

Faculté des sciences

Toric Varieties

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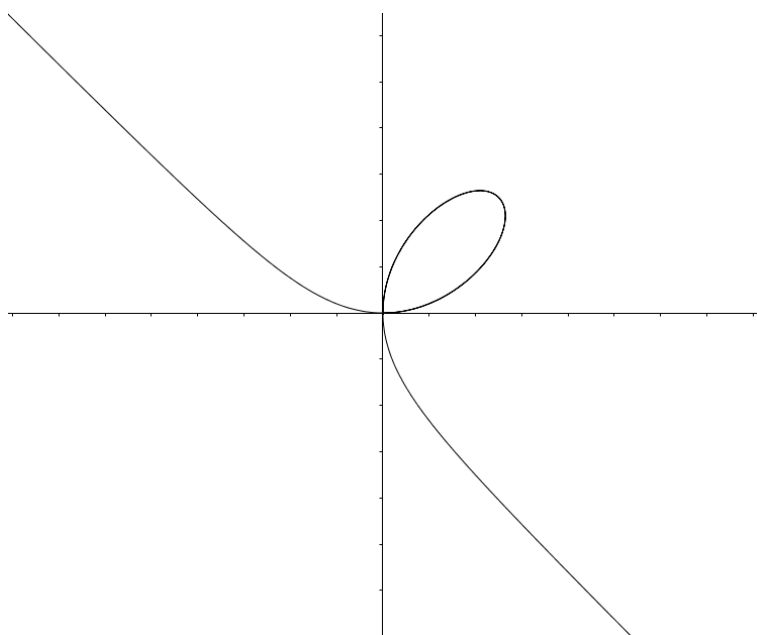
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Introduction

The study of polynomials started long ago with babylonians knowing how to find the solutions of quadratic polynomials. At the end of Middle Ages, the solutions for the cubic polynomials were found by Cardano and Tartaglia, followed by Ferrari solving quartic polynomials. The Abel-Ruffini theorem proves that is impossible for quintic polynomials. The problem of finding roots has lead Galois to developing Galois theory.

Polynomials have been also studied in analysis, using the Taylor polynomials to approximate differentiable functions. Those Taylor polynomials are the founding stone of complex analysis. Then the tools of analysis have been used to prove the fundamental theorem of algebra, stating that all nonconstant complex polynomials have roots.

The most famous examples of curves and surfaces are defined using polynomials. For example, the folium of Descartes, a curve known for looping on itself, is the set of solutions of the equation $x^3 + y^3 = 3xy$.



Those are just few examples of the interest mathematicians have for polynomials. The domain of mathematics that is the most interested in polynomials is algebraic geometry. It is an old domain of mathematics that still attracts excellent mathematicians,

such as Caucher Birkar who got the Fields Medal in 2018. Pierre Deligne, one of the Belgian mathematician we can be the most proud of, which is one of the few persons having received both the Abel prize and the Fields medal, was studying algebraic geometry.

This thesis focuses on a subdomain of algebraic geometry: toric varieties. It is a subdomain that focuses on the study of a class of varieties that have a really nice combinatorial description. Even though it is quite recent (it is considered that the first definition of a toric variety comes from [5] in 1970), it has rapidly gained in popularity.

This work divides itself in two chapters, the first one is about the main concepts from algebraic geometry that we need to study. The second chapter is developing these concepts for toric varieties, in the goal to make the reader able to introduce himself to geometric invariant theory. We decided to put the emphasis on understanding the concepts, rather than proving everything. The main reference for this work is [4].

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Chapter 1

General algebraic geometry

1.1 Affine varieties

We begin our study by going through the basics of algebraic geometry, beginning with affine algebraic geometry. Since toric varieties are endemic to complex varieties. Let us make a small precision on terminology.

Remark 1.1. We suppose the \mathbb{C} -algebras to always be associative and commutative and we suppose all rings to be unitary and commutative.

When doing geometry, we first need to know what a variety is. Basically, it is a subset of the affine space that corresponds to an ideal of a polynomial ring.

Definition 1.2. *Let I be an ideal $I \subseteq \mathbb{C}[X_1, \dots, X_n]$, there is a corresponding affine subset $\mathcal{V}(I) \subset \mathbb{C}^n$ defined by*

$$\mathcal{V}(I) := \{x \in \mathbb{C}^n \mid \forall P \in I, P(x) = 0\}.$$

*We call sets generated this way **affine algebraic varieties**.*

To make the text smoother, we omit the word *algebraic* since there is no confusion with other kind of varieties.

Dually, we can also associate an ideal to an affine variety.

Definition 1.3. *Let V be an affine subset $V \subseteq \mathbb{C}^n$ there is a corresponding ideal $\mathcal{I}(V)$ defined by*

$$\mathcal{I}(V) := \{P \in \mathbb{C}[X_1, \dots, X_n] \mid \forall x \in V, P(x) = 0\}.$$

Affine varieties have the required properties to form the closed sets of a topology. This result and the next one can be found in [3, Thm 2 p.173] or [13].

Definition 1.4. The **Zariski topology** on \mathbb{C}^n is the topology where the closed sets are affine varieties. The Zariski topology on an affine variety is the subspace Zariski topology. When an affine variety is connected in Zariski topology, we say it is **irreducible**.

The next result is Hilbert's Nullstellensatz, a central result in commutative algebra and algebraic geometry. It makes the link between ideals of polynomial rings and affine varieties.

Theorem 1.5. (Nullstellensatz) Let I be an ideal of $\mathbb{C}[X_1, \dots, X_n]$, then

$$\mathcal{I}(\mathcal{V}(I)) := \sqrt{I} = \{P \in \mathbb{C}[X_1, \dots, X_n] \mid P^n \in I \text{ for some } n \in \mathbb{N}^*\}.$$

From the Nullstellensatz we know that radical ideals are in bijective correspondence with affine varieties. Since a ring can be quotiented by an ideal, it makes sense to associate to an affine variety the quotient of the polynomial ring by the ideal associated to this variety.

Definition 1.6. Let $V \subset \mathbb{C}^n$ be an affine variety, the **coordinate ring** of V is:

$$\mathbb{C}[V] := \mathbb{C}[X_1, \dots, X_n] / \mathcal{I}(V).$$

In some references such as [7, p.35] or [11, Def 2.1 p.12], the coordinate ring is written $\Gamma[V]$.

Two polynomials have the same image in the coordinate ring if and only if their difference is zero on V . Therefore, one can think of the elements of the coordinate ring as polynomial functions from V . Such intuition is reinforced by the fact that $\mathbb{C}[\mathbb{C}^n] = \mathbb{C}[X_1, \dots, X_n]$.

Example 1.7. The cuspidal curve is the curve with equation $Y^2 = X^3$. Using this equation, we may describe the affine variety with the following parametrization: $\{(t^2, t^3) \mid t \in \mathbb{C}\}$. The ideal is directly described by the defining equation.

The coordinate ring is $\mathbb{C}[X, Y] / \langle Y^2 - X^3 \rangle$. Using the parametrization, we rewrite the coordinate ring as $\mathbb{C}[t^2, t^3]$.

Another important notion is the notion of morphism between varieties.

Definition 1.8. If U, V are two affine varieties, a function $\phi : U \rightarrow V$ is a **regular map** or **morphism** if we can find $f_1, \dots, f_m \in \mathbb{C}[X_1, \dots, X_n]$ such that :

$$\phi(P) = (f_1(P), \dots, f_m(P)).$$

Regular maps are strongly related to coordinate rings. Using the composition of functions, we are able to induce a \mathbb{C} -algebra morphism from a regular map.

Definition 1.9. *A regular map $\phi : U \rightarrow V$ induces a \mathbb{C} -algebra morphism:*

$$\phi^* : \mathbb{C}[V] \rightarrow \mathbb{C}[U], f \mapsto f \circ \phi.$$

This can be represented by the following diagram:

$$\begin{array}{ccc} U & \xrightarrow{\phi} & V \\ & \searrow \phi^*(f)=f \circ \phi & \downarrow f \\ & & \mathbb{C}. \end{array}$$

The next theorem establishes the link between affine varieties and their coordinate rings. It was discussed in [14].

Theorem 1.10. *There is an antiequivalence of categories between the category of affine varieties with regular maps and the category of finitely generated with no nonzero nilpotents \mathbb{C} -algebras with \mathbb{C} -algebra morphisms.*

The functor from affine varieties to \mathbb{C} -algebras is defined by: $V \mapsto \mathbb{C}[V]$ and $\phi \mapsto \phi^$.*

A finitely generated torsion-free algebra is called a **reduced** algebra.

Given R a reduced \mathbb{C} -algebra, the previous theorems tells us that there is a unique, up to isomorphism, affine variety V such that R is the coordinate ring of V . In other words, $R = \mathbb{C}[V]$.

Definition 1.11. *Let R be a reduced algebra. We define the **spectrum** $\text{Spec}(R)$ of a \mathbb{C} -algebra as the unique affine variety such that:*

$$\mathbb{C}[\text{Spec}(R)] := R.$$

To make things clearer, let us see how to compute this spectrum. Let $\{a_1, \dots, a_n\}$ be a generating set of R . Then the map $f : \mathbb{C}[X_1, \dots, X_n] \rightarrow R$, induced by $f(X_i) = a_i$ is a surjective \mathbb{C} -algebra morphism. By the first isomorphism theorem, we know that $\mathbb{C}[X_1, \dots, X_n]/\ker(f) \simeq \text{Im}(f) = R$. Since $\ker(f)$ is an ideal of $\mathbb{C}[X_1, \dots, X_n]$, this proves that $\mathcal{V}(\ker(f)) = \text{Spec}(R)$.

