C\times X}, \widehat{f})$ where $\pi_1^{C\times X}$ stands for the first projection of the product $C \times X$.

$$\begin{tikzcd}
C \times X \arrow[rr, "(\pi_1^{C\times X}, \widehat{f})"] \arrow[dr, two heads] & & C \times Y \\
& \mathrm{Im}_C(f) \arrow[ur, hook] &
\end{tikzcd}$$

In particular, $\mathrm{Im}_C(f)$ is subobject of $C \times Y$. Define $\mathrm{im}_C(f) : C \to \Omega^Y$ as the transpose of the characteristic morphism of $\mathrm{Im}_C(f)$. This construction yields a map

$$\mathrm{im}_C : \mathbf{Hom}(C, Y^X) \to \mathbf{Hom}(C, \Omega^Y).$$

If we get rid of the dependency on the object parameter C, we would get a nice morphism $Y^X \to \Omega^Y$. This is the goal of the next lemma :

Lemma 3.2.14. *The map* $\mathrm{im}_C : \mathbf{Hom}(C, Y^X) \to \mathbf{Hom}(C, \Omega^Y)$ *is natural in* C *and determines a unique morphism* $\mathrm{im} : Y^X \to \Omega^Y$*.*

Proof. Let $\alpha : C' \to C$ be an arbitrary morphism of $\mathscr{E}$. Naturality means the following diagram commutes

$$\begin{tikzcd}
\mathbf{Hom}(C, \Omega^Y) \arrow[r, "\mathrm{im}_C"] \arrow[d, "-\circ\alpha"] & \mathbf{Hom}(C, \Omega^Y) \arrow[d, "-\circ\alpha"] \\
\mathbf{Hom}(C', Y^X) \arrow[r, "\mathrm{im}_{C'}"'] & \mathbf{Hom}(C', \Omega^Y)
\end{tikzcd}$$

i.e $\mathrm{im}_{C'}(f\alpha) = \mathrm{im}_C(f) \circ \alpha$. Now first notice that by definition of cartesian closeness $\widehat{f\alpha} = \widehat{f}(\alpha \times 1_X)$.

$$\begin{tikzcd}
\mathbf{Hom}(C, Y^X) \arrow[d, "-\circ\alpha"] & \cong & \mathbf{Hom}(C \times X, Y) \arrow[d, "-\circ(\alpha\times 1_X)"] \\
\mathbf{Hom}(C', Y^X) & \cong & \mathbf{Hom}(C' \times X, Y)
\end{tikzcd}$$

Hence the following diagram commutes

$$\begin{tikzcd}
C' \times X \arrow[r, "\alpha\times 1_X"] \arrow[d, "(\pi_1^{C'\times X}{,}\widehat{f\alpha})"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \times X \arrow[d, "(\pi_1^{C\times X}{,}\widehat{f})"] \\
C' \times Y \arrow[r, "\alpha\times 1_Y"'] & C \times Y
\end{tikzcd}$$

and moreover it is a pullback. Since image factorization is pullback stable, $\mathrm{Im}_{C'}(f\alpha)$ is the pullback of $\mathrm{Im}_C(f)$ along $\alpha \times 1_Y$, yielding a morphism $(\alpha \times 1_Y)^{-1} : \mathrm{Sub}(C \times Y) \to \mathrm{Sub}(C' \times Y)$ that restricts on the images, $(\alpha \times 1_Y)^{-1} : \mathrm{Im}_{C'}(f\alpha) \to \mathrm{Im}_C(f)$. Hence since $\mathscr{C}$ is cartesian closed and by property of the subobject classifier

$$\begin{tikzcd}
\mathrm{Sub}(C \times Y) \arrow[d, "(\alpha\times 1_Y)^{-1}"'] & \cong & \mathbf{Hom}(C \times Y, \Omega) \arrow[d, "-\circ(\alpha\times 1_Y)"] & \cong & \mathbf{Hom}(C, \Omega^Y) \arrow[d, "-\circ\alpha"] \\
\mathrm{Sub}(C' \times Y) & \cong & \mathbf{Hom}(C' \times Y, \Omega) & \cong & \mathbf{Hom}(C', \Omega^Y)
\end{tikzcd}$$

thus $\mathrm{im}_{C'}(f\alpha) = \mathrm{im}_C(f) \circ \alpha$. Hence im_C is natural in C and $\mathrm{im} : \mathbf{Hom}(-, Y^X) \Rightarrow \mathbf{Hom}(-, \Omega^Y)$ is a natural transformation. Then, by the Yoneda lemma, this natural transformation correspond to a unique morphism $\mathrm{im} : Y^X \to \Omega^Y$. This natural transformation is determined for any $f : C \to Y^X$ by $\mathrm{im} \circ f = \mathrm{im}_C(f)$. $\square$

Now, let $\top_Y : 1 \to \Omega^Y$ be the transpose of $1 \times Y \to 1 \xrightarrow{\top} \Omega$ and $\mathrm{Epi}(X, Y)$ be the pullback of im along $\top_Y$.

$$\begin{tikzcd}
\mathrm{Epi}(X, Y) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, hook, "\top_Y"] \\
Y^X \arrow[r, "\mathrm{im}"'] & \Omega^Y
\end{tikzcd}$$

Notice $\top_Y$ is a mono, hence $\mathrm{Epi}(X, Y)$ is indeed a subobject of the object of morphisms.

Let us explain a little bit the intuition behind this pullback, as if we were in **Set**. We can see Ω^Y as the power of Y. Each morphism in Y^X is sent by im to its corresponding image in Ω^Y. The morphism $\top_Y$ is the morphism sending 1 to Y. Hence the pullback is precisely the object computing all morphisms $X \to Y$ such that their image is Y.

Let us show how in a topos with distinct initial and terminal object, such an object $\mathrm{Epi}(X, Y)$ represent the collection of epimorphisms form X to Y. But before that, we need a little lemma:

Lemma 3.2.15. *For any object C in a topos $\mathcal{E}$, a morphism $f : C \to Y^X$ factors through $\mathrm{Epi}(X, Y) \hookrightarrow Y^X$ if and only if $(\pi_1, \widehat{f}) : C \times X \to C \times Y$ is an epimorphism.*

Proof.

$$\begin{tikzcd}
C \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right, "f"'] & & \\
& \mathrm{Epi}(X,Y) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, hook, "\top_Y"] \\
& Y^X \arrow[r, "\mathrm{im}"'] & \Omega^Y
\end{tikzcd}$$

By universal property of the pullback, f factors through $\mathrm{Epi}(X, Y)$ if and only if $\mathrm{im} \circ f = \mathrm{im}_C(f)$ is the morphism $C \to 1 \xrightarrow{\top_Y} \Omega^Y$. Since $\top_Y$ is the transpose of $1 \times Y \to 1 \xrightarrow{\top} \Omega$ and by naturality of the cartesian closeness property, the morphism $C \to 1 \xrightarrow{\top_Y} \Omega^Y$ is the transpose of $C \times Y \to 1 \xrightarrow{\top} \Omega$. Remember that $\mathrm{im}_C(f)$ is the transpose of the characteristic morphism of the subobject $\mathrm{Im}_C(f) \hookrightarrow C \times Y$. Thus the characteristic morphism is precisely the morphism $C \times Y \to 1 \xrightarrow{\top} \Omega$. Now consider the following diagram,

$$\begin{tikzcd}
\mathrm{Im}_C(f) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, hook, "\top"] \\
C \times Y \arrow[r] & 1 \arrow[r, hook, "\top"'] & \Omega
\end{tikzcd}$$

By definition of characteristic morphism, the outer diagram is a pullback. It is easy to check to right diagram is a pullback, thus the left diagram is a pullback. Since the pullback of an isomorphism is an isomorphism, we get that $\mathrm{Im}_C(f) \cong C \times Y$. And thus since $\mathrm{Im}_C(f)$ is the image of $(\pi_1, \widehat{f})$, this morphism is an epimorphism. The converse can be proved exactly by running this argument backwards. $\square$

In the proof of the next proposition, we will use the fact that any morphism in a topos, with codomain the initial object, is an isomorphism. We shall prove that fact in the next chapter, see 3.3.9

Proposition 3.2.16. *In $\mathscr{E}$ a topos with distinct initial and terminal objects* ($0 \ncong 1$), *if* $\mathrm{Epi}(X, Y) = 0$ *then there are no epimorphisms* $X \to Y$.

Proof. Arguing by contradiction, let $g : X \to Y$ be an epimorphism in $\mathscr{E}$. Since $1 \times X \cong X$, g is the transpose of some $f : 1 \to Y^X$ and g can be seen as $(\pi_1, \widehat{f}) : 1 \times X \to 1 \times Y$. Thus by lemma 3.2.15 with $C = 1$, f factors through $\mathrm{Epi}(X, Y) = 0$. But any morphism to 0 is an isomorphism. Hence $1 \cong 0$, which by assumption is a contradiction. □

3.3 Subobjects in a topos

A topos is a category where its morphisms and subobjects can be manipulated and understood internally as objects of the topos. Thus, theories built on objects of a topos can be applied on its morphisms, allowing us to construct mathematical theories while staying inside a topos, as we do with sets. But one could object that the constructions we have being doing are not easy to manipulate. We have a notion of object of epimorphisms, but how this object is defined is not easy. It is more convenient to work with our usual language embedded in Set theory such as $(\forall g, h, k \in G)(g + (h + k) = (g + h) + k)$, than to work with complex definitions with diagrams. Fortunately we can actually recover our common language in the context of topoi. Remember in the first section we highlighted the fact that the logical operators are highly linked to the Heyting algebra structure on subsets. In this section we will study the subobjects in a topos, and show there is also an Heyting algebra structure on it.

Definition 3.3.1. Let $\mathscr{C}$ be a category and let be an object $C \in \mathscr{C}$. The *slice* category $\mathscr{C}/C$ is the category with

- Objects : the morphisms $f : A \to C$ in $\mathscr{C}$ for any $A \in \mathscr{C}$.
- Morphism : for any two objects $f : A \to C$, $g : B \to C$, a morphism $h : f \to g$ is an arrow $h : A \to B$ in $\mathscr{C}$ such that $g \circ h = f$.

Now remark that any morphism $f : C \to D$ in a arbitrary category with pullbacks $\mathscr{C}$ yields a functor on the slice categories $f^{-1} : \mathscr{C}/D \to \mathscr{C}/C$ defined as follows: for any morphism $a : A \to D$, the morphism $f^{-1}(a) : f^{-1}(A) \to C$ is the second projection of the pullback $f^{-1}(A)$ in $\mathscr{C}$ of a along f. If $h : a \to b$ is an arrow (i.e such that $a = b \circ h$) in $\mathscr{C}/D$ with $a : A \to D$ and $b : B \to D$ objects in $\mathscr{C}/D$, then $f^{-1}(h) : f^{-1}(a) \to f^{-1}(b)$ is the unique arrow $f^{-1}(A) \to f^{-1}(B)$ induced by the pullback $f^{-1}(B)$:

$$\begin{tikzcd}
f^{-1}(A) \arrow[rr] \arrow[dr, dashed, "f^{-1}(h)"] \arrow[ddr, bend right, "f^{-1}(a)"'] & & A \arrow[d, "h"'] \\
& f^{-1}(B) \arrow[r] \arrow[d, "f^{-1}(b)"] & B \arrow[d, "b"'] \\
& C \arrow[r, "f"] & D
\end{tikzcd}$$

The unicity of the arrows, makes f^{-1} a well defined functor. Remark also that any morphism $f : C \to D$ yields by definition an isomorphism of categories

$$(\mathscr{C}/D)/(f) \cong \mathscr{C}/C.$$

Since an object in $h \in (\mathscr{C}/D)/(f)$ is a morphism $h : g \to f$ in $\mathscr{C}/D$ with $g : A \to D$ that is defined a morphism $h : A \to C$ in $\mathscr{C}/C$. Conversely for any morphism $h : A \to C$, composing with f yields a morphism $h : f \circ h \to f$ in $\mathscr{C}/D$ and thus an object in $(\mathscr{C}/D)/(f)$. Thus the objects of both categories are in bijection. The same trick applies to the morphisms of both categories.

Proposition 3.3.2. *Let be a category $\mathscr{C}$ and an object $C \in \mathscr{C}$.*

- *If $\mathscr{C}$ is (finitely) complete then $\mathscr{C}/C$ is (finitely) complete.*
- *If $\mathscr{C}$ is (finitely) cocomplete then $\mathscr{C}/C$ is (finitely) cocomplete.*

Proof. It is easy to see that the terminal object in $\mathscr{C}/C$ is just the identity Id_C. Let $a : A \to C$ and $b : B \to C$ be objects in $\mathscr{C}/C$. Their product $(a \times b, p_1, p_2)$ is given by the pullback of a along b in $\mathscr{C}$ with $a \times b : A \times_C B \to C$, $a \times b = a \circ p_1 = b \circ p_2$ (p_1,p_2 the two projections of the pullback). For any object $d : D \to C$ and any two morphisms $f : d \to a$ and $g : d \to b$, we can see those morphisms as arrows $f : D \to A$ and $g : D \to B$ such that $a \circ f = b \circ g$. By definition of the pullback, this yields a unique arrow $(a, b) : D \to A \times_C B$ such that $a \times b \circ (a, b) = d$. Hence $\mathscr{C}/C$ has products. Let be two morphisms $f, g : a \to b$ in $\mathscr{C}/C$. Those morphisms seen as $f, g : A \to B$ have an equalizer $e : E \to A$ in $\mathscr{C}$. Thus it is immediate that $e : a \circ e \to a$ is the equalizer of f and g in $\mathscr{C}/C$. Since $\mathscr{C}/C$ has products and equalizers, it is complete.

Now in the case of colimits, assume we have a small category $\mathscr{D}$ and a functor $F : \mathscr{D} \to \mathscr{C}/C$ where, for any arrow $d : D \to D'$ in $\mathscr{D}$, $F(D) : G(D) \to C$ for a given $G(D) \in \mathscr{C}$ and $F(d) : F(D) \to F(D')$ for a given arrow $F(d) = G(d) : G(D) \to G(D')$ in $\mathscr{C}$. This induces a functor $G : \mathscr{D} \to \mathscr{C}$. Let $(L, \{s_D : G(D) \to L\}_{D \in \mathscr{D}})$ be the colimit of G. Notice that the arrows $F(D) : G(D) \to C$ form a cocone in $\mathscr{C}$ since $F(D') \circ F(d) = F(D)$. Thus there exists a unique morphism $\lambda : L \to C$ such that

$\lambda \circ s_D = F(D)$ for any $D \in \mathscr{D}$. Then $(\lambda, \{s_D : F(D) \to \lambda\}_{D\in\mathscr{D}})$ is the colimit of F. Indeed if $(\lambda' : L' \to C, \{s'_D : F(D) \to \lambda'\}_{D\in\mathscr{D}})$ is a cocone on F, then the $s'_D : G(D) \to L'$ form a cocone on G. Hence there exists a unique arrow $l : L \to L'$ such that $l \circ s_D = s'_D$ for every $D \in \mathscr{D}$. This yields a unique arrow $l : \lambda' \to \lambda$ such that $l \circ s_D = s'_D$. □

There is an asymmetry between limits and colimits to remark here. We showed that the colimits are equivalently computed in $\mathscr{C}$ and in $\mathscr{C}/C$, i.e a coproduct in $\mathscr{C}/C$ is a coproduct in C etc.... Whereas this is not the case with limits, a product in $\mathscr{C}/C$ is a pullback in $\mathscr{C}$. Notice, for colimits, that as in a particular case a colimit $(L, \{s_D\})$ in $\mathscr{C}$ corresponds to the colimit $(\mathrm{Id}_L, \{s_D\})$ in $\mathscr{C}/L$.

