Economics School of Louvain - ESL

Economics School of Namur - ESN

A Proposal for an Excessivist Social Welfare Ordering

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Academic Year 2023-2024

Master in Economics, Research focus - 120 credits (acces direct – 60 credits)

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Contents

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Master Thesis

Abstract

In this paper, we introduce a sub-class of rank-weighted social welfare orderings that we call "Excessivist". The Excessivist Social Welfare Ordering (eSWO) judges incomes above a fixed threshold as detrimental for the society. To accomplish this, the identification of a richness or affluence line is necessary; we employ a fixed, exogenous line of excess. We define an eSWO in form of a weighted sum of individual's incomes. This requires to introduces n+1 vectors of weights, one for all possible number of individuals below the threshold. To do this, the paper introduces some modifications of the class of rank-weighted social welfare orderings. Indeed, in our proposal, we allow the weights to be both positive (for individuals below the line) and negative (for individuals above); moreover, they assure that the higher the number of excessively rich individuals, the greater the negative impact the society has and vice versa. Then we introduce ethical concerns through an axiomatic approach. We proved that the proposed ordering satisfies the following axioms: absolute Aversion to Excessive Richness (AER), Pigou Dalton positive weights preserving Transfer (PDwpT), sign weight preserving Ratio Scale Comparability (swpRSC) and Strong Pareto below the threshold (SPb).

*Acknowledgments: I would like to express great appreciation to my supervisor Professor François Maniquet for his support and valuable insights, I would also like to thank Professor Gregory Ponthiere as reader of the thesis.

1 Introduction

The social welfare function approach has been largely implemented to rank alternative social states. This is due to the fact that any SWF subsumes some ethical values, crystallized in axioms and used to define a society as just. According to Sen's (2018) definition, social welfare functions can be described as mappings that establish the social ordering of a set of alternatives using a comprehensive list of individual numerical indicators, which may not necessarily be construed as utility functions. Since each social welfare function subsumes a specific complete and transitive binary relation on the set of alternatives (social welfare ordering) we will focus on this latter concept hereafter.

The social welfare ordering framework that will be implemented is consequentialist because it derives the rank judgement on the outcome which comes from an appropriate aggregation of individual well-being (Adler, 2019). Thus, it is also welfarist in the sense that social welfare comes from an aggregation of individual welfare indices (Fleurbaey and Maniquet, 2011). This latter argument necessarily raises the need of an appropriate variable for normative judgment. Such a dispute is well known. As declared by Rawls (1971), Dworkin (1981) and other philosophers, the distributional justice should merely involve resources, income or primary goods, but this is contestable see Bossert, Cato, and Kamaga (2022). We do not intend to enter in such dispute, in our framework we focus on individual income as representative indicator of individual well-being.

As said before, different SWO's embody dissimilar, often divergent ethical concerns. The view of justice investigated in this work and embodied in our excessivist ordering, asserts that a more equal society should be preferred to a more unequal one, that the level of income above a certain threshold procures a negative impact on the social welfare and that the reduction of this excess of income will be considered a social improvement whatsoever. This view is connected with, at least, two principles of fairness: the egalitarianism and sufficintarianism. At first sight these two principles have many facets and they are often opposed. Following Temkin (2003), by the term egalitarian we mean the ethics that attaches some value to equality state itself, and it will be discussed later on.

Moreover, our proposal will adopt the implementation of a threshold, like in the Sufficientist approach. Sufficientarianism is an ethical doctrine that has been recently analysed axiomatically by Alcantud, Mariotti, and Veneziani (2022) and Bossert et al (2022), but it traces back the thought of Frankfurt (1987). In its original formulation this principle assigns absolute priority to individuals below a certain threshold and it cares only that everybody has enough.

In this work the threshold will be of excess rather than of sufficiency. The identification of such line is an important and difficult task, we propose an exogenous fixed richness line, which even if simplistic seems a good starting point for our purposes. [1]

However, what these social ethics have been overlooking, together with the vast array of different rules, is the concern for extreme richness. Indeed, they would judge as positive an increase in the income of a super-rich individual, other things being equal. As argued by Atkinson (2007), income distribution of the rich need to be taken more into consideration for several reasons. An highly concentrated income can lead to a polarization, and it can generate social conflict. Moreover, an uneven distribution of political power driven by an uneven distribution of income would become a severe danger for our democracy. All this should bring as to treat the extreme richness as an issue.

In order to address our ideal of justice, the paper introduces a slight modification of the class of rank weighted of social welfare ordering (Weymark, 1981; Bossert, 1990) or rank weighted (dependent) utilitarianism (Sakamoto, 2020). Introduced by Bentham (1996), utilitarianism is a welfarist concept employed for ranking social alternatives such that alternative x surpasses alternative y only if the total utility in x exceeds that in y. This principle stays silent about income inequality and societal provisions for special needs (sufficiency or excess) which are major concerns for this work. A family of principles that has the same structure of the utilitarianism but is a better candidate is the rank weighted (dependent) utilitarian family of principles, which includes utilitarianism as a special case. The way in which the weights are assigned can ensure the observance of specific needs and ethical concerns.

