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Introduction

In the first year of university, we are introduced to the notion of vector spaces, along with linear maps, which serve as "functions" between them. In the second year, we encounter groups and their homomorphisms, followed by topological spaces and continuous maps. Later on, during our master, we study Lie algebras and Lie algebra morphisms. A recurring pattern emerges: many relevant mathematical objects, despite the fact that they are distinct, often share some structural similarities and, sometimes, the difference between two notions mainly lies in the amount of structure they possess. For instance, Lie algebras are vector spaces equipped with an additional operation—the Lie bracket.

These similarities between the structures were observed by S. Eilenberg and S. Mac Lane in their 1945 paper [30]. They introduced the fundamental concepts of category, functor, natural transformations, . . . , focusing on the relations and the morphisms between objects, more than on the objects themselves. Here, the terms "object" and "morphism" are to be understood from a categorical perspective; they can mean vector space/linear map, group/group homomorphism or even topological spaces/continuous maps, depending on the category one is considering. The strength of category theory lies in its ability to capture and formalize structural patterns across seemingly disparate mathematical domains.

For example, the Cartesian product of sets and the direct product of groups share a common categorical structure, usually referred to as the categorical product of objects in a given category. This level of abstraction allows mathematicians to define and study specific types of categories. One of the most well-known is the notion of abelian category, introduced in the 1950s by D. Buchsbaum in his Ph.D. thesis and popularized by A. Grothendieck under the name "Abelian category" in his famous article [39], with the typical example being the category of abelian groups. Abelian categories provide a powerful framework for developing homological algebra and understanding the structural properties of abelian groups and modules over a ring.

A category $\mathcal{C}$ is called an abelian category if it has finite products, a zero object and any morphism factors into a normal epimorphism followed by a normal monomorphism. For instance, the category of abelian groups satisfies this definition since:

1. the zero object is the trival group $\{e\}$;

2. the finite product of two groups G and H is

$$G \times H = \{(g, h) \mid g \in G \text{ and } h \in H\};$$

3. for any abelian group homomorphism $f : A \to B$, the image of f, $f(A)$, is a normal subgroup of B.

Note that, however, for groups, the third point does not hold in general, meaning that the category of groups is not abelian. Nevertheless, groups and their morphisms play a central role in mathematics, and are also related to abelian groups.

For a long time, mathematicians sought a categorical framework that could capture the properties of groups in the same way abelian categories capture the properties of abelian groups. After several attempts by mathematicians, G. Janelidze, L. Márki and W. Tholen introduced, in [48], a very interesting notion: namely, the one of **semi-abelian category**. This provides a good framework for studying properties of groups, such as the validity of Noether's isomorphism theorems, the Snake Lemma, the Split Short Five Lemma, and more [10]. Moreover, it enables the study of commutator theory, torsion theory, actions, and (co)homology of algebraic structures.

In the first chapter of this master thesis, we will define semi-abelian categories in terms of their "old" axioms and then present their "new" axioms, proving the equivalence of these formulations. Foundational concepts from category theory will be assumed; for which we refer the reader to [51].

Once the notion of semi-abelian category is established, we will start the second chapter, which explores the definitions which are relevent to study to the category of cocommutative Hopf algebras over an arbitrary field. Specifically, we will show that this category is semi-abelian. A proof of this result under the additional assumption that the base field has characteristic zero is given in [35]. Our goal is to prove this result in full generality as in [37], and describe, within the framework of cocommutative Hopf algebras, important structures such as the semi-direct product, commutators, and actions defined, more generally, in the setting of semi-abelian categories.

The concept of a Hopf algebra was first introduced in 1941 by Heinz Hopf in [43] while studying the homology of Lie groups. Hopf algebras now appear in numerous areas of mathematics and physics, including Lie theory, Galois theory ([56]), quantum physics ([52]), and knot theory ([50]). One of the fundamental idea behind Hopf algebras is that they describe symmetries of non-commutative spaces, much like groups describe the symmetries of classical spaces. For example, the symmetric group S_n describes permutations of elements in the set $K = \{1, 2, \dots, n\}$, capturing its symmetries. In the case of Hopf algebras, the symmetries apply to non-commutative spaces, a notion that goes beyond the scope of this work. From the algebraic point of view, Hopf algebras can be regarded as a kind of "linearization" of groups.

In this thesis, we will focus on **cocommutative** Hopf algebras rather than general Hopf algebras. The main reason for this restriction comes from the fact that the category of cocommutative Hopf algebras enjoys remarkable categorical properties. Indeed, proving that the category of general Hopf algebras is exact is really hard since limits and colimits are difficult to explicitly describe. Adding the cocommutativity condition simplifies the situation, as the tensor product becomes the categorical product. In this case, explicit descriptions of limits and equalizers are available.

In the third and final chapter, we will use the semi-abelian nature of the category of cocommutative Hopf algebras to define and study various objects. For instance, we will recall the semi-direct product and generalize the notions of characteristic subgroups and characteristic ideals (from Lie theory) to the framework of cocommutative Hopf algebras. Additionally, we will prove that the category of cocommutative Hopf algebras satisfies some other important categorical properties. These properties will allow one to establish further results, leading to the proof of the last result presented in this thesis, asserting that the commutator of two characteristic subobjects in the category of cocommutative Hopf algebras is itself characteristic.

Contents

Chapter 1

Semi-abelian categories: "old" and "new" axioms

If one opens a book on category theory and looks for the definition of a semi-abelian category, one will almost always encounter a definition involving the notion of protomodularity, exactness, coproducts, and zero object. This definition is quite beautiful because of its clarity and the way it breaks down into four simpler properties. However, as you can imagine, such definition took some time to develop. Indeed, the first version dates back to the 1950s/1960s and was expressed in terms of normal epimorphisms/monomorphisms.

In this section, I will present both the so-called "old" and "new" definitions of a semi-abelian category, and prove their equivalence following [48].

There are several reasons to explain the equivalence between those approaches. First, it is always interesting to know the historical development of a mathematical theory. Then, working with the "old axioms" allows for a better understanding of the basic idea behind the concept of a semi-abelian category. Finally, from a more practical point of view, having equivalent ways to define a concept can be useful depending on the context.

1.1 Semi-abelian in terms of "new axioms"

The notion of a semi-abelian category was introduced with the aim of providing a common framework for the study of algebraic properties valid for groups, rings, and algebras in the same way as abelian categories provide a general framework for the study of abelian groups and modules on a given ring.

As mentioned above, the definition of a semi-abelian category consists of four properties. To make all of this as clear as possible, I plan to divide this chapter into subsections, one for each of the properties, and, to illustrate them, I will prove that the category *Grp* of groups, the most typical example of a semi-abelian category, satisfies them.

1.1.1 Regular categories

To understand the main definition of this section, several concepts need to be recalled.

1. a) An object I in a category $\mathcal{C}$ is called **initial** if, for any object $X \in \mathcal{C}$, there exists a unique morphism from I to X. Dually, an object $T \in \mathcal{C}$ is called **terminal** if, for any object $X \in \mathcal{C}$, there exists a unique morphism from X to T.

   b) An object which is both initial and terminal is called a **zero object**. If such object exists in a category $\mathcal{C}$, we say that $\mathcal{C}$ is **pointed**.

   c) A morphism $0 : A \to B$ is the **zero morphism** if it is the unique morphism that factors through the zero object.

2. A category is **finitely complete** if it admits all finite limits. There are other equivalent ways to express this property. Indeed, the following conditions are equivalent:

   a) $\mathcal{C}$ is finitely complete;

   b) $\mathcal{C}$ has a terminal object, binary products and equalizers;

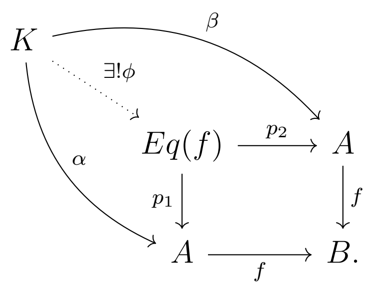
   c) $\mathcal{C}$ has a terminal object and pullbacks.

   A proof of this result is given in [8][Proposition 2.8.2.].

3. The **kernel pair** $p_1, p_2 : Eq(f) \rightrightarrows A$ of morphism $f : A \to B$ in a category $\mathcal{C}$ corresponds to the pullback of f with itself. In other words, this means that

   a) in the diagram below, $f \circ p_1 = f \circ p_2$;

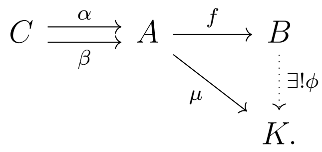
   b) for every object $K \in \mathcal{C}$ and for every pair of arrows $\alpha, \beta : K \to A$ such that $f \circ \alpha = f \circ \beta$, there exists a unique arrow $\phi : K \to Eq(f)$ such that $p_1 \circ \phi = \alpha$ and $p_2 \circ \phi = \beta$



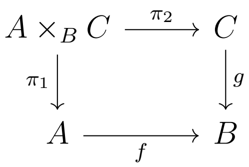
4. A morphism $f : A \to B$ in a category $\mathcal{C}$ is a **regular epimorphism** if it is the coequalizer of some parallel pair of morphisms. This means that there exists a pair of arrows $\alpha, \beta : C \rightrightarrows A$ such that

   a) in the diagram below, $f \circ \alpha = f \circ \beta$;

b) for every object $K \in \mathcal{C}$ and for every arrow $\mu : A \to K$ such that $\mu \circ \alpha = \mu \circ \beta$, there exists a unique arrow $\phi : B \to K$ such that $\phi \circ f = \mu$



5. To say that a class Σ of morphisms satisfying a certain property is **pullback stable** means that in any pullback of the form

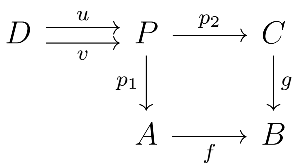


if f belongs to the class Σ, then π_2 does as well (we say that f is pullback stable along g). It is equivalent to ask that if g belongs to Σ, then π_1 does as well.

Let us see a useful morphism that is pullback stable:

Proposition 1.1. *Monomorphisms are pullback stable in any category with pullbacks.*

Proof. Consider the following pullback



where f is a monomorphism. We have to show that p_2 is a monomorphism as well.

Let $u, v : D \to P$ be two parallel morphisms such that $p_2 \circ u = p_2 \circ v$. We can compose each side with g and use that $g \circ p_2 = f \circ p_1$ to get that

$$f \circ p_1 \circ u = g \circ p_2 \circ u = g \circ p_2 \circ v = f \circ p_1 \circ v.$$

Then, we can deduce that $p_1 \circ u = p_1 \circ v$ since f is a monomorphism.

We can conclude the proof by using the fact that, in a pullback, the projections are jointly monic (i.e. having $p_1 \circ u = p_1 \circ v$ and $p_2 \circ u = p_2 \circ v$ imply $u = v$), meaning that p_2 is a monomorphism. $\square$

Now that all these definitions have been recalled, we can define the main notion of this subsection:

Definition 1.2. *A category $\mathcal{C}$ is* ***regular*** *(introduced in [7]) if*

1. *it is finitely complete;*
2. *the kernel pair $p_1, p_2 : Eq(f) \rightrightarrows A$ of any morphism $f : A \to B$ admits a coequalizer (i.e. there exists an arrow g such that g is the coequalizer of p_1 and p_2);*
3. *regular epimorphisms are pullback stable.*

There exists a theorem that provides an alternative version of the last two points of Definition 1.2 in terms of a certain factorization that will be useful:

Theorem 1.3. *Let $\mathcal{C}$ be a finitely complete category. Then, $\mathcal{C}$ is a regular category if and only if*

1. *any arrow in $\mathcal{C}$ factorizes as a regular epimorphism followed by a monomorphism;*
2. *these factorizations are pullback stable: if $m \circ q$ is the factorization of an arrow $p : E \to B$, $f : A \to B$ any arrow and the squares*

$$\begin{array}{ccccc} E \times_B A & \xrightarrow{q'} & E' \times_B A & \xrightarrow{m'} & A \\ {\scriptstyle \pi_1}\downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\ E & \underset{q}{\twoheadrightarrow} & E' & \underset{m}{\rightarrowtail} & B \end{array}$$

   are pullbacks, then $m' \circ q'$ is the (regular epi,mono)-factorization of the pullback projection $\pi_2 : E \times_B A \to A$.

Proof. [33][Theorem 1.14.] □

It is also possible to express the property that a category is finitely complete by using the notion of split monomorphism and the following remark. Recall that a morphism $g : C \to D$ is a **split monomorphism** if there exists an arrow $s : D \to C$ such that $1_C = s \circ g$.

Remark 1.4. We already know that the product of n objects $A_1, \ldots, A_n$ in $\mathcal{C}$ is $A_1 \times \ldots \times A_n$ together with n applications $\pi_i : A_1 \times \ldots \times A_n \to A_i$ satisfying the universal property of the product.

If $n = 0$, we get the empty product T. In this particular case, the universal property states that for every object $X \in \mathcal{C}$, there exists a unique arrow from X to T. Therefore, by definition, it is a terminal object.

Proposition 1.5. *A category $\mathcal{C}$ is finitely complete if and only if $\mathcal{C}$ has finite products and pullbacks of pairs of split monomorphisms.*

Proof. The idea of the proof is that the following two diagrams

$$
\begin{array}{ccc} & & B \\ & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array}
\qquad\qquad
\begin{array}{ccc} & & A \times B \\ & & \downarrow{\scriptstyle 1_A \times <g, 1_B>} \\ A \times B & \xrightarrow{<1_A, f> \times 1_B} & A \times C \times B \end{array}
$$

have isomorphic limits and that the empty product gives the terminal object (explained in Remark 1.4). □

An alternative way of stating regularity will prove particularly useful in Chapter 2 when discussing the central objects of this thesis: the category of cocommutative Hopf algebras.

Lemma 1.6. *Let $\mathcal{C}$ be a finitely complete category. Then, $\mathcal{C}$ is a regular category if and only if*

1. *any arrow in $\mathcal{C}$ factors as a regular epimorphism followed by a monomorphism;*
2. *given any regular epimorphism $f : A \to B$ and any object E, the induced arrow $1_E \times f : E \times A \to E \times B$ is a regular epimorphism;*
3. *regular epimorphisms are stable under pullbacks along split monomorphisms.*

Proof. $\Rightarrow$: The first point follows from Theorem 1.3. For the second point, if f is a regular epimorphism, this means that there exists a pair of parallel arrows $h, w : D \to A$ such that f is the coequalizer of h and w. We can deduce that $1_E \times f$ is the coequalizer of the pair of morphisms $(1_E \times h), (1_E \times w) : E \times D \to E \times A$. The third point follows from Definition 1.2 since pullback stability of regular epimorphisms implies pullback stability along split monomorphisms.

$\Leftarrow$: Using Theorem 1.3, it remains to show that regular epimorphisms are pullback stable. Indeed, computing the pullback along a composable morphism amounts to consecutively calculating two pullbacks along each morphism in the composition and Proposition 1.1 tells us that monomorphisms are pullback stable.

Given the pullback

$$
\begin{array}{ccc} E \times_B A & \xrightarrow{p_2} & A \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f} \\ E & \xrightarrow[p]{} & B \end{array}
$$

where f is a regular epimorphism. We have to show that p_1 is a regular epimorphism. To do so, we can consider the following diagram

$$\begin{array}{ccc} E \times_B A & \xrightarrow{e} & E \times A \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle 1_E \times f} \\ E & \xrightarrow[(1_E,p)]{} & E \times B \end{array} \tag{1.1}$$

where e is the equalizer of $p \circ \pi_1$ and $f \circ \pi_2$ (π_1,π_2 are the product projections of $E \times A$). Since e is an equalizer, diagram 1.1 is a pullback, meaning that p_1 is a regular epimorphism since it is the pullback of the regular epimorphism $1_E \times f$ along the split monomorphism $(1_E, p)$. □

Let us now illustrate the concept of regular category using the example of the category *Grp* of groups and that of the category $LieAlg_K$ of Lie algebras over a field K.

Example 1.7. We will show that the three conditions of Definition 1.2 are satisfied.

By using the second point of 1.1.1, it is possible to show that *Grp* is finitely complete. Indeed, the zero object (hence terminal) is the trivial group {e}, the equalizer of two morphisms f,g :$A \rightrightarrows B$ is given by the inclusion of $\{a \in A \mid f(a) = g(a)\}$ in A and the binary product of two groups A, B is their cartesian product $A \times B$ defined as $\{(x, y) \mid x \in A, y \in B\}$ provided with the pointwise group operation defined as follows $(x, y) \circ (x', y') = (x \circ x', y \circ y')$.

The kernel pair of a group homomorphism $f : A \to B$ is given by the set

$$Eq(f) = A \times_B A = \{(a, a') \in A \times A \mid f(a) = f(a')\}$$

together with two projection maps p_1 and p_2 defined as $p_1(a, a') = a$ and $p_2(a, a') = a'$. Moreover, $Eq(f)$ is endowed with the group structure induced by $A \times A$ (where we look at $Eq(f)$ as a subgroup of $A \times A$).

Then, using the well-known fact that $Eq(f)$ is an internal equivalence relation on A (it will be explained in Example 1.12), we can take the quotient of A by this internal equivalence relation and define the canonical quotient $q : A \to A/Eq(f)$ which maps an element a to its equivalence class $[a]$. If we assume that $[a].[b] = [a.b]$, we get a group structure on $A/Eq(f)$ such that q is a group homomorphism.

Let us now show that q is the coequalizer of p_1 and p_2.

First, we can see that $q(p_1(a, a')) = q(a) = [a] = [a'] = q(a') = q(p_2(a, a'))$ for every $(a, a') \in Eq(f)$, hence $q \circ p_1 = q \circ p_2$.

Now, let $g : A \to C$ be a group morphism such that $g \circ p_1 = g \circ p_2$. We can define $\phi : A/Eq(f) \to C$ by $\phi([a]) = g(a)$ such that it satisfies $\phi \circ q = g$. Thus, ϕ is a group homomorphism since g is one, and furthermore, it is unique. Indeed, if we have another group homomorphism $\tilde{\phi}$ such that $\tilde{\phi} \circ q = g$, we have $\phi \circ q = \tilde{\phi} \circ q$, but q is surjective morphism (and therefore an epimorphism in the particular case of groups, it is explained in [51]), implying that $\phi = \tilde{\phi}$. We can conclude that the kernel pair of every group homomorphism has a coequalizer.

Now, let us prove that regular epimorphisms are pullback stable.

Let us consider the pullback

$$\begin{array}{ccc} E \times_B A & \xrightarrow{p_2} & A \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f} \\ E & \xrightarrow[p]{} & B \end{array}$$

where all the arrows are group homomorphisms and p is a regular epimorphism.

In the category of groups, regular epimorphisms are surjective group homomorphisms (also explained in [51]). This means that we just have to show that p_2 is surjective.

Starting with $a \in A$ we know that p is surjective, i.e. there exists $e \in E$ such that $p(e) = f(a)$. But by definition of the pullback in the category of groups, we have that the object part $E \times_B A$ is equal to $\{(k, k') \in E \times A \mid p(k) = f(k')\}$. This means that $(e, a) \in E \times_B A$ and by definition of the projection maps, $p_2(e, a) = a$. This shows that p_2 is surjective.

This completes the proof that the category of groups is regular.

Example 1.8. The category $LieAlg_K$ is also regular. Indeed, this can be shown in a very similar way to what has just been done. We will give an outline of the proof.

1. The category $LieAlg_K$ is finitely complete since:
   a) the terminal object is the zero Lie algebra, consisting of the zero vector space with the trivial Lie bracket;
   b) the binary product of Lie algebras is given by their direct sum equipped with a Lie bracket defined component by component;
   c) the equalizer of two morphisms f,g :$L \rightrightarrows L'$ is given by the inclusion of $E = \{x \in L \mid f(x) = g(x)\}$ in L where E is a Lie subalgebra of L.
2. The kernel pair of a Lie algebra homomorphism $f : L \to L'$ is given by

$$Eq(f) = L \times_{L'} L = \{(x, x') \in L \times L \mid f(x) = f(x')\}$$

together with two projection maps p_1 and p_2 . As for groups, the canonical quotient $q : L \to L/Eq(f)$ will be the coequalizer of p_1 and p_2. The rest of the proof is similar to the case of groups.

3. Regular epimorphisms are pullback stable since regular epimorphisms are surjective morphisms in $LieAlg_K$. The proof is therefore exactly the same as in the case of groups.

1.1.2 Exact categories

Before discussing the notion of an exact category, we need to recall the notion of internal equivalence relation:

Definition 1.9. *Let X and Y be two objects in a category $\mathcal{C}$ that has binary products. An **internal relation** $R \subset X \times Y$ from X to Y is a pair of arrows $r_1 : R \to X$, $r_2 : R \to Y$ such that the factorization $(r_1, r_2) : R \to X \times Y$ is a monomorphism. It can be represented by the following diagram*

$$\begin{array}{ccccc} & & R & & \\ & \swarrow^{r_1} & & \searrow^{r_2} & \\ X & & & & Y. \end{array}$$

Remark 1.10. In the previous definition, we asked the morphism $(r_1, r_2) : R \to X \times Y$ to be a monomorphism. This condition is equivalent to asking that r_1 and r_2 are **jointly monic** (i.e. if $h, d : A \to R$ are morphisms such that $r_1 \circ h = r_1 \circ d$ and $r_2 \circ h = r_2 \circ d$, then $h = d$).

Dually, asking that $(p_1, p_2) : X + Y \to R$ is an epimorphism is equivalent to asking that p_1 and p_2 are jointly epic (where $X + Y$ is the coproduct).

Definition 1.11. *Let X be an object from a category $\mathcal{C}$ that has binary products and R an internal relation from X to X ($r_1, r_2 : R \rightrightarrows X$). Then, R is said to be an **internal equivalence relation** if*

1. *it is **reflexive**: this means that there exists an arrow $\delta : X \to R$ such that $r_1 \circ \delta = 1_X = r_2 \circ \delta$;*
2. *it is **symmetric**: this means that there exists an arrow $\sigma : R \to R$ such that $r_1 \circ \sigma = r_2$ and $r_2 \circ \sigma = r_1$;*
3. *it is **transitive**: this means that there exists an arrow $\tau : R \times_X R \to R$ such that $r_1 \circ \tau = r_1 \circ p_1$ and $r_2 \circ \tau = r_2 \circ p_2$ where p_1 and p_2 are the pullback projections*

$$\begin{array}{ccc} R \times_X R & \xrightarrow{p_2} & R \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle r_1} \\ R & \xrightarrow[r_2]{} & X. \end{array}$$

Example 1.12. Let $(Eq(f), p_1, p_2)$ be the kernel pair of a morphism $f : A \to B$. The universal property of the pullback

$$\begin{array}{ccc} Eq(f) & \xrightarrow{p_2} & A \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[f]{} & B \end{array}$$

will imply that it is an internal equivalence relation.

Definition 1.13. *An internal equivalence relation $r_1, r_2 : R \rightrightarrows X$ in a category $\mathcal{C}$ is said to be **effective** if it is the kernel pair of an arrow in $\mathcal{C}$.*

Now that we have all the necessary concepts, we can define what an exact category is:

Definition 1.14. *A category $\mathcal{C}$ is **exact** (in the sense of Barr 1971, [6]) if*

1. *it is regular;*
2. *all internal equivalence relations in $\mathcal{C}$ are effective.*

To conclude the subsection, let us prove that the category Grp of groups and the category $LieAlg_K$ of Lie algebras over a field K are exact.

Example 1.15. We have previously seen that the category Grp is regular.

Now, let R be an internal equivalence relation on an arbitrary group G. We can consider the canonical quotient

$$q : G \to G/R$$
$$g \mapsto [g]$$

where $[g]$ is the equivalence class of g. This is a group homomorphism since $[g.h] = [g].[h]$ and it is also well defined since $q(h) = [h] = [g] = q(g)$ when $h = g$. We can see that the internal equivalence relation $r_1, r_2 : R \rightrightarrows G$ is the kernel pair of q. Indeed,

$$(g, g') \in R \Leftrightarrow [g] = [g'] \Leftrightarrow q(g) = q(g') \Leftrightarrow (g, g') \in Eq(q)$$

since the object part of the kernel pair of q is given by the set $Eq(q)$ defined as follows $\{(g, g') \in G \times G \mid q(g) = q(g')\}$.

This completes the proof that the category of groups is exact.

Example 1.16. We already showed that $LieAlg_K$ is a regular category.

Now, let R be an internal equivalence relation on an arbitrary Lie algebra L. As for groups, we can consider the canonical quotient $q : L \to L/R$, and $r_1, r_2 : R \rightrightarrows L$ will be the kernel pair of q.

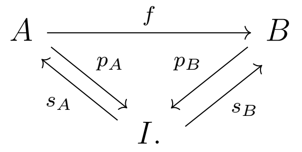
1.1.3 Protomodular categories

In order to define the notion of protomodular category, we need to define the so-called "category of points":

Definition 1.17. *Let $\mathcal{C}$ be an arbitrary category and let I be an object from $\mathcal{C}$.*

The ***catgegory of points*** *on I, written as $Pt_I(\mathcal{C})$, is a category where*

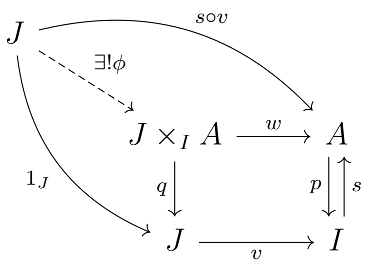
1. *objects are split epimorphisms with codomain I in $\mathcal{C}$. We will write them as $(A,\ p : A \to I,\ s : I \to A)$ where A is an object from $\mathcal{C}$ and s is a section of p (i.e. $p \circ s = 1_I$);*
2. *morphisms from $(A, p_A : A \to I, s_A : I \to A)$ to $(B, p_B : B \to I, s_B : I \to B)$ are mophisms $f : A \to B$ in $\mathcal{C}$ such that $p_B \circ f = p_A$ and $f \circ s_A = s_B$*



Remark 1.18. If I is the zero object, we have that $Pt_0(\mathcal{C}) \cong \mathcal{C}$ because of the unique morphisms coming from the definition of the zero object.