We won't expand too much on the properties and the technical details involving topoi, since our main focus is the specific case of Grothendieck topoi. Therefore we will admit the next fundamental result without proof. One can be found in [6, Section IV.7]. We will also admit topoi are finitely cocomplete, see [6, Section IV.5].

Theorem 3.3.3 (Fundamental theorem of topos theory)**.** *In a topos $\mathscr{E}$, for any object $E \in X$ the slice category $\mathscr{E}/X$ is a topos.*

Proposition 3.3.4. *Any topos is finitely cocomplete.*

Proposition 3.3.5. *In any topos $\mathscr{E}$, for any morphism $f : X \to Y$ in $\mathscr{E}$ the functor $f^{-1} : \mathscr{E}/Y \to \mathscr{E}/X$ on the slice topoi has both a left Σ_f and a right adjoint Π_f.*

Proof. The left adjoint Σ_f is defined as the composition by f, i.e for any object $b : B \to X$ in the slice topos $\mathscr{E}/X$, $\Sigma_f(b) = f \circ b$. A morphism $h : b \to b'$ in $\mathscr{E}/X$ is sent by Σ_f to itself but now seen as a morphism $\Sigma_f(h) : \Sigma_f(b) \to \Sigma_f(b')$.

Let us a prove this is an adjunction. Let $b : B \to X$ and $a : A \to Y$ and assume we have a morphism $h : \Sigma_f(b) \to a$. Then this yields a unique morphism $\overline{h} : b \to f^{-1}(a)$ by definition of the pullback.

$$\begin{tikzcd}
B \arrow[drr, bend left, "h"] \arrow[dr, dashed, "\overline{h}"'] \arrow[ddr, bend right, "b"'] & & \\
& f^{-1}(A) \arrow[r, "p"] \arrow[d, "f^{-1}(a)"] & A \arrow[d, "a"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

Conversely assume we have a morphism $\overline{h} : b \to f^{-1}(a)$. By composition with the first projection of the pullback $p : f^{-1}(A) \to A$, we get a morphism $p \circ \overline{h} : \Sigma_f(b) \to a$. It is easy to see we thus get a bijection

$$\mathbf{Hom}_{\mathscr{E}/Y}(\Sigma_f(b), a) \cong \mathbf{Hom}_{\mathscr{E}/X}(b, f^{-1}(a))$$

natural in a and b.

For the right adjoint Π_f, let us first consider the special case where $Y = 1$ the terminal object. By definition of terminal object $\mathscr{E}/1$ is isomorphic to $\mathscr{E}$. Taking the pullback of $a : A \to 1$ along $f : X \to 1$ is just the product $A \times X$. Thus the functor f^{-1} is the functor $- \times X : \mathscr{E} \to \mathscr{E}/X$ sending an object $A \in \mathscr{E}$ to the projection $\pi_2 : A \times X \to X$.

Now assume for any $A \in \mathscr{E}$ we have the projection $\pi_2 : A \times X \to X$ and an object $b : B \to X$ both in $\mathscr{E}/X$. A morphism between them is an arrow $h : A \times X \to B$ in $\mathscr{E}$ such that $b \circ h = \pi_2$. Since $\mathscr{E}$ is cartesian closed h corresponds to $\widehat{h} : A \to B^X$, π_2 corresponds to $\widehat{\pi_2} : A \to X^X$ such that, by naturality, $\widehat{\pi_2} = b^X \circ \widehat{h}$ (where b^X is the functor $(-)^X$ evaluation applied to b). Notice that since the unique arrow $a : A \to 1$ yields a morphism $j : A \times X \to 1 \times X \cong X$ and $\pi_2 = \mathrm{Id}_X \circ j$. Then by the naturality of the adjunction we get $\widehat{\pi_2} = \widehat{\mathrm{Id}_X} \circ a$ with $\widehat{\mathrm{Id}_X} : 1 \to X^X$. Let $\Pi_f(b)$ be the pullback of b^X along $\widehat{\mathrm{Id}_X}$. Since $b^X \circ \widehat{h} = \widehat{\mathrm{Id}_X} \circ a$, we get a unique arrow $\overline{h} : A \to \Pi_f(b)$ making the following diagram commute:

$$\begin{tikzcd}
A \arrow[dr, dashed, "\overline{h}"] \arrow[drr, bend left, "\widehat{h}"] \arrow[ddr, bend right, "a"'] & & \\
& \Pi_f(b) \arrow[r, "p"] \arrow[d] & B^X \arrow[d, "b^X"'] \\
& 1 \arrow[r, "\widehat{\mathrm{Id}_X}"] & X^X
\end{tikzcd}$$

This defines Π_f on the objects. The construction gives a bijection

$$\mathbf{Hom}_{\mathscr{E}/X}(f^{-1}(a), b) \cong \mathbf{Hom}_{\mathscr{E}/Y}(a, \Pi_f(b)).$$

Any morphism $h : A \times X \to B$ is sent to $\overline{h}$ and conversely any morphism $\overline{h} : A \to \Pi_f(b)$ is composed with the projection $p : \Pi_f(b) \to B^X$ and $p \circ \overline{h}$ correspond to a morphism $h : A \times X \to B$ by cartesian closeness. It is easy to see this is a bijection.

Assume we have a morphism $g : b \to c$ in $\mathscr{E}/X$ with $b : B \to X$ and $c : C \to X$. Since $(-)^X$ is a functor, $b^X = c^X \circ g^X$. Hence since $\Pi_f(c)$ is a pullback, we get a unique morphism $\Pi_f(b) \to \Pi_f(c)$. The uniqueness makes sure Π_f is a well defined functor. Moreover it quickly comes that the bijection is natural in a and b (we even used a bit of the naturality in a in the proof), since the naturality is induced by the naturality of the cartesian product.

Now in the general case with $f : X \to Y$, remember that $(\mathscr{E}/Y)/(f) \cong \mathscr{E}/X$ and $(\mathscr{E}/Y)/(\mathrm{Id}_Y) \cong \mathscr{E}/Y$. Thus the functor $f^{-1} : \mathscr{E}/Y \to \mathscr{E}/X$ can be seen as $f^{-1} : (\mathscr{E}/Y)/(\mathrm{Id}_Y) \to (\mathscr{E}/Y)/(f)$. But Id_Y is the terminal object in $\mathscr{E}/Y$, hence we can apply our previous case. □

Corollary 3.3.6. *In any topos $\mathscr{E}$, finite colimits are pullback stable.*

Proof. By 3.3.4 and 3.3.2 any colimit $(Y, \{s_D\})$ in $\mathscr{E}$ can be seen as a colimit $(\mathrm{Id}_Y, \{s_D\})$ in $\mathscr{E}/Y$. For any morphism $f : X \to Y$, since taking the pullback along f, $f^{-1} : \mathscr{E}/Y \to \mathscr{E}/X$, has a right adjoint by 3.3.5, it preserves colimits and thus $(\mathrm{Id}_X, \{f^{-1}(s_D)\})$ is a colimit in $\mathscr{E}/X$. This is the colimit $(X, \{f^{-1}(s_D)\})$ in $\mathscr{E}$. □

Corollary 3.3.7. *Topoi are regular categories.*

Proof. By definition of a topos and by 3.3.4, every morphism an a topos has a kernel pair, every kernel pair has a coequalizer and the pullback of any regular epimorphism along any morphism exists. Moreover by 3.3.6, coequalizers are pullback stable, so are regular epimorphisms. □

Remark that for two subobjects S, R of X in $\mathscr{E}$, they can be seen as objects in $\mathscr{E}/X$, and there is at most one morphism in $\mathscr{E}/X$ between two subobjects. Indeed assume we have two morphisms of subobjects h and g in $\mathscr{E}/X$. Then $s \circ h = r = s \circ g$, but since s is a monomorphism, we get that $h = g$. Thus let us define $\mathrm{Sub}(X)$ the category whose objects are the isomorphism classes of subobjects of X, and the morphisms are the same (up to isomorphism) as in $\mathscr{E}/X$. Since there is at most one morphism between to subobjects in $\mathrm{Sub}(X)$, we can see $\mathrm{Sub}(X)$ as a poset category.

Thus any morphism in a topos has an image factorization. This will come handy to understand the structure of subobjects. Indeed so far we know that for $\mathscr{E}$ a topos and an object $X \in \mathscr{E}$, the slice category $\mathscr{E}/X$ has finite products (pullbacks in $\mathscr{E}$) and finite coproducts (finite coproducts in $\mathscr{E}$). But what about the category of subobjects $\mathrm{Sub}(X)$? Monomorphisms are pullback stable, thus the product of two subobjects is again a subobject. But the coproduct of two monomorphisms is not always a monomorphism. Hence the coproduct in $\mathscr{E}/X$ is not the coproduct in $\mathrm{Sub}(X)$. But the image factorisation of the coproduct is.

Proposition 3.3.8. *Let be two subojects $s : S \hookrightarrow X$ and $t : T \hookrightarrow X$ of X in a topos $\mathscr{E}$. Let us define $s \cup t : S \cup T \to X$ where $s \cup t$ is the monomorphism part of the image factorization of $s \amalg t : S \amalg T \to X$ the unique arrow of the coproduct of s and t in $\mathscr{E}$. Then $s \cup t$ is the coproduct of s and t in* $\mathrm{Sub}(X)$.

Proof. Assume $d : D \hookrightarrow X$ is another subobject of X such that we have two morphisms of subobjects $f : s \to d$, $g : t \to d$, i.e two monomorphisms $f : S \hookrightarrow D$ and $g : T \hookrightarrow D$ in $\mathscr{E}$ such that $d \circ f = s$ and $d \circ g = t$. Then the coproduct $s \amalg t$ of s and t and the coproduct $f \amalg g$ of f and g in $\mathscr{E}$ are such that $d \circ (f \amalg g) = s \amalg t$ by unicity of the arrows. Let be $f \amalg h = m \circ e$ the image factorization of $f \amalg g$ where m is a mono an e a regular epi. Then $s \amalg t = d \circ m \circ e$ is an image factorization for $s \amalg t$ with $d \circ m$ a monomorphism and in particular m is $m : S \cup T \to D$ such that $s \cup t = d \circ m$. Since m is unique, by unicity of the corproduct and image factorization arrows, such that $f = m \circ e \circ i_S$ and $g = m \circ e \circ i_T$ with i_T, i_S the canonical inclusion of the coproduct $S \amalg T$, $S \cup T$ is the coproduct of the subobjects s and t in $\mathrm{Sub}(X)$. ☐

Proposition 3.3.9. *In a topos $\mathscr{E}$, any arrow $y : Y \to 0$ is an isomorphism.*

Proof. In any slice topos $\mathscr{E}/X$, we know the object $\mathrm{Id}_X : X \to X$ is the terminal object, and it is easy to show $0 \to X$ is the initial object. Thus in $\mathscr{E}/0$, the morphism $0 \to 0$ is both initial and terminal. Since by 3.3.5 the functor $y^{-1} : \mathscr{E}/0 \to \mathscr{E}/Y$ has both left and right adjoints, it preserves initial and terminal object. Thus $y^{-1}(\mathrm{Id}_0) : y^{-1}(0) \to Y$ is both terminal and initial in $\mathscr{E}/Y$. Thus there exist an isomorphism $t : y^{-1}(0) \to Y$ such that $y^{-1}(\mathrm{Id}_0) = \mathrm{Id}_Y \circ t$. Thus $y^{-1}(\mathrm{Id}_0)$ is an isomorphism in $\mathscr{E}$. Similarly $y^{-1}(0)$ is isomorphic to 0. Thus by unicity, $y \circ y^{-1}(\mathrm{Id}_0) = \mathrm{Id}_0$. Since both Id_0 and $y^{-1}(\mathrm{Id}_0)$ are isomorphisms, so is y. ☐

Corollary 3.3.10. *In a topos $\mathscr{E}$, for any object $X \in \mathscr{E}$, the unique morphism $0 \to X$ is a monomorphism.*

Proof. Assume we have to arrows $f, g : Y \to 0$. By 3.3.9, f and g are isomorphisms. Thus their inverse must coincide as the unique arrow $0 \to Y$. Hence $f = g$. ☐

Thus in a topos, the initial object is a subobject of any object.

Since it is more convenient to talk about a subobject $s : S \hookrightarrow X$ as just S, we denote the *union* $S \cup T$ for the coproduct of S and T, the *intersection* $S \cap T$ for the product of S and T and we denote the unique morphism between two subobjects by $S \subseteq T$. Those notations are motivated by the fact that in the case $\mathscr{E} = \mathbf{Set}$, the product of two subsets is their intersection, the coproduct is their union and the inclusion is a partially ordered relation. We the context needs to be clearer, we will add the subscript Y to the operations $\cap_Y, \dots$ to precise those are the operations on $\mathrm{Sub}(Y)$. When it is not necessary, we will omit it. We finally have everything we need to prove the subobjects of an object of a topos form a Heyting algebra. We will always for a subobject S, the letter s represents the associated monomorphism.

Proposition 3.3.11. *In a topos $\mathscr{E}$, for any object $X \in \mathscr{E}$, the category of subobject* $\mathrm{Sub}(X)$ *form a Heyting algebra.*

Proof. We show previously computed that $\mathrm{Sub}(X)$ has products $\cap$, coproducts $\cup$, the terminal object $\mathrm{Id}_X : X \to X$ and the initial object $0 \hookrightarrow X$. Thus $\mathrm{Sub}(X)$ is a bounded lattice. Let us define what will be the implication operation. For any two subobjects S, T in $\mathrm{Sub}(X)$, let $(S \Rightarrow T) \hookrightarrow X$ be the equalizer of the characteristic morphisms $\chi_{S\cap T} : X \to \Omega$ and $\chi_S : X \to \Omega$ in $\mathscr{E}$. Since an equalizer is a monomorphism, $S \Rightarrow T$ is a subobject of X. Consider the following diagram where both squares are pullbacks:

$$\begin{tikzcd}
S \cap T \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[r] \arrow[d, "s"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\top"'] \\
T \arrow[r, hook, "t"] & X \arrow[r, "\chi_S"] & \Omega
\end{tikzcd}$$

Thus the whole square is a pullback and by uniqueness of the characteristic morphism $\chi_S \circ t = \chi_{S\cap T} = \chi_T \circ s$. Notice that in a lattice, if $A \subseteq A \cap B$ then $A = A \cap B$, since $A \cap B \subseteq A$ is always true. Moreover since $A \cap A = A$, if $A \subseteq B$ then $A = A \cap A \subseteq A \cap B$. Now assume we have another subobject R in $\mathrm{Sub}(X)$. Then

$$\begin{aligned}
R \subseteq (S \Rightarrow T) &\iff \chi_S \circ r = \chi_{S\cap T} \circ r \\
&\iff R \cap S = R \cap S \cap T \\
&\iff R \cap S \subseteq R \cap S \cap T \\
&\iff R \cap S \subseteq T.
\end{aligned}$$

Therefore $\mathrm{Sub}(X)$ is a Heyting algebra. □

We know that any morphism $f : X \to Y$ in a topos $\mathscr{E}$ yields a functor $f^{-1} : \mathscr{E}/Y \to \mathscr{E}/X$ on the slice topoi with a right and left adjoint. Since f^{-1} is just the pullback and pullbacks preserve monomorphism, f^{-1} automatically restrict on the subobjects $f^{-1} : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$. We shall prove this restriction on the subobjects has again a right and left adjoint.