Our excessivist social welfare ordering (eSWO) is defined as the weighted sum of individual well-beings. This requires to introduces $n+1$ vectors of weights,

[1]See Medeiros and Souza (2014) for a review of affluence lines proposed in the literature

one for all possible number of individuals below the threshold. Higher weights are assigned to less well-beings, and they are positive (for individuals below the the richness line) and negative (for individuals above). Another condition (see section 2) on the weights is added due to the variable number of individuals above and below the threshold. This assures that the higher the number of individuals above the threshold, the greater the negative impact the society stands, and vice-versa for incomes below the threshold. To do so we relax the Strong Pareto axiom, which still holds for non rich individuals, and we introduce the novel axiom of Aversion to Excessive Richness that gathers our ideal of fairness. We add the sign weight preserving Ratio Scale Comparability for the informational foundation of our social welfare ordering and Pigou Dalton weight preserving Transfer as dominance axiom. The rest of the work is structured as follows: We briefly describe the ethics we are inspired by (section 1.1), then we justify our view of justice (section 1.2). Section 2 describes the theoretical framework and defines the excessivist social welfare ordering. Section 3 is devoted to the axioms that our ordering should satisfy. We focus on the partial characterization in section 4 (claim 1 to 5 are proven). Section 5 concludes.

1.1 Comparative Egalitarianism

According to this viewpoint, equality, seen as comparative fairness, holds inherent value, meaning it is valuable in itself, not just as a means to promote other ideals or goals. This has an important implication: a mean preserving transfer of the same amount from a relatively rich individual to a poorer one, is valued as just because it is inequality reducing, not because it helps the worst off individual [2]. Thus, Egalitarianism is transfer sensitive, a useful property embraced by our eSWO.

Moreover, as articulated by Adler (2019), egalitarians focus on relative considerations, examining how an individual's income (utility, well-being) aligns with the levels of others. Indeed, following the reasoning of Sen (1997) the rank-weighted family of social welfare orderings is contingent not on income magnitude but on the count of individuals positioned in between them. Additionally, in any pairwise comparisons, the individual with the lower income

[2]This is typical of prioritarian view, which has been largely developed by Parfit and it gives greater weight to transfer affecting worse-off individuals. "Benefit for the worse off matter more" (Parfit, 1984)

can be considered to be experiencing a degree of discontent [3], upon realizing their income is lower. Let this discontent be proportionate to the income difference. The cumulative sum of all such discontent in all feasible pairwise comparisons leads us to a rank-weighted SWO, which, henceforth, embodies the comparative egalitarian ethics.

Egalitarianism has faced at least one criticism: the "levelling down objection", which argues that simply reducing the well-being of a better-off person to the level of someone worse off does not lead to any ethical improvement. This objection, as noted by Temkin (2003), has strong intuitive appeal, but its strength stems largely from a position known as the "Slogan". It looks like the well-known Pareto principle which we are going to weaken for all excessively rich individuals.

The main replay to the objection is well expressed in the example that I report here and that cannot be better explained: "Egalitarians believe that it is unfair for some to be born blind, while others are not .... So they believe there is one respect in which an all-blind world would be better than one where some are blind and others sighted; it would be better regarding comparative fairness. But egalitarians do not believe that we should blind everyone; first...second and more importantly, because egalitarians are pluralists, and the all-blind world is surely worse than the partially-blind one, all things considered. Equality is not all that matters. Still, it matters some, and I see little reason for the egalitarian to forsake that conviction in the face of the levelling down objection" (Temkin, 2003). This is near to the conclusion of Walzer (1983) of the existence of several spheres of justice and the argument in favor of a "complex equality" concept.

To put it simple I align myself to what argued by Temkin: "I believe that it is terribly unfair that..., some must struggle to survive, and ultimately lose that struggle, while others indulge in extraordinary extravagance. That unfairness is only exacerbated, making an already bad situation even worse, when I imagine even more frivolous luxuries laid in the laps of the Rich or the Super-rich" (Temkin, 2003).

[3] Which is named complain in the terminology of Temkin (1993)

1.2 Income as Source of Power

We support our proposal through what as been argued by Atkinson (2007), and Barry (1998) about the role of income in getting command over people. Is this power linked with income? If the answer is yes, than of course we should refer to the top income owners as a threat.

However, are we able to conclude that this threat is due to having more than 1 million or with having 1 billion or some multiple of the mean income? The answer is no, nevertheless, the threat remains, because the last question falls back in the task of finding a richness line and does not yield us to abandon the "Income as Source of Power" position. Barry in particular identifies the link between the social exclusion and the top income individuals. He called this social exclusion as voluntary, and it is caused by the fact that the wealthiest fraction believes they can shield themselves from the shared destiny and escape the common institutions by purchasing their way out.

The theme that is crucial is to recognize that their wealth empowers them to construct barriers that prevent their others citizens from entering. Consequently, as Barry says: "this situation results in a minority having the ability to exclude the majority" (1998). Social exclusion, particularly in politics, is closely tied to exclusion from other social networks. Political networks often stem from social connections, influencing the workings of various associations and informal organizations. For instance, block associations, community groups, and informal organizations formed for specific purposes may be affected by this dynamic. If the interests of the socially excluded align with those who are more socially active, the outcomes may not differ significantly. However, it is essential to recognize that the socially excluded might have distinct interests, even in neighborhood politics.