If we now assume that the category $\mathcal{C}$ has pullbacks, we have that every morphism $v : J \to I$ in $\mathcal{C}$ induces a functor that will change the base object I of the category $Pt_I(\mathcal{C})$. Indeed, it is the **pullback functor** $v^* : Pt_I(\mathcal{C}) \to Pt_J(\mathcal{C})$ and it is defined as follows:

1. With any object $(A,\ p : A \to I,\ s : I \to A)$ in $Pt_I(\mathcal{C})$, the functor v^* associates $(J \times_I A,\ q : J \times_I A \to J,\ \phi : J \to J \times_I A)$

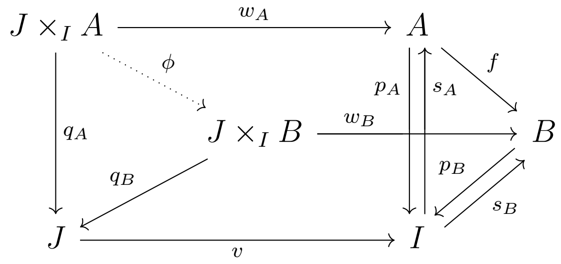


   where q is the pullback of p along v and the morphism ϕ is induced by the universal property of this pullback.

2. For each morphism

$$f : (A, p_A : A \to I, s_A : I \to A) \to (B, p_B : B \to I, s_B : I \to B)$$

in $Pt_I(\mathcal{C})$, we can define $v^*(f)$ as the morphism ϕ in the following diagram



such that $q_B \circ \phi = q_A$ and $w_B \circ \phi = f \circ w_A$. In other words, this means that $v^*(f) = \phi$ is defined as the unique arrow induced by the universal property of the pullback $(J \times_I B, q_B, w_B)$, since $v \circ q_A = p_A \circ w_A = p_B \circ f \circ w_A$.

We can now give the definition of a protomodular category:

Definition 1.19. *A category $\mathcal{C}$ is said to be* ***protomodular*** *(in the sense of Bourn [13]) if*

1. *the category $\mathcal{C}$ has pullbacks;*
2. *for every morphisms $v : J \to I$ in $\mathcal{C}$, the pullback functor $v^* : Pt_I(\mathcal{C}) \to Pt_J(\mathcal{C})$* ***reflects isomorphisms****. This means that for a morphism f in $Pt_I(\mathcal{C})$, if $v^*(f)$ is an isomorphism, then f is also an isomorphism.*

If we assume that $\mathcal{C}$ has a zero object 0 in addition to pullbacks, $\mathcal{C}$ becomes finitely complete (as explained in the second part of 1.1.1) and the Definition 1.19 simplifies. Indeed, we can drop the arbitrary morphisms $v : J \to I$ and only consider the morphisms $\iota_I : 0 \to I$. This works since, by definition, $\iota_I = v \circ \iota_J$ and if $\iota_I^* = \iota_J^* \circ v^*$ reflects isomorphisms, so does v^*.

We can therefore write a simplified version of Definition 1.19 if the category $\mathcal{C}$ has a zero object:

Definition 1.20. *A category $\mathcal{C}$ with a zero object and pullbacks is protomodular if for every object I in $\mathcal{C}$, the pullback functor $\iota_I^* : Pt_I(\mathcal{C}) \to Pt_0(\mathcal{C}) \cong \mathcal{C}$ reflects isomorphisms.*

In this particular case, the pullback functor is called the **kernel functor** and is written as Ker_I. This name comes from the fact that pulling back $p : A \to I$ along $\iota_I : 0 \to I$ is equivalent to taking the kernel of p.

It is also possible to translate Definition 1.20 using the so-called **Split Short Five Lemma**. This will allow one, in some cases, to easily prove that a category is protomodular.

Proposition 1.21. *Let $\mathcal{C}$ be a category with pullbacks and a zero object. Then, $\mathcal{C}$ is protomodular if and only if the Split Short Five Lemma holds in $\mathcal{C}$: given any diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \overset{k}{\rightarrowtail} & B & \underset{s}{\overset{p}{\rightleftarrows}} & C \\ & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ 0 & \longrightarrow & A' & \overset{k'}{\rightarrowtail} & B' & \underset{s'}{\overset{p'}{\rightleftarrows}} & C' \end{array} \tag{1.2}$$

in $\mathcal{C}$ where all squares are commutative, $k = ker(p)$, $k' = ker(p')$, and p, p' are split epimorphisms, if f and h are isomorphisms then g is an isomorphism as well.

Proof. [9][Proposition 3.1.2.] □

Remark 1.22. In the case where the category $\mathcal{C}$ is regular, it is possible to replace "split epimorphisms" with "regular epimorphisms" in Propositions 1.21 (this is explained in [13]).

If we now add the assumption that $\mathcal{C}$ has binary coproducts, in addition to pullbacks and a zero object, we can show that the kernel functor Ker_I has a left adjoint, namely, the functor

$$\begin{aligned} \sigma_I : \mathcal{C} &\to Pt_I(\mathcal{C}) \\ A &\to (I + A, [1, 0] : I + A \to I, i_I : I \to I + A) \end{aligned}$$

where i_I is the coproduct injection.

With this new assumption, it is possible to rewrite a new version of Definition 1.19 using the notion of **extremal epimorphism** (i.e. an epimorphism $e : C \to D$ such that if e factors as $e = m \circ g$, where m is a monomorphism and g is an arbitrary morphism, then m is an isomorphism).

Definition 1.23. *A category with pullbacks, binary coproducts and a zero object is protomodular if, for every diagram*

$$A \xrightarrow{k} E \underset{s}{\overset{p}{\rightleftarrows}} I \tag{1.3}$$

where p is a split epimorphism, s is a section of p and $k = ker(p)$, the morphism $[k, s] : A + I \to E$, coming from the universal property of the coproduct $A + I$, is an extremal epimorphism.

We have another way of expressing the property of protomodularity, which will be useful in the next subsection. However, before stating this, we need to clarify the notion of **relation between subobjects**:

Remark 1.24. If we have two monomorphisms $u : A \to B$ and $v : A' \to B$ with codomain B, we can define a relation between them. Indeed, $u \leqslant v$ if there exists a morphism $g : A \to A'$ such that $u = v \circ g$.

We can then define an internal equivalence relation between u and v as

$$u \sim v \text{ if } u \leqslant v \text{ and } v \leqslant u.$$

Using this internal equivalence relation, we can define a subobject of B as an equivalence class of these monomorphisms.

Moreover, $\leqslant$ induces a partial order on the collection of subobjects of B. We can then write $S(B)$ for the **ordered class of subobjects** of B (represented by a monomorphism with codomain B) with respect to this partial order.

Proposition 1.25. *A category $\mathcal{C}$ with pullbacks, binary coproducts, and a zero object is protomodular if and only if*

1. *for every diagram*

$$A \xrightarrow{k} E \underset{s}{\overset{p}{\rightleftarrows}} I \tag{1.4}$$

   where p is a split epimorphism, s is a section of p and $k = ker(p)$, the morphism $[k, s] : A + I \to E$ is an epimorphism;

2. *for every commutative diagram*

$$\begin{array}{ccccc} K & \xrightarrow{\leq} & F & \xrightarrow{q} & J \\ & \searrow^{ker(p)} & \downarrow g & & \downarrow h \\ & & E & \xrightarrow[p]{} & I \end{array}$$

   where g, h are monomorphisms and $ker(p) \leqslant g$, if q is a split epimorphism and h is a regular monomorphism, then g is also a regular monomorphism.

Proof. [48][Proposition 2.4.] □

Remark 1.26. In the second point of Proposition 1.25, we could have replaced regular monomorphisms with elements of a class of morphisms Σ that are stable under pullback and such that, if f is an epimorphism and belongs to Σ, then it is an isomorphism.

For example, if Σ is the class of all normal monomorphisms, we get the so-called "Hofmann Axiom" (this notion first appeared in [42]).

Let us now prove that the category *Grp* of groups and the category $LieAlg_K$ of Lie algebra over a field K are protomodular.

Example 1.27. We already know that *Grp* is finitely complete meaning that it has pullbacks.

Moreover, it has a zero object, the trivial group, denoted by $\{e\}$. Using Proposition 1.21, we have to show that the Split Short Five Lemma holds in *Grp*. Let us consider the following diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \overset{k}{\rightarrowtail} & B & \underset{s}{\overset{p}{\rightleftarrows}} & C \\ & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ 0 & \longrightarrow & A' & \underset{k'}{\rightarrowtail} & B' & \underset{s'}{\overset{p'}{\rightleftarrows}} & C' \end{array}$$

where all the squares commute, $k = ker(p)$, $k' = ker(p')$, p and p' are split epimorphisms with respective sections s, s' and f, h are isomorphisms. Without loss of generality we will assume that k and k' are both subgroup inclusions, meaning that if $a \in A$, then $k(a) = a \in B$.

We have to show that g is an isomorphism. In the present case, this amounts to showing that g is injective and surjective (it is explained in [51]). Let us start by proving the injectivity:

Let $b \in B$ such that $g(b) = e$. By the commutativity of the squares, we have

$$(h \circ p)(b) = (p' \circ g)(b) = p'(g(b)) = p'(e) = e = h(e)$$

since p' is a group homomorphism. But h is an isomorphism and thus it is injective so $p(b) = e$ meaning that b belongs to the kernel of p. We can now compute that

$$(k' \circ f)(b) = (g \circ k)(b) = g(b) = e = (k' \circ f)(e).$$

Since f and k' are injective, it follows that $b = e$, showing that g is injective.

For the proof of the surjectivity, let us consider an element $b' \in B'$, and $p'(b)$ its image by p'. We know that h and p are surjective, this means that there exists $b \in B$ such that $(h \circ p)(b) = p'(b')$. We can then compute that

$$p'(g(b)^{-1}.b') = p'(g(b))^{-1}.p'(b') = (h \circ p)(b)^{-1}.(h \circ p)(b) = e.$$

This means that $g(b)^{-1}.b'$ belongs to A', the kernel of p'. But f is surjective, so there exists $a \in A \hookrightarrow B$ such that $f(a) = g(b)^{-1}.b'$. We can end the proof of the surjectivity since

$$\begin{aligned} g(b.a) = g(b).g(a) = g(b).g(k(a)) = g(b).k'(f(a)) &= g(b).k'(g(b)^{-1}.b') \\ &= g(b).g(b)^{-1}.b' \\ &= b'. \end{aligned}$$

We conclude that g is injective and surjective, hence an isomorphism, meaning that the Split Short Five Lemma holds in the category of groups.

Example 1.28. We know that $LieAlg_K$ is finitely complete, this means that we have to show that the Split Short Five Lemma holds in $LieAlg_K$.

This can be done as in the previous proof, since, in the category $LieAlg_K$, a Lie algebra morphism is an isomorphism if it is bijective, and the group operation allows one to apply the proof given above to the case of Lie algebras.

1.1.4 Semi-abelian categories

We can now give the definition we are particulary interested in:

Definition 1.29. *A category $\mathcal{C}$ is said to be **semi-abelian** if*

1. *it is Barr-exact;*
2. *it is Bourn-protomodular;*
3. *it has a zero object;*
4. *it has binary coproducts.*

Thanks to the previous subsections, we can write an equivalent definition of semi-abelian category in terms of "new axioms", denoted by (NA):

Definition 1.30. *A category $\mathcal{C}$ is semi-abelian ("new" version) if and only if $\mathcal{C}$ is a category:*

$NA1)$ *that has binary products, binary coproducts and a zero object;*

$NA2)$ *that has pullbacks of split monomorphisms;*

$NA3)$ *that has coequalizer of kernel pairs;*

$NA4)$ *where the Split Short Five Lemma holds true;*

$NA5)$ *where regular epimorphisms are stable under pullbacks;*

$NA6)$ *where equivalences relations are effective.*

Example 1.31. The category *Grp* of groups is semi-abelian. Indeed, it is protomodular, exact, the trivial group is the zero object, and the so-called "free product" of groups is the binary coproduct.

Example 1.32. The category $LieAlg_K$ of Lie algebras over a field K is semi-abelian. The last thing we need is the coproduct of two Lie algebras. It is given by the free Lie algebra generated by their direct sum (see [46][Chapter 2, section 7] for more information about this concept).

Example 1.33. There is also a theorem characterizing semi-abelian algebraic varieties, in the sense of universal algebra, enabling us to find many examples through simple criteria (see [1] for more information about algebraic varieties).

Indeed, this theorem due to D. Bourn and G. Janelidze [18] states that an algebraic variety is semi-abelian if and only if the theory of the variety has

- a unique constant, that will be denoted by 0;
- n binary operations α_i such that $\alpha_i(x,x) = 0$;
- an $(n+1)$-ary operation θ such that $\theta(\alpha_1(x,y),\ldots,\alpha_n(x,y),y) = x$.

Using this theorem, it is simpler to prove that the category of groups is semi-abelian. Indeed, if we take $n = 1$ with $\alpha_1(x,y) = x.y^{-1}$ and $\theta(x,y) = x.y$, we obtain that $\alpha(x,x) = x.x^{-1} = e$ and $\theta(\alpha_1(x,y),y) = (x.y^{-1}).y = x$ where $e = 0$ is the unique constant. This makes the category of groups semi-abelian.

Using a similar argument, it is possible to prove that

1. the category Lie_K of Lie algebras on a field K, *Rng* of rings without unit, *Ab* of abelian groups, $Vect_K$ of vector spaces on a field K, $R - Mod$ of R-modules on a ring R,... are semi-abelian;
2. the category $URng$ of unitary rings is not semi-abelian, since it contains two distinct constants.

There is also a generalization of this theorem allowing one to prove that C^*-algebras are semi-abelian (see [36] for the theorem, and [57] for more information about C^*-algebras).

1.2 Semi-abelian in terms of "old" axioms

We will now see how the modern concept of semi-abelian category is related to the so-called "old axioms". These date from the 1950s/60s and differ from the new ones as they are written in terms of normal epimorphisms/monomorphisms. Recall that a monomorphism $f : A \to B$ is **normal** if it is the kernel of some morphism $g : B \to C$. In other words, this amounts to asking that f is the equalizer of g and the zero morphism from B to C, $0_{B,C} : B \to C$.

Dually, an epimorphism $f : B \to A$ is **normal** if it is the cokernel of some morphism $g : C \to B$. This means that f is the coequalizer of g and the zero morphism from C to B, $0_{C,B} : C \to B$.

Let us now define a semi-abelian category introducing the following "old axioms", denoted by (OA):

Definition 1.34. *A category $\mathcal{C}$ is semi-abelian ("old" version) if it satisfies the following axioms:*

$OA1)$ *$\mathcal{C}$ has binary products, binary coproducts and a zero object.*

$OA2)$ *$\mathcal{C}$ has binary intersections of monomorphisms.*

$OA3a)$ *Every product projection is a normal epimorphism.*

$OA3b)$ *(Images under normal epimorphisms) For every normal epimorphism $p : E \to B$ and every monomorphism $w : F \to E$, there is a commutative diagram*

$$\begin{array}{ccc} F & \xrightarrow{q} & C \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle v} \\ E & \xrightarrow[p]{} & B \end{array} \tag{1.5}$$

where v is a monomorphism and q is a normal epimorphism.

$OA4)$ *(Hofmann's Axiom) For every commutative diagram as 1.5 with p,q normal epimorphisms and v,w monomorphisms, if v is normal and $ker(p) \leqslant w$ then w is a normal monomorphism.*

$OA5)$ *(Inverse images under normal epimorphisms) For every normal epimorphism $p : E \to B$ and every monomorphism $v : C \to B$, there is a commutative diagram as 1.5 where w is a monomorphism and q is a normal epimorphism.*

$OA6)$ *For every commutatitve diagram as 1.5 with p,q normal epimorphisms and v, w monomorphisms, if w is normal then v is normal.*

Remark 1.35. 1. Note that $OA1)$ is $NA1)$.

2. Axiom $OA2)$ needs to be clarified. Indeed, it tells us that, for every $B \in \mathcal{C}$, the ordered class of subobjects of B, $S(B)$, has a binary infima

$$\begin{array}{ccc} A \cap C & \xrightarrow{q_2} & C \\ {\scriptstyle q_1}\downarrow & \searrow^{m \cap m'} & \downarrow{\scriptstyle m'} \\ A & \xrightarrow[m]{} & B \end{array}$$

where m and m' are representatives of their respective equivalence classes (i.e. representatives of their respective subobjects) of B.

We can already see that this "old" version is less concise and more complicated to use than Definition 1.30. That is the reason why, initially, we will work with a weaker version of the "old" axioms. This will allow us to prove the equivalence between Definition 1.30 and Definition 1.34.

The weaker axioms are as follows:

$OA^*1) = OA1) = NA1)$;

$OA^*2)$ $\mathcal{C}$ has binary intersection of split monomorphisms;

$OA^*3) = OA3a)$ & $OA3b)$ but p is assumed to be a normal split epimorphism;

$OA^*4) = OA4)$ where p and q are assumed to be normal split epimorphisms;

$OA^*5) = OA5)$;

$OA^*6) = OA6)$ where p and q are assumed to be normal split epimorphisms.

The purpose of this section is to prove the equivalence between the "old axioms" (OA) and the "new ones" (NA) as presented by G. Janelidze, L. Márki, and W. Tholen in [48]. This is done via the following theorem:

Theorem 1.36. *Let $\mathcal{C}$ be a category that satisfies $OA^*1)$ and $OA^*2)$. Then:*

1. *In $\mathcal{C}$, morphisms have a (normal epi, mono)-factorizations if and only if $OA^*3)$ is satisfied. In particular, $\mathcal{C}$ has all finite limits.*
2. *$\mathcal{C}$ is protomodular and morphisms have a (normal epi, mono)-factorization if and only if $OA^*3)$ and $OA^*4)$ are satisfied. Moreover, the canonical morphism*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : A + I \to A \times I$$

   is a normal epimorphism, for all objects A and I in $\mathcal{C}$.

3. *$\mathcal{C}$ is protomodular and morphisms have a pullback stable (normal epi, mono)-factorization if and only if $OA^*3)$-$OA^*5)$ are satisfied.*
4. *$\mathcal{C}$ is protomodular and Barr exact if and only if $OA^*3)$-$OA^*6)$, or equivalently if and only if $OA3)$-$OA6)$ are satisfied.*

To make the proof more readable, it will be subdivided into subsections for each part of the theorem.

1.2.1 First part of the theorem

We will fix $\mathcal{C}$ to be a category that satisfies $OA^*1)$ and $OA^*2)$.

Proposition 1.37. *The following assertions are equivalent:*

1. *In $\mathcal{C}$, any morphism has a (normal epi, mono)-factorization;*
2. *$\mathcal{C}$ satisfies $OA^*3)$;*
3. *$\mathcal{C}$ satisfies $OA3b)$ and split epimorphisms are normal;*

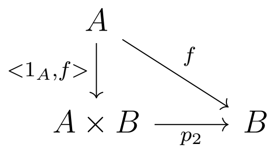
4. $\mathcal{C}$ satisfies $OA3b)$ and regular epimorphisms are normal.

Proof. 4)⇒3): This is trivial since every split epimorphism is, in particular, a regular epimorphism. Indeed, if $f : A \to B$ is a split epimorphism and $s : B \to A$ is a section (i.e. $f \circ s = 1_B$), f is the coequalizer of $s \circ f$ and 1_A.

3)⇒2): First, we can use the fact that every product projection in a pointed category is a split epimorphism. Indeed, if we have the product projection $\pi_1 : A \times B \to A$, it is a split epimorphism since $(1_A, 0)$ is a section of π_1 (the same reasoning applies for π_2, where a section would be $(0, 1_B)$). Therefore, it is normal by assumption.

Then, by definition, the "split version" of $OA3b)$ is weaker, as an axiom, than $OA3b)$, meaning that it is trivially satisfied.

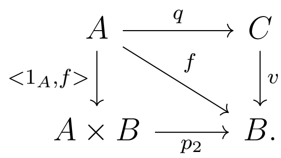
2)⇒1): Let $f : A \to B$ be an arbitrary morphism. It naturally factors as follows



(1.6)

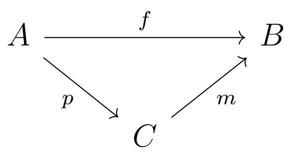
where $< 1_A, f >$ is a monomorphism and p_2, the second product projection, is a split epimorphism since we are in a pointed category.

The axiom $OA3a)$ tells us that p_2 is normal in addition to being split. We can then use the "split version" of $OA3b)$ to find the required factorization. Indeed,
$p_2 \circ < 1_A, f >= f = v \circ q$ where q is a normal epimorphism and v is a monomorphism



1)⇒4): We trivially have $OA3b)$ because of the (normal epi, mono)-factorisation of every morphism.

Moreover, we can show that in the presence of the (normal epi, mono)-factorization for morphisms, extremal epimorphisms are normal. Indeed, if $f : A \to B$ is an extremal epimorphism, we have that in the (normal epi, mono)-factorization $m \circ p$ of f



m is an isomorphism implying that f is also a normal epimorphism.

We can conclude by using the well known fact that any regular epimorphism is an extremal epimorphism (see [8][Proposition 4.3.3.] for a proof of this result). ☐

Proposition 1.38. *Each assertion of Proposition 1.37 implies that $\mathcal{C}$ is finitely complete and that*

$$\text{5) } \textit{kernels and their cokernels exist in } \mathcal{C}\textit{. Moreover, if } f \textit{ is a morphism with } Ker(f) = 0, \textit{ then } f \textit{ is a monomorphism.} \tag{1.7}$$

Proof. Axiom $OA^*2)$ tells us that binary intersections of split monomorphisms exists. In other words, this means that for every $B \in \mathcal{C}$, the ordered class of split subobjects of B has a binary infimum. It is depicted here:

$$\begin{array}{ccc} A\cap C & \xrightarrow{q_2} & C \\ {\scriptstyle q_1}\downarrow & \searrow^{m\cap m'} & \downarrow{\scriptstyle m'} \\ A & \underset{m}{\rightarrowtail} & B. \end{array}$$

We can see that this binary infimum enjoys the universal property of pullback. Indeed:

1. the (normal epi, mono)-factorization allows us to say that the square is commutative;

2. for a split subobject $K \overset{m_K}{\rightarrowtail} B$ with two arrows $\alpha : K \to A$ and $\beta : K \to C$ such that $m \circ \alpha = m' \circ \beta$, we have, via the definition of the relation that induces a partial order on the collection of split subobjects of B, that there exists an arrow $\phi : K \to A \cap C$ such that $q_1 \circ \phi = \alpha$ and $q_2 \circ \phi = \beta$. Moreover, this arrow is unique since in a partially ordered set, there is at most one arrow from an object to another one.

This means that we have pullbacks of split monomorphisms. If we now show that there is also finite products, we can use Proposition 1.5 to obtain the existence of all finite limits.

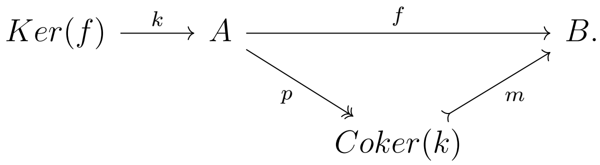
This is the case since axiom $OA^*1)$ guaranties the existence of the zero object and of binary products and we have that:

1. the zero object can be seen as the nullary product as explained in Remark 1.4;

2. a finite product can be seen as a finite iteration of of binary products.

This means in particular that kernels and cokernels exist, then any morphism $f : A \to B$ has a kernel $k : Ker(f) \to A$. Moreover, the (normal epi) part of the factorization of f

will be the cokernel of k

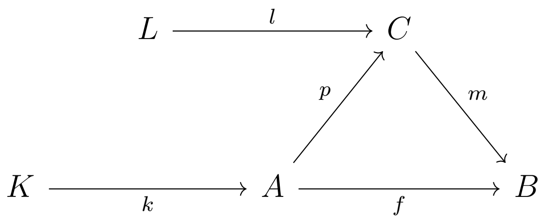


In case $Ker(f) = 0$, we have that $Coker(k) \cong A$, meaning that f coincides with the mono part of the factorization. □

Proposition 1.39. *If $\mathcal{C}$ also satisfies $OA^*5)$, the condition 5) in Proposition 1.38 is equivalent to those in Proposition 1.37.*

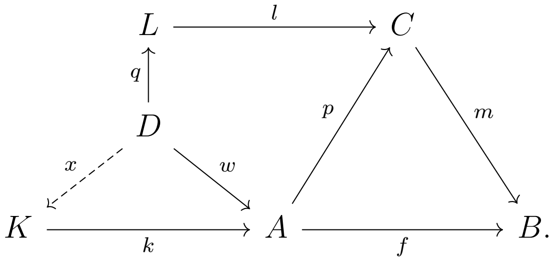
Proof. It remains to show that 5) in Proposition 1.38 implies those in Proposition 1.37. We will then show that 5) $\Rightarrow$ 1).

