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Geometric category, LS-category and strong category

An overview

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Contents

Introduction

Classifying and understanding the differences between objects of the same type has always been of major interest. In particular, homotopy invariants play a crucial role in the study of topological spaces, allowing us to classify them, to characterize their global properties, detect obstructions and establish links with other fields of mathematics. They offer a powerful tool for understanding the structure and properties of topological objects.

So, what is a homotopy invariant? It is a characteristic of topological spaces that remains unchanged when considering spaces of the *same homotopy type*. In this master thesis, we will first discuss a characteristic of topological spaces called the *geometric category*, originally defined by R. Fox in [Fox41]. Note that the term *category* here has no relation to the current category theory. This object is a natural number calculated for every topological space. Formally, if X is a topological space, it is the least natural number $n \in \mathbb{N}$ such that there exists a cover of X with $n+1$ contractible open sets. We denote it by $\text{gcat}(X)$. Unfortunately, we will prove that the geometric category is not a homotopy invariant. This raises the question of how to make it a homotopy invariant. What conditions could we add or relax?

One idea is to relax the conditions on the open sets. Instead of requiring contractible open sets, we will ask for open sets that can be contracted to a point *inside* X. This allows for more flexibility in the choice of open sets and leads to the definition of *Lusternik and Schnirelmann category* or *LS-category*, denoted by $\text{cat}(X)$. This is a well-known homotopy invariant. It was initially defined in 1934 by L. Lusternik and L. Schnirelmann in [LSK35]. The initial goal of this definition was to provide a lower bound on the number of critical points for any smooth function on a manifold. Although the primary intention was to contribute to analysis, particularly calculus of variations, this homotopy invariant has had consequences in many other fields. For example, in differential geometry, one use this invariant because it gives a lower bound for the minimal number of critical points of Morse functions; see [LSK35] and [Cor+03]. It is also used in computer science, in problems concerning the complexity of algorithms as explained in [Sma87].

A second idea we will explore is to force the invariance by considering the minimum value of the geometric category of all topological spaces that have the same homotopy type as X. This notion was introduced by T. Ganea in [Gan67] and is called the *strong category*. By definition, this is a homotopy invariant and it provides an upper bound for the LS-category.

In the first chapter of this thesis, we will establish the foundation for the following chapters by introducing the categories in which we will work. Not surprisingly, we will work in the category of topological spaces, denoted as Top, but we will also consider the category of topological spaces where homotopy classes of continuous maps serve as morphisms, denoted as hTop. In the latter category, we will observe that pushouts and pullbacks do not behave as expected.

The second chapter will be dedicated to finding a way to approximate pushouts and pullbacks in hTop. We will define the notions of *fibrations* and *cofibrations* that lead to the definition of *homotopy pushouts* and *homotopy pullbacks*. These constructions are used to study homotopic properties of topological spaces. They allow us to build new spaces from existing ones while preserving important homotopic properties. Homotopy pushouts allow to combine two topological spaces along a third one, while the homotopy pullbacks connect two topological spaces together thanks to two morphisms.

The third chapter marks the heart of this thesis. We define the *geometric category* and present some basic properties and examples. We will also show in details that it is not a homotopy invariant.

This third chapter naturally leads to the fourth and fifth chapters, where we define the two homotopy invariants: the *Lusternik and Schnirelmann category* and the *strong category* respectively. We provide examples and test these homotopy invariants on basic homotopical constructions like homotopy pushouts and pullbacks. Moreover, in Chapter 4, we present two other equivalent characterizations of the Lusternik and Schnirelmann category. The first one is the definition given by G. Whitehead in [Whi78], which concerns the existence of a factorization of the diagonal map in the *fat wedge* of the topological space. The second characterization, given by T. Ganea in [Gan67], utilizes the *fiber-cofiber* construction. Finally, in Chapter 5, we establish connections between the two homotopy invariants and seek a better understanding of the strong category. We will see that it equalizes the *cone-length* of the topological space we are analyzing.

The results of this thesis are not new, but the goal was to explain and understand the feasible constructions using topological spaces and how invariants act upon them. Another objective was to clarify some results that were not extensively developed in the reference book [Cor+03].

This thesis is intended to be accessible to master students having basic knowledge in algebraic topology and category theory.

Chapter 1

Fundamentals of homotopy theory

The aim of this chapter is to introduce all the notions that the reader might need to understand this thesis and define the categories in which we will work. In the first section, we recall some basic notions of homotopy theory and define the associated categories. The second section introduces the notion of category of pointed topological spaces. The following two sections are reminders of the notions of coproduct, product, pushout and pullback, with examples in the category of topological spaces and pointed topological spaces.

We introduce the key elements of this thesis in Section 1.5: the CW-complexes. Intuitively, they are spaces constructed by adding cells of different dimensions. As we shall see, it is possible to talk about the category of CW-complexes. We also explain how pullbacks and pushouts work with CW-complexes.

In the last section, we attempt to construct pullbacks and pushouts in the homotopy category of topological spaces. However, as one might anticipate, it will not work as easily as we want it to. Hence, we will discuss some solutions to this problem.

First, let $\mathcal{C}$ be a category with $\mathrm{Obj}(\mathcal{C})$ the objects and $\mathrm{Morph}(\mathcal{C})$ the collection of morphisms. Given $A, B \in \mathrm{Obj}(\mathcal{C})$, the class of arrows from A to B is $\hom_{\mathcal{C}}(A, B) \subseteq \mathrm{Morph}(\mathcal{C})$. Recall that in general, two objects in a category $\mathcal{C}$ are *isomorphic* if there are morphisms $f \in \hom(X, Y)$ and $g \in \hom(Y, X)$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$, f and g are called *isomorphisms*. Throughout this thesis, we will be working with topological spaces. In the category Top of topological spaces and continuous maps, two objects are isomorphic when they are homeomorphic.

1.1 The homotopy categories

In the first part of this section, we provide a brief review of basic homotopy theory. Later, we introduce some important categories in the field of algebraic topology, in which we will work throughout this thesis.

1.1.1 Homotopy Theory

The notions of homotopy and homotopy type are fundamental concepts in algebraic topology. Sometimes, in topology, requesting homeomorphisms is too restrictive. Therefore, we define the notion of homotopy type, which provides another way to express that two spaces have "the same shape". Before giving any definitions, let us recall that the compact-open topology on $Y^X = \{f : X \to Y \text{ continuous}\}$ is the topology generated by the sets

$$\mathcal{O}(K, U) = \{f \in Y^X | f(K) \subset U\},$$

where $K \subset X$ is compact and $U \subset Y$ is open. Here, "generated" means that the finite intersections of these sets form a basis for the open sets.

Definition 1.1. *Let X, Y be two topological spaces and $\mathcal{U} \subseteq Y^X$ a fixed subspace. A homotopy relative to $\mathcal{U}$ between two maps $f, g \in \mathcal{U}$ is a continuous map:*

$$H : X \times I \longrightarrow Y$$

such that,

(1) $H(x, 0) = f(x)$,

(2) $H(x, 1) = g(x)$,

(3) the path $\bar{H} : I \longrightarrow Y^X$ defined by $(\bar{H}(t))(x) = H(x, t)$ is such that $\bar{H}(I) \subseteq \mathcal{U}$.

We write in this case $f \simeq_{rel_{\mathcal{U}}} g$.

Note that we omit the subscript $\mathrm{rel}_{\mathcal{U}}$ when $\mathcal{U}$ is obvious and we simply use the notation $f \simeq g$.

Example 1.2. 1. Set $\mathcal{U} = Y^X$. In this case, we say that the maps f and g are freely homotopic because there are no restrictions on the homotopies.

2. Another example is given in the category PTop, where the objects are topological pairs (X, A), where A is a subspace of X and as morphisms we have continuous maps $f : (X, A) \to (Y, B)$ such that $f(A) \subseteq B$. Let Y, X be topological spaces and $A \subseteq X$ and $B \subseteq Y$ be subspaces. We can define $\mathcal{U} = (Y, B)^{(X,A)}$, the space of continuous maps $f : (X, A) \to (Y, B) \in$ PTop such that $f(A) \subseteq B$. In this case, we obtain homotopies preserving subsets.

3. Let $A \subseteq X$ and $f : A \to Y$ a continuous map. Consider the set

$$\mathcal{U} = \{g \in Y^X | g_{|A} = f\}$$

of maps extending f. We say that $H : X \times I \to Y$ is a *homotopy relative to* f if at any time t, $\bar{H}(t) \in \mathcal{U}$, i.e., $H(a, t) = f(a)$ for all $a \in A$, $t \in I$.

Homotopies are interesting also because the relation of being homotopic is an equivalence relation on the set $\mathcal{U}$ of chosen continuous maps.

Proposition 1.3. *Let $\mathcal{U} \subseteq Y^X$. The relation $\simeq_{rel_\mathcal{U}}$ of being homotopic relatively to $\mathcal{U}$ is an equivalence relation.*

Proof. Let $f, g, h : X \to Y$ three maps in $\mathcal{U}$. It is clear that $\simeq$ is reflexive because one can take the constant homotopy at f and obtain $f \simeq f$. If $f \simeq g$, then there is a homotopy $H : X \times [0,1] \to Y$ such that $H(x,0) = f(x)$, $H(x,1) = g(x)$ and $\bar{H}([0,1]) \subseteq \mathcal{U}$. One can define the homotopy

$$G : X \times [0,1] \to Y : (x,t) \mapsto H(x, 1-t)$$

such that $G(x,0) = g(x)$, $G(x,1) = f(x)$ and $\bar{G}([0,1]) \subseteq \mathcal{U}$ which gives $g \simeq f$. Therefore, the relation $\simeq$ is symmetric. Moreover, if $f \simeq g$ and $g \simeq h$, then there exist two homotopies

$$\begin{aligned} &K : X \times [0,1] \to Y, \quad \text{with } K(x,0) = f(x), K(x,1) = g(x), \quad \bar{K}([0,1]) \subseteq \mathcal{U} \\ &L : X \times [0,1] \to Y, \quad \text{with } L(x,0) = g(x), L(x,1) = h(x), \quad \bar{L}([0,1]) \subseteq \mathcal{U} \end{aligned}$$

and we define

$$\begin{aligned} M : X \times &[0,1] \to Y \\ &(x,t) \mapsto \begin{cases} K(x,2t) & \text{if } 0 \le t \le 1/2 \\ L(x,2t-1) & \text{if } 1/2 \le t \le 1 \end{cases} \end{aligned}$$

such that $M(x,0) = f(x)$, $M(x,1) = h(x)$ and $\bar{M}([0,1]) \subseteq \mathcal{U}$ which gives $f \simeq h$. We conclude that the relation $\simeq$ is transitive. □

Because being of the same homotopy type is an equivalence relation, the set of all continuous functions $f : X \to Y$ is partitioned into equivalence classes. We talk about *homotopy classes of maps* between topological spaces. The set of all freely homotopic classes of maps is denoted by $[X; Y]$. If one chooses the homotopies to be in $\mathcal{U} = (Y, B)^{(X,A)}$, we denote by $[(X, A); (Y, B)]$ the homotopy classes of maps preserving pairs in topological spaces.

1.1.2 Category of homotopy classes of maps

We previously discussed the category of topological spaces and continuous maps, and it is natural to consider taking homotopy classes of maps instead of the maps themselves. We define the category hTop where the objects are topological spaces and the morphisms between two objects X and Y are homotopy classes of maps $[X; Y]$. Given $[g] \in [X; Y]$,

$[f] \in [Y, Z]$, we define $[f] \circ [g] = [f \circ g]$ to be the composition and which is well-defined. Indeed, let $f, f' : X \to Y$ and $g, g' : Y \to Z$ be four continuous maps. If $f \simeq f'$ via the homotopy $F : X \times I \to Y$ and $g \simeq g'$ via a homotopy $G : Y \times I \to Z$ then, $g \circ f \simeq g' \circ f'$ via the homotopy

$$\begin{aligned} H : & X \times I \longrightarrow Z \\ & (x, t) \mapsto G\left(F(x, t), t\right) \end{aligned}$$

which is such that $H(x, 0) = g\left(f(x)\right)$ and $H(x, 1) = g'\left(f'(x)\right)$.

In this category, we work with homotopy classes of maps. Therefore, we say that $[f] : X \to Y$ is an isomorphism if there exists an equivalence class $[g] : Y \to X$ such that $[f \circ g] = [f] \circ [g] = [1_Y]$ and $[g \circ f] = [g] \circ [f] = [1_X]$. In this case, this means that $f \circ g \simeq 1_Y$ and $g \circ f \simeq 1_X$. This leads to the following definition:

Definition 1.4. *Let X, Y be two topological spaces. A continuous application $f : X \to Y$ is a homotopy equivalence if there exists $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$.*

In this case, we write $X \simeq Y$ and we say that X and Y have the same homotopy type. In hTop, morphisms are homotopy classes of maps $[f]$, but we will mostly speak about morphisms f up to homotopy instead of the classes and we will do almost all the constructions in Top. Also, we say that a diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \varphi} \\ C & \xrightarrow{\phi} & D \end{array}$$

commutes up to homotopy or is homotopy commutative if $\varphi \circ f \simeq \phi \circ g$.

1.1.3 Contractible spaces

The definition of homotopy type introduces a new type of topological spaces.

Definition 1.5. *A topological space X is said to be* contractible *if it is of the same homotopy type as a point, denoted as $X \simeq \{*\}$. A subspace A of X is contractible in X if the inclusion map $\iota : A \to X$ is homotopically trivial, meaning that $\iota \simeq c_*$, where c_* is a constant map.*

More generally, a map $f : X \to Y$ is called *nullhomotopic* or *homotopically trivial* if $f \simeq c_*$ where c_* is a constant map. Definition 1.5 means that a contractible space X is a space for which the identity map $\mathrm{id}_X : X \to X$ is homotopically trivial. Moreover, if we have a continuous map $f : Y \to X$ with codomain X contractible, then, $f \simeq c_*$.

Similarly, if $f : X \to Y$ is a continuous map with domain X contractible, then, $f \simeq c_*$. The following example illustrates the difference between being contractible and having an inclusion homotopically trivial.

Example 1.6. Injections from a contractible space, or inclusions into a contractible space are obviously contractible. However, there are also injections from non-contractible spaces to another non-contractible space that are nullhomotopic. For example, consider the circle S^1 and the sphere S^2, which are both non-contractible[1] spaces. Let the map $\iota : S^1 \to S^2$ be the canonical inclusion of the circle in the sphere (injection at the equator). It is possible to take arcs that continuously retract the circle to a point in the sphere. In fact, set $\{*\}$ to be the North pole and consider the arc going from $\iota(s) \in S^2$ to $\{*\}$,

$$\gamma_{s,*} : [0,1] \to S^2; t \mapsto \left(\cos\left(\frac{\pi t}{2}\right)\cos(\varphi), \cos\left(\frac{\pi t}{2}\right)\sin(\varphi), \sin\left(\frac{\pi t}{2}\right)\right),$$

where $s \in S^1$, and φ is the angle such that $(\cos\varphi, \sin\varphi, 0) = \iota(s)$. We now define the homotopy

$$\begin{aligned} H : S^1 \times [0,1] &\to S^2 \\ (s,t) &\mapsto \gamma_{s,*}(t) \end{aligned}$$

such that $H(s,0) = \gamma_{s,*}(0) = \iota(s)$ and $H(s,1) = \gamma_{s,*}(1) = \{*\}$ for all $s \in S^1$. This homotopy retracts $\iota(S^1)$ on the north pole. This shows that the circle is contractible in the sphere.

1.1.4 Equivalences of diagrams in Top

It is clear that if two continuous maps $f, g : X \to Y$ are homotopic, then, since morphisms in hTop are homotopy classes of maps, the following two diagrams in Top are the same in hTop :

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle u} \\ Z & \xrightarrow{v} & W \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{g} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle u} \\ Z & \xrightarrow{v} & W. \end{array}$$

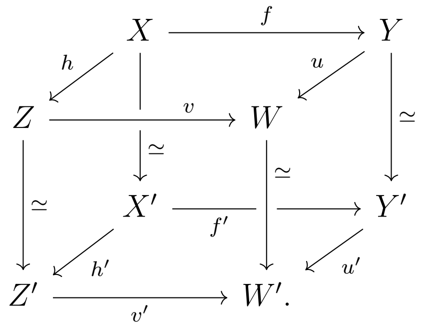
Now, we would like to understand, more generally, which diagrams in Top are considered the same in hTop. Therefore, we define the concept of *homotopy equivalence of diagrams.*

[1]The circle S^1 is not contractible because its fundamental group, $\pi_1(S^1)$, is not trivial (see [Hat02, Theorem 1.7]). To show that S^2 is not contractible, we use the homology. Since the homology is a homotopy invariant, and the homology groups of the sphere are not isomorphic to those of a point, we can conclude that S^2 is not contractible (see [Hat02, Proposition 2.8, Corollary 2.14]).

We say that two diagrams in Top,

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle u} \\ Z & \xrightarrow{v} & W \end{array} \qquad\qquad \begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle h'}\downarrow & & \downarrow{\scriptstyle u'} \\ Z' & \xrightarrow{v'} & W' \end{array}$$

are *homotopy equivalent* if there exist homotopy equivalences between the four topological spaces, i.e., $X \simeq X'$, $Y \simeq Y'$, $Z \simeq Z'$ and $W \simeq W'$. In addition, we require the vertical faces of the induced cube to be homotopy commutative:



In this case, the two diagrams represent the same class of diagrams in hTop.

Remark 1.7. (1) For readers familiar with category theory, we simply require a natural transformation $\alpha : \mathcal{D} \Rightarrow \mathcal{D}'$ such that α evaluated on an object is a homotopy equivalence. The natural transformation is between the diagrams $\mathcal{D} : I \to \text{Top}$ and $\mathcal{D}' : I \to \text{Top}$, considered as functors from an index category to Top.

(2) If we remove a space from the diagrams, the definition still holds, but we only need three equivalences. As an example, consider the diagrams:

$$X \xrightarrow{f} Y \xrightarrow{g} Z \qquad\qquad X' \xrightarrow{f'} Y' \xrightarrow{g'} Z'.$$

These two diagrams are be homotopy equivalent if $X \simeq X'$, $Y \simeq Y'$ and $Z \simeq Z'$ and the faces of the following diagram homotopy commutes:

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z'. \end{array}$$

We can have the same reasoning for other types of diagrams such as $X \xrightarrow{f} Y$, $X \xrightarrow{f} W \xleftarrow{g} Y$, $X \xleftarrow{g} W \xrightarrow{f} Y$ and so on.

1.2 Category of pointed topological spaces

In the category of pointed topological spaces, denoted Top_*, the objects are called *pointed topological space*. These are topological pairs such as (X, x_0) with a topological space X and a distinguished point $x_0 \in X$, namely the *base point*. In the category Top_*, the morphisms are base point preserving continuous maps, i.e., maps $f : (X, x_0) \to (Y, y_0)$ such that $f(x_0) = y_0$. These maps are called *pointed maps*. Considering such a map is equivalent to taking the base points as the subsets in Example 1.2 and considering the continuous maps relative to the base point.

Similarly, we can define the category hTop_* with homotopy classes of maps as morphisms. We work with maps in $(Y, y_0)^{(X,x_0)}$ and we require homotopies $F : X \times I \to Y$ to be relative to the base point, i.e, $F(x_0, t) = y_0$ for all $t \in I$. Therefore, hTop_* is the category of pointed topological spaces with homotopy classes of maps preserving base points as morphisms.

Note that the homotopy equivalences of diagram in Top_* are defined the same way as in Top, see Section 1.1.4.

1.3 Coproducts in Top and Top_*

The *coproduct* and its dual notion, the *product*, are fundamental constructions of category theory. Throughout this thesis, we will use many objects that require the coproduct and sometimes, the product. This is why we give the general definition of the coproduct here, which underlines its universal property.

Definition 1.8. *In a category $\mathcal{C}$, a coproduct (or sum) of two objects $A, B \in \mathrm{Obj}(\mathcal{C})$ is a triple $(A + B, \iota_A, \iota_B)$ with $A + B \in \mathrm{Obj}(\mathcal{C})$,*

$$\iota_A : A \to A + B, \qquad \iota_B : B \to A + B,$$

that satisfies the following universal property: given any pair of morphisms $\alpha : A \to X$, $\beta : B \to X$, there exists a unique morphism $\phi : A + B \to X$ such that $\phi \circ \iota_A = \alpha$, $\phi \circ \iota_B = \beta$:

$$\begin{array}{ccccc} B & \xrightarrow{\iota_B} & A + B & \xleftarrow{\iota_A} & A \\ & {\scriptstyle\beta}\searrow & \downarrow{\scriptstyle\phi} & \swarrow{\scriptstyle\alpha} & \\ & & X. & & \end{array}$$

We are mostly interested in understanding the coproduct in the categories Top and Top_*. In Top, the coproduct is the *disjoint union*. Given X and Y, two topological spaces, we define the disjoint union of X and Y as the space

$$X \amalg Y = (X \times \{0\}) \cup (Y \times \{1\}).$$

The topology on the coproduct in Top is such that a subspace $\mathcal{U}$ is open in $X \amalg Y$ if and only if $\mathcal{U} \cap X$ is open in X and $\mathcal{U} \cap Y$ is open in Y. If we have two continuous maps, $\alpha : X \to Z$, $\beta : Y \to Z$, there exists a unique continuous map $\phi : X \amalg Y \to Z$,

$$\phi(a, t) = \begin{cases} \alpha(a) & \text{if } t = 0 \\ \beta(a) & \text{if } t = 1 \end{cases}$$

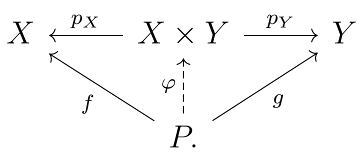
such that $\phi \circ \iota_A = \alpha$, $\phi \circ \iota_B = \beta$.

In Top_*, the coproduct is slightly different. In fact, because maps are base point preserving, we deduce that the coproduct of (X, x_0) and (Y, y_0) in Top_* is

$$X \vee Y = (X \amalg Y/ \sim, [x_0] = [y_0]),$$

where $\sim$ is the equivalence relation generated by $x_0 \sim y_0$, i.e., it identifies x_0 and y_0 together. This space is called the *wedge* of X and Y and it is equipped with the quotient topology.

The dual concept of the coproduct is the one of the product. In the category Top, any two objects X and Y have a product, which is the Cartesian product $X \times Y$ equipped with the product topology and the standard projection maps $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$. In fact, for all pairs of maps $f : P \to X$ and $g : P \to Y$, there is a unique map $\varphi : P \to X \times Y$ defined by $\varphi(p) = (f(p), g(p))$ which is continuous. This situation can be illustrated by the following diagram:



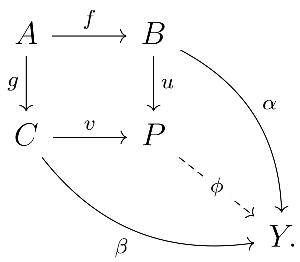
Remark 1.9. Notice that the products in Top and in Top_* are homeomorphic. However, this is not the case for coproducts. On one hand, we obtain the disjoint union, and on the other hand, we obtain the wedge.

1.4 Pushouts and pullbacks in Top and Top_*

The main goal of this section is to provide a method for creating new topological spaces using very common and abstract constructions, namely, pushouts and pullbacks. As a reminder, in a category $\mathcal{C}$, given a pair of arrows $f : A \to B$ and $g : A \to C$ with common domain A, a *pushout* of (f, g) is a commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle u} \\ C & \xrightarrow{v} & P \end{array}$$

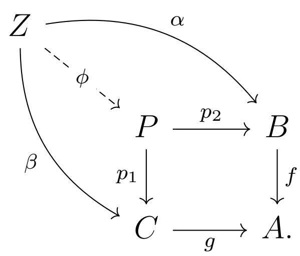
such that for any other pairs of arrows (α, β), $\alpha : B \to Y$, $\beta : C \to Y$ such that $\alpha \circ f = \beta \circ g$, there exists a unique map $\phi : P \to Y$ such that $\phi \circ u = \alpha$ and $\phi \circ v = \beta$. This is illustrated in the following diagram:



The *pullback* is the dual notion of the pushout. In fact, given in a category $\mathcal{C}$, a pair of arrows $f : B \to A$ and $g : C \to A$ with common codomain A, a pullback of (f, g) is a commutative square

$$\begin{array}{ccc} P & \xrightarrow{u} & B \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow{g} & A \end{array}$$

such that for any other pair of arrows (α, β), $\alpha : Z \to B$, $\beta : Z \to C$ such that $f \circ \alpha = g \circ \beta$, there exists a unique map $\phi : Z \to P$. This is represented in the following diagram:



In this section, we provide a detailed explanation of these two concepts in the categories Top and Top_*, and show how to construct topological spaces using pushouts and pullbacks.