Let us now learn about the concepts of normal ring and normal varieties. Working with normal varieties will be an important hypothesis for toric varieties as we will see later, for example at Theorem 2.65. One may note that normal varieties include smooth varieties, even though we won't see what smoothness means for algebraic varieties. Nonetheless, it corresponds to the notion of smoothness for differential manifolds.

Definition 1.12. Let R be an integral domain with field of fractions K . Then R is **normal** if all elements of K that are roots of monic polynomials in $R[x]$, those elements are in R .

An irreducible affine variety is **normal** if its coordinate ring is normal.

Example 1.13. A basic result of commutative algebra states that UFD are normal. Therefore, the variety \mathbb{C}^n is normal since its coordinate ring is $\mathbb{C}[X_1, \dots, X_n]$ which is an UFD.

The cuspidal curve is not normal: for that, we have to find a monic polynomial in $\mathbb{C}[t^2, t^3][X]$ that has a root in $\mathbb{C}(t^2, t^3) \setminus \mathbb{C}[t^2, t^3] = \mathbb{C}[t] \setminus \mathbb{C}[t^2, t^3]$. We know that t is an element satisfying this condition, therefore the monic polynomial $X^2 - t^2$ concludes the proof since it has t as a root.

We will next see the process of **localization**, that allows us to describe the process of boring holes in a variety. More rigorously, one could wish to see some Zariski open set U as an affine variety. Let us suppose that $U = \mathbb{C} \setminus \mathcal{V}(f)$ for some polynomial f . Then $f(x_1, \dots, x_n)$ is nonzero if and only if $(x_1, \dots, x_n) \in U$. In other words, the equation $Yf(x_1, \dots, x_n) = 1$ has solutions for Y only when $(x_1, \dots, x_n) \in U$.

Therefore, we can see U as an affine variety of \mathbb{C}^{n+1} . Instead of localizing open subsets of the affine space, we can do the same procedure for any affine variety. This leads to the next theorem.

Theorem 1.14. Let V be an affine algebraic variety, $f \in \mathbb{C}[V]$, then

$$V_f := V \setminus \mathcal{V}(f) \cong \mathcal{V}(\mathcal{I}(V), 1 - fY),$$

where $1 - fY \in \mathbb{C}[X_1, \dots, X_n, Y]$. Moreover, the coordinate ring of V_f is :

$$\mathbb{C}[V_f] = \{g/f^n | g \in \mathbb{C}[V], n \in \mathbb{N}\} =: \mathbb{C}[V]_f.$$

Example 1.15. The n -dimensional torus is defined as a variety isomorphic to $(\mathbb{C}^*)^n = \mathbb{C} \setminus \mathcal{V}(X_1 \cdots X_n)$. Using Theorem 1.14 we can compute that its coordinate ring is $\mathbb{C}[X_1, \dots, X_n, (X_1 \cdots X_n)^{-1}] = \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$.

The tori developed here are not the usual tori (which are $(S^1)^n$), but one can see that those two are homotopically equivalent for the Euclidean topology.

With this discussion about localization, we are now able to give a more general context of regularity of functions than the one of Definition 1.8. In particular, it allows us to work with rational functions instead of only polynomials.

Definition 1.16. Let V be an affine variety, $U \subset V$ a Zariski open subset and $\phi : U \rightarrow \mathbb{C}$, we say that ϕ is **regular** if for all $p \in U$, there exists $f_p \in \mathbb{C}[V]$ such that $p \in V_{f_p} \subseteq U$ and $\phi|_{V_{f_p}} \in \mathbb{C}[V_{f_p}]$. The set of regular functions $\phi : U \rightarrow \mathbb{C}$ is denoted $\mathcal{O}_V(U)$.

One can see that regular functions on a given set form a ring, taking pointwise addition and multiplication. The next proposition gives the computation of the ring of regular functions in a simple case and makes the connection with Definition 1.8.

Proposition 1.17. Let V be an affine variety and $f \in \mathbb{C}[V]$, then $\mathcal{O}_V(V_f) = \mathbb{C}[V_f]$. In particular, when $f = 1$ we obtain $\mathcal{O}_V(V) = \mathbb{C}[V]$.

Finally, we can extend the notion of morphism to open subsets of varieties.

Definition 1.18. For $i \in \{1, 2\}$, let U_i be open subsets of affine varieties V_i . A function $\phi : U_1 \rightarrow U_2$ is a **morphism** or **regular map** if the map $\varphi \mapsto \varphi \circ \phi$ defines a morphism $\phi^* : \mathcal{O}_{V_2}(U_2) \rightarrow \mathcal{O}_{V_1}(U_1)$.

Next we need to discuss local rings, which are nicely connected to the rings of regular functions.

Definition 1.19. Let p be a point of a variety X , the **local ring** of X at p is

$$\mathcal{O}_{X,p} := \{f : U \rightarrow \mathbb{C} \mid p \in U \subseteq X, f \in \mathcal{O}_X(U)\} / \sim$$

where for $i = 1, 2$ and $f_i : U_i \rightarrow \mathbb{C}$, $f_1 \sim f_2$ is equivalent to the existence of an open neighbourhood $U \subseteq U_1 \cap U_2$ of p such that $f_1|_U = f_2|_U$.

Since rational functions are holomorphic functions, we know from complex analysis that there is only one way to continue the functions on a larger domain if they are equal on a nonempty Euclidean open domain. Therefore we are sure that the quotient glues together only rational functions having the same expression but having different domain of definitions.

Since functions in the local rings are regular on a neighbourhood of p , if a function is regular on a whole open set U then it is in all local rings. Conversely, if a function is regular on U then it is regular on the neighbourhood U for all points $p \in U$. This can be expressed as

$$\bigcap_{p \in U} \mathcal{O}_{V,p} = \mathcal{O}_V(U).$$

To finish this small discussion/reminder about affine varieties, we finish by computing the coordinate ring of the product of affine varieties. Using Theorem 1.10, we know that we just need to compute the coproduct of the coordinate rings in the category of reduced algebras, which are tensor products, such as described in [9, Prop 6.1 p.630].

Lemma 1.20. *Let V_1, V_2 be affine algebraic varieties, then $\mathbb{C}[V_1 \times V_2] = \mathbb{C}[V_1] \otimes \mathbb{C}[V_2]$.*

1.2 Abstract varieties

Abstract varieties are varieties that do not come with a natural embedding in the affine (or projective) space. Those will be defined gluing back affine varieties together. We therefore need to find a notion able to describe the idea of gluing varieties together: sheaves of rings will work.

Sheaves of rings consists of a collection of rings indexed by the open sets of a topological space. It may help the intuition to consider the elements of the rings as functions defined on the corresponding open set, in the same way as the coordinate ring describes the functions on a variety.

One may ask to be able to restrict functions on smaller domains of definitions, the operation ρ will play this role. We also want that two functions are equal if and only if they are equal locally, this will be described by the exactness at $\mathcal{F}(U)$ in the exact sequence below. Finally, when we have functions defined locally and equal on the intersections of their domain, we may want to glue those functions defined locally to get one function defined globally. This will be described by the exactness at $\prod_{\alpha} \mathcal{F}(U_{\alpha})$.

Definition 1.21. *A **sheaf of rings** \mathcal{F} over a topological space X consists of*

- *for every open subset $U \subseteq X$, a ring $\mathcal{F}(U)$.*
- *for every couple of open subsets $V \subseteq U \subseteq X$, a morphism, called restriction map, $\rho_{U,V} : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$.*

such that

- *$\rho_{U,U}$ is the identity map.*
- *$\rho_{V,W} \circ \rho_{U,V} = \rho_{U,W}$ whenever $U \subseteq V \subseteq W$.*
- *for $\{U_{\alpha}\}$ an affine cover of $U \subseteq V$, the following sequence is exact:*

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{(\rho_{U,U_{\alpha}})} \prod_{\alpha} \mathcal{F}(U_{\alpha}) \xrightarrow[\begin{smallmatrix} (\rho_{U_{\beta},U_{\alpha} \cap U_{\beta}}) \\ (\rho_{U_{\beta},U_{\alpha} \cap U_{\beta}}) \end{smallmatrix}]{(\rho_{U_{\alpha},U_{\alpha} \cap U_{\beta}})} \prod_{\alpha, \beta} \mathcal{F}(U_{\alpha} \cap U_{\beta})$$

where exactness in $\prod_{\alpha} \mathcal{F}(U_{\alpha})$ means that an element (f_{α}) is in the image of $(\rho_{U,U_{\alpha}})$ if and only if $\rho_{U_{\alpha},U_{\alpha} \cap U_{\beta}}(f_{\alpha}) = \rho_{U_{\beta},U_{\alpha} \cap U_{\beta}}(f_{\beta})$ for all α, β .

The next proposition confirms our intuitions about the notion of sheaf.

Proposition 1.22. *Let V be an affine variety, then the mapping $U \mapsto \mathcal{O}_V(U)$ with the usual restriction map of functions forms a sheaf of rings over V . This sheaf is called the **structure sheaf** of the affine variety V .*

Let us now see the definition of an abstract variety. The main ingredients are affine varieties that are glued together along some open subsets. Topologies carry well with this procedure (disjoint unions and quotients), therefore the Euclidean and the Zariski topology of varieties are well-defined. The reader should take a particular look at the second bullet appearing in the definition, this bullet assures that the different gluings are compatible with each others. The definition finishes by embedding the affine parts of the variety into the variety itself.

Definition 1.23. *Let $\{V_\alpha\}_\alpha$ be a finite collection of affine varieties such that for each pair α, β we have Zariski open subsets $V_{\beta\alpha} \subseteq V_\alpha$ and isomorphisms $g_{\alpha\beta} : V_{\beta\alpha} \simeq V_{\alpha\beta}$ such that:*

- $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$ for all pairs α, β .
- $g_{\beta\alpha}(V_{\beta\alpha} \cap V_{\gamma\alpha}) = V_{\alpha\beta} \cap V_{\alpha\gamma}$ for all triplets α, β, γ .