Consider the following diagram:



where f is an arbitrary morphism, $f = m \circ p$, $k = ker(f)$, $p = coker(k)$ and $l = ker(m)$.

Since p is a normal epimorphism and l is a monomorphism by definition, we can use $OA^*5)$ to obtain a normal epimorphism $q : D \to L$ and a monomorphism $w : D \to A$ such that $l \circ q = p \circ w$. Moreover, since $f \circ w = m \circ p \circ w = m \circ l \circ q = 0 \circ q = 0$, the universal property of the kernel K tells us that there exists a unique morphism $x : D \to K$ such that $k \circ x = w$



However, since l is a monomorphism, $l \circ q = p \circ w = p \circ k \circ x = 0 \circ x = 0 = l \circ 0$ allows us to say that $q = 0$, meaning that $L \cong 0$. Thanks to 5) in Proposition 1.38, we can conclude that m is a monomorphism, completing the proof since $f = m \circ p$ is the (normal epi,mono)-factorization. □

Proposition 1.40. *If $\mathcal{C}$ satisfies $OA1), OA2), OA3a)$ and $OA3b)$, then $\mathcal{C}$ has finite limits and morphisms have a (normal epi,mono)-factorizations. Moreover, these factorizations are stable under pullback along monomorphisms if anf only if $OA5)$ is satisfied.*

Proof. The first assertion directly follows from Proposition 1.37 and Proposition 1.38 since $OA^*3)$ is weaker, as an axiom, than $OA3a)$ and $OA3b)$.

For the second assertion:

$\Rightarrow$: Let $p : E \to E'$ be a normal epimorphism, we can see it as part of the factorisation $p = 1_{E'} \circ p$ where $1_{E'}$ is trivially a monomorphism. We can pullback this factorization (recalled in Theorem 1.3) along any monomorphism $g : A \to E'$ to get

$$\begin{array}{ccccc}
E \times_{E'} A & \xrightarrow{p'} & E' \times_{E'} A & \xrightarrow{m'} & A \\
{\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_1'} & & \downarrow{\scriptstyle g} \\
E & \underset{p}{\twoheadrightarrow} & E' & \underset{1_{E'}}{\rightarrowtail} & E'
\end{array}$$

where $m' \circ p'$ is the factorization of the second pullback projection $\pi_2 : E \times_{E'} A \to A$. Moreover, monomorphisms are stable under pullbacks (Proposition 1.1), implying that π_1' and π_1 are also monomorphisms.

We conclude by saying that the left square is the diagram needed for $OA5)$ to be satisfied.

$\Leftarrow$: Suppose we have the following diagram:

$$\begin{array}{ccccc}
E \times_B A & & \xrightarrow{\qquad\pi_2\qquad} & & A \\
E \times_B A & \xrightarrow[p']{} & E' \times_B A & \xrightarrow[m']{} & A \\
{\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_1'} & & \downarrow{\scriptstyle g} \\
E & \xrightarrow{p} & E' & \xrightarrow{m} & B \\
E & & \xrightarrow[\qquad f\qquad]{} & & B
\end{array}$$

where $f = m \circ p$ is the (normal epi, mono)-factorization of f, the morphism g is a monomorphism and the squares are pullbacks.

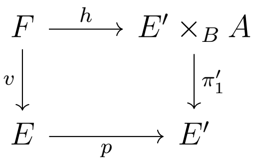
We will show that $m' \circ p'$ is the (normal epi, mono)-factorization of π_2. This will prove that factorizations of morphisms are stable under pullbacks along monomorphisms.

We then have to show that m' is a monomorphism and p' is a normal epimorphism:

1. Monomorphisms are stable under pullback (Proposition 1.1), meaning that m' is a monomorphism since m is one.

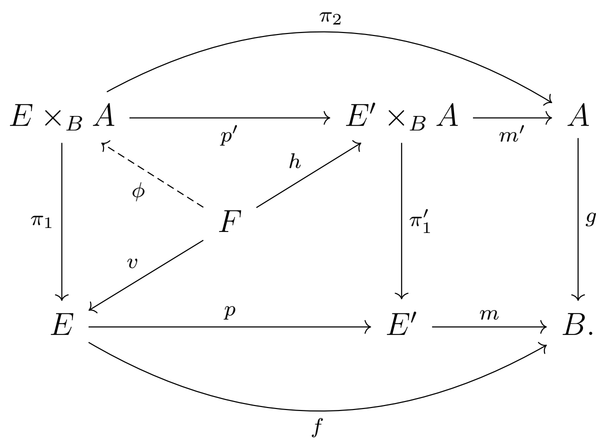
2. With the same type of reasoning as the previous point, π_1' and π_1 are monomorphisms since we assumed g to be a monomorphism.

   We can now apply $OA5)$ to p and π_1' in order to get the following commutative diagram:

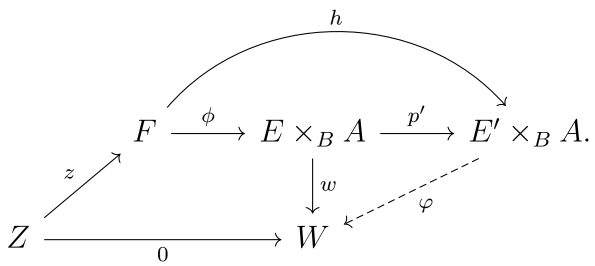


   where v is a monomorphism and h is a normal epimorphism.

   We can compute that $g \circ m' \circ h = m \circ \pi_1' \circ h = m \circ p \circ v = f \circ v$, meaning that, by the universal property of the pullback $E \times_B A$, there is a unique arrow $\phi : F \to E \times_B A$ such that $\pi_1 \circ \phi = v$ and $\pi_2 \circ \phi = m' \circ p' \circ \phi = m' \circ h$



   But m' is a monomorphism, hence $p' \circ \phi = h$. We will now show that p' is a normal epimorphism since $h = p' \circ \phi$ is one. Indeed, let $z : Z \to F$ be the morphism such that $h = \mathit{coker}(z)$, this means that $h \circ z = p' \circ (\phi \circ z) = 0$. If we moreover show that, for any $w : E \times_B A \to W$ such that $w \circ \phi \circ z = 0$, there exists a unique $\varphi : E' \times_B A \to W$ such that $\varphi \circ p' = w$, this will means that $p' = \mathit{coker}(\phi \circ z)$



   This is indeed the case since, if $(w \circ \phi) \circ z = 0$, there exists a unique morphism $\varphi : E' \times_B A \to W$ such that $\varphi \circ h = \varphi \circ p' \circ \phi = w \circ \phi$. Moreover, ϕ is a monormorphism since, if $\alpha, \beta : H \to F$ are morphisms such that $\phi \circ \alpha = \phi \circ \beta$, we can compute that

$$\pi_1 \circ \phi \circ \alpha = v \circ \alpha = v \circ \beta = \pi_1 \circ \phi \circ \beta,$$

meaning that $\alpha = \beta$ since v is a monomorphism. We can then conclude that p' is a normal epimorphism with $p' = coker(\phi \circ z)$.

□

1.2.2 Second part of the theorem

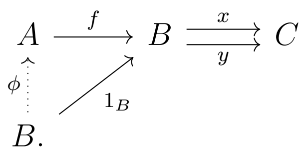
Before proving the first important result of this section, let us look at a lemma that will be used in it:

Lemma 1.41. *If an epimorphism $f : A \to B$ is also a regular monomorphism, then it is an isomorphism.*

Dually, a monomorphism that is also a regular epimorphism is an isomorphism.

Proof. By definition, f is the equalizer of two arrows $x, y : B \to C$ which means that $x \circ f = y \circ f$. But f is also an epimorphism, so that $x = y$.

Now, we can consider the identity morphism $1_B : B \to B$, via the universal property of the equalizer, there exists a unique arrow ϕ such that $1_B = f \circ \phi$



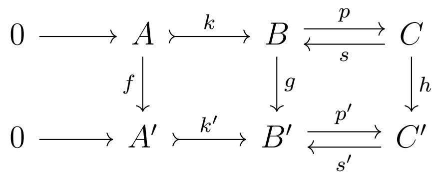
Then, we can now write $f = f \circ \phi \circ f$, but f is also a monomorphism since it is a regular monomorphism, meaning that $1_A = \phi \circ f$ and this concludes the proof. The second assertion can be proven dually. □

Proposition 1.42. *Let $\mathcal{C}$ be a category that satisfies $OA^*1)$ and $OA^*2)$, has pullbacks and morphisms have a (normal epi,mono)-factorization.*

The category $\mathcal{C}$ is protomodular if and only if $OA^*4)$ is satisfied.

Proof. $\Rightarrow$: This direction of the proof follows directly from Proposition 1.25 combined with Remark 1.26.

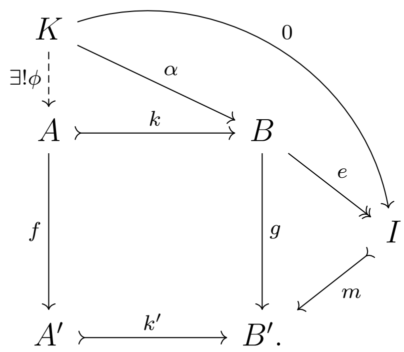
$\Leftarrow$: We will prove that the Short Split Five Lemma holds in $\mathcal{C}$. Let us then consider the following commutative diagram



(1.8)

in $\mathcal{C}$ where p and p' are split epimorphisms, f and h are isomorphisms, $k = ker(p)$ and $k' = ker(p')$. We have to show that g is an isomorphism.

First, we can write the (normal epi, mono)-factorization of g as $m{\circ}e$, where the morphism $e = coker(k) : B \to I$. Then, we can see that any morphism α with $e \circ \alpha = 0$ factors as $\alpha = k \circ \phi$ via the universal property of the cokernel I



Moreover, we can deduce from

$$k' \circ f \circ \phi = g \circ k \circ \phi = g \circ \alpha = m \circ e \circ \alpha = m \circ 0 = 0 = k' \circ f \circ 0$$

that $\phi = 0$ since k' and f are monomorphisms and the composite of two monomorphisms is again a monomorphism.

It is also possible to compute that $\alpha = k \circ \phi = k \circ 0 = 0$. This means, that e is the cokernel of a zero morphism and therefore it is an isomoprhism, meaning that g is a monomorphism

An application of Proposition 1.37 1)$\Rightarrow$3) shows that the split epimorphisms p and p' are normal. If we now apply $OA^{*}4$) to the right hand square of the diagram 1.8, we get that g is a normal monomorphism.

We can do the same construction dually and show that any morphism d with $d \circ f = 0$ must be 0. This allows, in particular, to prove that g is an epimorphism.

We then have that g is a normal monomorphism, thus a regular monomorphism, and moreover, it is epimorphic, so we can conclude by Lemma 1.41 that it is an isomorphism.

□

Corollary 1.43. *Let $\mathcal{C}$ be a category that satisfies $OA1)$, $OA2)$ and $OA3b)$. The following are equivalents:*

1. *the category $\mathcal{C}$ is protomodular;*
2. *axiom $OA^{*}4)$ is satisfied and for every diagram $K \xrightarrow{k} E \underset{s}{\overset{p}{\rightleftarrows}} I$ where p is a split epimorphism, s is a section of p and $k = ker(p)$, we have that the morphism $[k, s] : K + I \to E$ is an epimorphsim;*
3. *axiom $OA^{*}4)$ is satisfied and for every object $A, I \in \mathcal{C}$, the canonical morphism*
$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : A + I \to A \times I$$
is an epimorphism;
4. *axioms $OA3a)$ and $OA^{*}4)$ are satisfied.*

Proof. 1)⇒2): This follows from Proposition 1.42 and the first part of Proposition 1.25.

2)⇒3): We just have to consider the following diagram

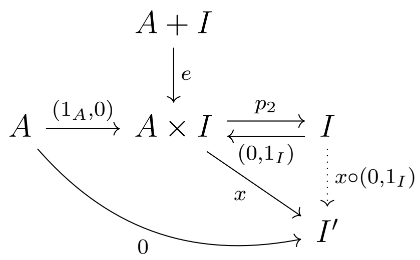
$$A \xrightarrow{(1_A,0)} A \times I \underset{(0,1_I)}{\overset{p_2}{\rightleftarrows}} I \tag{1.9}$$

where $p_2 \circ (0, 1_I) = 1_I$ and $(1_A, 0) = ker(p_2)$.

Indeed, 2) tells us that $[(1_A, 0), (0, 1_I)] : A + I \to A \times I$ is an epimorphism, which proves 3).

3)⇒4): We only have to show that p_2 in diagram 1.9 is the cokernel of the morphism $(1_A, 0)$.

First, we indeed have that $p_2 \circ (1_A, 0) = 0$ by definition of the morphisms. Then, for every morphism $x : A \times I \to I'$ such that $x \circ (1_A, 0) = 0$, we can compute $x \circ (0, 1_I) \circ p_2 \circ e = x \circ e$. This follows from the universal property of the coproduct $A + I$ since we have the arrow $(x \circ (1_A, 0))$ from A to I' and the arrow $x \circ (0, 1_I)$ from I to I'. Then, there exists a unique arrow from $A + I$ to I', meaning that $x \circ (0, 1_I) \circ p_2 \circ e = x \circ e$. Since e is an epimorphism, we get that $(x \circ (0, 1_I)) \circ p_2 = x$.



It remains to show that the morphism $(x \circ (0, 1_I))$ is the only one making the triangle commute. Let $y : I \to I'$ such that $y \circ p_2 = x$, we can conclude the proof by computing that $y = y \circ 1_I = y \circ p_2 \circ (0, 1_I) = x \circ (0, 1_I)$.

4)⇒1) Using Proposition 1.40, we get that $\mathcal{C}$ is finitely complete and has (normal epi, mono)-factorization. Using Proposition 1.42, we get that $\mathcal{C}$ is protomodular. □

Remark 1.44. Once we have establish the Barr-exactness of $\mathcal{C}$, we may trade OA^*4) for $OA4$).

We have then proved the first and the second part of the theorem.

1.2.3 Third part of the theorem

Thanks to the last two subsections, we only have two properties left to prove. Indeed, let $\mathcal{C}$ be a protomodular category such that morphisms have a (normal epi, mono)-factorization, we have to prove that:

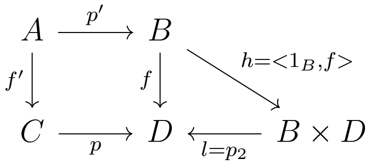
1. The category $\mathcal{C}$ satisfies axiom OA^*5) if the (normal epi, mono)-factorization is stable under pullback.
2. The (normal epi, mono)-factorization is stable under pullback if $\mathcal{C}$ satisfies axiom OA^*5).

The first assertion follows directly from Proposition 1.40. Indeed, if we have that the (normal epi, mono)-factorization is stable under pullback, we have it in particular along monomorphisms. We can thus use Proposition 1.40 to prove that OA^*5) is satisfied (recall that $OA5) = OA^*5)$).

The second assertion is more technical. Indeed, by using Proposition 1.40, we know that the factorization is stable along monomorphisms. Therefore, we need to prove that, in the presence of OA^*4), this factorization is stable along any morphism, not just along monomorphisms.

To do this, we can, in the first instance, use the fact that monomorphisms are stable along any morphism (Proposition 1.1). This will simplify the problem since now, we only need to show that normal epimorphisms are stable under pullback.

Let us then consider the following pullback diagram

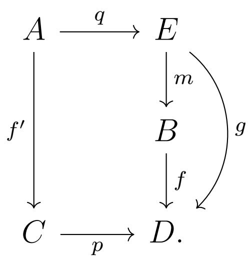


(1.10)

where p is a normal epimorphism and f in an arbitrary morphism. We have to prove that p' is also a normal epimorphism.

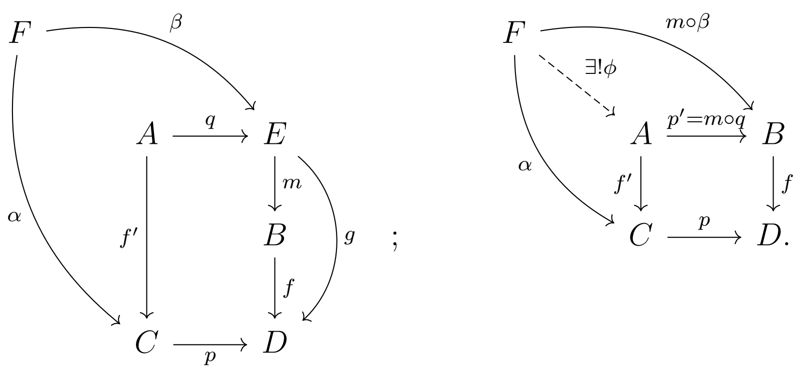
First, as was done in diagram 1.6, we can factor $f = l \circ h$ as as split monomorphism h followed by a split epimorphism l. However, the pullback along a composable morphism can be computed as two consecutive pullbacks. This means that the pullback of p along f can be computed as the pullback of p along a split epimorphism, then along a split monomorphism. Using Proposition 1.40, the pullback along the split monomorphism (hence monomorphism) is stable. It remains to show that the pullback along the split epimorphism is stable, hence, we may assume f to be a split epimorphism.

We can now considere the (normal epi, mono)-factorization $p' = m \circ q$ and define a new morphism $g := f \circ m$ in order to get a new pullback diagram



(1.11)

It is indeed a pullback because if $\alpha : F \to C$ and $\beta : F \to E$ are morphisms such that $p \circ \alpha = g \circ \beta$, we can deduce from the pullback diagram 1.10 that there exists a unique morphism $\phi : F \to A$ such that $f' \circ \phi = \alpha$ and $p' \circ \phi = m \circ q \circ \phi = (m \circ \beta)$. But m is a monomorphism, meaning that $q \circ \phi = \beta$, which thus proves that diagram 1.11 is a pullback since it satisfies the universal property with ϕ

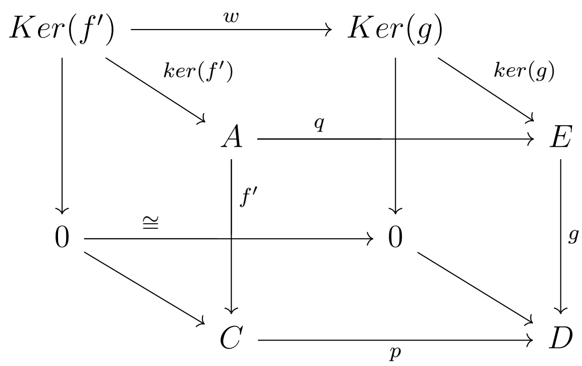


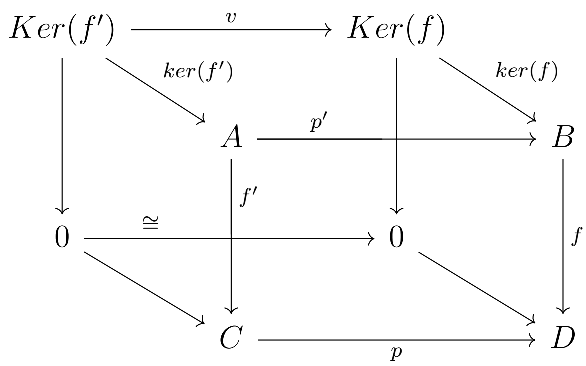
Following that same reasoning we have done above for f, we can assume that g is a split epimorphism.

Next, let us prove that

$$Ker(f) \cong Ker(f') \cong Ker(g). \tag{1.12}$$

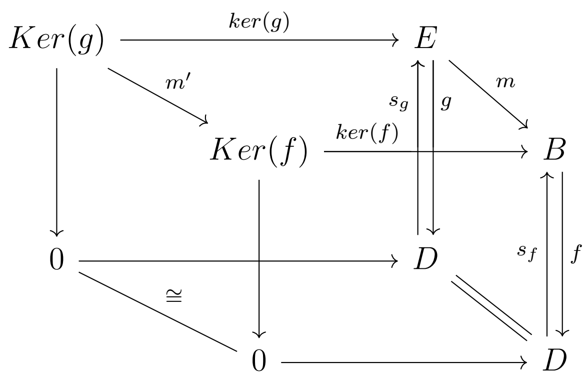
In the left-hand commutative square diagram





we have that the front, left and right face are pullbacks, implying that the back face is a pullback as well. Moreover, isomorphisms are stable under pullback, which means that the kernels are isomorphic, $Ker(f') \cong Ker(g)$. To prove that $Ker(f') \cong Ker(f)$, we can do the same computations witht the right-hand commutative square diagram.

To conclude the proof, we just need to examine the following square diagram:



where s_g and s_f are sections of f and g respectibely since they are assumed to be split epimorphisms.

We know that $\mathcal{C}$ is protomodular with a zero object, this means that the Kernel functor reflects isomorphisms. Since $Ker(m) = m'$ is an isomorphism, we can deduce that m is an isomorphism.

We know that $p' = m \circ q$ where q is a normal epimorphism and m is an isomorphism, meaning that p' is also a normal epimorphism.

1.2.4 Last part of the theorem

In order to show the last part of Theorem 1.36, we will first prove the following proposition:

Proposition 1.45. *Let $\mathcal{C}$ be a protomodular category with a zero object such that every morphism have a (normal epi, mono)-factorization and which satisfies OA^*6). Then, every reflexive relation is an effective internal equivalence relations.*

Proof. We first recall that a reflexive relation is a pair of jointly monic morphisms $r_1, r_2 : R \to A$ with a morphism $d : A \to R$ such that $r_1 \circ d = r_2 \circ d = 1_A$. Let $k = ker(r_1) : K \to R$ and $n := r_2 \circ k$.

The morphism n is a monomorphism. Indeed, let us first suppose that we have the following equality $n \circ x = r_2 \circ k \circ x = r_2 \circ k \circ y = n \circ y$ for any morphisms $x, y : I \to K$. We see that we also get $r_1 \circ k \circ x = 0 = r_1 \circ k \circ y$ since k is the kernel of r_1. As previously said, r_1 and r_2 are jointly monic, implying that $k \circ x = k \circ y$. Moreover, k is a (normal) monomorphism, so that $x = y$.

We can show that n is, more precisely, a normal monomorphism. Indeed, k and n are monomorphisms, 1_K is trivially a normal split monomorphism, r_2 is a split epimorphism with section $(0, 1_A)$ (since we are in a pointed category). Moreover, r_2 is even a normal epimorphism since $r_2 = coker(\tilde{q})$ where $\tilde{q} : A \to A/R$ is the canonical quotient map. We can then apply OA^*6) to

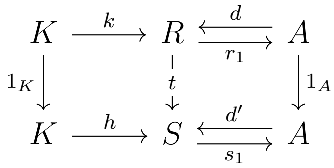
$$\begin{array}{ccc} K & \xrightarrow{1_K} & K \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle n} \\ R & \xrightarrow[r_2]{} & A \end{array}$$

in order to get that n is a normal monomorphism.

Now, let $q = coker(n) : A \to B$ and S be the induced internal equivalence relation $s_1, s_2 : S \to A$ (by taking the kernel pair of q). We have $q \circ r_1 \circ k = 0 = q \circ n = q \circ r_2 \circ k$ and $q \circ r_1 \circ d = q = q \circ r_2 \circ d$ by definition. But k and d are jointly epimorphic since we are in a protomodular category (follows from 1.25 and 1.10), thus $q \circ r_1 = q \circ r_2$. But the kernel pair S is a pullback by definition, so the previous equality allow us to say that there exists a unique $t : R \to S$ such that $s_2 \circ t = r_2$ and $s_1 \circ t = r_1$

$$\begin{array}{ccccc} R & \overset{r_1}{\underset{r_2}{\rightrightarrows}} & A & \xrightarrow{q} & B \\ & {\scriptstyle t}\searrow & {\scriptstyle s_2}\uparrow & & \uparrow{\scriptstyle q} \\ & & S & \xrightarrow[s_1]{} & A. \end{array}$$

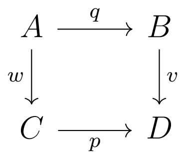
The last thing we have to show is that t is an isomorphism (this will prove that the reflexive relation R is an effective internal equivalence relation). We can define the morphism $h := ker(s_1)$ and finish by using the Split Short Five Lemma with the following diagram



where d' is the map coming from the reflexivity of S. □

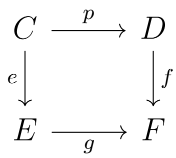
Proposition 1.46. *A semi-abelian category $\mathcal{C}$ satisfies $OA6$).*

Proof. Consider the following diagram

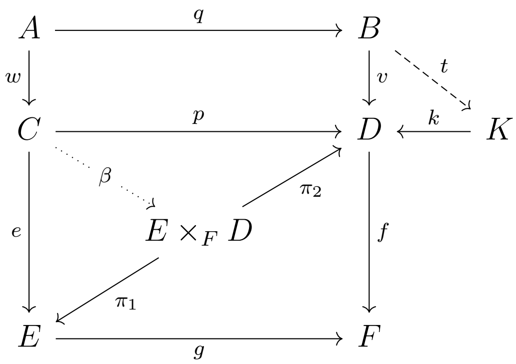


where p and q are normal epimorphisms, v is a monomorphism and w is a normal monomorphism. We have to prove that v is a normal monomorphism.

First, let us define $e = coker(w)$. We supposed that $\mathcal{C}$ is semi-abelian, meaning that we can use [9][Proposition 3.1.19.] to prove that $\mathcal{C}$ is also Mal'cev (i.e. a category where any reflexive relation is an internal equivalence relation). Knowing that, we can then use [22][Theorem 5.7.] to show that the following pushout diagram



exists. Moreover, this theorem tells us that the comparison map β coming from the universal property of the pullback $E \times_F D$



is a normal epimorphism.