Ultimately, preventing social exclusion becomes crucial in averting political exclusion. The link between one's position in a social network and political influence can be challenging to weaken, especially in inherently informal political contexts. Therefore, promoting inclusivity and addressing social exclusion are vital steps in ensuring equal participation and representation in politics and society.

"The ability to use 'exit' as a strategy is a clear manifestation of power" is expressed by Atkinson (2007) and, in addition, the author proposes to catch this power of income "in terms of number of people on average gross earnings that could be employed by a given income". It is an interesting proposal for our reasoning. Indeed, if income is a source of power and we measure this power in term of number of people that can be employed then: the more the richness the more the individuals employed, and thus, the negative effect that such link produces into the society. For instance, we can replicate the former reasoning in the political system. Here the capability to drive the social choice can be related to the number of votes that can be bought by a given income. The higher the income, the larger the amount of votes. And, we can add, if a restricted group of rich individual is able to buy the vote of the majority, then the democratic system itself is put in danger. In conclusion, if we consider just the society the one free of domination (Walzer, 1983) and if we treat the domination as mediated by some set of social goods (income), we owe to eradicate the dependence position of the poor, and to prevent, as well, the advantage or dominant status of the richer.

2 Theoretical Framework

2.1 Notation and definitions

A society is made up of a set $N = \{1, .., n\}$ of individuals of size n. An income distribution is denoted by $x = \{x_1, .., x_n\} \in D = \mathbb{R}^n_+$. Thus for any individual $i \in N$ corresponds an income $x_i \in \mathbb{R}_+$.

Let θ be a fixed threshold of income, which we call excessive richness line. We employ an absolutist notion of excessive richness where θ is assumed to be exogenously given. We can now define $\forall x \in D$ two sets:

$$h(x) = \{i \in N \,|\, x_i \geq \theta\}$$
$$l(x) = \{i \in N \,|\, x_i < \theta\}$$

Which represent the sets of individuals considered excessively rich and not considered excessively rich respectively.

For all distributions $x, x' \in D$ a social welfare ordering (SWO), denoted R, is a complete and transitive binary relation on D such that xRx' means that x distribution is at least as good as distribution x' and we denote the asymmetric and symmetric relation as P and I respectively.

2.2 Excessivist SWO

Our proposal for an excessivist SWO (eSWO) starts form the weighted utilitarian framework. Roughly speaking, for this kind of ordering the ranking of individuals is used to determine the weight (importance) of an individual. Once the weights have been assigned, they are then multiplied by the respective individual's incomes and the results are summed in a utilitarian way.

The specific representation of rank-weighted social welfare ordering and the ethics that it encapsulates may vary depending on the specific weights and their assignment rule. Our proposal differs form the standard approach because we do not employ a unique vector of weights and we allow for both positive and negative weights.

For our excessivist SWO, we need first to define $A = \{0, ..., n\}$ a set of non negative integers, which represents the set of all possible numbers of individuals below θ, and we denote by $j \in A$ the actual number of individuals below θ. We introduce now a family of sets:

$$\Lambda = \{\lambda^0, \ldots, \lambda^n\} \in \mathbb{R}^{n(n+1)}$$

Where any $\lambda^j \in \mathbb{R}^n$ is a vector of weights for a distribution with j numbers of individuals below the threshold. More formally, $\forall j \in A$ and $\forall x \in D$:

$$\lambda^j = \{\lambda_i^j \in \mathbb{R} \,|\, j = \text{card}(l(x) = \{i \in N \,|\, x_i < \theta\})\}$$

In summary we have introduced one vector of weights for all possible number of individuals below the threshold, and since they vary between 0 and n, we end up with n+1 possible vectors of weights.

Once defined the topological structure for our weights, we need to specify the way in which the weights are assigned for each individual. We can reach the solution by imposing some requirements in the assignment rule. Here will be only introduced the conditions of the weights since a larger discussion follows in section 4 after having defined the axioms.

If we denote by $k \in \mathbb{N}_0$ the rank of individuals ordered form the poorest to the richest, we define

$$f : \{1, .., n\} \subset \mathbb{N}_0 \to \mathbb{R} \qquad (1)$$

as a weight function which assigns a weight for each individual's rank $k \in \mathbb{N}_0$.

This function should assign weights such that: $\forall x \in D$ and $\forall j \in A$

Condition 1.

$$\lambda_i^j \leq 0 \;\forall i \in h(x), \;\; and \; \lambda_i^j \geq 0 \;\forall i \in l(x)$$

Since we argue that incomes above θ have negative impact for the society, then we should have negative weights for these incomes. However, below θ the weights are positive because any increasing income is welfare improving.

Condition 2.

$$\lambda_i^j \leq \lambda_{i+1}^j \; if \, x_i \geq x_{i+1} \forall i \in N$$

Condition 2 means that we are giving priority to poorer individuals whose income has more weight.

Condition 3

$$\forall x, x' \in D, \;\; with \; card(l(x)) < card(l(x')).$$

$$\sum_{i \in l(x)} \lambda_i^j \leq \sum_{i \in l(x')} \lambda_i^{j+1}, \sum_{i \in h(x)} \lambda_i^j \leq \sum_{i \in h(x')} \lambda_i^{j+1}$$

$$and \; |\lambda_i^j| \geq |\lambda_i^{j+1}|$$

This condition is introduced for all that cases of transfers which make one individual cross the threshold. The less the amount of individuals above the threshold, the grater the positive impact on the society and vice-versa.

