École polytechnique de Louvain

High order boundary conditions for flow simulations in vorticity-velocity formulation

Author: **Umut Inan**
Supervisor: **Grégoire Winckelmans**
Readers: **Philippe Chatelain, Vincent Legat**
Academic year 2018–2019
Master [120] in Mechanical Engineering

First of all, I would like to thanks my supervisor, G. Winckelmans and P. Chatelain, for guiding me along the year in order to make this thesis. I also thanks Thomas Gillis for giving me a code that efficiently creates matrices used for 2D case. And finally, I thanks Philippe Billuart for his help and for the code that allows me to use the *Petsy* library and Thomas algorithm.

Contents

List of Figures

Introduction

Context

In many application, fluids are an important aspect of a design. To analyze them, we have the well-known Navier-Stokes equations. They can be expressed in terms of velocity-pressure or vorticity-velocity. In this paper, we will focus our analyses on the second formulation. To integrate these partial differential equations we have to set boundary conditions. For the velocity-pressure formulation, we just set the value of the velocity (e.g. zero at the wall). But for the vorticity-velocity formulation, we have to compute the right amount of vorticity at the wall to ensure the proper velocity. For that, many formulas have been developed for that effect. The aim of this thesis is to investigate high order boundary conditions and to compare it with others formulas over a simple case in 1D for which the analytical solution is known. Then we will test this formula on a 2D case.

Navier-Stokes equations

As our thesis is about the vorticity-velocity formulation, we will demonstrate how they are obtained. We consider Navier-Stokes equations for incompressible fluids in velocity-pressure formulation. The $\mathbf{v}$ variable is the flow velocity vector, P the kinematic pressure and ν the kinematic viscosity :

$$\nabla \cdot \mathbf{v} = 0 \tag{1}$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\nabla \mathbf{v}) \cdot \mathbf{v} = -\nabla P + \nu \nabla^2 \mathbf{v} \tag{2}$$

with the Laplacian operator ∇^2 being defined by the divergence of the gradient. We define the vorticity $\boldsymbol{\omega}$ as :

$$\boldsymbol{\omega} = \nabla \times \mathbf{v} \tag{3}$$

The equation (1) expresses the conservation of mass for incompressible fluids. As the flow is divergence-free, the velocity vector can be written as the curl of a function. This define the stream function $\boldsymbol{\psi}$:

$$\mathbf{v} = \nabla \times \boldsymbol{\psi} \tag{4}$$

By re-injecting this expression in the definition of the vorticity (3), we get the first equation for Navier-Stokes equations :

$$\begin{aligned} \boldsymbol{\omega} = \nabla \times \nabla \times \boldsymbol{\psi} &= \nabla(\nabla \cdot \boldsymbol{\psi}) - \nabla^2 \boldsymbol{\psi} \\ &= -\nabla^2 \boldsymbol{\psi} \end{aligned} \tag{5}$$

As the stream function is taken divergence-free (choice of Lorentz).
The equation (2) expresses the momentum conversation of incompressible viscous fluids. Before taking the curl of this equation, we will first rewrite some terms to make appear the vorticity. For the viscous term, taking the curl of the vorticity allow us to rewritten it.

$$\begin{aligned} \nabla \times \boldsymbol{\omega} = \nabla \times \nabla \times \mathbf{v} &= \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v} \\ &= -\nabla^2 \mathbf{v} \end{aligned} \tag{6}$$

And for the convective term, we use this vectorial equality :

$$\nabla \left(\frac{\mathbf{v} \cdot \mathbf{v}}{2} \right) = (\nabla \mathbf{v}) \cdot \mathbf{v} + \mathbf{v} \times \boldsymbol{\omega} \tag{7}$$

We can then rewrite the equation (2) :

$$\frac{\partial \mathbf{v}}{\partial t} + \boldsymbol{\omega} \times \mathbf{v} = -\nabla \left(P + \frac{\mathbf{v} \cdot \mathbf{v}}{2} \right) - \nu \nabla \times \boldsymbol{\omega} \tag{8}$$

By applying the curl to equation (8) we get an equation for the vorticity :

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times (\boldsymbol{\omega} \times \mathbf{v}) = 0 - \nu \nabla \times (\nabla \times \boldsymbol{\omega}) \tag{9}$$

By following the same principle as equation (6), we can rewrite the viscous term :

$$\begin{aligned} \nabla \times (\nabla \times \boldsymbol{\omega}) &= \nabla(\nabla \cdot \boldsymbol{\omega}) - \nabla^2 \boldsymbol{\omega} \\ &= -\nabla^2 \boldsymbol{\omega} \end{aligned} \tag{10}$$

The second term can also be rewritten as :

$$\begin{aligned}\nabla \times (\boldsymbol{\omega} \times \mathbf{v}) &= \boldsymbol{\omega}(\nabla \cdot \mathbf{v}) - (\nabla \mathbf{v}) \cdot \boldsymbol{\omega} - \mathbf{v}(\nabla \cdot \boldsymbol{\omega}) + (\nabla \boldsymbol{\omega}) \cdot \mathbf{v} \\ &= -(\nabla \mathbf{v}) \cdot \boldsymbol{\omega} + (\nabla \boldsymbol{\omega}) \cdot \mathbf{v}\end{aligned} \tag{11}$$

This finally gives us the second Navier-Stokes equation in vorticity-velocity formulation :

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\nabla \boldsymbol{\omega}) \cdot \mathbf{v} = (\nabla \mathbf{v}) \cdot \boldsymbol{\omega} + \nu \nabla^2 \boldsymbol{\omega} \tag{12}$$

This thesis will focus on this formulation for 2D cartesian domain ($\widehat{e}_x$,$\widehat{e}_y$). In this case, the only non-zero component of the vorticity and stream function is in $\widehat{e}_z$ direction.

$$\boldsymbol{\psi} = (0, 0, \psi) \qquad \boldsymbol{\omega} = (0, 0, \omega) \qquad \mathbf{v} = (u, v, 0)$$

In imcompressible fluids, 2D Navier-Stokes equations in vorticity-velocity formulation are then :

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega \tag{13}$$

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \nu \left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right) \tag{14}$$

As the velocity is divergence-free, we can rewrite equation (15) and obtain the divergence form :

$$\frac{\partial \omega}{\partial t} + \frac{\partial(\omega u)}{\partial x} + \frac{\partial(\omega v)}{\partial y} = \nu \left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right) \tag{15}$$

And velocities are obtained from the equation (4).

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

Chapter 1

Boundary condition

To integrate Navier-Stokes equation (15), one must define boundary conditions. Let's assume that the boundary is aligned with the $\widehat{e}_x$ direction.

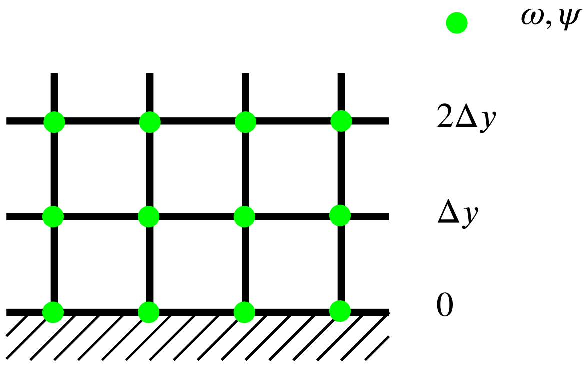

Figure 1.1: Example of a boundary.

The accuracy of a formula depend on the spatial discretization Δy. One of this boundary formula is derived by Woods and gives an error of $\mathcal{O}((\Delta y)^2)$ by only using $\psi(x, \Delta y)$ and $\omega(x, \Delta y)$ to evaluate $\omega(x, 0)$. Our goal is to derive a formula that uses one level more of information, $\psi(x, 2\Delta y)$ and $\omega(x, 2\Delta y)$. But first, as we will compare it with others formulas, we will demonstrate different methods and put in evidence their truncation errors in red.

We begin with the Poisson's equation (13) at the wall :

$$\nabla^2\psi = -\omega \qquad \Rightarrow \qquad \omega_{wall} = \omega(x,0) = -\frac{\partial^2\psi(x,0)}{\partial x^2} - \frac{\partial^2\psi(x,0)}{\partial y^2} \tag{1.1}$$

$$\text{On the boundary,} \quad \psi(x,0) = f(x) \quad \text{and} \quad \frac{\partial\psi(x,0)}{\partial y} = g(x)$$

$$\Rightarrow \omega(x,0) = -f''(x) - \frac{\partial^2\psi(x,0)}{\partial y^2} \tag{1.2}$$

This equation (1.2) shows that we need to find an expression for the second derivative of ψ at the wall to obtain the formula for the boundary condition.

1.1 1^{st} order condition with ψ

The only information that we have for this formulation is $\psi(x,\Delta y)$. By applying Taylor series to $\psi(x,\Delta y)$ close to $\psi(x,0)$, we can derive the desired expression :

$$\psi(x,\Delta y) = \psi(x,0) + \Delta y\frac{\partial\psi(x,0)}{\partial y} + \frac{(\Delta y)^2}{2}\frac{\partial^2\psi(x,0)}{\partial y^2} + \frac{(\Delta y)^3}{6}\frac{\partial^3\psi(x,0)}{\partial y^3} + \mathcal{O}((\Delta y)^4) \tag{1.3}$$

$$\frac{\partial^2\psi(x,0)}{\partial y^2} = \frac{2}{(\Delta y)^2}[\psi(x,\Delta y) - f(x)] - \frac{2}{\Delta y}g(x) - \frac{\Delta y}{3}\frac{\partial^3\psi(x,0)}{\partial y^3} + \mathcal{O}\left((\Delta y)^2\right) \tag{1.4}$$

And re-injecting it to equation (1.2) gives us the boundary condition that only uses $\psi(x,\Delta y)$:

$$\begin{aligned}\omega(x,0) = -f''(x) - \frac{2}{(\Delta y)^2}[\psi(x,\Delta y) - f(x)] + \frac{2}{\Delta y}g(x) \\ + \frac{\Delta y}{3}\frac{\partial^3\psi(x,0)}{\partial y^3} + \mathcal{O}\left((\Delta y)^2\right)\end{aligned} \tag{1.5}$$

This formula is known as Thom-Burggraf formula.

1.2 2^{nd} order condition with ψ

For this formula, we use one more information, $\psi(x, 2\Delta y)$, to increase the order of the error. So we write Taylor series close to $\psi(x, 0)$ for $\psi(x, \Delta y)$ and $\psi(x, 2\Delta y)$.

$$\psi(x, \Delta y) = \psi(x, 0) + \Delta y \frac{\partial \psi(x, 0)}{\partial y} + \frac{(\Delta y)^2}{2} \frac{\partial^2 \psi(x, 0)}{\partial y^2} + \frac{(\Delta y)^3}{6} \frac{\partial^3 \psi(x, 0)}{\partial y^3} + \frac{(\Delta y)^4}{24} \frac{\partial^4 \psi(x, 0)}{\partial y^4} + \mathcal{O}\left((\Delta y)^5\right) \quad (1.6)$$

$$\psi(x, 2\Delta y) = \psi(x, 0) + 2\Delta y \frac{\partial \psi(x, 0)}{\partial y} + \frac{4(\Delta y)^2}{2} \frac{\partial^2 \psi(x, 0)}{\partial y^2} + \frac{8(\Delta y)^3}{6} \frac{\partial^3 \psi(x, 0)}{\partial y^3} + \frac{16(\Delta y)^4}{24} \frac{\partial^4 \psi(x, 0)}{\partial y^4} + \mathcal{O}\left((\Delta y)^5\right) \quad (1.7)$$

By combining these two expressions, we can nullify the contribution of $(\Delta y)^3$ term here :

$$8\psi(x, \Delta y) - \psi(x, 2\Delta) = 7f(x) + 6\Delta y\, g(x) + \frac{4(\Delta y)^2}{2} \frac{\partial^2 \psi(x, 0)}{\partial y^2} + 0 - \frac{8(\Delta y)^4}{24} \frac{\partial^4 \psi(x, 0)}{\partial y^4} + \mathcal{O}\left((\Delta y)^5\right) \quad (1.8)$$

$$\frac{\partial^2 \psi(x, 0)}{\partial y^2} = \frac{1}{2(\Delta y)^2} [8\psi(x, \Delta y) - \psi(x, 2\Delta) - 7f(x)] - \frac{3}{\Delta y} g(x) + \frac{(\Delta y)^2}{6} \frac{\partial^4 \psi(x, 0)}{\partial y^4} + \mathcal{O}\left((\Delta y)^3\right) \quad (1.9)$$

This gives us the final expression that uses $\psi(x, \Delta y)$ and $\psi(x, 2\Delta y)$. This expression is Jensen's formula :

$$\omega(x, 0) = -f''(x) - \frac{1}{2(\Delta y)^2} [8\psi(x, \Delta y) - \psi(x, 2\Delta) - 7f(x)] + \frac{3}{\Delta y} g(x) - \frac{(\Delta y)^2}{6} \frac{\partial^4 \psi(x, 0)}{\partial y^4} + \mathcal{O}\left((\Delta y)^3\right) \quad (1.10)$$