1.4.1 Pushouts in Top

Pushouts exist in Top. If $f_1 : A \to X_1$ and $f_2 : A \to X_2$ are two continuous maps, the pushout of the pair (f_1, f_2) is given by

$$\begin{array}{ccc} A & \xrightarrow{f_1} & X_1 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle \iota_1} \\ X_2 & \xrightarrow{\iota_2} & X_1 \amalg X_2/\sim, \end{array}$$

where ι_1, ι_2 are the inclusions in the coproduct $X_1 \amalg X_2$, and $\sim$ is the equivalence relation on $X_1 \amalg X_2$ generated by $f_1(a) \sim f_2(a)$.

Example 1.10. The pushout is often thought of as a way to glue two spaces together. For example, if X and Y are two topological spaces and $A \subseteq X$ is a closed subset of X, then, if $f : A \to Y$ is continuous, *the space obtained from Y by attaching X via f* is given by

$$\frac{X \amalg Y}{\sim},$$

where $\amalg$ is the coproduct in Top and $\sim$ is the equivalence relation generated by $a \sim f(a)$. This space is denoted $Y \cup_f X$ and f is called the *attaching map.* This space is a pushout where in the definition, f_2 is an inclusion. In fact, take $X_1 = Y$, $X_2 = X$, and $f_2 = \iota : A \hookrightarrow X$. Hence, we obtain

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle \iota_1} \\ X & \xrightarrow{\iota_2} & Y \cup_f X = X \amalg Y/\sim . \end{array}$$

This construction forces the attachment of X to Y via the space A. This is what we call *gluing* or *attaching.*

This is not the only thing we can do with the pushout. The rest of this part of the section is devoted to examples of specific pushouts in Top that we will be using throughout this thesis.

Example 1.11. We have the following examples:

(1) *Mapping cylinder.*
Let $f : X \to Y$ be a map. We define the mapping cylinder of f as the pushout of the pair (f, ι_X), where $\iota_X : X \hookrightarrow X \times I; x \mapsto (x, 0)$. This gives the space:

$$M_f = \frac{Y \amalg (X \times I)}{\sim},$$

where $\sim$ is the equivalence relation given by $\iota_X(x) = (x, 0) \sim f(x)$ for all $x \in X$. The associated diagram is the following:

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ \Big\downarrow{\scriptstyle \iota_X} & & \Big\downarrow{\scriptstyle \iota_1} \\ X \times I & \xrightarrow{\iota_2} & M_f = \frac{Y \amalg (X \times I)}{\sim}. \end{array}$$

This is equivalent to attaching the cylinder of X to Y. Moreover, we note $M_f = Y \cup_f (X \times I)$, where $\cup_f$ means that we glue the cylinder $X \times I$ to Y using the image $f(X)$, i.e., $(x, 0)$ is identified with $f(x)$.

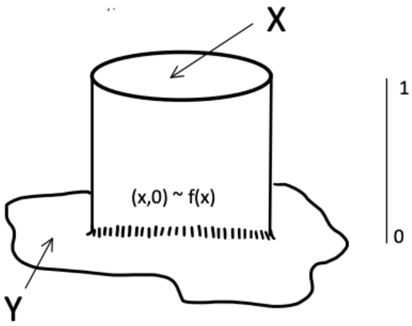


Figure 1.1: Mapping cylinder M_f.

(2) *Cone of a topological space X.*
Let X be a topological space. The cone of X is the pushout of the maps (c_*, ι_x) in Top, where $c_* : X \to \{*\}$ is the constant map and $\iota_X : X \hookrightarrow X \times I; x \mapsto (x, 1)$ is the inclusion of X at $t = 1$. We obtain

$$CX = \frac{(X \times I)}{\sim},$$

where $\sim$ is the equivalence relation given by $\iota_X(x) = (x, 1) \sim \{*\}$ for all $x \in X$.

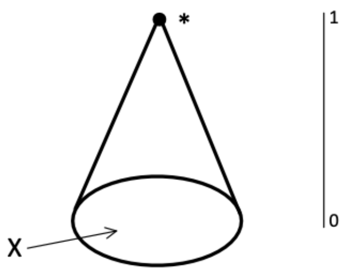


Figure 1.2: Cone CX.

We have the following diagram:

$$\begin{array}{ccc} X & \xhookrightarrow{\iota_X} & X \times I \\ \downarrow & & \downarrow \iota_1 \\ \{*\} & \xrightarrow{\iota_2} & CX. \end{array}$$

In other words, the cone of a topological space X is $CX = (X \times I)/(X \times \{1\})$.

(3) *Mapping cone.*
The mapping cone of a map $f : X \to Y$ is the pushout of f and $\iota_{CX} : X \hookrightarrow CX; x \mapsto (x, 0)$,

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \iota_{CX} \downarrow & & \downarrow \iota_1 \\ CX & \xrightarrow{\iota_2} & \frac{Y \amalg CX}{\sim}, \end{array}$$

where $\sim$ is the equivalence relation generated by $f(x) \sim (x, 0)$.

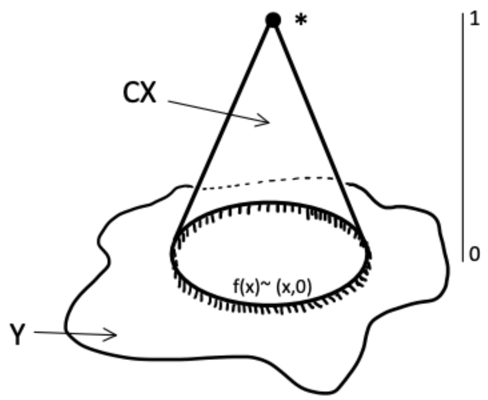


Figure 1.3: Mapping cone, C_f.

One observes that this equivalence relation simply attaches the cone of X to Y with the attaching map f. Thus, using the same notation as in Remark 1.10, we denote the mapping cone by $C_f = Y \cup_f CX$. Moreover, it is clear that $C_f = Y \cup_f CX = \frac{M_f}{X \times \{1\}}$.

(4) *Suspension.*
Let X be a topological space and CX the cone of X. Then, the suspension of X is the pushout of the inclusion map $\iota_{CX} : X \hookrightarrow CX; x \mapsto (x, 0)$ and the constant

map $c_* : X \to \{*\}$,

$$\begin{array}{ccc} X & \overset{\iota_{CX}}{\hookrightarrow} & CX \\ {\scriptstyle c_*}\downarrow & & \downarrow{\scriptstyle \iota_1} \\ \{*\} & \xrightarrow{\iota_2} & \frac{CX}{\sim}, \end{array}$$

where $\sim$ is the equivalence relation generated by $(x, 0) \sim \{*\}$, i.e., the relation collapsing $X \times \{0\}$ to a single point. We note the suspension as ΣX.

Intuitively, the suspension is a double cone. In fact, if X is a non-empty topological space, the suspension of X is the quotient

$$\Sigma X := \frac{X \times [0, 1]}{\sim},$$

where $\sim$ is the equivalence relation given by $(x, 1) \sim (x', 1)$, and $(x, 0) \sim (x', 0)$ for all $x, x' \in X$.

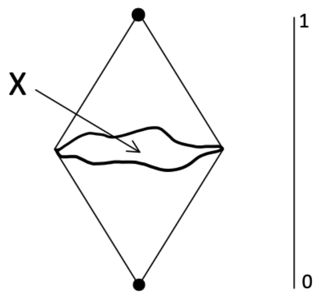


Figure 1.4: Suspension of X, ΣX.

As an example, consider $S^0 := \{-1, 1\}$. The suspension ΣS^0 is homeomorphic to the circle S^1. Similarly, ΣS^1 is homeomorphic to S^2 with the north and south pole being the two points where S^1 is collapsed. More generally, $\Sigma S^n = S^{n+1}$ for $n \in \mathbb{N}$.

1.4.2 Pushouts in Top_*

In Top_* the pushout is essentially the same, but we need to take care of the base points. In fact, Section 1.3 explains that in Top_*, the coproduct is the wedge sum of (X, x_0) and (Y, y_0). Therefore, if we consider two continuous maps $f_1 : (A, a_0) \to (X_1, x_1)$ and

$f_2 : (A, a_0) \to (X_2, x_2)$, the pushout of the pair (f_1, f_2) is given by :

$$\begin{array}{ccc} (A, a_0) & \xrightarrow{\quad f_1 \quad} & (X_1, x_1) \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle \iota_1} \\ (X_2, x_2) & \xrightarrow{\iota_2} & (X_1 \vee X_2 / \sim, [x_1] = [x_2]) \,, \end{array}$$

where ι_1, ι_2 are the inclusions in the wedge sum $X \vee Y$ and $\sim$ is the equivalence relation generated by $f_1(a) \sim f_2(a)$. Thanks to this construction, we can derive other useful topological spaces in Top_*.

Example 1.12. (1) *Quotient of X by a subspace A.*
Let X be a topological space and $A \subseteq X$ a subset. Then, the pushout of the inclusion $\iota : A \hookrightarrow X$ and of the constant map $c_* : A \to \{*\}$ is given by

$$\begin{array}{ccc} (A, x_0) & \xrightarrow{\quad c_* \quad} & \{*\} \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota_1} \\ (X, x_0) & \xrightarrow{\iota_2} & (X \vee \{*\}/ \sim, [x_0] = [*]) \end{array}$$

where $\sim$ identifies every point of the subset A with $\{*\}$. Therefore,

$$(X \vee \{*\}/ \sim, [x_0] = [*]) = (X/A, [x_0]).$$

This is equivalent to taking the quotient of a topological space by a subspace.

(2) *Smash Product.*
The smash product of (X, x_0) and (Y, y_0), denoted $X \wedge Y$, is the pushout of the obvious inclusion

$$\iota : (X \vee Y, [x_0] = [y_0]) \hookrightarrow (X \times Y, (x_0, y_0))$$

and the constant map

$$c_* : (X \vee Y, x_0 = y_0) \to \{*\}.$$

This construction gives the following commutative diagram:

$$\begin{array}{ccc} (X \vee Y, [x_0] = [y_0]) & \xhookrightarrow{\quad \iota \quad} & (X \times Y, (x_0, y_0)) \\ {\scriptstyle c_*}\downarrow & & \downarrow{\scriptstyle \iota_1} \\ \{*\} & \xrightarrow{\quad \iota_2 \quad} & \left(\frac{(X \times Y)}{X \vee Y}, [(x_0, y_0)] \right) . \end{array}$$

(3) *Pointed (or reduced) suspension of (X, x_0).*

Let (X, x_0) be a topological space. We define the *reduced cone of* X as

$$C_*X = \frac{(X \times I, (x_0, 0))}{\sim},$$

where $\sim$ is the equivalence relation generated by $(x, 1) \sim (x', 1)$ and $(x_0, t) \sim (x_0, t')$, $x, x' \in X$, $t, t' \in I$.

Then, the pushout of the map $\iota_{C_*X} : X \hookrightarrow C_*X; x \mapsto (x, 0)$, with $c_* : (X, x_0) \to \{*\}$, is given in the following diagram:

$$\begin{array}{ccc} (X, x_0) & \xhookrightarrow{\iota_{C_*X}} & C_*X \\ {\scriptstyle c_*}\downarrow & & \downarrow{\scriptstyle \iota_1} \\ \{*\} & \xrightarrow{\iota_2} & \frac{CX_*}{\sim}, \end{array}$$

where $\sim$ is the equivalence relation identifying $X \times \{0\}$ to a single point but also identifying the base point, i.e., $(x_0, t) \sim (x_0, t')$ for all $t, t' \in I$. We note Σ_*X for the pointed suspension of X. This construction implies that the pointed suspension of X can be seen as the following quotient:

$$(\Sigma_*X, [x_0, -]) := \frac{X \times [0, 1]}{\approx},$$

where the base point $[x_0, -]$ is the equivalence class of the base point of X and $\approx$ is the equivalence generated by $(x, 1) \approx (x', 1)$, $(x, 0) \approx (x', 0)$ and $(x_0, t) \approx (x_0, t')$ for $x, x' \in X$ and $t \in I$. In other words, the pointed suspension is the suspension where we also identify $\{x_0\} \times I$. As for the case in Top, see Example 1.11 , we also have that $\Sigma_*S^n = S^{n+1}$.

1.4.3 Pullbacks in Top and Top$_*$

The pullbacks are the same in Top and in Top$_*$ and they are a subspace of the product (see Section 1.3). Specifically, if we have a pair (f, g) of continuous maps in Top with the same codomain A, then the pullback of the pair is given by the diagram

$$\begin{array}{ccc} X \times_{f,g} Y & \xrightarrow{p_X} & X \\ {\scriptstyle p_Y}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & A, \end{array}$$

where

$$X \times_{f,g} Y = \{(x, y) \in X \times Y | f(x) = g(y)\}$$

and p_X, p_Y are the projections associated with the product $X \times Y$. The space $X \times_{f,g} Y$ is also called the *fibered product* of X and Y.

In Top_*, we obtain a similar construction:

$$\begin{array}{ccc} (X \times_{f,g} Y, (x_0, y_0)) & \xrightarrow{p_X} & (X, x_0) \\ {\scriptstyle p_Y}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ (Y, y_0) & \xrightarrow{\quad g \quad} & (A, a_0), \end{array}$$

and since we have base point preserving maps, $g(y_0) = a_0 = f(x_0)$. Hence, the point (x_0, y_0) is the natural base point of $X \times_{f,g} Y$.

1.5 Category of CW-complexes

Willingness to decompose complicated spaces into smaller and simpler ones is common in mathematics. Many interesting topological spaces can be represented by a decomposition into subsets that are glued together 'nicely' along their boundaries, making them easier to analyze. This is precisely what we aim to achieve with CW-complexes. In the first part of this section, we define precisely what a CW-complex is and discuss some of their intuitive properties. Every subsequent result in this thesis will take as hypothesis that the topological spaces are CW-complexes. One might wonder if this is overly restrictive, but in reality, we will see that it is not the case for what we aim to do in this thesis. In the second part of this section, we define the categories associated with CW-complexes and describe the pullbacks and pushouts in these categories.

1.5.1 CW-complexes and cellular spaces

We will begin by introducing the concept of *cell attachment*, and then proceed to define CW-complexes. Intuitively, CW-complexes are spaces that are built up by attaching cells of different dimensions, one at a time. The definition of CW-complexes can be found in almost every book on algebraic topology, including [Rot88, Chapter 8], [Hat02, Chapter 0], and [Bre93, Chapter 8] (as well as many others). There are many different ways to define CW-complexes, but for the purposes of this thesis, we will stick to the definition presented in this section, which describes the process by which they are constructed.

As explained in Remark 1.10, if X and Y are two topological spaces with $A \subseteq X$ a closed subset of X and $f : A \to Y$ a continuous map, *the space obtained from* Y *by* *attaching X via f* is the pushout of the pair (f, ι),

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle \iota}\big\downarrow & & \big\downarrow{\scriptstyle \iota_1} \\ X & \xrightarrow{\iota_2} & Y \cup_f X = X \amalg Y/\sim, \end{array}$$

with $\iota : A \hookrightarrow X$ being the inclusion, ι_1 and ι_2 the inclusions in the coproduct in Top, and $\sim$ being the equivalence relation generated by $a \sim f(a)$. Thanks to this gluing method, we can easily explain what it means to attach a *cell*. If $f : S^{n-1} \to Y$ is a continuous map from the $(n-1)$-sphere into Y, the space $Y \cup_f D^n$ obtained by attaching the n-disk to Y via the attaching map f is called the *space obtained from Y by attaching an n-cell via f*. In general, we write this space $Y \cup_f e^n$. We can illustrate the situation with the following pushout:

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{\quad f \quad} & Y \\ \big\downarrow{\scriptstyle \iota} & & \big\downarrow{\scriptstyle \iota_1} \\ D^n & \xrightarrow{\iota_2} & Y \cup_f e^n = \frac{X \amalg D^n}{\sim}, \end{array}$$

where $\sim$ is the equivalence relation identifying $f(S^{n-1})$ and the boundary of the disk D^n which is $\iota(S^{n-1})$. Moreover, $e^n = \iota_2(D^n)$ is called a *n-cell*. Note that the interior of e^n is homeomorphic to the interior of the disk D^n and generally, e^n is not homeomorphic to D^n since there are identifications on the boundary.

Example 1.13.

(1) The circle S^1 can be seen as one point with an added 1-cell,

$$S^1 \cong \{*\} \cup_f e^1$$

where $f : S^0 \to \{*\}$ is the attaching map and $S^0 = \{-1, 1\} \subseteq [-1, 1]$.

(2) Consider the sphere S^2 and the attaching map $f : S^0 \to S^2$, that sends -1 on $(0, 0, -1)$ and 1 on $(0, 0, 1)$. Then, the space $S^2 \cup_f e^1$ is the one on Figure 1.5.

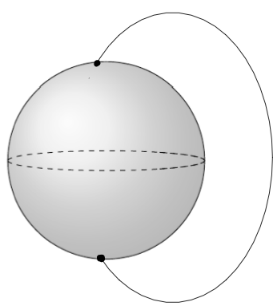


Figure 1.5: $S^2 \cup_f e^1$.

It is now time to define the important notion of this section: CW-complexes. We define them by describing the process by which they are constructed. Intuitively, these are spaces built step by step by attaching cells of different dimensions. The definition is inspired by [Hat02, Chapter 0, p.5] and [Bre93, Chapter 4, Section 8].

A CW-complex is a space X constructed in the following way:

1. Let $X^{(0)}$ be a discrete set of points. These points are the 0-cells.
2. Suppose that $X^{(n-1)}$ has been defined. We form $X^{(n)}$ from $X^{(n-1)}$ by attaching n-cells e^n_α via a collection of maps $\phi_\alpha : S^{n-1} \to X^{(n-1)}$, where α ranges over some indexing set. Let $B = \amalg_\alpha D^n_\alpha$ be a disjoint union of copies D^n_α of the disc D^n, one for each α. Let ∂B be the corresponding union of the boundaries S^{n-1}_α of these discs, and put together the maps $\phi_\alpha : S^{n-1} \to X^{(n-1)}$ to produce a map $\Phi : \partial B \to X^{(n-1)}$. Then define

$$X^{(n)} = X^{(n-1)} \cup_\Phi B = \frac{X^{(n-1)} \amalg_\alpha D^n_\alpha}{\sim},$$

   where the equivalence relation generated by $\sim$ identifies $\phi_\alpha(S^{n-1})$ and the boundary of the disk D^n_α for all α's.
3. One can either stop this inductive process at a finite stage, setting $X = X^{(n)}$ for $n \in \mathbb{N}$, or one can continue indefinitely. In the latter case, X is given the *weak topology*, i.e., a set A is open (or closed) if and only if $A \cap X^{(n)}$ is open (or closed) in $X^{(n)}$ for each n.

Note that every cell e^n_α is homeomorphic to the image of the disk D^n_α in the space $X^{(n)} = \frac{X^{(n-1)} \amalg_\alpha D^n_\alpha}{\sim}$. Moreover, we say that a CW-complex is *finite* if it has a finite number of cells.

Example 1.14. (1) A 1-dimensional cell complex is a *graph*. In fact, we have the points in $X^{(0)}$ called vertices, and the maximal dimension of the cells is one, so we attach 1-cells that we call edges. In these graphs, we allow for multiple edges between two vertices, as well as edges that have the same vertices (loops), as illustrated in Figure 1.6.

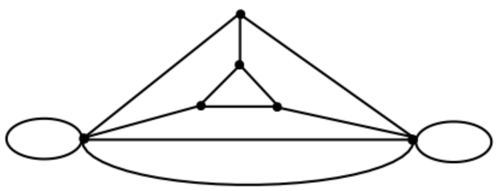


Figure 1.6: CW-complex with $\dim(X) = 1$.

(2) A CW-complex can, of course, admit more than one cellular decomposition. For example, the circle can be seen as one point with a 1-cell as explained in Example 1.13. Another way of constructing the circle is to take $X^0 = S^0 = \{-1, 1\}$ and attach two 1-cells,

$$S^1 \cong (\{-1, 1\} \cup_f e^1) \cup_f e^1,$$

where $f = \mathrm{id}_{S^0}$.

(3) More generally, the n-sphere is obtained from a point $\{*\}$ by attaching an n-cell. We have

$$S^n \cong \{*\} \cup_f e^n,$$

where $f : S^{n-1} \to \{*\}$ is the constant map. Hence, the sphere is a CW-complex.

(4) The projective plane is also a classic example of cellular space. Recall that $\mathbb{R}P^n$ is the space of all lines through the origin in $\mathbb{R}^{n+1}$. Each line can be represented by a vector of length one, so $\mathbb{R}P^n$ can also be seen as the sphere in dimension n with antipodal points identified. Using the quotient map $q : S^{n-1} \to \mathbb{R}P^{n-1}$ as an attaching map, we observe that

$$\mathbb{R}P^n \cong \mathbb{R}P^{n-1} \cup_q e^n,$$

which can be illustrated with the following diagram:

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{\quad q \quad} & \mathbb{R}P^{n-1} \\ \big\downarrow \iota & & \big\downarrow \\ D^n & \longrightarrow & \mathbb{R}P^{n-1} \cup_q e^n = \frac{X \amalg D^n}{\sim}. \end{array}$$

Intuitively, we have that $\mathbb{R}P^n \cong \mathbb{R}P^{n-1} \cup_q e^n$ because S^n with antipodal points identified is the same as considering D^n with the antipodal points on the boundary identified. Moreover, because $\partial D^n \cong S^{n-1}$, we have that ∂D^n with antipodal points identified is homeomorphic to $\mathbb{R}P^{n-1}$. Finally, we attach an n-disk to ∂D^n via the quotient map q. Therefore, the projective n-space $\mathbb{R}P^n$ is a CW-complex with one cell of each dimension. This result is proven in [Rot88, Theorem 8.8(i)] but is also explained in detail in [Hat02, Example 0.4].

(5) In [Man16], it is explained that every smooth manifold is triangulable. Recall that a triangulation is a way of cutting a geometric shape into a collection of triangles. Intuitively, cutting a manifold into triangles is the same as having a cell decomposition. Therefore, there is a wild class of manifolds having the homotopy type of a CW-Complex. In [Mil73], Milnor gives also a proof of this fact using morse functions. Let M be a manifold. Theorem 3.5 of [Mil73] states that if $f : M \to \mathbb{R}$ is a differentiable function on a manifold M with no degenerate critical points and if each set $M^a = \{x \in M : f(x) \leq a\}$ is compact, then, M has the homotopy type of a CW-complex with one cell of dimension n for each critical

point of index n. Consequently, if we restrict our next results to spaces that have the same homotopy type as CW-complexes it will not be a significant limitation.

Definition 1.15. *A subcomplex of a CW-complex X is a closed subspace $A \subseteq X$ which is a union of cells of X.*

It is easy to see that a subcomplex of a CW-complex is itself a CW-complex.

Example 1.16. If we consider the n-projective space again, the only subcomplexes of $\mathbb{R}P^n$ are $\mathbb{R}P^k$ for $k < n$. With one cell in each dimension, we have that

$$e^n = \mathbb{R}P^{n-1} \cup e^n = e^0 \cup e^1 \cup \cdots \cup e^n = \mathbb{R}P^n.$$

Therefore, any union of cells of $\mathbb{R}P^n$ is simply the cell with the greatest dimension.

A pair (X, A) consisting of a CW-complex and a subcomplex A is called a CW-pair. Moreover we have the following interesting result:

Proposition 1.17. *[Rot88, Theorem 8.27] If (X, A) is a CW-pair, then, the quotient space X/A is a CW-complex.*

We now give some nice properties of CW-complexes. The first one states that if we change, up to homotopy, our way of gluing the cells, then the resulting CW-complexes are still in the same homotopy class.

Proposition 1.18. *[Hat02, Proposition 0.18] If $f, g : S^{n-1} \to X$ are homotopic, then, $X \cup_f e^n \simeq X \cup_g e^n$.*

This proposition implies that the homotopy type of a CW-complex relies only on the homotopy class of the attaching maps. The following two theorems give information about the connectedness of CW-complexes.

Theorem 1.19. *[Rot88, Theorem 8.25] Every CW-complex is locally path-connected.*

Theorem 1.20. *[Rot88, Theorem 8.23] If X is a CW-complex, then it is connected if and only if it is path-connected.*

More properties and concepts about CW-complexes will be given in Chapter 2.