Now, let $k = ker(f) : K \to D$, we can compute that

$$f \circ v \circ q = f \circ p \circ w = g \circ e \circ w = g \circ 0 = 0 = 0 \circ q$$

since $e = coker(w)$, but q is a (normal) epimorphism, meaning that $f \circ v = 0$. By the universal property of the kernel K, there exists a unique $t : B \to K$ such that $k \circ t = v$. Moreover, since k and v are monomorphisms, t is one as well. We can also define $k' = ker(\pi_1)$ which satisfies $\pi_2 \circ k' = k$ since π_1 and f have the same kernel (this is proven using the same reasoning as for Equation 1.12).

Since $\pi_1 \circ \beta = e = coker(w)$ and $k' = ker(\pi_1)$, the following commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{w} & C & \xrightarrow{e} & E \\ {\scriptstyle t\circ q}\downarrow & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle 1_E} \\ K & \xrightarrow[k']{} & E \times_F D & \xrightarrow[\pi_1]{} & E \end{array}$$

shows that the left hand square is a pullback, implying that the morphism $t \circ q$ is a normal epimorphism since normal epimorphisms are pullback stable (subsection 1.2.3) and β is one. Using the same type of computation as in the "$\Leftarrow$" part of Proposition 1.40, we get that t is also a normal epimorphism.

The morphism t is then a monomorphism and a normal epimorphism, we can use the dual property of Lemma 1.41 to deduce that t is an isomorphism. But we said that $k \circ t = v$, meaning that v is a normal monomorphism. $\square$

This proposition concludes the proof of the equivalence between the "old" and the "new" axioms.

Chapter 2

The category of cocommutatives Hopf algebras is semi-abelian

The goal of this chapter will be, first of all, to define the central object of the thesis: the **category of cocommutative Hopf algebras**, denoted as $\mathsf{Hopf}_{K,coc}$. First introduced in 1941 by the mathematician Heinz Hopf ([43]), the concept of Hopf algebra appears in several fields of mathematics and physics, such as Lie theory, Galois theory ([56]), quantum physics ([52]), and even knot theory ([50]).

The intuitive idea of this object is that it describes the symmetries of non-commutative spaces, in the same way as groups describe the symmetries of classical spaces. Hopf algebras can be seen as a linearization of the notion of a group.

Once this concept is clear, we will use what we have done in the first chapter to show that $\mathsf{Hopf}_{K,coc}$ is a semi-abelian category for any field K ([37]).

Note that a proof of this result was given in [35] with the constraint that the field K had characteristic 0. This proof depends on the Milnor-Moore Theorem (see, for example, [64]) that only holds when the characteristic of the field K is zero.

2.1 Symmetric monoidal category

In this subsection, we will define what a symmetric monoidal category is (for more information about this concept, see, for example, [51][Chapter 7.1.]). Thanks to this type of category, we will be able to define cocommutative Hopf algebras in a general context.

However, in order to prove that $\mathsf{Hopf}_{K,coc}$ is semi-abeliban, we will have to restrict to a special case of symmetric monoidal category. This restriction comes from the fact that the theorem used in the proof only works, as far as we know, for this very specific type of categories.

Definition 2.1. *A* ***monoidal category*** *(also called tensor category) is a category $\mathcal{C}$ equipped with a unit object I, a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (called the tensor product) and three natural isomorphisms (i.e. a natural transformation with a two-sided inverse)*

1. *the associator $a : \otimes \circ (1_{\mathcal{C}} \times \otimes) \to \otimes \circ (\otimes \times 1_{C})$;*
2. *the left unit $l : \otimes \circ (I \times 1_{\mathcal{C}}) \to 1_{\mathcal{C}}$;*
3. *the right unit $r : \otimes \circ (1_{\mathcal{C}} \times I) \to 1_{\mathcal{C}}$;*

making the following diagrams commute

$$\begin{array}{ccccc} A \otimes (B \otimes (C \otimes D))) & \xrightarrow{a_{A,B,C\otimes D}} & (A \otimes B) \otimes (C \otimes D) & \xrightarrow{a_{A\otimes B,C,D}} & ((A \otimes B) \otimes C) \otimes D \\ {\scriptstyle 1_A \otimes a_{B,C,D}} \downarrow & & & & \downarrow {\scriptstyle a_{A,B,C} \otimes 1_D} \\ A \otimes ((B \otimes C) \otimes D) & & \xrightarrow[a_{A,B\otimes C,D}]{} & & (A \otimes (B \otimes C)) \otimes D \end{array}$$

$$\begin{array}{ccc} A \otimes (I \otimes B) & \xrightarrow{a_{A,I,B}} & (A \otimes I) \otimes B \\ & {\scriptstyle 1_A \otimes l_B} \searrow \quad \swarrow {\scriptstyle r_A \otimes 1_B} & \\ & A \otimes B. & \end{array}$$

We will will write it as $(\mathcal{C}, \otimes, I, a, l, r)$.

Definition 2.2. *A* ***braided monoidal category*** *is a monoidal category $(\mathcal{C}, \otimes, I, a, l, r)$ with a family of natural isomorphisms $\sigma_{X,Y} : X \otimes Y \to Y \otimes X$ such that*

$$\sigma_{X\otimes Y,Z} = (\sigma_{X,Z} \otimes 1_Y) \circ (1_X \otimes \sigma_{Y,Z})$$
$$\sigma_{X,Y\otimes Z} = (1_Y \otimes \sigma_{X,Z}) \circ (\sigma_{X,Y} \otimes 1_Z).$$

This family of natural isomorphisms is called a ***braiding*** *and is denoted by σ. Hence, this category will be denoted by $(\mathcal{C}, \otimes, I, \sigma, a, l, r)$.*

Definition 2.3. *A* ***symmetric monoidal category*** *is a braided monoidal category where $\sigma_{X,Y} = \sigma^{-1}_{Y,X}$ for each element of the braiding σ.*

Example 2.4. Any cartesian category (i.e. a category with binary products and a terminal object) is a symmetric monoidal category.

In particular, the category *Set* of sets is a symmetric monoidal category where $\otimes$ is the cartesian product, $I = \{*\}$ and the braiding is the "twist morphism" defined by $\sigma_{X,Y}(x \times y) = y \times x$ for all $x \in X$ and $y \in Y$.

The following example is important for us, since the theorem that will be used in the proof that $\mathsf{Hopf}_{K,coc}$ is semi-abelian only works, as far as we know, in this symmetric monoidal category.

Example 2.5. The category of vector spaces on an arbitrary field K is a symmetric monoidal category where $\otimes$ is the tensor product of vector spaces, $I = K$ and the braiding is the "twist morphism" defined by $\sigma_{X,Y}(x \otimes y) = y \otimes x$ for all $x \in X$ and $y \in Y$. This category will therefore be denoted by $(Vect_K, \otimes, K, \sigma, a, l, r)$.

2.2 Cocommutative Hopf algebras

We will now define the notion of a cocommutative Hopf algebra in an arbitrary symmetric monoidal category. Then we will restrict ourselves to $(Vect_K, \otimes, K, \sigma, a, l, r)$ to prove the main result.

To reach our goal, we will need to define some other notions, such as algebras, coalgebras, algebra morphisms, etc. For more information on these concepts, see, for example, [4]. Let us start with the most familiar one:

Definition 2.6. *An **algebra** in a symmetric monoidal category $(\mathcal{C}, \otimes, I, \sigma, a, l, r)$ is given by $A \in \mathcal{C}$ and two morphisms*

1. *the multiplication $m : A \otimes A \to A$;*
2. *the unit $u : I \to A$.*

It will be written as (A, m, u).

*Moreover, we say that this algebra is **associative** (resp. **unital**) if the left-hand (resp. the right-hand) diagram commutes*

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{1_A \otimes m} & A \otimes A \\ {\scriptstyle m \otimes 1_A}\downarrow & & \downarrow{\scriptstyle m} \\ A \otimes A & \xrightarrow[m]{} & A \end{array} \qquad \begin{array}{ccccc} A \otimes I & \xrightarrow{1_A \otimes u} & A \otimes A & \xleftarrow{u \otimes 1_A} & I \otimes A. \\ & {\scriptstyle r_A}\searrow & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle l_A} & \\ & & A & & \end{array}$$

Remark 2.7. In our case, by abuse of language, whenever we refer to an algebra, it will always be understood that we are talking about an associative and unital algebra.

Algebras behave well under the tensor product:

Proposition 2.8. *If A and B are algebras, their tensor product is also an algebra with $m_{A \otimes B}((a \otimes b) \otimes (a' \otimes b')) := aa' \otimes bb'$ and $u_{A \otimes B} := u_A \otimes u_B$.*

Definition 2.9. *A **subalgebra** of A is an object $S \subset A$ from $\mathcal{C}$ such that $m_A(S \otimes S) \subseteq S$. Moreover, m_S and u_S are restrictions of m_A and u_A to S.*

It is also possible to define morphisms between two such algebras:

Definition 2.10. *Let $A = (A, m_A, u_A)$ and $B = (B, m_B, u_B)$ be two algebras in a symmetric monoidal category $(\mathcal{C}, \otimes, I, \sigma, a, l, r)$. A **morphism of algebras** is a morphism $f : A \to B$ in $\mathcal{C}$ such that the following diagrams commute*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle m_A}\uparrow & & \uparrow{\scriptstyle m_B} \\ A \otimes A & \xrightarrow[f\otimes f]{} & B \otimes B \end{array} \qquad \begin{array}{ccccc} A & & \xrightarrow{f} & & B. \\ & {\scriptstyle u_A}\nwarrow & & \nearrow{\scriptstyle u_B} & \\ & & I & & \end{array}$$

Definition 2.11. *A **left** (resp. **right**) **ideal** J in an algebra A is a linear subspace $J \subset A$ such that $aJ = \{m_A(a \otimes j) = aj \mid j \in J\}$ (resp. $Ja = \{m_A(j \otimes a) = ja \mid j \in J\}$) belong to J for all $a \in A$. A **two-sided ideal** is a left and right ideal.*

Theorem 2.12. *If J is a two-sided ideal, the linear quotient A/J is an algebra such that the canonical projection $\pi : A \to A/J$ is an algebra morphism where, for all $a \in A$,*

$$\pi \circ m_A(a \otimes a) = m_{A/J} \circ (\pi \otimes \pi)(a \otimes a) \text{ and } \pi \circ u_A(a) = u_{A/J}(a).$$

Proof. Can be found in [4]. □

In a completely analogous way, it is possible to define coalgebras (the dual notion of algebras) as well as coalgebra morphisms:

Definition 2.13. *A **coalgebra** in a symmetric monoidal category $(\mathcal{C}, \otimes, I, \sigma, a, l, r)$ is given by $C \in \mathcal{C}$ and two morphisms*

1. *the comultiplication $\Delta : C \to C \otimes C$;*
2. *The counit $\epsilon : C \to I$.*

It will be written as (C, Δ, ϵ).

Remark 2.14. As for algebras, in our work, coalgebras will always be coassociative and counital. This means that, respectively, the left-hand and the right-hand diagrams commute

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \Delta\otimes 1_C} \\ C \otimes C & \xrightarrow[1_C\otimes\Delta]{} & C \otimes C \otimes C \end{array} \qquad \begin{array}{ccccc} C \otimes I & \xleftarrow{1_C\otimes\epsilon} & C \otimes C & \xrightarrow{\epsilon\otimes 1_C} & I \otimes C. \\ & {\scriptstyle r_C^{-1}}\nwarrow & \uparrow{\scriptstyle \Delta} & \nearrow{\scriptstyle l_C^{-1}} & \\ & & C. & & \end{array}$$

Remark 2.15. There exists a way to simplify the expression of the comultiplication, which is usually called the "**Sweedler notation**". Indeed, for every $c \in C$, we can write

$$\Delta(c) = \sum_i c_{1i} \otimes c_{2i} =: c_1 \otimes c_2.$$

For example, the coassociativity of a coalgebra can be written, for every $c \in C$, as

$$\begin{aligned}(\Delta \otimes 1_C) \circ \Delta(c) = (\Delta \otimes 1_C)(c_1 \otimes c_2) &= \Delta(c_1) \otimes c_2 \\ &= (c_1)_1 \otimes (c_1)_2 \otimes c_2 \\ &= c_1 \otimes c_2 \otimes c_3 \\ &= c_1 \otimes (c_2)_1 \otimes (c_2)_2 \\ &= (1_C \otimes \Delta) \circ \Delta(c).\end{aligned}$$

As for algebras, coalgebras behave well under the tensor product:

Proposition 2.16. *If C and D are coalgebras, their tensor product is also a coalgebra with $\Delta_{C\otimes D}(c \otimes d) := (c_1 \otimes d_1) \otimes (c_2 \otimes d_2)$ and $\epsilon_{C\otimes D}(c \otimes d) := \epsilon_C(c)\epsilon_D(d)$.*

Definition 2.17. *A **subcoalgebra** L of a coalgebra C is an object $L \subset C$ from $\mathcal{C}$ that has the following property: $\Delta_C(L) \subseteq L \otimes L$. Moreover, Δ_L and ϵ_L are restrictions of Δ_C and ϵ_C to L.*

Definition 2.18. *Let $C = (C, \Delta_C, \epsilon_C)$ and $D = (D, \Delta_D, \epsilon_D)$ be two coalgebras in a symmetric monoidal category $(\mathcal{C}, \otimes, I, \sigma, a, l, r)$. A **morphism of coalgebras** is a morphism $g : C \to D$ in $\mathcal{C}$ such that the following diagrams commute*

$$\begin{array}{ccc} C & \xrightarrow{\quad g \quad} & D \\ {\scriptstyle \Delta_C}\downarrow & & \downarrow{\scriptstyle \Delta_D} \\ C \otimes C & \xrightarrow{g\otimes g} & D \otimes D \end{array} \qquad\qquad \begin{array}{ccc} C & \xrightarrow{\qquad g \qquad} & D. \\ & {\scriptstyle \epsilon_C}\searrow \quad \swarrow{\scriptstyle \epsilon_D} & \\ & I. & \end{array}$$

For example, in Sweedler's notation, the left-hand diagram in Definition 2.18 can be written, for every $c \in \mathcal{C}$, as

$$(g(c))_1 \otimes (g(c))_2 = g(c_1) \otimes g(c_2).$$

Definition 2.19. *A **left** (resp. **right**) **coideal** J in a coalgebra C is a linear subspace $J \subset C$ such that $\Delta_C(J) \subset J \otimes C$ (resp. $\Delta_C(J) \subset C \otimes J$).*

*A **two-sided coideal** is a linear subspace $J \subset C$ such that $\Delta_C(J) \subset J \otimes C + C \otimes J$ and $\epsilon_C(J) = 0$.*

Remark 2.20. Note that a left and right coideal J is not necessarly a two-sided ideal. It is the case only if $J = 0$.

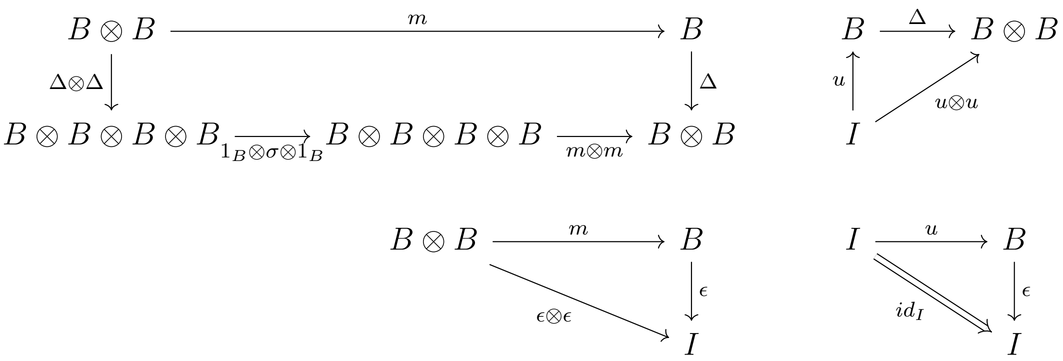
Theorem 2.21. *If J is a two-sided coideal, the linear quotient C/J is a coalgebra such that the canonical projection $\pi : C \to C/J$ is a coalgebra morphism where, for all $c \in C$,*

$$\Delta_{C/J} \circ \pi(c) = (\pi \otimes \pi) \circ \Delta_C(c) \text{ and } \epsilon_{C/J} \circ \pi(c) = \epsilon_C(c).$$

Proof. [4][Theorem 1.5.2.]. □

It is also possible to be both an algebra and a coalgebra at the same time:

Definition 2.22. *A **bialgebra** in a symmetric monoidal category $(\mathcal{C}, \otimes, I, \sigma, a, l, r)$ is given by a 5-tuple $(B, m, u, \Delta, \epsilon)$ where (B, m, u) is an algebra, (B, Δ, ϵ) is a coalgebra and Δ, ϵ are algebra morphisms. This means that the following diagrams commute*



where σ is the braiding of the symmetric monoidal category.

Remark 2.23. In the Definition 2.22, it is equivalent to define a bialgebra as the 5-tuple $(B, m, u, \Delta, \epsilon)$ where (B, m, u) is an algebra, (B, Δ, ϵ) is a coalgebra and m, u are coalgebra morphisms. This directly follows from the diagrams in the definitions.

Definition 2.24. *A morphism in $\mathcal{C}$ is a **morphism of bialgebras** if it is both a morphism of algebras and of coalgebras.*

Now that all these notions are clear, we can give the definition that mainly interests us. To do this, we recall that $S : A \to A$ is an anti-morphism of algebras if

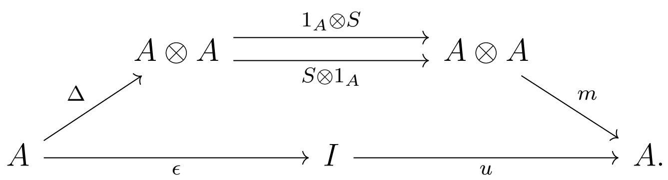
$$S(xy) = S \circ m(x \otimes y) = m \circ \sigma \circ (S \otimes S)(x \otimes y) = S(y)S(x)$$

and an anti-morphism of coalgebras if

$$(S(x))_1 \otimes (S(x))_2 = \Delta \circ S(x) = (S \otimes S) \circ \sigma \circ \Delta(x) = S(x_2) \otimes S(x_1)$$

for all $x \in A$ and for all $x \otimes y \in A \otimes A$.

Definition 2.25. *A **Hopf algebra** in a symmetric monoidal category* $(\mathcal{C}, \otimes, I, \sigma, a, l, r)$ *is a bialgebra* $(A, m, u, \Delta, \epsilon)$ *with an anti-morphism of algebras and coalgebras* $S : A \to A$, *called the **antipode**, such that the following diagram commutes*



(2.1)

Using Sweedler's notation, diagram 2.1 can be written as $\epsilon(a)1_A = a_1 S(a_2) = S(a_1)a_2$.

Remark 2.26. In some references (such as [4]), the antipode S is equivalently defined as the convolution inverse of Id_A in $End_K(A)$.

Definition 2.27. *A **morphism of Hopf algebras** is a morphism of bialgebras* $f : A \to B$ *such that the antipode is preserved. This means that for all* $a \in A$, *we have* $S_B \circ f(a) = f \circ S_A(a)$.

Remark 2.28. If the underlying algebra of the Hopf algebra is associative (we assumed that in this thesis), then S is automatically an anti-morphism of both algebras and coalgebras. Moreover, a morphism of associatives bialgebras always preserves the antipode (see, for example, [63]).

This means that, in our case, morphisms of Hopf algebras are morphisms of bialgebras.

Definition 2.29. *A **Hopf subalgebra** of* H *is an object* $H' \subset H$ *from* $\mathcal{C}$ *which is both a subalgebra and a subcoalgebra of* H *that is stable under the antipode: this means that* $S_H(H') \subseteq H'$.

Definition 2.30. *A **Hopf ideal*** J *in a Hopf algebra* H *is a two-sided ideal and two-sided coideal such that* $S_H(J) \subset J$.

Theorem 2.31. *If* J *is a Hopf ideal, the linear quotient* H/J *is a Hopf algebra such that the canonical projection* $\pi : H \to H/J$ *is a Hopf algebra morphism.*

Proof. [4][Theorem 4.1.11.]. □

It remains to explain what "cocommutative" means:

Definition 2.32. *A Hopf algebra $A = (A, m, u, \Delta, \epsilon, S)$ is* ***cocommutative*** *when its comultiplication Δ makes the following diagram commute*

$$\begin{array}{ccc} & A & \\ {}^{\Delta}\swarrow & & \searrow^{\Delta} \\ A \otimes A & \xrightarrow{\quad\sigma\quad} & A \otimes A \end{array}$$

where σ is the braiding coming from the symmetric monidal category $(\mathcal{C}, \otimes, I, \sigma, a, l, r)$.

In Sweedler's notation, the cocommutativity can be written as $\Delta(a) = a_1 \otimes a_2 = a_2 \otimes a_1$ for every $a \in A$.

We will denote by $\mathsf{Hopf}_{K,coc}$ the category of cocommutative Hopf algebras and Hopf algebra morphisms (=bialgebra morphisms by Remark 2.28).

To illustrate this important notion, we will provide examples related to groups and Lie algebras:

Example 2.33. Let $(Vect_K, \otimes, K, \sigma, a, l, r)$ be the symmetric monoidal category of vector spaces. First, let us recall what the group-algebra is (see [28] for more information about this notion). Any group $(G, \circ, e)$ gives rise to a **group-alegbra**, denoted $K[G]$. It is the free vector space on G over the field K. This means that

$$K[G] = \{\sum_{g \in G} \alpha_g g \mid (\alpha_g)_{g \in G} \text{ is a family of scalars with only}$$
$$\text{a finite number of them being non zero}\}$$

where $\{g \in G\}$ is then a basis of $K[G]$.

We can equip $K[G]$ with a cocommutative Hopf algebra structure. Indeed, it is a structure where

1. the multiplication $m : K[G] \otimes K[G] \to K[G]$ is defined as $m(g \otimes h) = g \circ h$;
2. the comultiplication $\Delta : K[G] \to K[G] \otimes K[G]$ is defined as $\Delta(g) = g \otimes g$;
3. the unit $u : K \to K[G]$ is defined as $u(k) = e$;
4. the counit $\epsilon : K[G] \to K$ is defined as $\epsilon(g) = 1_K$, the unit element ok K;
5. the antipode $S : K[G] \to K[G]$ is defined as $S(g) = g^{-1}$.

   It is clearly cocommutative by definition of Δ.

The following example will allow us to define another example, which is the one that is of interest for us.

Example 2.34. Let $(Vect_K, \otimes, K, \sigma, a, l, r)$ the symmetric monoidal category of vector spaces. We have to recall what the **tensor algebra** is (see [44][Chapter 5] for more information about this notion). Let V be a vector space, we can define $T^0(V) = K$, $T^1(V) = V$ and, for $n \geqslant 2$, $T^n(V) = V \otimes V \ldots \otimes V$ is the tensor product of n copies of V.

We can then define the tensor algebra, denoted by $T(V)$, as the graded algebra

$$T(V) = \bigoplus_{n \geqslant 0} T^n(V)$$

where the multiplication is defined as

$$m(x \otimes y) = x_1 \otimes x_2 \ldots \otimes x_n \otimes y_1 \otimes y_2 \ldots \otimes y_m \in T^{n+m}(V)$$

for any $x = x_1 \otimes x_2 \ldots \otimes x_n \in T^n(V)$ and $y = y_1 \otimes y_2 \ldots \otimes y_m \in T^m(V)$ (the multiplication of two arbitrary elements of $T(V)$ is obtained by extending linearly this formula) and the unit is $u : K \to T(V)$ such that $u(k) = 1_K \in T^0(V) = K$.

Furthermore, we can equip $T(V)$ with a cocommutative Hopf algebra structure by defining $\Delta(v) = v \otimes 1_K + 1_K \otimes v$, $\epsilon(v) = 0$ and $S(v) = -v$ for all $v \in V$ (all these maps extend linearly to all $T(V)$).

By definition of the comultiplication, it is cocommutative.

Example 2.35. Let $(Vect_K, \otimes, K, \sigma, a, l, r)$ be the symmetric monoidal category of vector spaces. First, let us recall the **universal enveloping algebra** of a Lie algebra (see [44][Chapter 5] for more information about this notion). If L is a Lie algebra, we can define the universal enveloping algebra of L, denoted $\mathcal{U}(L)$, as the quotient

$$\mathcal{U}(L) = T(L)/I$$

where $T(L)$ is the tensor algebra on the vector space underlying L and I is the two-sided ideal of $T(L)$ generated by the set $\{x \otimes y - y \otimes x - [x, y] \mid x, y \in L\}$. We have that the elements of L generates $\mathcal{U}(L)$ as an algebra.

We can equip $\mathcal{U}(L)$ with a cocommutative Hopf algebra structure by defining the multiplication as the concatenation, $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\epsilon(x) = 0$ and $S(x) = -x$ for all $x \in L$ (again, all these maps extend linearly to all $\mathcal{U}(L)$).

2.3 The category of cocommutative Hopf algebras is semi-abelian

From now on, the symmetric monoidal category underlying the cocommutative Hopf algebras will be $(Vect_K, \otimes, K, \sigma, a, l, r)$. As mentioned earlier, the reason of this restriction comes from the fact that the theorem used in the proof that $\mathsf{Hopf}_{K,coc}$ is semi-abelian only works, as far as we know, in the case of vector spaces.