1.3 2^{nd} order Woods condition

This formula is obtained by using a different approach. Instead of only using information on ψ, we can also use the vorticity close to the wall. So for this formula, we use $\psi(x,\Delta y)$ and $\omega(x,\Delta y)$. From the equation (1.6), we can get an expression for the second derivative of $\psi(x,0)$ at higher order.

$$\begin{aligned}\frac{\partial^2\psi(x,0)}{\partial y^2} = \frac{2}{(\Delta y)^2}[\psi(x,\Delta y) - f(x)] - \frac{2}{\Delta y}\,g(x) - \frac{\Delta y}{3}\frac{\partial^3\psi(x,0)}{\partial y^3} \\ - \frac{(\Delta y)^2}{12}\frac{\partial^4\psi(x,0)}{\partial y^4} + \mathcal{O}\left((\Delta y)^3\right)\end{aligned} \tag{1.11}$$

We now want to find an expression for the third derivative of $\psi(x,0)$. This is done by deriving Poisson's equation (1.2) :

$$\frac{\partial\omega(x,0)}{\partial y} = -\frac{\partial^2}{\partial x^2}\left(\frac{\partial\psi(x,0)}{\partial y}\right) - \frac{\partial^3\psi(x,0)}{\partial y^3} \tag{1.12}$$

$$\begin{aligned}\frac{\partial^3\psi(x,0)}{\partial y^3} = -g''(x) - \frac{\partial\omega(x,0)}{\partial y} = -g''(x) - \frac{\omega(x,\Delta y)-\omega(x,0)}{\Delta y} \\ + \frac{\Delta y}{2}\frac{\partial^2\omega(x,0)}{\partial y^2} + \mathcal{O}\left((\Delta y)^2\right)\end{aligned} \tag{1.13}$$

The derivative of $\omega(x,0)$ has been obtained by upwelling (see Appendix A.1 for details). By re-injecting this expression in equation (1.15), we have :

$$\begin{aligned}\frac{\partial^2\psi(x,0)}{\partial y^2} = \frac{2}{(\Delta y)^2}[\psi(x,\Delta y) - f(x)] - \frac{2}{\Delta y}\,g(x) \\ - \frac{\Delta y}{3}\left(-g''(x) - \frac{\omega(x,\Delta y)-\omega(x,0)}{\Delta y} + \frac{\Delta y}{2}\frac{\partial^2\omega}{\partial y^2}\right) \\ - \frac{(\Delta y)^2}{12}\frac{\partial^4\psi(x,0)}{\partial y^4} + \mathcal{O}\left((\Delta y)^3\right)\end{aligned} \tag{1.14}$$

$$\begin{aligned}\frac{\partial^2\psi(x,0)}{\partial y^2} = \frac{2}{(\Delta y)^2}[\psi(x,\Delta y) - f(x)] - \frac{2}{\Delta y}\,g(x) + \frac{\Delta y}{3}g''(x) \\ + \frac{1}{3}\omega(x,\Delta y) - \frac{1}{3}\omega(x,0) - \frac{(\Delta y)^2}{6}\frac{\partial^2\omega}{\partial y^2} - \frac{(\Delta y)^2}{12}\frac{\partial^4\psi(x,0)}{\partial y^4} + \mathcal{O}\left((\Delta y)^3\right)\end{aligned} \tag{1.15}$$

This finally allow us to calculate $\omega(x,0)$ by re-injecting equation (1.15) in equation (1.2) and isolating it :

$$\omega(x,0) = -f''(x) - \frac{2}{(\Delta y)^2}[\psi(x,\Delta y) - f(x)] + \frac{2}{\Delta y} g(x) - \frac{\Delta y}{3} g''(x) \\ - \frac{1}{3}\omega(x,\Delta y) + \frac{1}{3}\omega(x,0) + \frac{(\Delta y)^2}{6}\frac{\partial^2 \omega}{\partial y^2} + \frac{(\Delta y)^2}{12}\frac{\partial^4 \psi(x,0)}{\partial y^4} + \mathcal{O}\left((\Delta y)^3\right) \quad (1.16)$$

$$\omega(x,0) = -\frac{3}{2} f''(x) - \frac{3}{(\Delta y)^2}[\psi(x,\Delta y) - f(x)] + \frac{3}{\Delta y} g(x) - \frac{\Delta y}{2} g''(x) \\ - \frac{1}{2}\omega(x,\Delta y) + \left(\frac{1}{4}\frac{\partial^2 \omega(x,0)}{\partial y^2} + \frac{1}{8}\frac{\partial^4 \psi(x,0)}{\partial y^4}\right)(\Delta y)^2 + \mathcal{O}\left((\Delta y)^3\right) \quad (1.17)$$

1.4 3^{rd} order Woods condition

We now look for higher order boundary condition. To do so, we use two information in both ψ and ω. Like others formulas, we apply Taylor series to $\psi(x,\Delta y)$ and $\psi(x,2\Delta y)$ but with one higher order term :

$$\psi(x,\Delta y) = \psi(x,0) + \Delta y \frac{\partial \psi(x,0)}{\partial y} + \frac{(\Delta y)^2}{2}\frac{\partial^2 \psi(x,0)}{\partial y^2} + \frac{(\Delta y)^3}{6}\frac{\partial^3 \psi(x,0)}{\partial y^3} \\ + \frac{(\Delta y)^4}{24}\frac{\partial^4 \psi(x,0)}{\partial y^4} + \frac{(\Delta y)^5}{120}\frac{\partial^5 \psi(x,0)}{\partial y^5} + \mathcal{O}\left((\Delta y)^6\right) \quad (1.18)$$

$$\psi(x,2\Delta y) = \psi(x,0) + 2\Delta y \frac{\partial \psi(x,0)}{\partial y} + \frac{4(\Delta y)^2}{2}\frac{\partial^2 \psi(x,0)}{\partial y^2} + \frac{8(\Delta y)^3}{6}\frac{\partial^3 \psi(x,0)}{\partial y^3} \\ + \frac{16(\Delta y)^4}{24}\frac{\partial^4 \psi(x,0)}{\partial y^4} + \frac{32(\Delta y)^5}{120}\frac{\partial^5 \psi(x,0)}{\partial y^5} + \mathcal{O}\left((\Delta y)^6\right) \quad (1.19)$$

By combining these two equations we can nullify the contribution of $(\Delta y)^4$ term and obtain an expression for the second derivative of $\psi(x,0)$:

$$16\psi(x,\Delta y) - \psi(x,2\Delta y) = 15f(x) + 14\Delta y\, g(x) + 12\frac{(\Delta y)^2}{2}\frac{\partial^2\psi(x,0)}{\partial y^2}$$
$$+ 8\frac{(\Delta y)^3}{6}\frac{\partial^3\psi(x,0)}{\partial y^3} + 0 - 16\frac{(\Delta y)^5}{120}\frac{\partial^5\psi(x,0)}{\partial y^5} + \mathcal{O}\left((\Delta y)^6\right) \quad (1.20)$$

$$-6(\Delta y)^2\frac{\partial^2\psi(x,0)}{\partial y^2} = -16\psi(x,\Delta y) + \psi(x,2\Delta y) + 15f(x) + 14\Delta y\, g(x)$$
$$+ \frac{4}{3}(\Delta y)^3\frac{\partial^3\psi(x,0)}{\partial y^3} - \frac{2}{15}(\Delta y)^5\frac{\partial^5\psi(x,0)}{\partial y^5} + \mathcal{O}\left((\Delta y)^6\right) \quad (1.21)$$

$$\frac{\partial^2\psi(x,0)}{\partial y^2} = \frac{1}{6(\Delta y)^2}[16\psi(x,\Delta y) - \psi(x,2\Delta y) - 15f(x)] - \frac{7}{3}\frac{1}{\Delta y}g(x)$$
$$- \frac{2}{9}\Delta y\frac{\partial^3\psi(x,0)}{\partial y^3} + \frac{1}{45}(\Delta y)^3\frac{\partial^5\psi(x,0)}{\partial y^5} + \mathcal{O}\left((\Delta y)^4\right) \quad (1.22)$$

Just like the 2^{nd} order Woods formula, we use equation (1.12) to obtain an expression for the third derivative of $\psi(x,0)$. But, as we have two informations on ω, we can use the 2^{nd} order fully decentered discretization (see Appendix A.2 for details).

$$\frac{\partial\omega(x,0)}{\partial y} = \frac{-3\omega(x,0) + 4\omega(x,\Delta y) - \omega(x,2\Delta y)}{2\Delta y}$$
$$+ \frac{1}{3}(\Delta y)^2\frac{\partial^3\omega(x,0)}{\partial y^3} + \mathcal{O}\left((\Delta y)^3\right) \quad (1.23)$$

$$\Rightarrow \frac{\partial^3\psi(x,0)}{\partial y^3} = -g''(x) - \frac{-3\omega(x,0) + 4\omega(x,\Delta y) - \omega(x,2\Delta y)}{2\Delta y}$$
$$- \frac{1}{3}(\Delta y)^2\frac{\partial^3\omega(x,0)}{\partial y^3} + \mathcal{O}\left((\Delta y)^3\right) \quad (1.24)$$

We can now have final expression for the second derivative of $\psi(x,0)$:

$$\frac{\partial^2\psi(x,0)}{\partial y^2} = \frac{1}{6(\Delta y)^2}[16\psi(x,\Delta y) - \psi(x,2\Delta y) - 15f(x)] - \frac{7}{3}\frac{1}{\Delta y}g(x)$$
$$- \frac{2}{9}\Delta y\left(-g''(x) + \frac{3\omega(x,0) - 4\omega(x,\Delta y) + \omega(x,2\Delta y)}{2\Delta y} - \frac{1}{3}(\Delta y)^2\frac{\partial^3\omega(x,0)}{\partial y^3}\right)$$
$$+ \frac{1}{45}(\Delta y)^3\frac{\partial^5\psi(x,0)}{\partial y^5} + \mathcal{O}\left((\Delta y)^4\right) \quad (1.25)$$

By re-injecting it in the Poisson's equation (1.2) and by isolating $\omega(x,0)$, we get :

$$\begin{aligned}\omega(x,0) = -f''(x) - \frac{1}{6(\Delta y)^2}[16\psi(x,\Delta y) - \psi(x,2\Delta y) - 15f(x)] \\ + \frac{7}{3}\frac{1}{\Delta y}g(x) - \frac{2}{9}\Delta y\, g''(x) + \frac{1}{3}\omega(x,0) - \frac{4}{9}\omega(x,\Delta y) + \frac{1}{9}\omega(x,2\Delta y) \\ - \left(\frac{2}{27}\frac{\partial^3\omega(x,0)}{\partial y^3} + \frac{1}{45}\frac{\partial^5\psi(x,0)}{\partial y^5}\right)(\Delta y)^3 + \mathcal{O}\left((\Delta y)^4\right)\end{aligned} \quad (1.26)$$

$$\begin{aligned}\omega(x,0) = -\frac{3}{2}f''(x) - \frac{1}{4(\Delta y)^2}[16\psi(x,\Delta y) - \psi(x,2\Delta y) - 15f(x)] \\ + \frac{7}{2}\frac{1}{\Delta y}g(x) - \frac{1}{3}\Delta y\, g''(x) - \frac{2}{3}\omega(x,\Delta y) + \frac{1}{6}\omega(x,2\Delta y) \\ - \left(\frac{1}{9}\frac{\partial^3\omega(x,0)}{\partial y^3} + \frac{1}{30}\frac{\partial^5\psi(x,0)}{\partial y^5}\right)(\Delta y)^3 + \mathcal{O}\left((\Delta y)^4\right)\end{aligned} \quad (1.27)$$

Chapter 2

Analytical solution

2.1 Channel flow with constant pressure gradient

The case on which we will numerically integrate must be simple enough to have an analytical solution and be bounded on the boundary. This analytical solution will be used to compare different formulas that express the vorticity at the wall. This simple case is a flow between two plates separated by a distance H with a constant pressure gradient along $\hat{e}_x$ direction. This problem was already solved in the notes [1] but in pipe flow.

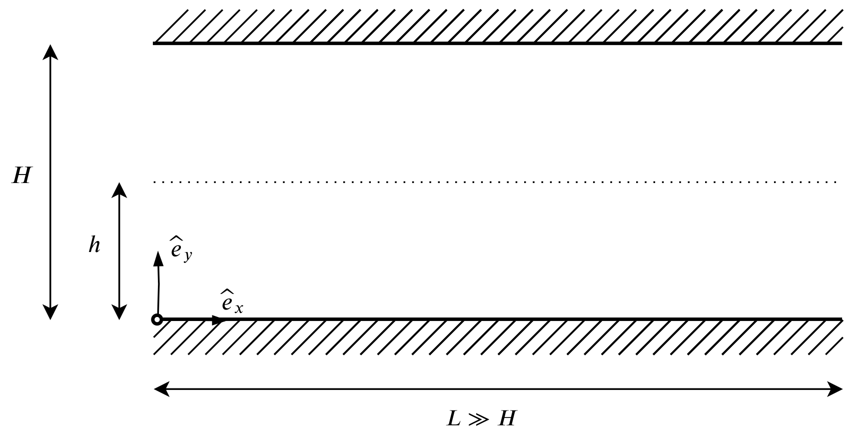

Figure 2.1: Channel flow.