Now it is possible to define our rule for ranking alternative income distributions: the excessivist social welfare ordering.

Definition 1.

A class of social welfare orderings is defined excessivist (R^e) if and only if:

$$\forall x, x' \in D, \;\forall i \in N, \;\forall j, h \in A,$$

$$\exists\Lambda = \{\lambda^0, \ldots, \lambda^n\} \in \mathbb{R}^{n(n+1)}$$

with

$$\lambda^j, \lambda^h = \{\lambda_i^j, \lambda_i^h \in \mathbb{R} \,|\, j = \text{card}(l(x)),\ h = \text{card}(l(x'))\},$$

$$\lambda_i^j, \lambda_i^h \geq 0 \text{ if } x_i, x_i' < \theta,\ \lambda_i^j, \lambda_i^h \leq 0 \text{ if } x_i, x_i' \geq \theta,$$

$$\lambda_i^{j,h} \geq \lambda_{i+1}^{j,h} \textit{ if } x_i \leq x_{i+1}, x_i' \leq x_{i+1}',$$

$$\sum_{i \in l(x)} \lambda_i^j \leq \sum_{i \in l(x')} \lambda_i^{j+1}, \sum_{i \in h(x)} \lambda_i^j \leq \sum_{i \in h(x')} \lambda_i^{j+1}$$

and

$$|\lambda_i^j| \geq |\lambda_i^{j+1}|$$

such that

$$xR^e x' \iff \sum_{i \in N} \lambda_i^j x_i \geq \sum_{i \in N} \lambda_i^h x_i'.$$

3 Axiomatization

We turn now the attention on the axioms which the eSWO should satisfy, we define first a general axiom of robustness, and then the dominance and invariance requirements will be presented. For almost each axiom a brief discussion and an example follow the formal definition.

Continuity Above and Below the Threshold

$$\forall x, x' \in l(x), l(x') \textit{ if the sets } B = \{x \in l(x) \,|\, xR^e x'\} \textit{ and}$$
$$C = \{x \in l(x) \,|\, x' R^e x\} \textit{ are closed } \implies R^e \textit{ is continuous below } \theta$$

and

$$\forall x, x' \in h(x), h(x') \textit{ if the sets } D = \{x \in h(x) \,|\, xR^e x'\} \textit{ and}$$
$$E = \{x \in h(x) \,|\, x' R^e x\} \textit{ are closed } \implies R^e \textit{ is continuous above } \theta$$

First we have introduced an axiom of robustness. It requires that small changes in two income distributions above or below the threshold do not lead to changes in their ordering R^e.

Absolute Aversion for Excessive Richness

$$\forall x, x' \in D : card(l(x)) = card(l(x')),\ \textit{if}\ x_i = x'_i\ \forall i \in N,$$

$$\textit{but}\ x_j \leq x'_j\ \textit{for any}\ j \in h(x), h(x') \implies xR^e x'$$

This is the core dominance axiom for the eSWF. Roughly speaking, the axiom suggests that if two distributions $x, x' \in D$ are identical for any respect but for one individual above the threshold, who is richer in x', then x should be preferred to x' by the society. For instance looking at the table where θ is the threshold we should rank $xR^e x'$:

Table 1: $\theta = 10$

x	1	3	5	35	50
x'	1	3	5	40	50

This means that we do not care about waste of resources above the threshold, the only priority is the reduction of excessive income. To understand better the implication of this axiom, let's recall the milestone strong Pareto axiom which is a widely accepted principle. Formally:

$$\forall x, x' \in D\ \textit{if}\ x_i \geq x'_i\ \forall i \in N,\ \textit{but for one}\ j \in N, x_j > x'_j \implies xPx'$$

It means that if everyone is at least as well off in one distribution compared to another, with at least one individual being better off, the former distribution is considered better than the latter.

Our axiom is a clear violation of strong Pareto. The absence of such principle opens to some critiques, however "there is no a priori reason to reject a non paretian welfare function, nor does our every day experience of human judgements make it implausible" as expressed by Kondor (1989).

Pigou Dalton Positive Weights Preserving Transfer

$$\forall x, x' \in D,\ \textit{if}\ \exists\, \delta \in \mathbb{R}_+ \,|\, x'_i = x_i + \delta,\ x'_j = x_j - \delta,$$

$$\textit{with}\ \ x_i \leq x'_i \leq x'_j \leq x_j\ \ \forall i, j \in N,$$

$$j = card(l(x)) \leq k = card(l(x')),$$

$$and\ x_h = x'_h\ \forall h \in N \setminus \{i,j\} \implies x' R^e x$$

According to this principle, a transfer from a better-off individual to a worse-off individual despite their relative position to the threshold, without reversing their ranks, leads to an improved distribution if:

1. the number of individuals below the threshold is the same after the transfer

2. the number of individuals below the threshold has increased

This means that are allowed all transfers that bring "excessive" rich individuals below θ but not the opposite. Such choice comes form the fact that we desire a distribution free of excessive richness. Hence, a Pigou Dalton Transfer which reduces the distance among two individuals but that makes one of them excessively rich, could cause a struggle with the AER axiom previously defined.