In order to prove that $\mathsf{Hopf}_{K,coc}$ is semi-abelian for any field K, we will follow what have been done by M. Gran, F. Sterck and J. Vercruysse in [37].

2.3.1 $\mathsf{Hopf}_{K,coc}$ is finitely complete and finitely cocomplete

It is shown, in [64] and [60], that the category of Hopf algebras over an arbitrary field, Hopf_K, is complete and cocomplete and, in [2], a description of these limits and colimits is given.

However, in the case of cocommutative Hopf algebras, descriptions of limits and colimits become simpler since we have an explicit description of the binary product: the **tensor product**. In order to become more familiar will all these concepts, we will show that $\mathsf{Hopf}_{K,coc}$ is finitely complete by giving an explicit description of the limits. In particular, we will show that $\mathsf{Hopf}_{K,coc}$ has a zero object, binary products and equalizers.

The zero object

The base field K is the zero object. Indeed, we can define the comultiplication as $\Delta : K \cong K \otimes K$ (the natural isomorphism) and ϵ, m, u as the identity maps. Moreover, for any cocommutative Hopf algebra A, there is a unique morphism $u_A : K \to A$ and a unique morphism $\epsilon_A : A \to K$.

This means, among other things, that a zero morphism between cocommutative Hopf algebras A and B is of the form

$$A \xrightarrow{\epsilon_A} K \xrightarrow{u_B} B.$$

The binary product

The binary product of two cocommutative Hopf algebras A and B is their tensor product $A \otimes B$ together with two morphisms $\pi_1 : A \otimes B \to A$ and $\pi_2 : A \otimes B \to B$ defined as

$$\pi_1(a \otimes b) = r_A \circ (1_A \otimes \epsilon_B)(a \otimes b) = a\epsilon_B(b)$$

and

$$\pi_2(a \otimes b) = l_B \circ (\epsilon_A \otimes 1_B)(a \otimes b) = \epsilon_A(a)b$$

for all $a \otimes b \in A \otimes B$.

First, using Proposition 2.8, Proposition 2.16 and the definition of the cocommutative Hopf algebras, we see that $A \otimes B$ is a cocommutative bialgebra. It is also a Hopf algebra by defining the antipode as $S_{A\otimes B} = S_A \otimes S_B$.

Let us now prove that π_1 and π_2 are morphisms of Hopf algebras.

To avoid doing the same type of calculation four times, we will only show that π_1 is a morphism of algebras, the rest of the calculations are left to the reader, since they are completely similar.

For all $a \otimes b$, $a' \otimes b'$ in $A \otimes B$ and $k \in K$, we have

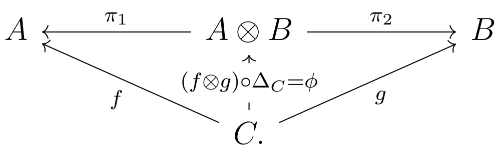
$$\begin{aligned} m_A \circ (\pi_1 \otimes \pi_1)((a \otimes b) \otimes (a' \otimes b')) &= m_A(a\epsilon_B(b) \otimes a'\epsilon_B(b')) \\ &= a\epsilon_B(b)a'\epsilon_B(b') \\ &= aa'\epsilon_B(bb') \\ &= \pi_1(aa' \otimes bb') \\ &= \pi_1 \circ m_{A\otimes B}((a \otimes b) \otimes (a' \otimes b')) \end{aligned}$$

where, in the third equality, we used that fact that $\epsilon_B(b) \in K$ and that ϵ_B is an algebra morphism since $A \otimes B$ is a bialgebra.

Moreover, we have that

$$\pi_1 \circ u_{A\otimes B}(k) = \pi_1(u_A(k) \otimes u_B(k)) = u_A(k)\epsilon_B(u_B(k)) = u_A(k).$$

Now, we will show that the tensor product satisfies the universal property of the product. Let C be an object of $\mathsf{Hopf}_{K,coc}$ and $f : C \to A$, $g : C \to B$ two morphisms of Hopf algebras



We will show that $\phi = (f \otimes g) \circ \Delta_C$ is the unique Hopf algebra morphism such that $\pi_1 \circ \phi = f$ and $\pi_2 \circ \phi = g$.

First, let us show that it is a coalgebra morphism:

$$\begin{aligned}(\phi \otimes \phi) \circ \Delta_C(c) &= (f \otimes g)(\Delta_C(c_1)) \otimes (f \otimes g)(\Delta_C(c_2)) \\ &= f((c_1)_1) \otimes g((c_1)_2) \otimes f((c_2)_1) \otimes g((c_2)_2) \\ &= f(c_1) \otimes g(c_2) \otimes f(c_3) \otimes g(c_4) \\ &= f(c_1) \otimes g(c_3) \otimes f(c_2) \otimes g(c_4) \\ &= f((c_1)_1) \otimes g((c_2)_1) \otimes f((c_1)_2) \otimes g((c_2)_2) \\ &= f((c_1))_1 \otimes g((c_2))_1 \otimes f((c_1))_2 \otimes g((c_2))_2 \\ &= \Delta_{A\otimes B}(f(c_1) \otimes g(c_2)) \\ &= \Delta_{A\otimes B} \circ (f \otimes g) \circ \Delta_C(c) \\ &= \Delta_{A\otimes B} \circ \phi(c)\end{aligned}$$

where we used, for the fourth equality, the fact that the Hopf algebras are cocommutatives ($\sigma \circ \Delta = \Delta$). Moreover

$$\begin{aligned}\epsilon_{A\otimes B} \circ \phi(c) &= \epsilon_{A\otimes B}(f(c_1) \otimes g(c_2)) \\ &= \epsilon_A(f(c_1))\epsilon_B(g(c_2)) \\ &= \epsilon_C(c_1)\epsilon_C(c_2) \\ &= \epsilon_C(c).\end{aligned}$$

Then, we can show that it is also an algebra morphism:

$$\begin{aligned}m_{A\otimes B} \circ (\phi \otimes \phi)(c \otimes d) &= m_{A\otimes B}((f(c_1) \otimes g(c_2)) \otimes (f(d_1) \otimes g(d_2))) \\ &= f(c_1)f(d_1) \otimes g(c_2)g(d_2) \\ &= f(c_1 d_1) \otimes g(c_2 d_2) \\ &= (f \otimes g)(c_1 d_1 \otimes c_2 d_2) \\ &= (f \otimes g) \circ \Delta_C \circ m_C(c \otimes d) \\ &= \phi \circ m_C(c \otimes d)\end{aligned}$$

and

$$\begin{aligned}\phi \circ (u_C(k)) &= f(u_C(k)_1) \otimes g(u_C(k)_2) \\ &= u_A(k_1) \otimes u_B(k_2) \\ &= u_{A\otimes B}(k).\end{aligned}$$

It remains to show that it is the only one such that $\pi_1 \circ \phi = f$ and $\pi_2 \circ \phi = g$. Let us suppose that there exists another morphism of Hopf algebras $\iota : C \to A \otimes B$ such that

$\pi_1 \circ \iota = f$ and $\pi_2 \circ \iota = g$, then

$$\begin{aligned}
(f \otimes g) \circ \Delta_C &= ((\pi_1 \circ \iota) \otimes (\pi_2 \circ \iota)) \circ \Delta_C \\
&= (\pi_1 \otimes \pi_2) \circ (\iota \otimes \iota) \circ \Delta_C \\
&= (\pi_1 \otimes \pi_2) \circ \Delta_{A \otimes B} \circ \iota \\
&= (\pi_1 \otimes \pi_2) \circ (1_A \otimes \sigma \otimes 1_B) \circ (\Delta_A \otimes \Delta_B) \circ \iota \\
&= (r_A \circ (1_A \otimes \epsilon_B) \otimes l_B \circ (\epsilon_A \otimes 1_B)) \circ (1_A \otimes \sigma \otimes 1_B) \circ (\Delta_A \otimes \Delta_B) \circ \iota \\
&= (r_A \circ (1_A \otimes \epsilon_A) \otimes l_B \circ (\epsilon_B \otimes 1_B)) \circ (\Delta_A \otimes \Delta_B) \circ \iota \\
&= (r_A \circ (1_A \otimes \epsilon_A) \circ \Delta_A \otimes l_B \circ (\epsilon_B \otimes 1_B) \circ \Delta_B) \circ \iota \\
&= r_A \otimes l_B \circ \iota \\
&= \iota.
\end{aligned}$$

At the third equality we used the fact that ι is a coalgebra morphism and at the eighth equality we used the fact that the Hopf algebra is counital. Moreover, recall that r_A and l_B are the maps from the symmetric monoidal category $(Vect_K, \otimes, K, \sigma, a, l, r)$.

This concludes the proof that the tensor product is the binary product in $\mathsf{Hopf}_{K,coc}$. This is something important, and it is what makes all the difference with general Hopf algebras in the proof that $\mathsf{Hopf}_{K,coc}$ is semi-abelian.

The equalizer

In [3], N. Andruskiewitsch and J. Devoto gave an explicit description of equalizers and coequalizers of arbitrary Hopf algebra. Here, we are going to give the description of equalizers in $\mathsf{Hopf}_{K,coc}$. To do this, let us start with three constructions in the category of Hopf algebras.

Definition 2.36. *Let $f, g : A \to B$ two Hopf algebra morphisms. Let us define the following subalgebras of A:*

$$LEqual(f,g) = \{a \in A \mid (f \otimes id_A) \circ \Delta_A(a) = (g \otimes id_A) \circ \Delta_A(a)\}$$

$$REqual(f,g) = \{a \in A \mid (id_A \otimes f) \circ \Delta_A(a) = (id_A \otimes g) \circ \Delta_A(a)\}$$

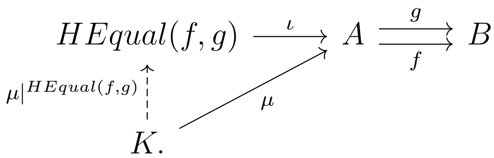
$$\begin{aligned}
HEqual(f,g) = \{a \in A \mid (id_A \otimes f \otimes id_A) \circ (\Delta_A \otimes id_A) \circ \Delta_A(a) \\
= (id_A \otimes g \otimes id_A) \circ (\Delta_A \otimes id_A) \circ \Delta_A(a)\}.
\end{aligned}$$

Remark 2.37. If a belongs to one these three sets, we have $f(a) = g(a)$.

Proposition 2.38. *In $\mathsf{Hopf}_{K,coc}$, the three subalgebras from Definition 2.36 are equal. Moreover, if $f, g : A \to B$ are morphisms of Hopf algebras, the inclusion of $HEqual(f,g)$ in A is the equalizer of f and g.*

Proof. Let us write $\iota : HEqual(f,g) \to A$ for the inclusion. We can first compute that $f \circ \iota(a) = f(a) = g(a) = g \circ \iota(a)$ thanks to Remark 2.37.

Now, suppose that there exists a morphism $\mu : K \to A$ such that $f \circ \mu = g \circ \mu$. Then, there exists a morphism $\phi = \mu|^{HEqual(f,g)}$, the corestriction of μ (i.e. the restriction of μ on its codomain because $\mu(K) \subseteq HEqual(f,g)$ since $f \circ \mu = g \circ \mu$), such that $\iota \circ \phi = \mu$



Moreover, this morphism is unique. Indeed, if ψ is another morphism such that $\iota \circ \psi = \mu$, we have that $\iota \circ \psi = \mu = \iota \circ \phi$. But the inclusion ι is a monormophism, meaning that $\phi = \psi$. ∎

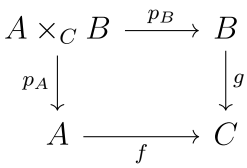
The definition of the kernel of a morphism directly follows from Proposition 2.38:

Proposition 2.39. *The kernel of a morphism $f : A \to B$ in $\mathsf{Hopf}_{K,coc}$ is the inclusion of $HKer(f)$ in A where*

$$HKer(f) = \{a \in A \mid (f \otimes id_A) \circ \Delta_A(a) = 1_B \otimes a\}.$$

$\mathsf{Hopf}_{K,coc}$ is finitely complete

Using what have been done previously, we have that $\mathsf{Hopf}_{K,coc}$ is finitely complete. This means that we thus have pullbacks. Indeed, the pullback of $f : A \to C$ along $g : B \to C$ in $\mathsf{Hopf}_{K,coc}$ is expressed as



where $A \times_C B = \{a \otimes b \in A \otimes B \mid a_1 \otimes f(a_2) \otimes b = a \otimes g(b_1) \otimes b_2\}$ and the pullback projections are $p_A = r_A \circ (1_A \otimes \epsilon_B)$ and $p_B = l_B \circ (\epsilon_A \otimes 1_B)$.

However, there is a way to express the pullback of an inclusion along any morphism that will be useful to prove that $\mathsf{Hopf}_{K,coc}$ is regular. To do so, it is necessary to define several objects:

Definition 2.40. *Given any coalgebra C, we can define C^+ as*

$$C^+ = \{c \in C \mid \epsilon_C(x) = 0\}.$$

Remark 2.41. For every coalgebra morphism $f : C \to D$, we have that $f(C^+) = f(C)^+$. Indeed, $y \in f(C)^+$ if and only if there is a $x \in C$ such that $f(x) = y$ and $0 = \epsilon_D(y)$. But $0 = \epsilon_D(y) = \epsilon_D(f(x)) = \epsilon_C(x)$ since f is a coalgebra morphism. This means that $x \in C^+$ and $y = f(x)$.

Proposition 2.42. *Given any morphism $p : A \to B$ in $\mathsf{Hopf}_{K,coc}$ and a Hopf subalgebra C of B, then, the subset*

$$p^{-1}(C) = \{x \in A \mid (p \otimes 1_A)\Delta(x) \in C \otimes A\}$$

is a Hopf subalgebra of A. It is called the ***h-inverse*** *of C along p.*

Proof. [68][Proposition 1 & Corollary 2]. □

Lemma 2.43. *Let $p : A \to B$ be a morphism in $\mathsf{Hopf}_{K,coc}$. Then, for all Hopf subalgebra C of B, $p(p^{-1}(C)) \subseteq C$.*

Proof. [37][Lemma 2.5.]. □

We will now prove that this h-inverse is in fact the (object part of the) pullback of the inclusion of C in B along $p : A \to B$.

Lemma 2.44. *Given an inclusion of a Hopf subalgebra $\iota : C \to B$ in a cocommutative Hopf algebra B, we have that the following diagram*

$$\begin{array}{ccc} p^{-1}(C) & \xrightarrow{\tilde{p}} & C \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \iota} \\ A & \xrightarrow[p]{} & B \end{array} \tag{2.2}$$

is a pullback where j is the inclusion of $p^{-1}(C)$ in A and $\tilde{p}$ is the restriction of p on $p^{-1}(C)$.

Proof. First, Lemma 2.43 tells us that if $x \in p^{-1}(C)$, then $p(x) \in C$. This allows us to say that the diagram 2.2 commutes.

Then, let $\alpha : Z \to A$ and $\beta : Z \to C$ be two Hopf algebra morphisms such that $p \circ \alpha = \iota \circ \beta$. We will prove that the unique morphism $\phi : Z \to p^{-1}(C)$ such that $j \circ \phi = \alpha$ and $\tilde{p} \circ \phi = \beta$ is the morphism α.

Because of the inclusions, the only thing we have to prove is that $\alpha(z) \in p^{-1}(C)$ for all $z \in Z$:

$$\begin{aligned}(p \otimes 1_A) \circ \Delta_A(\alpha(z)) &= (p \otimes 1_A) \circ (\alpha \otimes \alpha) \circ \Delta_Z(z) \\ &= (p \otimes 1_A)(\alpha(z_1) \otimes \alpha(z_2)) \\ &= p(\alpha(z_1)) \otimes \alpha(z_2) \\ &= \iota(\beta(z_1)) \otimes \alpha(z_2)\end{aligned}$$

which belongs to $C \otimes A$ since $\beta(\tilde{z}) \in C$ for all $\tilde{z} \in Z$. □

$\mathsf{Hopf}_{K,coc}$ is finitely cocomplete

As previously mentioned, it is shown in [64] and [60] that $\mathsf{Hopf}_{K,coc}$ is also cocomplete. We will not give a proof of this result, but we will see the key elements. First, we already know that the zero object is K. Then, the binary coproduct in $\mathsf{Hopf}_{K,coc}$ is the same as the one in the category of algebras Alg_K and is explicitly described in [67].

For the coequalizers (implying cokernels), we will provide an explicit description because it will be used in the next subsections.

Proposition 2.45. *Let $f, g : A \to B$ be two Hopf algebra morphisms. Then, BJB, the two-sided ideal from B generated by $J = \{f(x) - g(x) \mid x \in A\}$ is a Hopf ideal. Moreover, the coequalizer of f and g is given by the canonical projection $\pi : B \to B/BJB$.*

From this proposition, it is easy to get cokernels:

Proposition 2.46. *Let $f : A \to B$ be an arrow in $\mathsf{Hopf}_{K,coc}$. Its cokernel is given by the canonical quotient $q : B \to B/B(f(A)^+)B$.*

2.3.2 $\mathsf{Hopf}_{K,coc}$ is protomodular

First, we have to define the important concept of internal group in a category:

Definition 2.47. *Let $\mathcal{C}$ be a category with binary products and a terminal object T. An **internal group** in $\mathcal{C}$ is an object $G \in \mathcal{C}$ together with three morphisms*

1. *the unit $1 : T \to G$;*
2. *the inversion $s : G \to G$;*
3. *the multiplication $m : G \times G \to G$;*

such that the following squares commute

$$
\begin{array}{ccc}
G \times G \times G & \xrightarrow{id_G \times m} & G \times G \\
{\scriptstyle m \times id_G}\downarrow & & \downarrow{\scriptstyle m} \\
G \times G & \xrightarrow[m]{} & G
\end{array}
\qquad
\begin{array}{ccc}
G & \xrightarrow{(1, id_G)} & G \times G \\
{\scriptstyle (id_G, 1)}\downarrow & \searrow^{id_G} & \downarrow{\scriptstyle m} \\
G \times G & \xrightarrow[m]{} & G
\end{array}
\tag{2.3}
$$

$$
\begin{array}{ccc}
G & \xrightarrow{(id_G, s) \circ \delta} & G \times G \\
{\scriptstyle (s, id_G) \circ \delta}\downarrow & \searrow^{id_G} & \downarrow{\scriptstyle m} \\
G \times G & \xrightarrow[m]{} & G
\end{array}
$$

where $\delta = (id_G, id_G) : G \to G \times G$ is the diagonal of the binary product.

Example 2.48. The category of internal groups in the category of sets, $Grp(Set)$, is the category Grp of groups.

Example 2.49. If Top is the category of topological spaces and continuous maps, then, $Grp(Top)$ is the category of topological groups.

The example that interests us here is the following:

Example 2.50. The category of internal groups in the category of cocommutative coalgebras, $Grp(CoAlg_{K,coc})$, is the category of cocommutative Hopf algebras $\mathsf{Hopf}_{K,coc}$. Indeed, the two first diagrams of Definition 2.3 give the algebra structure, and the third one gives the antipode.

Now, let us see a theorem that will allow us to prove that $\mathsf{Hopf}_{K,coc}$ is protomodular:

Theorem 2.51. *Let $\mathcal{C}$ be a finitely complete category. Then, the category of internal groups in $\mathcal{C}$, $Grp(\mathcal{C})$, is a protomodular category.*

The detailed proof can be found in [14][Proposition 3.24.]. However, we will provide the main ideas of it:

Proof. 1. Any internal group X induces a group structure on the hom-set $\mathcal{C}(Z, X)$ for all $Z \in \mathcal{C}$. Indeed, if m is the multiplication of the internal group, the binary operation $*$ defined as $f * g = m \circ (f, g)$ for all $f, g \in \mathcal{C}(Z, X)$ defines a group structure on $\mathcal{C}(Z, X)$.

2. The functor $\mathcal{C}(-, X)$ factors as

$$\begin{array}{ccc} \mathcal{C}^{op} & \xrightarrow{\mathcal{C}(-,X)} & Sets \\ & {\scriptstyle \tilde{\mathcal{C}}(-,X)} \searrow \quad \nearrow {\scriptstyle U} & \\ & Grp & \end{array}$$

   where U is the forgetful functor and $\tilde{\mathcal{C}}(-, X)$ takes an object $Z \in \mathcal{C}$ and associates $\mathcal{C}(Z, X)$ with the group structure $*$ described above.

3. The following functor

$$F : Grp(\mathcal{C}) \to Grp^{\mathcal{C}^{op}}$$
$$X \to \tilde{\mathcal{C}}(-, X)$$

   preserves finite limits and is conservative (i.e. it reflects isomorphisms). This implies that the protomodularity of $Grp^{\mathcal{C}^{op}}$ can be lifted to $Grp(\mathcal{C})$.

□

Moreover, thanks to what has been proven in [60] and [40], we can assume that the category of cocommutative coalgebras $CoAlg_{K,coc}$ is finitely complete.

Theorem 2.52. *The category of cocommutative Hopf algebras is protomodular*

Proof. Using Example 2.50, we get that $Grp(CoAlg_{K,coc})$ is $\mathsf{Hopf}_{K,coc}$. Moreover, we assumed that $CoAlg_{K,coc}$ is a finitely complete category.

We can then use Theorem 2.51 to deduce that $\mathsf{Hopf}_{K,coc}$ is protomodular. □

2.3.3 $\mathsf{Hopf}_{K,coc}$ is regular

In order to prove that $\mathsf{Hopf}_{K,coc}$ is regular, we will use an important theorem from K. Newman, [58]. Indeed, this theorem is an important step in the proof as it allows us to establish an equivalence between Hopf subalgebras and ideals that are also two-sided coideals. Before stating it, let us clarify a few objects:

Definition 2.53. *Let A be an algebra (or a Hopf algebra) in a symmetric monoidal category $(\mathcal{C}, \otimes, I, \sigma, a, l, r)$. An **A-module** is an object $B \in \mathcal{C}$ with a morphism that is called the **action**, $\triangleright : A \otimes B \to B$, such that the following diagrams commute*

$$\begin{array}{ccc} A \otimes A \otimes B & \xrightarrow{id_A \otimes \triangleright} & A \otimes B \\ {\scriptstyle m \otimes id_B} \downarrow & & \downarrow {\scriptstyle \triangleright} \\ A \otimes B & \xrightarrow[\triangleright]{} & B \end{array} \qquad \begin{array}{ccc} K \otimes B & \xrightarrow{u_A \otimes id_B} & A \otimes B \\ & {\scriptstyle l_B} \searrow & \downarrow {\scriptstyle \triangleright} \\ & & B. \end{array}$$

This means that for every a, $a' \in A$ and for every $b \in B$ we have that $a \triangleright (a' \triangleright b) = aa' \triangleright b$ and $b = u_A \triangleright b$ (where, with a slight abuse of notation, we write that $u_A : K \to A$ belongs to A).

Example 2.54. A cocommutative algebra A can be seen as an A-module where the action is the multiplication m. We can do that since, in this thesis, A is assumed to be associative and unital.

Definition 2.55. *A* ***morphism between two A-modules*** *$(B, \triangleright : A \otimes B \to B)$ and $(B', \triangleright' : A \otimes B' \to B')$ is a morphism $f : B \to B'$ in $\mathcal{C}$ such that the following diagram commutes*

$$\begin{array}{ccc} A \otimes B & \xrightarrow{id_A \otimes f} & A \otimes B' \\ \downarrow{\scriptstyle \triangleright} & & \downarrow{\scriptstyle \triangleright'} \\ B & \xrightarrow[f]{} & B'. \end{array}$$

Definition 2.56. *Let A be a Hopf algebra. An* ***A-module coalgebra*** *is a coalgebra C which is an A-module with an action $\triangleright$ satisfying $\Delta_C(a \triangleright c) = (a_1 \triangleright c_1) \otimes (a_2 \triangleright c_2)$ and $\epsilon_C(a \triangleright c) = \epsilon_A(a)\epsilon_C(c)$ for all $a \in A$ and for all $c \in C$.*

Let us now state the result, due to K. Newman [58], which plays a key role in the proof that $\mathsf{Hopf}_{K,coc}$ is regular:

Theorem 2.57. *Let A be a cocommutative Hopf algebra in the symmetric monoidal category $(Vect_K, \otimes, K, \sigma, a, l, r)$. There exists a bijective correspondence between*

$$\mathcal{S} = \{D \subset A \mid D \text{ is a Hopf subalgebra of } A\}$$

and

$$\mathcal{I} = \{I \subset A \mid I \text{ is a left ideal and a two-sided coideal of } A\}.$$

More precisely:

1. *The correspondence $\Phi_A : \mathcal{S} \to \mathcal{I}$ takes a Hopf subalgebra D of A and it associates with D the corresponding left ideal two-sided coideal*

$$\Phi_A(D) = AD^+$$

   which is the A-module generated by D^+ (i.e. AD^+ is equal to the vector subspace of A whose elements are products of elements of A and D^+).

2. *The inverse correspondence $\Psi_A : \mathcal{I} \to \mathcal{S}$ takes a left ideal two-sided coideal I of A to the corresponding Hopf subalgebra of A*

$$\Psi_A(I) = \{x \in A \mid (1_A \otimes \pi) \circ \Delta(x) = x \otimes [u_A]\}$$

where $\pi : A \to A/I$ is the canonical quotient and $[u_A] = \pi(u_A)$ is the equivalence class of the unit u_A (seen as an element of A).

Moreover, when A is a cocommutative Hopf algebra (seen as an A-module via Example 2.54), B is an A-module cocommutative coalgebra and $f : A \to B$ is a morphism of A-module, the vector space kernel $ker(f)$ is a left ideal two-sided coideal in A, and $ker(f) = A\Psi_A(HKer(f))^+$.