At time $t \leq 0$, there is no pressure gradient, so the velocity is zero. When $t > 0$, a constant pressure gradient is imposed and the flow starts to accelerate.

The velocity field should tend to a stationary solution when $t \to \infty$. To obtain the analytical velocity field of that problem, we begin with Navier-Stokes equation in velocity-pressure formulation for horizontal velocity in 2D cartesian domain :

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{dP}{dx} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{2.1}$$

As the pressure gradient is only in $\widehat{e}_x$ direction, the vertical velocity is null. Plus, there is no variation of u along x axis. Meaning that Navier-Stokes equation reduces to :

$$\frac{\partial u(y,t)}{\partial t} = -\frac{dP}{dx} + \nu\frac{\partial^2 u(y,t)}{\partial y^2} \tag{2.2}$$

To obtain the time dependant solution of the problem, we assume that the horizontal velocity is the sum of a transitional solution $\Theta(y,t)$ and a stationary solution $R(y)$.

$$u(y,t) = \Theta(y,t) + R(y) \tag{2.3}$$

This last one is the well-known Poiseuille flow and it equals to :

$$R(y) = -\frac{dP}{dx}\frac{h^2}{2\nu}\left(1 - \left(\frac{y}{h} - 1\right)^2\right) = u_c\left(1 - \left(\frac{y}{h} - 1\right)^2\right) \tag{2.4}$$

With u_c being the velocity at the center of the channel. This function is obtained by integrating this equation :

$$\frac{\partial R(y)}{\partial t} = 0 = -\frac{dP}{dx} + \nu\frac{\partial^2 R(y)}{\partial y^2} \tag{2.5}$$

By subtracting the equation (2.2) with (2.5), we can obtain a differential equation for $\Theta(y,t)$.

$$\frac{\partial \Theta(y,t)}{\partial t} = \nu\frac{\partial^2 \Theta(y,t)}{\partial y^2} \tag{2.6}$$

We now introduce a change of variable to ease the integration :

$$\eta = \frac{y}{h} \qquad \xi = \frac{\nu t}{h^2}$$

The partial differential equation (2.6) become :

$$\frac{\partial \Theta(\eta,\xi)}{\partial \xi} = \frac{\partial^2 \Theta(\eta,\xi)}{\partial \eta^2} \tag{2.7}$$

To integrate it, we need to identify the boundary and the initial conditions in term of $\Theta(\eta,\xi)$. On the bottom and top plate, it's a solid wall. This means that the horizontal velocity is zero at all time. For the initial condition, as there is no pressure gradient when t $\leq$ 0, there is no horizontal velocity.

$$\Theta(0,\xi) = 0 \qquad \Theta(2,\xi) = 0 \qquad \Theta(\eta,0) = u(\eta,0) - R(\eta) = -R(\eta)$$

Using the method of separated variables, we can obtain the solution of that equation. So we assume that $\Theta(\eta,\xi)$ is the product of two functions, $\Phi(\eta)$ and $\Gamma(\xi)$ and we re-inject it in the equation (2.7).

$$\Phi\Gamma' = \Phi''\Gamma \tag{2.8}$$

$$\frac{\Gamma'}{\Gamma} = \frac{\Phi''}{\Phi} = -\lambda^2 \tag{2.9}$$

We now have two ordinary differential equations for Φ and Γ that can be integrated separately.

$$\begin{cases} \Gamma' + \lambda^2\Gamma = 0 \\ \Phi'' + \lambda^2\Phi = 0 \end{cases} \tag{2.10}$$

Integrating these two equations give these solutions :

$$\begin{cases} \Gamma(\xi) = Ce^{-\lambda^2\xi} \\ \Phi(\eta) = A\cos(\lambda\eta) + B\sin(\lambda\eta) \end{cases} \tag{2.11}$$

By using boundary conditions, we know that $\Theta(0,\xi) = 0$ which implies that the A constant must be equals to 0. Moreover, as $\Theta(2,\xi) = 0$, we can obtain an expression for λ :

$$\lambda = \frac{k\pi}{2} \qquad \text{with k = 0,1,2,3...} \tag{2.12}$$

We can now recover the expression of $\Theta(\eta,\xi)$ by doing the sum over all values of k :

$$\Theta(\eta,\xi) = \Phi(\eta)\Gamma(\xi) = \sum_{k=0}^{\infty} D_k \sin(\lambda_k\eta) e^{-\lambda_k^2\xi} \tag{2.13}$$

with D = BC being the last unknown of the equation. To obtain his expression, we use the last condition still unused :

$$\Theta(\eta, 0) = -R(\eta) \tag{2.14}$$

By multiplying both side with $\sin(\lambda_l \eta)$ and then integrating the result, we can use sinus orthogonality propriety :

$$\int_0^2 \Theta(\eta, 0) \sin(\lambda_l \eta) d\eta = -\int_0^2 R(\eta) \sin(\lambda_l \eta) d\eta \tag{2.15}$$

This propriety indicate that :

$$\int_0^L \sin(\lambda_k \eta) \sin(\lambda_l \eta) d\eta = \begin{cases} 0, & \text{if } k \neq l \\ \frac{L}{2}, & \text{if } k = l \end{cases} \tag{2.16}$$

This finally gives us :

$$D_k = -\int_0^2 R(\eta) \sin(\lambda_k \eta) d\eta \tag{2.17}$$

Since the mode $k = 0$ is null, the summation can begin at $k = 1$. We now have the final expression of the horizontal velocity :

$$u(\eta, \xi) = u_c \left(1 - (\eta - 1)^2\right) + \sum_{k=1}^{\infty} D_k \sin(\lambda_k \eta) e^{-\lambda_k^2 \xi} \tag{2.18}$$

With this expression, it's possible to get different quantities like the mass flow by integrating equation (2.18) on the domain :

$$h \int_0^2 u(\eta, \xi) d\eta = Q(\xi) = \frac{4}{3} u_c h + \sum_{k=1}^{\infty} 2 D_k \frac{h}{\lambda_k} \sin^2(\lambda_k) e^{-\lambda_k^2 \xi} \tag{2.19}$$

As the vertical velocity is null, we can obtain an expression for the vorticity by deriving the horizontal velocity :

$$\begin{aligned} \omega(\eta, \xi) &= \frac{\partial v(\eta, \xi)}{\partial x} - \frac{\partial u(\eta, \xi)}{\partial y} = -\frac{\partial u(\eta, \xi)}{\partial \eta} \frac{\partial \eta}{\partial y} \\ &= 2 u_c (\eta - 1) \frac{1}{h} - \sum_{k=1}^{\infty} D_k \cos(\lambda_k \eta) \frac{\lambda_k}{h} e^{-\lambda_k^2 \xi} \end{aligned} \tag{2.20}$$

The expression for the stream function can also be obtain by this integral :

$$\begin{aligned}\psi(\eta,\xi) &= h\int_0^\eta u(\eta',\xi)d\eta' \\ &= u_c\eta^2\left(1-\frac{1}{3}\eta\right)h + \sum_{k=1}^{\infty} 2D_k\frac{h}{\lambda_k}\sin^2\left(\frac{\lambda_k}{2}\eta\right)e^{-\lambda_k^2\xi}\end{aligned} \tag{2.21}$$

We can know plot these quantities to see if our analytical solution effectively gives the right solution.

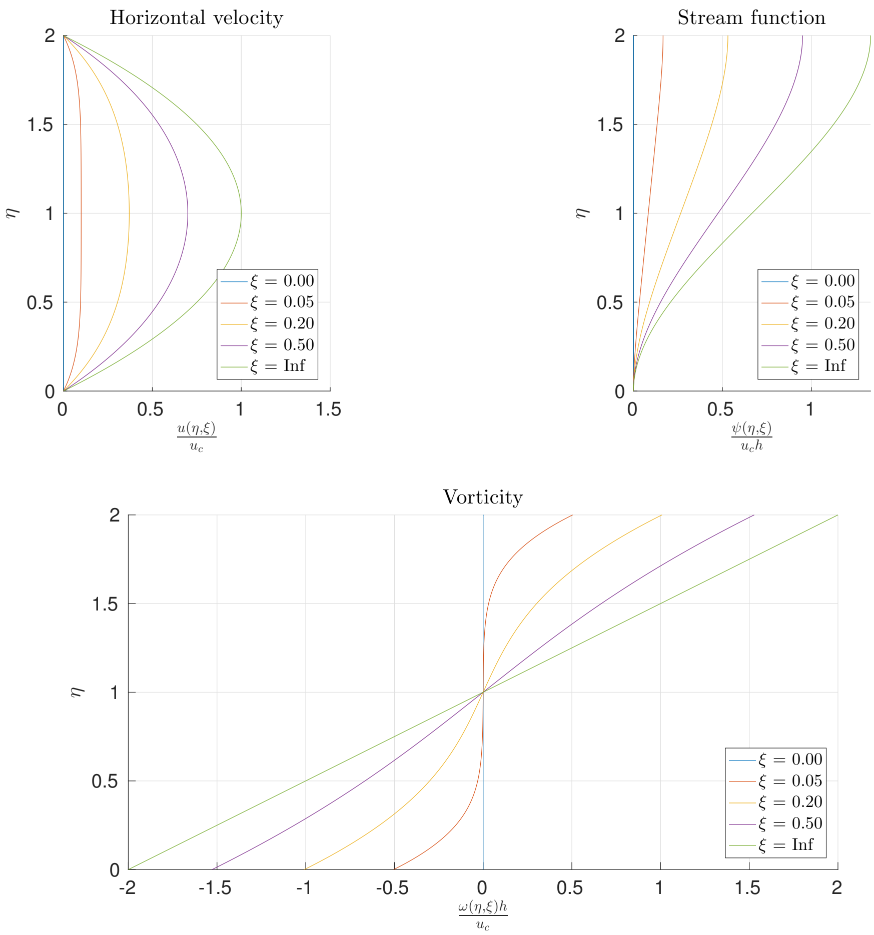

Figure 2.2: Evolution of dimensionless velocity, vorticity and stream function in the channel with constant pressure gradient for different times.

This analytical solution will be use as a reference solution in order to calculate the error of numerical results.
Another problem that could be used for investigating the convergence in 1D is the case of a periodic pressure gradient in a channel flow (see Appendix B for the developed analytical solution for this case).

Chapter 3

Numerical resolution of 1D problem

3.1 Numerical integrator

Now that the analycal solution and formulas for boundary conditions have been developed, we can begin to analyze numerical methods in order to solve the problem. As explained before, the case is purely 1D. This means that all quantities depend only on time and y axis. There is no variation in x axis and vertical velocity is always zero. Equation (15) reduces to :

$$\frac{\partial \omega}{\partial t} = \nu \frac{\partial^2 \omega}{\partial y^2} \tag{3.1}$$

This is only a diffusion equation. Therefore an explicit Euler scheme can solve this equation :

$$\omega^{n+1} = \omega^n + \Delta t \, f(\omega^n, t^n) \tag{3.2}$$

And we will also analyze results given with 2^{nd} order Runge-Kutta scheme :

$$\begin{aligned} k_1 &= \Delta t \, f(\omega^n, t^n) \\ k_2 &= \Delta t \, f(\omega^n + k_1, t^n + \Delta t) \\ \omega^{n+1} &= \omega^n + \frac{1}{2} k_1 + \frac{1}{2} k_2 \end{aligned} \tag{3.3}$$

Here $f(\omega, t)$ is the discretized form of equation (3.1). By using 2^{nd} order finite difference for diffusive part, we have :

$$f(\omega,t) = \frac{\partial\omega}{\partial t}|_i = \nu\,\frac{\partial^2\omega}{\partial y^2}|_i = \nu\,\frac{(\omega_{i+1} - 2\,\omega_i + \omega_{i-1})}{(\Delta y)^2} \tag{3.4}$$

3.2 Poisson solver

Now that we used the equation of conservation of momentum, we have to impose the conservation of mass. This is done by solving equation (13), known as the Poisson's equation. To solve this, one must impose boundary conditions on the top and bottom plates for ψ. We know that :

$$\int_0^{2h} u(y,t)\,dy = Q(t) = \int_0^{2h} \frac{\partial\psi(y,t)}{\partial y}\,dy = \psi(2h,t) - \psi(0,t) \tag{3.5}$$

We consider that the mass flow is given by equation (2.19). This means that, by imposing $\psi(0,t) = 0$, the above condition implies $\psi(2h,t) = Q(t)$. To easily impose this condition, we make a change of variable. We first solve Poisson's equation with $\bar{\psi}(0,t) = \bar{\psi}(2h,t) = 0$. Then we add along y axis a linear term that passes through 0 at $y = 0$ and through $Q(t)$ at $y = 2h$. As this term is linear, its second derivative is zero and thus, doesn't change Poisson's equation.

$$\nabla^2\psi(y,t) = \nabla^2\bar{\psi}(y,t) = -\omega(y,t) \tag{3.6}$$

$$\psi(y,t) = \bar{\psi}(y,t) + \frac{Q(t)}{2}y \tag{3.7}$$

As said before, ψ doesn't depend on x axis. We can then use again the 2^{nd} order finite difference and obtain :

$$\frac{\partial^2\bar{\psi}}{\partial y^2}|_i = -\omega_i = \frac{\left(\bar{\psi}_{i+1} - 2\,\bar{\psi}_i + \bar{\psi}_{i-1}\right)}{(\Delta y)^2} \tag{3.8}$$

$\bar{\psi}$ being the unknown. This gives us a tridiagonal system to solve.

$$\frac{1}{(\Delta y)^2}\begin{bmatrix} -2 & 1 & 0 & & & & & & 0 \\ 1 & -2 & 1 & 0 & & & & & \\ 0 & 1 & -2 & 1 & 0 & & & & \\ & . & . & . & . & . & & & \\ & & 0 & 1 & -2 & 1 & 0 & & \\ & & & . & . & . & . & . & \\ & & & & 0 & 1 & -2 & 1 & 0 \\ & & & & & 0 & 1 & -2 & 1 \\ 0 & & & & & & 0 & 1 & -2 \end{bmatrix} \times \begin{bmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \\ \bar{\psi}_3 \\ . \\ \bar{\psi}_i \\ . \\ \bar{\psi}_{N-4} \\ \bar{\psi}_{N-3} \\ \bar{\psi}_{N-2} \end{bmatrix} = - \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ . \\ \omega_i \\ . \\ \omega_{N-4} \\ \omega_{N-3} \\ \omega_{N-2} \end{bmatrix} \tag{3.9}$$

We use Thomas tridiagonal algorithm to obtain the values of ψ efficiently. When the matrix is solved, we apply the change of variable by adding the linear term to obtain the solution for ψ with correct boundary conditions.