An example follows, where the distribution x' is obtained from distribution x after a transfer below θ of $\delta = 1$, whereas x'' shows another possible cross transfer which brings an individual below θ, and x''' involves a transfer only above θ (in red the threshold, in green the transfers).

Table 2: $\theta = 10$

x	1	3	/	10	14	35	50
x'	1	4	/	14	9	35	50
x''	1	8	9	10	/	35	50
x'''	1	3	/	10	15	34	50

This axiom represents a slightly modification of the standard Pigou Dalton Transfer principle, which is defined formally:

$$\forall x, x' \in D, \forall i, j \in N\ if\ \exists\, \delta \in \mathbb{R}_+ : x_i \leq x'_i = x_i + \delta \leq x'_j = x_j - \delta \leq x'_j$$

$$and\ \forall h \in N \setminus \{i,j\},\ x_h = x'_h \implies x' R x$$

Strong Pareto Below Threshold

$$\forall x, x' \in D \; if \; x_i = x'_i \; \forall i \in N \;\; but$$

$$x_j < x'_j < \theta \; for \; one \; j \in l(x), l(x') \implies x' R^e x$$

This requirement simply stands to avoid waste of resources. Indeed, an increase in the income of an individual below the threshold is a social improvement, as in table 3.

Table 3: $\theta = 10$

x	1	3	10	35	50
x'	1	4	10	35	50

Since it holds only for individual below θ, a hierarchy of principle, where Aversion to Excessive Richness gets priority, appears clear. Indeed, our formulation is a clear drop of the standard Strong Pareto.

Anonymity

$$\forall x, x' \in D \;, and \; permutation \; \pi, \;\; if \; x' = \pi x \implies x I x'$$

This requirement crystallizes the commitment for impartial deliberation. If two distributions are ordered in the same pattern and they differ only in the identity of the individuals then, the rank should not change. For example, the two distributions x, x' are equally good:

Table 4:

x	1	3	5	35	50
x'	35	3	1	5	50

Sign Weight Preserving Ratio Scale Comparability

$$\forall x, x', y, y' \in D \; if \; \exists \, b \in \mathbb{R}_+ : y_i = bx_i, \;\; y'_i = bx'_i, \; \forall i \in N,$$

$$and\ card(l(x)) = card(l(y)),\ card(l(x')) = card(l(y')),\ then$$
$$xR^e x' \iff yR^e y'$$

Now we address the question of the informational basis and invariance for our SWO. The concept of invariance is typically applied to a principle aimed at limiting the measurability and comparability aspects of individual evaluation profiles (d'Aspremont and Gevers, 2002). These aspects constitute the informational foundation of a SWO, as described by Sen (2018).

The social ranking linked to a certain profile must remain unchanged when the original profile is substituted with an equivalent one.

Our axiom is more demanding than a standard Ratio Scale Comparability axiom. Indeed, we need to assure that after a transformation one individual does not end to become excessively rich in only one distribution and not in the other; indeed, since we have introduced both positive and negative weights, this could cause a change in the ordering between distributions. The issue is reported in the table below:

Table 5: $\theta = 20$

x	x'	y	y'	$\lambda^{j=3}$	$\lambda^{h=2}$
1	3	2	6	0.52	0.54
2	5	4	10	0.32	0.34
/	15	/	/	0.12	/
21	20	42	30	-0.07	-0.14
25	60	50	40	-0.27	-0.25
50	/	100	120	/	-0.45

Here $x, x' \in D$ are two distributions ranked such that $x'Rx$, λ^j is the vector of weights that we assign to the distributions of income with $card(l(x)) = j = 3$ and λ^h is the vector assigned when $card(l(x)) = h = 2$. Whereas y and y' are two distributions obtained by x and x' through a transformation ($b = 2$), such that, one non excessive rich individual in x becomes "super rich" (in green) after the transformation. Applying the excessivist SWO as by definition 1 we end to rank $x'R^e x$ and $yR^e y'$ violating the axiom. Whereas, in the example below, where the individuals positions respect to θ are preserved, the sign weight preserving Ratio Scale Comparability holds.

Table 6: $\theta = 20$

x	x'	y	y'	$\lambda^{j=3}$	$\lambda^{h=2}$
1	3	2	6	0.52	0.54
2	5	4	10	0.32	0.34
/	6	/	12	0.12	/
21	20	42	40	-0.07	-0.14
25	60	50	120	-0.27	-0.25
50	/	100	/	/	-0.45

4 Partial Characterization

This section is devoted to analyse the conditions imposed by the axioms above discussed on the weight function. Indeed, we have defined in section 1 the class of eSWO through specific conditions on the weights, these, as we will see, allow our eSWO to satisfy the axioms.

We make clear under what conditions the eSWO satisfies, Aversion to Excessive Richness (AER) and Pareto below threshold (Pbt), Pigou Dalton positive weights preserving Transfer (PDpwpT) and sign weight preserving Ratio Scale Comparability (swpRSC).

Hence, $\forall x \in D$, $\forall j \in A$, and $\forall i \in N$, the individual's weights λ_i^j for the i-th individual in the distributions with j individuals below the threshold are determined as follows:

Condition 1.

$$\lambda_i^j \leq 0 \ \forall i \in h(x), \quad and \ \ \lambda_i^j \geq 0 \ \forall i \in l(x)$$

Condition 1 is a necessary and sufficient condition for the AER and Pbt axioms (the proofs are straightforward). Since we argue that incomes above θ have negative impact for the society, then we should have negative weights for these incomes. However, below θ the weights are positive because any increasing income is welfare improving. Hence, we refuse a standard monotonicity assumption.