In summary, the theorem tells us the following two things:

1. For any surjective morphism $g : A \to B$, where A is a cocommutative Hopf algebra and B is a (cocommutative) A-module coalgebra, the vector space kernel of g naturally has a structure of left ideal two-sided coideal.

2. When I is a left ideal two-sided coideal, the quotient A/I has the structure of an A-module coalgebra and the canonical projection $\pi : A \to A/I$ is an A-module coalgebra morphism.

Remark 2.58. An interesting consequence of Newman's theorem (proven in [67][Lemma 3.5.]) is that the monomorphisms in $\mathsf{Hopf}_{K,coc}$ are the injective morphisms of Hopf algebras.

We can restrict the Theorem 2.57 to normal Hopf subalgebras in order to get a new correspondence. Indeed, the restricted correspondence is between normal Hopf subalgebra and Hopf ideals.

Definition 2.59. *Let B be a Hopf subalgebra of a cocommutative Hopf algebra A. One says that B is a **normal Hopf subalgebra** if, for any $a \in A$ and any $b \in B$, we have that $a_1 b S(a_2) \in B$.*

Corollary 2.60. *Let B be a Hopf subalgebra of a cocommutative Hopf algebra A. The following are equivalent:*

1. *B is a normal Hopf subalgebra;*

2. *$\Phi_A(B)$ is a Hopf ideal (meaning that $A/\Phi_A(B)$ is a Hopf algebra via Theorem 2.31);*

3. *the inclusion of B in A is a normal monomorphism. This means that it is the kernel of some morphism in $\mathsf{Hopf}_{K,coc}$.*

Proof. [37][Corollary 2.3.] □

We are now ready to prove that $\mathsf{Hopf}_{K,coc}$ is regular. To do so, we will prove the assumptions needed to apply Lemma 1.6 one by one, which will then establish the regularity.

Remark 2.61. Nota that in $\mathsf{Hopf}_{K,coc}$, it is shown in [23] that regular epimorphisms are exactly surjective Hopf algebra morphisms.

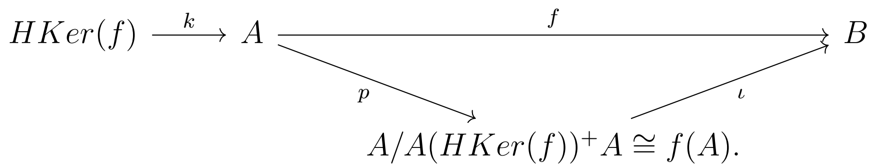
Proposition 2.62. *Any arrow in* $\mathsf{Hopf}_{K,coc}$ *factors as a regular epimorphism followed by a monomorphism. Using Remark 2.61 and Remark 2.58, this amounts to saying that any arrow in* $\mathsf{Hopf}_{K,coc}$ *factors as a surjective morphism followed by an injective morphism.*

Proof. In the proof that, for a field K of characteristic 0, $\mathsf{Hopf}_{K,coc}$ is semi-abelian ([35]), it was already shown that the (regular epi, mono) factorisation of a morphism $f : A \to B$ is obtained by taking the cokernel of $ker(f)$, the vector space kernel of f.

Using the restricted version of Newman's theorem, we know that $ker(f)$ is a Hopf ideal whenever $f : A \to B$ belongs to $\mathsf{Hopf}_{K,coc}$. This means that $ker(f) = A(HKer(f))^+$. But $A(HKer(f))^+ = A(HKer(f))^+A$ since kernels are always normal.

In particular, we get that $A/A(HKer(f))^+A = A/ker(f) \cong f(A)$. This means that the (regular epi, mono)-factorisation in $\mathsf{Hopf}_{K,coc}$ is obtained from the one in the category of vector spaces $Vect_K$.

We can then conclude that the factorisation of f is $\iota \circ p$ where p is the cokernel of k and ι is the inclusion of $f(A)$ in B



□

Proposition 2.63. *Given any regular epimorphism* $f : A \to B$ *and any object* E *from* $\mathsf{Hopf}_{K,coc}$*, the induced arrow*

$$1_E \times f : E \times A \to E \times B$$

is a regular epimorphism.

Proof. Using Remark 2.61 and the fact that the binary product in $\mathsf{Hopf}_{K,coc}$ is the tensor product, it remains to show that the induced arrow

$$1_E \otimes f : E \otimes A \to E \otimes B$$

is surjective. For any $e \otimes b \in E \otimes B$, we can use the fact that f is surjective to get an element $a \in A$ such that $f(a) = b$, and conclude by saying that

$$(1_E \otimes f)(e \otimes a) = e \otimes f(a) = e \otimes b.$$

□

In order to prove the third point of Lemma 1.6, we will have to use the two following lemmas:

Lemma 2.64. *Let $p : A \to B$ be a morphism in $\mathsf{Hopf}_{K,coc}$. Then, for any Hopf subalgebra C of B, $C = p(p^{-1}(C))$ if and only if $C = p(D)$ for some Hopf subalgebra D of A.*

Proof. [37][Lemma 2.5.]. □

Lemma 2.65. *Let $p : A \to B$ be a surjective morphism in $\mathsf{Hopf}_{K,coc}$. Then, for any Hopf subalgebra D from A, we have that*

1. *if D is normal, $p(D)$ is a normal Hopf subalgebra of B;*
2. *we have the following identity*

$$\Phi_B \circ p(D) = p \circ \Phi_A(D).$$

Proof. [37][Lemma 2.7.] □

Proposition 2.66. *In $\mathsf{Hopf}_{K,coc}$, regular epimorphisms are stable under pullbacks along split monomorphisms. Using what has been previously observed, this means that, in the following pullback diagram*

$$\begin{array}{ccc} p^{-1}(C) & \xrightarrow{\tilde{p}} & C \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \iota} \\ A & \xrightarrow[p]{} & B \end{array}$$

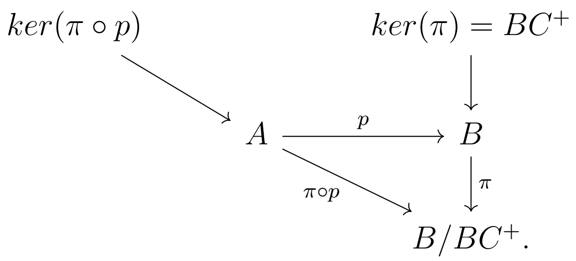
where ι is the inclusion of a Hopf subalgebra C in B, the morphism $\tilde{p}$ is surjective whenever p is a surjective morphism in $\mathsf{Hopf}_{K,coc}$.

Proof. Recall that from Lemma 2.44, $\tilde{p}$ is defined as the restriction of p. This means that $\tilde{p}$ is surjective if and only if $\tilde{p}(p^{-1}(C)) = p(p^{-1}(C))$ is equal to C, but Lemma 2.64 tells us that this is the case if and only if $C = p(D)$ for some Hopf subalgebra D of A.

We will have to build this D in order to prove the pullback stability of regular epimorphisms (= surjective morphisms).

Using Newman's theorem, we can build $\Phi_B(C) = BC^+$, the left ideal two-sided coideal defined as the B-module generated by C^+. Using Newman's theorem once again, we can build the quotient B/BC^+ which is a B-module coalgebra. By restriction of scalars along p (i.e. we define an action $a \triangleright_A m := p(a) \triangleright_B m$ for all $a \in A$ and for all $m \in B/BC^+$), B/BC^+ becomes an A-module coalgebra.

In the category $Vect_K$, we get the following diagram



where π and p are surjectives, meaning that $p(ker(\pi \circ p)) = ker(\pi) = BC^+$.

Moreover, $\pi \circ p$ is an A-module coalgebra morphism, meaning that the kernel of $\pi \circ p$ (seen as a vector space) is a left ideal two-sided coideal of A.

We can use Newman's theorem once again to define D, the Hopf subalgebra of A, as $D = \Psi_A(ker(\pi \circ p))$. Then, by Lemma 2.65 2), we can compute that

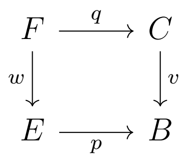
$$\Phi_B(C) = ker(\pi) = p(ker(\pi \circ p)) = p(\Phi_A(\Psi_A(ker(\pi \circ p)))) = p(\Phi_A(D)) = \Phi_B(p(D)).$$

We can complete the proof by using Newman's theorem one last time to get the needed equality $C = p(D)$. □

It follows that $\mathsf{Hopf}_{K,coc}$ is a regular category since we have shown that the conditions of Lemma 1.6 are satisfied.

2.3.4 $\mathsf{Hopf}_{K,coc}$ is exact

Using what has been done in subsection 1.2.4, it remains to show that the axiom $OA6$) holds in $\mathsf{Hopf}_{K,coc}$ to prove that it is exact. This means that for every commutative diagram



where p, q are normal epimorphisms and v, w are monomorphisms, then v is a normal monomorphism whenever w is one. In other words, the image of a normal monomorphism along a normal epimorphism has to be again a normal monomorphism. This is the case thanks to Lemma 2.65 1).

Thus, $\mathsf{Hopf}_{K,coc}$ is and exact category.

2.3.5 $\mathsf{Hopf}_{K,coc}$ is semi-abelian

Using the previous subsections, we can conclude this chapter with the following theorem:

Theorem 2.67. *The category* $\mathsf{Hopf}_{K,coc}$ *of cocommutatives Hopf algebras is semi-abelian.*

Proof. The category $\mathsf{Hopf}_{K,coc}$ has a zero object and binary coproducts via Subsection 2.3.1, is Barr-exact via Subsection 2.3.4 and is Bourn-protomodular via Subsection 2.3.2.

We can then conclude that it is semi-abelian. □

Remark 2.68. This result has been extended by A. Sciandra in [61] for cocommutative color Hopf algebras (i.e. cocommutative Hopf monoids in the category $Vect_G$ of G-graded vector spaces) if G is a finitely generated abelian group and has characteristic different from 2.

From Theorem 2.67, we can deduce the following theorem of M. Takeuchi ([65]):

Theorem 2.69. *The category* $Hopf_{K,comm,coc}$ *of commutative (i.e.* $m \circ \sigma = m$*) and cocommutatives Hopf algebras over a field* K *is an abelian category.*

Proof. [37][Theorem 2.11.]. □

Remark 2.70. Note that, as outlined in [64], A. Grothendieck first stated that the category of finite-dimensional commutative and cocommutative Hopf algebras over a field K is abelian. Theorem 2.69 is then an extension of what A. Grothendieck has done.

Chapter 3

Characteristic subobjects in semi-abelian categories

In the category of groups, the notion of characteristic subgroup, i.e. a subgroup invariant under automorphisms of the group, plays an important role. Indeed, the center and the derived subgroup of a group are examples of characteristic subgroups and they appear in several interesting results in group theory.

Moreover, the characteristic subgroup has the following properties used, for example, to deal with solvable and nilpotent groups:

1. If H is a characteristic subgroup of K, and K is a characteristic subgroup of G, then H is a characteristic subgroup of G, this is the transitivity of the property of being characteristic.
2. If H is a characteristic subgroup of K and K is a normal subgroup of G, then H is a normal subgroup of G.

Similarly, in the category of Lie algebras, there is the notion of characteristic ideal, i.e. a Lie subalgebra invariant under derivations. This similar notion also satisfies the analogue properties for Lie algebras of the ones recalled above for groups.

The aim of this chapter will be to generalize the notions of characteristic subgroup and of characteristic ideal to the context of semi-abelian categories. This is done via the notion of characteristic subobject, introduced by A. S. Cigoli and A. Montoli in [27]. In addition to that, we will have other equivalent definitions of the characteristic subobject thanks to an equivalence of categories that will be established in the following section.

3.1 Monads

In order to define the notion of characteristic subobject, we will need monads. For more details about this notion and proofs of the propositions that will be cited in this section, see [51][Chapter 6], for instance.

Definition 3.1. *Let $\mathcal{C}$ be a category. A **monad** $T = (T, \eta, \mu)$ in $\mathcal{C}$ consists of an endofunctor $T : \mathcal{C} \to \mathcal{C}$ along with two natural transformations:*

1. *the unit $\eta : 1_{\mathcal{C}} \to T$ (where $1_{\mathcal{C}}$ is the identity functor);*
2. *the multiplication $\mu : T^2 = T \circ T \to T$;*

such that the following diagrams commute:

$$\begin{array}{ccc} T^3 & \xrightarrow{T\mu} & T^2 \\ {\scriptstyle \mu T}\downarrow & & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow[\mu]{} & T \end{array} \qquad\qquad \begin{array}{ccccc} 1_{\mathcal{C}}T & \xrightarrow{\eta T} & T^2 & \xleftarrow{T\eta} & T1_{\mathcal{C}} \\ & \searrow\!\!\!\!\searrow & \downarrow{\scriptstyle \mu} & \swarrow\!\!\!\!\swarrow & \\ & & T. & & \end{array}$$

Remark 3.2. Note that a monad can be seen as monoid in the category of endofunctors of $\mathcal{C}$.

Given an adjunction, it is possible to construct a monad. Let us denote by $(F, G, \eta, \epsilon) : \mathcal{C} \to \mathcal{A}$ the adjunction where $F : \mathcal{C} \to \mathcal{A}$ is the left adjoint, $G : \mathcal{A} \to \mathcal{C}$ is the right adjoint, $\eta : 1_{\mathcal{C}} \Rightarrow G \circ F$ is the unit and $\epsilon : F \circ G \Rightarrow 1_{\mathcal{A}}$ is the counit.

Proposition 3.3. *Every adjunction $(F, G, \eta, \epsilon) : \mathcal{C} \to \mathcal{A}$ between two categories gives rise to a monad in $\mathcal{C}$. Indeed, we can define the endofunctor T as the composition $T = G \circ F : \mathcal{C} \to \mathcal{C}$, the unit on T is the natural transformation $\eta : 1_{\mathcal{C}} \to G \circ F = T$ and the composition of ϵ with F and G yields a natural transformation $\mu = G \circ \epsilon \circ F : T^2 \to T$.*

We then get $(G \circ F, \eta, G \circ \epsilon \circ F)$, the monad defined from the following adjunction $(F, G, \eta, \epsilon) : \mathcal{C} \to \mathcal{A}$.

The previous proposition shows that adjunctions and monads are connected. One can ask ourself if every monad can be defined by an appropriate adjunction.

There are two ways to answer this question. The first one, that is the one needed for our work, is due to Eilenberg-Moore, and it uses the notion of T-algebra. The second one is due to Kleisli, and it uses the notion of free algebra for a monad (for more information about this, see [51][Section 6.5.]). Let us first define what T-algebras are:

Definition 3.4. *Let $T = (T, \eta, \mu)$ be a monad in the category $\mathcal{C}$.*

*A **T-algebra** (A, h) consists of an object $A \in \mathcal{C}$ together with a map $h : T(A) \to A$ such that the following diagrams commute*

$$\begin{array}{ccc} T^2(A) = (T \circ T)(A) & \xrightarrow{Th} & T(A) \\ {\scriptstyle \mu_A}\downarrow & & \downarrow{\scriptstyle h} \\ T(A) & \xrightarrow[h]{} & A \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\eta_A} & T(A) \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle h} \\ & & A. \end{array}$$

Definition 3.5. *Let $T = (T, \eta, \mu)$ be a monad in $\mathcal{C}$ and (A, h),(A', h') two T-algebras. A **morphism of T-algebras** $f : (A, h) \to (A', h')$ is a morphism $f : A \to A'$ in $\mathcal{C}$ such that the following diagram commutes*

$$\begin{array}{ccc} T(A) & \xrightarrow{h} & A \\ {\scriptstyle T(f)}\downarrow & & \downarrow{\scriptstyle f} \\ T(A') & \xrightarrow[h']{} & A'. \end{array}$$

*We will denote the **category of T-algebra and their morphisms** by $\mathcal{C}^T$.*

We can now answer the question concerning the relationship between monads and adjunctions that we mentioned above:

Proposition 3.6. *Let $T = (T, \eta, \mu)$ be a monad in $\mathcal{C}$. There exists an adjunction $(F^T, G^T, \eta^T, \epsilon^T) : \mathcal{C} \to \mathcal{C}^T$, such that the monad $G^T \circ F^T$ defined by Proposition 3.3 from this adjunction is precisely T.*

Now, let us assume that we have an adjunction $(F, G, \eta, \epsilon) : \mathcal{C} \to \mathcal{A}$, we can build the monad $T = G \circ F$ from it. Then, the previous proposition tells us that there is an adjunction $(F^T, G^T, \eta^T, \epsilon^T) : \mathcal{C} \to \mathcal{C}^T$ such that $T = G^T \circ F^T$. We can see that the categories $\mathcal{A}$ and $\mathcal{C}^T$ are connected in a certain way. The following theorem, called **theorem of comparison of adjunctions with algebras**, clarifies their relationship:

Theorem 3.7. *Let $(F, G, \eta, \epsilon) : \mathcal{C} \to \mathcal{A}$ be an adjunction, $T = (G \circ F, \eta, G \circ \epsilon \circ F)$ the monad in $\mathcal{C}$ defined by this adjunction, and $(F^T, G^T, \eta^T, \epsilon^T) : \mathcal{C} \to \mathcal{C}^T$ the adjunction given by Proposition 3.6 for the monad T.*

*Then, there exists a unique functor $Q : \mathcal{A} \to \mathcal{C}^T$, called the **comparison functor**, such that $G^T \circ Q = G$ and $G \circ F = F^T$.*

*Moreover, if Q is an equivalence of categories (resp. isomorphism of categories), we say that the functor G is **monadic** (resp. **strictly monadic**).*

Example 3.8. It is shown in [55] that, if $\mathcal{C}$ is a semi-abelian category, the pullback functor $v^* : Pt_I(\mathcal{C}) \to Pt_J(\mathcal{C})$, induced by $v : J \to I$, is monadic with a left adjoint $v_! : Pt_J(\mathcal{C}) \to Pt_I(\mathcal{C})$ given by the pushout along v.

The example that will be of interest for us comes from Example 3.8. Indeed, in a semi-abelian category, the kernel functor $Ker_I : Pt_I(\mathcal{C}) \to \mathcal{C}$ is also monadic.

Let us elaborate further on this example, which will form the basis of everything that will follow in this chapter. In a semi-abelian category $\mathcal{C}$, the functor Ker_I has a left adjoint, given by

$$\begin{aligned}\sigma_I : \mathcal{C} &\to Pt_I(\mathcal{C})\\ A &\to (I + A, [1, 0] : I + A \to I, i_I : I \to I + A)\end{aligned}$$

where i_I is the coproduct injection. From this adjunction, Proposition 3.3 tells us that we can define a monad where the endofunctor is $Ker_I \circ \sigma_I$ and will be denoted by $I\flat(-)$. This means that for all $A \in \mathcal{C}$, the object $I\flat A$ is the kernel of the morphism $[1, 0] : I + A \to I$. Then, using Proposition 3.6, we get an adjunction

$$\mathcal{C} \underset{G^{I\flat(-)}}{\overset{F^{I\flat(-)}}{\rightleftarrows}} \mathcal{C}^{I\flat(-)} \quad (\perp)$$

where $\mathcal{C}^{I\flat(-)}$ is the category of $I\flat(-)$-algebras. The fact that Ker_I is monadic means that the categories $Pt_I(\mathcal{C})$ and $\mathcal{C}^{I\flat(-)}$ are equivalent.

Remark 3.9. As in [11], it is common to refer to the $I\flat(-)$-algebras of $\mathcal{C}^{I\flat(-)}$ as **internal I-actions**.

This equivalence of categories is important, because it will be used to define the notion of categorical semi-direct product (in the sense of [17]) and will provide equivalent definitions of characteristic subobjects.

Definition 3.10. *Let $\mathcal{C}$ be a semi-abelian category, $I \in \mathcal{C}$ and $(A, h) \in \mathcal{C}^{I\flat(-)}$.*

The ***semi-direct product*** *of (A, h) and I, denoted by $A \rtimes_h I$, is the domain H of the point $(H, p : H \to I, s : I \to H)$ corresponding to (A, h) via the equivalence $\mathcal{C}^{I\flat(-)} \cong Pt_I(\mathcal{C})$.*

Remark 3.11. Note, however, that it is possible to define the semi-direct product in every pointed finitely complete category with binary coproducts and coequalisers. Indeed, this is explained in [11][3.4], but the idea is that the semi-direct product $A \rtimes_h I$

is defined as the object part of the coequaliser of $k = ker([1,0])$ and $\iota_A \circ h$ where $[1,0] : I + A \to I$ and ι_A is the coproduct injection of A

$$I \flat A \underset{\iota_A \circ h}{\overset{k}{\rightrightarrows}} I + A \longrightarrow A \rtimes_h I.$$

Let us see, without proving it, a property linked to the semi-direct product that will be useful in the next section:

Proposition 3.12. *Let $\mathcal{C}$ be a semi-abelian category. For all $I \in \mathcal{C}$ and for all $(A, h) \in \mathcal{C}^{I\flat(-)}$, the following sequence is exact*

$$0 \longrightarrow A \xrightarrow{k} A \rtimes_h I \underset{s}{\overset{p}{\rightleftarrows}} I \longrightarrow 0.$$

This means that $k = ker(p)$ and $p = coker(k)$.

Proof. [55][Proposition 3.16.]. □

Remark 3.13. The semidirect product is an interesting notion. Indeed, it is proved in [17] that, in semi-abelian categories, internal I-actions (i.e. $I\flat(-)$-algebras by Remark 3.9) are equivalent to split extensions, via the semi-direct product construction. Recall that a split extension is a diagram

$$G \overset{l}{\rightarrowtail} X \underset{t}{\overset{q}{\rightleftarrows}} I$$

where $(X, q, t) \in Pt_I(\mathcal{C})$ and $l = ker(q)$.

3.2 The characteristic subobject

In the category of groups, a subgroup H of a group G is said to be characteristic if H is invariant under automorphisms of G, i.e. for all $\phi \in Aut(G)$, $\phi(H)$ is a subgroup of H. In other words, this means that every automorphism of G restricts to an automorphism of H. Moreover, it is well known that $Aut(G)$ classifies all the group actions on G. Recall that an action of a group B on a group G can be described as a group homomorphism $B \to Aut(G)$, meaning that that subgroup H of G is characteristic if and only if all group actions on G restrict to a group action on H.

In [27], A. S. Cigoli and A. Montoli introduced the notion of **characteristic subobject** in semi-abelian categories via the notion of internal actions (corresponding to $I\flat(-)$-algebras), that extends the classical definition for groups.

Definition 3.14. *Let $\mathcal{C}$ be a semi-abelian category, G an object from $\mathcal{C}$ and $h : H \rightarrowtail G$ a subobject of G. We say that H is **characteristic** in G if, for any object $I \in \mathcal{C}$ and for any internal action ξ of I on G (i.e. for any $I\flat(-)$-algebra (G, ξ)), the internal action ξ resticts to H. This means that there exists a (unique) internal action $\tilde{\xi}$ of I on H such that the following square commutes*

$$\begin{array}{ccc} I\flat H & \xrightarrow{1\flat h} & I\flat G \\ {\scriptstyle\tilde{\xi}}\downarrow & & \downarrow{\scriptstyle\xi} \\ H & \underset{h}{\rightarrowtail} & G. \end{array}$$

From this definition, it is straightforward to show the transitivity of the property of being characteristic:

Proposition 3.15. *Let $\mathcal{C}$ be a semi-abelian category and $h : H \rightarrowtail K$, $k : K \rightarrowtail G$ be two subobjects of G. If H is characterstic in K and K is characteristic in G, then H is characteristic in G.*

*We then say that $\mathcal{C}$ has the **transitivity of the property of being characteristic**.*

Proof. Let I be an object from $\mathcal{C}$ and ξ an internal action of I on G. We want to show that ξ restricts to H.

We know that K is characteristic in G, meaning that there exists a (unique) internal action $\tilde{\xi}$ of I on K making the right-hand square of the diagram below commute. Moreover, H is characteristic in K, meaning that there exists a (unique) internal action $\tilde{\xi}'$ making the left-hand square of the same diagram commute

$$\begin{array}{ccccc} & & \overset{1\flat k\circ 1\flat h}{\frown} & & \\ I\flat H & \xrightarrow[1\flat h]{} & I\flat K & \xrightarrow[1\flat k]{} & I\flat G \\ {\scriptstyle\tilde{\xi}'}\downarrow & & \downarrow{\scriptstyle\tilde{\xi}} & & \downarrow{\scriptstyle\xi} \\ H & \xrightarrow{h} & K & \xrightarrow{k} & G. \\ & & \underset{k\circ h}{\smile} & & \end{array}$$

The composite of two subobjects is again a subobject, so that $k \circ h : H \rightarrowtail G$ is a subobject.

We can then conclude that the arbitrary internal action ξ of I on G restricts to H via $\tilde{\xi}'$ since the outer square commutes. □

The categories we are currently working with are semi-abelian, implying that the correspondence between actions and points explained in section 3.1 holds. This allows us to write an equivalent version of Definition 3.14 using "points".

This will have the advantage of simplifying the use of the characteristic subobject. Indeed, we will be able to deduce properties of characteristic subobjects from properties of the kernel functor.