Let $Y := \coprod_\alpha V_\alpha$ and \sim a relation on Y defined by $x \sim y$ if and only if for $x \in V_\alpha$ and $y \in V_\beta$, $x = g_{\alpha\beta}(y)$.

Then let $X := Y / \sim$ equipped with the quotient topology. The set X with all the above data is called an **abstract variety**.

For each α let $U_\alpha := \{[a] \in X | a \in V_\alpha\}$ and $h_\alpha : V_\alpha \rightarrow X, a \mapsto [a]$. Then h_α is an homeomorphism $V_\alpha \simeq U_\alpha$.

Let us develop a famous example: the projective plane.

Example 1.24. To cover the projective plane $\mathbb{P}^n(\mathbb{C})$, we will need $n + 1$ copies of \mathbb{C}^n , indexed by $i \in \{0, \dots, n\}$. Coordinates in \mathbb{C}_i^n are expressed as $(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$. One can think \mathbb{C}_i^n as the hyperplane of equation $x_i = 1$ in \mathbb{C}^{n+1} .

The usual definition requires us to glue together elements that are in the same one-dimensional vector space. Therefore $(x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n)$ must be glued to $(\frac{x_0}{x_j}, \dots, \frac{x_{i-1}}{x_j}, \frac{1}{x_j}, \frac{x_{i+1}}{x_j}, \dots, \frac{x_{j-1}}{x_j}, 1, \frac{x_{j+1}}{x_j}, \dots, \frac{x_n}{x_j})$. Let us now define the gluing g_{ij} and the open subsets \mathbb{C}_{ji}^n . To be able to divide by x_j , we need to assume $x_j \neq 0$. The particular case implies $x_i \neq 0$ for the image.

Thus we define $\mathbb{C}_{ji}^n := \mathbb{C}_i^n \setminus \mathcal{V}(x_j)$ and

$$g_{ji} : \mathbb{C}_{ji}^n \simeq \mathbb{C}_{ij}^n, (x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \mapsto \left(\frac{x_0}{x_j}, \dots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \dots, \frac{x_{i-1}}{x_j}, \frac{1}{x_j}, \frac{x_{i+1}}{x_j}, \dots, \frac{x_n}{x_j} \right).$$

It is clear that $g_{ji} = g_{ij}^{-1}$. For the second condition, it is equivalent to show $g_{ji}(\mathbb{C}_i^n \setminus \mathcal{V}(x_j x_k)) = \mathbb{C}_j^n \setminus \mathcal{V}(x_i x_k)$ which is evident.

Let us now see an example of variety that is **not** Hausdorff in Euclidean topology.

Example 1.25. Let $U_1 = U_2 = \mathbb{C}$ and $U_{12} = U_{21} = \mathbb{C}^*$. Now let the isomorphism be the identity. The resulting topological space will be the complex plane with the origin splitted in two points. The Hausdorff condition for the variety fails to work when tested with the two special points.

In the rest of this thesis we will assume that the variety we are working with are Hausdorff in the Euclidean topology.

We may now extend the notion of normality to abstract varieties.

Definition 1.26. A variety X with a cover consisting of V_α is **normal** if all V_α are normal.

Now let us finish this section by adapting the notions related to regular maps to abstract varieties.

Definition 1.27. Let $X = \bigcup_\alpha U_\alpha$ be a variety with affine open cover by U_α and $Y = \bigcup_\beta V_\beta$ be a variety with affine open cover by V_β . A morphism $\varphi : X \rightarrow Y$ is a Zariski continuous mapping such that the restrictions

$$\varphi|_{U_\alpha \cap \varphi^{-1}(V_\beta)} : U_\alpha \cap \varphi^{-1}(V_\beta) \rightarrow V_\beta$$

are morphisms in the sense of Definition 1.18.

In simple words, a morphism of abstract varieties is a function which is locally a morphism. The notion of regular map will be adapted in the same way.

Definition 1.28. Let X be a variety, $U \subset X$ a Zariski open subset and $\phi : U \rightarrow \mathbb{C}$. Let $W_\alpha := h_\alpha^{-1}(U \cap U_\alpha)$ then we say that ϕ is **regular** if $\phi \circ h_\alpha|_{W_\alpha} : W_\alpha \rightarrow \mathbb{C}$ is regular in the sense of Definition 1.18 for all alpha. The set of regular functions $\phi : U \rightarrow \mathbb{C}$ is denoted $\mathcal{O}_X(U)$.

Now we may see why the notion of sheaf is the good adaptation of the notion of coordinate ring.

Proposition 1.29. An abstract variety X with the mapping $U \mapsto \mathcal{O}_X(U)$ with the usual restriction map of function forms a sheaf of rings over X . This sheaf is called the **structure sheaf** of the variety X .

Finally, we adapt the notion of local ring, which in fact is defined in exactly the same words as for affine varieties.

Definition 1.30. Let p be a point of a variety X , the **local ring** of X at p is

$$\mathcal{O}_{X,p} := \{f : U \rightarrow \mathbb{C} \mid p \in U \subseteq X, f \in \mathcal{O}_X(U)\} / \sim$$

where for $i = 1, 2$ and $f_i : U_i \rightarrow \mathbb{C}$, $f_1 \sim f_2$ is equivalent to the existence of $U \subseteq U_1 \cap U_2$ open neighbourhood of p such that $f_1|_U = f_2|_U$.

1.3 Good categorical quotient and (almost) geometric quotient

A classical procedure in mathematics is taking quotients. In fact we used quotients for many definitions in this work: coordinate ring, abstract variety, local ring, ... One may ask how quotients work for algebraic varieties. For example, the usual definition of $\mathbb{P}^n(\mathbb{C})$ involves quotienting $\mathbb{C}^{n+1} \setminus \{0\}$ under the scalar multiplication action of \mathbb{C}^* . In other words, $\mathbb{P}^n(\mathbb{C}) = (\mathbb{C}^{n+1} \setminus \{0\}) / (x \sim \lambda x)$ where $\lambda \in \mathbb{C}^*$.

In this section we will take a look at the different notions of quotient in algebraic geometry.

Remark 1.31. For this section, when we suppose that a group G acts on a variety X , we assume that for all $g \in G$, $\phi_g(x) := g \cdot x$ defines a morphism $\phi_g : X \rightarrow X$.

We will first see the notion of induced action, which is transforming the action on the variety into an action on the coordinate ring.

Definition 1.32. Let G act on an affine variety $X = \text{Spec}(R)$, then for all $g \in G$, ϕ_g induces an homomorphism $\phi_g^* : R \rightarrow R$. The **induced action** of G on R is given by:

$$g \cdot f := \phi_{g^{-1}}^*(f).$$

Therefore, $g \cdot f(x) = \phi_{g^{-1}}^*(f)(x) = f(\phi_{g^{-1}}(x)) = f(g^{-1} \cdot x)$.

Notation 1.33. The set of G -orbits is written $X/G := \{G \cdot x \mid x \in X\}$.

Definition 1.34. The ring of polynomials that are constant on G -orbits is called the **ring of invariants** $R^G := \{f \mid g \cdot f = f \text{ for all } g \in G\}$.

Naively, one may want to define X/G as the quotient of X by the action of G , with coordinate ring R^G since elements of R^G are elements of R that are constant on G -orbits, in other words those are the functions on X that are well-defined in the quotient X/G .

First of all, the ring R^G may not be finitely generated. A paper from Nagata [10] explicitly constructs such a counterexample. Therefore $\text{Spec}(R^G)$ may not be well-defined. Let us see now an example where $\text{Spec}(R^G)$ differs from X/G .

Example 1.35. Let $X = \mathbb{C}^4$ and \mathbb{C}^* act on \mathbb{C}^4 via $\lambda \cdot (a_1, a_2, a_3, a_4) = (\lambda a_1, \lambda a_2, \lambda^{-1} a_3, \lambda^{-1} a_4)$. It is fairly evident that the ring of invariants is $\mathbb{C}[x_1 x_3, x_2 x_4, x_1 x_4, x_2 x_3]$. Multiplying together the first two generators of this ring gives the same result as multiplying the two others and no other relation appears. Therefore $\text{Spec}(\mathbb{C}[x_1 x_3, x_2 x_4, x_1 x_4, x_2 x_3]) = \mathcal{V}(xy - zw) \subset \mathbb{C}^4$.

This description of the ring of invariants allows us to define a map

$$\phi : \mathbb{C}^4 / \mathbb{C}^* \rightarrow \mathcal{V}(xy - zw), (a_1, a_2, a_3, a_4) \mapsto (a_1 a_3, a_2 a_4, a_1 a_4, a_2 a_3).$$

It is easy to see that this map is invariant on \mathbb{C}^* -orbits. Let us take (x, y, z, w) such that $xy - zw = 0$, then one may compute that $\phi(x, w, 1, \frac{z}{x}) = (x, y, z, w)$ which proves that ϕ is surjective.

Nonetheless, multiple orbits are sent to 0, for example $\phi(1, 0, 0, 0) = (0, 0, 0, 0) = \phi(0, 0, 0, 0)$ but it is clear that $(1, 0, 0, 0)$ is not in the same orbit as 0! This proves that ϕ is not injective.

Now that we know that the spectrum of the ring of invariants may differ from the set of G -orbits, we may try another approach starting from $\text{Spec}(R^G)$.

