Annex n°1 : Interest rates theory

In order to understand the duration's modelling of non-maturing liabilities or different interest models, the interest rates' theory need to be briefly introduced. This section will begin by explaining the yield curve with spot and forward rates. Then, we will see the determinant of the interest rates level to understand how we reached a low interest rate environment. Finally, we will end this section by explaining the fundamentals of duration, convexity and immunization.

1. Interest rates and yield curves

1.1. The yield curve

According to the ECB, a yield curve represents "the relationship between market remuneration (interest rates) and the remaining time to maturity of debt securities" [1]. The yield curve is also known as the *term structure of interest rates* which is a representation of interest rates as a function of time to maturity.
Yield to maturity (YTM) is the return an investor get by holding his bond until it matures. For instance, the YTM of a fixed coupon bond is computed in Equation (69).

$$PV = \sum_{t=1}^{M} \frac{C_t}{(1+y)^t} \quad \textbf{(69)}$$

with $PV = \text{present, and } C_t = \text{bond cash flow, } \text{y} = \text{annualized yield to maturity and M} = \text{maturity}$.

The YTM is found by solving y in the equation. For instance, a 10y-bond having a 9% annual coupon which is trading $938.5 has a YTM of 10%.[2]

The Yield curve risk is the risk that the general level of interest rates may change, and that this consequently results in some adverse financial impact on the bank. This is the most common type of IRR affecting banks. A bank can only be subject to yield curve risk if there is a mismatch existing between assets and liabilities. Moreover, the yield curve risk is most represented by considering a parallel change in interest rates. The *parallel yield curve risk* is the parallel change that the interest rates for all fixed maturities move by the same amount. On the contrary, a *non-parallel yield curve risk* is the risk of some change to the shape of the yield curve, *i.e.* the yield curve become steeper if longer maturities go up more than the shorter one and flatter if the opposite happen.

In order to understand the different interest rate models, we have to differ the spot and the forward rates. Let us have a tenor structure with possible maturities $T_1, T_2, \ldots, T_N$ and a tenor is defined as $\tau_i = T_{i+1} - T_i$. Then, the *discount rates*, meaning the price at time *t* of a zero-coupon bond paying 1 unit of currency at maturity time T_i, is denoted by $P(t, T_i)$. The *spot interest rate* at time *t* for maturity T_i is denoted by $R(t, T_i)$. The relationship between the discount and the spot rates is defined in Equation (70).

$$P(t, T_i) = \frac{1}{\left(1+\frac{R(t,T_i)}{m}\right)^{T,m}} \quad \textbf{(70)}$$

where *m* is the compounding frequency[3] of the spot rate

[1] http://faculty.baruch.cuny.edu/ryao/fin3710/pimco_yield_curve_primer.pdf
[2] https://www.fidelity.com/learning-center/investment-products/fixed-income-bonds/yield-maturity-bond
[3] The compounding frequency is for example *annually* or *semi-annually*. A semi-annually compounding frequency is 2 while a *monthly* compounding frequency is 12.

There subsists one important interest rate to complement these concepts which is the *forward rate*. This is the interest rate which will be contract for a time *t* for a loan between future time T_i and T_{i+1}. Given a set of zero-coupon bonds, we get Equation (71).

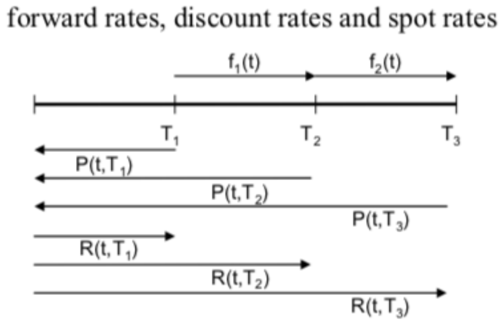
$$f(t, T_i, \tau_i) = \frac{\frac{P(t,T_i)}{P(t,T_i+\tau_i)} - 1}{\tau_i} \quad \textbf{(71)}$$

T_i and T_{i+1} are called the reset and the maturity of the *I*'th forward rate. (71) can be illustrated as it is the case in (72)

$$f(T_i, T_i, \tau_i) = R(T_i, T_i + \tau_i) \quad \textbf{(72)}$$

The relationship above relates that the forward rate for a given period can change as time passes. But it is fixed at the reset time, *i.e.* resets, and equal to the spot rate for the same time period. The interest is due for payment at the end of the period which is the maturity time of the forward rate. The timing convention and the relations between the forward rates, discount rates and spot rates are showed in the Figure 15.