Proposition 3.3.12. *Let $f : X \to Y$ be a morphism of a topos. Then pulling back along subobjects gives a functor $f^{-1} : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ and this functor has both a left adjoint $\exists_f$ and a right adjoint $\forall_f$.*

$$\begin{tikzcd}
\mathrm{Sub}(X) \arrow[r, shift left=3, "\forall_f"] \arrow[r, shift right=3, "\exists_f"'] & \mathrm{Sub}(Y) \arrow[l, "\top"', "\top"]
\end{tikzcd}$$

Proof. Let be the right adjoint $\Pi_f : \mathscr{E}/X \to \mathscr{E}/Y$. As a right adjoint, it preserves terminal objects and monomorphisms. Let $s : S \hookrightarrow X$ be a subobject of X in $\mathscr{E}$. s can be equivalently seen as a subobject $s \hookrightarrow \mathrm{Id}_X$ of Id_X in $\mathscr{E}/X$, where Id_X is the terminal object of $\mathscr{E}/X$. Pi_f maps this subobject to a subobject $Pi_f(s) \hookrightarrow \mathrm{Id}_Y$ that is again equivalent to $\Pi_f \hookrightarrow Y$ a subobject of Y in $\mathscr{E}$. Hence Π_f preserves subobjects and then the same adjunction as in 3.3.5 still holds for any $R \in \mathrm{Sub}(Y)$ and $S \in \mathrm{Sub}(X)$,

$$\mathbf{Hom}_{\mathrm{Sub}(X)}(f^{-1}(R), S) \cong \mathbf{Hom}_{\mathrm{Sub}(Y)}(R, \Pi_f(S)).$$

For further motivated reasons we will note the restriction of Π_f to the subobjects by $\forall_f$.

Now let be the left adjoint $\Sigma_f : \mathscr{E}/X \to \mathscr{E}/Y$. This functor does not preserves subobjects. If $s : S \hookrightarrow X$ is a monomorphism, then the composition $\Sigma_f(s) = f \circ s$ does not have to be a monomorphism. To correct that, let us take the monomorphism $\exists_f(s) : \exists_f(S) \hookrightarrow Y$ in the image factorization of $f \circ s$, where $\exists_f(S)$ denotes the image.

Now assume we have a morphism between subobject $h : S \to R$. The strong epimorphism in the image factorisation of $f \circ s$ induces an arrow $\exists_f(h) : \exists_f(S) \to \exists_f(R)$ as in the right diagram below. One can verify the uniqueness of the construction yields a well defined functor.

$$\begin{tikzcd}
 & & \exists_f S \arrow[rr, dashed, "\exists_f(h)"] \arrow[dr, hook] & & \exists_f R \arrow[dl, hook] \\
S \arrow[urr, two heads] \arrow[rr, hook, "g"] \arrow[dr, hook, "s"'] & & R \arrow[urr, two heads] \arrow[dl, hook, "r"'] & Y & \\
 & X \arrow[urr, "f"'] & & &
\end{tikzcd}
\qquad
\begin{tikzcd}
S \arrow[r, two heads] \arrow[d] & \exists_f S \arrow[d, hook] \arrow[dl, dashed, "\exists_f(h)"] \\
\exists_f R \arrow[r, hook] & Y
\end{tikzcd}$$

Let be two subobjects $r : R \hookrightarrow Y$ and $s : S \hookrightarrow X$. Assume we have an arrow $h : \exists_f(s) \to r$. Since it is a morphism of subobjects, it is a monomorphism $h : \exists_f(S) \to R$. Thus composing with the regular epimorphism $e : S \to \exists_f(S)$ of the image factorization of $f \circ s$ yields an image factorization for the morphism $h \circ e : S \to R$. In particular $r \circ h \circ e = \exists_f(s) \circ e = f \circ s$. Thus $h \circ e : \Sigma_f(s) \to r$ is a morphism in $\mathscr{E}/Y$. Conversely assume we have a morphism $\overline{h} : \Sigma_f(s) \to r$. Take its image factorization in $\mathscr{E}$ $h \circ e : B \to H \hookrightarrow A$, with e a regular epimorphism and h a monomorphism. In particular $r \circ h \circ e = f \circ s$ and since h and r are both monomorphisms, $H = \exists_f(S)$ and $h : \exists_f(S) \hookrightarrow R$ and we get the subobject $r \circ h : \exists_f(S) \hookrightarrow Y$. Hence this define a morphism $h : r \circ h \to r$ in $\mathscr{E}/Y$.

This construction yields a bijection

$$\mathbf{Hom}_{\mathrm{Sub}(Y)}(\exists_f(s), r) \cong \mathbf{Hom}_{\mathscr{E}/Y}(\Sigma_f(s), r),$$

for any subobjects $s : S \hookrightarrow X$ and $r : R \hookrightarrow Y$. Moreover it is easy to check this bijection is natural on the subobjects r and s. Therefore the bijection brought by the adjunction of f^{-1} and Σ_f in 3.3.5, "restricts" in the case of subobjects to the adjunction

$$\mathbf{Hom}_{\mathrm{Sub}(Y)}(\exists_f(S), R) \cong \mathbf{Hom}_{\mathrm{Sub}(X)}(S, f^{-1}(R)),$$

natural in S and R. $\square$

Let us take a little bit of time to explain the motivation behind the notations $\exists_f$ and $\forall_f$. Assume we are working in **Set** ans take a map $f : X \to Y$. Since $s : S \hookrightarrow X$ is just the inclusion of subset $S \subseteq X$, the composition $\Sigma_f(S) = f \circ s$ is just the restriction $f|_S$ of f to $S \subseteq X$. The image in the image factorization in **Set** is literally the image of $f|_S$. Thus

$$\exists_f(S) = \{y \in Y | \exists x \in S \text{ such that } f(x) = y\}.$$

The case of $\forall_f$ is trickier and unfortunately uses explicitly the structure of a slice topos induced by the original topos (here **Set**), property we assumed without a proof. Thus computing by hand the structure of a slice $\mathbf{Set}/Y$ would take time that is not really of any worth beyond the next equality. Such a work can be found scattered in [3, Example 5.8.6] and some remarks after [3, Proposition 6.2.3], the result is

$$\forall_f(S) = \{y \in Y | \forall x \in X \text{ such that } (f(x) = y) \Rightarrow x \in S\}.$$

The notations $\exists_f$ and $\forall_f$ become even clearer when f is the projection $\pi : X \times Y \to Y$ and $S \subseteq X \times Y$,

$$\begin{aligned}
\exists_\pi(S) &= \{y \in Y | \exists (x, y') \in S \text{ such that } y' = y\} \\
&= \{y \in Y | \exists x \in X \text{ such that } (x, y) \in S\}. \\
\forall_\pi(S) &= \{y \in Y | \forall (x, y') \in X \times Y \text{ such that } (y' = y) \Rightarrow (x, y') \in S\} \\
&= \{y \in Y | \forall x \in X \text{ such that } (x, y) \in S\}.
\end{aligned}$$

Corollary 3.3.13. *Let $\mathscr{E}$ be a topos and $f : X \to Y$ a morphism in $\mathscr{E}$. $f^{-1} : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ is a homomorphism of Heyting algebras.*

Proof. Since by 3.3.12 $f^{-1} : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ has both left and right adjoints, it preserves all finite limits and finite colimits. Thus it preserves the intersection, the union (as product and coproduct) and the top and bottom element (as terminal and initial objects). It also preserves the inclusion since it is a functor. Let us prove it also preserves the implication.

Let $S, T \in \mathrm{Sub}(Y)$ be two subobjects, their implication $S \Rightarrow_Y T$ is the equalizer of χ_S and $\chi_{S \cap_Y T}$. For reasons strictly similar to the equality $\chi_{S \cap T} = \chi_S \circ t$ (see proof of 3.3.11), we compute

$$\chi_{f^{-1}(S)} = \chi_S \circ f, \quad \chi_{f^{-1}(S \cap_Y T)} = \chi_{S \cap_Y T} \circ f.$$

Thus $\alpha : f^{-1}(S) \Rightarrow_X f^{-1}(T) \hookrightarrow X$ is the equalizer of $\chi_S \circ f$ and $\chi_{S \cap_Y T} \circ f$. Now consider the following diagram

$$
\begin{tikzcd}
f^{-1}(S) \Rightarrow_X f^{-1}(T) \arrow[dr, dashed, shift left, "\lambda"] \arrow[drr, dashed, bend left, "\beta"] \arrow[ddr, hook, bend right, "\alpha"'] & & \\
& f^{-1}(S \Rightarrow_Y T) \arrow[ul, dashed, shift left, "\epsilon"] \arrow[r, "p"] \arrow[d, hook, "f^{-1}(e)"'] \arrow[dr, phantom, "\lrcorner", very near start] & S \Rightarrow_Y T \arrow[d, hook, "e"] & \\
& X \arrow[r, "f"'] & Y \arrow[r, shift left, "\chi_S"] \arrow[r, shift right, "\chi_{S \cap_Y T}"'] & \Omega.
\end{tikzcd}
$$

$\chi_S \circ f \circ f^{-1}(e) = \chi_S \circ e \circ p = \chi_{S \cap_Y T} \circ e \circ p = \chi_{S \cap_Y T} \circ f \circ f^{-1}(e)$. Since α is an equalizer, there exists a unique arrow $\epsilon : f^{-1}(S \Rightarrow_Y T) \to f^{-1}(S) \Rightarrow_X f^{-1}(T)$ such that $\alpha \circ \epsilon = f^{-1}(e)$. Since $\chi_S \circ f \circ \alpha = \chi_{S \cap_Y T} \circ f \circ \alpha$ and e is an equalizer, there exists a unique arrow $\beta : f^{-1}(S) \Rightarrow_X f^{-1}(T) \to S \Rightarrow_Y T$ such that $e \circ \beta = f \circ \alpha$. Thus, since $f^{-1}(S \Rightarrow_Y T)$ is a pullback, there exists a unique arrow $\lambda : f^{-1}(S) \Rightarrow_X f^{-1}(T) \to f^{-1}(S \Rightarrow_Y T)$ such that $\alpha = f^{-1} \circ \lambda$ and $\beta = p \circ \lambda$. Again by universal properties of the pullback and the equalizer, λ and ϵ are inverse to each other. Thus $f^{-1}(S \Rightarrow_Y T) = f^{-1}(S) \Rightarrow_X f^{-1}(T)$. $\square$

In practice, the fact that f^{-1} is a homomorphism of Heyting algebra, for any $f : X \to Y$ in a topos $\mathscr{E}$, especially means that the binary operations $\cap_Y, \cup_Y, \Rightarrow_Y \colon \mathrm{Sub}(Y) \times \mathrm{Sub}(Y) \to \mathrm{Sub}(Y)$ are natural in $Y \in \mathscr{E}$. Now recall that since $\mathscr{E}$ is a topos, the subobjects are naturaly in bijection with the characteristic morphisms, i.e $\mathrm{Sub}(Y) \cong \mathbf{Hom}_{\mathscr{E}}(Y, \Omega)$ is bijection natural in $Y \in \mathscr{E}$. Then $\mathbf{Hom}_{\mathscr{E}}(Y, \Omega)$ is a Heyting algebra with greater and least elements respectively the characteristic morphisms of 1 and 0 and with binary operations $\wedge_Y, \vee_Y, \Rightarrow_Y \colon \mathbf{Hom}_{\mathscr{E}}(Y, \Omega \times \Omega) \to \mathbf{Hom}_{\mathscr{E}}(Y, \Omega)$ defined as the following diagram

$$
\begin{tikzcd}
\mathrm{Sub}(Y) \times \mathrm{Sub}(Y) \arrow[d, "\cap_Y, \cup_Y, \Rightarrow_Y"'] \arrow[r, phantom, "\cong"] & \mathbf{Hom}_{\mathscr{E}}(Y, \Omega) \times \mathbf{Hom}_{\mathscr{E}}(Y, \Omega) \arrow[r, phantom, "\cong"] & \mathbf{Hom}_{\mathscr{E}}(Y, \Omega \times \Omega) \arrow[d, "\wedge_Y, \vee_Y, \Rightarrow_Y"'] \\
\mathrm{Sub}(Y) \arrow[rr, phantom, "\cong"] & & \mathbf{Hom}_{\mathscr{E}}(Y, \Omega).
\end{tikzcd}
$$

Since those operations are natural in $Y \in \mathscr{E}$, they yield natural transformations $\wedge, \vee, \Rightarrow \colon \mathbf{Hom}_{\mathscr{E}}(-, \Omega \times \Omega) \to \mathbf{Hom}_{\mathscr{E}}(-, \Omega)$. By the Yoneda lemma (A.0.1), each one of those natural transformations is uniquely determined by an element in $\mathbf{Hom}_{\mathscr{E}}(\Omega \times \Omega, \Omega)$, we will denote those elements $\wedge, \vee, \Rightarrow \colon \Omega \times \Omega \to \Omega$. For any two subobjects $S, T \in \mathrm{Sub}(Y)$

and the product (χ_S, χ_T) of their characteristic morphism, those three arrows are in particular determined by the composition:

$$\begin{aligned}\wedge \circ (\chi_S, \chi_T) &= \wedge_Y(\chi_S, \chi_T) = \chi_{S \cap_Y T},\\ \vee \circ (\chi_S, \chi_T) &= \vee_Y(\chi_S, \chi_T) = \chi_{S \cup_Y T},\\ \Rightarrow \circ (\chi_S, \chi_T) &= \Rightarrow_Y (\chi_S, \chi_T) = \chi_{S \Rightarrow_Y T}.\end{aligned}$$

From now on, for any characteristic morphisms φ, ψ, we will denote their meet, join and implication by $\varphi \wedge \psi$, $\varphi \vee \psi$ and $\varphi \Rightarrow \psi$ for the composition of their product with the arrows $\wedge, \vee, \Rightarrow$.

Notice that in $\mathbf{Hom}_{\mathscr{E}}(1, \Omega)$ the top element is the characteristic morphism of $\mathrm{Id}_1 : 1 \to 1$ which is precisely $\top : 1 \to \Omega$. The bottom element, the characteristic morphism of $0 \hookrightarrow 1$ is the *false* morphism $\bot : 1 \to \Omega$. Let be for any object $X \in \mathscr{E}$ the unique arrow $X \to 1$. Since the pullback $\mathrm{Sub}(1) \to \mathrm{Sub}(X)$ along this arrow preserves the top and bottom elements, the top and bottom elements in $\mathbf{Hom}_{\mathscr{E}}(X, \Omega)$ are $\top_X : X \to 1 \xrightarrow{\top} \Omega$ and $\bot_X : X \to 1 \xrightarrow{\bot} \Omega$.

Finally, the negation operator $\neg$, by exactly the same argument as for the other binary operators, yields an arrow $\neg : \Omega \to \Omega$ such that the composition with χ_S for any $S \in \mathrm{Sub}(Y)$ is

$$\neg \circ \chi_S = \Rightarrow_Y \circ(\chi_S, \bot_Y) = \chi_{S \Rightarrow_Y 0} = \chi_{\neg S}.$$

3.4 Internal language

Since we have a notion of logical operators in a topos that behave very much like the one in **Set**, i.e they are encoded by the underlying Heyting algebra of the subobjects, we have all the ingredients we needed to interpret a theory in a topos. In the case of **Set** we usually formulate sentences like "let $x \in \mathbb{R} \ldots$". In other words any object in **Set** is a type in the set theoristic language and in such language we dispose of variables, such as x is a variable of type $\mathbb{R}$. Then a term such as $x + y$ for both $x, y \in \mathbb{R}$ in the language can be interpreted in the category **Set** as the morphisms

$$\mathbb{R} \times \mathbb{R} \xrightarrow{\mathrm{Id} \times \mathrm{Id}} \mathbb{R} \times \mathbb{R} \xrightarrow{\quad + \quad} \mathbb{R}.$$

That would be "how to understand" $x + y$ in the category **Set**. Let us generalise this language to a topos and let give us the tools to interpret this language in the topos.