Proposition 3.16. *Let $\mathcal{C}$ be a semi-abelian category. A subobject $h : H \rightarrowtail G$ is characteristic in G if and only if for every split extension with kernel G*

$$G \overset{l}{\rightarrowtail} X \underset{t}{\overset{q}{\rightleftarrows}} I$$

(i.e. $(X, q, t) \in Pt_I(\mathcal{C})$ and $l = ker(q)$) there exists a split extension with kernel H

$$H \overset{k}{\rightarrowtail} Y \underset{s}{\overset{p}{\rightleftarrows}} I$$

and a unique morphism $(h, \phi, 1_I)$ of split extensions making the following diagram commute

$$\begin{array}{ccccc} H & \overset{k}{\rightarrowtail} & Y & \underset{s}{\overset{p}{\rightleftarrows}} & I \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \phi} & & \| \\ G & \overset{l}{\rightarrowtail} & X & \underset{t}{\overset{q}{\rightleftarrows}} & I. \end{array} \tag{3.1}$$

Proof. $\Rightarrow$: Let us consider a split extension with kernel G

$$G \overset{l}{\rightarrowtail} X \underset{t}{\overset{q}{\rightleftarrows}} I.$$

Since $\mathcal{C}$ is a semi-abelian category, we can use the equivalence between points and actions in order to get, by definition of the comparison functor, the internal action ξ of I on G (corresponding to (X, q, t)) making the following diagram commute

$$\begin{array}{ccccc} I\flat G & \xrightarrow{ker([1,0]^G)} & I + G & \underset{\iota_I^G}{\overset{[1,0]^G}{\rightleftarrows}} & I \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle [t,l]} & & \| \\ G & \xrightarrow[l]{} & X & \underset{t}{\overset{q}{\rightleftarrows}} & I \end{array}$$

where $[t, l]$ comes from the universal property of the coproduct $I + G$ and ι_I^G is the coproduct inclusion of I.

Since H is characteristic in G, we can restrict the internal action ξ to an internal action $\tilde{\xi}$ of I on H.

Once again, via the comparison functor, we can associate to the internal action $\tilde{\xi}$, the semi-direct product $Y = H \rtimes_{\tilde{\xi}} I$ with a morphism of split extensions making the following diagram commute

$$\begin{array}{ccccc} I\flat H & \xrightarrow{ker([1,0]^H)} & I+H & \underset{\iota_I^H}{\overset{[1,0]^H}{\rightleftarrows}} & I \\ {\scriptstyle\tilde{\xi}}\downarrow & & \downarrow{\scriptstyle\pi} & & \| \\ H & \underset{k}{\rightarrowtail} & H \rtimes_{\tilde{\xi}} I = Y & \underset{s}{\overset{p}{\rightleftarrows}} & I. \end{array} \tag{3.2}$$

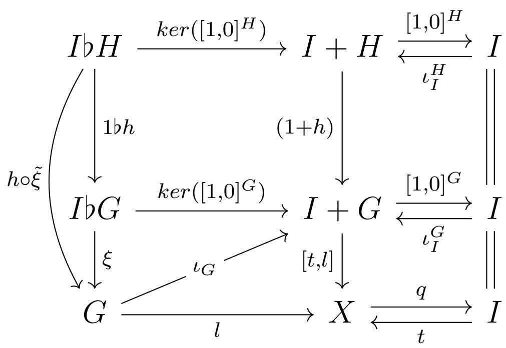
Note that, thanks to Proposition 3.12, the bottom line of the diagram is a split extension since $k = ker(p)$. Let us now show that it is the needed split extension, this means that it remains to show that there exists a unique morphism of split extensions $(h, \phi, 1_I)$ making diagram 3.1 commute. To do this, we will use Remark 3.11, where the semi-direct product was presented as the object part of the coequaliser

$$I\flat H \underset{\iota_H\circ\tilde{\xi}}{\overset{ker([1,0]^H)}{\rightrightarrows}} I+H \xrightarrow{\pi} H \rtimes_{\tilde{\xi}} I = Y.$$

We can do that since, in a semi-abelian category, we have coequalizers of kernel pairs and $(I\flat H, ker([1,0]^H), \iota_H \circ \tilde{\xi})$ is the kernel pair of $[1,0]^H$.

Let us now consider the arrow $\psi = [t,l]\circ(1{+}h) : I{+}H \to X$ where $(1{+}h) : I{+}H \to I{+}G$ is the canonical morphism. We will show that $\psi \circ ker([1,0]^H) = \psi \circ (\iota_H \circ \tilde{\xi})$, and then use the universal property of the coequaliser π to get a unique arrow from Y to X.

To do this, let us consider the following commutative diagram



and let us compute:

$$\begin{aligned} \psi \circ ker([1,0]^H) &= [t,l] \circ (1+h) \circ ker([1,0]^H) \\ &= [t,l] \circ ker([1,0]^G) \circ 1\flat h \\ &= l \circ \xi \circ 1\flat h \\ &= l \circ (h \circ \tilde{\xi}) \\ &= [t,l] \circ \iota_G \circ h \circ \tilde{\xi} \\ &= [t,l] \circ (1+h) \circ \iota_H \circ \tilde{\xi} \end{aligned}$$

where the last equality holds since the square

$$\begin{array}{ccc} H & \xrightarrow{\iota_H} & I+H \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle 1+h} \\ G & \xrightarrow[\iota_G]{} & I+G \end{array} \tag{3.3}$$

commutes.

From the universal property of the coequaliser, we have that there exists a unique morphism $\phi : Y \to X$ such that $\phi \circ \pi = \psi$. This means that we can compute

$$\phi \circ k \circ \tilde{\xi} = \phi \circ \pi \circ ker([1,0]^H) = \psi \circ ker([1,0]^H) = l \circ h \circ \tilde{\xi}.$$

But the left-hand square in the diagram 3.2 is a pullback, so that $\tilde{\xi}$ is a regular epimorphism since $\pi : I + H \to H \rtimes_{\tilde{\xi}} I$ is a regular epimorphism. Knowing this, the previous equality implies that $\phi \circ k = l \circ h$.

Moreover,

$$q \circ \phi \circ \pi = q \circ \psi = q \circ [t,l] \circ (1+h) = [1,0]^H = p \circ \pi = 1_I \circ p \circ \pi \tag{3.4}$$

and π is a (regular) epimorphism, so $q \circ \phi = 1_I \circ p$.

Finally, $\phi \circ s = \phi \circ \pi \circ \iota_I^H = \psi \circ \iota_I^H = [t,l] \circ (1+h) \circ \iota_I^H = t = t \circ 1_I$.

All these equalities allow us to say that that diagram 3.1 is commutative.

Thus, uniqueness of the morphism of split extensions comes from the fact that the pair (k, s) is jointly epimorphic since we are in a protomodular category (this follows from Propositon 1.25 and Remark 1.10).

$\Leftarrow$: Let I be an object from $\mathcal{C}$ and $\xi : I \flat G \to G$ an internal action of I on G. The equivalence between actions and points associates to this action the semi-direct product and a morphism of split extensions such that the following diagram commutes:

$$\begin{array}{ccccc} I\flat G & \xrightarrow{ker([1,0]^G)} & I+G & \underset{\iota_I^G}{\overset{[1,0]^G}{\rightleftarrows}} & I \\ {\scriptstyle \xi}\downarrow & & \downarrow & & \parallel \\ G & \xrightarrow[l]{} & G \rtimes_\xi I & \underset{t}{\overset{q}{\rightleftarrows}} & I. \end{array}$$

Again, thanks to Proposition 3.12, the bottom line of the diagram is a split extension since $l = ker(q)$. By assumption, there exists a split extension

$$H \xrightarrow{k} Y \underset{s}{\overset{p}{\rightleftarrows}} I$$

and a unique morphism of split extensions such that the following diagram commutes

$$\begin{array}{ccccc} H & \xrightarrow{k} & Y & \underset{s}{\overset{p}{\rightleftarrows}} & I \\ {\scriptstyle h}\downarrow & & \downarrow & & \| \\ G & \xrightarrow[l]{} & G \rtimes_{\xi} I & \underset{t}{\overset{q}{\rightleftarrows}} & I. \end{array}$$

Using the equivalence between points and actions, we can associate to the point (Y, p, s) an internal action of I on H. In other words, ξ restricts to H. □

Now, let us recall that in the category of groups, we have the following property: if H is a characteristic subgroup of a group K, and K is a normal subgroup of a group G, then H is a normal subgroup of G. We will see that, as for the transitivity property seen in Proposition 3.15, semi-abelian categories satisfy an analogous property.

Proposition 3.17. *Let $\mathcal{C}$ be a semi-abelian category. If H is a characteristic subobject of K and K is a normal subobjet of G, then, H is a normal subobject in G.*

Proof. [27][Proposition 2.4.]. □

Note that since, in a semi-abelian category, any object is characteristic in itself, we get the following corollary out of the previous proposition:

Corollary 3.18. *In a semi-abelian category $\mathcal{C}$, if H is characteristic in G, then H is also normal in G.*

There exists another equivalent definition for the characteristic subobject. The proof of the equivalence between this new version and Definition 3.14 relies on the following result:

Lemma 3.19. *Let $\mathcal{C}$ be a semi-abelian category. Given a split extension with kernel G*

$$G \xrightarrow{k} X \underset{s}{\overset{p}{\rightleftarrows}} I$$

and a subobject $h : H \rightarrowtail G$ such that the morphism $k \circ h$ is normal, there exists a split extension with kernel H

$$H \xrightarrow{i} Y \underset{t}{\overset{q}{\rightleftarrows}} I$$

and a normal monomorphism of split extensions such that the following diagram commutes

$$\begin{array}{ccccc} H & \overset{i}{\dashrightarrow} & Y & \underset{s}{\overset{q}{\dashrightarrow\!\!\!\!\dashleftarrow}} & I \\ \downarrow{\scriptstyle h} & & \vdots & & \| \\ G & \xrightarrow{k} & X & \underset{s}{\overset{p}{\rightleftarrows}} & I. \end{array} \tag{3.5}$$

Proof. [26][Lemma 2.6.]. □

Proposition 3.20. *Let $\mathcal{C}$ be a semi-abelian category and $h : H \rightarrowtail G$ a subobject. Then, H is characteristic in G if and only if, for all object $X \in \mathcal{C}$ such that G is normal in X, H is normal in X.*

Proof. $\Rightarrow$: If H is characteristic in G and G is normal in X, then H is normal in X by Proposition 3.17.

$\Leftarrow$: Using Lemma 3.19, we get a split extension with kernel H and a normal monomorphism of split extensions (as in diagram 3.5). This means that we can use Proposition 3.16 to get that H is characteristic in G. □

Let us now see some properties of characteristic subobjects. These will be stated without proofs, that can be found in [27].

Proposition 3.21. *Let $h : H \rightarrowtail K$ and $k : K \rightarrowtail G$ two subobjects of G in a semi-abelian category $\mathcal{C}$.*

1. *If H is characteristic in G, then every action on G induces an action on the quotient G/H.*
2. *If H is characteristic in G and K/H is characteristic in G/H, then K is characteristic in G.*
3. *If H and K are characteristic in G, then their intersection $H \wedge K$ is characteristic in G.*
4. *If H and K are characteristic in G, then their join $H \vee K$ are characteristic in G.*

3.3 Categorical properties of $\mathsf{Hopf}_{K,coc}$

We already know, from the second chapter, that the category of cocommutative Hopf algebras is semi-abelian. However, it is possible to prove that $\mathsf{Hopf}_{K,coc}$ satisfies other interesting properties, allowing us to simplify and find some new results.

3.3.1 $\mathsf{Hopf}_{K,coc}$ is action representable

This concept was introduced by F. Borceux, G. Janelidze and G.M. Kelly in [11] with the aim of providing a common categorical description to objects such as $Aut(G)$, the automorphism group of a group G and $Der(L)$, the derivations of a Lie algebra L.

Definition 3.22. *Let $\mathcal{C}$ be a pointed protomodular category. We say that $\mathcal{C}$ is **action representable** if, given any objet $X \in \mathcal{C}$, there exists a split extension*

$$0 \longrightarrow X \xrightarrow{\iota_1} \tilde{X} \underset{p_2}{\overset{\iota_2}{\rightleftarrows}} [X] \longrightarrow 0$$

*with kernel X that is **universal**. This means that given any other split extension in $\mathcal{C}$ with X as kernel*

$$0 \longrightarrow X \xrightarrow{k} A \underset{f}{\overset{s}{\rightleftarrows}} B \longrightarrow 0$$

there is a unique arrow (up to isomorphism) $\chi : B \to [X]$ (and $\tilde{\chi} : A \to \tilde{X}$) such that the following diagram commutes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{k} & A & \underset{f}{\overset{s}{\rightleftarrows}} & B & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \tilde{\chi}} & & \downarrow{\scriptstyle \chi} & & \\
0 & \longrightarrow & X & \xrightarrow{\iota_1} & \tilde{X} & \underset{p_2}{\overset{\iota_2}{\rightleftarrows}} & [X] & \longrightarrow & 0.
\end{array}$$

*The objet $[X]$ is called the **split extension classifiers** for X.*

Example 3.23. In the category Grp of groups, the split extension classifiers $[G]$ is given by the group $Aut(G)$ of automorphisms of the group G.

Before looking at the second example, let us recall that the Lie algebra derivations of a Lie algebra L is a linear map $D : L \to L$ satisfying the Leibniz rule (this means that $D(xy) = D(x)y + xD(y)$ for all $x,y \in L$). Given two derivations D_1, D_2, their commutator $[D_1, D_2] = D_1D_2 - D_2D_1$ is again a derivation. This operation makes the space $Der(L)$ of all derivations of L into a Lie algebra.

Example 3.24. In the category Lie_K of Lie algebras over a field K, the split extension classifiers $[L]$ is given by $Der(L)$, the Lie algebra derivations of the Lie algebra L.

In order to show that $\mathsf{Hopf}_{K,coc}$ is action representable, we have to introduce the notion of cartesian closed category.

Definition 3.25. *A category $\mathcal{C}$ is **cartesian closed** if it is a cartesian category (i.e. a category with binary products and a terminal object) such that, for every object $C \in \mathcal{C}$, the functors $- \times C : \mathcal{C} \to \mathcal{C}$ has a right adjoint, usually denoted by $(-)^{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ (this functor is often called the exponentiation).*

Example 3.26. The category of cocommutative coalgebras $CoAlg_{K,coc}$ is a cartesian closed category with products given by tensor product (see [40][2.3 in chapter 2]).

Proposition 3.27. *The category $\mathsf{Hopf}_{K,coc}$ is action representable.*

Proof. We will use [11][Theorem 4.4.] to prove the proposition. This theorem tells us that the category of internal groups in a cartesian closed category is always action representable, provided that this category is semi-abelian.

The proof is then complete thanks to what have been showed in chapter 2: $\mathsf{Hopf}_{K,coc}$ is semi-abelian and $\mathsf{Hopf}_{K,coc} = Grp(CoAlg_{K,coc})$ is a category of internal groups in a cartesian closed category (see Example 3.26). □

Remark 3.28. If we assume that the field K is algebraically closed and has characteristic 0, there is an explicit description of the split extension classifiers in $\mathsf{Hopf}_{K,coc}$. It is obtained by introducing the so-called "Hopf derivations" (see [34]).

3.3.2 $\mathsf{Hopf}_{K,coc}$ is locally algebraically cartesian closed

This new notion, introduced by D. Bourn, J. R. A. Gray in [16], will allow us to prove that $\mathsf{Hopf}_{K,coc}$ is algebraically coherent in the next subsection.

Definition 3.29. *A category $\mathcal{C}$ is said to be **locally algebraically cartesian closed** when, for any morphism $v : J \to I$ in $\mathcal{C}$, the pullback functor $v^* : Pt_J(\mathcal{C}) \to Pt_I(\mathcal{C})$ has a right adjoint.*

In order to show that $\mathsf{Hopf}_{K,coc}$ satisfies this definition, we will use the same type of theorem as in the proof of Proposition 3.27 (that is, a theorem using the fact that $\mathsf{Hopf}_{K,coc}$ can be seen as an internal group in the category of cocommutative coalgebras).

Theorem 3.30. *Let $\mathcal{C}$ be a cartesian closed category with pullbacks. The pullback functor along any morphism $v : J \to I$ in the category of internal groups in $\mathcal{C}$, $Grp(\mathcal{C})$, has a right adjoint.*

Proof. [38][Theorem 5.3.] □

Proposition 3.31. *The category* $\mathsf{Hopf}_{K,coc}$ *is locally algebraically cartesian closed.*

Proof. First, Example 3.26 tells us that the category $CoAlg_{K,coc}$ of cocommutatives coalgebras over the field K is a cartesian closed category. Moreover, thanks to what has been proven in [60] and [40], we know that $CoAlg_{K,coc}$ is finitely complete (in particular, it has pullbacks).

Then, Example 2.50 tells us that the category $Grp(CoAlg_{K,coc})$ is the category $\mathsf{Hopf}_{K,coc}$.

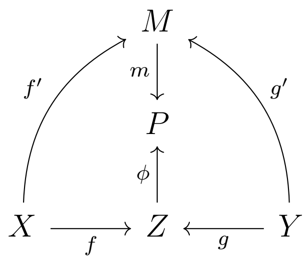
We can then conclude by Theorem 3.30 that for any morphism $v : J \to I$ in $\mathsf{Hopf}_{K,coc}$, the pullback functor v^* has a right adjoint, i.e. $\mathsf{Hopf}_{K,coc}$ is locally algebraically cartesian closed. □

3.3.3 $\mathsf{Hopf}_{K,coc}$ is algebraically coherent

The last categorical property we will see in this chapter is algebraically coherence in the sence of A. S. Cigoli, J. R. A. Gray and T. Van der Linden [25]. It will guarantee the validity of some good properties of commutators (it will be showed in the next section).

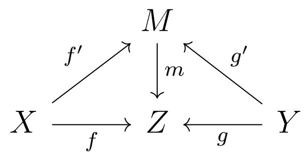
In order to define this notion, we will first recall the following:

Definition 3.32. *Let* $f : X \to Z$ *and* $g : Y \to Z$ *be morphisms in a category* $\mathcal{C}$. *We say that* f *and* g *are* ***jointly strongly epimorphic*** *if, for each commutative diagram*



where m *is a monomorphism, there exists a unique morphism* $\varphi : Z \to M$ *such that* $m \circ \varphi = \phi$.

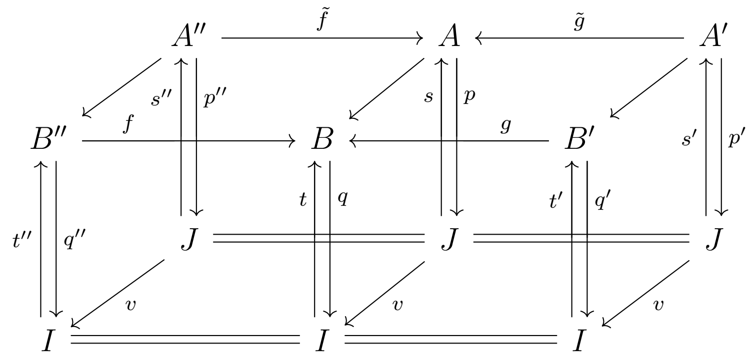
Remark 3.33. Note that, similarly to strong and extremal epimorphisms, if the underlying category $\mathcal{C}$ has pullback, f and g are jointly strongly epimorphic if and only if for each commutative diagram



where m is a monomorphism, then m is an isomorphism (we say that f and g are **jointly extremally epimorphic**).

Definition 3.34. *A category with finite limits is **algebraically coherent** if and only if, for each morphism $v : J \to I$ in $\mathcal{C}$, the pullback functor $v^* : Pt_I(\mathcal{C}) \to Pt_J(\mathcal{C})$ preserves finite limits and jointly strongly epimorphic pairs.*

Note that, saying "v^* preserves jointly strongly epimorphic pairs" means that for each diagram



where $v^*((B, q, t)) = (A, p, s)$ is the pullback of q along v ((A', p', s') and (A'', p'', s'') are constructed in the same way) and the pair (f, g) is strongly epimorphic in $Pt_I(\mathcal{C})$, then the pair $(\tilde{f}, \tilde{g}) = (v^*(f), v^*(g))$ is strongly epimorphic in $Pt_J(\mathcal{C})$.

If we now assume that the category $\mathcal{C}$ is pointed and protomodular category, the definition 3.34 can be expressed in terms of kernel functors $Ker_I : Pt_I(\mathcal{C}) \to \mathcal{C}$.

Proposition 3.35. *A pointed protomodular category $\mathcal{C}$ is algebraically coherent if and only if the kernel functor preserves finite limits and jointly strongly epimorphic pairs.*

Proof. This follows from Proposition 3.12. in [25]. □

Now, if we also assume that $\mathcal{C}$ has the binary coproducts, we get a new way of expressing the property of being algebraically coherent.

Proposition 3.36. *A pointed protomodular category with binary coproducts $\mathcal{C}$ is algebraically coherent if and only if, for any $I \in \mathcal{C}$ and for any pair $((Y, p_1, s_1), (Z, p_2, s_2))$ of objects from $Pt_I(\mathcal{C})$, the canonical arrow*

$$Ker((Y, p_1, s_1)) + Ker((Z, p_2, s_2)) \to Ker((Y, p_1, s_1) + (Z, p_2, s_2))$$

is a regular epimorphism.

Proof. In a protomodular category with binary coproducts, the functor Ker_I has a left adjoint. This means that Ker_I is a right adjoint and then it preserves finite limits.

Moreover, in a regular category a morphism f is a regular epimorphism if and only if (f, f) is a jointly strongly epimorphic pair.

This arrow means that the kernel functor preserves jointly strongly epimorphic pairs. □

We will now show that $\mathsf{Hopf}_{K,coc}$ is algebraically coherent. To do so, we have to show the following.

Theorem 3.37. *Any locally algebraically cartesian closed category is algebraically coherent.*

Proof. The pullback functors (hence, in particular, the kernel functors) always preserve limits. Moreover, since we are in a locally algebraically cartesian closed category, these functors are left adjoints, meaning that they preserve colimits, in particular they preserve jointly strongly epimorphic pairs. □

Proposition 3.38. *The category* $\mathsf{Hopf}_{K,coc}$ *is algebraically coherent.*

Proof. We know from Proposition 3.31 that $\mathsf{Hopf}_{K,coc}$ is a locally algebraically cartesian closed category. By Theorem 3.37, we can then deduce that it is algebraically coherent. □

3.4 Commutators in $\mathsf{Hopf}_{K,coc}$

In the category of groups, one can define the commutator of two normal subgroups G, G' as the (normal) subgroup $[G, G']$ generated by all elements of the form

$$[g, g'] = g^{-1}.g'^{-1}.g.g'$$

where $g \in G$ and $g' \in G'$.

Analogously, in the category of rings, for J and K two ideals of a ring A, we can define the commutator $[J, K]$ as the ideal generated by

$$J.K + K.J = \{j.k + k.j \mid j \in J, k \in K\}.$$

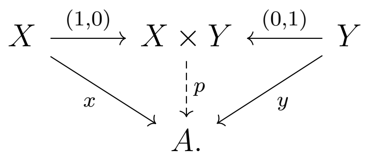
We will see that it is possible to generalize this notion, as we did for the notion of characteristic subobject. In doing so, several commutators have appeared in the literature: the Huq commutator, the Higgins commutator, the Smith commutator . . . . In some cases, these commutators coincide, allowing one to simplify certain problems in different areas of mathematics, such as, for example, in (co)homological algebra.

3.4.1 Commutators

In this section, we will fix $\mathcal{C}$ to be a semi-abelian category so that all the notions of commutators that will be introduced can be defined.

Let us start with the Huq commutator. To define it, we must first define centrality (in the sense of Huq):

Definition 3.39. *Let $x : X \to A$ and $y : Y \to A$ be two subobjects of A. We say that they* ***centralize*** *(in the sense of* ***Huq*** *[45]) if and only if x and y* ***commute****: this means that there exists an arrow p such that the following diagram commutes*

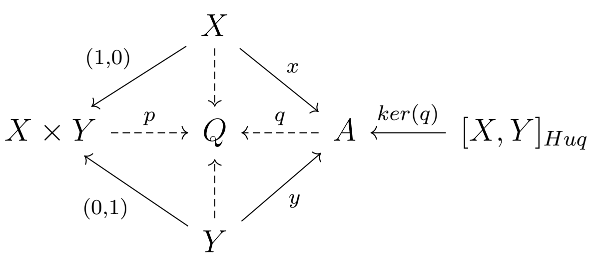


Following [9], we will call p the ***cooperator*** *of x and y.*

Example 3.40. In the category *Grp* of groups, if H and J are subgroups of a group G, then the cooperator $p : H \times J \to G$ exist if and only if $hj = jh$ for any $h \in H$ and $j \in J$.

Definition 3.41. *Let $x : X \to A$ and $y : Y \to A$ be two subobjects of A. The* ***Huq commutator*** *$[X, Y]_{Huq}$ (introduced in [45]) of x and y is the smallest normal subobject N of A such that the image of x and y commute in the quotient A/N.*

Moreover, it can be constructed as the kernel of q in the following diagram diagram



where Q is the colimit of the square of solid arrows.

Being defined as a kernel, it is always a normal subobject, even if X and Y are not normal. There is an equivalent way to construct the Huq commutator.

Proposition 3.42. *The Huq commutator can be equivalently defined as the kernel of the morphism q in*

$$\begin{array}{ccccc} & & X+Y & \xrightarrow{\Sigma_{X,Y}} & X \times Y \\ & & {\scriptstyle [x,y]}\downarrow & & \downarrow \\ [X,Y]_{Huq} & \longrightarrow & A & \xrightarrow[q]{} & Q \end{array}$$

where $\Sigma_{X,Y} = [(1,0),(0,1)] : X+Y \to X \times Y$ is the canonical map, and the commutative square is a pushout.