Figure 15. – Discount, spot and forward rates.



1.2. Determinants of interest rates

First, the interest rate is fundamentally a price at which the lender charges a borrower for the temporary use of funds that he lent to him. The interest rate will not only reflect the credit risk of the borrower and the time value of money, that also represents any operational cost that the lender will support by making this loan. If the credit risks are subtracted, then it will be called "risk-free rate", which mean the theoretical rate that the lender would charge the borrower if he had no credit risk such that the repayment was assured. Theoretically, the level of risk-free rates is determined by demand and supply with the perfect market's assumption.

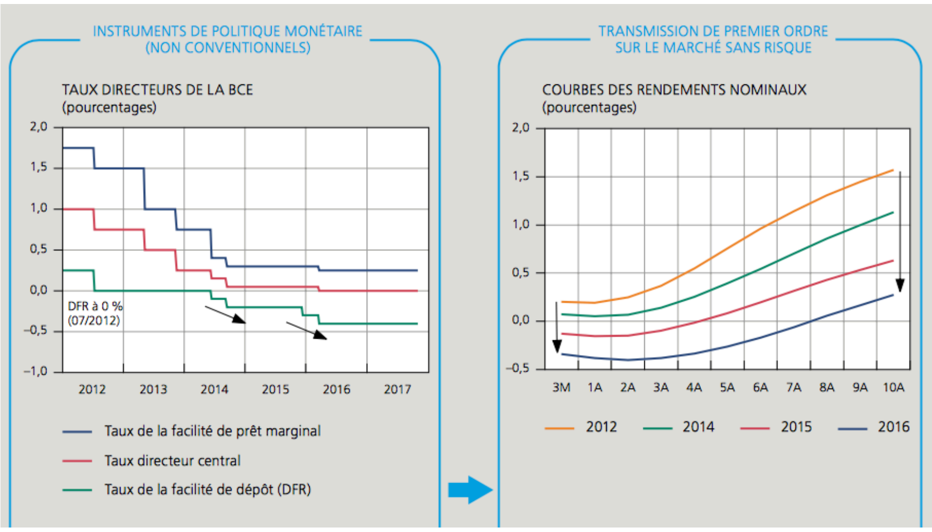
In this thesis, we speak about "low interest rates environment" but we have to know why we reached this level. When bunch of banks such Lehmann Brothers bankrupted[4], there was a hit on the countries such GDP growth decline and a surge in risk-aversion. A distrust against international investors resulted in overnight lending. Moreover, because huge default happened, the banks restrict their credit access to protect their balance sheets against the collapse. The banking balance sheet was affected mainly by the stock prices shrink, sharp declines in house prices, which cause an even worse balance sheet for firms and households which reinforce the dynamics. As consequence, in the Eurozone the interest rates were decreased by the ECB in order to stimulate activity.
Then came a second main negative shock in the Eurozone : the sovereign debt crisis. The sovereign borrowing costs spiked because of a surge in credit spread for bond yields, the banks exposures to the selected government was under scrutiny. In this case, with already

[4] https://www.ecb.europa.eu/pub/pdf/scpops/ecbocp130.pdf

inefficient low interest rates effect on the economy, the ECB put some measures in place to directly influence the effect of interest rates. The first measure is the *negative interest rates* on deposit facility. In a context of excess reserves and a long low inflation period, this policy has brought overnight rates even more down to provide positive stimuli for the economic activity. This effect is that the short term and medium term was flattened to give easier credit conditions.