1.5.2 Category of cellular spaces

As already explained in Examples 1.14, many useful topological spaces, such as graphs and manifolds, are CW-complexes. This is why focusing on results mainly for CW-complexes is not too restrictive. Moreover, we know a lot about their structure, which is highly convenient as it allows us to understand things step by step. As a consequence, we will work with topological spaces having the homotopy type of CW-complexes.

Intuitively, we want maps that preserve the cellular structure, i.e., we require that a map $f : X \to Y$ between CW-complexes maps cells to cells of the same or lower dimension : $f(X^{(n)}) \subseteq Y^{(n)}$. Such a map is called a *cellular map*. According to the *cellular approximation theorem*, every map $f : X \to Y$ of CW-complexes is homotopic to a cellular map (see [Hat02, Theorem 4.8.]). Therefore, in every homotopy class of map between CW-complexes, there will be a cellular map. We denote by CWTop the subcategory of Top whose objects are the topological spaces having the homotopy type of a CW-complex, and the morphisms are continuous maps $f : X \to Y$ such that there is a homotopy equivalent diagram $f' : X' \to Y'$, where f' is a cellular maps between the CW-complexes X' and Y'.

Obviously, we can consider CWTop_* the pointed version of the category CWTop, which is a subcategory of Top_*. The category hCWTop is the homotopy category associated to CWTop. Pushout and pullbacks in CWTop and CWTop_* are defined as in Top and Top_* respectively.

If W, X and Y are CW-complexes and $f : W \to X$, $g : W \to Y$ are continuous cellular maps, the pushout $X \amalg Y/\sim$ where $\sim$ is the equivalence relation generated by $f(x) \sim g(x)$ is still a CW-complex. This situation is illustrated by the following diagram:

$$\begin{array}{ccc} W & \xrightarrow{\quad f \quad} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \iota_1} \\ X & \xrightarrow{\iota_2} & Y \cup_f X = X \amalg Y/\sim . \end{array}$$

Observe that $X \amalg Y$ is still a CW-complex, and that f and g preserve the cellular structure. This implies that the images of f and g are subcomplexes of X and Y respectively. Hence, the relation $\sim$ consists of identifying a subcomplex of X with another subcomplex of Y, identifying cells with cells. Therefore, we preserve a certain cell decomposition. In conclusion, $X \amalg Y/\sim$ is still a CW-complex. The same reasoning applies to CWTop_* since the wedge of two CW-complexes corresponds to the disjoint product of the two complexes, where two 0-cells are identified.

The pullback of two cellular maps between (*locally finite*, see Remark 1.21) CW-complexes is given by

$$\begin{array}{ccc} X \times_{f,g} Y & \xrightarrow{p_X} & X \\ {\scriptstyle p_Y}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & A, \end{array}$$

where

$$X \times_{f,g} Y = \{(x, y) \in X \times Y | f(x) = g(y)\}.$$

If X and Y are CW-complexes, then $X \times Y$ has the structure of a CW-complex with cells being the product $e^i_\alpha \times e^j_\beta$ where e^i_α ranges over the cells of X and e^j_β ranges over the cells of Y ([Hat02, Chapter 0, p.8]). Because f and g are cellular, if there is a point $(x, y) \in X \times Y$ such that $f(x) = g(y)$, we can consider the cell $e^i_X \times e^j_Y$ such that e^i_X is a cell containing x, e^j_Y is a cell containing y, and $f(e^i_X)$ and $g(e^j_Y)$ are contained in a cell of A containing $f(x) = g(y)$. Moreover, if X and Y are CW-complexes with either X or Y locally finite, then the weak topology coincides with the product topology on $X \times Y$ (see [Bre93, Theorem 12.3]). This gives a cellular decomposition for $X \times_{f,g} Y$.

Remark 1.21. We say that a CW-complex is *locally finite* if every point $x \in X$ has a neighbourhood that intersects, in a non-trivial manner, only a finite number of the cells of X. Understanding the product of CW-complexes in the completely general case can sometimes be difficult. This is due to the fact that the product topology and the weak topology might not coincide. However, it is proved in [Hat02, Theorem A.6] that if either X or Y is locally finite, then $X \times Y$ has a CW-complex structure. In this thesis, we exclusively work with locally finite CW-complexes. Therefore, we will maintain our definition of CWTop, allowing us to perform pullbacks as in Top.

Furthermore, we may wonder about the ideal subcategory of Top in which we should work in order to have well-defined pushouts and pullbacks. For readers who are interested in this topic, the answer is given in the work of N. E. Steenrod, [Ste67].

1.6 Obstructions to the existence of pullbacks and pushouts in hTop and hTop$_*$

In previous sections, we presented pullbacks and pushouts in the categories Top and Top$_*$, but what can we say about hTop and hTop$_*$? We recall that these categories are built from Top and Top$_*$ by identifying maps under the relation of homotopy equivalence. However, as we will show, this is a source of difficulty when constructing pullbacks and pushouts, since these constructions may depend on the choice of representative in a homotopy class.

Recall that a *path* in a topological space X is a continuous map $\alpha : [0, 1] \to X$. We denote by X^I the *free path space on* X which is endowed with the compact-open topology (see Section 1.1.1). Now, let $(X, \{*\})$ be a pointed topological space. We can define the following set of paths all beginning at the same point $\{*\}$,

$$PX = \{\alpha : I \to X | \alpha(0) = *\} \subset X^I,$$

on which we consider the subspace topology. Note that this space is contractible because every path can be contracted to the constant path at $\{*\}$, namely c_*. We also define the subspace of PX consisting of loops based at $\{*\}$,

$$\Omega X = \{\alpha : I \to X | \alpha(0) = * = \alpha(1)\}.$$

Now, consider the pullback of the maps $\iota : \{*\} \hookrightarrow X$ and $\mathrm{ev} : PX \to X; \alpha \mapsto \alpha(1)$ in Top_* which is given by

$$PX \times_{\iota,\mathrm{ev}} \{*\} = \{(\alpha, *) | \mathrm{ev}(\alpha) = *\} = \{\alpha \in PX | \alpha(1) = *\} = \Omega X.$$

Hence, we have the following pullback diagram:

$$\begin{array}{ccc} \Omega X & \longrightarrow & PX \\ \downarrow & & \downarrow{\scriptstyle \mathrm{ev}} \\ \{*\} & \xrightarrow{\iota} & X. \end{array}$$

Similarly, if one takes a pair of maps (f, ι) in Top, where $f = \iota : * \hookrightarrow X$, we obtain the following pullback:

$$\begin{array}{ccc} \{(*,*)\} & \longrightarrow & \{*\} \\ \downarrow & & \downarrow{\scriptstyle f} \\ \{*\} & \xrightarrow{\iota} & X. \end{array}$$

Since PX is of the same homotopy type as $\{*\}$, i.e it is contractible, the two following diagrams are homotopy equivalent (see Section 1.1.4) :

$$\begin{array}{ccc} & & PX \\ & & \downarrow{\scriptstyle \mathrm{ev}} \\ \{*\} & \xrightarrow{\iota} & X \end{array} \qquad\qquad \begin{array}{ccc} & & \{*\} \\ & & \downarrow{\scriptstyle f} \\ \{*\} & \xrightarrow{\iota} & X. \end{array}$$

Therefore, the two last pullbacks should also represent the same diagram in hTop and thus, should be of the same homotopy type, but they are not. In fact, ΩX is not always contractible. Stating that ΩX is contractible is equivalent to saying that every loop $\gamma \in \Omega X$ can be contracted to a constant path c_* but it is not always the case (it suffice that the fundamental group of X is not trivial). This example emphasizes the fact that pullbacks in hTop cannot be defined using those in Top_* without careful choices of representatives.

Dually, let us consider suspensions. Recall that the suspension of X is the pushout of the map $\iota_{CX} : X \to CX; x \mapsto (x, 0)$ and the constant map $c_* : X \to *$,

$$\begin{array}{ccc} X & \xrightarrow{\iota_{CX}} & CX \\ {\scriptstyle c_*}\big\downarrow & & \big\downarrow \\ \{*\} & \longrightarrow & \Sigma X, \end{array}$$

where $\sim$ is the equivalence relation collapsing $X \times \{0\}$ to a single point. Additionally, we can construct the pushout of (c_*, c_*) given by the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{c_*} & \{*\} \\ {\scriptstyle c_*}\big\downarrow & & \big\downarrow \\ \{*\} & \longrightarrow & \{*\}. \end{array}$$

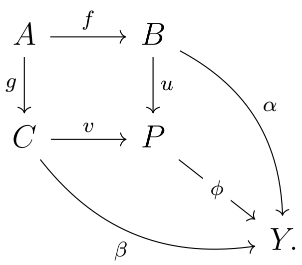
Since the cone of a space is contractible, we have that the two upper parts of the diagrams below are equivalent. Therefore, the two pushouts should be of the same homotopy type. However, it is not true that the suspension of X is contractible for all X. As an example, consider the circle, $\Sigma S^1 = S^2$ which is not contractible.

Since we cannot perform pushouts in hTop and hTop_* using the constructions already known in Top, we are forced to find solutions to approximate pushouts and pullbacks in the homotopy categories. The following definition gives a first idea of what we aim to construct.

Definition 1.22. *A weak pushout of (f, g) is a commutative square*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle u} \\ C & \xrightarrow{v} & P \end{array}$$

such that for any other pairs of arrows (α, β), $\alpha : B \to Y$, $\beta : C \to Y$ such that $\alpha \circ f = \beta \circ g$, there exists a ***not necessarily*** *unique map $\phi : P \to Y$ such that $\phi \circ u = \alpha$ and $\phi \circ v = \beta$.*



The *weak pullback* is the dual notion, i.e, it satisfies the existence property of a map but not necessarily the uniqueness.

Remark 1.23. Here, we do not consider pushouts and pullbacks directly in hTop and hTop_* because they do not exist. For readers familiar with category theory, it is important to note that not only do pullbacks and pushouts not exist, but the same applies to limits and colimits in general, except for the product and coproduct. This limitation arises from the fact that hTop is not a complete category. At the beginning of Section 2 in [Cut20], an argument against the existence of pushouts in Top_* is presented. He also mentions that the story is the same without base points.

A solution to the problem will be to construct *homotopy pushouts* and *homotopy pullbacks* in Top that always exist and are unique up to homotopy for all pairs of maps, and that will induce weak pushouts and pullbacks in hTop. In fact, considering arrows up to homotopy gives a flexibility that makes us lose the uniqueness of the map. Hence, a first way of recovering some kind of structure will be to restrict ourselves to a type of maps called *fibrations* and *cofibrations* and take the pullbacks and pushouts, respectively, in Top. Another technique will be to define two spaces that will be the homotopy pushouts and pullbacks of the maps without changing their nature.

In the next chapter, we will define all of these structures in detail, show how they behave, and finally, show that the two techniques of finding homotopy pullbacks and pushouts are equivalent.

Chapter 2

Homotopy pushouts and homotopy pullbacks

As we have seen in the last chapter, pullbacks and pushouts work very well in Top and Top_* but not in hTop and hTop_*. In fact, up to homotopy, we are not able to obtain the same pushouts or pullbacks if we change the representative in the homotopy class. Moreover, being able to consider maps and spaces up to homotopy is really interesting. For example, the mapping cylinder M_f with $f : X \to Y$ has the same homotopy type as Y. Also, the cone CX has the same homotopy type as a point. These are convenient features that we can use only up to homotopy.

The aim of this section is to define objects that always exist in hTop and hTop_* and that behave similarly to pullbacks and pushouts. In the first section, we will introduce fibrations and cofibrations, which are key elements in the construction of the homotopy pushouts and pullbacks. In the second section, we will define the most important notions. In fact, as already explained, homotopy pushouts and homotopy pullbacks are the constructions that we will use to approximate pushouts and pullbacks in hTop.

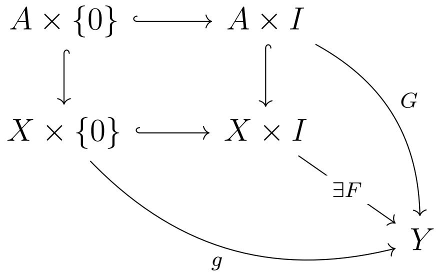
2.1 Fibrations and cofibrations

A way of approximating pushouts and pullbacks in hTop would be to restrict the maps to *fibrations* and *cofibrations*. In this section, we introduce these new concepts, give examples, and construct the pushouts and pullbacks associated with them. We will be defining all of this for topological spaces, thus we are working in Top. This section is inspired from the book [Spa81].

2.1.1 Cofibrations

We say that the inclusion map $\iota : A \to X$ has the *homotopy extension property* with respect to Y if, given maps $g : X \to Y$ and $G : A \times I \to Y$ such that $g(x) = G(x, 0)$

for $x \in A$, there is a map $F : X \times I \to Y$ such that $F(x, 0) = g(x)$ for $x \in X$ and $F|_{A \times I} = G$. If g is regarded as a map of $X \times \{0\}$ to Y, this is equivalent to having the following commuting diagram:



with the four maps in the square being inclusions.

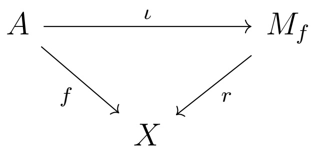
Definition 2.1. *The map $\iota : A \to X$ is a* cofibration *if it has the homotopy extension property with respect to any space. The quotient map $q : X \to \frac{X}{\iota(A)}$ is called the* cofiber *of ι.*

A nice observation is that if $f : A \to X$ is any map (not necessarily an inclusion), then it is homotopy equivalent to a cofibration. In fact, we can observe that the mapping cylinder M_f is of the same homotopy type as X and that there exists an inclusion map

$$\iota : A \to M_f; a \mapsto (a, 1).$$

The following theorem states that this inclusion is a cofibration.

Theorem 2.2. *[Spa81, Theorem 5.12, Chapter 1] Given a map $f : A \to X$, there is a homotopy-commutative diagram*



such that r is a homotopy equivalence and ι is a cofibration.

Moreover, ι is a cofibration of cofiber

$$q_f : M_f \to \frac{M_f}{\iota(A)} = X \cup_f CA = C_f,$$

where CA is the cone of A and C_f is the mapping cone of $f : X \to Y$.

Definition 2.3. *Any sequence* $A \xrightarrow{f} X \xrightarrow{p} C$ *is called a* cofiber sequence *if it is homotopy equivalent to the following sequence:*

$$A \hookrightarrow M_f \xrightarrow{q_f} C_f.$$

This can be summed up in the following homotopy commutative diagram:

$$\begin{array}{ccccc} A & \xrightarrow{f} & X & \xrightarrow{p} & C \\ & \searrow^{\iota} & \downarrow \simeq & & \downarrow \simeq \\ & & M_f & \xrightarrow{q_f} & C_f. \end{array}$$

The following figure provides an illustration of the cofiber sequence with the inclusion of the subspace A and the identification highlighted in blue.

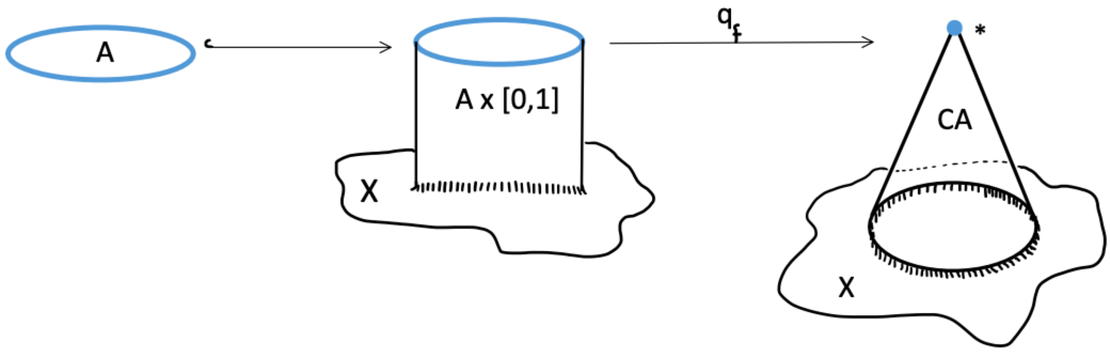


Figure 2.1: Cofiber sequence $A \hookrightarrow M_f \xrightarrow{q_f} C_f$.

As explained at the end of the last chapter, the goal is to approximate pushouts in hTop. Let $f : W \to X$ and $g : W \to Y$ be two maps in Top. Theorem 2.2 states that every continuous map f is homotopy equivalent to a cofibration. Then instead of f, consider ι the associated cofibration such that we have the following homotopy commuting diagram:

$$\begin{array}{ccc} W & \xrightarrow{\iota} & M_f \\ & {}_{f}\searrow \quad \swarrow_{\simeq} & \\ & X. & \end{array}$$

Therefore, one way to approximate pushouts in hTop is to construct the pushout of g and ι. This pushout is given by the following homotopy commutative diagram:

$$
\begin{array}{ccccc}
 & & W & \xrightarrow{\;g\;} & Y \\
 & \overset{f}{\swarrow} & & & \\
X & & \downarrow{\scriptstyle \iota} & & \downarrow \\
 & \underset{\simeq}{\searrow} & & & \\
 & & M_f & \longrightarrow & \frac{M_f \amalg Y}{\sim},
\end{array}
$$

where $\sim$ is the equivalence relation given by $\iota(w) \sim g(w)$ for all $w \in W$. This equivalence relation glues the space Y on top of the mapping cylinder using the attaching map $g : W \to Y$. This situation is illustrated on Figure 2.2.

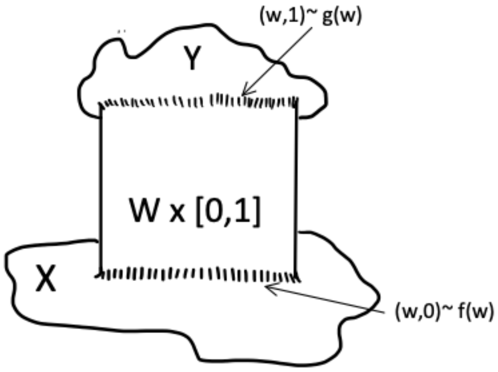


Figure 2.2: Pushout $\frac{M_f \amalg Y}{\sim}$.

In section 2.2 we will show that this construction is equivalent to the homotopy pushout which is a weak pushout in hTop.

2.1.2 CW-complexes and cofibrations

Many nice properties of CW-complexes are related to the notion of cofibration. First, recall that a CW-pair is a pair (X, A) where X is a CW-complex and A is a subcomplex of X. The following important property is proved in [Hat02, Proposition 0.16], and [Rot88, Theorem 8.33].

Proposition 2.4. *If (X, A) is a CW-pair, then it satisfies the homotopy extension property, i.e., the inclusion $\iota : A \to X$ is a cofibration.*

This proposition implies that if $x_0 \in X$, then the inclusion $x_0 \hookrightarrow X$ is a cofibration. In this case, we say that X *has a non-degenerate base point* x_0.

The following result explains that if there is a contractible subspace in a CW-complex, then taking the quotient results in the same homotopy class. This proposition is showed in [Hat02, Proposition 0.17].

Proposition 2.5. *If the pair (X, A) satisfies the homotopy extension property (i.e $A \hookrightarrow X$ is a cofibration) and A is contractible, then the quotient map $q : X \to X/A$ is a homotopy equivalence.*

In particular, if the pair (X, A) is such that X is a CW complex and A is a subcomplex of X that is contractible, then the quotient map $q : X \to X/A$ is a homotopy equivalence.

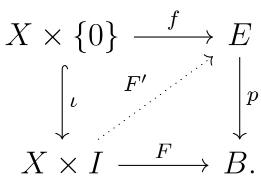
Example 2.6. Let B be a CW-complex. The cone CB is equivalent to the reduced cone CB_*. In the reduced cone, all points (b_0, t) are identified for all $t \in I$ representing a contractible path in the cone. Hence, by Proposition 2.5,

$$C_*B = \frac{CB}{\sim},$$

where $\sim$ is the equivalence relation generated by $(b_0, t) \sim (b_0, t')$ for all $t \in [0, 1]$, is of the same homotopy type as the cone CB. Notice that a similar argument can be made for the suspension ΣB and the reduced suspension $\Sigma_* B$.

2.1.3 Fibrations

To define the concept of fibration, we need to understand the *homotopy lifting property*. We say that a map $p : E \to B$ has the homotopy lifting property with respect to a space X if given maps $f : X \to E$ and $F : X \times I \to B$ such that $F(x, 0) = p \circ f(x)$ for $x \in X$, there exists a map $F' : X \times I \to E$ such that $F'(x, 0) = f(x)$ for $x \in X$ and $p \circ F' = F$. If f is regarded as a map of $X \times \{0\}$ to E, this is equivalent to the existence of the dotted arrow in the following diagram:



Note that if $p : E \to B$ has the homotopy lifting property with respect to a space X and $f, g : X \to B$ are homotopic, we can observe that f can be lifted to E if and only if g can be lifted to E. Therefore, determining whether a map $f : X \to B$ can be lifted to E depends solely on the homotopy class of the map.

Definition 2.7. *A map $p : E \to B$ is a* Hurewicz fibration *if it has the homotopy lifting property with respect to any space. We name E the* total space *and B the* base space *of the fibration. For $b \in B$, $p^{-1}(b)$ is called the* fiber over b.

In the sequel, we will not refer to $p : E \to B$ as a Hurewicz fibration, but simply as a fibration.

Proposition 2.8. *If $p : E \to B$ is a fibration and $f : B' \to B$ is a map, then the map*

$$p' : E \times_{p,f} B' \to B' : (b', e) \mapsto b'$$

with

$$E \times_{p,f} B' = \{(e, b') \in E \times B' | f(b') = p(e)\}$$

is a fibration and it is called the fibration induced from p by f.

The space $E \times_{p,f} B'$ is the fibered product of the spaces B' and E. Specifically, this constitutes a pullback in the category Top, as explained in Section 1.4.3. Here, we provide the proof of the proposition. In [Spa81], it corresponds to Property 6 in Section 8.

Proof. Let X be a topological space. We want to prove that $p' : E \times_{p,f} B' \to B'$ has the homotopy lifting property with respect to X. Let $g : X \to E \times_{p,f} B'$ and $G : X \times I \to B'$ be two maps such that $G(x, 0) = p' \circ g(x)$ for $x \in X$. Consider the following commutative diagram:

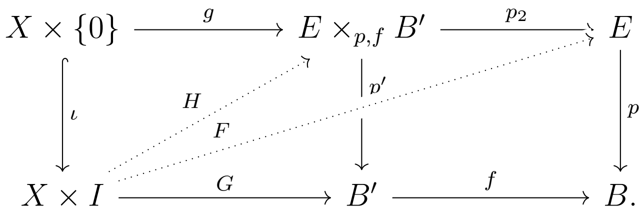
$$\begin{array}{ccccc} X \times \{0\} & \xrightarrow{\quad g \quad} & E \times_{p,f} B' & \xrightarrow{\quad p_2 \quad} & E \\ \big\downarrow{\scriptstyle \iota} & & \big\downarrow{\scriptstyle p'} & & \big\downarrow{\scriptstyle p} \\ X \times I & \xrightarrow{\quad G \quad} & B' & \xrightarrow{\quad f \quad} & B. \end{array}$$

Since $p : E \to B$ is a fibration, there exists a map $F : X \times I \to E$ such that $F(x, 0) = p_2 \circ g(x)$ for $x \in X$ and $p \circ F = f \circ G$. Define the homotopy

$$\begin{aligned} H :& X \times I \to E \times_{p,f} B' \\ & (x, t) \mapsto (G(x, t), F(x, t)), \end{aligned}$$

which satisfies $p' \circ H = G$ since p' is the projection on the first component. Moreover, $H(x, 0) = (G(x, 0), F(x, 0))$. By the commutativity of the first diagram, we have $G(x, 0) = p' \circ g(x)$ and by the homotopy lifting property of F, we have that $F(x, 0) = p_2 \circ g(x)$. Since p' and p_2 are the projections on the first and second component respectively, we obtain that

$$H(x, 0) = (G(x, 0), F(x, 0)) = (p' \circ g(x), p_2 \circ g(x)) = g(x).$$



Therefore, $p' : E \times_{p,f} B' \to B'$ has the homotopy lifting property with respect to any space, i.e., it is a fibration. □

Example 2.9 (Mapping path fibration)**.** If Y is a space, then $Y^I = \{\omega : [0,1] \to Y\}$ represents the free path space on Y. Let $p' : Y^I \to Y$ be defined by $p'(\omega) = \omega(0)$ for $\omega \in Y^I$. It can be shown that p' is a fibration (refer to [Spa81, Corollary 3, Section 8]). Let $f : X \to Y$ be any continuous map. Given that p' is a fibration, we can define the *mapping path fibration* which is the fibration induced from p' by f, denoted as $p : P_f \to X$ with

$$P_f = \{(x, \omega) \in X \times Y^I | f(x) = \omega(0)\}.$$

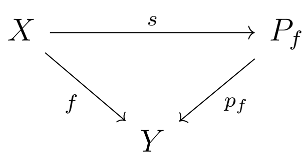
Notice that the fiber over $x \in X$ is $PX = \{\omega \in Y^I | f(x) = \omega(0)\}$.