3.3 Convergence test

Now, we will use boundary conditions formulas that we developed to study and compare their order of convergence. The boundary is made of two solid walls. This means that the velocities are null at walls.

$$u(0,t) = u(2h,t) = \frac{\partial \psi(0,t)}{\partial y} = g(x) = 0 \tag{3.10}$$

$$v(0,t) = v(2h,t) = -\frac{\partial \psi(0,t)}{\partial x} = -f'(x) = 0 \tag{3.11}$$

So $f(x)$ function must be a constant equal to ψ_0, and $g(x)$ is set to zero.

$$\omega_0 = -\frac{2}{(\Delta y)^2}(\psi_1 - \psi_0) \tag{3.12}$$

$$\omega_0 = -\frac{1}{2(\Delta y)^2}(8\psi_1 - \psi_2 - 7\psi_0) \tag{3.13}$$

$$\omega_0 = -\frac{3}{(\Delta y)^2}(\psi_1 - \psi_0) - \frac{1}{2}\omega_1 \tag{3.14}$$

$$\omega_0 = -\frac{1}{4(\Delta y)^2}(16\psi_1 - \psi_2 - 15\psi_0) - \frac{1}{6}(4\omega_1 - \omega_2) \tag{3.15}$$

If we want to test the order of convergence of a numerical method, we should set a constant time step Δt and decrease the spatial discretization Δy at each simulation.

Unfortunately, this is not possible when computing fluid dynamic due to Fourier number r. Depending on the time integration scheme, this number can't exceed certain value (like 0.5 for the Euler scheme) otherwise the integration will become unstable and eventually blow up. But keeping Δt constant and decreasing Δy will increase the dimensionless Fourier number until the integration goes unstable. Instead, we will run the convergence test with constant r :

$$r = \frac{\nu \Delta t}{(\Delta y)^2} \Rightarrow \Delta t = \frac{r\,(\Delta y)^2}{\nu} \tag{3.16}$$

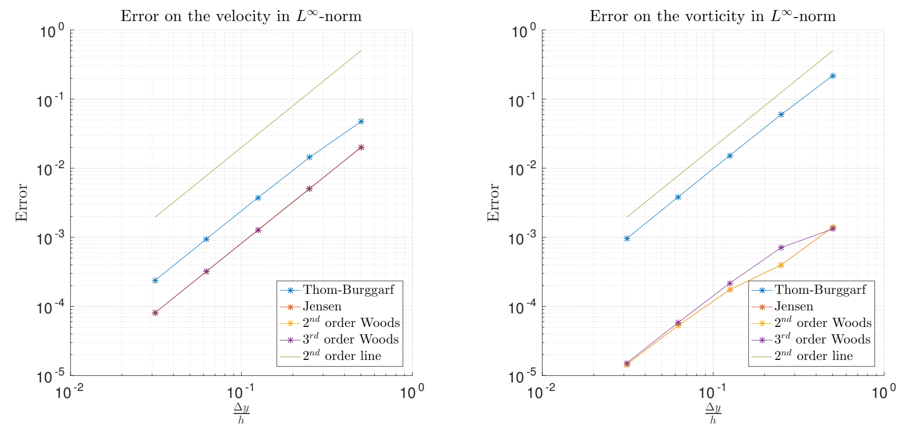

Figure 3.1: Error obtained by using an Explicit Euler scheme.

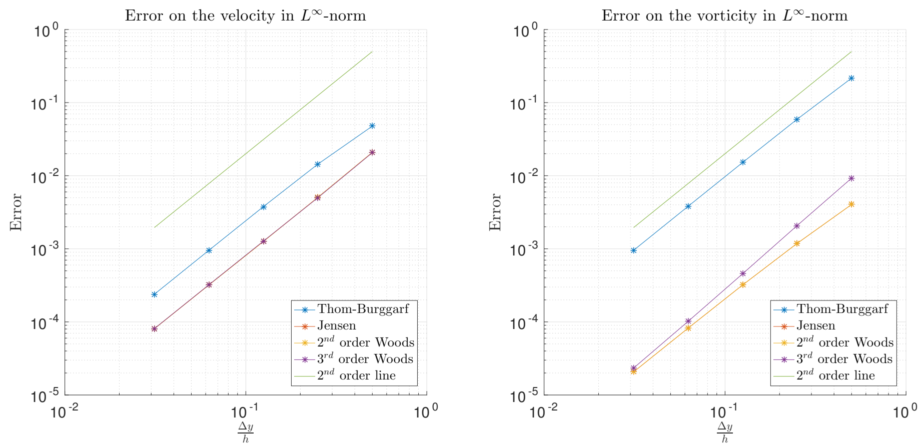

Figure 3.2: Error obtained by using a 2^{nd} order Runge-Kutta scheme.

All these formulas give an error of second order. We see that higher order formulas are more accurate than Thom-Burggarf formula, giving then better result for

vorticity at the wall.
As explained in the paper [2], it is interesting to see that the error obtained with the formula (3.13) and (3.14) are exactly the same. This is because these two formulas lead to the same expression for one-dimensional problems and used with second-order Poisson solver. To prove that, we take the formula (3.14) and we replace ω_1 by the discretized Poisson equation in one-dimension.

$$\omega_1 = -\frac{(\psi_2 - 2\psi_1 + \psi_1)}{(\Delta y)^2} \tag{3.17}$$

$$\downarrow$$

$$\omega_0 = -\frac{3}{(\Delta y)^2}(\psi_1 - \psi_0) + \frac{1}{2}\frac{(\psi_2 - 2\psi_1 + \psi_1)}{(\Delta y)^2} \tag{3.18}$$

$$\omega_0 = -\frac{1}{(2\Delta y)^2}(6\psi_1 - 6\psi_0 - \psi_0 + 2\psi_1 - \psi_2) \tag{3.19}$$

$$\omega_0 = -\frac{1}{(2\Delta y)^2}(8\psi_1 - \psi_2 - 7\psi_0) \tag{3.20}$$

Which is exactly the formula (3.13). This proves that the errors given by formulas (3.13) and (3.14) are the same for one-dimensional problem with second-order Poisson solver.

Chapter 4

Numerical resolution of 2D problem

For 2D case, we want to test the order of convergence for 3^{rd} order Woods formula in a case where the solution depends on x and y axis. So we are looking for a 2D problem that has a bounded boundary condition and has an analytical solution. Unfortunately, we couldn't find one. So we decided to do a self-converging test. The principle is to run a simulation with very small spatial discretization, and then consider this result as the reference solution. We can now run simulations with coarser discretization and calculate the error relatively to the reference simulation.

4.1 Description of the problem

The selected case that we will analyze is the backward facing step. We consider a channel flow that get an abrupt enlargement as shown in the figure 4.1.

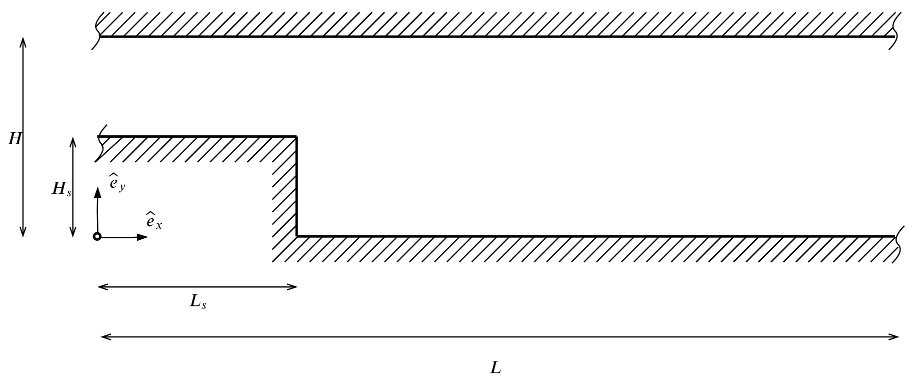

Figure 4.1: Backward facing step.

The inlet is a Poiseuille flow. After some distance, the channel gets abruptly enlarged. The flow behaves differently depending on the Reynolds number. This number is defined as follow :

$$Re = \frac{Q}{\nu} \tag{4.1}$$

If the Reynolds number is not too high, a stationary solution exists. In this case, the flow reattaches to the bottom plate and there is a recirculating cell next to the step. If we rise the Reynolds number high enough, the flow won't reattach to the bottom plate like the stationary solution. It becomes unsteady and creates vortex shedding past the step.

4.2 Numerical integrator

As this problem is 2D, the vertical velocity is no longer zero and the vorticity is function of x axis, we can not neglect the convection term of equation (15).

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \nu \left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right) \tag{4.2}$$

Now, we have a contribution from convective and diffusive terms in the equation. So if we want to integrate Navier-Stokes equation, a simple explicit Euler or a 2^{nd} order Runge-Kutta scheme might become unstable, as the marginal stability curve of such schemes does not cross the imaginary axis. We hence chose a 3^{rd} order Runge-Kutta scheme as its marginal stability curve contains a part of the imaginary axis (crossing it at $\sqrt{3}\, i$).

$$\begin{aligned} k_1 &= \Delta t\, f(\omega^n, t^n) \\ k_2 &= \Delta t\, f\left(\omega^n + \frac{1}{3}k_1, t^n + \frac{1}{3}\Delta t\right) \\ k_3 &= \Delta t\, f\left(\omega^n + \frac{2}{3}k_2, t^n + \frac{2}{3}\Delta t\right) \\ \omega^{n+1} &= \omega^n + \frac{1}{4}k_1 + \frac{3}{4}k_3 \end{aligned} \tag{4.3}$$

4.3 Convective term

The convective term can be written in two equivalent forms; the advective and divergence form, as shown in the following equation :

$$u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \frac{\partial(\omega u)}{\partial x} + \frac{\partial(\omega v)}{\partial y} \tag{4.4}$$

This equality is true due to the fluid being incompressible; it's divergence free. Considering a Marker And Cell mesh, we can obtain the discretized form of these terms.

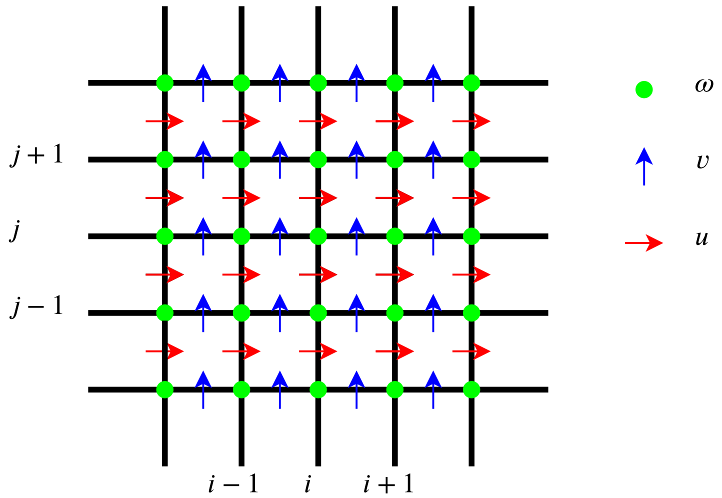

Figure 4.2: Marker And Cell mesh.