Definition 3.4.1. The *equality* on an object A of an topos $\mathscr{E}$ is the characteristic morphism

$$=_A : A \times A \to \Omega$$

of the diagonal $\Delta_A : A \to A \times A$.

Definition 3.4.2. Let $\mathscr{E}$ be a topos. A *term* is inductively:

1. A variable x of type X is a term of type X of free variable x and its interpretation is the identity $\mathrm{Id} : X \to X$.
2. A constant c of type X is a term of type X without free variables and its interpretation is a morphism $1 \to X$.
3. If σ, τ are terms of types A_1 and A_2 with the same free variables $x_1, \ldots, x_n$ interpreted as $\sigma : X_1 \times \cdots \times X_n \to A_1$ and $\tau : X_1 \times \cdots \times X_n \to A_2$, then (σ, τ) is a term of type $A_1 \times A_2$ with free variables $x_1, \ldots, x_n$ interpreted as
$$(\sigma, \tau) : X_1 \times \cdots \times X_n \to A_1 \times A_2.$$
4. If $f : A \to B$ is a morphism and τ is a term of type A with free variable $x_1, \ldots, x_n$ and with interpretation $\tau : X_1 \times \cdots \times X_n \to A$, then $f(\tau)$ is a term of type B with free variable x and its interpretation is
$$f \circ \tau : X_1 \times \cdots \times X_n \to B.$$
In particular if $\tau : X_1 \times \cdots \times X_n$ is a term of type A with free variables $x_1, \ldots, x_n$ and $\sigma_1, \ldots, \sigma_n$ are terms of types $X_1, \ldots, X_n$ with the same free variables $y_1, \ldots, y_m$ of type $Y_1, \ldots, Y_m$, then $\tau(\sigma_1, \ldots, \sigma_n)$ is a term of type A with free variables $y_1, \ldots, y_m$.
5. If $x_1, \ldots, x_m$ are free variables of type $X_1, \ldots, X_n$ and τ is a term of type A with free variables $x_1, \ldots, x_n$ with $n \leq m$ then $\tau_{(x_1,\ldots,x_m)}$ is the term of type A and free variables $x_1, \ldots, x_m$ interpreted as
$$\tau_{(x_1,\ldots,x_m)} : X_1 \times \cdots \times X_m \xrightarrow{\pi} X_1 \times \cdots \times X_n \xrightarrow{\tau} A,$$
where π is the canonical projection. In practice we will ignore the subscript $(x_1, \ldots, x_m)$ and write τ for $\tau_{(x_1,\ldots,x_m)}$. The variables from x_n to x_m are called *ghost* variables for obvious reasons.

A *formula* is a term of type Ω. If φ is a formula with free variables $x_1, \ldots, x_n$, we will denote by
$$\{(x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n | \varphi(x_1, \ldots, x_n)\},$$
the subobject classified by φ. We have the following specific formulas :

1. $\top : 1 \hookrightarrow \Omega$ and $\perp : 1 \hookrightarrow \Omega$ are formulas with no free variables.
2. If σ, τ are terms both of type A with the same free variables $x_1, \ldots, x_n$ interpreted as $\sigma : X_1 \times \cdots \times X_n \to A$ and $\tau : X_1 \times \cdots \times X_n \to A$, then $\sigma = \tau$ is the formula $=_A (\tau, \sigma)$ with free variables $x_1, \ldots, x_n$.

3. If τ is a term of type A and σ is a term of type Ω^A both with the same free variables $x_1, \ldots, x_n$, then $\tau \in \sigma$ is the formula $\mathrm{ev}_{A,\Omega}(\sigma, \tau)$ with free variables $x_1, \ldots, x_n$.
4. If φ and ψ are formulas with the same free variables $x_1, \ldots, x_n$, then the formulas $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \Rightarrow \psi$ are respectively the formulas $\wedge(\varphi, \psi)$, $\vee(\varphi, \psi)$ and $\Rightarrow(\varphi, \psi)$ with free variables $x_1, \ldots, x_n$.
5. If φ is a formula with free variables $x_1, \ldots, x_n$, then $\neg\varphi$ is the formula $\neg(\varphi)$ with free variables $x_1, \ldots, x_n$.
6. If φ is a formula with free variables $x, x_1, \ldots, x_n$ (x distinct from the other variables), then $(\exists x)\varphi$ and $(\forall x)\varphi$ are formulas with free variables $x_1, \ldots, x_n$ and their interpretations are respectively the characteristic morphisms of the subobjects $\exists_\pi(\{(x, x_1, \ldots, x_n) \in X \times X_1 \times \cdots \times X_n | \varphi(x_1, \ldots, x_n)\})$ and $\forall_\pi(\{(x, x_1, \ldots, x_n) \in X \times X_1 \times \cdots \times X_n | \varphi(x_1, \ldots, x_n)\})$ with π the projection $X \times X_1 \times \cdots \times X_n \to X_1 \times \cdots \times X_n$.

When the context needs to be clear, we will write $x \in X$ for a variable x of type X. As an example in the language of a topos, the object of epimorphisms becomes :

$$\mathrm{Epi}(X, Y) = \{f \in Y^X | (\forall y \in Y)(\exists x \in X) f(x) = y\}.$$

To this long definition, we add the notion of validity.

Definition 3.4.3. In a topos $\mathscr{E}$, a formula $\varphi : X_1 \times \cdots \times X_n \to \Omega$ is *universally valid*, written $\models \varphi$, when $\varphi = \top_{X_1 \times \cdots \times X_n}$.

Proposition 3.4.4. *In a topos $\mathscr{E}$, the following are equivalent :*

(1) A formula $\varphi : X \to \Omega$ is universally valid,

(2) $\{x \in X | \varphi(x)\} = X$,

(3) The formula $(\forall x)\varphi : 1 \to \Omega$ is universally valid.

Proof. The proof (1) $\iff$ (2) is straight forward considering the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, hook, "\top"] \\ X \arrow[r] & 1 \arrow[r, hook, "\top"] & \Omega. \end{tikzcd}$$

The right and left squares are pullbacks. Thus the outer square is a pullback. Hence $\{x \in X | \varphi(x)\} = X$ if and only if the morphism $X \to 1 \xrightarrow{\top} \Omega$ is precisely φ.

The proof (2) $\iff$ (3) is straight forward too : we look back a the definition of the adjoint Π_f with $b : \{x \in X | \varphi(x)\} \hookrightarrow X$ the subobject classified by φ. Applying the previous result of the proof, we know that $\models (\forall x)\varphi$ if and only if $1 = \{|(\forall x)\varphi\}$. The following diagram is a pullback

$$\begin{tikzcd}
1 \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner", very near start] & \{x \in X|\varphi(x)\}^X \arrow[d, "b^X"] \\
1 \arrow[r] & X^X
\end{tikzcd}$$

if and only if b^X is an isomorphism, that is if and only if b is an isomorphism, and thus if and only if $\{x \in X|\varphi(x)\} = X$. Thus $1 = \{|(\forall x)\varphi\} \iff \{x \in X|\varphi(x)\} = X$. □

Now that we have a good language in our topos, it is finally time to interpret a theory in a topos.

Definition 3.4.5. Let T be a theory on a signature Σ. A *model* of the theory in a topos $\mathscr{E}$ is the following data :

1. for each type s in the signature, an object $A \in \mathscr{E}$,
2. for each constant $c :\to s$ of type s, a constant $c : 1 \to A$ in $\mathscr{E}$,
3. for each operation symbol $\sigma : s_1 \times \cdots \times s_n \to s$, a morphism $\sigma : A_1 \times \cdots \times A_n \to A$ in $\mathscr{E}$,
4. for each relation symbol $R \subseteq s_1 \times \cdots \times s_n$, a subobject $r : R \hookrightarrow A_1 \times \cdots \times A_n$ in $\mathscr{E}$,

such that the axioms of the theory are universally valid in $\mathscr{E}$.

Here, we just need to specify that for a given relation $R \hookrightarrow A_1 \times \cdots \times A_n$ with free variables $a_1, \ldots, a_n$, if $\sigma_1, \ldots, \sigma_n$ are terms of type $A_1, \ldots, A_n$ with same free variables $b_1, \ldots, b_m$ of type $B_1, \ldots, B_m$, then the relation $R(\sigma_1, \ldots, \sigma_n)$ is the pullback of r along $(\sigma_1, \ldots, \sigma_n)$.

Definition 3.4.6. Let M_1, M_2 be two models of a theory T in a topos $\mathscr{E}$ and denotes their type as $A_1^1, \ldots, A_n^1$ and $A_1^2, \ldots, A_n^2$ and same for the symbols and constants. A *morphism of models* $\mu : M_1 \to M_2$ is the data of a morphism $\mu_{A_i} : A_i^1 \to A_i^2$ for each type such that :

1. for each constant c of type A,
$$\models \mu_A(c^1) = c^2,$$
2. for each operation σ of type A and free variables $x_1, \ldots, x_n$ of type $X_1^1, \ldots, X_n^1$,
$$\models \mu_A(\sigma^1(x_1, \ldots, x_n)) = \sigma^2(\mu_{X_1}(x_1), \ldots, \mu_{X_n}(x_n)),$$

3. for each relation R of type $A_1 \times \cdots \times A_n$ and free variables $a_1, \ldots, a_n$ of type $A_1^1, \ldots, A_n^1$,
$$\models R^1(a_1, \ldots, a_n) \Rightarrow R^2(\mu_{A_1}(a_1), \ldots, \mu_{A_n}(a_n)).$$

One can observe that as expected the two definitions above in the case of **Set** are the definitions of structure and homomorphism of structure.

The models of a theory T in a topos $\mathcal{E}$ and the morphisms of models constitute a category $\mathbf{Mod}_{\mathcal{E}}T$.

When we defined the notion of classifying topos, morphisms between topoi we used are the geometric morphisms. If one works with an arbitrary theory, geometric morphisms are not the right to study a theory trough different topoi. Geometric morphisms do not in general preserve the subobject classifier. But if someone works with a geometric theory, then geometric morphisms become more than interesting. Let us recall here that geometric formulas are formulas built up from $=$, relations, terms, $\top$, $\bot$, $\exists$, finite $\wedge$ and small $\vee$. Remark that we don't know if a given topos $\mathcal{E}$ has small coproducts and thus infinitary many $\vee$. But since the main focus in the case of the classifying topos and the geometric morphisms are the Grothendieck topoi, who are cocomplete, we can assume our topos $\mathcal{E}$ is cocomplete and then our whole construction for finite $\vee$ holds for an infinite number.

Proposition 3.4.7. *Let $\mathcal{E}, \mathcal{F}$ be topoi and $(f_*, f^*) : \mathcal{E} \to \mathcal{F}$ be a geometric morphism, i.e f^* is a left adjoint to f_* such that f^* is left exact. Then $f^* : \mathcal{F} \to \mathcal{E}$ preserves the subobject classified by any geometric formula.*

Proof. Since f^* preserves finite limits and small colimits, it preserves the image factorization, the union and the intersection of subobject, any bottom subobject and any top subobject. Since $\wedge$,$\vee$,$\top$ and $\bot$ are directly constructed from intersection, union, bottom and top, they all are preserved by f^*. Since the subobject of $\exists x \varphi$ is defined as the the image factorization of $\{(x, y)|\varphi(x, y)\} \to X \times Y \xrightarrow{\pi} Y$, as long as f^* preserves the subobject of φ, by preserving image factorization, it preserves the subobject of $\exists x \varphi$. Finally, for the equality, assume τ and σ are two terms of type A and free variables $x_1, \ldots, x_n$ of types $X_1, \ldots, X_n$ interpreted in $\mathcal{E}$. Then by definition of $\sigma = \tau$, the outer and right squares in the following diagram are pullbacks :

$$\begin{tikzcd}
\{(x_1, \ldots, x_n)|\tau(x_1, \ldots, x_n) = \sigma(x_1, \ldots, x_n)\} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[r] \arrow[d, hook, "\Delta_A"] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, hook, "\top"] \\
X_1 \times \cdots \times X_n \arrow[r] & A \times A \arrow[r, hook, "\top"] & \Omega.
\end{tikzcd}$$

Hence the left square is also a pullback, that is preserved by f^* and Δ_A is sent to $\Delta_{f^*(A)}$ (since Δ_A is defined as the product of identities, both products and identities are preserved by f^*). Thus the subobject of $\tau = \sigma$ is preserved. $\square$

We can not expect the subobjects of $\varphi \Rightarrow \psi$ or $\neg\varphi$ to be preserved since looking back at their definition, the implication of subobjects is defined as the equalizer of characteristic morphisms, thus if the subobject classifier is not preserved there is no chance the implication and the negation are. Nevertheless a striking result is that although the implication is not preserved, its universal validity is.

Proposition 3.4.8. *Let $\mathscr{E}, \mathscr{F}$ be topoi and $(f_*, f^*) : \mathscr{E} \to \mathscr{F}$ be a geometric morphism. Then $f^* : \mathscr{F} \to \mathscr{E}$ preserves the universal validity of the formula $\varphi \Rightarrow \psi$ for any two geometric formulas φ and ψ.*

Proof. Assume $x \in X$ the free variable of φ and ψ interpreted in $\mathscr{F}$ and recall that X is the top element in the Heyting algebra $\mathrm{Sub}(X)$. Then

$$\begin{aligned} \models \varphi \Rightarrow \psi &\iff X = \{x \in X | \varphi(x) \Rightarrow \psi(x)\} && \text{by 3.4.4} \\ &\iff X \subseteq \{x \in X | \varphi(x) \Rightarrow \psi(x)\} \\ &\iff X \subseteq \{x \in X | \varphi(x)\} \Rightarrow \{x \in X | \psi(x)\} \\ &\iff X \cap \{x \in X | \varphi(x)\} \subseteq \{x \in X | \psi(x)\} && \text{by 3.1.16} \\ &\iff \{x \in X | \varphi(x)\} \subseteq \{x \in X | \psi(x)\}. \end{aligned}$$

Since by 3.4.7, f^* preserves $\{x \in X | \varphi(x)\} \subseteq \{x \in X | \psi(x)\}$, it preserves $\models \varphi \Rightarrow \psi$. □

Corollary 3.4.9. *Let $\mathscr{E}, \mathscr{F}$ be topoi and $(f_*, f^*) : \mathscr{E} \to \mathscr{F}$ be a geometric morphism. Then $f^* : \mathscr{F} \to \mathscr{E}$ preserves the universal validity of φ and $\neg\varphi$ for any geometric formula φ.*

Proof. By 3.1.19

$$\{x \in X | \varphi(x)\} = (X \Rightarrow \{x \in X | \varphi(x)\}) = \{x \in X | \top_X(x) \Rightarrow \varphi(x)\}.$$

And by definition of the negation

$$\{x \in X | \neg\varphi(x)\} = \{x \in X | \varphi(x) \Rightarrow \bot_X(x)\}.$$

Therefore by 3.4.4 $\models \varphi$ if and only if $\models \top_X \Rightarrow \varphi$. Similarly $\models \neg\varphi$ if and only if $\models \varphi \Rightarrow \bot_X$. Hence by 3.4.8, f^* preserves both $\models \varphi$ and $\models \neg\varphi$. □

This corollary motivates the whole study of theories in Grothendieck topoi and the study of the classifying topoi. If we find a Grothendieck topos $\mathscr{E}$ where a theory T is easy to manipulate and with a geometric morphism from **Set** to $\mathscr{E}$, maybe the validity of a statement of that theory in the Grothendieck topos would be easier to prove and thus proving it automatically in our set theoretic language. We need to understand better the models of a theory in a Grothendieck topos to find the right one to work with. Hence the classifying topos.