Intuitively, this is the dual notion of the mapping cylinder. A really nice fact is that every map is homotopy equivalent to a fibration. Indeed, the space P_f constructed in the last example is the mapping path fibration associated with $f : X \to Y$. One can derive another fibration with total space P_f and with base Y:

$$p_f : P_f \to Y : (x, \omega) \mapsto \omega(1).$$

One can prove that this is a fibration with a similar argument as in Example 2.9. Thus, for any map, we have the following theorem.

Theorem 2.10. *[Spa81, Theorem 8.9 Chapter 2] Given a map $f : X \to Y$, there is a homotopy commutative diagram*



such that s is a homotopy equivalence and P_f is the mapping path fibration with the fibration $p_f : P_f \to Y : (x, \omega) \mapsto \omega(1)$.

Proof. We define $s : X \to P_f$ as the map sending a point $x \in X$ to the pair $(x, c_{f(x)}) \in P_f$ where $c_{f(x)}$ is the constant path at $f(x)$. First, we prove that s is a homotopy equivalence. Consider as homotopy inverse the projection, $p : P_f \to X; (x, \omega) \mapsto x$. We have that

$p \circ s = \mathrm{id}_X$. Now, we want a homotopy between $s \circ p(x, \omega) = (x, c_{f(x)})$ and $\mathrm{id}_{P_f} : P_f \to P_f$. Because every path begins at $f(x)$ in P_f, we can define the following homotopy:

$$\begin{aligned} H: \quad P_f \times I \quad &\to \quad P_f \\ ((x, \omega(t)), k) &\mapsto (x, \omega(kt)). \end{aligned}$$

Observe that $H((x, \omega), 0) = (x, c_{f(x)}) = s \circ p(x, \omega)$ and $H((x, \omega), 1) = (x, \omega) = \mathrm{id}_{P_f}(x, \omega)$. Therefore, s is a homotopy equivalence. Now, notice that the diagram commutes because we have chosen s such that it commutes. □

Furthermore, the fiber of this fibration p_f over the point $y_0 \in Y$ is given by

$$F_f = p_f^{-1}(y_0) = \{(x, \omega) \in X \times Y^I | f(x) = \omega(0), \omega(1) = y_0\}$$

which we call the *homotopy fiber*.

Remark 2.11. Note that the pullback in Top of the maps $\iota : \{*\} \hookrightarrow B$ such that $\iota(*) = b_0$ and $p : E \to B$ is given by

$$\{*\} \times_{\iota, p} E = \{(*, e) \in \{*\} \times E | \iota(*) = p(e)\} = \{e \in E | b_0 = p(e)\} = p^{-1}(b_0).$$

In the case where p is a fibration, we have that the preimage is the fiber, i.e., $p^{-1}(b_0) = F$. Therefore, the following commutative diagram is a pullback:

$$\begin{array}{ccc} F & \hookrightarrow & E \\ \downarrow & & \downarrow{\scriptstyle p} \\ \{*\} & \overset{\iota}{\hookrightarrow} & B. \end{array}$$

Definition 2.12. *A sequence $F \xrightarrow{j} E \xrightarrow{f} B$ is called a fibration sequence if it is homotopy equivalent to the following sequence:*

$$F_f \hookrightarrow P_f \xrightarrow{p_f} B.$$

This can be summed up in the following homotopy-commutative diagram:

$$\begin{array}{ccccc} F & \xrightarrow{j} & E & \xrightarrow{f} & B \\ \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} & \nearrow_{p_f} & \\ F_f & \xrightarrow{\iota} & P_f. & & \end{array}$$

To construct the homotopy pullbacks, we use a similar approach to what has been done for cofibrations. Let $f : X \to W$ and $g : Y \to W$ be two maps in Top. Theorem 2.10

states that every map f is homotopy equivalent to a fibration. Therefore, one way to approximate pullbacks in hTop will be to construct the pullback in Top of g and instead of f, consider p_f the equivalent fibration, resulting in the following homotopy commuting diagram:

$$\begin{array}{ccc} X & \xrightarrow{\simeq} & P_f \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle p_f} & \\ & W. & \end{array}$$

The pullback in Top is given by the following homotopy commutative diagram:

$$\begin{array}{ccc} Y \times_{p_f, g} P_f & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle g} \\ P_f & \xrightarrow{p_f} & W \\ & {\scriptstyle \simeq}\searrow \quad \nearrow{\scriptstyle f} & \\ & X & \end{array}$$

where

$$\begin{aligned} P_f \times_{p_f, g} Y &= \{(p, y) \in P_f \times Y | p_f(p) = g(y)\} \\ &= \Big\{((x, \omega), y) \in X \times W^I \times Y | f(x) = \omega(0), g(y) = \omega(1)\Big\}. \end{aligned}$$

2.1.4 Serre fibration

In this section, we introduce a particular case of fibration that can help us better understand CW-complexes.

Definition 2.13. *A map $p : E \to B$ is called a* Serre fibration *if it satisfies the homotopy lifting property with respect to all n-discs D^n.*

This definition implies that if $p : E \to B$ is a Serre fibration, the following dotted arrow exists for all $n \in \mathbb{N}$:

$$\begin{array}{ccc} D^n \times \{0\} & \xrightarrow{f} & E \\ {\scriptstyle \iota}\big\downarrow & \overset{F'}{\nearrow} & \big\downarrow{\scriptstyle p} \\ D^n \times I & \xrightarrow{F} & B. \end{array}$$

Note that, in particular, a Hurewicz fibration is a Serre fibration. However, the opposite implication does not hold in general. Moreover, a map $p : E \to B$ is a Serre fibration if it satisfies the homotopy lifting property with respect to all CW-complexes. This is not difficult to see. In fact, every CW-complex is constructed by attaching n-discs. If we can lift every disc, then we can lift the CW-complex.

2.2 Definition of homotopy pushout and pullback

While defining the notions of fibration and cofibration, we have seen a first way to approximate pushouts and pullbacks in hTop, each time considering the associated cofibration or fibration respectively.

In this section, we will define what is meant by *homotopy pushout* and *homotopy pullback* but we will also see that it is the same as the approximation that we have seen before. Many examples will also be given throughout the section.

The main reference for this section is the paper of Michael Mather, [Mat76].

2.2.1 Homotopy pushout

In this section, we define the homotopy pushout as it is done in [Cor+03] and [Mat76], and we provide some clarifying examples. Let $f : W \to X$ and $g : W \to Y$ be two maps. The *standard homotopy pushout* of these two maps is the double mapping cylinder of f and g, i.e.,

$$D_{f,g} = (X \amalg (W \times I) \amalg Y)/\sim,$$

where $(w, 0) \sim f(w)$ and $(w, 1) \sim g(w)$, $w \in W$, together with two injections $\iota_X : X \hookrightarrow D_{f,g}$ and $\iota_Y : Y \hookrightarrow D_{f,g}$ such that the square

$$\begin{array}{ccc} W & \xrightarrow{f} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \iota_X} \\ Y & \xrightarrow{\iota_Y} & D_{f,g} \end{array}$$

commutes up to a homotopy $G : W \times I \to D_{f,g}$ with $G(w, 0) = \iota_X \circ f(w)$ and $G(w, 1) = \iota_Y \circ g(w)$. Moreover, given another homotopy commutative square

$$\begin{array}{ccc} W & \xrightarrow{f} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ Y & \xrightarrow{k} & A, \end{array}$$

there exists a map

$$\phi : D_{f,g} \to A : u \mapsto \begin{cases} h(u) & \text{if } u \in X \\ H(w,t) & \text{if } u = (w,t) \in W \times I \\ k(u) & \text{if } u \in Y \end{cases},$$

where $H : W \times I \to A$ is the homotopy between $h \circ f$ and $k \circ g$. This map ϕ is such that $\phi \circ \iota_Y = k$, $\phi \circ \iota_X = h$ and $\phi \circ G = H$. We say that A is a *homotopy pushout* if the map ϕ is a homotopy equivalence. Intuitively, this map measures the deviation of the last square from being the standard homotopy pushout.

Remark 2.14. This map ϕ is not unique (not even up to homotopy). However having this map still implies that the space $D_{f,g}$ is unique up to homotopy equivalence by definition. This gives a weak pushout in hTop. Furthermore, we want to show that if we have two homotopy equivalent diagrams $Y \overset{g}{\leftarrow} W \overset{f'}{\to} X$ and $Y' \overset{g'}{\leftarrow} W' \overset{f}{\to} X'$, then $D_{f,g} \simeq D_{f',g'}$. The situation can be illustrated by

$$\begin{array}{ccccc} X & \xleftarrow{f} & W & \xrightarrow{g} & Y \\ \downarrow{\scriptstyle\psi_1} & & \downarrow{\scriptstyle\psi_2} & & \downarrow{\scriptstyle\psi_3} \\ X' & \xleftarrow{f'} & W' & \xrightarrow{g'} & Y' \end{array}$$

where the two square homotopy commutes via two homotopies $H : W \times I \to X'$ and $H' : W \times I \to Y'$ respectively and the maps ψ_1, ψ_2 and ψ_3 are homotopy equivalences. Define the following map:

$$\Psi : D_{f,g} \to D_{f',g'} : u \mapsto \begin{cases} \psi_1(u) & \text{if } u \in X \\ H(w,3t) & \text{if } u = (w,t) \in W \times I, 0 \le t \le 1/3 \\ (\psi_2(w), 3t-1) & \text{if } u = (w,t) \in W \times I, 1/3 \le t \le 2/3 \\ H'(w,3t-2) & \text{if } u = (w,t) \in W \times I, 2/3 \le t \le 1 \\ \psi_3(u) & \text{if } u \in Y. \end{cases}$$

Then, Theorem 6.2.8 of [Ark11] states that if ψ_1, ψ_2 and ψ_3 are homotopy equivalences, then, Ψ is a homotopy equivalence. This implies that $D_{f,g} \simeq D_{f',g'}$. Therefore, any two homotopy equivalent diagrams will have the same homotopy pushout.

Remark 2.15. The double mapping cylinder

$$D_{f,g} = (X \amalg (W \times I) \amalg Y)/\sim,$$

where the equivalence relation $\sim$ is generated by $(w,0) \sim f(w)$ and $(w,1) \sim g(w)$, $w \in W$, is exactly the topological space obtained in Section 2.1.1. In fact, this equivalence relation glues the space Y on top of the cylinder of W and glues X on the bottom of the cylinder. But this is exactly the same as gluing Y to the top of the mapping cylinder using the attaching map $g : W \to Y$. This situation is illustrated on Figure 2.2.

Example 2.16. We define here several special topological spaces that are obtained using homotopy pushouts and that we will encounter many times in the following pages of this thesis.

(1) In Top_*, we can construct the homotopy pushout of the pair of maps $(\{*\} \hookrightarrow X, \{*\} \hookrightarrow Y)$ which is

$$D = \frac{X \amalg (* \times I) \amalg Y}{\sim}.$$

We have the following diagram:

$$\begin{array}{ccc} \{*\} & \hookrightarrow & (X, x_0) \\ \downarrow & & \downarrow \\ (Y, y_0) & \longrightarrow & D. \end{array}$$

Moreover, consider the following homotopy commutative diagram

$$\begin{array}{ccc} \{*\} & \hookrightarrow & (X, x_0) \\ \downarrow & & \downarrow \iota_1 \\ (Y, y_0) & \xrightarrow{\iota_2} & (X \vee Y, x_0 = y_0), \end{array}$$

where ι_1 and ι_2 are the inclusions in the wedge. We have that the map

$$\phi : D \to X \vee Y : u \mapsto \begin{cases} \iota_1(u) & \text{if } u \in X \\ x_0 = y_0 & \text{if } u = (*, t) \in \{*\} \times I \\ \iota_2(u) & \text{if } u \in Y \end{cases}$$

is a homotopy equivalence. Hence, $X \vee Y$ is a homotopy pushout of the pair of maps $(\{*\} \hookrightarrow X, \{*\} \hookrightarrow Y)$.

(2) The mapping cone of a map in Top is simply the homotopy pushout of $f : X \to Y$ and $c_* : X \to \{*\}$. In fact, we obtain the following pushout square :

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ c_* \downarrow & & \downarrow \\ \{*\} & \longrightarrow & Y \cup_f CX. \end{array}$$

The cell attachment is the same construction as the mapping cone up to homotopy. In fact, with $X = S^{n-1}$, a cell attachment is obtained with the following pushout in Top :

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{f} & X \\ \big\downarrow & & \big\downarrow \\ D^n & \longrightarrow & X \cup_f e^n. \end{array}$$

Since $D^n \simeq \{*\} \simeq CS^n$, we have $Y \cup_f e^n \simeq Y \cup_f CS^n$.

(3) Let $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ be the projections in Top. Their homotopy pushout will be given by

$$D_{p_X, p_Y} = (X \amalg ((X \times Y) \times I) \amalg Y)/\sim,$$

where $((x, y), 0) \sim x$ and $((x, y), 1) \sim y$, $(x, y) \in X \times Y$. This pushout is denoted by $X * Y$ and named the *join of* X *and* Y. We obtain the following commutative diagram:

$$\begin{array}{ccc} X \times Y & \xrightarrow{p_X} & X \\ {\scriptstyle p_Y}\big\downarrow & & \big\downarrow{\scriptstyle \iota_X} \\ Y & \xrightarrow{\iota_Y} & X * Y. \end{array}$$

(4) If $f = g : B \to \{*\}$ are maps in Top, then $D_{f,g}$ is the cylinder $B \times I$ with all points in $t = 0$ and $t = 1$ identified. We recognize this as the suspension of B, ΣB. This implies that the square

$$\begin{array}{ccc} B & \xrightarrow{f} & \{*\} \\ {\scriptstyle g}\big\downarrow & & \big\downarrow \\ \{*\} & \longrightarrow & \Sigma B \end{array}$$

is a homotopy pushout. Note that this was the counterexample we gave in Section 1.6, but now the construction works great because if we take the pushout with the cone of B, we obtain

$$\begin{array}{ccc} B & \xrightarrow{f} & CB \\ {\scriptstyle g}\big\downarrow & & \big\downarrow \\ \{*\} & \longrightarrow & \Sigma B \end{array}$$

which is considered as same diagram as the previous one in hTop because they are homotopy equivalent in Top. In particular, we still have that $\Sigma S^n \simeq S^{n+1}$ with this construction. Moreover, consider the following homotopy pushout:

$$\begin{array}{ccc} \emptyset & \longrightarrow & \{*\} \\ \big\downarrow & & \big\downarrow \\ \{*\} & \longrightarrow & \Sigma\emptyset. \end{array}$$

By definition,

$$D = \frac{\{*\} \amalg (\emptyset \times I) \amalg \{*\}}{\sim} = \{*\} \amalg \{*\}.$$

Therefore, one concludes that the suspension of the empty set is the disjoint sum of two singletons and we see that $\Sigma\emptyset \simeq \{*\} \amalg \{*\} \simeq S^0$. Notice that in Top, we do not obtain the same result. Suspension where defined as the pushout using the cone and the cone of the empty set is still the empty set.

Finally, Remark 2.6 explains that $\Sigma X \simeq \Sigma_* X$. It is important to note that these spaces are isomorphic in hTop and that the reduced suspension is also the homotopy pushout of $f = g : B \to \{*\}$.

(5) We have that $Z \xrightarrow{f} X \to Y$ is a cofiber sequence if and only if the following square is a homotopy pushout:

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ \{*\} & \longrightarrow & Y. \end{array}$$

In fact, if the sequence $Z \xrightarrow{f} X \to Y$ is a cofiber sequence, then $Y \simeq X \cup_f CZ = C_f$. Moreover, the homotopy pushout of f and $Z \to \{*\}$ is given by

$$D_{f,*} = (X \amalg (Z \times I) \amalg \{*\}) / \sim$$

where $(z, 0) \sim f(x)$ and $(z, 1) \sim \{*\}$ for $z \in Z$. This is equivalent to $C_f = X \cup_f CZ$. Therefore, if the square is a homotopy pushout, then $Y \simeq D_{f,*} \simeq C_f$ and we have a cofiber sequence. Conversely, if $Y \simeq C_f$, then since $D_{f,*} \simeq C_f$, we have that the square is a homotopy pushout.

2.2.2 Homotopy pullback

We finally arrive at the definition of homotopy pullback. We will explain it as it is done in [Cor+03] and in [Mat76].

Let $f : B \to D$ and $g : C \to D$ be two maps. The *standard homotopy pullback* of these two maps is a space

$$P_{f,g} = \{(b, \omega, c) \in B \times D^I \times C | f(b) = \omega(0), g(c) = \omega(1)\}$$

together with two projections $pr_B : P_{f,g} \to B$ and $pr_C : P_{f,g} \to C$ such that the square

$$\begin{array}{ccc} P_{f,g} & \xrightarrow{pr_B} & B \\ {\scriptstyle pr_C}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow{g} & D \end{array}$$

commutes up to a homotopy $G : P_{f,g} \times I \to D$ such that $G((b, \omega, c), 0) = f(b)$ and $G((b, \omega, c), 1) = g(c)$ for $(b, \omega, c) \in P_{f,g}$. Moreover, given another homotopy commutative square

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow{g} & D \end{array}$$

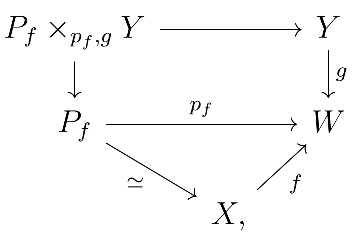
there exists a map

$$\phi : A \to P_{f,g} : a \mapsto (h(a), H(a,t), k(a)),$$

where $H : A \times I \to D$ is the homotopy between $f \circ h$ and $g \circ k$. This map ϕ satisfies $pr_B \circ \phi = h$, $pr_C \circ \phi = k$, and $G \circ (\phi \times \mathrm{id}) = H$. We say that A is a *homotopy pullback* if the map ϕ is an equivalence. In [Mat76], this map is called *whisker map*.

Remark 2.17. Note that this map ϕ is not unique (not even up to homotopy). However, it implies that the space $P_{f,g}$ is unique up to homotopy equivalence. This gives a weak pullback in hTop. Furthermore, we want to show that if we have two homotopy-equivalent diagrams, namely $Y \xleftarrow{g} W \xrightarrow{f} X$ and $Y' \xleftarrow{g'} W' \xrightarrow{f'} X'$, then $P_{f,g} \simeq P_{f',g'}$. This statement is true, and the reasoning behind it is analogous to the one presented in Remark 2.14. The proof can be found in [Ark11, Theorem 6.2.16].

Remark 2.18. The standard homotopy pullback of two maps $f : X \to W$ and $g : Y \to W$ is equivalent to the pullback obtained at the end of section 2.1.3. In fact, the pullback in Top is given by the following homotopy commutative diagram:



where

$$\begin{aligned} P_f \times_{p_f,g} Y &= \{(p, y) \in P_f \times Y | p_f(p) = g(y)\} \\ &= \left\{((x, \omega), y) \in X \times W^I \times Y | f(x) = \omega(0), \omega(1) = g(y)\right\} \\ &= P_{f,g} \end{aligned}$$

Therefore, taking the pullback considering the associated fibration is equivalent to applying the definition of the standard homotopy pullback.

Example 2.19. (1) The homotopy pullback of the maps $(X \to \{*\}, Y \to \{*\})$ is given by

$$P = \left\{(x, \omega, y) \in X \times \{*\}^I \times Y | \omega(0) = *, \omega(1) = *\right\}.$$

Since $\{*\}^I$ implies that we only have the constant path at $\{*\}$, the conditions $* = \omega(0)$ and $\omega(1) = *$ are trivial. We obtain the following diagram:

$$\begin{array}{ccc} X \times Y & \xrightarrow{pr_X} & X \\ {\scriptstyle pr_Y}\big\downarrow & & \big\downarrow \\ Y & \longrightarrow & \{*\}. \end{array}$$

(2) In Top, we have that if $F \to E \xrightarrow{p} X$ is a fibration sequence if and only if the following square is a homotopy pullback:

$$\begin{array}{ccc} F & \longrightarrow & E \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ \{*\} & \xrightarrow{\iota} & X \end{array}$$

In fact, if $F \to E \xrightarrow{p} X$, then $F \simeq F_p$. Moreover, the homotopy pullback of p and the inclusion of $\{*\}$ is

$$\begin{aligned} P_{p,\iota} &= \{(e, \theta, *) \in E \times X^I \times \{*\} | \theta(0) = p(e), \theta(1) = *\} \\ &= \{(e, \theta) \in E \times X^I | p(e) = \theta(0), * = \theta(1)\} \\ &= F_p. \end{aligned}$$

Hence, if the square is a homotopy pullback, $F_p \simeq F$. Conversely, if we have a fibration sequence, then $F \simeq F_p = P_{p,\iota}$ and the square is a pullback.

(3) Let (B, b_0) be a pointed space. If $f = g : \{*\} \to B$ in Top_*, then

$$P_{f,g} = \left\{(*, \omega, *) \in \{*\} \times B^I \times \{*\} | \omega(1) = * = \omega(0)\right\}.$$

Thus the square

$$\begin{array}{ccc} \Omega B & \longrightarrow & \{*\} \\ \big\downarrow & & \big\downarrow{\scriptstyle f} \\ \{*\} & \xrightarrow{g} & B \end{array}$$

is a homotopy pullback. Note that this was the counterexample used in Section 1.6, and now this homotopy pullback is working perfectly fine. In fact, recall that

$$PB = \{\gamma \in B^I | \gamma(0) = *\}$$

and because $PB \simeq \{c_*\}$, we obtain the following homotopy pullback in Top_*:

$$\begin{array}{ccc} \Omega B & \longrightarrow & PB \\ \big\downarrow & & \big\downarrow{\scriptstyle p_\iota} \\ \{*\} & \xrightarrow{\ g\ } & B, \end{array}$$

where $\mathrm{ev} : PX \to X; \gamma \mapsto \gamma(1)$ is the evaluation map. This pushout is, in particular, a fibration by (2). We name it *the path fibration* .

Chapter 3

Geometric category

Once the context is established, we can begin looking for homotopy invariants. The goal of this thesis is to provide an introduction to what a homotopy invariant is, as well as demonstrate how we might infer homotopy invariants from other features that are not. In this chapter, we will be working in the same categories that were introduced in Chapter 1.

In keeping with the idea of using the simplest spaces to analyze other topological spaces, a good starting point for finding homotopy invariants is to focus on contractible spaces, which are the simplest spaces in homotopy theory. This is the goal of the *geometric category*, first defined by Fox in 1941 in [Fox41]. We will use contractible spaces to find covers of other topological spaces. In the first section of this chapter, we will define this concept and then proceed to show some of its properties. This will help the reader understand how to use or find it. The second section of the chapter is devoted to proving that the geometric category is not a homotopy invariant. In fact, the first definition given will not be a homotopy invariant, and in the two subsequent chapters, we will present solutions for transforming it into a homotopy invariant.

3.1 Definition and properties

First of all, let us formalize the definition of the geometric category of a topological space X. Recall that a contractible space is one that has the same homotopy type as a point. One might ask how many contractible spaces are needed to create a cover of X? More precisely, what is the *least* natural number of contractible open sets we need to obtain a cover of X? This is precisely the definition of the geometric category:

Definition 3.1. *Let X be a topological space. The* geometric category *of X, denoted* $\mathrm{gcat}(X)$*, is the minimal $m \in \mathbb{N}$ such that there exists a cover of X with $m+1$ open, contractible subsets.*

If there is not such cover of X, we write $\mathrm{gcat}(X) = \infty$.

Example 3.2. Here are a few examples:

1) If X is contractible, then we can take X itself as an open cover, and thus $\mathrm{gcat}(X) = 0$. Conversely, if $\mathrm{gcat}(X) = 0$, then X is contractible because it can be covered with only one open set. Therefore, X is contractible if and only if $\mathrm{gcat}(X) = 0$.