Condition 2.

$$\lambda_i^j \leq \lambda_{i+1}^j \ \textit{if}\ x_i \geq x_{i+1} \forall i \in N$$

Condition 2 is necessary for Pigou Dalton Positive weights preserving Transfer to hold. Condition 2 means that we are giving priority to poorer individuals whose income has more weight.

Claim 1. *R^e satisfies PDpwpT $\implies$ Condition 2*

Proof. We show first the proof that Condition 2 is a necessary condition for PDpwpT to hold with R^e as ordering rule. This equals to prove that if R^e satisfies PDpwpT then $\forall i \in N$, $\forall j \in A$, $\lambda_i^j \geq \lambda_{i+1}^j$ whenever $x_i \leq x_{i+1}$.

Suppose by contradiction that R^e indeed satisfies PDpwpT but $\lambda_i^j \leq \lambda_{i+1}^j \forall i \in l(x), \lambda_i^j \leq \lambda_{i+1}^j \forall i \in h(x)$ whenever $x_i \leq x_{i+1}$

We reach the proof by contradiction.

If R^e satisfies PDpwpT then $\exists \delta \in \mathbb{R}_+$: for any $x, x' \in D$ with $card(l(x)) = card(l(x'))$ and for any $l, h \in N$,

$$x'_l = x_l + \delta,\ x'_h = x_h - \delta,\ x_l \leq x'_l \leq x'_h \leq x_h,$$

$$x_m = x'_m \forall m \in N/\{l, h\}$$

and

$$x_h, x_l \in l(x)\ \ or\ x_h, x_l \in h(x)\ :$$

$$x' R^e x \iff \sum_{i \in N} \lambda_i^k x'_i \geq \sum_{i \in N} \lambda_i^j x_i \iff$$

$$\sum_{m \in N} \lambda_m^k x'_m + \lambda_l^k x'_l + \lambda_h^k x'_h \geq \sum_{m \in N} \lambda_m^j x_m + \lambda_l^j x_l + \lambda_h^j x_h.$$

Case 1. Suppose the ranks $\forall i \in N$ are preserved after the transfer, and that $card(l(x)) = card(l(x'))$, then

$$\sum_{m \in N} \lambda_m^k x'_m = \sum_{m \in N} \lambda_m^j x_m$$

Now, by PDpwpT

$$x_l \leq x'_l \leq x'_h \leq x_h$$

by hypothesis

$$\lambda_l^j \leq \lambda_h^k$$

and by construction

$$\lambda_l^j = \lambda_l^k,\ \lambda_h^j = \lambda_h^k$$

then from

$$\sum_{m \in N} \lambda_m^k x'_m + \lambda_l^k x'_l + \lambda_h^k x'_h \geq \sum_{m \in N} \lambda_m^j x_m + \lambda_l^j x_l + \lambda_h^j x_h$$

and after simple calculation we get

$$\lambda_l^k x'_l + \lambda_h^k x'_h \geq \lambda_l^j x_l + \lambda_h^j x_h \iff$$

$$\lambda_l^k x'_l + \lambda_h^k x'_h - \lambda_l^j x_l - \lambda_h^j x_h \geq 0 \iff$$

$$\lambda_l^{k=j}(x'_l - x_l) + \lambda_h^{k=j}(x'_h - x_h) \geq 0 \iff$$

$$\lambda_l^{k=j}\delta - \lambda_h^{k=j}\delta \geq 0 \implies \delta(\lambda_l^{k=j} - \lambda_h^{k=j}) \geq 0$$

By PDpwpT $\delta \in \mathbb{R}_+$ is positive.

By assumption

$$\lambda_l^{k=j} \leq \lambda_h^{k=j} \implies \lambda_l^{k=j} - \lambda_h^{k=j} \leq 0 \implies \delta(\lambda_l^{k=j} - \lambda_h^{k=j}) \leq 0$$

Leading to a contradiction.

Case 2. Suppose now the ranks are not preserved after the transfer.

As in case 1,

$$card(l(x)) = card(l(x')) \implies j = k, \forall j, k \in A.$$

If the ranks are not preserved then

$$\exists\, x'_p\, \forall p \in P(x') = \{i \in N | x_l \leq x'_p \leq x'_l\}$$

and

$$\exists\, x'_q\, \forall q \in Q(x') = \{i \in N | x'_h \leq x'_q \leq x_h\}$$

By PDpwpT we know that

$$x' R^e x$$

Let now $x'' \in D$ be another distribution such that $x''_h = x_h - \delta'$, $x''_l = x_l + \delta'$ with $\delta' \leq \delta$, $x_l \leq x''_l \leq x''_h \leq x_h$ and $card(P(x')) = card(P(x''))$, $card(Q(x')) = card(Q(x''))$.