Advective form

The discretization of the convective term in advective form is :

$$u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} \Rightarrow u_{i,j}\frac{(\omega_{i+1,j} - \omega_{i-1,j})}{2\Delta x} + v_{i,j}\frac{(\omega_{i,j+1} - \omega_{i,j-1})}{2\Delta y} \tag{4.5}$$

As the mesh is staggered, the velocity at i, j are unknown. We calculate them by mean value of adjacent velocities :

$$u_{i,j} = \frac{1}{2}\left(u_{i,j+\frac{1}{2}} + u_{i,j-\frac{1}{2}}\right) \tag{4.6}$$

$$v_{i,j} = \frac{1}{2}\left(v_{i+\frac{1}{2},j} + v_{i-\frac{1}{2},j}\right) \tag{4.7}$$

Divergence form

And for the divergence form, the discretization gives :

$$\frac{\partial(\omega u)}{\partial x} + \frac{\partial(\omega v)}{\partial y} \Rightarrow \frac{(\omega u)_{i+1,j} - (\omega u)_{i-1,j}}{2\Delta x} + \frac{(\omega v)_{i,j+1} - (\omega v)_{i,j-1}}{2\Delta y} \tag{4.8}$$

As for the advective form, as the mesh is staggered, the velocities are also calculated by mean value of adjacent velocities.

Skew-symmetric form

There is also a third form for discretizing these terms; the Skew-symmetric form. It consist of taking half of each previous forms. By doing so, we obtain the skew-symmetric form for the vorticity-velocity formulation :

$$\frac{1}{2\Delta x}\left(u_{i+\frac{1}{2},j}\omega_{i+1,j} - u_{i-\frac{1}{2},j}\omega_{i-1,j}\right) + \frac{1}{2\Delta y}\left(v_{i,j+\frac{1}{2}}\omega_{i,j+1} - v_{i,j-\frac{1}{2}}\omega_{i,j-1}\right) \tag{4.9}$$

If there is no viscosity, this form has the advantage of conserving the kinetic energy and the momentum. (See Appendix C for a different manners to discretize these terms).

4.4 Poisson solver

To impose a free-divergence flow, we have to solve the Poisson equation. As the flow is not 1D problem, the solution to that equation depends of x and y. A simple tridiagonal system will not work for this case. Instead we use a sparse solver given by the *Petsy* library. We just have to assemble the sparse matrix and the right hand side vector.
The discretized Poisson's equation for any points out of the boundary is :

$$\frac{(\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j})}{(\Delta x)^2} + \frac{(\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1})}{(\Delta y)^2} = -\omega_{i,j} \tag{4.10}$$

$$\frac{\psi_{i,j+1}}{(\Delta y)^2} + \frac{\psi_{i+1,j}}{(\Delta x)^2} - \left(\frac{2}{(\Delta x)^2} + \frac{2}{(\Delta y)^2}\right)\psi_{i,j} + \frac{\psi_{i-1,j}}{(\Delta x)^2} + \frac{\psi_{i,j-1}}{(\Delta y)^2} = -\omega_{i,j} \tag{4.11}$$

Equation (4.11) gives us the coefficients to put in the sparse matrix.

To solve this equation, we have to impose boundary condition for ψ. Like 1D problem, we impose ψ at the top plate to be equal to the mass flow $\left(Q = \frac{3}{2}(H - H_s)u_c\right)$ and the bottom plate to be equal to zero. For inlet and outlet, we impose a natural condition both on ψ and ω. The derivative of these two quantities with respect to the normal (in this case, the x axis) is null.

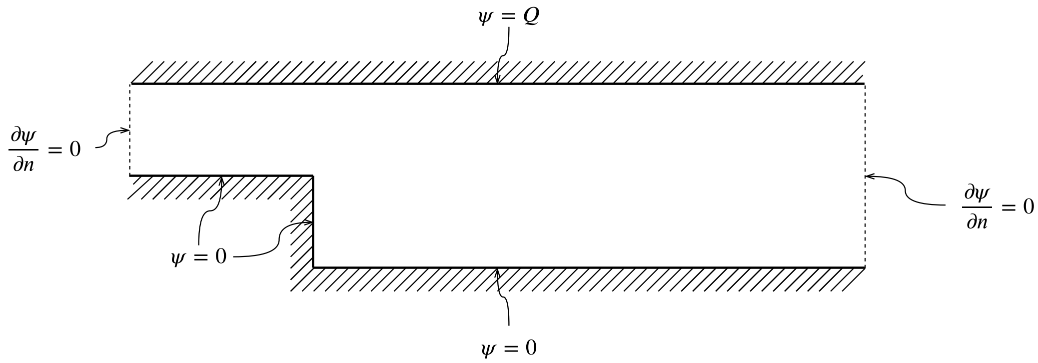

Figure 4.3: Boundary condition for ψ.

Therefore coefficients at inlet and outlet are sightly different :

$$\frac{\psi_{i,j+1}}{(\Delta y)^2} + \frac{\psi_{i+1,j}}{(\Delta x)^2} - \left(\frac{1}{(\Delta x)^2} + \frac{2}{(\Delta y)^2}\right)\psi_{i,j} + \frac{\psi_{i,j-1}}{(\Delta y)^2} = -\omega_{i,j} \qquad \textit{for inlet} \tag{4.12}$$

$$\frac{\psi_{i,j+1}}{(\Delta y)^2} - \left(\frac{1}{(\Delta x)^2} + \frac{2}{(\Delta y)^2}\right)\psi_{i,j} + \frac{\psi_{i-1,j}}{(\Delta x)^2} + \frac{\psi_{i,j-1}}{(\Delta y)^2} = -\omega_{i,j} \qquad \textit{for outlet} \tag{4.13}$$

4.5 Results

When the Reynolds number is not too high, the solution should be stationary as the flow would reattach after the step. After some distance, we should recover a Poiseuille flow. At high Reynolds number, the flow becomes unstable and can not reattach to the bottom plate.
The simulations for figures 4.4, 4.5 and 4.6 are done with the same 512×512 mesh grid. But as the computation domain is 1×8, the ratio $\frac{\Delta x}{\Delta y}$ equals to 8. This mesh ensure a mesh Reynolds number based on the vorticity below 5 for the worst case; the simulation at $Re = 1000$ (See Appendix D for the mesh Reynolds number in 2D). For this case, the maximum mesh Reynolds number is 3.82. We can see the consequence of this number at figure 4.6; the simulation is a bit under-resolved at

the center of the channel. The ratio we took for these simulation are :

$$\frac{H_s}{H} = \frac{1}{2} \qquad\qquad \frac{L_s}{L} = \frac{1}{4}$$

As said before, when the stationary solution exists, we recover a Poiseuille flow on the outlet of the domain. This is the case for figure 4.4 but not for figure 4.5. We don't recover Poiseuille flow as the computational domain is not long enough; The outlet condition interfere with the flow.

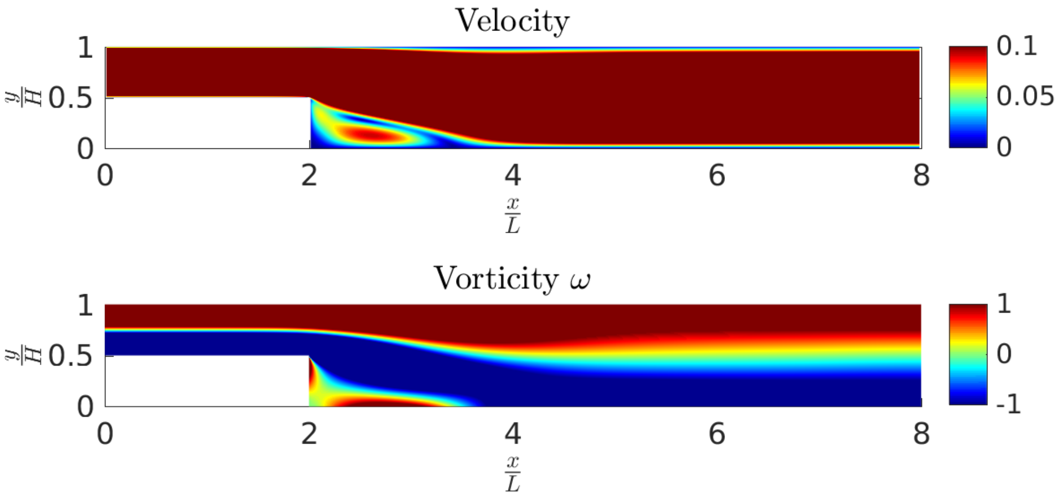

Figure 4.4: Velocity norm and vorticity for the backward facing step solved at $Re = 50$.

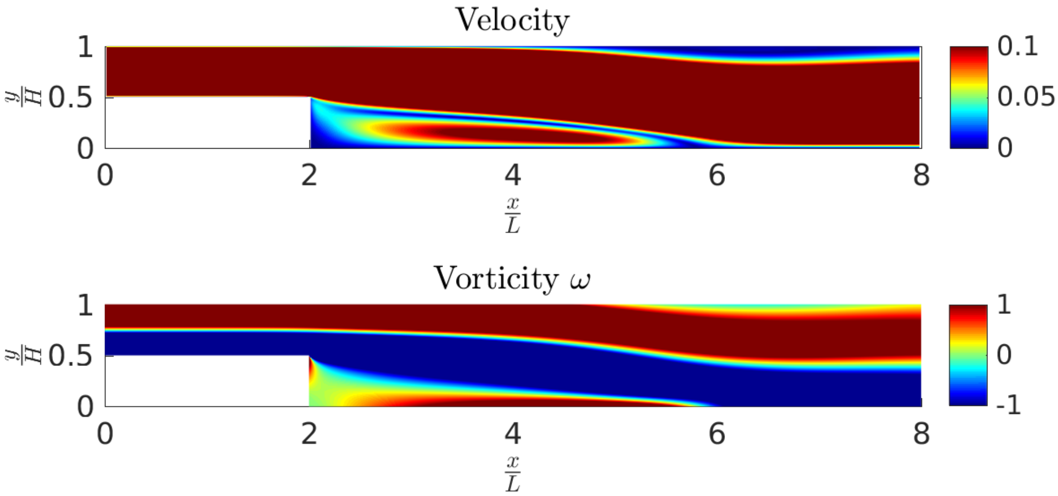

Figure 4.5: Velocity norm and vorticity for the backward facing step solved at $Re = 200$.

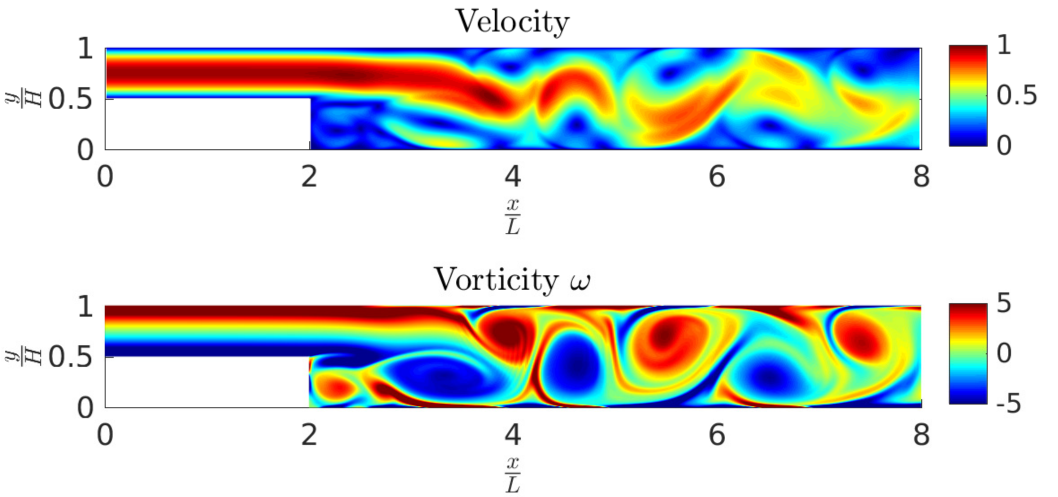

Figure 4.6: Velocity norm and vorticity for the backward facing step solved at $Re = 1000$ at some arbitrary time.

4.6 Convergence test

As there is no analytical solution for this 2D problem, we have to do a self convergence test. We run a simulation with a very fine spatial discretization and we consider it as reference solution. Then we compare it to coarser simulations. Like 1D problem, we should see that the error decreases with second order.