2) We know that we need two discs to cover the sphere. Indeed, we can take one for the northern hemisphere and one for the southern hemisphere, as shown in the following Figure:

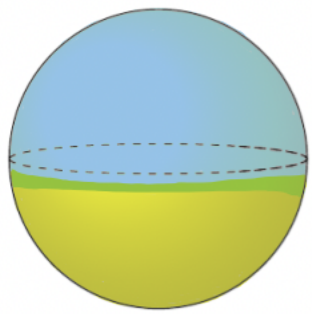


Figure 3.1: Cover of the sphere.

This implies that $\mathrm{gcat}(S^2) \leq 1$. However, since the sphere is not contractible, we conclude that $\mathrm{gcat}(S^2) = 1$.

Intuitively, we want that if a space is the union of two other open spaces, then the geometric category is less than or equal to the sum of the geometric category of the two spaces. This is what the following proposition states:

Proposition 3.3. *[Fox41, Proposition 1.27] Let $Y_1, Y_2 \subseteq X$ be open subspaces of X such that $X = Y_1 \cup Y_2$. Then*

$$\mathrm{gcat}(X) \leq \mathrm{gcat}(Y_1) + \mathrm{gcat}(Y_2) + 1.$$

Proof. Suppose $\mathrm{gcat}(Y_1) = k$ and $\mathrm{gcat}(Y_2) = l$, with associated contractible open covers $\{U_i\}_{i \in \{1,\dots,k+1\}}$ and $\{V_j\}_{j \in \{1,\dots,l+1\}}$ of Y_1 and Y_2 respectively. Then $\{U_i, V_j\}$ with $i = 1, \dots, k+1$, and $j = 1, \dots, l+1$ is a contractible cover of the union $X = Y_1 \cup Y_2$. Because Y_1, Y_2 are open in X, the V_i and U_i are also open in X. Hence, $\mathrm{gcat}(X) \leq k + 1 + p + 1 - 1 = \mathrm{gcat}(V_1) + \mathrm{gcat}(V_2) + 1$. □

Some other features of the geometric category follow from this proposition.

Proposition 3.4. *[Cor+03, Proposition 3.2] The following properties are valid:*

1. *Let $D_{g,f}$ be the homotopy pushout of the maps $g : W \to Y$ and $f : W \to X$. Then,*

$$\mathrm{gcat}(D_{g,f}) \leq \mathrm{gcat}(Y) + \mathrm{gcat}(X) + 1.$$

2. *If X is a path-connected, finite CW-complex of dimension n, then*

$$\mathrm{gcat}(X) \leq n.$$

Proof. 1. Recall that the homotopy pushout of the maps $g : W \to Y$ and $f : W \to X$ is the double mapping cylinder given by

$$D_{f,g} = (X \amalg (W \times I) \amalg Y)/\sim,$$

where $(w, 0) \sim f(w)$ and $(w, 1) \sim g(w)$, $w \in W$. Let $U = X \cup_f (W \times [0, 2/3))$ and $V = Y \cup_g (W \times (1/3, 1])$ be two open sets in $D_{f,g}$ such that $D_{f,g} = U \cup V$.

Now, if $\{\mathcal{U}_i\}_{i\in\{0,\ldots,k\}}$ is a contractible cover of X, we want to enlarge the open sets to obtain a cover of U. Suppose that $\mathrm{Im}(f) \cap \mathcal{U}_i \neq \emptyset$. Then, consider the set $W|_{\mathcal{U}_i} = \{w \in W | f(w) \in \mathcal{U}_i\}$. Now, we simply enlarge the initial open sets in the following way :

$$\mathcal{U}'_i = \mathcal{U}_i \cup_{\mathrm{Im}(f|_{\mathcal{U}_i})} (W|_{\mathcal{U}_i} \times [0, 2/3)) .$$

These new open sets are contractible because $\mathcal{U}_i \cup_{\mathrm{Im}(f|_{\mathcal{U}_i})} (W|_{\mathcal{U}_i} \times [0, 2/3))$ can be contracted to $\mathcal{U}_i$, which is contractible by assumption. Note that if $\mathrm{Im}(f) \cap \mathcal{U}_i = \emptyset$, then $\mathcal{U}'_i = \mathcal{U}_i$. Hence, $\{\mathcal{U}'_i\}_{i\in\{0,\ldots,k\}}$ is a contractible cover for U and the quantity of open sets in this cover is the same as in the cover for X. A similar argument hold for V.

Therefore, $\mathrm{gcat}(U) \leq \mathrm{gcat}(X)$ and $\mathrm{gcat}(V) \leq \mathrm{gcat}(Y)$. We conclude that

$$\mathrm{gcat}(D_{f,g}) \leq \mathrm{gcat}(U) + \mathrm{gcat}(V) + 1 \leq \mathrm{gcat}(X) + \mathrm{gcat}(Y) + 1.$$

2. First, let us observe that even if X is a path-connected CW-complex, the set $X^{(0)}$ consists of points and is not connected. However, $X^{(1)}$ will be connected if we want X to be connected in the end. This is because cells of higher dimensions are attached only to one connected component by construction. Now, let us assume that $\dim(X) = 1$. Recall Example 1.14(1) where we explained that a 1-dimensional CW-complex is a graph. In this graph, we allow for multiple edges between two vertices and also for edges that have the same vertices (loops), as illustrated in Figure 1.6. In every graph G, there exists a maximal tree, i.e., a subgraph that contains all the vertices of G (see [Bre93, Lemma 7.10]). It is important to note that a tree is a contractible subset of X. Let T_1 and T_2 be two such trees in X. As an example, we have represented two maximal trees T_1 in red and T_2 in green in Figure 3.2 which is the graph of Example 1.14(1).

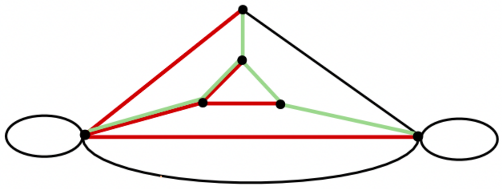


Figure 3.2: Two maximal trees of X.

Now, we are going to extend T_1 and T_2 in a way that they remain contractible. Let $E = G \backslash \{T_1, T_2\}$ be the set of edges that are not part of the trees. We will extend T_1 slightly at the endpoints of each edge $e \in E$. In other words, we add a small segment within each edge of E, starting at a vertex $v \in T_1$ and ensuring that the segments for the same edge do not intersect. If $e \in E$ is a loop, we add two segments starting at the same vertex. Intuitively, to extend T_2, we need to cover all the parts that are not already covered. More formally, let S be the set of all the added segments. For every $s \in S$, we choose a point p_s on s. Now, for a vertex $v \in T_2$ and an edge $e \in E$ with vertices v and w, and with associated added segments s_1 and s_2, we extend T_2 at v by adding a segment that includes p_{s_1} and p_{s_2}, starting at v but not including w. If $e \in E$ is a loop, we add a segment that starts at v, contains p_{s_1} and p_{s_2}, and ends before reaching v again. These two ways of extending the trees are illustrated in Figure 3.3.

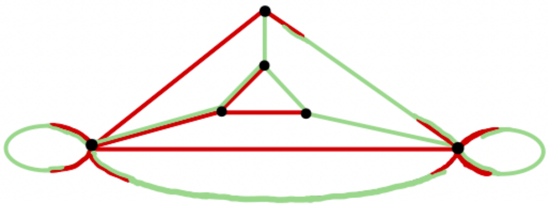


Figure 3.3: Extended trees.

Because all we did was to add some segments that can be contracted, the extended trees T_1' and T_2' remain contractible. Moreover, their union covers X. By taking a small open neighbourhood of T_1' and T_2', we obtain two contractible open subsets such that their union covers X. This implies that $\text{gcat}(X) \leq 1$.

Now, suppose we have a path-connected CW-complex X with $\dim(X) = n \geq 2$. Let U_i be the disjoint union of the interiors of all the i-dimensional cells of X, $2 \leq i \leq n$. This is a disjoint union of contractible sets. Consider the graph $X^{(1)}$ associated with X. We choose a maximal tree, denoted as T, contained in X, and extend it in the same manner as T_1 mentioned earlier. Next, we magnify the edges in X until we find that $U_i \cap T' \neq \emptyset$, where T' is the extended and enlarged tree. In this case, $U_i \cup T'$ is a contractible subset of X. By taking a small open neighbourhood of this union we obtain a contractible open subset of X, denoted as N_i. Now, if $T_1', T_2' \subseteq X^{(1)}$ are contractible and form a covering of $X^{(1)}$ (as done in the first part of the proof), we have that

$$\{T_1, T_2, N_2, \ldots, N_n\}$$

is a cover of X containing contractible open subsets. Therefore, $\mathrm{gcat}(X) \leq n$.

□

From this last result, we can deduce the geometric category of a suspension.

Proposition 3.5. *Let X be a topological space, then,* $\mathrm{gcat}(\Sigma X) \leq 1$.

Proof. Recall that $\Sigma X = D_{c_*,c_*}$ where D_{c_*,c_*} is the homotopy pushout in hTop of twice the constant map $c_* : X \to \{*\}$,

$$\begin{array}{ccc} X & \xrightarrow{c_*} & \{*\} \\ {\scriptstyle c_*}\downarrow & & \downarrow \\ \{*\} & \longrightarrow & \Sigma X. \end{array}$$

Moreover, the geometric category of a point is zero. Finally, by Proposition 3.4 we obtain

$$\mathrm{gcat}(\Sigma X) \leq \mathrm{gcat}(\{*\}) + \mathrm{gcat}(\{*\}) + 1 = 1.$$

□

Remark: If X is a contractible space, then, the suspension of X is also contractible. Therefore, $\mathrm{gcat}(\Sigma X) = 0$.

We now consider the case of fibrations.

Proposition 3.6. *[Cor+03, Proposition 3.2] If $F \to E \xrightarrow{p} B$ is a locally-trivial fibration, then,*

$$\mathrm{gcat}(E) \leq (\mathrm{gcat}(F) + 1)(\mathrm{gcat}(B) + 1) - 1.$$

Moreover, $\mathrm{gcat}(E/F) \leq \mathrm{gcat}(B)$.

The proof is very similar to those done for Theorem 1.41 and Proposition 1.44 in [Cor+03].

Sketch of the proof. The idea behind the proof of the first inequality is as follows: if we have $\text{gcat}(F) = n$ with $\{V_j\}_{j\in\{0,\dots,n\}}$ and $\text{gcat}(B) = m$ with $\{\mathcal{U}_i\}_{i\in\{0,\dots,m\}}$ contractible open covers, we can use the homotopy lifting property to construct contractible covers of $p^{-1}(\mathcal{U}_i)$ for all $i = 0,\dots,m$. The cover will be obtained from the contractible open sets V_j and will be of size $n+1$. Hence, we have a cover for each $\mathcal{U}_i$, i.e., $m+1$ covers of size $n+1$, and their union is a cover of E because $\{\mathcal{U}_i\}$ is a cover of B. Moreover, one show that each open is contractible. Consequently, it is a cover. The second inequality is proven by showing that $\{q(p^{-1}(\mathcal{U}_i))\}$, with $q : E \to E/F$, is a cover of E/F. □

3.2 The geometric category is not a homotopy invariant

In order to show that $\text{gcat}(X)$ is not a homotopy invariant, we will give an example of topological spaces that are of the same homotopy type but that do not have the same geometric category. This counterexample is inspired by [Fox41, §40] and [Cor+03, Proposition 3.11]. We define the topological space J_1 to be the wedge product of two circles and a sphere,

$$J_1 = S^2 \vee S^1 \vee S^1.$$

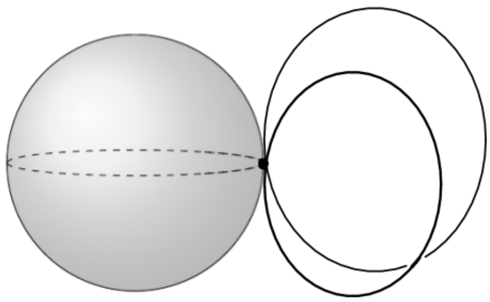


Figure 3.4: J_1.

We take J_2 to be the sphere with three distinct points identified,

$$J_2 = \frac{S^2}{\{p_1, p_2, p_3\}}.$$

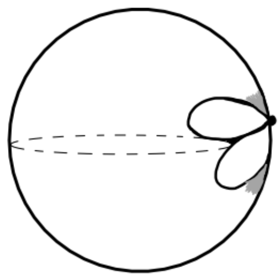


Figure 3.5: J_2.

The first part of this section is devoted to showing that J_1 and J_2 have the same homotopy type. In the second part, we prove that $\text{gcat}(J_1) = 1$ and $\text{gcat}(J_2) = 2$.

3.2.1 Identification of points in CW-complexes

Understanding the obstruction that we add when we identify points in CW-complexes is an interesting problem. We first analyze this for one of the simplest CW-complexes: the sphere.

Proposition 3.7. *Take $m \in \mathbb{N}^*$. The quotient space*

$$S^m/\{p_1, p_2, \ldots, p_n\}$$

i.e., the m-dimensional sphere with n identified points, is of the same homotopy type as

$$S^m \vee \overbrace{S^1 \vee S^1 \vee \cdots \vee S^1}^{n-1}$$

namely, the wedge sum of an m-sphere with $n-1$ circles.

Proof. If $n = 1$, then the space $S^m/\{p_1\}$ is homeomorphic to S^m. We consider the case $n = 2$. Set

$$\mathcal{A} = \frac{S^m}{\{p_1, p_2\}}, \qquad \mathcal{B} = S^m \vee S^1,$$

and

$$\mathcal{C} = S^m \cup_\iota e^1$$

where $\iota : S^0 \to S^m$ is the inclusion of -1 and 1 at $(0, \ldots, 0, -1)$ and $(0, \ldots, 0, 1)$ respectively.

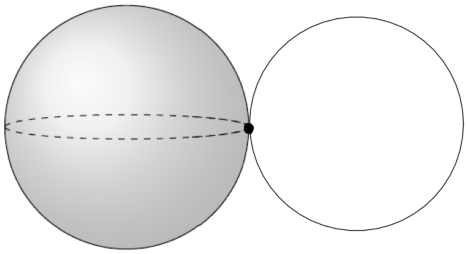


Figure 3.6: Space $\mathcal{B}$, $m = 2$.

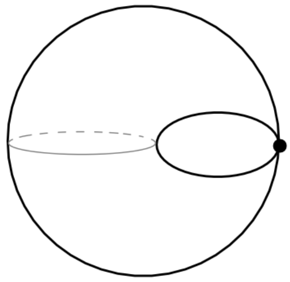


Figure 3.7: Space $\mathcal{A}$, $m = 2$.

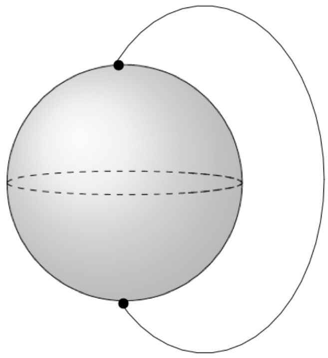


Figure 3.8: Space $\mathcal{C}$, $m = 2$.

In order to show that $\mathcal{A} \simeq \mathcal{C}$, we use the fact that e^1 is a 1-cell and $\mathcal{C}$ a CW-complex. Because e^1 is a contractible subcomplex of $\mathcal{C}$, Proposition 2.5 implies that the quotient map $q : \mathcal{C} \to \mathcal{C}/e^1$ is a homotopy equivalence. Thus $\mathcal{A} \simeq \mathcal{C}$.
We show that $\mathcal{B} \simeq \mathcal{C}$ in a similar way. Consider γ_{p_1,p_2} as an arc in the sphere such that $\gamma_{p_1,p_2}(0) = p_1$ and $\gamma_{p_1,p_2}(1) = p_2$ (note in the case where p_1 is the south pole and p_2 is the north pole, then the proof is independent of the choice of the arc). Again, the image of γ_{p_1,p_2} is a 1-cell in the sphere and, in particular, a subcomplex of $\mathcal{C}$. Therefore, the quotient map $q : \mathcal{C} \to \frac{\mathcal{C}}{\gamma_{p_1,p_2}([0,1])}$ is a homotopy equivalence. Because $\frac{\mathcal{C}}{\gamma_{p_1,p_2}([0,1])}$ is homeomorphic to $\mathcal{B}$, we conclude that $\mathcal{B} \simeq \mathcal{C}$. By the transitivity of the equivalence relation $\simeq$, we can conclude that $\mathcal{A} \simeq \mathcal{B}$.

Suppose that the proposition is true for n points on the sphere. We now show that it is still true for $n+1$ identified points. Note that by the induction hypothesis :

$$S^m/\{p_1, p_2, \ldots, p_n, p_{n+1}\} \simeq S^m/\{p_*, p_{n+1}\} \vee \overbrace{S^1 \vee S^1 \vee \cdots \vee S^1}^{n-1}$$

where p_* is such that $[p^*] = [p_1] = \cdots = [p_{n+1}]$ in $S^m/\{p_1, p_2, \ldots, p_n, p_{n+1}\}$, i.e., p^* is the point where all the $p_i, i = 1, \ldots, n+1$ are collapsed. To complete the proof, we just need to show that:

$$S^m/\{p_*, p_{n+1}\} \vee \overbrace{S^1 \vee S^1 \vee \cdots \vee S^1}^{n-1} \simeq S^m \vee \overbrace{S^1 \vee S^1 \vee \cdots \vee S^1}^{n}.$$

We define the spaces

$$\mathcal{A} = \frac{S^m}{\{p^*, p_{n+1}\}} \vee \overbrace{S^1 \vee S^1 \vee \cdots \vee S^1}^{n-1}, \qquad \mathcal{B} = S^m \vee \overbrace{S^1 \vee S^1 \vee \cdots \vee S^1}^{n},$$

and

$$\mathcal{C} = S^m \cup_\iota e^1 \vee \overbrace{S^1 \vee S^1 \vee \cdots \vee S^1}^{n-1}$$

where $\iota : S^0 \to S^m$ is the inclusion of -1 and 1 at $(0, \dots, 0, -1)$ and $(0, \dots, 0, 1)$ respectively. We can prove that $\mathcal{B} \simeq \mathcal{C}$ using an analogous proof to the case $n = 2$, by taking the arc $\gamma_{p^*, p_{n+1}}$ as a contractible subcomplex in $\mathcal{C}$ and then applying Proposition 2.5. Similarly, we can also prove that $\mathcal{A} \simeq \mathcal{C}$ by taking I as a subcomplex in $\mathcal{C}$ and applying Proposition 2.5. Finally, we get

$$S^m/\{p_1, p_2, \dots, p_{n+1}\} \simeq S^m \vee \overbrace{S^1 \vee S^1 \vee \cdots \vee S^1}^{n},$$

using the transitivity of the relation $\simeq$ (see 1.3). □

With Proposition 3.7, we can see that

$$J_2 = \frac{S^2}{\{p_1, p_2, p_3\}} \simeq S^2 \vee S^1 \vee S^1 = J_1.$$

Remark 3.8. We have proved this for the m-dimensional sphere, which can be viewed as a point with an attached m-cell. Now, we suppose that we have X, a path-connected CW-complex, and we consider $Y = X \cup_f e^n$, which is X with an attached n-cell. The interior of the n-cell is homeomorphic to a hemisphere of an n-sphere, and in Proposition 3.7, the chosen points are arbitrary. Thus, if we identify two points on the interior of the cell, $y_1, y_0 \in e^n$, then $Y/\{y_1 \sim y_0\} \simeq Y \vee S^1$.

3.2.2 Geometric category of J_1 and J_2

The second step is to show that gcat(J_2)=2 and gcat(J_1)=1. It is easy to see that J_1 can be covered with two contractible open sets each of which is the union of an open half-sphere and two open semicircles containing the base point. Because J_1 is not contractible, we conclude that gcat(J_1)=1.

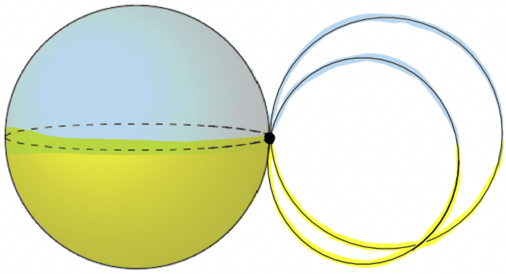


Figure 3.9: Categorical cover of J_1.

However, showing that $\mathrm{gcat}(J_2) = 2$ requires more work.

Proposition 3.9. *The geometric category of J_2 is 2.*

In order to prove this proposition, we require a lemma that Fox showed in [Fox41] for the case $n = 2$.

Lemma 3.10. *[Fox41, (37)] Let Y and X be CW-complexes, where X is obtained from Y by identifying n points $y_1, \dots, y_n \in Y$ with the quotient map $q : Y \to X : y \mapsto [y]$. If B is a contractible subset of X, then $q^{-1}(B)$ is either a contractible subset that does not contain any of the y_i, or it is the union of disjoint contractible subsets A_1 to A_n such that $y_1 \in A_1$, $y_2 \in A_2$, $\dots$, $y_n \in A_n$.*

Proof. If $n = 1$, then X is homeomorphic to Y, so there is nothing to prove. Suppose $n = 2$ and set $* = q(y_1) = q(y_2)$. Since q is just identifying two points, $q|_{Y\setminus\{y_1,y_2\}} : Y \to X$ is a homeomorphism onto its image. Therefore, if $* \notin B$, $q^{-1}(B)$ is homeomorphic to B. In this case, since B is contractible, $q^{-1}(B) \subseteq Y$ is also contractible.

From now, let us assume that $* \in B$. First, suppose that $q^{-1}(B)$ has a connected component $\mathcal{C}$ that does contain neither y_1 nor y_2. Then, $q(\mathcal{C}) \cong \mathcal{C} \in B$. Let $\mathcal{C}'$ be another connected component of $q^{-1}(B)$. Since B is connected, $q(\mathcal{C}) \cap q(\mathcal{C}'\setminus\{*\}) \neq \emptyset$. This means there exists $b_0 \in q(\mathcal{C}) \cap q(\mathcal{C}'\setminus\{*\})$. However, $q^{-1}(b_0) \in q^{-1}(q(\mathcal{C}) \cap q(\mathcal{C}'\setminus\{*\})) \cong \mathcal{C} \cap (\mathcal{C}'\setminus\{*\})$ because q is a homeomorphism on $Y\setminus\{y_1, y_2\}$, but this is impossible because $\mathcal{C} \cap \mathcal{C}' = \emptyset$. Thus, $q^{-1}(B)$ has at most two components, one containing y_1 and another containing y_2.

The next step is to show that there are exactly two path-connected components. Suppose by contradiction that $q^{-1}(B)$ has only one path-connected component. This means that there is a path between y_1 and y_2, which is impossible because B must be contractible, and this path would be a non-contractible loop in B.

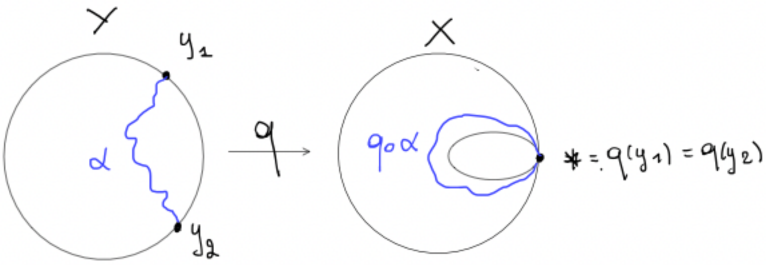


Figure 3.10

In fact, suppose by contradiction that $q^{-1}(B) \subseteq Y$ is path-connected, and let $\alpha : [0,1] \to Y$ be such that $\alpha(0) = y_1$ and $\alpha(1) = y_2$ as shown in Figure 3.10. Since $\{*\} \simeq \{*\} \times [0,1]$, we have $X \simeq X^*$ where X^* is the topological space obtained from X by replacing $\{*\}$ by $\{*\} \times [0,1]$.

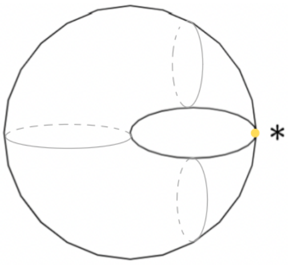


Figure 3.11: Space X.

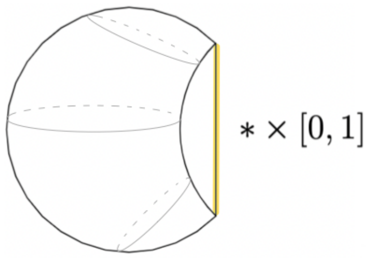


Figure 3.12: Space X^*.