By PDpwpT

$$x'' R x$$

By construction

$$x''_l \leq x'_l,\ x''_h \geq x'_h,\ x''_i = x'_i \forall i \in N/\{h, l\}.$$

Moreover, since $card(P(x')) = card(P(x''))$, $card(Q(x')) = card(Q(x''))$ the ranks of x', x'' are preserved, then λ^j_i are the same in the two distributions $\forall i \in N$ then

$$\sum_{i \in N} \lambda^j_i x''_i \geq \sum_{i \in N} \lambda^j_i x'_i \implies$$

$$x'' R x'$$

However, x' is also obtained from x'' after a rank preserving transfer $\delta - \delta'$, thus, by PDpwpT

$$x' R x''$$

Henceforth, the hypothesis that $\lambda^j_i \leq \lambda^j_{i+1} \forall i \in l(x), \lambda^j_i \leq \lambda^j_{i+1} \forall i \in h(x)$ whenever $x_i \leq x_{i+1}$ leads to a contradiction, we should have instead $\lambda^j_i \geq \lambda^j_{i+1}$ whenever $x_i \leq x_{i+1}$ □

Claim 2. *Condition 1 + Condition 2 are not sufficient for R^e to satisfies PDpwpT*

Proof. We prove that the two conditions are not sufficient for R^e to satisfies PDpwpT by the simple counterexample shown in the table below. Let $\delta \in \mathbb{R}_+$: be a transfer such that for any $x, x' \in D$ with $card(l(x)) < card(l(x'))$ and for any $l, h \in N$,

$$x'_l = x_l + \delta,\ x'_h = x_h - \delta,\ x_l \leq x'_l \leq x'_h \leq x_h,$$

$$x_m = x'_m \forall m \in N/\{l, h\}$$

and

$$x_l \in l(x), x'_h \in l(x')$$

Table 7: $\theta = 20$

x	x'	$\lambda^{k=3}$	$\lambda^{j=2}$
1	2	1	1
2	5	0.8	0.5
/	18	0.5	/
22	25	-0.5	-0.05
25	50	-1	-0.06
50	/	/	-0.07

Since

$$\sum_{i \in N} \lambda_i^{j=2} x_i = -4.1 > \sum_{i \in N} \lambda_i^{k=3} x'_i = -47.5 \implies$$

$$x R^e x'$$

In spite the decreasing weights, a new vector is assigned to the new distribution (x') after the transfer, hence, the PDpwpT might not hold when one individual crosses the threshold. □

One solution is to make the axiom stronger such that it remains silent whenever one individual, crossing the threshold, is no more super rich. In that case *Condition 1 + Condition 2* are a necessary condition for R^e to satisfy $PDpwpT$ in his stronger version.

The second possibility, on which we focus here, is to introduce another condition on the weights.

Condition 3.

$$\forall x, x' \in D, \; with \; card(l(x)) < card(l(x')).$$

$$\sum_{i\in l(x)} \lambda_i^j \leq \sum_{i\in l(x')} \lambda_i^{j+1}, \sum_{i\in h(x)} \lambda_i^j \leq \sum_{i\in h(x')} \lambda_i^{j+1}$$

and

$$|\lambda_i^j| \geq |\lambda_i^{j+1}|$$

This condition is introduced for all that cases of transfers which make one individual cross the threshold.

Claim 3 *R^e satisfies $PDpwpT$*

Proof. Suppose $\exists\delta \in \mathbb{R}_+$: for any $x, x' \in D$ with $card(l(x)) \leq card(l(x'))$ and for any $l, h \in N$,

$$x'_l = x_l + \delta,\ x'_h = x_h - \delta,\ x_l \leq x'_l \leq x'_h \leq x_h,$$

$$x_m = x'_m \forall m \in N/\{l, h\}$$

and

$$x_h, x_l \in l(x)\ \ or\ x_h, x_l \in h(x)\ \ or\ x_l \in l(x), x_h \in h(x) :$$

$$x' R^e x$$

It is straightforward to prove the claim in the case $card(l(x)) = card(l(x'))$ either when the transfer is realised below or above the threshold, then, we focus in the other possibility $card(l(x)) < card(l(x'))$.

To prove that R^e satisfies the axiom we need to show that

$$\sum_{i\in N} \lambda_i^j x_i \leq \sum_{i\in N} \lambda_i^k x'_i \iff$$

$$\sum_{m\in N/\{l,h\}} \lambda_m^j x_m + \lambda_l^j x_l + \lambda_h^j x_h \leq \sum_{m\in N/\{l,h\}} \lambda_m^k x'_m + \lambda_l^k x'_l + \lambda_h^k x'_h \iff$$

$$\sum_{q\in h(x)} \lambda_q^j x_q + \lambda_h^j x_h - \sum_{q\in h(x')} \lambda_q^k x'_q \leq \sum_{p\in l(x')} \lambda_p^k x'_p - \sum_{p\in l(x)} \lambda_p^j x_p + \lambda_l^k x'_l + \lambda_h^k x'_h - \lambda_l^j x_l$$