Lemma 1.36. *Let G act on $X = \text{Spec}(R)$ such that R^G is a finitely generated \mathbb{C} -algebra and let $\pi : X \rightarrow Y = \text{Spec}(R^G)$ be the morphism induced by the inclusion $R^G \subset R$. Then:*

- *If X is irreducible then Y is irreducible.*
- *Given any diagram*

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Z \\ & \searrow \pi & \nearrow \bar{\phi} \\ & & Y \end{array}$$

where ϕ is a morphism of varieties such that $\phi(g \cdot x) = \phi(x)$ for all $g \in G$ and $x \in X$, then there exists a unique morphism $\bar{\phi}$ such that the diagram commutes.

Proof. For the first part of the proof, let us remark that the coordinate ring of a variety is an integral domain if and only if the variety is irreducible. The proof follows since $R^G \subset R$.

For the second part of the proof, let us write $Z = \text{Spec}(S)$. Then $\phi^* : S \rightarrow R$ is constant on G -orbits since ϕ is. Thus we have $\phi^*(S) \subseteq R^G$. Since the inclusion of R^G in R is

given by π^* , we may decompose $\phi^* = \pi^* \circ \bar{\phi}^*$. When we look at the induced morphisms of varieties, we get the expected commutative diagram. \square

We are now ready to introduce ourselves to the notion of a good categorical quotient, a construction that looks like a quotient in the category of algebraic varieties.

Definition 1.37. *Let G act on a variety X and $\pi : X \rightarrow Y$ be a morphism constant on G -orbits. Then π is a **good categorical quotient** if:*

- *For $U \subseteq Y$ open, the natural map $\mathcal{O}_Y(U) \rightarrow \mathcal{O}_X(\pi^{-1}(U))$ induces an isomorphism $\mathcal{O}_Y(U) \simeq \mathcal{O}_X(\pi^{-1}(U))^G$.*
- *If W is closed and G -invariant, then $\pi(W)$ is closed.*
- *If W_1, W_2 are closed, disjoint and G -invariant in X , then $\pi(W_1)$ and $\pi(W_2)$ are disjoint.*

If those conditions are satisfied, we write $Y = X // G$.

The first bullet is the generalization of the assumption of Lemma 1.36 for abstract varieties, where the coordinate ring is not defined. The second bullet assures that the good categorical quotient behaves like a quotient for Zariski's topology and the last bullet avoid quotienting too much.

Let us now establish some properties of good categorical quotients .

Lemma 1.38. *Let $\pi : X \rightarrow X // G$ be a good categorical quotient . Then:*

- *Given any diagram*

$$\begin{array}{ccc}
 X & \xrightarrow{\phi} & Z \\
 \searrow \pi & & \nearrow \bar{\phi} \\
 & X // G &
 \end{array}$$

where ϕ is a morphism of varieties such that $\phi(g \cdot x) = \phi(x)$ for all $g \in G$ and $x \in X$, then there exists a unique morphism $\bar{\phi}$ such that the diagram commutes.

- *π is surjective.*
- *A subset $U \subseteq X // G$ is open if and only if $\pi^{-1}(U) \subseteq X$ is open.*
- *For all $x, y \in X$, we have $\pi(x) = \pi(y) \iff \overline{G \cdot x} \cap \overline{G \cdot y} \neq \emptyset$.*

Proof. Proof of the first bullet is Proposition 6.2 of [6].

For the second bullet, first remark that we only need to prove that $\pi^* : \text{Spec}(X // G) \rightarrow X$ is injective, which follows from the first bullet of the definition of a good categorical quotient.

For the third bullet, one direction is trivial since morphisms are continuous. The other follows from the second bullet of Definition 1.37 because $\pi^{-1}(U)$ is G -invariant.

For the fourth bullet, one direction comes from the third bullet of Definition 1.37 taken with $\overline{G \cdot x}$ and $\overline{G \cdot y}$, which are disjoint and for the other direction let us suppose $\pi(x) \neq \pi(y)$, then $G \cdot x$ is disjoint from $G \cdot y$. But since π is continuous, $\pi^{-1}(\pi(x))$ is closed, therefore $\overline{G \cdot x} \subset \pi^{-1}(\pi(x))$ and $\overline{G \cdot y} \subset \pi^{-1}(\pi(y))$, but by assumption

$$\pi^{-1}(\pi(x)) \cap \pi^{-1}(\pi(y)) = \emptyset.$$

□

One can show that the Example 1.36 is a good categorical quotient . From Lemma 1.38, we know that if we ask G -orbits to be closed, then quotients would be like the geometric intuitions we have.

Definition 1.39. *Let $\pi : X \rightarrow X // G$ be a good categorical quotient , then we call $X // G$ a **geometric quotient** if all G -orbits are closed. Then this quotient is written $\pi : X \rightarrow X/G$.*

There are other characterizations of geometric quotient. The following theorem is about a fiber product, a reader which is not accustomed to working with fiber products can learn about that in [9, p.61].

Theorem 1.40. *Let $\pi : X \rightarrow X // G$ be a good categorical quotient , then the following are equivalent:*

1. $\pi : X \rightarrow X/G$ is a geometric quotient.
2. For all $x, y \in X$, we have

$$\pi(x) = \pi(y) \iff G \cdot x = G \cdot y.$$

3. π induces a bijection between G -orbits in X and $X // G$.
4. The image of the morphism $G \times X \rightarrow X \times X, (g, x) \mapsto (g \cdot x, x)$ is the fiber product $X \times_{X // G} X$.

Finally, let us see an in-between definition of quotient that will turn up to be useful to characterize quotient constructions of toric varieties.

Definition 1.41. Let $\pi : X \rightarrow X // G$ be a good categorical quotient, it is an **almost geometric quotient** if there exists a G -invariant Zariski dense open subset U such that $G \cdot x$ is closed in X for all $x \in U$.

For example, Example 1.35 is an almost geometric quotient: taking

$$U = \mathbb{C}^4 \setminus (\mathbb{C}^2 \times \{(0, 0)\} \cup \{(0, 0)\} \times \mathbb{C}^2).$$

Therefore our "bad" example is rather good. Let us see an example of a good categorical quotient which is not an almost geometric quotient.

Example 1.42. Let \mathbb{C}^* act on \mathbb{C}^n by scalar multiplication. The ring of invariants is the set of polynomials of $\mathbb{C}[X_1, \dots, X_n]$ such that $f(\lambda x_1, \dots, \lambda x_n) = f(x_1, \dots, x_n)$, and the only polynomials to have this property are constants, therefore the ring of invariants is \mathbb{C} . So the quotient is just a point. One may show this forms a good categorical quotient but this is not an almost geometric quotient since the only closed G -orbit is the orbit of $\{(0, 0, 0, 0)\}$ and it is not dense in \mathbb{C}^4 .

Chapter 2

Toric varieties

We may now learn about the central concept of this work: toric varieties.

Just like for general algebraic geometry, we are going to see toric varieties from an affine point of view, then from an abstract point of view and we will finish by taking a look at quotients.

2.1 Affine toric varieties

Let us extend Example 1.15, the tori $(\mathbb{C}^*)^n$ have a linear group structure given by componentwise multiplication. Any variety T isomorphic to $(\mathbb{C}^*)^n$ inherits a group structure from the isomorphism and is therefore called a **torus**.

Let us begin by seeing the definition of the main object of this thesis.

Definition 2.1. *A variety V is a **toric variety** if it is an irreducible variety V having a torus T as a dense Zariski open subset and such that the action of T on itself can be extended to an algebraic action on V (an algebraic action is an action given by a morphism $T \times V \rightarrow V$).*

Since a toric variety contains a torus as a large Zariski open subset, it means that the complementary is an affine variety of smaller dimension.

Example 2.2. The cuspidal curve from Example 1.7 is an affine toric variety. Let us localize the variety at the point 0, the coordinate ring becomes $\mathbb{C}[X, Y, Z]/\langle X^3 - Y^2, XZ - 1 \rangle$. Let us check that the morphism

$$\phi : \mathbb{C}[t^{\pm 1}] \rightarrow \mathbb{C}[X, Y, Z]/\langle X^3 - Y^2, XZ - 1 \rangle, t \mapsto Y^2Z$$

is an isomorphism. We compute $\phi(t^2) = Y^4Z^2 = Y$, $\phi(t^3) = Y^6Z^3 = X^4Z^3 = X$ and $\phi(t^{-3}) = Z$, therefore it is surjective. The injectivity part comes from the existence of a (left) inverse ψ defined by $\psi(Y) = t^2$, $\psi(X) = t$ and $\psi(Z) = t^{-3}$ (it is clear that ψ

is well-defined because it is compatible with the quotient). We remark $\psi(X/Y) = t$, hence we conclude that the map ϕ is an isomorphism. It implies that the cuspidal curve localized at 0 is isomorphic to \mathbb{C}^* , hence we have our torus.

We need to check the action of the torus on itself, we can compute it is given by $(t^2, t^3) \cdot (u^2, u^3) = (t^2u^2, t^3u^3)$, therefore making the point $(0,0)$ trivial under the torus action makes an algebraic action, therefore the cuspidal curve is a toric variety.

We will use notions of linear algebraic groups, that will allow us to learn informations about the tori of toric varieties. More details can be found in Paragraphs 5 and 8 of [1].

Definition 2.3. *Let T be a torus, a **character** of T is a group homomorphism $T \rightarrow \mathbb{C}^*$ that is also a morphism. A **one-parameter subgroup** of T is a group homomorphism $\mathbb{C}^* \rightarrow T$ that is also a morphism.*

The set of characters and the set of one-parameter subgroups are well described, as we can see in the next two propositions.

Proposition 2.4. *Let $m = (a_1, \dots, a_n) \in \mathbb{Z}^n$ and let*

$$\chi^m : (\mathbb{C}^*)^n \rightarrow \mathbb{C}^*, (t_1, \dots, t_n) \mapsto t_1^{a_1} \cdots t_n^{a_n}.$$

Then χ^m is a character and all characters arise this way.