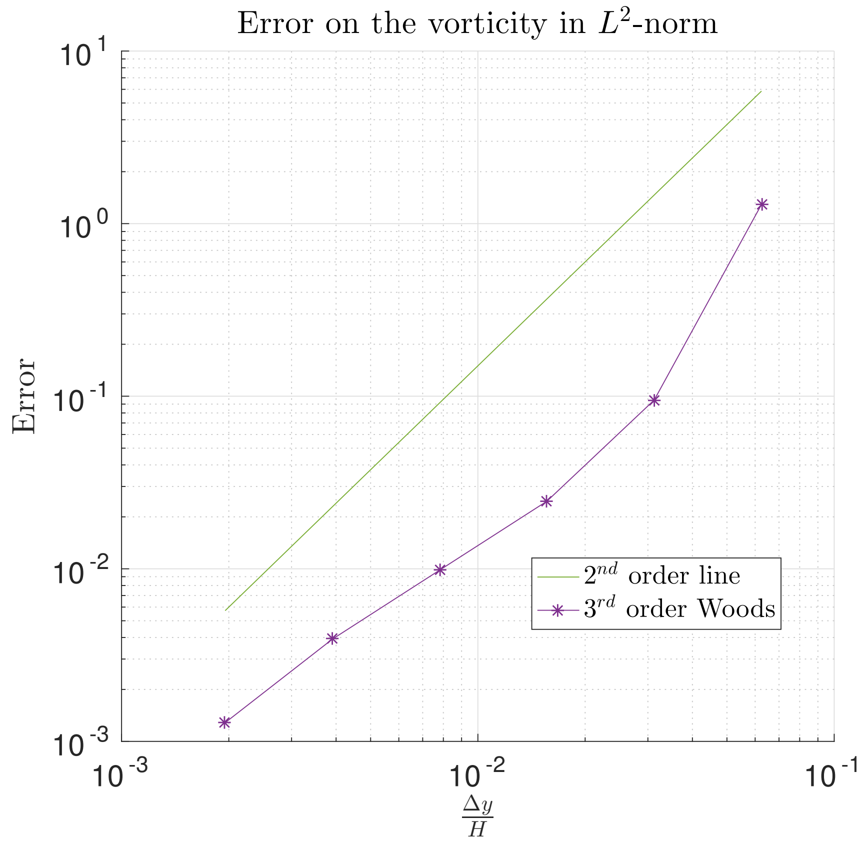

Figure 4.7: Error obtained in L^2-norm by using a 3^{rd} order Runge-Kutta.

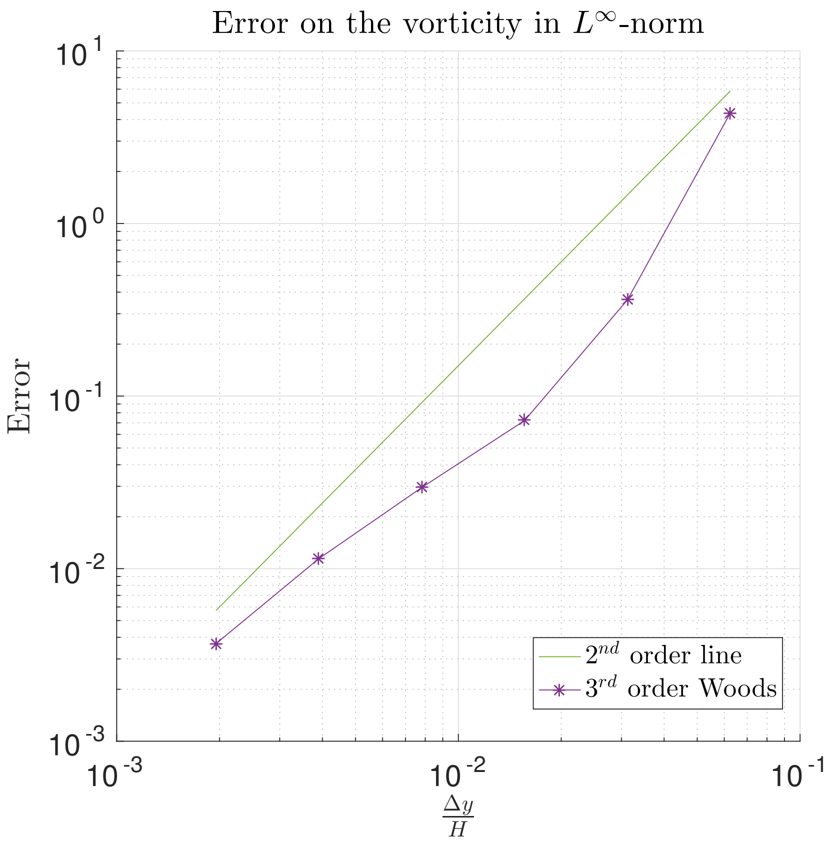

Figure 4.8: Error obtained in L^∞-norm by using a 3^{rd} order Runge-Kutta.

The reference solution is calculated over a 1024×1024 mesh grid and mesh Reynolds number associated to it is 0.04. We then use coarser spatial discretization by taking a small mesh grid; 512×512, 256×256, 128×128 ,64×64 ,32×32, 16×16. Each simulations are then compared to the 1024×1024. By doing so, we can obtain the error for each simulation. When using the 3^{rd} order Woods formula, we can see on figure 4.8 and 4.7 that the error converges to zero with second order when doing convergence test at constant r.

Dipole in a Poiseuille flow

An interesting problem that we can simulate is the case of Poiseuille flow with a dipole. In 2D cartesian domain, at $t < 0$, we consider a long channel with Poiseuille flow. At $t = 0$, we introduce a dipole close to the center of the channel. This simulation was done so as to provide to a professor of Caltech, Anthony Leonard, a reference solution of the Navier-Stokes equations for that problem. Indeed, he has developed a method to obtain the exact solution of the linearized Navier-Stokes equations for any small perturbation of the Poiseuille flow. Our solution will be considered as the reference to compare to.

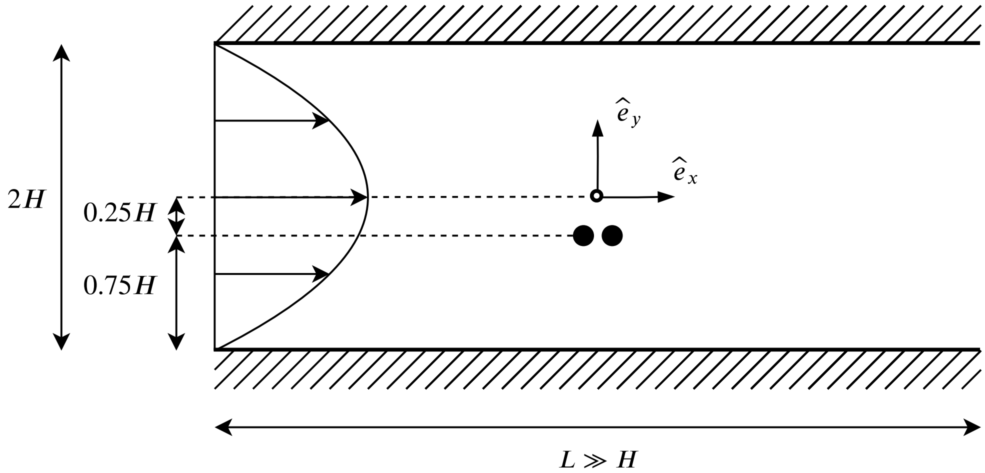

Figure 4.9: Description of the case.

The channel has to be considered as infinite. We will then use a very long domain and assume that the flow is periodic in order to compute the solution. The computational domain is 512×128 grid points in a $8H \times 2H$ domain. The spacial discretization is thus $\Delta x = \Delta y = h = \frac{H}{64}$.
The vorticity of the dipole is given in the PhD thesis [4] :

$$\omega_{dip} = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{1}{2}\frac{r^2}{\sigma^2}\right)\frac{r}{\sigma}\sin(\theta)\frac{\gamma}{\sigma} \tag{4.14}$$

Where γ is the strength of the dipole, r is the distance between the evaluated point to the center of the dipole and θ the angle between the evaluated point and a vertical line. As the linearized theory involved small perturbation, the dipole has to be weak.

$$\gamma = 0.01 u_c H^2 \tag{4.15}$$

As we want our dipole well resolved, we set σ to be equal to $3h$.
Now that we define all quantities in equation (4.14), we still have to define the viscosity of the fluid. It is calculated by the Reynolds number :

$$Re = \frac{u_c(2H)}{\nu} = 5000 \tag{4.16}$$

For the time integration, we use the same 3^{rd} order Runge-Kutta scheme as the 2D problem; equation (4.3). The CFL used in the simulation is taken low to be accurate in time.

$$CFL = \frac{\Delta t\, u_c}{h} = 0.128 \tag{4.17}$$

We output the solution fields every 5 time step. We thus have the solution at every $\frac{tu_c}{h} = \frac{1}{100}$. As the theory is only valid for small time, we stop the integration 250 time step later; when $\frac{tu_c}{h} = \frac{250}{5}\frac{1}{100} = 0.5$.
The aim of this case is to confront a linear theory with numerical results. We are mainly interested in the vertical velocity. The results are presented as the perturbation of the quantities. This means that from the solution obtained by numerical integration we subtract the Poiseuille flow.

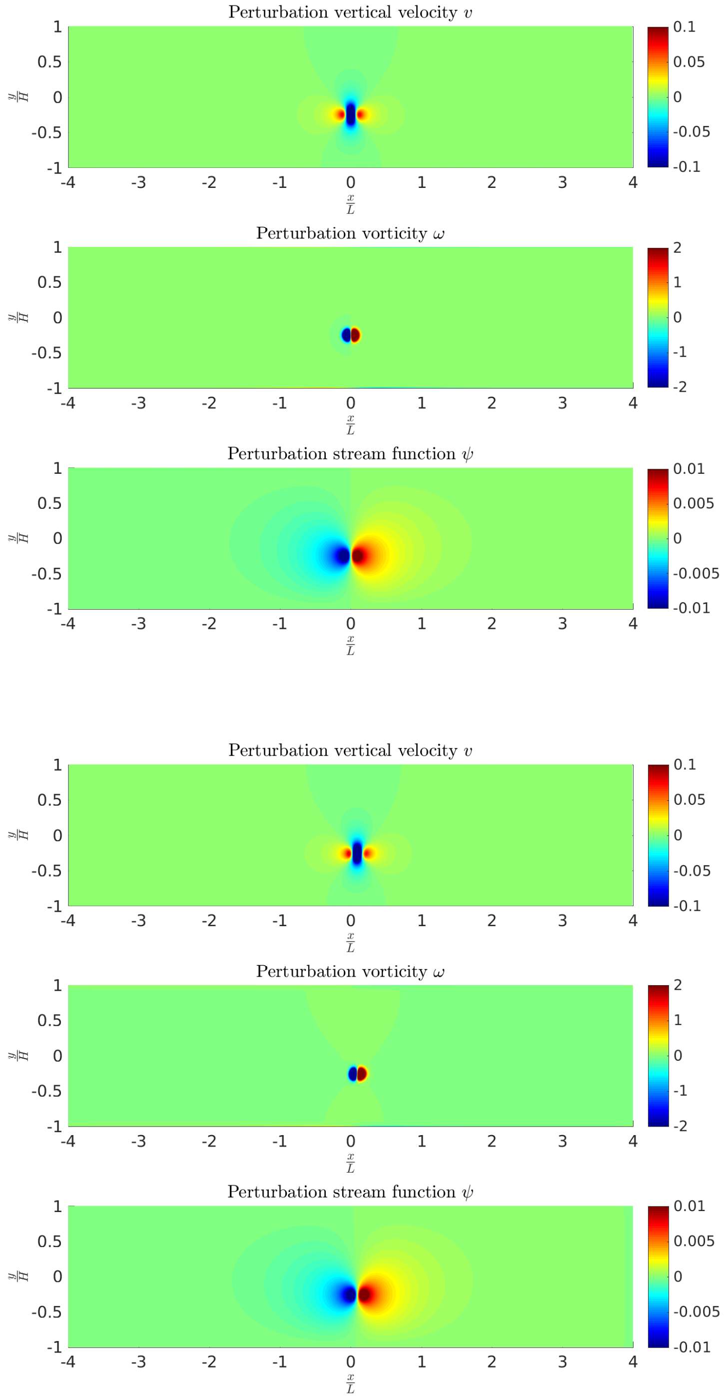

Figure 4.10: Numerical solution of the flow at $t = 0$ and $t = 0.1$.