The second measure is the *asset purchase programme* which has helped on depress the term structure of the yield curve by compressing the risk premium along it. That lead to lower interest rates for borrower and higher assets prices for banks which stimulate the economy on the both sides. In the Figure 16, we can observe the effect of the negative interest rate policy on the yield curve we spoke in the previous section. On the left side, there are three different rates: the lending facility rate[5] is in blue and represents the ceiling rate; the deposit facility rate[6] in green which represents the floor rate and finally the key interest rate[7] in red with which the European Central Bank influences the economy. In this sense, we call "negative interest rate policy" if the deposit facility rate is below the zero-bound. Then on the right side, we have the risk-free yield curve between 2012 (orange) and 2016 (blue). It appeared that the measures not only lowered the curve, but also flattened it. The flattening is due to the sensibility of the long-term rates to the assets purchases program.

Figure 16. – Effect of unconventional monetary policy measures on yield curve[8].



Then, in a depositor's point of view, we can observe that the deposits rate goes not under the zero-bound. The rigidity of the retail deposit rates is mainly due to the existance of fiat money – such coins and banknotes – which has a rate of return equal to 0%. If the deposit rate goes below zero, the households will withdraw their money from deposits to get it in fiat money such that they look after the highest return. The consequence for the banks is dramatic: when there is an erosion of the deposit volumes, this is problematic since the deposits are the main stable funding source of the banks. Nevertheless, the assumption of money withdrawing is not realistic. Indeed, the costs related to the possession of only physical cash at home is high for obvious reasons. According to a Royal Decree, the minimum legal rate that a bank can apply is 0.11% with 0.01% of base rate and 0.10% of fidelity.

[5] Permits to the banks to borrow liquidities to the Central Bank
[6] Permits to the banks to put their liquidity surplus to the Central Bank
[7] This is the rate for the main refinancing operations
[8] National Bank of Belgium, December 2017

2. Duration and convexity

We saw in the previous section dedicated to IRRBB the different risks that the ALM has to face. The duration the convexity are the main risk indicators to help taking decisions. This section will be presented as the way Bergendhal and Janssen did (1999).

First, we consider $CF = \{CF_j, t_j\}$ the financial future cash flow of an asset or liability where CF_j, t_j are the amount and the time of the *j*th movement of this flow, on a time scale [0, ∞) with non-negative amounts. With a fixed annual rate *i,* the present value of the flow *PV* is given by Equation (73).

$$PV(i) = \sum_{j=1}^{n} CF_j \,(1+i)^{-t_j} \quad \textbf{(73)}$$

Now let us take the same example as they took. Let's take the case of a bank which has customer loans for a total of € 100,000,000 generating an annual interest income of 5% for 3 years. The administrative costs amount 1% per year and, due to credit losses, the loans will be repaid at 98%. These loans are refunded on the interbank at rate of 3.5%. In this sense, the results are presented in Table 7.

Table 7. – Example for customer loans (in €million)

Year	Interest income	Administrative costs	Repayment
1	0.05 x 100	-0.01 x 100	4
2	0.05 x 100	-0.01 x 100	4
3	0.05 x 100	-0.01 x 100	98+4=102

Then the market value of the flow at time 0 is computed as in Equation (74).

$$PV = \frac{4}{1.035} + \frac{4}{(1.035)^2} + \frac{102}{(1.035)^3} = €99{,}596{,}533 \quad \textbf{(74)}$$

which gives us a loss of € 403,467, that is 4.04%.

Now let's consider that the interbank rate decreases by 1 basis point (BPS), then the new market value at time 0 is represented in Equation (75)

$$PV = \frac{4}{1.025} + \frac{4}{(1.025)^2} + \frac{102}{(1.025)^3} = €102{,}426{,}836 \quad \textbf{(75)}$$

which gives us a profit of €2,426,836, that is 2.42%.