In the rest of the text, characters will be written with the χ^m notation of last Proposition.

Proposition 2.5. *Let $u = (b_1, \dots, b_n) \in \mathbb{Z}^n$ and let*

$$\lambda^u : \mathbb{C}^* \rightarrow (\mathbb{C}^*)^n, t \mapsto (t^{a_1}, \dots, t^{a_n}).$$

Then λ^u is a one-parameter subgroup and all one-parameter subgroups arise this way.

In the rest of the text, one-parameter subgroups will be written with the λ^u notation of last Proposition.

Since we know that the set of one-parameter subgroups and the set of characters both have a \mathbb{Z}^n structure, we can summarize that into a common notion, allowing us to fluidify further definitions.

Definition 2.6. *In this text a **lattice** is a free abelian group of finite rank (ie \mathbb{Z}^n).*

One shall be careful with the word lattice, since in certain contexts it is a partially ordered set with some other properties and in other contexts, such as here, it is some abelian group. Nonetheless, in French the notions cannot be confused, since the word **treillis** is used in the context of partially ordered sets and the word **réseau** is used in the context of abelian groups.

Remark 2.7. From Proposition 2.4, we know that the set of characters of a given torus forms a lattice that we will usually denote M . Similarly, from Proposition 2.5 we know that the set of one-parameter subgroups of a given torus forms a lattice that we will usually denote N .

We may now take a look at the duality between the two lattices we have defined.

Definition 2.8. *Let M and N be the character and the one-parameter subgroup lattices of the same torus T , then we define the **bilinear pairing** of M and N :*

$$\langle \cdot, \cdot \rangle : M \times N \rightarrow \mathbb{Z}, (m, u) = ((a_1, \dots, a_n), (b_1, \dots, b_n)) \mapsto \langle m, u \rangle := \sum_{i=1}^n a_i b_i.$$

This bilinear pairing can be seen in a more intrinsic way: the composition of a character χ^m and a one-parameter subgroup is a map $\chi^m \circ \lambda^u : \mathbb{C}^* \rightarrow \mathbb{C}^*$. A direct computation allows us to see that $\chi^m \circ \lambda^u(t) = t^{\langle m, u \rangle}$.

Let us now see a way to generate toric varieties. A motivation for next definition is that when we have a morphism which is a group homomorphism between two tori, then the image of the map is always a torus.

Definition 2.9. *Let M be a character lattice of a torus T and let \mathcal{A} be a finite subset of M , let us write $\mathcal{A} = \{m_1, \dots, m_s\}$, we define:*

$$\Phi_{\mathcal{A}} : T \rightarrow (\mathbb{C}^*)^s, t \mapsto \Phi_{\mathcal{A}}(\chi^{m_1}(t), \dots, \chi^{m_s}(t)).$$

We also define $Y_{\mathcal{A}}$ to be the Zariski closure of $\Phi_{\mathcal{A}}(T)$ in \mathbb{C}^s .

We know that the image of this map is a torus. Taking the Zariski closure of the image is common for parametrizations, to transform a Zariski open subset into a closed one, therefore an affine variety. The next proposition tells us that the variety we get is indeed toric.

Proposition 2.10. *Let M be a character lattice of a torus T and let $\mathcal{A} = \{m_1, \dots, m_s\} \subseteq M$, then $Y_{\mathcal{A}}$ is an affine toric variety whose torus has character lattice $\mathbb{Z}\mathcal{A}$, the lattice generated by \mathcal{A} . In particular the rank of $\mathbb{Z}\mathcal{A}$ is the dimension of $Y_{\mathcal{A}}$.*

One may want to know the ideals of the toric varieties we can generate this way. It will only depend on the set of linear combinations of characters that equals the zero character.

Notation 2.11. Let $l = (l_1, \dots, l_s) \in \mathbb{Z}^n$, we denote $l_+ = \sum_{l_i > 0} l_i e_i$ and $l_- = \sum_{l_i < 0} l_i e_i$ where (e_i) is the canonical basis.

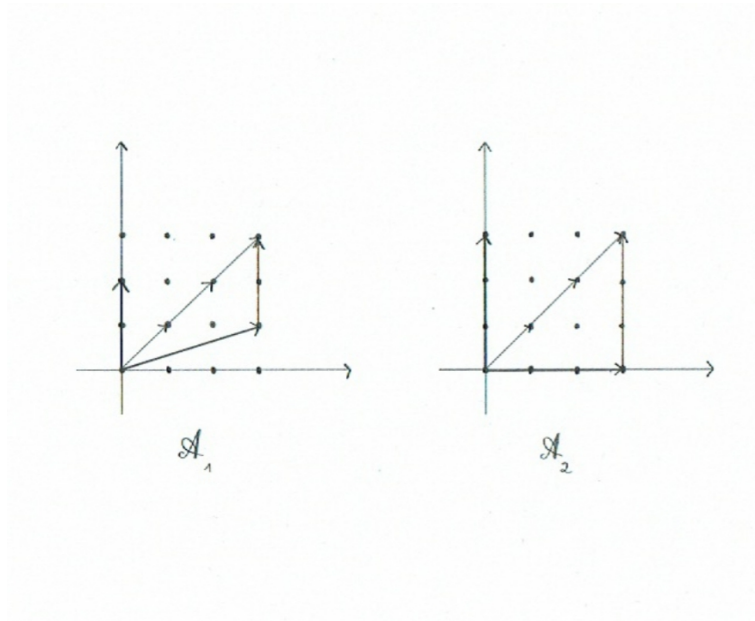
Proposition 2.12. Let $Y_{\mathcal{A}}$ be the toric variety generated by $\mathcal{A} = \{m_1, \dots, m_s\} \subseteq M$. Let $L = \{l = (l_1, \dots, l_s) \in \mathbb{Z}^s \mid \sum l_i m_i = 0\}$, then

$$I(Y_{\mathcal{A}}) = \langle x^{l_+} - x^{l_-} \mid l \in L \rangle = \langle x^a - x^b \mid a - b \in L \text{ and } a, b \in \mathbb{N}^s \rangle.$$

Example 2.13. Let us illustrate this proposition with two varieties. Let us fix our character lattice to be \mathbb{Z}^2 and let us take $\mathcal{A}_1 = ((2, 0), (1, 1), (1, 3))$ and $\mathcal{A}_2 = ((3, 0), (1, 1), (0, 3))$. To compute the ideal of $Y_{\mathcal{A}_1}$, we need to find the set of solutions of the integer equation $(2a + b + c, a + 3c) = (0, 0)$. We can compute that this set is generated by the element $(1, -3, 1)$, leading to the ideal $I(Y_{\mathcal{A}_1}) = \langle xz - y^3 \rangle$.

For $Y_{\mathcal{A}_2}$, we get the integer equation $(3a + b, a + 3c) = (0, 0)$, and the set of solutions is generated by $(1, -3, 1)$, leading to the conclusion that $I(Y_{\mathcal{A}_1}) = I(Y_{\mathcal{A}_2})$, implying $Y_{\mathcal{A}_1} = Y_{\mathcal{A}_2}$.

The next image represents this discussion.



We now give a name to ideals generated this way.

Definition 2.14. Let $L \subseteq \mathbb{Z}^s$ be a sublattice.

We define the **lattice ideal** $I_L := \langle x^a - x^b \mid a - b \in L \text{ and } a, b \in \mathbb{N}^s \rangle$. A **toric ideal** is a prime lattice ideal.

The next proposition is a characterization of toric ideals.

Proposition 2.15. Let $I \subseteq \mathbb{C}[X_1, \dots, X_n]$ be an ideal, then I is toric if and only if it is prime and generated by binomials.

We now move on to another way of generating affine toric varieties, which will later allow us to introduce polyhedral cones. Polyhedral cones will turn out to be important when working with abstract toric varieties.

Since we can generate affine toric varieties via parametrizations, we may want to generate those varieties via their coordinate ring, since a parametrization of a variety can be used to describe the coordinate ring without any quotient, such as illustrated in Example 1.7.

Definition 2.16. Let \mathbf{S} be a semigroup (that is a set with an associative binary operation with an identity), we say that \mathbf{S} is an **affine semigroup** if it is commutative, finitely generated (as a semigroup) and embedded in a lattice M .

We denote the binary operation with $+$ and for a finite set $\mathcal{A} \subseteq \mathbf{S}$ we define

$$\mathbb{N}\mathcal{A} := \left\{ \sum_{m \in \mathcal{A}} a_m m \mid a_m \in \mathbb{N} \right\}.$$

Finally, given an affine semigroup \mathbf{S} embedded in an character lattice M , we define the **semigroup algebra** $\mathbb{C}[\mathbf{S}]$ as:

$$\mathbb{C}[\mathbf{S}] := \left\{ \sum_{m \in \mathbf{S}} c_m \chi^m \mid c_m \in \mathbb{C} \text{ and } c_m = 0 \text{ for all but finitely many } m \right\},$$

where the multiplication is induced by $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$.

To clarify the definition of semigroup algebra, if $\mathbf{S} = \mathbb{N}\mathcal{A}$ where $\mathcal{A} = \{m_1, \dots, m_s\}$ then $\mathbb{C}[\mathbf{S}] = \mathbb{C}[\chi^{m_1}, \dots, \chi^{m_s}]$.

For example, if $\mathcal{A} = (e_1, \dots, e_n)$ then $\mathbb{N}\mathcal{A} = \mathbb{N}^n$ hence $\mathbb{C}[\mathbb{N}\mathcal{A}] = \mathbb{C}[X_1, \dots, X_n]$.

It is obvious that semigroup algebras generated this way are finitely generated and without nonzero nilpotents, allowing us to regard their spectrum.