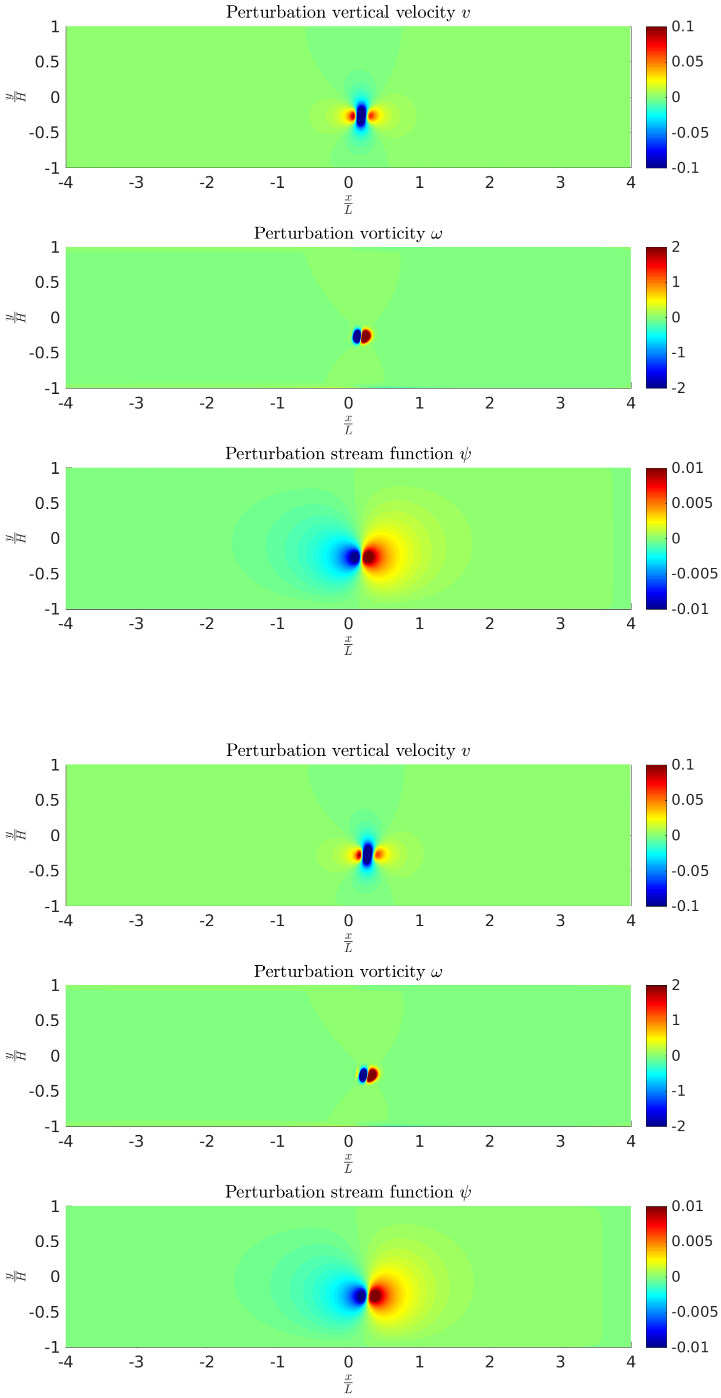

Figure 4.11: Numerical solution of the flow at $t = 0.2$ and $t = 0.3$.

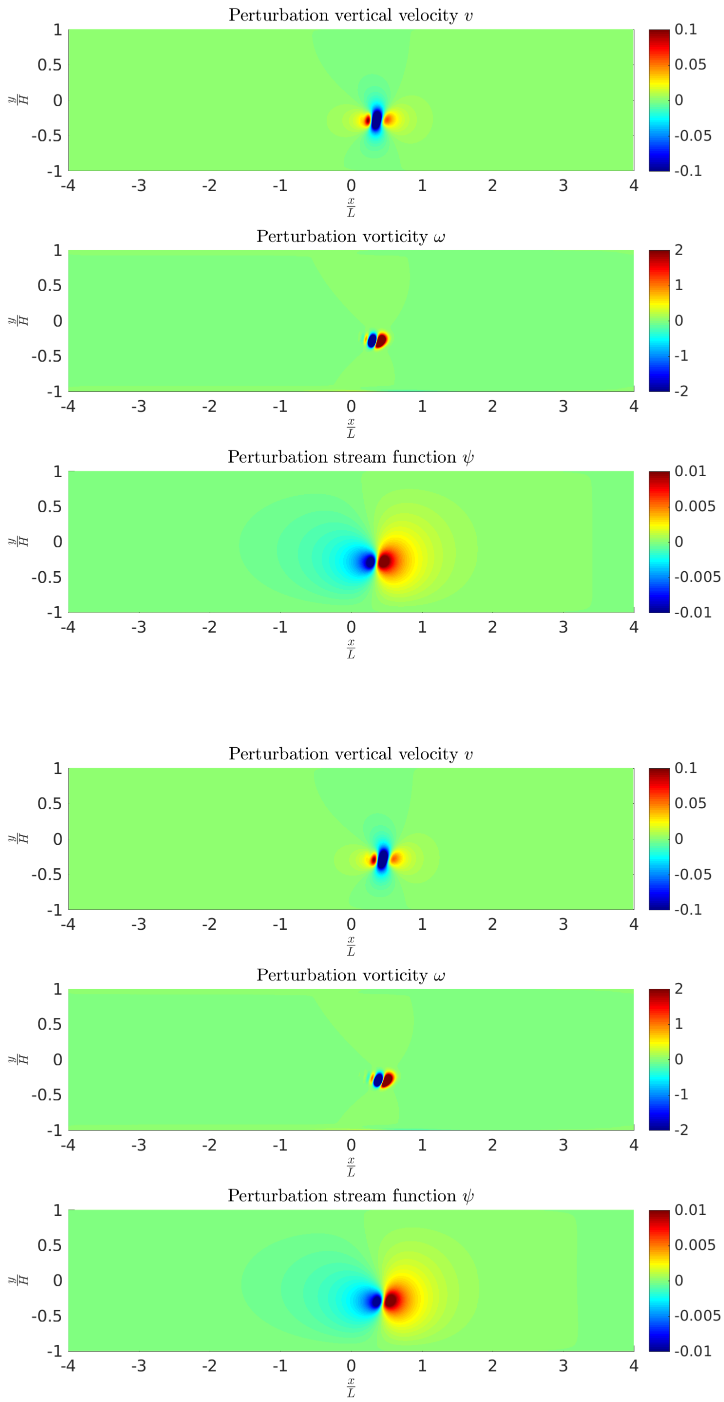

Figure 4.12: Numerical solution of the flow at $t = 0.4$ and $t = 0.5$.

Conclusion

The aim of this thesis was to investigate a high order boundary conditions for the Navier-Stokes equations in vorticity-velocity formulation and to compare it to other formulas that evaluate the vorticity at the wall. After we obtained these formulas, we developed analytical solutions for constant and periodic pressure gradient cases. With these reference solution implemented, we tested the order of convergence for Thom-Burggraf, Jensen, 2^{nd} order Woods and 3^{nd} order Woods boundary conditions. These tests were first made in 1D case by considering constant Fourier number. We made the same tests in a 2D problem; the backward facing step. Finally, to confront a linear theory with numerical results, we simulated the evolution of a dipole in Poiseuille flow between two infinite plates.
It would be interesting to test other formulas, like Briley's formula (uses 3 level of information in ψ).

Bibliography

[1] V. Legat and G. Winckelmans. “Mecanique des fluides et transferts I”. 2018.

[2] M. Napolitano, G. Pascazio, and L. Quartapelle. “A review of vorticity conditions in the numerical solution of the $\zeta - \psi$ equations”. PhD thesis. Istituto di Macchine ed Energetica, Politecnico di Bar, 1998, p. 162.

[3] G. Winckelmans. “LMECA2660 incompressible flows”. 2018.

[4] G. Winckelmans. “Topics in vortex methods for the computation of thee- and two-dimensional incompressible unsteady flow”. PhD thesis. Graduate Aeronautical Laboratories, California Institute of thecnology (Advisor A. Leonard), 1989.

Appendix A

Fully decentered discretization

Let's assume that we want to discretize the first derivative of a random quantity U by only using information that comes next. This is called upwelling.

A.1 1^{st} order

For this discretization, we only use U_{i+1} to obtain a formula that have an error of 1^{st} order. We begin with Taylor series of U_{i+1} close to U_i :

$$U_{i+1} = U_i + \Delta x \frac{\partial U}{\partial x} + \frac{(\Delta x)^2}{2} \frac{\partial^2 U}{\partial x^2} + \mathcal{O}\left((\Delta x)^3\right) \tag{A.1}$$

By isolating the desired term, we have :

$$\frac{\partial U}{\partial x} = \frac{U_{i+1} - U_i}{\Delta x} - \frac{\Delta x}{2} \frac{\partial^2 U}{\partial x^2} + \mathcal{O}\left((\Delta x)^2\right) \tag{A.2}$$

A.2 2^{nd} order

For this formula, we use U_{i+1} and U_{i+2}. Like the 1^{st} order error, we write Taylor series :

$$U_{i+1} = U_i + \Delta x \frac{\partial U}{\partial x} + \frac{(\Delta x)^2}{2} \frac{\partial^2 U}{\partial x^2} + \frac{(\Delta x)^3}{6} \frac{\partial^3 U}{\partial y^3} + \mathcal{O}\left((\Delta x)^4\right) \tag{A.3}$$

$$U_{i+2} = U_i + 2\Delta x \frac{\partial U}{\partial x} + 4\frac{(\Delta x)^2}{2} \frac{\partial^2 U}{\partial x^2} + 8\frac{(\Delta x)^3}{6} \frac{\partial^3 U}{\partial y^3} + \mathcal{O}\left((\Delta x)^4\right) \tag{A.4}$$

We combine these two expressions in order to nullify the second derivative :

$$4U_{i+1} - U_{i+2} = 3U_i + 2\Delta x \frac{\partial U}{\partial x} + 0 - 4\frac{(\Delta x)^3}{6} \frac{\partial^3 U}{\partial x^3} + \mathcal{O}\left((\Delta x)^4\right) \tag{A.5}$$

This gives the 2^{nd} order error formula :

$$\frac{\partial U}{\partial x} = \frac{-3U_i + 4U_{i+1} - U_{i+2}}{2\Delta x} + \frac{(\Delta x)^2}{3} \frac{\partial^3 U}{\partial x^3} + \mathcal{O}\left((\Delta x)^3\right) \tag{A.6}$$

Appendix B

Channel flow with periodic pressure gradient

A other interesting 1D problem is the case of a flow between two infinite plates with periodic pressure gradient. Like the constant pressure gradient, there is no vertical velocity and the horizontal velocity does not change along x axis. Navier-Stokes equation (2.1) reduces to :

$$\frac{\partial u(y,t)}{\partial t} = -\frac{dP}{dx} + \nu \frac{\partial^2 u(y,t)}{\partial y^2} \tag{B.1}$$

As the pressure gradient is periodic, it can be written like :

$$\frac{dP}{dx} = \frac{dP}{dx}|_0 \cos(\bar{\omega} t) = \Re \left\{ \frac{dP}{dx}|_0 e^{i\bar{\omega} t} \right\} \tag{B.2}$$

The maximal velocity u_c of this case is obtained when $\cos(\bar{\omega} t) = 1$ and its expression comes from constant pressure gradient stationary solution at the center of the channel. By equation (2.4), we can express the maximal pressure gradient with u_c :

$$u_c = -\frac{dP}{dx}|_0 \frac{h^2}{2\nu} \tag{B.3}$$

The equation (B.1) can be rewritten without pressure gradient.

$$\frac{\partial u(y,t)}{\partial t} = u_c \frac{2\nu}{h^2} e^{i\bar{\omega} t} + \nu \frac{\partial^2 u(y,t)}{\partial y^2} \tag{B.4}$$

We then introduce a change of variable to ease the integration :

$$\eta = \frac{y}{h} \qquad \xi = \frac{\nu t}{h^2} \qquad \bar{\omega}^* = \frac{\bar{\omega} h^2}{\nu}$$

The equation (B.4) becomes :

$$\frac{\partial u(\eta,\xi)}{\partial \xi} = u_c 2 e^{i\bar{\omega}t} + \frac{\partial^2 u(\eta,\xi)}{\partial \eta^2} \tag{B.5}$$

We assume that the solution to that equation is the product of a periodic term and a function only depending on the distance to the wall, expressed by η.

$$\frac{u(\eta,\xi)}{u_c} = H(\eta) e^{i\bar{\omega}t} = H(\eta) e^{i\bar{\omega}^*\xi} \tag{B.6}$$

$$\downarrow$$

$$\frac{\partial u(\eta,\xi)}{\partial \xi} = u_c H(\eta) i\bar{\omega}^* e^{i\bar{\omega}^*\xi} \tag{B.7}$$

$$\frac{\partial^2 u(\eta,\xi)}{\partial \eta^2} = u_c H''(\eta) e^{i\bar{\omega}^*\xi} \tag{B.8}$$

By re-injecting these expressions in equation (B.5), we can obtain an ordinary differential equation :

$$u_c H(\eta) i\bar{\omega}^* e^{i\bar{\omega}^*\xi} = u_c 2 e^{i\bar{\omega}^*\xi} + u_c H''(\eta) e^{i\bar{\omega}^*\xi} \tag{B.9}$$

$$-H''(\eta) + i\bar{\omega}^* H(\eta) = 2 \tag{B.10}$$

To solve this equation, we assume that the solution is a sum of two terms, one homogeneous $H_h(\eta)$ and one particular $H_p(\eta)$ solutions.
The homogeneous solution is obtained by solving this equation :

$$-H_h''(\eta) + i\bar{\omega}^* H_h(\eta) = 0 \tag{B.11}$$

Integration this ODE gives :

$$H_h(\eta) = A\cos\left(\sqrt{i\bar{\omega}^*}\eta\right) + B\sin\left(\sqrt{i\bar{\omega}^*}\eta\right) \tag{B.12}$$

For the particular solution, as the right hand side of equation (B.10) is constant, we can assume that H_p is also constant. The particular solution is easily obtained as its derivative is null.

$$H_p = \frac{2}{i\bar{\omega}^*} \tag{B.13}$$

The final solution to equation (B.10) is :

$$H(\eta) = A\cos\left(\sqrt{i\bar{\omega}^*}\eta\right) + B\sin\left(\sqrt{i\bar{\omega}^*}\eta\right) + \frac{2}{i\bar{\omega}^*} \tag{B.14}$$