What I want to illustrate is the consequence of the profit caused by the movements in interest rates. This influence is called the *duration*. To define it, let us assume that the interest rate has a variation (positive or negative) Δi; using the Taylor formula to the value $PV(i)$ defined by the relation (73) we obtain Equation (76)

$$\Delta PV = PV(i + \Delta i) - PV(i) \cong PV'(i)\Delta i + \frac{1}{2}PV''(i)(\Delta i)^2 \quad \textbf{(76)}$$

and for the relative increment:

$$\frac{PV(i+\Delta i)-PV(i)}{PV(i)} \cong -D_m(i)\Delta i + \frac{1}{2}CV(i)(\Delta i)^2 \quad \textbf{(77)}$$

with $-D_m = \frac{PV\prime(i)}{PV(i)}$, $(1+i)D_m(i) = D(i)$, $CV(i) = \frac{PV\prime\prime(i)}{PV(i)}$ where $D(i), D_m(i),$ and $CV(i)$ are respectively *duration (Macaulay), modified duration and convexity*.

From the definition of Equation (77), we can define each indicator as follow:

$$D(i) = \frac{1}{PV(i)}\sum_{j=1}^{n} t_j\, CF_j(1+i)^{-t_j} \quad \textbf{(78)}$$

$$D_m(i) = \frac{1}{(1+i)PV(i)}\sum_{j=1}^{n} t_j\, CF_j(1+i)^{-t_j} \quad \textbf{(79)}$$

$$CV(i) = \frac{1}{PV(i)}\sum_{j=1}^{n} t_j\, CF_j(1+i)^{-t_j-2} \quad \textbf{(80)}$$

The main difference between the duration and the modified duration is that the former is quoted in years although the latter is quoted in percentage change in price. Moreover, the duration is an absolute measure while the modified duration is a relative measure. To take the case of a bond, the coupon rate, the interest rates and the maturity contribute to the duration measures. If we referred to the definitions of duration, modified duration and convexity, when the maturity increases, the duration increases and the bond's price becomes more sensitive to interest rate changes. Then, if the coupon increases, the duration will decrease and in consequence the bond becomes less sensitive to interest rate changes. Finally, if the interest rates increase, the duration will decrease and the bond becomes less sensitive.
What is important to understand is, these concepts represent the influence of interest rates movements and can be used as a measure of risk.

3. Duration matching : immunization

A solution to the reduce the interest rate sensitivity is to match the duration as well as present values of the portfolio and the future cash obligations. This process is called *immunization*, *i.e.* protection against change in yield. By matching duration, portfolio value and present value of cash obligations will respond identically to a change in yield.
Let us consider two flows : an asset flow *A* and a liability flow *L* represented by the following future cash flows:

$$A = \{(A_j, t_j), j = 1, \dots, n\} \qquad L = \{(L_j, t_j, j = 1, \dots, n\} \quad \textbf{(81)}$$

Here we don't consider that there is an equity cash-flow.
With a constant interest rate, the present market value called $S(i)$ is given by Equation (82).

$$S(i) = \sum_{j=1}^{n}(A_j - L_j)(1+i)^{-t_j} \quad \textbf{(82)}$$

Let define $D_{m,A}$(i), $D_{m,L}$(i), CV_A(i), and CV_L(i) be respectively the modified duration of asset and liabilities, and the convexity of the assets and the liabilities. Then, from (76) and (82), $\Delta S(i) = S(i+\Delta i)) - S(i)$, *i.e.* the variation of present market value is defined in Equation (83).

$$\Delta S(i) \cong -[A(i)D_{m,A}(i) - L(i)D_{m,L}(i)]\Delta i + \frac{1}{2}[CV_A(i)A(i) - CV_L L(i)](\Delta i)^2 \quad \textbf{(83)}$$

Finally, at the first-order, the position {A,B} will be immunized against small variations of the interest rate if and only if:

$$A(i)D_{m,A}(i) = L(i)D_{m,L}(i) \quad \textbf{(84)}$$

which says that the cash flows of the assets hedge the cash flow of the liabilities. If we are at time 0 with a null position, *i.e.* $A(i) = L(i)$, then the last condition is that the duration of assets corresponds to the duration of the liabilities as in Equation (85).

$$D_{m,A}(i) = D_{m,L}(i). \quad \textbf{(85)}$$

From (28), this immunization is reinforced if the condition in (86) is respected.