Proposition 2.17. *Let $\mathbf{S} \subseteq M$ be an affine semigroup, then $\mathbb{C}[\mathbf{S}]$ is a reduced algebra and $\text{Spec}(\mathbb{C}[\mathbf{S}])$ is an affine toric variety.*

The character lattice of $\text{Spec}(\mathbb{C}[\mathbf{S}])$ is $\mathbb{Z}\mathbf{S}$, and if $\mathbf{S} = \mathbb{N}\mathcal{A}$ for a finite set $\mathcal{A} \subseteq M$, then $\text{Spec}(\mathbb{C}[\mathbf{S}]) = Y_{\mathcal{A}}$.

Let us remind Definition 1.32, which describes the induced action on the coordinate ring: if T is a torus acting on $\text{Spec}(\mathbb{C}[M])$, where $\mathbb{C}[M]$ is a semigroup algebra, then for $t \in T$, $f \in \mathbb{C}[M]$, $t \cdot f(x) = f(t^{-1} \cdot x)$.

The next lemma is an important lemma about the structure of subspaces invariant under the action of a torus.

Lemma 2.18. *Let T be a torus, M its character lattice and $A \subseteq \mathbb{C}[M]$ stable under the action of T , then*

$$A = \bigoplus_{\chi^m \in A} \mathbb{C} \cdot \chi^m.$$

The next result summarizes the results we have seen so far and adds an important result: we can generate all affine toric varieties with our techniques.

Theorem 2.19. *Let V be an affine variety, the following conditions are equivalent:*

1. *V is an affine toric variety.*
2. *There exists a finite set \mathcal{A} in lattice such that $V = Y_{\mathcal{A}}$.*
3. *The ideal $\mathcal{I}(V)$ is a toric ideal.*
4. *There exists an affine semigroup such that $V = \text{Spec}(\mathbb{C}[\mathbf{S}])$.*

Let us now move on to polyhedral cones, which is another way to generate toric varieties. Unfortunately, those cones cannot generate all affine toric varieties but they can be used to get more information about the varieties.

We need first to embed our lattices in vector spaces, since our cones need vector spaces to be defined.

Definition 2.20. *Let N be a lattice, we define $N_{\mathbb{R}}$ to be the real vector space generated by N , concretely:*

$$N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}.$$

Remark 2.21. We now assume that M is always a character lattice and N the one-parameter subgroup lattice of the same torus T . Moreover, if we have a lattice L , then T_L is the torus having L as one-parameter subgroup lattice.

Reminding the bilinear pairing of Definition 2.8, we deduce that $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$ are dual vector spaces.

We can now move on to the definition of polyhedral cone. We will see a lot of results about their geometry without going too much into the details, since discrete geometry is not the focus of this thesis. An interested reader can look at [2].

Definition 2.22. A *polyhedral cone* in $N_{\mathbb{R}}$ is a set σ such that there exists a finite set $S \subseteq N_{\mathbb{R}}$ such that

$$\sigma = \text{Cone}(S) := \left\{ \sum_{u \in S} \lambda_u u \mid \lambda_u \geq 0 \right\}.$$

We say that S generates σ .

We use the convention $\text{Cone}(\emptyset) = \{0\}$.

The **dimension** of a polyhedral cone is the dimension of the smallest linear subspace containing σ .

Note that polyhedral cones are convex.

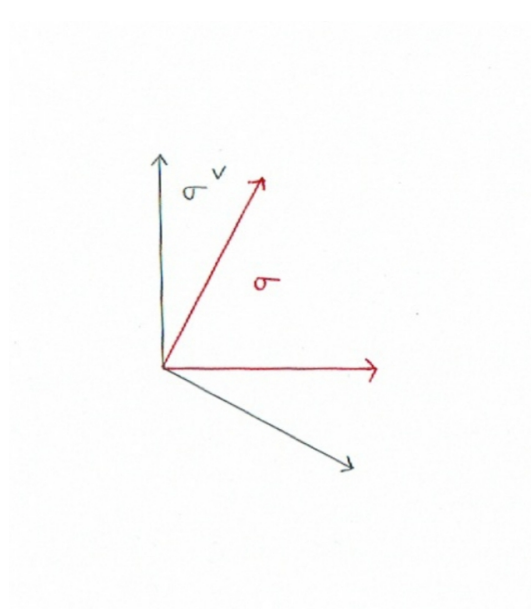
We define the notion of dual of a cone, which is in the dual space, which has the property we often ask for a dual: taking the dual is an involution and the dual is a polyhedral cone.

Definition 2.23. Let σ be a polyhedral cone in $N_{\mathbb{R}}$, its **dual cone** σ^\vee is

$$\sigma^\vee := \{m \in M_{\mathbb{R}} \mid \langle m, u \rangle \geq 0 \text{ for all } u \in \sigma\}.$$

Proposition 2.24. Let σ be a polyhedral cone in $N_{\mathbb{R}}$, then its dual cone σ^\vee is a polyhedral cone and $(\sigma^\vee)^\vee$.

Example 2.25. Here is an image of a polyhedral cone and its dual.



We need to develop a notion of supporting hyperplane, corresponding to an hyperplane that intersects a cone at the border of this cone. This will be useful when making links between the cone and its dual.

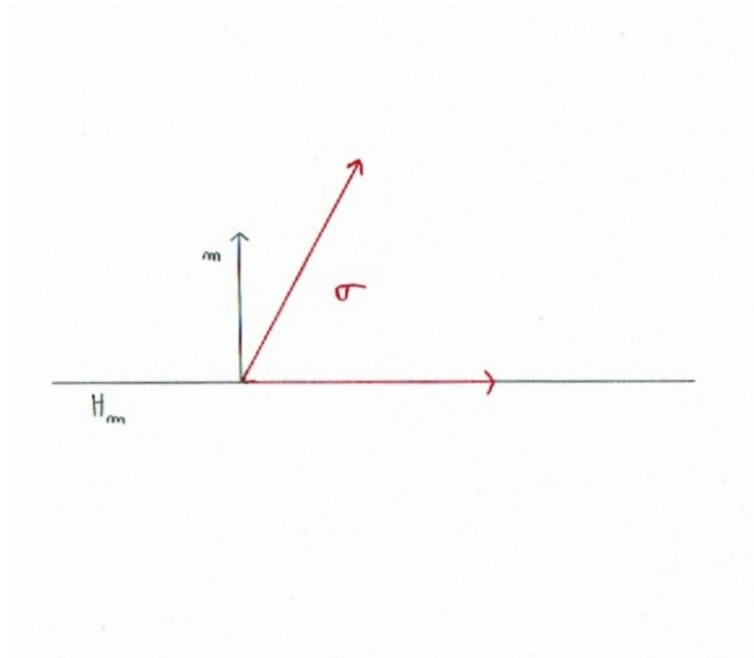
Definition 2.26. Given $m \in M_{\mathbb{R}}$, we define two subsets of $N_{\mathbb{R}}$:

$$H_m := \{u \in N_{\mathbb{R}} \mid \langle m, u \rangle = 0\}$$

$$H_m^+ := \{u \in N_{\mathbb{R}} \mid \langle m, u \rangle \geq 0\}.$$

Let σ be a polyhedral cone in $N_{\mathbb{R}}$ and $m \in M_{\mathbb{R}}$, then we say that H_m is a **supporting hyperplane** of σ if $\sigma \subseteq H_m^+$. In that case, we say that H_m^+ is a **supporting half-space** of σ .

Example 2.27. Here is a representation of a cone with a supporting hyperplane.



Remark 2.28. Let us remark that H_0 is not an hyperplane. Also, H_m is a supporting hyperplane if and only if $m \in \sigma^\vee$, and using the fact that the dual of the dual of a cone is the cone itself, we establish that if m_1, \dots, m_s generate σ^\vee , then:

$$\sigma = H_{m_1}^+ \cap \dots \cap H_{m_s}^+.$$

Let us now remark that our cones are said to be **polyhedral**, meaning that it may be interesting to give a notion of face of the cone.

Definition 2.29. A **face** of a polyhedral cone σ is a set τ such that there exists $m \in \sigma^\vee$ such that $\tau = H_m \cap \sigma$. We use the notation $\tau \preceq \sigma$ to say that τ is a face of σ . Faces $\tau \neq \sigma$ are called **proper faces** of σ and it is written $\tau \prec \sigma$.

The next lemma confirms the intuitions we can have about the faces of a cone.

Lemma 2.30. Let σ be a polyhedral cone in $N_{\mathbb{R}}$, then:

- Each face of σ is a polyhedral cone.
- The intersection of any collection of faces of σ is a face of σ .
- If $\eta \preceq \tau \preceq \sigma$ then $\eta \preceq \sigma$.
- If τ is a face of σ and $u, v \in \sigma$ are such that $u + v \in \tau$, then $u, v \in \tau$.

Remark 2.31. One may see that the set of faces of a given polyhedral cone σ is a partially ordered set with the order \preceq . It is even a lattice (in the sense of partially ordered sets).

There are two special cases of faces we need to highlight: the faces of dimension 1 and of codimension 1.

Definition 2.32. Let σ be a polyhedral cone, a face τ is a **facet** if $\dim(\sigma) - \dim(\tau) = 1$ and a face ρ is an **edge** if $\dim(\rho) = 1$.

The next proposition makes the link between faces, supporting hyperplanes and the dual cone.

Proposition 2.33. Let $\sigma \subseteq N_{\mathbb{R}}$ be a polyhedral cone and let $\dim(N_{\mathbb{R}}) = n$. Then:

- If $m_i \in \sigma^\vee, i \in \{1, \dots, s\}$ are such that $\sigma = H_{m_1}^+ \cap \dots \cap H_{m_s}^+$, then $\sigma^\vee = \text{Cone}(m_1, \dots, m_s)$. Moreover, if $\dim(\sigma) = n$ then the $\tau_i := H_{m_i} \cap \sigma$ are the facets of σ .
- The intersection of all the facets of σ containing a proper face τ is τ itself.