Note that the loop $q \circ \alpha \in X$ based at $*$ is now extended in X^* because we "enlarged" the point $\{*\}$. The subspace B being contractible, we can contract $(q \circ \alpha)(I) \cap X$ in X^*. The remaining part of $q \circ \alpha$ is the added part $* \times [0,1]$ with identified boundaries. This is homeomorphic to the circle. Since the circle is not contractible, we conclude that $q \circ \alpha$ is not contractible in B. This contradicts the fact that B is contractible. Therefore, $q^{-1}(B)$ has two connected components, one containing y_1 and the other containing y_2.

Suppose now that the lemma is true for $n-1$ points. Let X be the CW-complex obtained from Y by identifying n points $y_1, \ldots, y_n \in Y$, $n > 0$ by the quotient map $q : Y \to X : y \mapsto [y]$, with $* = q(y_1) = \cdots = q(y_n)$. If $* \notin B$, then $q^{-1}(B)$ is homeomorphic to B, and so it is contractible.

It is possible to decompose the map q as the composition of two maps,

$$k : Y \to \frac{Y}{\{y_1, \ldots, y_{n-1}\}}, \quad \text{and} \quad l : \frac{Y}{\{y_1, \ldots, y_{n-1}\}} \to X,$$

such that $q = l \circ k$. If $* \in B$, by the case $n = 2$, we know that $l^{-1}(B)$ contains 2 disjoint contractible components, A'_n such that $y_n \in A'_n$ and B' containing the point $k(y_1) =$

$\cdots = k(y_{n-1})$. By induction hypothesis, $k^{-1}(B')$ is the union of disjoint contractible subsets A_1 to A_{n-1} such that $y_1 \in A_1$, $y_2 \in A_2$, ..., $y_{n-1} \in A_{n-1}$. Note also that $k^{-1}(A'_n)$ is a contractible open not containing the y_i for $i = 1, \ldots, n-1$, and set $A_n = k^{-1}(A'_n)$. It is just left to see that A_n is disjoint from the other sets A_i, $i = 1, \ldots, n-1$. This is true because if $A_n \cap A_i$ for a chosen $i \in \{1, \ldots, n-1\}$, then, there is a path between y_n and y_i but with a similar argument as done before, this is impossible because it will create a non contractible loop in B via the quotient map. As a consequence, $q^{-1}(B) = k^{-1}(l^{-1}(B))$ is the union of disjoint contractible subsets A_1 to A_n such that $y_1 \in A_1$, $y_2 \in A_2$, ..., $y_n \in A_n$. This implies our claim. □

The following proof is inspired from [Fox41].

Proof of Proposition 3.9 . First, J_2 can be covered with three contractible sets. Let $q : S^2 \to J_2$ be the quotient map with $q(p_1) = q(p_2) = q(p_3) = *$, i.e., the map identifying the three points $p_1, p_2, p_3 \in S^2$. Now, consider for each point $p_1, p_2, p_3 \in S^2$, an open disk in S^2, denoted U_{p_1}, U_{p_2} and U_{p_3}, each of them containing only one of the points, and such that they cover the sphere, $S^2 = \bigcup_{i=1,2,3} U_{p_i}$. Then, the three opens $q(U_{p_1}), q(U_{p_2})$, and $q(U_{p_3})$ form a cover of J_2.

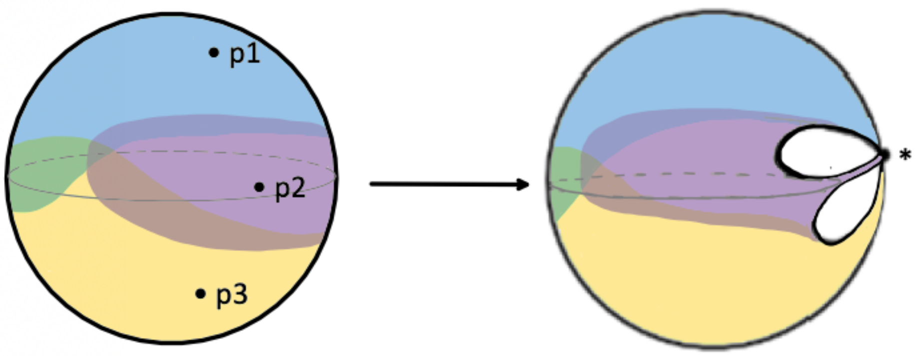


Figure 3.13: Covering of J_2.

Therefore, $\text{gcat}(J_2) \leq 2$. Now, we prove that $\text{gcat}(J_2) \geq 2$. Suppose on the contrary that there exist two contractible open sets Y_0 and Y_1 such that $J_2 = Y_0 \cup Y_1$. Note that $S^2 = q^{-1}(Y_1) \cup q^{-1}(Y_0)$. Without loss of generality, we need to solve two cases: first, $* \in Y_0 \setminus Y_1$ and second, $* \in Y_0 \cap Y_1$.

If $* \in Y_0 \setminus Y_1$, Lemma 3.10 implies that $q^{-1}(Y_1)$ is contractible and does not contain any of the p_i's. Also, $q^{-1}(Y_0) = \bigcup_{i=1,2,3} A_i$, which is the union of three disjoint contractible components such that $p_i \in A_i$. Now, if $q^{-1}(Y_1)$ is contractible, according to

Alexander duality [1], the complement $S^2 \setminus q^{-1}(Y_1)$ must also be connected. Therefore, $S^2 \setminus q^{-1}(Y_1) \subseteq A_i$ for some $i \in 1, 2, 3$. Let us assume $S^2 \setminus q^{-1}(Y_1) \subseteq A_1$. Hence, the other two components, A_2 and A_3, are contained within $q^{-1}(Y_1)$. However, this leads to a contradiction since it implies that both p_2 and p_3 are in $q^{-1}(Y_1)$ but $* \notin Y_1$.

If $* \in Y_0 \cap Y_1$, then $q^{-1}(Y_0) = \bigcup_{i=1,2,3} A_i$, which is the union of three disjoint contractible components, A_1, A_2 and A_3 such that $p_i \in A_i$. Since each of these components is connected, Alexander duality implies that $S^2 \setminus q^{-1}(Y_0)$ is connected. Note that $q^{-1}(Y_1)$ is also the disjoint union of three contractible components B_1, B_2 and B_3 such that $p_i \in B_i$. Since $S^2 \setminus q^{-1}(Y_0)$ is contained in $q^{-1}(Y_1)$ and is connected, it is contained in one of the contractible opens B_1, B_2 and B_3. Suppose that B_1 contains $S^2 \setminus q^{-1}(Y_0)$. This implies that $B_2 \subseteq A_2$ and $B_3 \subseteq A_3$ because $p_i \in A_i \cap B_i$.

Similarly, $S^2 \setminus q^{-1}(Y_1)$ is connected and has to be contained in only one connected component of $q^{-1}(Y_0)$. If $S^2 \setminus q^{-1}(Y_1) \subseteq A_1$, then $A_2 \subseteq B_2$ and $A_3 \subseteq B_3$ which imply that $A_2 = B_2$, $A_3 = B_3$. If $S^2 \setminus q^{-1}(Y_1) \subseteq A_2$ or $S^2 \setminus q^{-1}(Y_1) \subseteq A_3$, then $A_1 \subseteq B_1$, and/or $A_2 = B_2$, or $A_3 = B_3$. Suppose $A_2 = B_2$, then all other components A_i and B_i, $i = 1, 3$ are disjoint from $A_2 = B_2$. But this contradicts the fact that $\{q^{-1}(Y_1), q^{-1}(Y_0)\}$ is a cover of S^2 which is connected. The same holds for the case $A_3 = B_3$. □

So we have two topological spaces J_1 and J_2, that share the same homotopy type but have different geometric categories, i.e, $\mathrm{gcat}(J_2) = 2 \leq 1 = \mathrm{gcat}(J_1)$. This proves that gcat is not a homotopy invariant number. Clapp and Montejano explained in [CM87] that the geometric category is far from being a homotopy invariant. In fact, they proved the following proposition:

Proposition 3.11. *[CM87, Theorem 2] For every positive integer n, there is a space K_n such that*

$$\mathrm{gcat}(K_n) - \mathrm{gcat}(K_n \times [0, 1]) \geq n.$$

Therefore, the goal of the following chapters is to find solutions to obtain a homotopy invariant from the geometric category. In Chapter 4, we will relax the condition on the open sets in the definition of the geometric category of X, and consider open sets that are contractible *in* the space X. In Chapter 5, we will simply force the homotopy invariance of the geometric category of a topological space X by taking the minimum of the geometric categories for all the spaces that have the same homotopy type as X.

[1] Alexander duality is a way of understanding the homology groups of $S^2 \setminus K$, i.e., the sphere minus a well behaved subset K, using higher cohomology groups. In our case, it implies that $\tilde{H}_0(S^2 \setminus K; G)$ is isomorphic to $\tilde{H}^1(K; G) = 0$ because K is contractible. This implies that $S^2 \setminus K$ has only one connected component. More information can be found in [Hat02, Corollary 3.45].

Chapter 4

Lusternik-Schnirelmann category

In this chapter, we define our first homotopy invariant. We may wonder what would change if, in the definition of the geometric category, we consider open sets that are contractible in the space X and not necessarily contractible *per se*. We will see that by relaxing the condition that each $\mathcal{U}_i$ must be contractible and by considering instead the inclusion $\mathcal{U}_i \hookrightarrow X$ to be nullhomotopic, i.e., $\mathcal{U}_i$ is contractible in X, we obtain a homotopy invariant named the *Lusternik-Schnirelmann (LS) category*. In the first section, we provide examples and general properties of the LS-category. We also focus on the LS-category of the homotopy pushout and the LS-category of the homotopy pullback. In the second section, we explore the definition of the LS-category given by George Whitehead in 1978, [Whi78]. We show that it is equivalent to the first definition of the LS-category. Similarly, in the last section of this chapter, we explain the reformulation of the definition of the category given by Tudor Ganea in [Gan67] and show how it is equivalent to the formulation of George Whitehead.

4.1 Definition and properties of the LS-category

If we relax the condition that each $\mathcal{U}_i$ must be contractible and instead we only ask the inclusions $\mathcal{U}_i \hookrightarrow X$ to be nullhomotopic, we obtain the following definition:

Definition 4.1. *Let X be a topological space. The* Lusternik-Schnirelmann (LS) category *of X, denoted by* $\mathrm{cat}(X)$*, is the least $n \in \mathbb{N}$ such that there exists an open cover $\{\mathcal{U}_0, \ldots, \mathcal{U}_n\}$ of X with each $\mathcal{U}_i$ contractible in X.*

We say that such a cover $\{\mathcal{U}_0, \ldots, \mathcal{U}_n\}$ is *categorical*. If there is not such cover, we write $\mathrm{cat}(X) = \infty$. Note that the main difference between the definitions of $\mathrm{gcat}(X)$ and $\mathrm{cat}(X)$ is that the former is obtained by considering covers of open subsets that are contractible, while the latter is defined based on covers of open subsets that are contractible *in the topological space* X i.e, such that the inclusion $\mathcal{U}_i \hookrightarrow X$ is nullhomotopic for $\mathcal{U}_i$ in the cover of X. Note that Example 1.6 shows that the circle is not a contractible space, but in the sphere, the circle can be contracted to a point, and the inclusion $S^1 \hookrightarrow S^2$ is nullhomotopic. We provide some examples.

Example 4.2. (1) If X is contractible, we obtain a result similar to that of the geometric category. Indeed, if X is contractible, then X is a categorical cover of itself. Hence $\text{cat}(X) = 0$ if and only if X is contractible.

(2) If X is a compact CW-complex (i.e., a finite CW-complex), then $\text{cat}(X)$ will be finite. In fact, any CW-complex admits a categorical cover. This is due to the fact that each cell has a contractible closure, and we can consider, for each closure, a slightly bigger open in X that contains it and is also contractible because X is a CW-complex (see [Cor+03, Theorem A.2]). Moreover, if X is compact there is always a finite subcover of any categorical cover. This subcover is also categorical.

The hypothesis of being a CW-complex is important because there exist compact spaces that have an infinite category. For example, the Figure 4.1 represents a variant of the topologist's sin curve, the Warsaw circle.

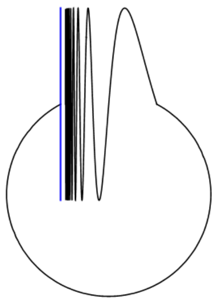


Figure 4.1: Warsaw Circle.

The Warsaw Circle, denoted by W, is a planar, 1-dimensional, path-connected, and compact space. However, it is not contractible and it is not locally path-connected at any point on the arc $\{0\} \times [-1, 1]$ (i.e., the blue line on Figure 4.1). This topological space is not a CW-complex since Proposition 1.19 states that every connected CW-complex is locally path-connected. Moreover, we cannot find a categorical covering for this topological space. Intuitively, if we suppose that there is a contractible open set U that contains a point x on the arc $\{0\} \times [-1, 1]$. The subset U will also contain other points on the sinusoidal part since it is open. In particular, because U is contractible, and therefore path-connected, it must contain the entire sinusoidal part and the circle. In other words, U would be of the same homotopy type as the Warsaw circle. However, since W is not contractible, U cannot be contractible either. This leads to a contradiction, suggesting that there is no open set in the Warsaw Circle that is both contractible and contains a point of the arc $\{0\} \times [-1, 1]$. Hence, there is no possible categorical covering, and $\text{cat}(W) = \infty$.

(3) The sphere S^n, $n > 1$ can be covered with two hemispheres that have been slightly extended so that they form an open cover of S^n, see Figure 4.2.

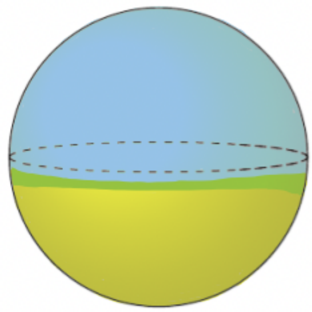


Figure 4.2: Categorical cover of S^2.

The hemispheres are contractible. Moreover, there is no categorical cover of the sphere containing only one open (the sphere is not contractible). We conclude that $\text{cat}(S^n) = 1$.

Remark 4.3. Taking a refinement of a categorical cover still gives a categorical cover, but this is not the case for covers containing contractible open sets. In the case of a categorical cover, the inclusion $\mathcal{U}_i \to X$ is null-homotopic, and this implies that for $\mathcal{V}_j \subseteq \mathcal{U}_i$, the inclusion $\mathcal{V}_j \to X$ is also null-homotopic. However, refinements of contractible covers are not necessarily contractible anymore. For instance, consider $D^2 = \{(x,y)|x^2 + y^2 \leq 1\}$ the disc, and $\mathcal{U} = \{D^2\}$ as a contractible cover. A refinement of the cover $\mathcal{U}$ of the disc is given by $\mathcal{V} = \{V_0, V_1\}$, with $V_0 = \{(x,y)|x^2 + y^2 < 2/3\}$ and $V_1 = \{(x,y)|x^2 + y^2 > 1/3\}$ as shown in Figure 4.3.

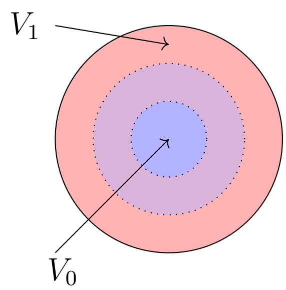


Figure 4.3: Refinement $\mathcal{V} = \{V_0, V_1\}$ of $\mathcal{U}$.

This refinement is not a contractible cover because V_1 is not contractible. In fact, V_1 has the homotopy type of a circle. Intuitively, this result explains that we have more flexibility finding categorical covers than contractible covers.

From this remark, we understand that asking for a cover containing contractible open sets is more restrictive than asking for a categorical cover. Therefore, in general, categorical covers gives us more flexibility in the choice of the open sets. Hence it is clear that $\text{gcat}(X) \geq \text{cat}(X)$ because if $\text{gcat}(X) = n$, we have a cover by contractible open sets $\{\mathcal{U}_1, \ldots, \mathcal{U}_{n+1}\}$, and this implies that $\text{cat}(X) \leq n$.

4.1.1 General properties

In this subsection, we explain and summarize some important features of the LS-category. The properties are given for CW-complexes, thus we are working in CWTop. The reference book for this section is by [Cor+03]. The first proposition gives a good intuition of the LS-category. Recall that the dimension $\dim(X)$ of a CW-complex X is the biggest $n \in \mathbb{N}$ such that there exists an n-cell in X.

Proposition 4.4. *[Cor+03, Theorem 1.7] For a path-connected CW-complex X, we have* $\mathrm{cat}(X) \leq \dim(X)$.

Proof. By Proposition 3.4, we know that $\mathrm{gcat}(X) \leq \dim(X)$. Moreover, as explained earlier, $\mathrm{cat}(X) \leq \mathrm{gcat}(X)$. Therefore, it follows that $\mathrm{cat}(X) \leq \dim(X)$. □

This proposition is also proven in [Cor+03] using the fact that if X is a CW-complex of dimension n and $\{\mathcal{U}_0, \ldots, \mathcal{U}_k\}$ is a categorical cover, there exists a refinement with $n+1$ open sets. As Explained in 4.3, this refinement is still categorical.

Example 4.5. As described in Chapter 1, the real projective n-space is a CW-complex obtained by adding one cell in each dimension. Hence, by the last proposition, $\mathrm{cat}(\mathbb{R}P^n) \leq n$. Similarly, for the complex projective n-space $\mathbb{C}P^n$, where we attach $2i$-cells for $i \in \{0, \ldots, n\}$, we obtain $\mathrm{cat}(\mathbb{C}P^n) \leq 2n$. The infinite real projective space is an infinite CW-complex. In fact, because $\mathbb{R}P^n$ is obtained from $\mathbb{R}P^{n-1}$, we can keep attaching cells infinitely. The infinite union

$$\mathbb{R}P^\infty = \bigcup_n \mathbb{R}P^n$$

is a cellular space with one cell in each dimension. In this case, the proposition is not very helpful because it only gives $\mathrm{cat}(\mathbb{R}P^\infty) \leq \infty$. Similarly, one obtains $\mathrm{cat}(\mathbb{C}P^\infty) \leq \infty$. We now need a lower bound to have a better approximation of the LS-category of the projective spaces.

In order to find a lower bound for the LS-category, we need to recall the following definition:

Definition 4.6. *Let R be a commutative ring and X be a topological space. The* cup-length *of X with coefficients in R is the supremum over $n \in \mathbb{N}$ such that for $a_1, \ldots, a_n \in \widetilde{H}^*(X; R)$*[1]*, the cup-product $a_1 \cup a_2 \cup \cdots \cup a_n$ is non-zero. We denote the cup-length as* $\mathrm{cup}_R(X)$.

[1]The ring $\widetilde{H}^*(X; R)$ is the reduced cohomology ring of X with coefficients in R, which is the direct sum of the cohomology groups $\widetilde{H}^n(X; R)$, endowed with the cup product. For readers who are interested, more information can be found in Section 3.2 of [Hat02].

In other words, the cup-length is the word-length of the largest word that can be written in the cohomology ring. If the reader is interested in more details, a great reference is [Mac63], where the cup product is explained in detail. The next result expresses a nice link between the cup-length and the category:

Proposition 4.7. *[Cor+03, Proposition 1.5] The cup-length of X with coefficients in R is less than or equal to the LS-category of X for all coefficients R:*

$$\mathrm{cup}_R(X) \leq \mathrm{cat}(X).$$

Example 4.8. For example, take the commutative ring to be $R = \mathbb{Z}/2\mathbb{Z}$. Theorem 3.19 in [Hat02] implies that the cohomology ring of $\mathbb{R}P^n$ is $\tilde{H}^*(\mathbb{R}P^n; R) = \frac{R[\alpha]}{\alpha^{n+1}}$, the truncated polynomial ring on a generator α of degree 1. Hence, the cup-length of the real projective n-space is $\mathrm{cup}_R(\mathbb{R}P^n) = n$. Proposition 4.7 and Example 4.5 give :

$$\mathrm{cup}_R(\mathbb{R}P^n) = n \leq \mathrm{cat}(\mathbb{R}P^n) \leq n.$$

The LS-category of the real projective space of dimension n is n. Similarly, $\tilde{H}^*(\mathbb{C}P^n; R) = \frac{R[\alpha]}{\alpha^{n+1}}$, the truncated polynomial ring on a generator α of degree 2. The cup-length is $\mathrm{cup}_R(\mathbb{C}P^n) = n$ and we deduce that $\mathrm{cat}(\mathbb{C}P^n) \in \{n, \ldots, 2n\}$. This is more precise than before but we still have a lot of possibilities.

Finally, $\mathrm{cup}_R(\mathbb{R}P^\infty) = \infty = \mathrm{cup}_R(\mathbb{C}P^\infty)$, whence $\mathrm{cat}(\mathbb{R}P^\infty) = \infty = \mathrm{cat}(\mathbb{C}P^\infty)$.

After establishing a lower bound and an upper bound, we can prove one of the most significant properties of cat(X): its homotopy invariance. In order to demonstrate this, we will introduce the following lemma and deduce the invariance directly from it.

Lemma 4.9. *[Cor+03, Theorem 1.29] If $f : X \to Y$ has a right homotopy inverse, then $\mathrm{cat}(X) \geq \mathrm{cat}(Y)$.*

We present here the proof from [Cor+03].

Proof. Suppose that $\mathrm{cat}(X) = n$ with a categorical cover $\{\mathcal{U}_0, \ldots, \mathcal{U}_n\}$. Each $\mathcal{U}_i$ is contractible in X via the homotopy $G_i : \mathcal{U}_i \times I \to X$ such that $G_i(u, 0) = u$ and $G_i(u, 1) = x_0$, where x_0 is a chosen point in X. Moreover, let $g : Y \to X$ be the right homotopy inverse such that $f \circ g \simeq \mathrm{id}_Y$. Let $H : Y \times I \to Y$ be the homotopy such that $H(y, 0) = \mathrm{id}_Y(y) = y$ and $H(y, 1) = f \circ g(y)$ and let $\mathcal{V}_i = g^{-1}(\mathcal{U}_i)$ be open sets in

Y. Since $\{\mathcal{U}_i\}$ is a cover of X, $\{\mathcal{V}_i\}$ is a cover for Y, and we want it to be categorical. Define the contracting homotopy

$$\begin{aligned} F : & \mathcal{V}_i \times I \longrightarrow Y \\ & (v,t) \mapsto \begin{cases} H(v, 2t) & \text{if } 0 \leq t \leq 1/2 \\ f(G_i(g(v), 2t-1)) & \text{if } 1/2 \leq t \leq 1. \end{cases} \end{aligned}$$

This homotopy is well-defined because $g(v) \in \mathcal{U}_i$ when $v \in \mathcal{V}_i$ and at $t = 1/2$, we have

$$F(v, 1/2) = H(v, 1) = f \circ g(v) = f \circ G_i(g(v), 0).$$

But also, this homotopy contracts $\mathcal{V}_i$ in Y because

$$F(v, 0) = H(v, 0) = v \text{ and } F(v, 1) = f \circ G_i(g(v), 1) = f(x_0) = y_0.$$

Therefore, $\{\mathcal{V}_i\}$ is a categorical cover of Y with $n+1$ open sets. This implies that $\operatorname{cat}(X) \geq \operatorname{cat}(Y)$. □

From this lemma, it follows immediately that the LS-category is a homotopy invariant.

Theorem 4.10. *[Cor+03, Theorem 1.30] If $f : X \to Y$ is a homotopy equivalence, then* $\operatorname{cat}(X) = \operatorname{cat}(Y)$.

Proof. If $f : X \to Y$ is a homotopy equivalence, there exists a homotopy inverse $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$. Lemma 4.9 implies that $\operatorname{cat}(X) \geq \operatorname{cat}(Y)$. Applying the same argument to $g : Y \to X$ and its homotopy inverse $f : X \to Y$ yields $\operatorname{cat}(Y) \geq \operatorname{cat}(X)$. Therefore, $\operatorname{cat}(X) = \operatorname{cat}(Y)$. □

We just showed that the LS-category is a homotopy invariant. However, this was not the case for the geometric category, and we now realize more and more that those two numbers are really different. Additionally, in [CM87], the following theorem is proven:

Theorem 4.11. *[CM87, Theorem 1] For every positive integer n, there is a space K_n such that*

$$\operatorname{gcat}(K_n) - \operatorname{cat}(K_n) \geq n.$$

This theorem shows that the difference between the two numbers is unbounded. However, we still have some results and properties of the LS-category that are similar to those of the geometric category, which we will present in the following subsections.