3.5 Classifying topos of a coherent theory

We end our chapter with the construction of a classifying topos for any geometric theory.

Theorem 3.5.1. *A geometric theory is sketchable and admits a classifying topos.*

Proof. The goal is to construct a "sketch" as in 2.2.6 from a geometric theory $\mathcal{T}$ with the same category of models.
The first step is to construct a graph $\mathcal{G}$ out of $\mathcal{T}$:

1. For each finite sequence of types $T_1, \ldots, T_n$, $\mathcal{G}$ has an object $(T_1, \ldots, T_n)$ and morphisms $p_i : (T_1, \ldots, T_n) \to T_i$ for $1 < i < n$. For the empty sequence of types the corresponding object in $\mathcal{G}$ is $(\,)$.
2. For any operation $\sigma : T_1 \times \cdots \times T_n \to T$, $\mathcal{G}$ has a morphism $\sigma : (T_1, \ldots, T_n) \to T$.
3. For any constant $c :\to T$, $\mathcal{G}$ has a morphism $\gamma : (\,) \to T$.
4. For any $R \subseteq T_1 \times \cdots \times T_n$, $\mathcal{G}$ has an object R and a morphism $i_R : R \to (T_1, \ldots, T_n)$.

By construction $\mathcal{G}$ has a set of objects and a set of morphisms, but it is not a category. We have objects and morphisms but no composition law. A composition law can be forced with paths. A *path* in $\mathcal{G}$ is a non-empty and finite alternating sequence $(X_1, f_1, X_2, f_2, \ldots, X_n)$ of objects and morphisms of $\mathcal{G}$ with first and last terms being objects and f_i has domain X_i and codomain X_{i+1}. In particular any object X admits a canonical path (X). The path category $\mathcal{P}$ of $\mathcal{G}$:

1. has the same class of objects as $\mathcal{G}$,
2. for any two objects X and Y, its morphisms $\mathbf{Hom}_{\mathcal{P}}(X, Y)$ are the paths starting at X and ending at Y,
3. the composition of morphisms is defined as
$$(X_n, f_n, X_{n+1}, f_{n+1}, \ldots, X_m) \circ (X_1, f_1, X_2, f_2, \ldots, X_n) = (X_1, f_1, X_2, f_2, \ldots, X_m).$$
4. the identity of $X \in \mathcal{P}$ is the canonical path (X).

Since $\mathcal{G}$ has a set of objects and a set of morphisms, $\mathcal{P}$ is a small category. Now let $\mathcal{T}_0$ be the theory $\mathcal{T}$ without its axioms. Following the definition of a model in a category $\mathcal{E}$, 3.4.5, one can remark models of $\mathcal{T}_0$ are the graph morphisms $\mathcal{G} \to \mathcal{E}$, which are equivalent to functors $F : \mathcal{P} \to \mathcal{E}$ (see [2, Proposition 5.1.4]), satisfying that:

1. F maps $(T_1, \ldots, T_n)$ and the family $\{p_i\}_{1 \le i \le n}$ to a product cone (note that $F()$ is then the terminal object),

2. F maps a relation $i_R : R \to (T_1, \dots, T_n)$ to a monomorphism, or equivalently, F maps the following diagram to a pullback

$$\begin{tikzcd} R \arrow[r, equal] \arrow[d, equal] & R \arrow[d] \\ R \arrow[r] & (T_1, \dots, T_n) \end{tikzcd}$$

Hence, the models of $\mathcal{T}_0$ are precisely the functors $F : \mathscr{P} \to \mathscr{E}$ mapping a specific set of cones to limit cones in $\mathscr{E}$. This means the study of the theory $\mathcal{T}_0$ reduces to the study of $\mathscr{P}$ seen as a sketch with the corresponding cones and no cocones. Applying lemma 2.2.1, we then get a small finitely complete category $\mathscr{C}$ and a map $\tau : \mathscr{P} \to \mathscr{C}$ such that the models of our sketch are equivalent to the left exact functors $\mathscr{C} \to \mathscr{E}$.

Now we have a nice category $\mathscr{C}$ with limit cones representing our theory $\mathcal{T}_0$. The next step is to specify some additional data, families of morphisms, in $\mathscr{C}$ such that we recover the whole theory $\mathcal{T}$ in $\mathscr{C}$. For any formula φ of free variables $t_1, \dots, t_n$ of type $T_1, \dots, T_n$, we will associate a family of morphisms

$$\{f_i : X_i \to T_1 \times \cdots \times T_n\}_{1 \leq i \leq k}$$

in $\mathscr{C}$ to the formula (we denote T instead of $\tau(T)$ for the types as objects in $\mathscr{C}$, same notation for the operations and relations), such that for any left exact functor $F : \mathscr{C} \to \mathscr{E}$

$$\{(t_1, \dots, t_n) | \varphi(t_1, \dots, t_n)\} = \mathrm{Im}F(f_1) \cup \cdots \cup \mathrm{Im}F(f_k),$$

where $\mathrm{Im}F(f_i)$ is the image factorisation of $F(f_i)$ in $\mathscr{E}$. The construction of this family is inductive, let us fix $F : \mathscr{C} \to \mathscr{E}$ a left exact functor :

1. We associate to the true formula $\top$ the identity $\mathrm{Id}_1 : 1 \to 1$ in $\mathscr{C}$. The image of $F(\mathrm{Id}_1) = \mathrm{Id}_1 : 1 \to 1$ in $\mathscr{E}$ is the subobject $1 \hookrightarrow$ and is indeed the subobject classified by $\top : 1 \to \Omega$.

2. We associate to the false formula $\perp$ the empty family of morphisms of codomain 1, $\emptyset \subseteq \{f | \mathrm{codom}(f) = 1\}$. Applying F gives the empty family of codomain 1 in $\mathscr{E}$, $\emptyset \subseteq \{f | \mathrm{codom}(f) = F(1) = 1\}$. Since the empty family factors through every subobject of 1 (empty condition), the image is the subobject $0 \hookrightarrow 1$ classified by $\perp : 1 \to \Omega$.

3. Notice that since the only thing we need in order to interpret terms (not formulas) are finite products. Thus we can already interpret terms in $\mathscr{C}$. Now let $\sigma, \tau : T_1 \times \cdots \times T_n \to T$ be the interpretations of two terms in $\mathscr{C}$ with same free variables. Since $\mathscr{C}$ is finitely complete, take the pullback D of the product (τ, σ) along the diagonal Δ_T. This pullback gives a morphism $d : D \to T_1 \times \cdots \times T_n$. Now applying

F preserving the limits, we get that left square is a pullback

$$\begin{tikzcd}
F(D) \arrow[r] \arrow[d, "F(d)"] \arrow[dr, phantom, "\lrcorner", very near start] & F(T) \arrow[r] \arrow[d, "\Delta_{F(T)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\top"'] \\
F(T_1) \times \cdots \times F(T_n) \arrow[r, "(\tau,\sigma)"] & F(T) \times F(T) \arrow[r, "=_{F(T)}"] & \Omega
\end{tikzcd}$$

Since the right square is a pullback, the outer one is a pullback too. Hence, the subobject classified by $\sigma = \tau$ is precisely $F(D)$. Moreover $\mathrm{Im}(F(d)) = F(D)$ since $F(d)$ is a monomorphism. Therefore we associate to the formula $\sigma = \tau$ the single element $d : D \to T_1 \times \cdots \times T_n$ in $\mathscr{C}$.

Now assume that for two geometric formulas φ, ψ of same free variables $t_1, \ldots, t_n$ of types $T_1, \ldots, T_n$, we are given two families of morphisms $\{f_i : X_i \to T_1 \times \cdots \times T_n\}_{1 \le i \le k}$, $\{g_j : Y_j \to T_1 \times \cdots \times T_n\}_{1 \le j \le l}$ in $\mathscr{C}$ such that $\{(t_1, \ldots, t_n) | \varphi(t_1, \ldots, t_n)\} = \mathrm{Im}F(f_1) \cup \cdots \cup \mathrm{Im}F(f_k)$ and $\{(t_1, \ldots, t_n) | \psi(t_1, \ldots, t_n)\} = \mathrm{Im}F(g_1) \cup \cdots \cup \mathrm{Im}F(g_l)$.

4. By straightforward computation we get that

$$\begin{aligned}
\{(t_1, \ldots, t_n) | \varphi \vee \psi(t_1, \ldots, t_n)\} &= \{(t_1, \ldots, t_n) | \varphi(t_1, \ldots, t_n)\} \cup \{(t_1, \ldots, t_n) | \psi(t_1, \ldots, t_n)\} \\
&= \mathrm{Im}F(f_1) \cup \cdots \cup \mathrm{Im}F(f_k) \cup \mathrm{Im}F(g_1) \cup \cdots \cup \mathrm{Im}F(g_l).
\end{aligned}$$

Thus we associate to $\varphi \vee \psi$ the family

$$\{f_i : X_i \to T_1 \times \cdots \times T_n\}_{1 \le i \le k} \cup \{g_j : Y_j \to T_1 \times \cdots \times T_n\}_{1 \le j \le l}.$$

The case of infinitary many joins is analogous.

5. Let $\{h_{ij} : Z_{ij} \to T_1 \times \cdots \times T_n\}_{1 \le i \le k;\, 1 \le j \le l}$ be the family defined by $h_{ij} = f_i \circ x_{ij} = g_j \circ y_{ij}$, where

$$\begin{tikzcd}
Z_{ij} \arrow[r, "y_{ij}"] \arrow[d, "x_{ij}"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y_j \arrow[d, "g_j"] \\
X_i \arrow[r, "f_i"'] & T_1 \times \cdots \times T_n.
\end{tikzcd}$$

Since image factorizations are pullback stable in $\mathscr{E}$, one gets the following diagram where each square is a pullback

$$\begin{tikzcd}
F(Z_{ij}) \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Im}F(y_{ij}) \arrow[r, hook] \arrow[d, two heads] & F(Y_j) \arrow[d, two heads] \\
\mathrm{Im}F(x_{ij}) \arrow[r, two heads] \arrow[d, hook] & \mathrm{Im}F(f_i) \cap \mathrm{Im}F(g_j) \arrow[r, hook] \arrow[d, hook] & \mathrm{Im}F(g_j) \arrow[d, hook] \\
F(X_i) \arrow[r, two heads] & \mathrm{Im}F(f_i) \arrow[r, hook] & F(T_1) \times \cdots \times F(T_n)
\end{tikzcd}$$

Thus, $\mathrm{Im}F(h_{ij}) = \mathrm{Im}F(f_i) \cap \mathrm{Im}F(g_j)$. Then :

$$\begin{aligned}
\{(t_1,\dots,t_n)|\varphi \wedge \psi(t_1,\dots,t_n)\} &= \{(t_1,\dots,t_n)|\varphi(t_1,\dots,t_n)\} \cap \{(t_1,\dots,t_n)|\psi(t_1,\dots,t_n)\} \\
&= (\mathrm{Im}F(f_1) \cup \cdots \cup \mathrm{Im}F(f_k)) \cap (\mathrm{Im}F(g_1) \cup \cdots \cup \mathrm{Im}F(g_l)) \\
&= \bigcup_{i,j} (\mathrm{Im}F(f_i) \cap \mathrm{Im}F(g_j)) \\
&= \bigcup_{i,j} \mathrm{Im}F(h_{ij}).
\end{aligned}$$

Hence to the formula $\varphi \wedge \psi$, we associate the family

$$\{h_{ij} : Z_{ij} \to T_1 \times \cdots \times T_n\}_{1 \leq i \leq k;\ 1 \leq j \leq l}.$$

6. Let $\pi : T_1 \times \cdots \times T_n \to T_2 \times \cdots \times T_n$ be the canonical projection. By 3.3.12, we get the following commutative diagram

$$\begin{tikzcd}
F(X_i) \arrow[d, two heads] \arrow[drr, "F(f_i)"] & & \\
\mathrm{Im}F(f_i) \arrow[rr, hook] \arrow[d, two heads] & & F(T_1) \times \cdots \times F(T_n) \arrow[d, "F(\pi)"] \\
\exists_{F(\pi)}\mathrm{Im}F(f_i) \arrow[rr, hook] & & F(T_2) \times \cdots \times F(T_n)
\end{tikzcd}$$

thus $\exists_{F(\pi)}\mathrm{Im}F(f_i) = \mathrm{Im}F(\pi \circ f_i)$ for all $1 \leq i \leq k$. Therefore, since left adjoint functors preserve colimits, we get the computation

$$\begin{aligned}
\{(t_2,\dots,t_n)|(\exists t_1)\varphi(t_1,\dots,t_n))\} &= \exists_{F(\pi)} (\mathrm{Im}F(f_1) \cup \cdots \cup \mathrm{Im}F(f_k)) \\
&= \Big(\exists_{F(\pi)} (\mathrm{Im}F(f_1))\Big) \cup \cdots \cup \Big(\exists_{F(\pi)} (\mathrm{Im}F(f_k))\Big) \\
&= \mathrm{Im}F(\pi \circ f_1) \cup \cdots \cup \mathrm{Im}F(\pi \circ f_k).
\end{aligned}$$

Hence, we associate to $(\exists t_1)\varphi(t_1,\dots,t_n)$ the family in $\mathscr{C}$

$$\{\pi \circ f_i : X_i \to T_2 \times \cdots \times T_n\}_{1 \leq i \leq k}.$$

7. Let $t_1,\dots,t_m$ be free variable of type $T_1,\dots,T_m$ with $n \leq m$ and let $\pi : T_1 \times \cdots \times T_m \to T_1 \times \cdots \times T_n$ be the canonical projection in $\mathscr{C}$. Consider the following pullback in C

$$\begin{tikzcd}
Y_i \arrow[r] \arrow[d, "g_i"] \arrow[dr, phantom, "\lrcorner", very near start] & X_i \arrow[d, "f_i"] \\
T_1 \times \cdots \times T_m \arrow[r, "\pi"'] & T_1 \times \cdots \times T_n.
\end{tikzcd}$$

F as a left exact functor preserves this pullback. Thus, since the pullback functor 3.3.12 $(F(\pi))^{-1}$ small colimits and image factorization,

$$\begin{aligned}
\bigcup_{i=1}^{k} \mathrm{Im}F(g_i) &= \bigcup_{i=1}^{k} (F(\pi))^{-1}(\mathrm{Im}F(f_i)) \\
&= (F(\pi)^{-1}\left(\bigcup_{i=1}^{k} \mathrm{Im}F(f_i)\right) \\
&= (F(\pi))^{-1}(\{(t_1,\dots,t_n)|\varphi(t_1,\dots,t_n)\}) \\
&= \{(t_1,\dots,t_m)|\varphi(t_1,\dots,t_m)\}.
\end{aligned}$$

8. Let $R \subseteq T_1 \times \cdots \times T_n$ be a relation and $\tau_i : X_1 \times \cdots \times X_m \to T_i$ terms with same free variables, for $1 \le i \le n$ in $\mathscr{C}$ an let S be the pullback of the subobject R along $(\tau_1,\dots,\tau_n)$.

$$\begin{tikzcd}
S \arrow[r] \arrow[d, hook, "s"] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, hook, "r"] \\
X_1 \times \cdots \times X_m \arrow[r, "(\tau_1,\dots,\tau_n)"'] & T_1 \times \cdots \times T_n
\end{tikzcd}$$