We still have to find the values of A and B. For that, we use the boundary conditions. We know that the velocity is null at the wall. This means that $H(\eta)$ must be null when $\eta = 0$ and at $\eta = 2$

$$\begin{cases} H(0) = 0 \\ H(2) = 0 \end{cases} \Rightarrow \begin{cases} A + \frac{2}{i\bar{\omega}^*} = 0 \\ A\cos\left(2\sqrt{i\bar{\omega}^*}\right) + B\sin\left(2\sqrt{i\bar{\omega}^*}\right) + \frac{2}{i\bar{\omega}^*} = 0 \end{cases} \tag{B.15}$$

When solving these equations for A and B, we obtain :

$$A = -\frac{2}{i\bar{\omega}^*} \qquad B = -\frac{2}{i\bar{\omega}^*}\tan\left(\sqrt{i\bar{\omega}^*}\right)$$

Now, we have everything to recover the horizontal velocity $u(\eta, \xi)$:

$$\frac{u(\eta,\xi)}{u_c} = \Re\left\{-\frac{2}{i\bar{\omega}^*}\left[\cos\left(\sqrt{i\bar{\omega}^*}\eta\right) + \tan\left(\sqrt{i\bar{\omega}^*}\right)\sin\left(\sqrt{i\bar{\omega}^*}\eta\right) - 1\right]e^{i\bar{\omega}^*\xi}\right\} \tag{B.16}$$

Same as we did for constant pressure gradient, we can also recover the expressions of the vorticity and the stream function for periodic pressure gradient case :

$$\begin{aligned} \omega(\eta,\xi) &= \frac{\partial v(\eta,\xi)}{\partial x} - \frac{\partial u(\eta,\xi)}{\partial y} = -\frac{\partial u(\eta,\xi)}{\partial \eta}\frac{\partial \eta}{\partial y} \\ &= \Re\left\{\frac{2u_c}{hi\bar{\omega}^*}\left[-\sin\left(\sqrt{i\bar{\omega}^*}\right)\sqrt{i\bar{\omega}^*} + \tan\left(\sqrt{i\bar{\omega}^*}\right)\cos\left(\sqrt{i\bar{\omega}^*}\eta\right)\sqrt{i\bar{\omega}^*}\right]e^{i\bar{\omega}^*\xi}\right\} \end{aligned} \tag{B.17}$$

$$\begin{aligned} \psi(\eta,\xi) &= h\int_0^\eta u(\eta',\xi)d\eta' \\ &= \Re\left\{\frac{-2u_c h}{i\bar{\omega}^*}\left[\frac{\sin\left(\sqrt{i\bar{\omega}^*}\eta\right)}{\sqrt{i\bar{\omega}^*}} + \frac{2\tan\left(\sqrt{i\bar{\omega}^*}\right)}{\sqrt{i\bar{\omega}^*}}\sin^2\left(\frac{\sqrt{i\bar{\omega}^*}\eta}{2}\right) - \eta\right]e^{i\bar{\omega}^*\xi}\right\} \end{aligned} \tag{B.18}$$

It is interesting to see that when $\bar{\omega}^*$ is low, the flow follows the pressure gradient. The figure B.1 shows that the velocity profile is nearly a Poiseuille flow when $\xi = 2k\pi$ with $k = 0, 1, 2....$

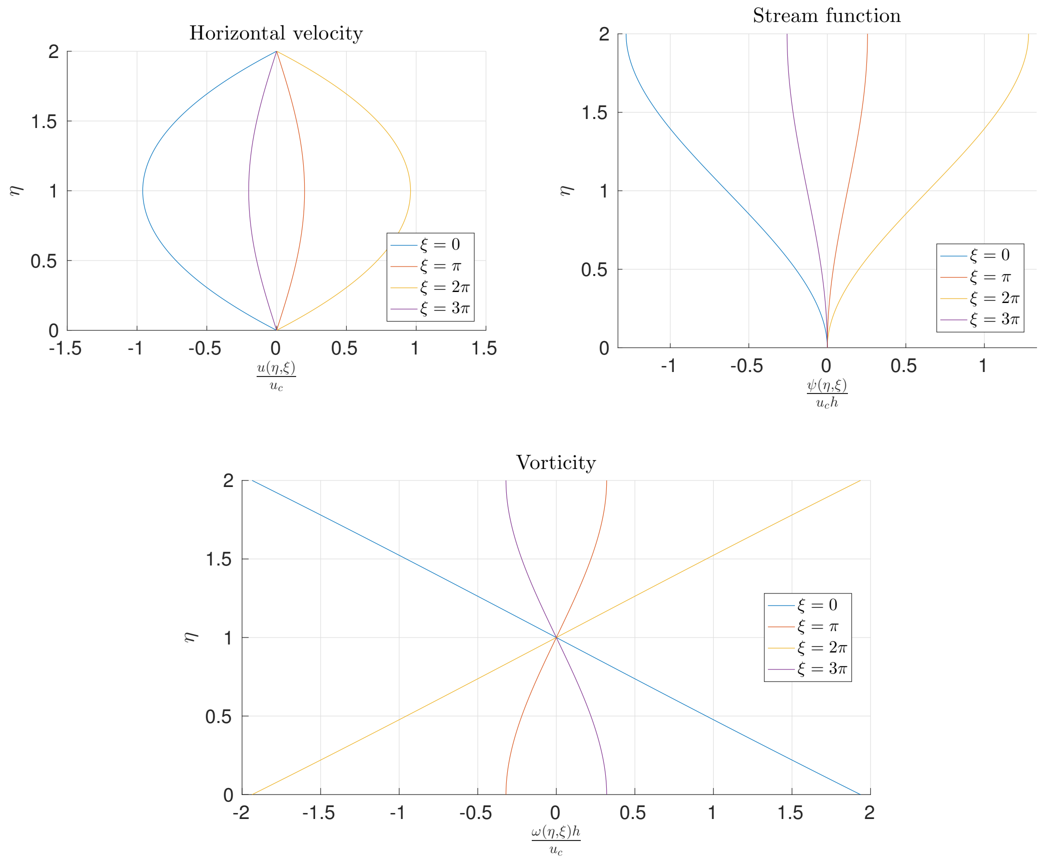

Figure B.1: Evolution of dimensionless velocity, vorticity and stream function in the channel with periodic pressure gradient for different times at $\bar{\omega}^* = \frac{1}{2}$.

If we increase $\bar{\omega}^*$, we should see an interesting behaviour of the flow. The sign of velocity profile may change along η. This is shown at figure 2.4.

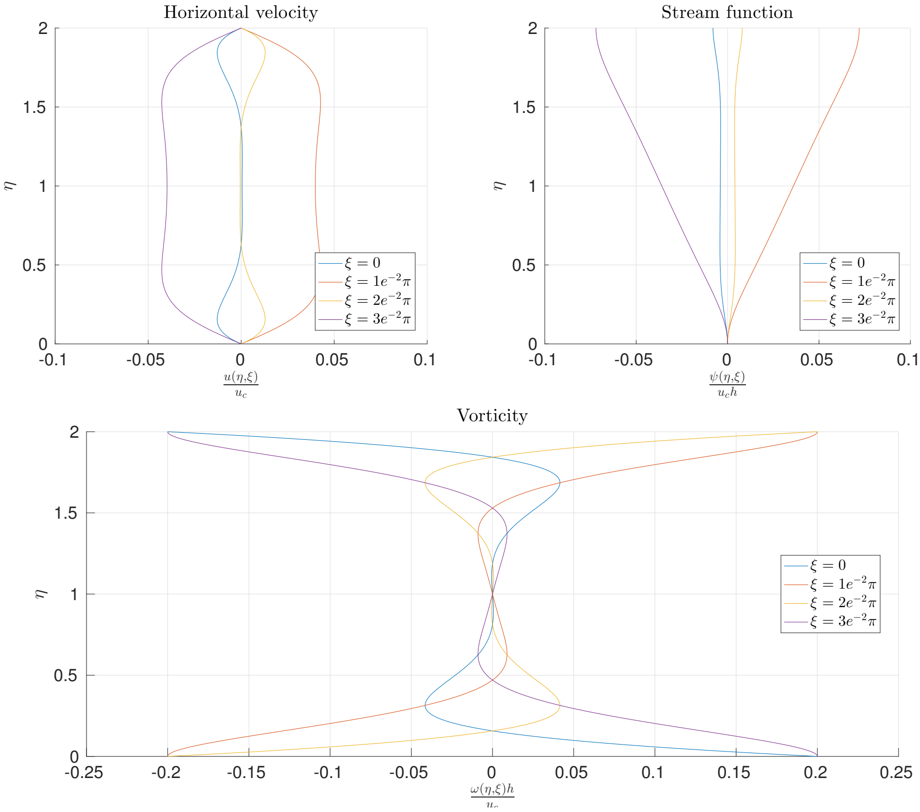

Figure B.2: Evolution of dimensionless velocity, vorticity and stream function in the channel with periodic pressure gradient for different times at $\bar{\omega}^* = 50$

Unfortunately, equation (B.16) does not consider the flow at rest for initial condition. This means that when doing the numerical integration of this problem, the initial condition will be the value given by analytical solution at a defined time t.

Appendix C

Discretization of convective term

The discretization can also be written like we do in velocity-pressure formulation; discretize the term in $i + \frac{1}{2}$ and in $i - \frac{1}{2}$ and calculate the mean value of them.

Advective form

$$\frac{1}{2}\left[u_{i+\frac{1}{2},j}\frac{\omega_{i+1,j}-\omega_{i,j}}{\Delta x} + u_{i-\frac{1}{2},j}\frac{\omega_{i,j}-\omega_{i-1,j}}{\Delta x}\right] + \frac{1}{2}\left[v_{i,j+\frac{1}{2}}\frac{\omega_{i,j+1}-\omega_{i,j}}{\Delta x} + v_{i,j-\frac{1}{2}}\frac{\omega_{i,j}-\omega_{i,j-1}}{\Delta x}\right] \quad (C.1)$$

Divergence form

$$\frac{1}{2}\left[\frac{u_{i+1,j}\omega_{i+1,j}-u_{i,j}\omega_{i,j}}{\Delta x} + \frac{u_{i,j}\omega_{i,j}-u_{i-1,j}\omega_{i-1,j}}{\Delta x}\right] + \frac{1}{2}\left[\frac{v_{i,j+1}\omega_{i,j+1}-v_{i,j}\omega_{i,j}}{\Delta x} + \frac{v_{i,j}\omega_{i,j}-v_{i,j-1}\omega_{i,j-1}}{\Delta x}\right] \quad (C.2)$$

Appendix D

Mesh Reynolds number

The mesh Reynolds number is an dimensionless value used to determine the correctness of a simulation. When $\Delta x = \Delta y = h$, its expression is :

$$Re_h = \frac{(|u| + |v|)h}{\nu} = \frac{\beta}{r} \tag{D.1}$$

with β being the Courant–Friedrichs–Lewy condition (CFL), and r the Fourier number.
When $\Delta x \neq \Delta y$ an expression for Re_h can be obtain by using β and r. In 2D, their expressions are :

$$\begin{aligned} \beta &= \beta_x + \beta_y \\ &= \Delta t \left(\frac{|u|}{\Delta x} + \frac{|v|}{\Delta y} \right) = \Delta t \frac{|u|\Delta y + |v|\Delta x}{\Delta x \Delta y} \end{aligned} \tag{D.2}$$

$$\begin{aligned} r &= r_x + r_y \\ &= \frac{\nu \Delta t}{(\Delta x)^2} + \frac{\nu \Delta t}{(\Delta y)^2} = \nu \Delta t \frac{(\Delta x)^2 + (\Delta y)^2}{(\Delta x)^2 (\Delta y)^2} \end{aligned} \tag{D.3}$$

Now, we can finally get an expression for mesh Reynolds number based on the velocity :

$$\begin{aligned} Re_h = \frac{\beta}{r} &= \Delta t \frac{|u|\Delta y + |v|\Delta x}{\Delta x \Delta y} \frac{1}{\nu \Delta t} \frac{(\Delta x)^2 (\Delta y)^2}{(\Delta x)^2 + (\Delta y)^2} \\ &= \frac{|u|\Delta y + |v|\Delta x}{\nu} \frac{(\Delta x)^2 (\Delta y)^2}{(\Delta x)^2 + (\Delta y)^2} \end{aligned} \tag{D.4}$$

It should be noted that considering $\Delta x = \Delta y$ with this formula gives twice the definition of D.1. This is due to different definitions of Fourier number in 1D and 2D.
There is also another mesh Reynolds number based on the vorticity. In 2D, its expression is :

$$Re_\omega = \frac{|\omega| h^2}{\nu} \Rightarrow \frac{|\omega| \Delta x \Delta y}{\nu} \tag{D.5}$$

UNIVERSITÉ CATHOLIQUE DE LOUVAIN
École polytechnique de Louvain
Rue Archimède, 1 bte L6.11.01, 1348 Louvain-la-Neuve, Belgique | **www.uclouvain.be/epl**