4.1.2 Cofibrations and homotopy pushout

We aim to calculate the LS-category of the double mapping cylinder, i.e., the homotopy pushout. To do so, we need to determine the LS-category of the union of two topological spaces.

Proposition 4.12. *[Cor+03, Proposition 1.27] Let $Z = X \cup Y$ be a topological space that can be written as the union of two open sets X and Y. Then, we have*

$$\text{cat}(Z) \leq \text{cat}(X) + \text{cat}(Y) + 1.$$

This argument is the same as that for the geometric category. Suppose $\text{cat}(X) = n$ and $\text{cat}(Y) = k$ with two categorical covers $\{\mathcal{U}_i\}_{i=0,\ldots,n}$ and $\{\mathcal{V}_i\}_{i=0,\ldots,k}$ of X and Y, respectively. Then, $\{\mathcal{U}_i, \mathcal{V}_i\}$ is a categorical cover of Z with $k+n+2$ open sets. Hence, $\text{cat}(Z) \leq k+n+1 = \text{cat}(Y) + \text{cat}(X) + 1$.

Now, we observe that the double mapping cylinder $D_{f,g}$, where $f : W \to X$ and $g : W \to Y$, is the union of two simple mapping cylinders, M_f and M_g shown in yellow and blue in Figure 4.4, respectively.

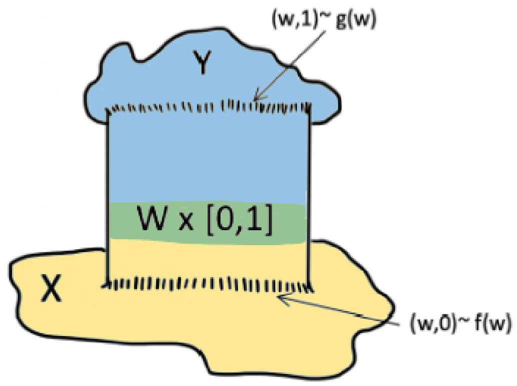


Figure 4.4: Double mapping cylinder.

The two parts are such that $M_g \simeq Y$ and $M_f \simeq X$. Therefore, combining Proposition 4.12 and Proposition 4.10, we obtain the following result:

Proposition 4.13. *[Cor+03, Proposition 1.34] For mappings $f : W \to X$, $g : W \to Y$, we have*

$$\text{cat}(D_{f,g}) \leq \text{cat}(X) + \text{cat}(Y) + 1.$$

Using Proposition 4.13, we can calculate the category of the mapping cone. As explained in Example 2.16, the mapping cone is obtained by taking the homotopy pushout of $f : X \to Y$ and $c_* : X \to \{*\}$. The associated diagram is the following:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle c_*}\downarrow & & \downarrow \\ \{*\} & \longrightarrow & Y \cup_f CX. \end{array}$$

Moreover, $\operatorname{cat}(\{*\}) = 0$. From Proposition 4.13 we obtain the following theorem:

Theorem 4.14. *[Cor+03, Theorem 1.32] If $f : X \to Y$ is a map with mapping cone $C_f = Y \cup_f CX$, then*

$$\operatorname{cat}(C_f) \leq \operatorname{cat}(Y) + 1.$$

Remark 4.15. If we take $X = S^{n-1}$, the mapping cone of $f : S^{n-1} \to Y$ is the result of attaching an n-cell to Y, as stated in Example 2.16. From this point of view, Theorem 4.14 implies that adding a cell increases the category by at most 1. In fact,

$$\operatorname{cat}(C_f) = \operatorname{cat}(Y \cup_f e^n) \leq \operatorname{cat}(Y) + 1.$$

Note that it could reduce the category. For instance, take $Y = S^1$, $\operatorname{cat}(S^1) = 1$. Now, consider the identity map $\operatorname{id}_{S^1} : S^1 \to S^1$ and attach a 2-cell:

$$\begin{array}{ccc} S^1 & \xrightarrow{f} & S^1 \\ {\scriptstyle \iota}\downarrow & & \downarrow \\ D^2 & \longrightarrow & S^1 \cup_{\operatorname{id}_{S^1}} e^2 \simeq D^2. \end{array}$$

By Proposition 4.10, $\operatorname{cat}(S^1 \cup_{\operatorname{id}_{S^1}} e^2) = \operatorname{cat}(D^2) = 0$. Therefore, $\operatorname{cat}(S^1 \cup_{\operatorname{id}_{S^1}} e^2) < \operatorname{cat}(S^1)$.

Since the wedge sum of two topological spaces is obtained as the homotopy pushout in CWTop_* of the maps $\{*\} \hookrightarrow (X, x_0)$ and $\{*\} \hookrightarrow (Y, y_0)$, we deduce the upper bound via Proposition 4.13,

$$\operatorname{cat}(X \vee Y) \leq \operatorname{cat}(X) + \operatorname{cat}(Y) + 1.$$

But we can actually do better than that by constructing a categorical cover using the categorical covers of X and Y, as done in [Cor+03], where the reader can find a proof of the following proposition:

Proposition 4.16. *[Cor+03, Proposition 1.27] If (X, x_0) and (Y, y_0) are in CWTop_*, then,*

$$\operatorname{cat}(X \vee Y) = \max\{\operatorname{cat}(X), \operatorname{cat}(Y)\}.$$

This same Proposition 4.13 allows us to easily calculate the category of a suspension:

Proposition 4.17. *Let X be a topological space. Then* $\text{cat}(\Sigma X) \leq 1$.

Proof. Recall that the suspension of X is the homotopy pushout of twice the constant map $c_* : X \to \{*\}$,

$$\begin{array}{ccc} X & \xrightarrow{c_*} & \{*\} \\ {\scriptstyle c_*}\downarrow & & \downarrow{\scriptstyle \iota_1} \\ \{*\} & \xrightarrow{\iota_2} & \Sigma X. \end{array}$$

Proposition 4.13 implies that

$$\text{cat}(\Sigma X) \leq \text{cat}(\{*\}) + \text{cat}(\{*\}) + 1 = 1$$

since the LS-category of a point is 0. □

Remark 4.18. If X is a contractible space, then the suspension of X is also contractible. Hence, in this case $\text{cat}(\Sigma X) = 0$.

4.1.3 Fibrations and homotopy pullback

To identify the category of a product in Top, more work is needed. A proof of the following result can be found in [Fox41, Theorem 9]. For a more recent proof, the reader can also refer to [Cor+03, Theorem 1.37].

Proposition 4.19. *[Fox41, Theorem 9] If $X = X_1 \times X_2$ is a CW-complex and X_1, X_2 are finite CW-complex, then*

$$\max\{\text{cat}(X_1), \text{cat}(X_2)\} \leq \text{cat}(X) \leq \text{cat}(X_1) + \text{cat}(X_2).$$

The first inequality is obtained because the projection $p : X_1 \times X_2 \to X_i$ has the inclusion $\iota : X_i \hookrightarrow X_1 \times X_2$ as a right homotopy inverse, such that $p \circ \iota = \text{id}_{X_i}$ for $i = 1, 2$. Therefore, by Lemma 4.9, $\text{cat}(X_i) \leq \text{cat}(X_1 \times X_2)$, for $i = 1, 2$. The second inequality is harder to prove. However, by considering categorical covers $\mathcal{U}_i$ of X_1 and $\mathcal{V}_j$ of X_2, it is possible to construct a categorical cover of X by taking the product of different open sets $\mathcal{U}_i \times \mathcal{V}_j$.

Example 4.20. This allows us to calculate the LS-category of the n-torus,

$$T^n = \overbrace{S^1 \times S^1 \times \cdots \times S^1}^{n}.$$

Using the last result, we obtain

$$\text{cat}(T^{n-1}) \leq \text{cat}(T^n) \leq n.$$

Moreover, as explained in [Hat02, Example 3.13], the n-torus has a cohomology ring $H^*(T^n, \mathbb{Z})$ which is an exterior algebra on n generators. Thus, $\text{cup}_{\mathbb{Z}}(T^n) = n$. Whence, by Proposition 4.7, we get $\text{cat}(T^n) = n$.

In this section, we aim to describe the relationship between fibrations and the LS-category. In 1965, Varadarajan was the first to prove the following proposition in [Var65]. Over the years, the result has been improved, and we present the proposition from [Cor+03] here:

Proposition 4.21. *[Cor+03, Theorem 1.41] Let $F \overset{\iota}{\hookrightarrow} E \overset{p}{\to} B$ be a fibration in Top$_*$. Then,*

$$\text{cat}(E) \leq (\text{cat}(F) + 1)(\text{cat}(B) + 1) - 1.$$

The idea of the proof is exactly the same as the one explained for the geometric category. Moreover, as for the geometric category, we have the following result coming from a paper of Octavian Cornea, [Cor94].

Proposition 4.22. *[Cor94, Proposition 2.1] If $F \overset{\iota}{\hookrightarrow} E \overset{p}{\to} B$ is a fibration in CWTop$_*$, with E path-connected, and B has a non degenerate basepoint b_0, then*

$$\text{cat}(E/F) \leq \text{cat}(B).$$

In [Cor+03], the authors explore further and provide the following corollary:

Corollary 4.23. *[Cor+03, Corollary 1.45] If $F \overset{\iota}{\hookrightarrow} E \overset{p}{\to} B$ is a fibration in CWTop$_*$, E is path-connected, B has a non degenerate basepoint b_0, and the inclusion ι is nullhomotopic, then*

$$\text{cat}(E) \leq \text{cat}(B).$$

In particular, if $p : E \to B$ is a cover map with E path-connected, then $\text{cat}(E) \leq \text{cat}(B)$.

The proof is as follows: if $\iota : F \hookrightarrow E$ is nullhomotopic, then, $E/F \simeq E \cup_\iota CF \simeq E \vee \Sigma F$. This implies that $\text{cat}(E/F) = \text{cat}(E \vee \Sigma F)$. Furthermore, according to Proposition 4.16, we have $\text{cat}(E \vee \Sigma F) = \max\{\text{cat}(E), \text{cat}(\Sigma F)\} = \text{cat}(E)$. The last equality holds because if E is not contractible, then $\text{cat}(E) \geq 1$, and the LS-category of a suspension is always less than or equal to 1 (see Proposition 4.17). Hence, we can conclude that $\text{cat}(E/F) = \text{cat}(E)$. Therefore, applying Proposition 4.22, we obtain $\text{cat}(E) \leq \text{cat}(B)$. Moreover, in the case of a cover map, the inclusion of a discrete set of points into a path-connected total sapce E is nullhomotopic.

4.2 The Whitehead definition

This section is devoted to exposing a new way of defining the LS-category given by G. Whitehead in [Whi78, Section 1, Chapter 10]. We will state the equivalence between the LS-category and Whitehead's definition of category in Theorem 4.29. Moreover, as noted in [Whi78], Whitehead's definition appears to be more appropriate for the homotopy theory of spaces with non-degenerate base points. Therefore, in this section, we will be working in CWTop_*. To present Whitehead's formulation of category, we first need to introduce a new topological space.

Definition 4.24. *Let $k \in \mathbb{N}^*$. In CWTop_*, the* fat wedge *of a pointed topological space (X, x_0) is defined by*

$$T^k(X) = \Big\{(x_1, \dots, x_k) \in X^k | \text{ at least one } x_j \text{ is the basepoint } x_0 \Big\}.$$

Clearly, $T^2(X) = X \vee X$, because another way of thinking of the wedge sum is to take the following space:

$$X \vee Y = \{(x, y) \in X \times Y | x = x_0 \text{ or } y = y_0, \text{ with } x_0 \in X, y_0 \in Y\}.$$

Let $j : T^k(X) \hookrightarrow X^k$ be the inclusion of the fat wedge into the product X^k and let $\Delta : X \to X \times X; x \mapsto (x, x)$, be the diagonal map.

Definition 4.25. *[Cor+03, Whitehead's Definition of Category] The* Whitehead category *of X, denoted $\mathrm{cat}^{Wh}(X)$, is the least integer n such that there exists a map $\Delta' : X \to T^{n+1}(X)$, which makes the following diagram commute up to homotopy*

$$\begin{array}{ccc} X & \xrightarrow{\Delta'} & T^{n+1}(X) \\ & \searrow^{\Delta} & \downarrow j \\ & & X^{n+1}. \end{array}$$

Example 4.26. Let X be a non-contractible topological space. As for the geometric category and the first definition of LS-category, we want to show that Whitehead's definition of category also yields $\mathrm{cat}^{Wh}(\Sigma X) \leq 1$. To achieve this, we need to find a map $\Delta' : \Sigma X \to \Sigma X \vee \Sigma X$ for which the diagram

$$\begin{array}{ccc} \Sigma X & \xrightarrow{\Delta'} & \Sigma X \vee \Sigma X \\ & \searrow^{\Delta} & \downarrow j \\ & & \Sigma X \times \Sigma X. \end{array}$$

is homotopy-commutative. We denote by $*$ the base point of the wedge $\Sigma X \vee \Sigma X$. We define the map

$$\Delta' : \Sigma X \to \Sigma X \vee \Sigma X; [x,t] \mapsto \begin{cases} ([x,2t],*) & \text{if } 0 \leq t \leq 1/2 \\ (*,[x,2t-1]) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

Intuitively, the map Δ' takes points of the suspension and sends them to a point in the wedge, considering it as a subset of $\Sigma X \times \Sigma X$. This map is represented in Figure 4.5.

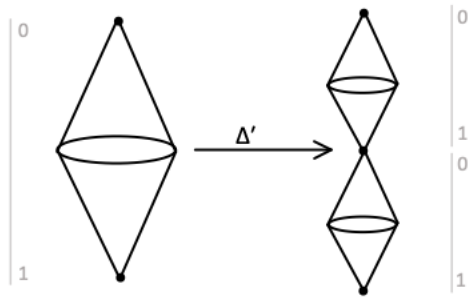


Figure 4.5: $\Delta' : \Sigma X \to \Sigma X \vee \Sigma X$

Now, our goal is to obtain a homotopy between $j \circ \Delta'$, where j simply denotes the inclusion in $\Sigma X \times \Sigma X$, and the map $\Delta : \Sigma X \to \Sigma X \times \Sigma X$, which is the diagonal map sending $[x,t]$ to $([x,t],[x,t])$. Let p_1 and p_2 be the projections from $\Sigma X \times \Sigma X$ to ΣX, representing the first and second components respectively, such that

$$p_1 \circ j \circ \Delta'([x,t]) = \begin{cases} [x,2t] & \text{if } 0 \leq t \leq 1/2 \\ * & \text{if } 1/2 \leq t \leq 1, \end{cases}$$

and

$$p_2 \circ j \circ \Delta'([x,t]) = \begin{cases} * & \text{if } 0 \leq t \leq 1/2 \\ [x,2t-1] & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

We are going to provide homotopies that will give us $p_1 \circ j \circ \Delta' \simeq \mathrm{id}_{\Sigma X}$ and $p_2 \circ j \circ \Delta' \simeq \mathrm{id}_{\Sigma X}$. Once we have these, it will be easy to find a homotopy such that $j \circ \Delta' \simeq \Delta$. We define the following:

$$\begin{aligned} H_1 :& \Sigma X \times I \to \Sigma X \\ & ([x,t],s) \mapsto \begin{cases} [x,(2-s)t] & \text{if } 0 \leq t \leq \frac{1}{(2-s)} \\ * & \text{if } \frac{1}{(2-s)} \leq t \leq 1, \end{cases} \end{aligned}$$

so that

$$H_1([x,t],0) = \begin{cases} [x,2t] & \text{if } t \leq 1/2 \\ * & \text{if } 1/2 \leq t \end{cases} \quad \text{and} \quad H_1([x,t],1) = \begin{cases} [x,t] & \text{if } t \in [0,1] \\ * & \text{if } t = 1. \end{cases}$$

We define also

$$\begin{aligned} H_2 :& \Sigma X \times I \to \Sigma X \\ & ([x,t],s) \mapsto \begin{cases} * & \text{if } 0 \le t \le \frac{1-s}{(2-s)} \\ [x,(2-s)t-(1-s)] & \text{if } \frac{1-s}{(2-s)} \le t \le 1, \end{cases} \end{aligned}$$

so that

$$H_2([x,t],0) = \begin{cases} * & \text{if } t \le 1/2 \\ [x,2t-1] & \text{if } 1/2 \le t \end{cases} \quad \text{and} \quad H_1([x,t],1) = \begin{cases} * & \text{if } t = 0 \\ [x,t] & \text{if } t \in [0,1]. \end{cases}$$

Therefore, the homotopy

$$\begin{aligned} H :& \Sigma X \times I \to \Sigma X \times \Sigma X \\ & ([x,t],s) \mapsto (H_1([x,t],s), H_2([x,t],s)) \end{aligned}$$

is such that

$$H([x,t],0) = (p_1 \circ j \circ \Delta'([x,t]), p_2 \circ j \circ \Delta'([x,t])) = j \circ \Delta'([x,t]),$$

and

$$H([x,t],1) = ([x,t],[x,t]) = \Delta([x,t]).$$

This proves that $j \circ \Delta' \simeq \Delta$ and therefore, $\mathrm{cat}^{Wh}(\Sigma X) \le 1$.

After this example, one might ask whether this definition is more useful than the previous definition of LS-category we had. To answer the question, with the help of this new definition, we obtain a better approximation in the calculations of the category of finite CW-complexes. Recall that a topological space is said to be n-connected if every continuous map from the k-sphere $S^k \to X$ extends to a continuous map from the $k+1$-disk $D^{k+1} \to X$ for all $k = 0, \ldots, n$.

Proposition 4.27. *[Cor+03, Theorem 1.50] If X is an $(n-1)$-connected CW-complex for $n \geq 1$, then*

$$\mathrm{cat}^{Wh}(X) \le \frac{\dim(X)}{n}.$$

Example 4.28. In Example 4.8, we showed that $\mathrm{cat}(\mathbb{C}P^n) \in \{n, \ldots, 2n\}$. However, we can determine the exact value of the category of the complex projective n-space by using that it is simply connected, and thus 1-connected. By the last proposition we obtain $\mathrm{cat}^{Wh}(\mathbb{C}P^n) \le \dim(\mathbb{C}P^n)/2 = n$. If the two definitions of category are equivalent, we will have $\mathrm{cat}(\mathbb{C}P^n) = n$. However we need to show consistency in the definitions.

The following theorem from [Cor+03], gives consistency with Definition 4.1:

Theorem 4.29. *[Cor+03, Theorem 1.55] For a path-connected CW-complex (X, x_0) in* CWTop_**, Whitehead's definition of category coincides with the LS-category :* $\mathrm{cat}(X) = \mathrm{cat}^{Wh}(X)$.

4.3 Ganea formulation of category

This section presents an alternative characterization of the LS-category given by T. Ganea in [Gan67]. We will demonstrate the equivalence between Whitehead's and Ganea's definitions of category in Theorem 4.36. This interpretation of category relies on the concept of the *fibre-cofibre construction*, also known as the *Ganea construction*.

Definition 4.30. *[Cor+03, Definition 1.59] The* fiber-cofiber construction *is the following induction process in* $CWTop_*$*:*

(1) Let $F_0(X) \xrightarrow{i_0} G_0(X) \xrightarrow{p_0} X$ denote the path fibration on (X, x_0) (see Example 2.19), i.e.,

$$\Omega X \to PX \xrightarrow{\mathrm{ev}} X,$$

where ΩX is the space of loops in X at x_0 and PX is the space of paths beginning at x_0.

(2) Suppose a fibration $F_n(X) \xrightarrow{i_n} G_n(X) \xrightarrow{p_n} X$ has been constructed. Let $C_{i_n} = G_n(X) \cup CF_n(X)$ be the mapping cone of $F_n(X) \xrightarrow{i_n} G_n(X)$. We may extend $p_n : G_n(X) \to X$ to a map $q_n : C_{i_n} \to X$ by defining $q_n(x) = p_n(x)$ if $x \in G_n(X)$ and $q_n([y, t]) = *$ if $[y, t] \in CF_n(X)$.

(3) Now, convert q_n into a fibration,

$$p_{n+1} : G_{n+1}(X) \to X,$$

using Theorem 2.10 that gives the following accompanying commutative diagram :

$$\begin{array}{ccccc} G_n(X) & \xrightarrow{\iota} & C_{i_n} & \xrightarrow{\simeq} & G_{n+1}(X). \\ & \searrow^{p_n} & \downarrow{\scriptstyle q_n} & \swarrow_{p_{n+1}} & \\ & & X & & \end{array}$$

(4) Continuing inductively, one obtains the following commutative diagram of Ganea fibrations:

$$
\begin{array}{ccccccccccc}
\Omega X = F_0(X) & \longrightarrow & F_1(X) & \longrightarrow & F_2(X) & \longrightarrow & \dots & \longrightarrow & F_n(X) & \longrightarrow & \dots \\
\downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_2} & & & & \downarrow{\scriptstyle i_n} & & \\
PX = G_0(X) & \longrightarrow & G_1(X) & \longrightarrow & G_2(X) & \longrightarrow & \dots & \longrightarrow & G_n(X) & \longrightarrow & \dots \\
\downarrow{\scriptstyle p_0} & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} & & & & \downarrow{\scriptstyle p_n} & & \\
X & \xrightarrow{\mathrm{id}_X} & X & \xrightarrow{\mathrm{id}_X} & X & \xrightarrow{\mathrm{id}_X} & \dots & \xrightarrow{\mathrm{id}_X} & X & \xrightarrow{\mathrm{id}_X} & \dots
\end{array}
$$

Example 4.31. Let (X, x_0) be a pointed topological space. Let us try to understand what $G_1(X)$ is. By definition, the first step of the fiber-cofiber construction is as follows:

$$\Omega X \xrightarrow{i_0} PX \xrightarrow{p_0} X,$$

where $G_0(X) = PX$. We obtain the mapping cone $C_{i_0} = PX \cup_{i_0} C\Omega X$. Since PX is contractible, the space C_{i_0}/PX is equivalent to $C\Omega X$ where we have identified the points at $t = 0$. This is exactly the suspension of ΩX. Hence, we have $C_{i_0}/PX \simeq \Sigma\Omega X$ and $G_1(X) \simeq \Sigma\Omega X$.

Remark 4.32. Let $p : E \to B$ be a fibration with fiber $F = p^{-1}(b_0)$, and let $E \cup_\iota CF$ the mapping cone of the inclusion $\iota : F \hookrightarrow E$. As in the fiber-cofiber construction, we can extend the map p to a map $r : E \cup CF \to B$. By Theorem 2.10, there exists a homotopy equivalent fibration $p' : P_r \to B$. In [Gan64], it is proved that the fiber of this map has the homotopy type of the join $F * \Omega B$. This result is also demonstrated in [Cor+03, Corollary B.32.]. This implies that in the fiber-cofiber construction, $F_n(X) \simeq *^{n+1}\Omega X$.

We will now define, for the third time, the notion of the category of a topological space, as formulated by Ganea.

Definition 4.33. *[Cor+03, Definition 1.64] The category of a connected space X is n, denoted by $\mathrm{cat}^{Gan}(X) = n$, if and only if n is the least integer such that there exists a section $s : X \to G_n(X)$ to p_n, where the sequence*

$$F_n(X) \to G_n(X) \xrightarrow{p_n} X$$

is the n-th Ganea fibration in the fiber-cofiber construction.

At first glance, it may not be clear that this definition is equivalent to the other definitions we have already analyzed. However, the next three results will show the equivalence between cat^{Wh} and cat^{Gan}. Consider the homotopy pullback of the map $T^{n+1} \xrightarrow{j_{n+1}} X^{n+1}$, i.e., the inclusion in the Whitehead definition, and the diagonal map $\Delta : X \to X^{n+1}$:

$$\begin{array}{ccc} \tilde{G}_n(X) & \xrightarrow{\delta} & T^{n+1}(X) \\ \downarrow{\scriptstyle \tilde{p}_n} & & \downarrow{\scriptstyle j_{n+1}} \\ X & \xrightarrow{\Delta} & X^{n+1}, \end{array}$$

where

$$\tilde{G}_n(X) = \{(x, \theta, y) \in X \times (X^{n+1})^I \times T^{n+1}(X) | \Delta(x) = \theta(0), j_{n+1}(y) = \theta(1)\},$$

and $\delta : \tilde{G}_n(X) \to T^{n+1}(X)$ and $\tilde{p}_n : \tilde{G}_n(X) \to X$ are the two projections.