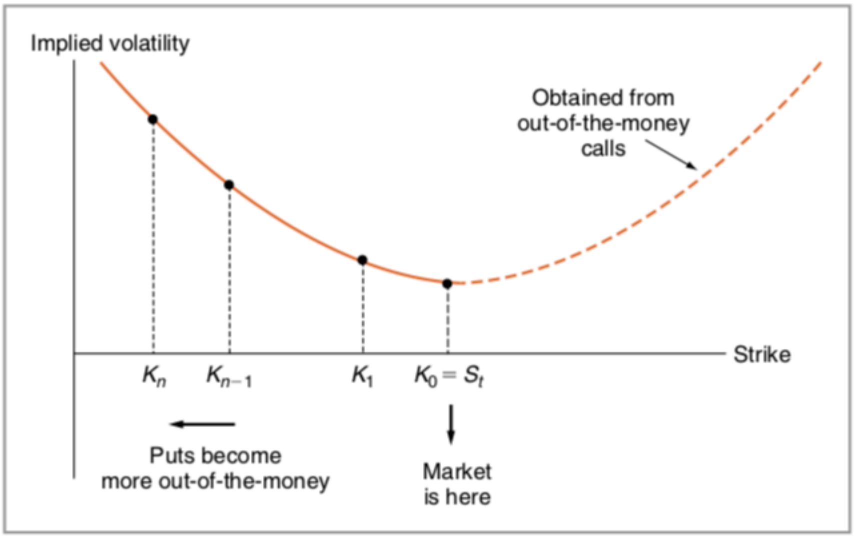
$$[CV_A(i)A(i) - CV_L L(i)] \geq 0 \quad \textbf{(86)}$$

There are difficulties with immunization procedure in practice. First, it is necessary to rebalance or re-immunize the portfolio from time to time since the duration depends on yield. Second, the immunization method assumes that all yields are equal. This is not quite realistic to have bonds with different maturities to have the same yield. Last, when the prevailing interest rate changes, it is unlikely that the yields on all bonds all change by the same amount.

Annex n°2: the volatility smile

In the section of interest rate models in negative interest rate environment, we saw that the volatility smile need to be correctly modelled in order to get a relevant model. First, the *implied volatility* need be explained. In the options, there is the observed market price which relies on the Black-Scholes formula to price them. Then the implied volatility is the volatility which must be plugged into the formula in order to recover the fair market price. It differs from the realized volatility if the volatility is stochastic and if a risk premium needs to be added to volatility quotes. Before explaining the volatility smile, I need to differentiate for options in-the-money, at-the-money, and out-of-the-money. For a call/put option, being *in-the-money* (ITM) means that the strike price is lower/higher than the underlying price. On the contrary, being *out-of-the-money* (OTM) means that the strike price is higher/lower than the underlying price. Finally, being *at-the-money* (ATM) means that the strike price is equal than the underlying price. It turns out that everything else being the same, an OTM put or call has a higher implied volatility than an ATM call/put. The more out-of-the-money you a call/put option is, the higher is the corresponding implied volatility. This ensures that there is an empirical fact which is called *volatility smile* or *volatility skew* which is represented in the Figure 17.

Figure 17. – Representation of the volatility smile/skew



On the left side, more you have an OTM put, more the implied volatility will increase. Then on the right side, this is the same reasoning but for a call option. On the x-axis, the K_0 is ATM and as K_i decreases[9], the puts go deeper OTM.

The volatility smile is an empirical phenomenon that violates the assumptions of the Black-Scholes world. Taking in account a new volatility process such the volatility smile is more relevant for the modelling. First, the volatility smile dynamics offer trade opportunities for market professionals since the volatility smile is associated with all the risk factors and offers to traders the opportunity to take spread positions and arbitrage it. Then, the volatility smile contain information about the dynamics of the underlying realized volatility processes.

[9] K_i represents the *i*th strike of the option series and S_t is the equity price index