Since last proposition makes a link between the facets of a cone and the generator of its dual, we have a clue that there is a link between the partially ordered sets of faces of a cone and the one of its dual.

Definition 2.34. Let σ be a polyhedral cone in $N_{\mathbb{R}}$ and let τ be a face of σ . We define:

$$\tau^\perp := \{m \in M_{\mathbb{R}} \mid \langle m, u \rangle = 0 \text{ for all } u \in \tau\}$$

$$\tau^* := \{m \in \sigma^\vee \mid \langle m, u \rangle = 0 \text{ for all } u \in \tau\} = \sigma^\vee \cap \tau^\perp.$$

Proposition 2.35. *Let τ be a face of a polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$, then:*

- *The set τ^* is a face of σ^\vee .*
- *The map $\tau \mapsto \tau^*$ is a bijective correspondence, which reverses the inclusions, between faces of σ and faces of σ^\vee .*
- *We have $\dim \tau + \dim \tau^* = n$.*

The next notion corresponds to removing the faces of a cone. For a cone which has the same dimension as the space it is embedded in, we can use the topological interior to define this notion. For cones of smaller dimensions, we use a trick to go back to the previous case.

Definition 2.36. *Let σ be a polyhedral cone in $N_{\mathbb{R}}$ and let $\text{Span}(\sigma)$ be the smallest linear subspace containing σ . The **relative interior** of σ , denoted $\text{Relint}(\sigma)$, is the interior of σ as a subset of $\text{Span}(\sigma)$.*

One might remark that polyhedral cones can take the look of something that doesn't suit our intuition of a cone. For example, the affine space \mathbb{C}^n is a polyhedral cone. The next proposition identifies the cones that we want to name cones. It will turn up to be useful to only use those cones.

Proposition 2.37. *Let σ be a polyhedral cone in $N_{\mathbb{R}}$, let $n = \dim(N_{\mathbb{R}})$, then the following conditions are equivalent:*

- *The set $\{0\}$ is a face of σ .*
- *The cone σ contains no nontrivial linear subspace of $N_{\mathbb{R}}$.*
- *The equation $\sigma \cap (-\sigma) = \{0\}$ holds.*
- *The equation $\dim(\sigma^\vee) = n$ holds.*

*Any polyhedral cone satisfying one of those conditions is called **strongly convex**.*

The next lemma allows us to describe the supporting hyperplane of a common face of two cones.

Lemma 2.38. *(Separation lemma) Let σ_1, σ_2 be two cones in $N_{\mathbb{R}}$ that intersect on a common face $\tau = \sigma_1 \cap \sigma_2$. Then for all $m \in \text{Relint}(\sigma_1^\vee \cap (-\sigma_2)^\vee)$, we have*

$$\tau = H_m \cap \sigma_1 = H_m \cap \sigma_2.$$

We now take a look at a property that cones can have that assures that the semigroup of lattice points of a polyhedral cone is finitely generated (a lattice point is a point on the lattice underlying the vector space).

Definition 2.39. A polyhedral cone σ in $N_{\mathbb{R}}$ is **rational** if there exists $S \subseteq N$ such that $\text{Cone}(S) = \sigma$.

Definition 2.40. Let σ be a strongly convex rational polyhedral cone in $N_{\mathbb{R}}$, then for each edge ρ of σ , there exists a unique generator u_{ρ} of the semigroup $\rho \cap N$, such generator is called the **ray generator** of ρ . The collection of the ray generators of σ is called **minimal generators** of σ .

The hypothesis of strong convexity is need otherwise the semigroup corresponding to an edge can need two elements to be generated.

Strongly convex rational polyhedral cones are generated by the lattice points of their edges, as expressed in the following lemma.

Lemma 2.41. Let S be the set of ray generators of a strongly convex rational polyhedral cone σ , then we have

$$\text{Cone}(S) = \sigma.$$

The next two notions are especially important notions for strongly convex polyhedral cones.

Definition 2.42. Let σ be a strongly convex rational polyhedral cone in $N_{\mathbb{R}}$, we say that σ is **smooth** (or **regular**) if its minimal generators are a subcollection of a \mathbb{Z} -basis of N and we say that σ is **simplicial** if its collection of minimal generators is linearly independent over $N_{\mathbb{R}}$.

Example 2.43. The cone $\sigma = \text{Cone}(2e_1 + e_2, 2e_1 + 3e_2)$ is simplicial since is two generators are linearly independent but is not smooth since all integer linear combinations of those generators have even abscissa.

We now define the semigroup associated to a cone. It will be used to generate varieties

Definition 2.44. Let σ be a rational polyhedral cone in $N_{\mathbb{R}}$, we define the semigroup $\mathbf{S}_{\sigma} := \sigma^{\vee} \cap M$.

The reader may ask the reason to take the dual and not simply work with $\sigma \cap N$.
T

Lemma 2.45. (Gordan's lemma) The above construction is finitely generated which implies it is an affine semigroup.

The next theorem assures that our choice of definition of affine semigroup works. Moreover, it makes a link between the dimension of the variety and the strong convexity of the corresponding cone.

Theorem 2.46. *Let σ be a rational polyhedral cone in $N_{\mathbb{R}}$ where $\dim(N_{\mathbb{R}}) = n$, then*

$$U_{\sigma} := \text{Spec}(\mathbb{C}[\mathbf{S}_{\sigma}]) = \text{Spec}(\mathbb{C}[\sigma^{\vee} \cap M])$$

*is an affine toric variety. We call this variety the **variety generated** by σ . Moreover,*

$$\dim(U_{\sigma}) = n \iff \text{the torus of } U_{\sigma} \text{ is } T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^* \iff \sigma \text{ is strongly convex.}$$

When necessary, we might precise the lattice N as an index: $U_{\sigma} = U_{\sigma, N}$.

Remark 2.47. We now always assume that our cones are strongly convex.

The next theorem is a really important theorem because it allows to associate to points some semigroup homomorphisms, which is key ingredient for off the most important theorems of this thesis: the orbit-cone correspondence (Theorem 2.78). Also note that the correspondence between maximal ideals and points is not specific to the toric case.

Theorem 2.48. *Let \mathbf{S} be an affine semigroup and let $V = \text{Spec}(\mathbb{C}[\mathbf{S}])$ be the affine toric variety generated by \mathbf{S} . Then the three following sets are in bijective correspondence:*

- *The set of points $p \in V$.*
- *The set of maximal ideals $\mathfrak{m} \subseteq \mathbb{C}[\mathbf{S}]$.*
- *The set of semigroup homomorphisms $\mathbf{S} \rightarrow \mathbb{C}$, where the binary operation on \mathbb{C} is the multiplication.*

To associate to a point p a maximal ideal, it is in fact fairly easy: we only need to take $\mathfrak{m} = \{P \in \mathbb{C}[\mathbf{S}] \mid P(p) = 0\}$. The other way is trickier.

The semigroup homomorphism associated to p is the one mapping $m \mapsto \chi^m(p)$. To go the other way around, we need a set $\mathcal{A} = \{m_1, \dots, m_s\}$ such that the variety V is $Y_{\mathcal{A}}$ and such that $\mathbb{N}\mathcal{A} = \mathbf{S}$, then the point p is $p = (\gamma(m_1), \dots, \gamma(m_s))$.

We now see the notion of pointed semigroup, which means that the semigroup has only 0 as invertible element.

Definition 2.49. *An affine semigroup \mathbf{S} is **pointed** if for all $a, b \in \mathbf{S}$, $a + b = 0$ implies $a = 0$ or $b = 0$.*

This notion corresponds to the fact that 0 is an element of the variety corresponding variety and to the existence of a point fixed by the action of the torus.

Proposition 2.50. *Let $V = \text{Spec}(\mathbb{C}[\mathbf{S}])$ be an affine toric variety, then:*

- *The torus action has a fixed point if and only if \mathbf{S} is pointed. In that case, the fixed point is unique and corresponds to the semigroup homomorphism $\mathbf{S} \rightarrow \mathbb{C}$ defined by:*

$$m \mapsto \begin{cases} 1 & m = 0 \\ 0 & m \neq 0. \end{cases}$$

- *Let $\mathcal{A} \subseteq \mathbf{S} \setminus \{0\}$ be such that $V = Y_{\mathcal{A}}$, the only point that can be a fixed point for the torus action is 0 , and if $0 \in V$, then it is a fixed point for this action.*

Let us link the two parts of this proposition. One can see that with Theorem 2.48, if 0 is in the variety V , we can compute the semigroup homomorphism corresponding to 0 , and this semigroup sends $m = (m_1, \dots, m_n)$ to $\chi^m(0) = \prod_{i=1}^s 0^{m_i}$ which is 1 if $m = 0$ and 0 otherwise (using the convention $0^0 = 1$).

Let us now see the maximal ideal corresponding to the fixed point, when it exists.

Corollary 2.51. *Let σ be a strongly convex rational polyhedral cone on $N_{\mathbb{R}}$ and U_{σ} the variety corresponding to this cone. Then the torus action on U_{σ} has a fixed point if and only if $\dim(\sigma) = \dim(N_{\mathbb{R}})$. If there is a fixed point it is unique and it corresponds to the maximal ideal*

$$\langle \chi^m \mid m \in \mathbf{S}_{\sigma} \setminus \{0\} \rangle \subseteq \mathbb{C}[\mathbf{S}_{\sigma}].$$

We now see a property of semigroups that can characterize if an affine toric variety can be generated by a cone or not.