Suppose that $\text{cat}^{Wh}(X) \leq n$. This implies that we have a map $\Delta' : X \to T^{n+1}(X)$ such that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} X & \xrightarrow{\Delta'} & T^{n+1}(X) \\ & {\scriptstyle \Delta}\searrow & \downarrow{\scriptstyle j_{n+1}} \\ & & X^{n+1}. \end{array}$$

Then, we obtain a map $s : X \to \tilde{G}_n(X)$ via the property of the homotopy pushout, i.e., we have a homotopy commutative diagram:

$$\begin{array}{ccccc} X & & \xrightarrow{\Delta'} & & \\ & \overset{s}{\searrow} & & & \\ & & \tilde{G}_n(X) & \xrightarrow{\delta} & T^{n+1}(X) \\ & {\scriptstyle \text{id}_X}\searrow & \downarrow{\scriptstyle \tilde{p}_n} & & \downarrow{\scriptstyle j_{n+1}} \\ & & X & \xrightarrow[\Delta]{} & X^{n+1}. \end{array}$$

such that $\tilde{p}_n \circ s \simeq \text{id}_X$. We now better understand the idea of the section, because we just showed that if $\text{cat}^{Wh}(X) \leq n$, then there exists a section to $\tilde{p}_n$. Conversely, if there exists a section to $\tilde{p}_n$, then, define $\Delta' : X \to T^{n+1}(X)$ by $\Delta' = \delta \circ s$. This satisfies the condition of commutativity in the definition of Whitehead, $j_{n+1}\Delta' \simeq j_{n+1}\delta s \simeq \Delta\tilde{p}_n s \simeq \Delta$. Hence, $\text{cat}^{Wh}(X) \leq n$. This leads to the following proposition:

Proposition 4.34. *[Cor+03, Proposition 1.57] There exists a section, $s : X \to \tilde{G}_n(X)$, of $\tilde{p}_n$ if and only if $\text{cat}^{Wh}(X) \leq n$.*

The obvious question that comes to mind at this point is why use the fiber-cofiber construction when we can take the homotopy pullback? As one might have seen, homotopy pullbacks are complicated spaces, and it is difficult to work with them. Therefore, instead of using the homotopy pullbacks, we try to describe them in a more explicit manner via the fiber-cofiber construction. The following theorem allows us to use $G_n(X)$ instead of $\tilde{G}_n(X)$ in the last construction.

Theorem 4.35. *[Cor+03, Theorem 1.63] For all n, there is a homotopy commutative diagram*

$$\begin{array}{ccc} G_n(X) & \xrightarrow{\simeq} & \tilde{G}_n(X) \\ & {\scriptstyle p_n}\searrow \quad \swarrow{\scriptstyle \tilde{p}_n} & \\ & X. & \end{array}$$

Moreover, this theorem gives us the equivalence between Ganea's definition and Whitehead's definition of category.

Theorem 4.36. *Whitehead's definition of category coincides with Ganea's definition of category:* $\mathrm{cat}^{Gan}(X) = \mathrm{cat}^{Wh}(X)$.

Proof. If $\mathrm{cat}^{Gan}(X) = n$, then n is the smallest integer for which there exists a section $s : X \to G_n(X)$ of p_n. According to Theorem 4.35, this also implies the existence of a section for $\tilde{p}_n$. Hence, Proposition 4.34 implies that $\mathrm{cat}^{Wh}(X) \leq n = \mathrm{cat}^{Gan}(X)$. Moreover, if $\mathrm{cat}^{Wh}(X) = n$, there exists a section to $\tilde{p}_n$, and by Theorem 4.35, there is also a section to p_n. This implies that $\mathrm{cat}^{Gan}(X) \leq n = \mathrm{cat}^{Wh}(X)$. Therefore, $\mathrm{cat}^{Gan}(X) = n$ when $\mathrm{cat}^{Wh}(X) = n$. □

In conclusion, the definitions of Whitehead and Ganea are equivalent. Furthermore, since the definition of LS-category and Whitehead's definition of category coincide, i.e., $\mathrm{cat}^{Wh}(X) = \mathrm{cat}(X)$, we can deduce that the three definitions are equivalent.

Chapter 5

Strong Lusternik-Schnirelmann category

In this chapter, we will explore another approach to extract a homotopy invariant object from $\operatorname{gcat}(X)$. To achieve this, we will force invariance and take the minimum of all the values of $\operatorname{gcat}(Y)$ for every space $X \simeq Y$. This new homotopy invariant is known as the *strong category*. The properties of the strong category are similar to those given for the geometric category. We will review some of them in the first section. In the second section, we establish a connection between the LS-category and the strong category using suspensions, and also calculate the strong category of a suspension. The third section introduces the concept of *cone-length*, which is a homotopy invariant that provides valuable information about the homotopical structure of spaces.

5.1 Definition and first properties

As explained, we are interested in finding a new homotopy invariant. To turn $\operatorname{gcat}(X)$ into a homotopy invariant, we simply consider the value of $\operatorname{gcat}(Y)$ for all $Y \simeq X$. We define it as follows:

Definition 5.1. *The* strong Lusternik-Schnirelmann category *of a topological space* X *is given by*

$$\operatorname{Cat}(X) = \min_{Y}\{\operatorname{gcat}(X) | Y \simeq X\}.$$

The strong category was first defined by Ganea in 1967 in [Gan67]. By definition, the strong category is a homotopy invariant. Hence, the two topological spaces J_1 and J_2 having the same homotopy type that we used in Section 3.2 to prove that the geometric category is not a homotopy invariant have the same strong category, $\operatorname{Cat}(J_2) = \operatorname{Cat}(J_1) = 1$.

Note that since $\operatorname{gcat}(X)$ is defined in terms of contractible open sets and $\operatorname{cat}(X)$ in terms of open sets contractible in the space X, if we have a cover satisfying the conditions of the definition of gcat, then, it is also a categorical cover. Thus, $\mathrm{gcat}(X) \geq \mathrm{cat}(X)$. In particular,

$$\mathrm{cat}(X) \leq \mathrm{Cat}(X)$$

Moreover, the strong category has some similar features to $\mathrm{gcat}(X)$.

Proposition 5.2. *[Cor+03, Corollary 3.5] In the category of pointed spaces having the homotopy type of CW-complexes, we have the following:*

1. *Let $D_{g,f}$ be the homotopy pushout of the inclusions $g : Z \to Y$ and $f : Z \to X$, then,*
$$\mathrm{Cat}(D_{g,f}) \leq \mathrm{Cat}(Y) + \mathrm{Cat}(X) + 1.$$

2. *If $F \to E \xrightarrow{p} B$ is a Serre fibration, then,*
$$\mathrm{Cat}(E) \leq (\mathrm{Cat}(F) + 1)(\mathrm{Cat}(B) + 1) - 1.$$
Moreover, $\mathrm{Cat}(E/F) \leq \mathrm{Cat}(B)$.

3. *If X has homotopical dimension n, then,*
$$\mathrm{Cat}(X) \leq n,$$
where the homotopical dimension is the minimal dimension of a CW-complex homotopy equivalent to X.

The idea of the proof is exactly the same as for the geometric category. Furthermore, recall that the mapping cone is the following homotopy pushout:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle c_*}\downarrow & & \downarrow \\ \{*\} & \longrightarrow & Y \cup_f CX. \end{array}$$

Moreover, $\mathrm{Cat}(\{*\}) = 0$. Therefore, Proposition 5.2(1) implies that $\mathrm{Cat}(Y \cup_f CX) \leq \mathrm{Cat}(Y) + 1$.

Remark 5.3. We specifically require Serre fibrations (refer to Section 2.1.4) because we are working within the category of pointed spaces that have the homotopy type of CW-complexes. The strong category examines all spaces that have the homotopy type of X. Therefore, we can restrict our consideration to the case of fibrations between CW-complexes, which corresponds to Serre fibrations.

Another property of the strong category is proven by Takens in [Tak70].

Proposition 5.4. *[Tak70, Section 3] If X and Y are topological spaces having the homotopy type of a CW-complex, then,*

$$\mathrm{Cat}(X \times Y) \leq \mathrm{Cat}(X) + \mathrm{Cat}(Y).$$

5.2 Link between LS-category and strong category

Since LS-category and strong category are both homotopical invariants and share some similar properties, we could try to determine how far they are from each other. In this section, the goal is to understand this relationship. We will see that they differ by at most 1. To prove this result, we need to calculate the strong category of the suspension which is defined as the following homotopy pushout:

$$\begin{array}{ccc} X & \xrightarrow{c_*} & \{*\} \\ {\scriptstyle c_*}\downarrow & & \downarrow \\ \{*\} & \longrightarrow & \Sigma X. \end{array}$$

Proposition 5.2 implies that for all topological spaces X, $\mathrm{Cat}(\Sigma X) \leq 1$. This result is analogous to Proposition 4.17 and Proposition 3.5, where we have seen that the geometric category and the LS-category of suspensions are bounded by 1. For the strong category, we obtain a more precise result, explaining that only suspensions have a category less than 1.

Proposition 5.5. *[Cor+03, Proposition 3.16] Strong category characterizes suspensions. More precisely, we have $\mathrm{Cat}(X) \leq 1$ if and only if $X \simeq \Sigma Z$, for some Z.*

This result is not proven in [Cor+03], we give the proof here:

Proof. Suppose that $X \simeq \Sigma Z$ for some topological space Z. Then, $\mathrm{Cat}(\Sigma X) \leq 1$ by what is explained at the beginning of this section. For the reverse implication, suppose that $\mathrm{Cat}(X) = 0$. Then, X is contractible, and we can choose $Z \simeq \{*\}$ since the suspension of a contractible space is of the same homotopy type as the point itself. If $\mathrm{Cat}(X) = 1$, up to replacing X by a homotopically equivalent space, there exists a cover of X, $\{\mathcal{U}_1, \mathcal{U}_0\}$, such that $X = \mathcal{U}_1 \cup \mathcal{U}_0$, and each $\mathcal{U}_i$, for $i = 0, 1$, is contractible. Let $Y = \mathcal{U}_0/(\mathcal{U}_1 \cap \mathcal{U}_0)$.

First we observe that $\mathcal{U}_0/(\mathcal{U}_1 \cap \mathcal{U}_0)$ is homeomorphic to $X/\mathcal{U}_1$. This implies that

$$Y = \mathcal{U}_0/(\mathcal{U}_1 \cap \mathcal{U}_0) \simeq X/\mathcal{U}_1 \simeq X/\{*\} \simeq X.$$

Now, consider the cofiber sequence $\mathcal{U}_1 \cap \mathcal{U}_0 \hookrightarrow \mathcal{U}_0 \to Y$ which gives the following homotopy pushout:

$$\begin{array}{ccc} \mathcal{U}_1 \cap \mathcal{U}_0 & \longrightarrow & \mathcal{U}_0 \\ \downarrow & & \downarrow \\ \{*\} & \longrightarrow & Y. \end{array}$$

Since $\mathcal{U}_0$ is contractible, the last homotopy pushout diagram is homotopy equivalent to the following diagram:

$$\begin{array}{ccc} \mathcal{U}_1 \cap \mathcal{U}_0 & \longrightarrow & \{*\} \\ \downarrow & & \downarrow \\ \{*\} & \longrightarrow & \Sigma(\mathcal{U}_1 \cap \mathcal{U}_0) \end{array}$$

which implies that $X \simeq Y \simeq \Sigma(\mathcal{U}_1 \cap \mathcal{U}_0)$. □

Finally, we have the following proposition explaining the link between the strong category and the LS-category. The proposition comes from Taken's paper [Tak70], where he uses a proof thought by Ganea.

Proposition 5.6. *[Cor+03, Proposition 3.15] For any path-connected CW-complex X, there exists a suspension ΣZ such that* $\mathrm{Cat}(X \vee \Sigma Z) = \mathrm{cat}(X)$*. Moreover,* $\mathrm{cat}(X) \leq \mathrm{Cat}(X) \leq \mathrm{Cat}(X) + 1$.

5.3 Cone-length

In this section, we work in CWTop_*. Since the strong category is closely related to the LS-category, we aim to understand why the strong category is necessary and what its other characterizations are. Specifically, we will see that the strong category equalizes the *cone-length* of a topological space in Theorem 5.11. We first define this notion and provide some examples.

Recall that any sequence $A \xrightarrow{f} X \xrightarrow{p} C$ is called a cofiber sequence if it is homotopy equivalent to the following sequence:

$$A \hookrightarrow M_f \xrightarrow{q_f} C_f.$$

This can be summed up in the following homotopy commutative diagram:

$$\begin{array}{ccccc} A & \xrightarrow{f} & X & \xrightarrow{p} & C \\ & \overset{\iota}{\searrow} & \downarrow \simeq & & \downarrow \simeq \\ & & M_f & \xrightarrow{q_f} & C_f, \end{array}$$

where $C_f \simeq X \cup_f A$. The definition of the cone-length depends almost entirely on this concept.

Definition 5.7. *[Cor+03, Definition 3.19] The* cone-length *of a path-connected space X, denoted by $\mathrm{cl}(X)$, is 0 if X is contractible. Otherwise, it is equal to the minimal $n \in \mathbb{N}^*$ such that there are cofiber sequences*

$$Z_{i-1} \to Y_{i-1} \to Y_i,$$

for $1 \leq i \leq n$ with $Y_0 \simeq *$ and $Y_n \simeq X$.

The suspension cone-length of X, denoted $\mathrm{cl}_\Sigma(X)$ is defined similarly with the additional requirement that for each $j \geq 0$, the space Z_j is a suspension of order j (i.e., $Z_j = \Sigma^j W$ for some W).

Example 5.8. 1. Spheres have a cone length of 1. Consider the following cofibartion sequence :

$$\begin{array}{ccccc} S^n & \xrightarrow{c_*} & \{*\} & \longrightarrow & S^{n+1} \\ & \searrow & \downarrow\simeq & & \downarrow\simeq \\ & & CS^n & \xrightarrow{q_{c*}} & \Sigma S^n. \end{array}$$

This is a sequence such that $Y_0 \simeq \{*\}$ and $Y_1 \simeq S^{n+1}$. Therefore, $\mathrm{cl}(S^n) = 1$ for $n \geq 1$ (the sphere S^0 is not path-connected).

Moreover, since every sphere of dimension $n \geq 1$ is a suspension, i.e., $\Sigma S^n = S^{n+1}$, we deduce that $\mathrm{cl}_\Sigma(S^n) = 1$ for $n \geq 2$. In the case of S^1, the suspension length is also 1. To see that, it suffices to recall that $\Sigma\emptyset = \{*\} \amalg \{*\} = S^0$ (see Example 2.16). Therefore, we obtain the following homotopy commutative diagram:

$$\begin{array}{ccccc} S^0 & \xrightarrow{c_*} & \{*\} & \longrightarrow & S^1 \\ & \searrow & \downarrow\simeq & & \downarrow\simeq \\ & & CS^0 & \xrightarrow{q_{c*}} & \Sigma S^0. \end{array}$$

2. As already explained in Example 1.14, the real projective n-space is a CW-complex obtained by attaching one cell in each dimension. Furthermore, the following square is a homotopy pushout:

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{q} & \mathbb{R}P^{n-1} \\ \iota\downarrow & & \downarrow j_1 \\ D^n & \xrightarrow{j_2} & \mathbb{R}P^{n-1} \cup_q D^n. \end{array}$$

Since D^n has the homotopy type of a point, we deduce from Example 2.16(5) that the sequence

$$S^{n-1} \to \mathbb{R}P^{n-1} \to \mathbb{R}P^n$$

is a cofiber sequence. Using an induction argument, we can conclude that $\mathrm{cl}(\mathbb{R}P^n) = n$.

The following result is a property of the cone-length that helps us understand why this number is called the "cone-length."

Lemma 5.9. *For a path-connected X with $\mathrm{cl}(X) = k$ we have*

$$X \simeq Y_0 \cup CZ_0 \cup CZ_1 \cup \cdots \cup CZ_{k-2} \cup CZ_{k-1}$$

where Y_0 and the Z_i's come from the associated cofiber sequence.

This is called a cone decomposition of X.

Proof. If $\mathrm{cl}(X) = k$, then there exist cofiber sequences

$$Z_{i-1} \xhookrightarrow{j_{i-1}} Y_{i-1} \to Y_i, \qquad 1 \leq i \leq k,$$

with $Y_0 \simeq \{*\}$ and $Y_k \simeq X$. By definition, we have the following commutative diagram:

$$\begin{array}{ccccc} Z_{i-1} & \xrightarrow{j_{i-1}} & Y_{i-1} & \longrightarrow & Y_i \\ & \searrow & \downarrow\simeq & & \downarrow\simeq \\ & & M_{j_{i-1}} & \longrightarrow & C_{j_{i-1}}. \end{array}$$

We prove the statement by induction. If $\mathrm{cl}(X) = 0$, then $X \simeq \{*\}$, which implies we can take $X \simeq Y_0$. If $\mathrm{cl}(X) = 1$, then we only have one sequence

$$Z_0 \xhookrightarrow{j_0} Y_0 \to Y_1,$$

so that $Y_1 \simeq C_{j_0} = Y_0 \cup_{j_0} CZ_0$.

We have the following induction hypothesis: if $\mathrm{cl}(X) = k-1$, then, $X \simeq Y_0 \cup CZ_0 \cup CZ_1 \cup \cdots \cup CZ_{k-2}$. Let us assume now that $\mathrm{cl}(X) = k$. This implies that there exists cofiber sequences with $\mathrm{cl}(Y_{k-1}) = k-1$. If it were not the case, we would have a smaller cone-length. The induction hypothesis implies that

$$Y_{k-1} \simeq Y_0 \cup CZ_0 \cup CZ_1 \cup \cdots \cup CZ_{k-2}.$$

Now, the last cofiber sequence of X is $Z_{k-1} \xhookrightarrow{j_{k-1}} Y_{k-1} \to Y_k$, with $Y_k \simeq X$. Because it is a cofiber sequence, we have that

$$Y_k \simeq Y_{k-1} \cup_{j_{k-1}} CZ_{k-1} \simeq Y_0 \cup CZ_0 \cup CZ_1 \cup \cdots \cup CZ_{k-2} \cup CZ_{k-1}.$$

Therefore, $X \simeq Y_0 \cup CZ_0 \cup CZ_1 \cup \cdots \cup CZ_{k-2} \cup CZ_{k-1}$. □

Example 5.10. In the fiber-cofiber construction (see 4.30) we assume that we have a fibration $F_n(X) \xrightarrow{i_n} G_n(X) \xrightarrow{p_n} X$, and we take $C_{i_n} = G_n(X) \cup_{i_n} CF_n(X)$, which is the mapping cone of $F_n(X) \xrightarrow{i_n} G_n(X)$. This gives the following cofiber sequence:

$$\begin{array}{ccccc} F_n(X) & \xrightarrow{i_n} & G_n(X) & \longrightarrow & G_{n+1}(X) \\ & \searrow & \downarrow \simeq & & \downarrow \simeq \\ & & M_{i_n} & \longrightarrow & C_{i_n}. \end{array}$$

Therefore, for all $n \in \mathbb{N}$, $\mathrm{cl}(G_n(X)) \leq n$.

We now give the proof of the fact that the cone-length coincides with the strong category for a given topological space.

Theorem 5.11. *For a path-connected CW-complex X,* $\mathrm{Cat}(X) = \mathrm{cl}(X)$.

The proof is inspired from the proof of [Cor+03, Theorem 3.26].

Proof. We first show that $\mathrm{Cat}(X) \leq \mathrm{cl}(X)$. If $\mathrm{cl}(X) = k$, then by Lemma 5.9(1.), $X \simeq Y_0 \cup CZ_0 \cup CZ_1 \cup \cdots \cup CZ_{k-2} \cup CZ_{k-1}$. Since Y_0 has the homotopy type of a point and since a cone is a contractible space, Proposition 5.2 implies that $\mathrm{Cat}(X) \leq k = \mathrm{cl}(X)$. Now, for the reverse inequality, we use an induction argument. If $\mathrm{Cat}(X) = 1$, then Proposition 5.6 implies that $X \simeq \Sigma Z$ for a space Z and the cofiber sequence $Z \hookrightarrow CZ \to \Sigma Z$ is such that $CZ \simeq \{*\}$ and $\Sigma Z \simeq X$. Hence, $\mathrm{cl}(X) = 1$.

Now, let us assume the following induction hypothesis : if $\mathrm{Cat}(X) \leq n - 1$, then, $\mathrm{Cat}(X) = \mathrm{cl}(X)$. Suppose that $\mathrm{Cat}(X) = n$, with a categorical cover $\{\mathcal{U}_0, \ldots, \mathcal{U}_n\}$ such that $X = \bigcup_{i=0}^{n} \mathcal{U}_i$. Consider $X' = \bigcup_{i=0}^{n-1} \mathcal{U}_i$, which has a strong category less than or equal to $n-1$ and such that $X = X' \cup \mathcal{U}_n$. Let $V = X' \cap \mathcal{U}_n$. The homotopy cofiber of the inclusion $\iota : V \hookrightarrow X'$ is denoted by $C_\iota = X' \cup CV$. This space has the same homotopy type as X since $X = X' \cup \mathcal{U}_n$ and attaching CV to X' is homotopically equivalent to recreating the open $\mathcal{U}_n$ using the cone of the intersection $X' \cap \mathcal{U}_n$ and attaching it to X'. By the induction hypothesis, because $\mathrm{Cat}(X') = n - 1$, there is a cone decomposition of length $n-1$ with $Y_0 \simeq \{*\}$ and $Y_{n-1} \simeq X'$. That is, there is a homotopy equivalence $g : X' \to Y_{n-1}$. Consider the composition $g \circ \iota : V \to Y_{n-1}$ and take the homotopy cofiber:

$$\begin{array}{ccccc} V & \xrightarrow{\iota} & X & \xrightarrow{g} & Y_{n-1} \\ & & \downarrow{\scriptstyle q_\iota} & & \downarrow{\scriptstyle q_{g\circ\iota}} \\ & & C_\iota & \xrightarrow{\simeq} & C_{g\circ\iota}. \end{array}$$

Because g is a homotopy equivalence, we have $C_\iota \simeq C_{g\circ\iota}$. Therefore, $X \simeq C_\iota \simeq C_{g\circ\iota}$. Moreover, the following sequence is a cofiber sequence:

$$V \xrightarrow{g\circ\iota} Y_{n-1} \xrightarrow{q_{g\circ\iota}} C_{g\circ\iota}$$

with $X \simeq C_{g\circ\iota}$. By considering the cone decomposition of X' where we add this last cofiber sequence, we obtain a cone decomposition of X such that $Y_0 \simeq \{*\}$ and $X \simeq C_{g\circ\iota}$. Therefore, $\mathrm{cl}(X) \leq n = \mathrm{Cat}(X)$. □

This last theorem gives a precise result for the strong category of topological spaces for which we can calculate the cone length precisely. As explained in Example 5.8, the n-sphere has a cone length of 1, hence $\mathrm{Cat}(S^n) = 1$. The projective n-space has a cone length of n, thus $\mathrm{Cat}(\mathbb{R}P^n) = n$, which corresponds well with Example 4.8.

Conclusion

The goal of this thesis was to familiarize ourselves with specific homotopy invariants. In the first part, we had the opportunity to better understand how the hTop and hTop_* categories work, especially in terms of pushouts and pullbacks. Moving on to the second part, we introduced the geometric category. We then applied this new concept to the objects discussed in the first two chapters and proved that it is not a homotopy invariant. This raised questions about how we could obtain a homotopic invariant, which led to the definitions of LS-category and strong category. These definitions appeared to be homotopy invariants. In the last two chapters, we explored additional definitions and properties of the LS-category and the strong category.

Our main focus was on studying an upper bound of the LS-category by considering contractible open sets. Additionally, the work was designed to be accessible to a master's student with some knowledge of algebraic topology and category theory. Consequently, we did not delve into the details of homology theory (cohomology) while discussing the lower bound in Theorem 4.7, or when mentioning Alexander duality. However, it is worth noting that there are other homotopy invariants that can serve as lower bounds, such as the *Toomer invariant* (see [Cor+03, Definition 2.7]).

Moreover, there are other intriguing homotopy invariants of the same nature as the ones we studied. For instance, if X is a topological space, instead of seeking contractible open sets of X, we can investigate the existence of a cover that contains closed disks. This concept gives rise to the notion of the *ball-category* of X (see [Cor+03, Definition 3.1.]). Additionally, rather than focusing solely on covers of X, we can explore covers of $X \times X$ that satisfy certain conditions. This approach may lead to the definition of *topological complexity* (see [JMP12, Definition 1.1]). The study of these objects is therefore a natural extension of this thesis.

We hope that the reader, just as much as the writer, has enjoyed exploring this area of mathematics and has appreciated the ideas presented in this thesis.

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