Recall that condition 1 implies

$$\lambda_q^j, \lambda_q^k, \lambda_h^j \leq 0\ \forall q \in h(x), h(x'),\quad and\ \ \lambda_p^k, \lambda_p^j, \lambda_l^k, \lambda_h^k, \lambda_l^j \geq 0\ \forall p \in l(x), l(x')$$

condition 2 implies

$$\lambda^j_h \leq \lambda^j_l, \lambda^k_h \leq \lambda^k_l,$$

condition 3 implies

$$\lambda^j_h \leq \lambda^k_h, \lambda^j_l \leq \lambda^k_l, \lambda^j_m \leq \lambda^k_m, \forall m \in N/h, l$$

then

$$\lambda^j_h x_h \leq 0$$

and by condition 2 and 3

$$\sum_{q \in h(x)} \lambda^j_q x_q \leq \sum_{q \in h(x')} \lambda^k_q x'_q \implies$$

$$\sum_{q \in h(x)} \lambda^j_q x_q + \lambda^j_h x_h - \sum_{q \in h(x')} \lambda^k_q x'_q \leq 0$$

Now, since by condition 3

$$\sum_{p \in l(x)} \lambda^k_p + \lambda^k_l + \lambda^k_h \geq \sum_{p \in l(x')} \lambda^j_p + \lambda^j_l$$

and

$$\sum_{p \in l(x')} x'_p + x'_l + x'_h \geq \sum_{p \in l(x)} x_p + x_l$$

then

$$\sum_{p \in l(x')} \lambda^k_p x'_p + \lambda^k_l x'_l + \lambda^k_h x'_h \geq \sum_{p \in l(x)} \lambda^j_p x_p + \lambda^j_l x_l$$

hence

$$\sum_{q \in h(x)} \lambda^j_q x_q + \lambda^j_h x_h - \sum_{q \in h(x')} \lambda^k_q x'_q \leq \sum_{p \in l(x')} \lambda^k_p x'_p - \sum_{p \in l(x)} \lambda^j_p x_p + \lambda^k_l x'_l + \lambda^k_h x'_h - \lambda^j_l x_l \iff$$

$$x' R^e x$$

□

Before to show that our eSWO satisfies the scale invariance axiom, it would be interesting to argue what follows:

Claim 4. *R^e does not satisfy Cardinal Full Comparability*

Proof. We prove the claim by the simple counterexample. Let $\forall x, x', y, y' \in D$ be four distributions such that: $y_i = a + bx_i,\ y'_i = a + bx'_i, \forall i \in N, \forall a \in \mathbb{R}_+, b \in \mathbb{R}_{++}$. Let $b = 2, a = 1$

Table 8: $\theta = 3.5$

x	x'	y	y'	$\lambda^{k=1}$	$\lambda^{j=0}$
1	/	3	/	0	/
5	3	11	7	-6	-2
5.1	3.1	11.2	7.2	-9	-6
/	5	/	11	/	-10

Since

$$\sum_{i\in N} \lambda_i^{k=1} x_i = -75.9 < \sum_{i\in N} \lambda_i^{j=0} x'_i = -74.6 \implies$$

$$x' R^e x$$

However

$$\sum_{i\in N} \lambda_i^{k=1} y_i = -166.8 < \sum_{i\in N} \lambda_i^{j=0} y'_i = -167.2 \implies$$

$$y R^e y'$$

□

Differently from other rank depended SWO, the eSWO might violate the axiom in spite condition 1 + condition 2 + condition 3, hence we referred to a more demanding invariance axiom which of course requires a wider informational basis

We end this section by introducing our last result.

Claim 5. R^e *satisfies swpRSC*

Proof. The proof are quite straightforward. If R^e satisfies *swpRSC* then

$$\forall x, x', y, y' \in D \ \textit{if} \ \exists\, b \in \mathbb{R}_+ : y_i = bx_i,\ y'_i = bx'_i, \ \forall i \in N,$$

$$card(l(x)) = card(l(y)) = j,\ card(l(x')) = card(l(y')) = k,\ then$$
$$xR^e x' \iff yR^e y'$$

Let us suppose

$$xR^e x' \iff \sum_{i \in N} \lambda_i^j x_i \geq \sum_{i \in N} \lambda_i^k x'_i \iff$$

and suppose $card(l(x)) = card(l(y)) = j \leq k = card(l(x')) = card(l(y'))$ (we omit the opposite case).

We need to prove that

$$\sum_{i \in N} \lambda_i^j y_i \geq \sum_{i \in N} \lambda_i^k y'_i \iff$$
$$\sum_{i \in N} \lambda_i^j (bx_i) \geq \sum_{i \in N} \lambda_i^k (bx'_i) \iff$$
$$b \sum_{i \in N} \lambda_i^j x_i \geq b \sum_{i \in N} \lambda_i^k x'_i \iff$$
$$\sum_{i \in N} \lambda_i^j x_i \geq \sum_{i \in N} \lambda_i^k x'_i$$

Since the last inequality is true by hypothesis then $yR^e y'$

□

5 Conclusion

What the commonly implemented social welfare functions have been overlooking is the concern for extreme richness at the top. The Excessivist Social Welfare Ordering aims to fill this gap. The work has proven that by relaxing the Pareto principle and the Ratio Scale Comparability axiom, we can introduce a subclass of rank-weighted social welfare orderings that satisfies the axioms of Aversion to Excessive Richness, Strong Pareto below the threshold, Pigou Dalton positive weights preserving Transfer and sign weights preserving Ratio Scale Comparability and that judges the lowering of incomes above a certain threshold as social improvement whatsoever. Since imposing a fixed richness line as relevant economical content an interesting and necessary line of research would be also to expand this ordering for non-fixed richness lines.

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