Proof. [53][Proposition 5.5.] □

We will now define a new commutator, which is related to the Huq commutator. Indeed, the Huq commutator is the normal closure of the Higgins commutator (see [53][Proposition 5.7.] for a proof of this result), which is defined as follows:

Definition 3.43. *Let $x : X \to A$ and $y : Y \to A$ be two subobjects of the same object A. Let us consider the canonical morphism*

$$\Sigma_{X,Y} = [(1,0),(0,1)] : X + Y \to X \times Y$$

and let

$$\sigma_{X,Y} : X \diamond Y \to X + Y$$

be its kernel.

*The **Higgins commutator** (introduced in [41] and generalized in [53]) of X and Y, denoted by $[X,Y]_{Hig}$, is the regular image of the morphism $[x,y] \circ \sigma_{X,Y}$*

$$\begin{array}{ccc} X \diamond Y & \overset{\sigma_{X,Y}}{\rightarrowtail} & X+Y \\ \downarrow & & \downarrow {\scriptstyle [x,y]} \\ [X,Y]_{Hig} & \rightarrowtail & A. \end{array}$$

Remark 3.44. The object $X \diamond Y$ is called the **binary cosmash product**, and was introduced by A. Carboni and G. Janelidze in [21] as the dual of the smash product.

In general, the Huq and the Higgins commutators do not necessarily coincide (i.e. the Higgins commutator is not necessarily a normal subobject). However, in some categories, the Huq commutator of normal monomorphisms coincide with the Higgins commutator of two normal monomorphisms. This is the (NH) condition that was introduced by A. S. Cigoli in his Ph.D. thesis [24].

Definition 3.45. *Let $\mathcal{C}$ be a semi-abelian category. We say that $\mathcal{C}$ satisfies the condition "**Normality of Higgins commutator**", denoted by (NH), if, for all pair $x : X \to A$, $y : Y \to A$ of normal subobjects of the same object A, the Higgins commutator $[X, Y]_{Hig}$ is a normal subobject of A.*

In other words, the Huq and the Higgins commutators coincide.

The last notion we will see is the centralization (in the sense of Smith). It was first introduced in [62] for congruence and generalized in [59] for internal equivalence relation.

Definition 3.46. *Let (R, r_0, r_1) and (S, s_0, s_1) be two internal equivalence relations on the same object A. If we denote by*

$$\begin{array}{ccc} R \times_A S & \xrightarrow{\pi_2} & S \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle s_0} \\ R & \xrightarrow[r_1]{} & A \end{array} \tag{3.6}$$

*the pullback of s_0 along r_1, we say that **the relations R and S centralize** (in the sens of **Smith** [62]) if there exists an arrow $\tilde{p} : R \times_A S \to A$ (called the **connector** between R and S ([15])) sending an element $xRySz$ from the pullback to $\tilde{p}(x, y, z)$ such that*

1. *$xS\tilde{p}(x, y, z)$ and $zR\tilde{p}(x, y, z)$;*
2. *$\tilde{p}(x, x, y) = y$ and $\tilde{p}(x, y, y) = x$ (it is the Mal'cev axiom);*
3. *$\tilde{p}(x, y, \tilde{p}(y, u, v)) = \tilde{p}(x, u, v)$ and $\tilde{p}(\tilde{p}(x, y, u), u, v) = \tilde{p}(x, y, v)$.*

*When R and S centralize, we can define the **Smith commutator** $[R, S]_{Smith}$ as the smallest equivalence relation on A.*

Remark 3.47. Note that, in a Mal'cev categories, the morphism p from Definition 3.39 and the morphism $\tilde{p}$ from Definition 3.46 are unique, whenever they exist ([15]). This means that, in a Mal'cev category, centralization becomes a property.

Recall that any semi-abelian category is a mal'cev category ([9][Proposition 3.1.19.]).

The centrality we have just stated and the centrality from Definition 3.39 are not independent. Indeed, if two relations centralize (in the sense of Smith), their normalization centralize (in the sense of Huq).

To be more precise, let (R, r_0, r_1) and (S, s_0, s_2) be two equivalence relations over the same object A that centralize (in the sense of Smith). Then, their respective normalization $r_1 \circ ker(r_0) : Ker(r_0) \to A$ and $s_1 \circ ker(s_0) : Ker(s_0) \to A$ centralize (in the sense of Huq)

$$\begin{array}{ccccc} Ker(r_0) & \xrightarrow{(1,0)} & Ker(r_0) \times Ker(s_0) & \xleftarrow{(0,1)} & Ker(s_0) \\ & {\scriptstyle r_1 \circ ker(r_0)} \searrow & \downarrow {\scriptstyle p} & \swarrow {\scriptstyle s_1 \circ ker(s_0)} & \\ & & A. & & \end{array}$$

Indeed, it was shown in [15] that the arrow p is equal to $\tilde{p} \circ \psi$ where $\tilde{p}$ is the connector between R and S, and $\psi : Ker(r_0) \times Ker(s_0) \to R \times_A S$ is induced by the universal property of the pullback 3.6.

The converse is not true in general. That is why, in [15], they outlined a property that can distinguish categories where the two notions of centrality coincide.

Definition 3.48. *We say that a category $\mathcal{C}$ has the* ***"Smith is Huq"*** *property ([54]), denoted by (SH), if any two equivalence relations centralize as soon as their normalizations centralize (in the sense of Huq).*

Lemma 3.49. *Any action representable category has the (SH) property.*

Proof. The proof is given in [19][Theorem 5.4.]. □

3.4.2 The commutators in $\mathsf{Hopf}_{K,coc}$

To conclude this thesis, we will compute the commutator of characteristic subobjects in the category of cocommutative Hopf algebras, that is semi-abelian, action representable and algebraically coherent.

Let us first prove the following proposition, due to A. S. Cigoli and A. Montoli in [27].

Proposition 3.50. *Let $\mathcal{C}$ be a semi-abelian algebraically coherent category and $x : X \to A$, $y : Y \to A$ two subobjects. If X and Y are characteristic in A, then, their Higgins commutator $[X, Y]_{Hig}$ is characteristic in A.*

Proof. By assymption, let $\xi : I \flat A \to A$ be an action from an object $I \in \mathcal{C}$ on A. We have to show that this action restricts to the Higgins commutator.

By the equivalence between actions and points, ξ allows us to obtain the following diagram via the comparison functor:

$$
\begin{array}{ccccc}
A\flat I & \xrightarrow{ker([1,0])} & I+A & \underset{\iota_I}{\overset{[1,0]}{\rightleftarrows}} & I \\
{\scriptstyle \xi}\downarrow & & \downarrow & & \parallel \\
A & \xrightarrow[\iota_A]{} & A \rtimes_{\xi} I & \underset{s_I}{\overset{p_I}{\rightleftarrows}} & I.
\end{array}
$$

We know that the bottom line of the previous diagram is a split extension with kernel A and $\iota_A \circ x$, $\iota_A \circ y$ are normal subobjects. We can then use Lemma 3.19 in order to get two split extensions (with kernel X and Y respectively) and two normal monomorphism of split extensions such that the following diagram commutes

$$
\begin{array}{ccccc}
X & \xrightarrow{k_1} & W & \underset{s_1}{\overset{p_1}{\rightleftarrows}} & I \\
{\scriptstyle x}\downarrow & & \downarrow & & \parallel \\
A & \xrightarrow[\iota_A]{} & A \rtimes_{\xi} I & \underset{s_I}{\overset{p_I}{\rightleftarrows}} & I. \\
{\scriptstyle y}\uparrow & & \uparrow & & \parallel \\
Y & \xrightarrow{k_2} & Z & \underset{s_2}{\overset{p_2}{\rightleftarrows}} & I.
\end{array}
$$

We know that the product $(W, p_1, s_1) \times (Z, p_2, s_2)$ in $Pt_I(\mathcal{C})$ has

$$
Ker_I((W, p_1, s_1)) \times Ker_I((Z, p_2, s_2)) = X \times Y \tag{3.7}
$$

as kernel since the functor Ker_I preserves limits when $\mathcal{C}$ is a semi-abelian category.

Let us now consider the following commutative diagram where
$N = Ker_I((W, p_1, s_1) + (Z, p_2, s_2))$:

$$
\begin{array}{ccccc}
X \diamond Y & \xrightarrow{\sigma_{X,Y}} & X+Y & \xrightarrow{\Sigma_{X,Y}} & X \times Y \\
{\scriptstyle v}\downarrow & 1 & \downarrow{\scriptstyle u} & & \parallel \\
M & \xrightarrow[j]{} & N & \xrightarrow[\alpha]{} & X \times Y \\
\vdots & & \downarrow{\scriptstyle \beta} & & \\
[X,Y]_{Hig} & \dashrightarrow & A. & &
\end{array}
\tag{3.8}
$$

Let us explain this diagram:

1. The arrow $\Sigma_{X,Y}$ is the canonical morphism introduced in the definition of the Higgins commutator and $\sigma_{X,Y}$ is its kernel.
2. The arrow

$$
u : Ker_I((W, p_1, s_1)) + Ker_I((Z, p_2, s_2)) \to Ker_I((W, p_1, s_1) + (Z, p_2, s_2))
$$

is a regular epimorphism thanks to Proposition 3.36 since we are in an algebraically coherent category.

3. The arrow α is the induced arrow $Ker_I(\chi)$ where

$$\chi : (W, p_1, s_1) + (Z, p_2, s_2) \to (W, p_1, s_1) \times (Z, p_2, s_2)$$

is the canonical morphism in $Pt_I(\mathcal{C})$ and, thanks to equation 3.7, we get the required morphism. Moreover, $j = ker(\alpha)$.

4. The arrow β is the induced arrow $Ker(\rho)$ where

$$\rho : (W, p_1, s_1) + (Z, p_2, s_2) \to (A \rtimes_\xi I, p_I, s_I)$$

is the canonical morphism in $Pt_I(\mathcal{C})$.

5. The arrow v comes from the universal property of the kernel M (since $j = ker(\alpha)$) and is such that $j \circ v = u \circ \sigma_{X,Y}$. Indeed, $\alpha \circ u \circ \sigma_{X,Y} = \Sigma_{X,Y} \circ \sigma_{X,Y} = 0$ since $\sigma_{X,Y} = ker(\Sigma_{X,Y})$.

Now that diagram 3.8 is clear, we can continue.

The arrow v is a regular epimorphism. Indeed, the two rows in the diagram 3.8 are exact and the arrow $1_{X\times Y}$ is an isomorphism, meaning that the square 1 is a pullback. Moreover, in a semi-abelian category, regular epimorphisms are stable under pullback, and u is one.

Now, let us consider the Higgins commutator $[X, Y]_{Hig}$ of X and Y in A, i.e. the regular image of the morphism $\beta \circ u \circ \sigma_{X,Y}$. We know that $\beta \circ u \circ \sigma_{X,Y} = \beta \circ j \circ v$ and v is a regular epimorphism, implying that the needed regular image is the image of the morphism $\beta \circ j$.

In others words, the commutator $[X, Y]_{Hig}$ is the regular image of $\beta \circ j$ where $\beta \circ j$ is the restriction to the kernels of a certain arrow in $Pt_I(\mathcal{C})$. But the kernel functor preserve the (regular epi, mono)-factorization, implying that the monomorphism $[X, Y]_{Hig} \to A$ is the restriction on kernels of an arrow in $Pt_I\mathcal{C}$.

We can then use Lemma 3.19 in order to get a split extension with kernel $[X, Y]_{Hig}$ and a normal monomorphism of split extensions such that the following diagram commutes;

$$\begin{array}{ccccc}
[X, Y]_{Hig} & \longrightarrow & L & \rightleftarrows & I \\
\downarrow & & \downarrow & & \| \\
A & \xrightarrow[\iota_I]{} & A\flat_\xi I & \underset{s_I}{\overset{p_I}{\rightleftarrows}} & I.
\end{array}$$

In other words, $[X, Y]_{Hig}$ is characteristic in A. □

Theorem 3.51. *Any algebraically coherent semi-abelian category satisfies the condition (NH) and the (SH) property.*

Proof. The proof follows directly from the two following theorems: [25][Theorem 6.16 d)] and [26][Theorem 6.5] i) ⇔ v). □

In [69], H. Yanagihara defined the **commutator of normal Hopf subalgebras** in $\mathsf{Hopf}_{K,coc}$. Indeed, if B and B' are two normal Hopf subalgebras of a Hopf algebra A, we can define the commutator of B and B' as the subalgebra $[B, B']$ of A generated by the elements of the form $[b, b'] = b_1 b'_1 S(b_2) S(s'_2)$ for any $b \in B$ and $b' \in B'$.

Then, thanks to the description of the Huq commutator of two normal subobjects explained in the fourth section of [37], we can prove the following proposition:

Proposition 3.52. *Let B and B' be two normal Hopf subalgebras of a Hopf algebra A. Then, the commutator of Hopf algebras is equal to the Huq commutator*

$$[B, B'] = [B, B']_{Huq}.$$

Proof. [37][Proposition 4.3.]. □

We can end this section with the following theorem:

Theorem 3.53. *In the category of cocommutative Hopf algebras*

1. *the centrality (in the sens of Huq) and the centrality (in the sense of Smith) coincide;*
2. *the Huq, the Higgins, the Smith commutators and the commutator of Hopf algebras coincide for normal Hopf algebras.*

The second point implies that the commutator of two characteristic subobjects, in the category of cocommutative Hopf algebras, is characteristic.

Proof. The category $\mathsf{Hopf}_{K,coc}$ is semi-abelian (chapter 2), action representable (3.27) and algebraically coherent (3.38).

The first point can then be proven using either Lemma 3.49 or Theorem 3.51.

The second point can be proven using Proposition 3.51 and Proposition 3.52.

We can conclude the proof using Proposition 3.50 saying that the Higgins commutator of two characteristic subobjects is characteristic. □

Conclusion

In this thesis, we observed that the category of cocommutative Hopf algebras possesses remarkable categorical properties making them quite close to groups and Lie algebras. Indeed, it is semi-abelian, but it is also action representable, locally algebraically cartesian closed, and then algebraically coherent. All these remarkable properties made it possible to show that the Higgins, Smith, Huq commutators, and the commutator of Hopf algebras coincide, allowing us to deduce that the commutator of two characteristic cocommutative Hopf algebras is also characteristic. This coincidence of commutators make it possible to investigate the notion of nilpotent and solvable Hopf algebras, analgously to what has been done in the category of groups.

Something that could also be interesting to investigate is the notion of **relative commutator** in the sense of T. Everaert and T. Van der Linden [32] in the context of cocommutative Hopf algebras. We observe that the full subcategory of abelian objects in $\mathsf{Hopf}_{K,coc}$ is a Birkhoff subcategory so that this project is certainly feasable following the line of T. Everaert and M. Gran in [31].

Another interesting problem that could be considered is the description of the non-abelian tensor product of cocommutative Hopf algebras, which has been introduced in the case of groups by R. Brown, J-L. Loday in [20]. Given that $\mathsf{Hopf}_{K,coc}$ is semi-abelian, the description of the non-abelian tensor product would offer an example of the generalized non-abelian tensor product described by D. di Micco and T. Van der Linden in [29]. Such work could provide valuable insights into the interplay between algebraic and categorical structures within this framework.

It could also be interesting to look at what happens with the commutators in color Hopf algebras, as they have been proven to be semi-abelian, action representable and locally algebraically cartesian closed (thus algebraically coherent) by A. Sciandra in [61].

Bibliography

[1] J. Adámek, J. Rosický, E. Vitale, Algebraic Theories: A Categorical Introduction to General Algebra, Cambridge University Press (2010).

[2] A. Agore, Limits of Coalgebras, Bialgebras and Hopf algebras, Proc. Amer. Math. Soc. 139 3 (2011) 855–863.

[3] N. Andruskiewitsch and J. Devoto. Extensions of Hopf algebras. Algebra i Analiz, 7 (1):22 – 61, 1995.

[4] A. Ardizzoni, An Introduction to Hopf Algebras, Course notes, University of Torino, 2021.

[5] M. Barr, Coalgebras over a commutative ring, J. Algebra 32 (1974), 600–610

[6] M. Barr, Exact categories, Lecture Notes in Mathematics 236, Springer (1971), 1-120.

[7] M. Barr, P. Grillet and D. van Osdol, Exact Categories and Categories of Sheaves, Lec. Notes in Math. 236, Springer-Verlag, Berlin, 1971.

[8] F. Borceux, Handbook of Categorical Algebra, Vol. 1, Cambridge University Press, Cambridge, 1994.

[9] F. Borceux, D. Bourn, Mal'cev, protomodular, homological and semi-abelian categories, Mathematics and its applications, Vol. 566, Kluwer, 2004.

[10] F. Borceux, Handbook of Categorical Algebra, Volume 2 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1994.

[11] F. Borceux, G. Janelidze and G Kelly, Internal object actions, Comment. Math. Univ. Carolin. 46 (2005) 235–255.

[12] D. Bourn, Mal'cev categories and fibrations of pointed objects. Appl. Categ. Structure. 4, 307-327, 1996.

[13] D. Bourn, Normalization equivalence, kernel equivalence and affine categories, in: Category Theory, Como, 1990, in: Lect. Notes Math., Vol. 1488, Springer, Berlin, 1991, pp. 43-62.

[14] D. Bourn, M. Gran, Regular, protomodular, and abelian categories, Categorical foundations, Encyclopedia Math. Appl., Cambridge Univ. Press, Vol.97 (2004), 165-211.

[15] D. Bourn and M. Gran, Centrality and normality in protomodular categories, Theory Appl. Categ. 9 8 (2002) 151–165.

[16] D. Bourn, J. R. A. Gray, Aspects of algebraic exponentiation, Bull. Belg. Math. Soc. Simon Stevin 19 (2012) 823–846.

[17] D. Bourn, G. Janelidze, Protomoduralty, descent, and semi-direct products, Theory Appl. Categories 4 (1998) 37-46.

[18] D. Bourn, G. Janelidze, Characterisation of protomodular varieties of universal algebras, Theory and Applications of Categories, Vol.11, No. 6 (2003), 143-147.

[19] D. Bourn and G. Janelidze, Centralizers in action accessible categories, Cah. Topol. Géom. Differ. Catég. L (2009), no. 3, 211–232.

[20] R. Brown, J-L. Loday, Van Kampen theorems for diagrams of spaces, Topology, Volume 26, Issue 3, 1987, 311-335.

[21] A. Carboni and G. Janelidze. Smash product of pointed objects in lextensive categories. J. Pure Appl. Algebra, 183:27–43, 2003.

[22] A. Carboni, G.M. Kelly, M.C. Pedicchio, Some remarks on Maltsev and Goursat categories, Appli. Categorical Struct. 1 (1993) 385-421.

[23] A. Chirvasitu, On epimorphisms and monomorphisms of Hopf algebras, J. Algebra 323 (2013) 1593–1606.

[24] A. S. Cigoli, Centrality via internal actions and action accessibility via centralizers, Ph.D. thesis, Università degli Studi di Milano, 2009.

[25] A. S. Cigoli, J. R. A. Gray, T. Van der Linden, Algebraically coherent categories, (2014), arXiv:1409.4219.

[26] A. S. Cigoli, J. R. A. Gray, T. Van der Linden, On the normality of Higgins commutators, J. Pure Appl. Algebra 219 (2015) 897–912.

[27] A. S. Cigoli and A. Montoli, Characteristic subobjects in semi-abelian categories, Theory and Applications of Categories, Vol.30, No. 7 (2015), 206-228.

[28] C.W. Curtis, I. Reiner, Representation theory of finite groups and associative algebras, Pure and Applied Mathematics, Vol. XI, Interscience Publishers (1962).

[29] D. di Micco and T. Van der Linden, An intrinsic approach to the non-abelian tensor product via internal crossed squares, Theory Appl. categories, Vol. 35 (2020), 1268-1311.

[30] S. Eilenberg and S. Mac Lane, General theory of natural equivalences, Trans. Amer. Math. Soc. 58 2 (1945) 231–294.

[31] T. Everaert and M. Gran, Relative commutator associated with varieties of n-nilpotent and of n-solvable groups, Archivum Mathematicum, Vol.42(4) (2006), 387-396.

[32] T. Everaert, T. Van der Linden, Relative commutator theory in semi-abelian categories, Journal of Pure and Applied Algebra, Volume 216, Issues 8–9, 2012, 1791-1806.

[33] M. Gran, An introduction to regular categories, New Perspectives in Algebra, Topology and Categories, Coimbra Mathematical Texts, Springer (2021), 113-145.

[34] M. Gran, G. Kadjo and J. Vercruysse, Split extension classifiers in the category of cocommutative Hopf algebras, Bull. Belg. Math. Society Simon Stevin 25 3 (2018) 355–382.

[35] M. Gran, G. Kadjo and J. Vercruysse, A torsion theory in the category of cocommutative Hopf algebras, Appl. Categ. Structures 24 (2016) 269–282.

[36] M. Gran, J. Rosický, Semi-abelian monadic categories, Theory and Applications of Categories, Vol. 13 (2004), 106-113.

[37] M. Gran, F. Sterck, J. Vercruysse, A semi-abelian extension of a theorem by Takeuchi, Journal of Pure and Applied Algebra 223 (2019), 4171-4190.

[38] J.R.A. Gray, Algebraic exponentiation in general categories, Appl. Categ. Structures, Vol. 20, No. 6 (2012), 543-567.

[39] A. Grothendieck, Sur quelques points d'algèbre homologique, The Tohoku Mathematical Journal. Second Series, Vol.9, 1957, 119-221.

[40] L. Grunenfelder and R. Paré, Families parametrized by coalgebras, J. Algebra 107 2 (1987) 316–375.

[41] P. J. Higgins, Groups with multiple operators. Proc. Lond. Math. Soc. 6, 366-416, 1956.

[42] F. Hofmann, Uber eine die Kategorie der Gruppen umfassende Kategorie, Sitzungsberichte Bayerische A Akad. Wissenschaften, Math. Naturw. Klasse (1960) 163–204.

[43] H. Hopf. Fundamentalgruppe und zweite Bettische Gruppe. Comment. Math. Helv., 14:257–309, 1942.

[44] J.E. Humphreys, Introduction to Lie Algebras and Representation Theory. GTM, vol. 9, 1st edn (Springer, New York, 1972)

[45] S.A. Huq, Commutator, nilpotency, and solvability in categories, Quart. J. Math. Oxford Ser. 19 2 (1968) 363–389.

[46] N. Jacobson. Lie algebras. Republication of the 1962 original. Dover Publications, Inc., New York, 1979.

[47] Z. Janelidze, The pointed subobject functor, 3×3 lemmas, and subtractivity of spans. Theory and Application of categories 23, 221-242, 2010.

[48] G. Janelidze, L. Márki, and W. Tholen, Semi-abelian categories, Journal of Pure and Applied Algebra 168 (2002), 367-386.

[49] C. Kassel, Quantum Groups, Springer Publishing Company, Incorporated, 2012.

[50] R.J. Lawrence, A universal link invariant using quantum groups, Differential geometric methods in theoretical physics (1989) 55–63.

[51] S. Mac Lane, Categories for the working mathematician, Graduate texts in mathematics, Second edition, Springer-Verlag, 1997.

[52] S. Majid, Hopf algebras for physics at the Planck scale, Classical and Quantum Gravity 5 (1999) 1587.

[53] S. Mantovani and G. Metere, Normalities and commutators. Journal of Algebra 324, 2568-2588, 2010.

[54] N. Martins-Ferreira and T. Van der Linden, A note on the "Smith is Huq" condition, Appl. Categ. Structures 20 (2012), no. 2, 175–187.

[55] A. Michel, Sous-objets caractéristiques dans les catégories semi-abéliennes, Master Thesis at Université catholique de Louvain (2019).

[56] S. Montgomery, Hopf Galois theory: A survey, Geom. Topol. Monogr. 16 2 (2009) 367–400.

[57] G. J. Murphy, C*-Algebras and Operator Theory. Academic Press, 1990.

[58] K. Newman, A correspondence between bi-ideals and sub-Hopf algebras in cocommutative Hopf algebras, J.Algebra 36 (1975) 1–15.

[59] M.C. Pedicchio, A categorical approach to commutator theory. Journal of Algebra 177, 647-657, 1995.

[60] H.E. Porst, On subcategories of the category of Hopf algebras, Arab. J. Sci. Eng. 36 6 (2011) 1023–1039.

[61] A. Sciandra, Semi-abelian condition for color Hopf algebras, Journal of Pure and Applied Algebra, Volume 228, Issue 9, 2024.

[62] J. D. H. Smith, Mal'cev varieties, Lecture Notes in Math., vol. 554, Springer, 1976.

[63] R. Street, Quantum groups a path to current algebra, Australian Mathematical Society Lecture Series, Vol. 19, Cambridge University Press, Cambridge (2007).

[64] M.E. Sweedler, Hopf algebras, Benjamin New York (1969).

[65] M. Takeuchi, A correspondence between Hopf ideals and sub-Hopf algebras, Manuscripta Math. 7 (1972) 252–270.

[66] T. Van der Linder, LMAT2215, Homological Algebra, Course notes, UCLouvain, 2022-2023.

[67] C. Vespa and M. Wambst, On some properties of the category of cocommutative Hopf algebras, North-Western European J. Mathematics (2018) 21–37.

[68] H. Yanagihara, On group theoretic properties of cocommutative Hopf algebras, Hiroshima Math. J. 9 (1979) 179–200.

[69] H. Yanagihara, On isomorphism theorems of formal groups, J. Algebra 55 (1978) 341–347.

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