Since, for variables x_i of types $F(T_i)$ $\{(x_1,\dots,x_n)|R(x_1,\dots,x_n)\} = F(R)$ then, $\{(\tau_1,\dots,\tau_n)|R(\tau_1,\dots,\tau_n))\} = F(S)$. $F(S)$ is the image of $F(s)$. We, then, associate to $R(\tau_1,\dots,\tau_n)$ the arrow

$$s : S \to X_1 \times \cdots \times X_m.$$

9. Let $\tau_i : Y_1 \times \cdots \times Y_m \to T_i$ be terms with same free variables for $1 \le i \le n$ in $\mathscr{C}$ and let Z_i be the pullback of f_i along $(\tau_1,\dots,\tau_n)$

$$\begin{tikzcd}
Z_i \arrow[r] \arrow[d, "g_i"] \arrow[dr, phantom, "\lrcorner", very near start] & X_i \arrow[d, "f_i"] \\
Y_1 \times \cdots \times Y_m \arrow[r, "(\tau_1,\dots,\tau_n)"'] & T_1 \times \cdots \times T_n
\end{tikzcd}$$

Since by 3.3.12 the pullback functor $(F(\tau_1,\dots,\tau_n))^{-1} : \mathrm{Sub}(T_1 \times \cdots \times T_n) \to$

$\mathrm{Sub}(Y_1 \times \cdots \times Y_m)$ preserves small colimits and image factorization :

$$\begin{aligned}\bigcup_{i=1}^{k} \mathrm{Im}F(g_i) &= \bigcup_{i=1}^{k} (F(\tau_1,\ldots,\tau_n))^{-1}(\mathrm{Im}F(f_i)) \\ &= (F(\tau_1,\ldots,\tau_n))^{-1}\left(\bigcup_{i=1}^{k} \mathrm{Im}F(f_i)\right) \\ &= (F(\tau_1,\ldots,\tau_n))^{-1}(\{(t_1,\ldots,t_n)|\varphi(t_1,\ldots,t_n)\}) \\ &= \{(\tau_1,\ldots,\tau_n)|\varphi(\tau_1,\ldots,\tau_n)\}.\end{aligned}$$

Thus to the formula $\varphi(\tau_1,\ldots,\tau_n)$ we associate the family $\{g_i : Z_i \to X_1 \times \cdots \times X_n\}_{1\leq i\leq k}$.

Now that we have constructed a family of arrows in $\mathscr{C}$ for each coherent formula, let us show that any coherent axiom $\forall(x_1,\ldots,x_n)(\varphi(x_1,\ldots,x_n) \Rightarrow \psi(x_1,\ldots,x_n))$ holds if and only if the family $\{F(x_{ij}) : F(Z_{ij}) \to F(X_i)\}_{1\leq i\leq k, 1\leq j\leq l}$ is jointly epimorphic, where $\{x_{ij} : Z_{ij} \to X_i\}_{1\leq i\leq k,1\leq j\leq l}$ are morphisms obtained by the pullback of the families associated to φ and ψ (see paragraph 5). By 3.4.4 and the proof of 3.4.8 :

$$\begin{aligned}&\forall(x_1,\ldots,x_n)(\varphi(x_1,\ldots,x_n) \Rightarrow \psi(x_1,\ldots,x_n)) \\ &\iff \{(x_1,\ldots,x_n)|\varphi(x_1,\ldots,x_n)\} \subseteq \{(x_1,\ldots,x_n)|\psi(x_1,\ldots,x_n)\} \\ &\iff \bigcup_{1\leq i\leq k} \mathrm{Im}F(f_i) \subseteq \bigcup_{1\leq j\leq l} \mathrm{Im}F(g_j) \\ &\iff \forall i = 1,\ldots,k \quad \mathrm{Im}F(f_i) \subseteq \bigcup_{1\leq j\leq l} \mathrm{Im}F(g_j) \\ &\iff \forall i = 1,\ldots,k \quad \mathrm{Im}F(f_i) = \mathrm{Im}F(f_i) \cap \left(\bigcup_{1\leq j\leq l} \mathrm{Im}F(g_j)\right) \\ &\iff \forall i = 1,\ldots,k \quad \mathrm{Im}F(f_i) = \bigcup_{1\leq j\leq l} (\mathrm{Im}F(f_i) \cap \mathrm{Im}F(g_j)) \\ &\iff \forall i = 1,\ldots,k \quad F(X_i) = \bigcup_{1\leq j\leq l} \mathrm{Im}F(x_{ij}) \\ &\iff \forall i = 1,\ldots,k \quad \{F(x_{ij})\}_{1\leq j\leq l} \text{ a jointly epimorphic family.}\end{aligned}$$

The statement $\mathrm{Im}F(f_i) = \bigcup_{1\leq j\leq l} (\mathrm{Im}F(f_i) \cap \mathrm{Im}F(g_j)) \iff F(X_i) = \bigcup_{1\leq j\leq l} \mathrm{Im}F(x_{ij})$, for a fixed i, comes from the following consideration :

$$\begin{tikzcd}
q_i^{-1}\left(\bigcup_{1\leq j\leq l}\mathrm{Im}F(f_i)\cap\mathrm{Im}F(g_j)\right) \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner", very near start] & \bigcup_{1\leq j\leq l}\mathrm{Im}F(f_i)\cap\mathrm{Im}F(g_j) \arrow[d, equal] \\
F(X_i) \arrow[r, two heads, "q_i"'] & \mathrm{Im}F(f_i)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Im}F(x_{ij}) \arrow[r, two heads] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Im}F(f_i)\cap\mathrm{Im}F(g_j) \arrow[d, hook] \\
F(X_i) \arrow[r, two heads, "q_i"'] & \mathrm{Im}F(f_i)
\end{tikzcd}$$

By definition of pullback functor (3.3.12) the top diagram is a pullback and whenever one of the two vertical arrows is an iso, so is the other one. Now since q^{-1} preserves small colimits and image factorization, $q_i^{-1}\left(\bigcup_{1\leq j\leq l}\mathrm{Im}F(f_i)\cap\mathrm{Im}F(g_j)\right) = \bigcup_{1\leq j\leq l}q^{-1}\left(\mathrm{Im}F(f_i)\cap\mathrm{Im}F(g_j)\right)$. By the bottom diagram, that we proved to be a pullback in the $\varphi\wedge\psi$ paragraph, we get $q_i^{-1}\left(\mathrm{Im}F(f_i)\cap\mathrm{Im}F(g_j)\right) = \mathrm{Im}F(x_{ij})$.

Finally the last statement, for a fixed i, $F(X_i) = \bigcup_{1\leq j\leq l}\mathrm{Im}F(x_{ij}) \iff \{F(x_{ij})\}_{1\leq j\leq l}$ is a jointly epimorphic family, comes from the fact that $F(X_i) = \bigcup_{1\leq j\leq l}\mathrm{Im}F(x_{ij}) \iff$ the canonical arrow $\coprod_j \mathrm{Im}F(x_{ij}) \to F(X_i)$ is a strong epi $\iff \{F(x_{ij})\}_{1\leq j\leq l}$ is a jointly epimorphic family.

Therefore a left exact functor $F:\mathscr{C}\to\mathscr{E}$ to a Grothendieck topos is a model of $\mathcal{T}$ if and only if all the families $\{x_{ij}\}_j$ associated to the geometric axioms of the theory are jointly epimorphic families. Thus by 2.2.6, the geometric theory $\mathcal{T}$ admits a classifying topos.

□

Remark 3.5.2. *We end this chapter with rather a remark than an example of this last theorem. In [6, Section VIII.6], when Mac Lane and Moerdijk are proving the classifying topos of the theory of local rings is the big Zariski topos they are not using explicitly 3.5.1. But when looking at their proof, we can see they implicitly are. The whole point of theorem 3.5.1 is to associate to each axioms a jointly epimorphic family and this is precisely what Mac Lane and Moerdijk are doing. They start with the rings and the classifying topos of rings* $\mathbf{Pr}(\mathbf{fp\text{-}rings}^{op})$*. Then they state that a ring R is a local ring if for any element $a\in R$ either a or $1-a$ is invertible. By definition of universal validity, this is equivalent to the union of the following subobjects*

$$\{a\in R|\exists b, a.b=1\}\hookrightarrow R$$
$$\{a\in R|\exists b, (1-a).b=1\}\hookrightarrow R$$

being all R. They then prove this to be equivalent to the data of an epimorphism. They end the proof with the construction of a suitable Grothendieck topology J on $\mathbf{Pr}(\mathbf{fp\text{-}rings}^{op})$*, such that* $\mathbf{Sh}(\mathbf{fp\text{-}rings}^{op}, J)$ *(the Big Zariski topos) is the suitable classifying topos where the arrow associated to the union of the two subobjects is an epimorphism.*

3.6 Topoi and theories beyond

There is a lot more to discuss about topos theory. We will give here a list of references to check if one wants to get further in certain subjects mentioned during this chapter.

Concerning the logical aspect of a topos, we only scratched the surface here. A whole account for the technical aspects of logic can found be in [5].

When we were describing the language of a topos, we only talked about the syntax of the language. We show how to write formulas such as $(\forall x)\varphi(x, y)$ in a topos. But we never explained what $(\forall x)\varphi(x, y)$ does *mean*. For example let X be a topological space and let $\mathcal{O}(X)$ be the poset of opens in X with the inclusion as the order. One can then study the category of sheaves $A : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Set}$. Assume φ is a formula with one free variable a of type A, what does $(\exists a)\varphi(a)$ means ? Certainly not that "there exists an element $a \in A$ satisfying φ". It would rather something like "for each point $x \in X$, there exist a neighborhood $U \subseteq X$ of x and an element $a \in A(U)$". There is a way to put meanings into our formulas, it is called the Krypke-Joyal semantic. More information on that can be found in [6]. In particular the thesis [1] explores what is the internal language (Syntaxic and semantic) of the sheaves on a topological space and how the internal language allows for an easier understanding of some local ring sheaf properties

Finally we cannot end this section without mentioning the work of Olivia Caramello. Until now we have not mentioned it, but sometimes two distinct theories admit the same classifying topos. Those theories are said to be Morita-equivalent. If two theories are Morita-equivalent then their common classifying topos can be used as *bridge*, transferring properties and results across the two theories. It is an highly technical subject deeply explained in [4].

Conclusion

Let us summarize what we explored in this thesis. We first introduced the notion of Grothendieck topos and then we saw two different ways to study what are mathematical theories in Grothendieck topoi. The first approach was to sketch the theories while the second one was to interpret them in the internal language of a Grothendieck topos. The end of chapter 3 left us with the conclusion that the left adjoint f^* of a geometric morphism preserves the universal validity of any formula. Now, if a theory $\mathcal{T}$ admits a classifying topos $\mathscr{E}[\mathcal{T}]$, then for any Grothendieck topos $\mathscr{F}$, there is an equivalence between the geometric morphisms from $\mathscr{F}$ to $\mathscr{E}[\mathcal{T}]$ and the models of the theory in $\mathscr{F}$. Now assume we are studying a theory in **Set** with our usual set-theoretic language. Any models of the theory in **Set** corresponds to a geometric morphism $\mathbf{Set} \to \mathscr{E}[\mathcal{T}]$. Hence if we prove a formula to be valid in $\mathscr{E}[\mathcal{T}]$, it will be also valid in **Set**. Therefore, the classifying topoi are an excellent place to attempt to prove unsolved assumptions in our theories in the set-theoretic language.

Appendix A

Yoneda embedding

Theorem A.0.1 (Yoneda)**.** *Let be a functor $F : \mathscr{C} \to \mathbf{Set}$ for a small category $\mathscr{C}$. For every object $C \in \mathscr{C}$, there exists a bijection*

$$\mathbf{Nat}(\mathbf{Hom}_{\mathscr{C}}(C, -), F) \cong F(C),$$

that is natural in C and F.

Proof. Let us define $\theta_{F,C} : \mathbf{Nat}(\mathbf{Hom}_{\mathscr{C}}(C, -), F) \to F(C)$, $\alpha \mapsto \alpha(C)(\mathrm{Id}_C)$. For an element $x \in F(C)$, and for every object $D \in \mathscr{C}$, define $\tau_{F,C}(x)(D) : \mathbf{Hom}(C, D) \to F(D)$, $f \mapsto F(f)(x)$. For any morphism $g : D \to D'$ in $\mathscr{C}$, we have that

$$F(g) \circ \tau_{F,C}(x)(D)(f) = F(g) \circ F(f)(x) = F(g \circ f)(x) = \tau_{F,C}(x)(D')(g \circ f).$$

Thus $\tau_{F,C}(x) : \mathbf{Hom}_{\mathscr{C}}(C, -) \Rightarrow F$ is a natural transformation.

We shall prove τ and θ are inverse one to each other. We will not write the subscript F, C on τ and θ to avoid things to look heavy. It will be used only in case where the notation is needed to avoid confusions. We compute given an element $x \in F(C)$:

$$\theta(\tau(x)) = \tau(x)(C)(\mathrm{Id}_C) = F(\mathrm{Id}_C)(x) = x.$$

Given a natural transformation $\alpha : \mathbf{Hom}(C, -) \Rightarrow F$ and any morphism $f : C \to D$ in $\mathscr{C}$:

$$\begin{aligned}
\tau(\theta(\alpha))(D)(f) &= \tau(\alpha(C)(\mathrm{Id}_C))(D)(f) \\
&= F(f)(\alpha(C)(\mathrm{Id}_C)) \\
&= \alpha(D)(f \circ \mathrm{Id}_C) && \text{by naturality of } \alpha \\
&= \alpha(D)(f).
\end{aligned}$$

We now prove the naturality of the bijection. First the naturality on the objects. Let be an arrow $g : C \to D$ in $\mathscr{C}$ and fix a functor $F : \mathscr{C} \to \mathbf{Set}$. We denote by $N(g)$

the morphism $\mathbf{Nat}(\mathbf{Hom}_{\mathscr{C}}(g,-),F)$. $N(g)(\alpha) = \alpha \circ \mathbf{Hom}_{\mathscr{C}}(g,-)$. We compute for any $\alpha \in \mathbf{Nat}(\mathbf{Hom}_{\mathscr{C}}(C,-),F)$:

$$\begin{aligned}
(\theta_{F,D} \circ N(g))(\alpha) &= \theta_{F,D}(\alpha \circ \mathbf{Hom}_{\mathscr{C}}(g,-)) \\
&= (\alpha \circ \mathbf{Hom}_{\mathscr{C}}(g,-))(D)(\mathrm{Id}_D) \\
&= \alpha(D)(g) \\
&= ()\alpha(D) \circ \mathbf{Hom}_{\mathscr{C}}(C,g))(\mathrm{Id}_C) \\
&= F(g)(\alpha(C)(\mathrm{Id}_C)) && \text{by naturality of } \alpha \\
&= (F(g) \circ \theta_{F,C})(\alpha).
\end{aligned}$$

For the naturality on the functors, let us a fix a natural transformation $\gamma : F \Rightarrow G$ and an object $C \in \mathscr{C}$. We denote by $M(\gamma)$ the morphism $\mathbf{Nat}(\mathbf{Hom}_{\mathscr{C}}(C,-),\gamma)$, i.e for every $\alpha \in \mathbf{Nat}(\mathbf{Hom}_{\mathscr{C}}(C,-),F)$, $M(\gamma)(\alpha) = \gamma \circ \alpha$. Then

$$\begin{aligned}
(\theta_{G,C} \circ M(\gamma))(\alpha) &= \theta_{G,C}(\gamma \circ \alpha) \\
&= (\gamma \circ \alpha)(C)(\mathrm{Id}_C) \\
&= \gamma(C)(\alpha(C)(\mathrm{Id}_C)) \\
&= (\gamma(C) \circ \theta_{F,C})(\alpha).
\end{aligned}$$

□

Definition A.0.2. The *Yoneda embedding* for a small category $\mathscr{C}$ is the functor

$$Y : \mathscr{C} \to \mathbf{Set}^{\mathscr{C}^{op}}.$$

it is defined on the objects $C \in \mathscr{C}$ by $Y(C) = \mathbf{Hom}_{\mathscr{C}}(-,C)$ and defined by $Y(f) = \mathbf{Hom}_{\mathscr{C}}(-,f)$ on the morphisms.

There is a second Yoneda embedding

$$Y^* : \mathscr{C}^{op} \to \mathbf{Set}^{\mathscr{C}},$$

defined as $Y^*(C) = \mathbf{Hom}_{\mathscr{C}}(C,-)$ on the objects and in the obvious similar way on the morphisms. The Yoneda embedding does not require special treatment since it is just the Yoneda embedding $Y : \mathscr{C}^{op} \to \mathbf{Set}^{(\mathscr{C}^{op})^{op}}$. Thus we will not make, unless if necessary, any distinction between the two embeddings.

Proposition A.0.3. *Here are some properties of the Yoneda embedding Y:*

1. *Y is fully faithful;*
2. *Y preserves limits.*

Proof. 1. Apply Yoneda Lemma.

2. Since limits of $\mathbf{Set}^{\mathscr{C}^{op}}$ are computed pointwise (see further result B.0.1), $(\lim_{D\in\mathscr{D}} Y(D))(C) = \lim_{D\in\mathscr{D}}(Y(D)(C)) = \lim_{D\in\mathscr{D}}(\mathbf{Hom}(C,D))$. For any $C \in \mathscr{C}$, the representable functor $\mathbf{Hom}(C,-) : \mathscr{C} \to \mathbf{Set}$ preserves all limits existing, hence Y preserves limits.

$\square$

Appendix B

Limits

Proposition B.0.1. *Limits (and colimits) are computed pointwise in the category of presheaves* $\mathbf{Pr}(\mathscr{C})$*, for any small category* $\mathscr{C}$*. This means, given a diagram* $F : \mathscr{D} \to \mathbf{Pr}(\mathscr{C})$ *and an object* $C \in \mathscr{C}$

$$(\lim_{D\in\mathscr{D}} F(D))(C) = \lim_{D\in\mathscr{D}}(F(D)(C)).$$

Proof. Consider the functor $F(-)(C) : \mathscr{D} \to \mathbf{Set}$. For each object $C \in \mathscr{C}$, let be $\Big(L(C), \{p_D^C : L(C) \to F(D)(C)\}_{D\in\mathscr{D}}\Big)$ the limit of $F(-)(C)$ in $\mathbf{Set}$, it exists. For each morphism $f : C' \to C$ in $\mathscr{C}$ we get a natural transformation $F(-)(f)$, thus $\Big(L(C), \{F(D)(f) \circ p_D^C\}_{D\in\mathscr{D}}\Big)$ is a cone on $F(-)(C')$. It induces a unique arrow $L(f) : L(C) \to L(C')$ such that $p_D^{C'} \circ L(f) = F(D)(f) \circ p_D^C$ for any $D \in \mathscr{D}$. Thus $L : \mathscr{C}^{op} \to \mathbf{Set}$ is a well defined functor and $p_D := \{p_D^C\}_{C\in\mathscr{C}}$ is a natural transformation for any $D \in \mathscr{D}$.

Notice that $(L, \{p_D : L \to F(D)\}_{D\in\mathscr{D}})$ is a cone on F since for any $d : D \to D'$ in $\mathscr{D}$, $F(d)(C) \circ p_D^C = p_{D'}^C$ for any $C \in \mathscr{C}$. Now assume we have another cone $(M, \{q_D\}_{D\in\mathscr{D}})$ on F. Thus for any $C \in \mathscr{C}$, $\Big(M(C), \{q_D^C : M(C) \to F(D)(C)\}_{D\in\mathscr{D}}\Big)$ is a cone on $F(-)(C)$. Hence there exists a unique morphism $r(C) : M(C) \to L(C)$ such that $p_D^C \circ r(C) = q_D^C$. We compute for any morphism $f : C \to C'$ in $\mathscr{C}$:

$$\begin{aligned}
p_D^{C'} \circ L(f) \circ r(C) &= F(D)(f) \circ p_D^C \circ r(C) \\
&= F(D)(f) \circ q_D^C \\
&= q_D^{C'} \circ M(f) && \text{since } q_D \text{ is a natural transformation} \\
&= p_D^{C'} \circ r(C') \circ M(f).
\end{aligned}$$

And since as arrows of a cone, the $p_D^{C'}$'s for a fixed $C' \in \mathscr{C}$ are jointly monic, we get $L(f) \circ r(C) = r(C') \circ M(f)$. Hence $r : M \to L$ is a natural transformation such that $p_D \circ r = q_D$ for any $D \in \mathscr{D}$. r is unique since its components $r(C)$ are. Hence $(L, \{p_D : L \to F(D)\}_{D\in\mathscr{D}})$ is the limit of F. Therefore

$$(\lim_{D\in\mathscr{D}} F(D))(C) = \lim_{D\in\mathscr{D}}(F(D)(C)).$$

The exact same proof can be done for the case of colimits. $\square$

Proposition B.0.2. *Representable functors G (i.e those who are isomorphic to $\mathbf{Hom}(-,C)$ for an object in $\mathscr{C}$) in the category of presheaves $\mathbf{Pr}(\mathscr{C})$ for a given small category $\mathscr{C}$ transform existing colimits in $\mathscr{C}$ into limits in $\mathbf{Set}$. This means for a diagram $F : \mathscr{D} \to \mathscr{C}$ with $\mathscr{D}$ a small category:*

$$G(\mathrm{colim}_{D\in\mathscr{D}} F(D)) = \lim_{D\in\mathscr{D}} GF(D),\ G(\lim_{D\in\mathscr{D}} F(D)) = \mathrm{colim}_{D\in\mathscr{D}} GF(D).$$

Proof. Let us fix an object $C \in \mathscr{C}$, the contravariant functor $\mathbf{Hom}_{\mathscr{C}}(-,C)$ and a diagram $F : \mathscr{D} \to \mathscr{C}$ with $\mathscr{D}$ a small category. Let be $(L, \{s_D : F(D) \to L\}_{D\in\mathscr{D}})$ the colimit of F in $\mathscr{C}$. Applying the functor $\mathbf{Hom}_{\mathscr{C}}(-,C)$ on the colimit cone yields a cone $(\mathbf{Hom}_{\mathscr{C}}(L,C), \{\mathbf{Hom}_{\mathscr{C}}(s_D,C) : \mathbf{Hom}(L,C) \to \mathbf{Hom}_{\mathscr{C}}(F(D),C)\}_{D\in\mathscr{D}})$ in $\mathbf{Set}$. Now assume we have another cone $(M, \{p_D : M \to \mathbf{Hom}_{\mathscr{C}}(F(D),C)\}_{D\in\mathscr{D}})$. For any $m \in M$ and any $D \in \mathscr{D}$ we get an arrow $p_D(m) : F(D) \to C$, such that for any morphism $d : D \to D'$ in $\mathscr{D}$, $p_{D'}(m) \circ F(d) = p_D(m)$. Thus $(C, \{p_D(m)\}_{D\in\mathscr{D}})$ is a cocone in $\mathscr{C}$. Hence there exists a unique morphism $p(m) : L \to C$. Therefore we get a map $p : M \to \mathbf{Hom}_{\mathscr{C}}(L,C)$ uniquely determined by the $p(m)$'s. $(\mathbf{Hom}_{\mathscr{C}}(L,C), \{\mathbf{Hom}_{\mathscr{C}}(s_D,C) : \mathbf{Hom}_{\mathscr{C}}(L,C) \to \mathbf{Hom}_{\mathscr{C}}(F(D),C)$ is the limit of $\mathbf{Hom}_{\mathscr{C}}(F-,C)$. $\square$

It is important to note that in the equality $G(\mathrm{colim}_{D\in\mathscr{D}} F(D)) = \lim_{D\in\mathscr{D}} GF(D)$, the existence of the limit does not implies the existence of the colimit. This equality must be understood as "If the colimit exists in $\mathscr{C}$, then this equality holds".

Proposition B.0.3. *Let $\mathscr{C}$ be a small category and $F \in \mathbf{Fun}(\mathscr{C}, \mathbf{Set})$. F is the colimit of a diagram constituted of representable functors and representable natural transformations.*

Proof. Let $\phi_F : \mathbf{Elts}(F) \to \mathscr{C}$ be the forgetful functor (see C.0.3) and consider the contravariant diagram $Y \circ \phi_F : \mathbf{Elts}(F) \to \mathbf{Fun}(\mathscr{C}, \mathbf{Set})$ where Y is the contravariant Yoneda embedding $Y : \mathscr{C} \to \mathbf{Fun}(\mathscr{C}, \mathbf{Set})$ with, $\forall C \in \mathscr{C}$, $Y(C) = \mathbf{Hom}_{\mathscr{C}}(C,-)$. Let be $(A,a) \in \mathbf{Elts}(F)$ a pair $A \in \mathscr{C}$ and $a \in F(A)$. By the Yoneda lemma a correspond to the natural transformation $s_{(A,a)} : \mathbf{Hom}_{\mathscr{C}}(A,-) \Rightarrow F$, with for any object $B \in \mathscr{C}$ and arrow $f : A \to B$, $s_{(A,a)}(B)(f) = F(f)(a)$. If $f : (A,a) \to (B,b)$ is a morphism in $\mathbf{Elts}(F)$, then $F(f)(a) = b$, thus since the bijection in the Yoneda lemma is natural

$$\begin{tikzcd}
\mathbf{Nat}(\mathbf{Hom}_{\mathscr{C}}(A,-),F) \arrow[r, "\cong"] \arrow[d, "\mathbf{Hom}_{\mathscr{C}}(f,-)"] & F(A) \arrow[d, "F(f)"] \\
\mathbf{Nat}(\mathbf{Hom}_{\mathscr{C}}(B,-),F) \arrow[r, "\cong"] & F(B)
\end{tikzcd}$$

f is a morphism in $\mathbf{Elts}(F)$ if and only if $s_{(A,a)} \circ \mathbf{Hom}_{\mathscr{C}}(f,-) = s_{(B,b)}$. Therefore

$$\left(F, \left\{s_{(A,a)}\right\}_{(A,a)\in\mathbf{Elts}(F)}\right)$$

is a cocone on $Y \circ \phi_F$.

Now let $\left(G, \left\{t_{(A,a)}\right\}_{(A,a)\in\mathbf{Elts}(F)}\right)$ be another cocone on $Y \circ \phi_F$. For any object $C \in \mathscr{C}$ and any element $c \in F(C)$, we consider the natural transformation $t_{(C,c)} : \mathbf{Hom}_{\mathscr{C}}(C,-) \Rightarrow G$. By the Yoneda lemma, this natural transformation correspond to a unique element $t_{(C,c)}(\mathrm{Id}_C) \in G(C)$ and let us call this element $\alpha_C(c) := t_{(C,c)}(\mathrm{Id}_C)$. We now prove this defines a natural transformation $\alpha : F \Rightarrow G$. Let be a morphism $g : C \to D$ with $D \in \mathscr{C}$. By definition this yields a morphism $g : (C,c) \to (D, Fg(c))$ in $\mathbf{Elts}(F)$. By definition of the arrows $t_{(A,a)}$ we get that $t_{(C,c)} \circ \mathbf{Hom}_{\mathscr{C}}(g,-) = t_{(D,Fg(c))}$, thus $G(g)\,(\alpha_C(c)) = \alpha_D(Fg(c))$. This proves the naturality of α.

Given $(C,c) \in \mathbf{Elts}(F)$,

$$\begin{aligned}
\alpha \circ s_{(C,c)} &\cong (\alpha \circ s_{(C,c)})_C(\mathrm{Id}_C) && \text{by the Yoneda lemma}\\
&\cong \alpha_C(s_{(C,c)}(C)(\mathrm{Id}_C))\\
&\cong \alpha_C(c) && \text{by definition of } s_{(C,c)}\\
&\cong t_{(C,c)} && \text{by the Yoneda lemma.}
\end{aligned}$$

Moreover, if $\beta : F \Rightarrow G$ is another natural transformation such that for any $(C,c) \in \mathbf{Elts}(F)$ $\beta \circ s_{(C,c)} = t_{(C,c)}$, then by the Yoneda lemma $\alpha_C(c) = \beta_C(c)$. Thus α is the unique morphism such that $\alpha \circ s_{(C,c)} = t_{(C,c)}$. The cocone

$$\left(F, \left\{s_{(A,a)}\right\}_{(A,a)\in\mathbf{Elts}(F)}\right)$$

is the colimit of $Y \circ \phi_F$. $\square$

Remark that in the case of a contravariant functor $G : \mathscr{C} \to \mathbf{Set}$ the theorem still holds with $G = \mathrm{colim}_{(A,a)\in\mathbf{Elts}(G)}(Y \circ \phi_G)$ where Y is the other Yoneda embedding $Y(A) = \mathbf{Hom}_{\mathscr{C}}(-,A)$.

Proposition B.0.4. *Let $D : \mathscr{A} \to \mathscr{B}$ be a diagram. If 1 is a terminal object in $\mathscr{A}$ with the unique morphisms $1_i : i \to 1$, $\forall i \in \mathscr{A}$, then the colimit of D exists and is precisely $(D(1), (D(1_i))_{i\in\mathscr{A}})$.*

Proof. Since 1 is terminal, we have that for all $u : i \to j$ morphism in $\mathscr{A}$, $1_j \circ u = 1_i$. Hence, since D is a functor, $D(1_i) = D(1_j) \circ D(u)$. $(D(1), (D(1_i))_{i\in\mathscr{A}})$ is a cocone.

Assume $(L, (\rho_i : D_i \to L)_{i\in\mathscr{A}})$ is another cocone. Notice $\rho_1 : D(1) \to L$ is a morphism satisfying $\rho_1 \circ D(1_i) = \rho_i$, since $(L, (\rho_i : D_i \to L)_{i\in\mathscr{A}})$ is a cocone.

Now let $\alpha : D(1) \to L$ be another arrow such that $\alpha \circ D(1_i) = \rho_i$. Since the unique morphism $1 \to 1$ is the identity, we get in particular that $\alpha \circ D(\mathrm{Id}) = \rho_1$. Hence $\alpha = \rho_1$. ρ_1 is unique. $\square$

One can note, based on a similar proof, than for $D : \mathscr{A} \to \mathscr{B}$, if $\mathscr{A}$ has an initial object 0, then the limit of D exists and is $(D(0), (D(0_i))_{i \in \mathscr{A}})$, with $0_i : D(0) \to i$ for all object $i \in \mathscr{A}$.

Eventually, we will admit this last important result about limits (and its equivalent about colimits) withtout a proof. One can be found in [2, 2.12.1].

Proposition B.0.5. *Given a functor $F : \mathscr{C} \times \mathscr{D} \to \mathscr{A}$ with $\mathscr{A}$ complete and $\mathscr{C}, \mathscr{D}$ small. Then*

$$\lim_{C \in \mathscr{C}}(\lim_{D \in \mathscr{D}} F(C, D)) \cong \lim_{D \in \mathscr{D}}(\lim_{C \in \mathscr{C}} F(C, D)).$$

Appendix C

Kan extensions

Definition C.0.1. Let be two functors $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{A} \to \mathscr{C}$. The *left Kan extension* of G along F is a pair $(\mathbf{Lan}_F G, \alpha)$ where

1. $\mathbf{Lan}_F G : \mathscr{B} \to \mathscr{C}$ is a functor,
2. $\alpha : G \Rightarrow (\mathbf{Lan}_F G) \circ F$ is a natural transformation,

such that it satisfies the following universal property: for any other pair (H, β) there exists a unique natural transformation $\gamma : \mathbf{Lan}_F G \Rightarrow H$ such that $(\gamma \bullet 1_F) \circ \alpha = \beta$. Here $\bullet$ is the horizontal composition of natural transformations and $\circ$ the vertical one.

$$\begin{tikzcd}
\mathscr{A} \arrow[r, shift left, "F"] \arrow[r, shift right, "F"'] \arrow[r, phantom, "\Downarrow 1_F"] \arrow[rr, bend right, "G"'] & \mathscr{B} \arrow[r, shift left, "\mathrm{Lan}_F G"] \arrow[r, shift right, "H"'] \arrow[r, phantom, "\Downarrow \gamma"] & \mathscr{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
H \circ F & (\mathrm{Lan}_F G) \circ F \arrow[l, Rightarrow, "\gamma \bullet 1_F"'] \\
 & G \arrow[u, Rightarrow, "\alpha"'] \arrow[ul, Rightarrow, "\beta"]
\end{tikzcd}$$

The universal property of the Kan extension allows us to talk about "the" Kan extension of G along F.

Example C.0.2. The adjunction of two functors is a specific case of Kan extension.

If $\mathscr{C}$ as in the definition is cocomplete, then we can construct a Kan extension. The following definition is an ingredient for the proof of that affirmation.

Definition C.0.3. Let be a functor $F : \mathscr{A} \to \mathbf{Set}$ (resp. $F : \mathscr{A}^{op} \to \mathbf{Set}$). The category $\mathbf{Elts}(F)$ of elements of F is :

1. the objects are the pairs (A, a) with $A \in \mathrm{Obj}(\mathscr{A})$ and $a \in F(A)$.
2. the morphisms $f : (A, a) \to (A', a')$ are the arrows $f : A \to A'$ in $\mathscr{A}$ such that $Ff(a) = a'$ (resp. $Ff(a') = a$).

With the category of elements of F comes for free a forgetful functor $\phi_F : \mathbf{Elts}(F) \to \mathscr{A}$ sending $\phi_F(A,a) = A$ and $\phi_F(f) = f$ now seen as a morphism in $\mathscr{A}$.

Example C.0.4. Consider the functor $\mathbf{Hom}(-,B) : \mathscr{A}^{op} \to \mathbf{Set}$ for a fixed $B \in \mathscr{A}$. Then the objects of $\mathbf{Elts}(\mathbf{Hom}(-,B))$ are the pair (A,a) with $A \in \mathscr{A}$ and $a : A \to B$. let $f : A \to A'$ an arrow in $\mathscr{A}$, then $\mathbf{Hom}(f,B)(a') = a' \circ f$ for any $a' : A' \to B$. Hence the morphisms of $\mathbf{Elts}(\mathbf{Hom}(-,B)))$ are of the form $f : (A, a' \circ f) \to (A', a')$.

Theorem C.0.5. *Let be two functors $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{A} \to \mathscr{C}$, with $\mathscr{A}$ a small category and $\mathscr{C}$ a cocomplete category. Then the left Kan extension of G along F exists.*

Proof. Fix $B \in \mathscr{B}$, and define $\mathscr{E}_B := \mathbf{Elts}(\mathbf{Hom}(F-,B))$ and $\phi_B : \mathscr{E}_B \to \mathscr{A}$ the forgetful functor. By construction, since $\mathscr{A}$ is small, $\mathscr{E}_B$ is small too. Hence since $\mathscr{C}$ is cocomplete, the small diagram $G \circ \phi_B : \mathscr{E}_B \to \mathscr{C}$ has a colimit $(KB, (s^B_{(A,b)})_{(A,b)\in\mathscr{E}_B})$. We define a functor on the objects $K : \mathscr{B} \to \mathscr{C},\ B \mapsto K(B) = \operatorname{colim}(G \circ \phi_B)$.

Now we need to define K on the morphisms, that is to associate to each morphism $f : B \to B'$ in $\mathscr{B}$ an arrow $Kf : KB \to KB'$. Remark that for any (A,b) object in $\mathscr{E}_B$, then $(A, f \circ b)$ is an object in $\mathscr{E}_{B'}$ since if $b \in \mathbf{Hom}(FA,B)$ then $f \circ b \in \mathbf{Hom}(FA,B')$. Assume $a : (A,b) \to (A',b')$ is a morphism in $\mathscr{E}_B$ with $b = \mathbf{Hom}(Fa,B)(b') = b' \circ Fa$. Then it is immediate that $f \circ b = \mathbf{Hom}(Fa,B')(f \circ b')$. Hence, for a an arrow in $\mathscr{E}_B$, it gives an arrow $a : (A, f \circ b) \to (A', f \circ b')$ in $\mathscr{E}_{B'}$ based on the same $a : A \to A'$ in $\mathscr{A}$. Notice that $(G \circ \phi_B)(a) = (G \circ \phi_{B'})(a) = G(a)$. Hence $(KB', (s^{B'}_{(A,f\circ b)})_{(A,b)\in\mathscr{E}_B})$ is a cocone on $G \circ \phi_B$. By definition of a colimit, there exist a unique arrow $Kf : KB \to KB'$ such that $Kf \circ s^B_{(A,b)} = s^{B'}_{(A,f\circ b)}$. The uniqueness implies $K(1_B) = 1_{KB}$ and $K(f \circ g) = Kf \circ Kg$.

$$\begin{tikzcd}
 & & GA \arrow[dll, "s^B_{(A,b)}"'] \arrow[dl, "s^{B'}_{(A,f\circ b)}"] \arrow[dd, "Ga"] \\
KB \arrow[r, dashed, "Kf"] & KB' & \\
 & & GA' \arrow[ull, "s^B_{(A',b')}"] \arrow[ul, "s^{B'}_{(A',f\circ b')}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
GA \arrow[r, "s^{FA}_{(A,1_{FA})}"] \arrow[d, "Ga"'] \arrow[dr, "s^{FA'}_{(A,Fa)}"] & KFA \arrow[d, "KFa"] \\
GA' \arrow[r, "s^{FA'}_{(A',1_{FA'})}"'] & KFA'
\end{tikzcd}$$

Given a morphism $a : A \to A'$ in $\mathscr{A}$ we have established that $KFa \circ s^{FA}_{(A,1_{FA})} = s^{FA'}_{(A,Fa)}$. One can notice $a : (A, Fa) \to (A', 1_{FA'})$ is a morphism in $\mathscr{E}_{FA'}$, since $\mathbf{Hom}(Fa, FA')(1_{FA'}) = 1_{FA'} \circ Fa = Fa$. Then since by definition of cocone, $s^{FA'}_{(A',1_{FA'})} \circ Ga = s^{F'A}_{(A,Fa)}$. We have that $KFa \circ s^{FA}_{(A,1_{FA})} = s^{FA'}_{(A',1_{FA'})} \circ Ga$, this proves the natural transformation $\alpha : G \Rightarrow KF$ defined as $(\alpha_A = s^{FA}_{(A,1_{FA})})_{A\in\mathscr{A}}$ is natural in A.

Now consider another pair (H,β), $H : \mathscr{B} \to \mathscr{C}$ and $\beta : G \Rightarrow HF$. Let's create a unique natural transformation $\gamma : K \Rightarrow H$ such that $(\gamma \bullet 1_F) \circ \alpha = \beta$. Fix $B \in \mathscr{B}$, for any $(A, b) \in \mathscr{E}_B$, i.e $A \in \mathscr{B}$ and $b : FA \to B$, we have $Hb : HFA \to HB$ and $\beta_A : GA \to HFA$ such that, by naturality of β and since H is a functor, the following left diagram commutes

$$
\begin{tikzcd}
GA \arrow[r, "\beta_A"] \arrow[dd, "Ga"'] & HFA \arrow[dd, "HFa"] \arrow[dr, "Hb"] & \\
& & HB \\
GA' \arrow[r, "\beta_{A'}"'] & HFA' \arrow[ur, "Hb'"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
GA \arrow[rr, "Ga"] \arrow[dr, "s^B_{(A,b)}"] \arrow[ddr, bend right, "H(f\circ b)\circ\beta_A"'] & & GA' \arrow[dl, "s^B_{(A',b')}"'] \arrow[ddl, bend left, "H(f\circ b')\circ\beta_{A'}"] \\
& KB \arrow[d, shift right, "Hf\circ\gamma_B"'] \arrow[d, shift left, "\gamma_{B'}\circ Kf"] & \\
& HB' &
\end{tikzcd}
$$

Notice $(G \circ \phi_B)(A, b) = GA$ for all $(A, b) \in \mathscr{E}_B$ and $(G \circ \phi_B)(a) = Ga$ for all $a \in \mathscr{E}_B$. Hence $(HB, Hb \circ \beta_A)_{(A,b) \in \mathscr{E}_B}$ is a cocone for $G \circ \phi_B$. Since KB is a colimit, there exists a unique morphism $\gamma_B : KB \to HB$ such that $\gamma_B \circ s^B_{(A,b)} = Hb \circ \beta_A$. To prove the naturality of γ in B we need, for any $f : B \to B'$, the following computation

$$
\begin{aligned}
Hf \circ \gamma_B \circ s^B_{(A,b)} &= HF \circ Hb \circ \beta_A = H(f \circ b) \circ \beta_A \\
&= \gamma_{B'} \circ s^{B'}_{(A,f\circ b)} = \gamma_{B'} \circ Kf \circ s^B_{A,b}.
\end{aligned}
$$

By definition of a colimit (see up right diagram) we obtain the natural equality $Hf \circ \gamma_B = \gamma_{B'} \circ Kf$. γ is a natural transformation.

Last thing to check is that γ is such that $(\gamma \bullet 1_F) \circ \alpha = \beta$. We verify this equality pointwise, using the definition of γ_B :

$$
\gamma_{FA} \circ \alpha_A = \gamma_{FA} \circ s^{FA}_{(A,1_{FA})} = H(1_{FA}) \circ \beta_A = \beta_A.
$$

γ is unique since any other natural transformation γ' should also verify the equality $\gamma'_{FA} \circ s^{FA}_{(A,1_{FA})} = H(1_{FA}) \circ \beta_A$ and as shown earlier the γ_B's are the unique one verifying it. Hence the pair $(Lan_F G = K, \alpha)$ is a Kan extension. $\square$

Proposition C.0.6. *Let $F : \mathscr{A} \to \mathscr{B}$ be a fully faithful functor on a small category $\mathscr{A}$. For any functor $G : \mathscr{A} \to \mathscr{C}$ with $\mathscr{C}$ a cocomplete category, the natural transformation $\alpha : G \Rightarrow (\mathbf{Lan}_F G) \circ F$ is an isomorphism.*

Proof. Let us prove $(A, 1_{FA})$ is a terminal object in $\mathscr{E}_{FA}$. Let $(A', b) \in \mathscr{E}_{FA}$ be an object in $\mathscr{E}_{FA}$ with $b : FA' \to FA$. Since F is full, there exists a morphism $a : A' \to A$ in $\mathscr{A}$ such that $Fa = b$. Hence

$$
Fa = b \iff 1_{FA} \circ Fa = b \iff \mathbf{Hom}(Fa, FA)(1_{FA}) = b.
$$

Then $a : (A', b) \to (A, 1_{FA})$ is a morphism. Now assume we have another such morphism $a' : (A', b) \to (A, 1_{FA})$. Then by the same computation we get that $Fa' = b$. Hence $Fa = Fa'$, but since F is faithful we have that $a = a'$. $(A, 1_{FA})$ is a terminal object in $\mathscr{E}_{FA}$.

Then by B.0.4

$$(\mathbf{Lan}_F G) \circ F(A) = \operatorname{colim}(G \circ \phi_{FA}) \cong G \circ \phi_{FA}(A, 1_{FA}) = GA.$$

Since each component $\alpha_A : GA \to (\mathbf{Lan}_F G) \circ F(A)$ is an iso, so is α. □

The next proposition will play a key role in some of our proofs. We admit it without proof but one can be found in [2, Proposition 3.7.4]

Proposition C.0.7. *If L is left adjoint, then $L \circ \mathbf{Lan}_F G = \mathbf{Lan}_F LG$.*

Proposition C.0.8. *Let $F : \mathscr{C} \to \mathbf{Set}$ be a functor and $\mathscr{C}$ a small finitely complete category. Then the following conditions are equivalent:*

1. *F is left exact;*
2. *the left Kan extension $\mathbf{Lan}_Y F$ of F along the Yoneda embedding Y is left exact.*

Proof. By A.0.1, Y preserves limits and by C.0.6, $(\mathbf{Lan}_Y F) \circ Y \cong F$. Hence, if $\mathbf{Lan}_Y F$ is left exact, so is $(\mathbf{Lan}_Y F) \circ Y$, implying F is left exact.

Conversely, the proof is left to the reader. (There is a lot of technical details that need to be explained and they don't really serve any purpose for the rest of this work) □

Proposition C.0.9. *Let $\mathscr{C}$ be a small category and let be $F : \mathscr{C} \to \mathbf{Set}$ a covariant functor and $G : \mathscr{C} \to \mathbf{Set}$ a contravariant functor. Let be the Yoneda embeddings $Y : \mathscr{C} \to \mathbf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathbf{Set})$ and $Y^* : \mathscr{C}^{\mathrm{op}} \to \mathbf{Fun}(\mathscr{C}, \mathbf{Set})$ defined $\forall C \in \mathscr{C}$ as $Y(C) = \mathbf{Hom}_{\mathscr{C}}(-, C)$ and $Y^*(C) = \mathbf{Hom}_{\mathscr{C}}(C, -)$. Then*

$$(\mathbf{Lan}_{Y^*} G)\,(F) = (\mathbf{Lan}_Y F)\,(G).$$

Proof. By B.0.3 we have that

$$F = \operatorname{colim}_{(A,a) \in \mathbf{Elts}(F)}(\mathbf{Hom}_{\mathscr{C}}(A, -)),$$
$$G = \operatorname{colim}_{(B,b) \in \mathbf{Elts}(G)}(\mathbf{Hom}_{\mathscr{C}}(-, B)).$$

By B.0.5, B.0.1 and C.0.5, we compute

$$\begin{aligned}
(\mathbf{Lan}_{Y^*}G)(F) &= \operatorname{colim}_{(A,a)} G(A) \\
&= \operatorname{colim}_{(A,a)} \Big(\operatorname{colim}_{(B,b)} \mathbf{Hom}_{\mathscr{C}}(-,B)(A)\Big) \\
&= \operatorname{colim}_{(A,a)} \operatorname{colim}_{(B,b)} \mathbf{Hom}_{\mathscr{C}}(A,B) \\
&= \operatorname{colim}_{(B,b)} \operatorname{colim}_{(A,a)} \mathbf{Hom}_{\mathscr{C}}(A,B) \\
&= \operatorname{colim}_{(B,b)} \Big(\operatorname{colim}_{(A,a)} \mathbf{Hom}_{\mathscr{C}}(A,-)(B)\Big) \\
&= \operatorname{colim}_{(B,b)} F(B) \\
&= (\mathbf{Lan}_{Y}F)(G).
\end{aligned}$$

□

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