Definition 2.52. *We say that an affine semigroup $\mathbf{S} \subseteq M$ is **saturated** if there is no $k \in \mathbb{N} \setminus \{0\}$ and $m \in M$, such that $km \in \mathbf{S}$ and $m \notin \mathbf{S}$.*

Theorem 2.53. *Let V be an affine toric variety with torus T , then the following are equivalent:*

- *The variety V is normal.*
- *There is a saturated affine semigroup \mathbf{S} such that $V = \text{Spec}(\mathbb{C}[\mathbf{S}])$.*
- *There is a strongly convex rational polyhedral cone σ such that $V = U_{\sigma}$.*

Example 2.54. As we saw earlier, the cuspidal curve is not a normal variety and its coordinate ring is $\mathbb{C}[t^2, t^3]$, meaning that the affine semigroup $\mathbb{N}\{2, 3\}$ generates the cuspidal curve. One can see that this affine semigroup is not saturated. Finally, we must accept the impossibility to generate the cuspidal curve with a cone.

Now let us take a look at morphisms that preserve the toric structure. We first define this notion using the semigroups generating the varieties and then we make a link with the toric structure.

Definition 2.55. Let $\mathbf{S}_1, \mathbf{S}_2$ be two affine semigroups and let $V_1 = \text{Spec}(\mathbb{C}[\mathbf{S}_1]), V_2 = \text{Spec}(\mathbb{C}[\mathbf{S}_2])$. A morphism $\phi : V_1 \rightarrow V_2$ is **toric** if the corresponding homomorphism of coordinate rings $\phi^* : \mathbb{C}[\mathbf{S}_2] \rightarrow \mathbb{C}[\mathbf{S}_1]$ is such that there a semigroup homomorphism $\hat{\phi} : \mathbf{S}_2 \rightarrow \mathbf{S}_1$ satisfying $\phi(\chi^m) = \chi^{\hat{\phi}(m)}$.

Proposition 2.56. Let V_1, V_2 be two affine toric varieties with tori T_1, T_2 respectively. Then a morphism $\phi : V_1 \rightarrow V_2$ is toric if and only if $\phi|_{T_1}$ is a group homomorphism $\phi|_{T_1} : T_1 \rightarrow T_2$.

If ϕ is toric, then it is **equivariant**, which means that we have $\phi(t \cdot p) = \phi(t) \cdot \phi(p)$.

One can deduce from this theorem that if a morphism between two toric affine varieties preserve the group structure of the tori, then it preserves the action of these tori.

Remark 2.57. Let $\bar{\phi} : N_1 \rightarrow N_2$ be a homomorphism between lattices, then we can extend it uniquely to a linear map $\bar{\phi}_{\mathbb{R}} : (N_1)_{\mathbb{R}} \rightarrow (N_2)_{\mathbb{R}}$ such that $\bar{\phi}_{\mathbb{R}}(n \otimes 1) = \bar{\phi}(n)$ for all $n \in N_1$.

Moreover, $\bar{\phi}$ also extends to a morphism $\phi := T_{N_1} \rightarrow T_{N_2}$ satisfying $\phi(t^n) = t^{\bar{\phi}(n)}$ for all $n \in N_1$.

Let us now see a result that tell if a morphism between two tori can be extended into a morphism between two toric varieties.

Proposition 2.58. Let σ_1, σ_2 be two strongly rational polyhedral cones in $(N_1)_{\mathbb{R}}, (N_2)_{\mathbb{R}}$ respectively and let $\bar{\phi} : N_1 \rightarrow N_2$. Then $\phi : T_{N_1} \rightarrow T_{N_2}$ extends to a morphism $\phi : U_{\sigma_1} \rightarrow U_{\sigma_2}$ if and only if $\bar{\phi}_{\mathbb{R}}(\sigma_1) = \sigma_2$.

One may wonder if there is a link between the variety generated by a polyhedral cone and the variety generated by one of its faces. The next result tells that the variety generated by the face is a Zariski open subset of the variety generated by the original cone.

Proposition 2.59. Let σ be a strongly convex rational polyhedral cones in $N_{\mathbb{R}}, \tau$ a face of σ and let $m \in \sigma^{\vee} \cap M$ such that $\tau = H_m \cap \sigma$. Localizing the semigroup algebra $\mathbb{C}[\mathbf{S}_{\sigma}]$ at $\chi^m \in \mathbb{C}[\mathbf{S}_{\sigma}]$ gives $\mathbb{C}[\mathbf{S}_{\tau}]$, it means that

$$\mathbb{C}[\mathbf{S}_{\sigma}]_{\chi^m} = \mathbb{C}[\mathbf{S}_{\tau}].$$

It implies

$$U_\tau = \text{Spec}(\mathbb{C}[\mathbf{S}_\tau]) = \text{Spec}(\mathbb{C}[\mathbf{S}_\sigma]_{\chi^m}) = (\text{Spec}(\mathbb{C}[\mathbf{S}_\sigma]))_{\chi^m} = (U_\sigma)_{\chi^m} \subseteq U_\sigma,$$

which means that U_τ is an affine open subset of U_σ .

This last proposition about affine varieties describes the link between two varieties generated by the same cone but embedded in two different lattices.

Proposition 2.60. *Let N' be a sublattice of finite index of N , let $G = N/N'$ and let $\sigma \subseteq N'_\mathbb{R} \subseteq N_\mathbb{R}$ be a strongly convex rational polyhedral cone. Then:*

- *There are natural isomorphisms between the following groups:*

$$G \simeq \text{hom}_\mathbb{Z}(M'/M, \mathbb{C}^*) = \ker(T_{N'} \rightarrow T_N),$$

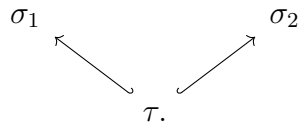
where M is the dual lattice of N and M' the dual lattice of N' .

- *There is an action of G on $\mathbb{C}[\sigma^\vee \cap M']$ with ring of invariants $\mathbb{C}[\sigma^\vee \cap M']^G = \mathbb{C}[\sigma^\vee \cap M]$.*
- *There is an action of G on $U_{\sigma, N'}$ and the morphism $\phi : U_{\sigma, N'} \rightarrow U_{\sigma, N}$ is constant on G -orbits, from which we deduce a bijection $U_{\sigma, N'}/G \simeq U_{\sigma, N}$.*

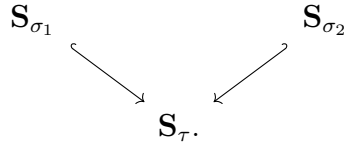
2.2 Abstract toric varieties

We may now start our study of abstract toric varieties. The main idea to start the study of those varieties is to find a way to transfer the notion of affine varieties glued together in a notion compatible with our combinatorics tools developed for affine toric varieties.

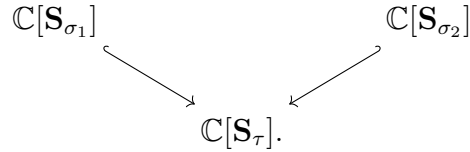
We saw how to generate varieties from strongly convex rational polyhedral cones. Let us imagine two such polyhedral cones σ_1, σ_2 intersecting in a common face τ and the affine toric varieties associated to these cones. Let us try to use the face τ to describe how we could glue the affine varieties. We have the following inclusions:



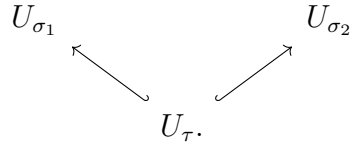
Flipping the arrows, we may now look at the affine semigroups of those cones:



This transfers directly into a diagram of semigroup algebras:



Finally, we flip the arrow once again by applying the spectrum:



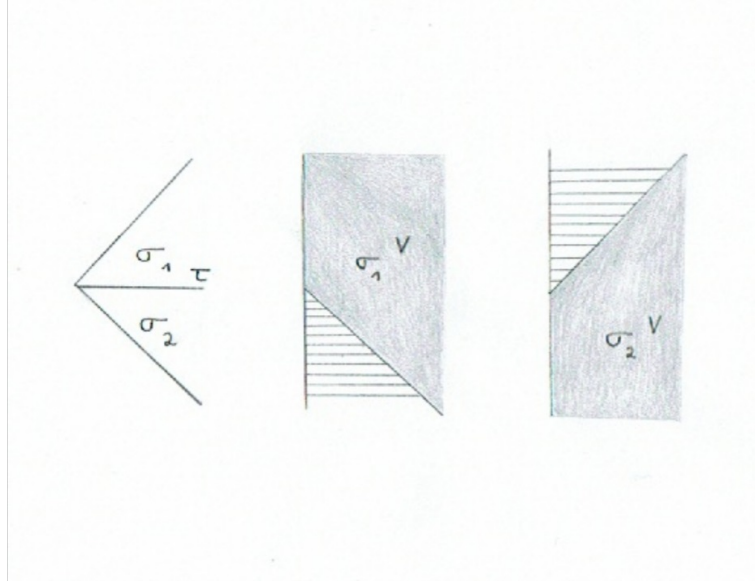
And this last diagram tells us intuitively that the intersection of U_{σ_1} and U_{σ_2} must be U_{τ} .

A similar reasoning was used in the proof of Proposition 2.59. This proposition also describes the inclusions $U_{\tau} \subseteq U_{\sigma_i}$:

$$U_{\sigma_1} \supseteq (U_{\sigma_1})_{\chi^m} = U_{\tau} = (U_{\sigma_2})_{\chi^{-m}} \subseteq U_{\sigma_2}, \quad (2.1)$$

where $m \in \sigma_1^{\vee} \cap (-\sigma_2)^{\vee}$ is such that $\tau = H_m \cap \sigma_1 = H_m \cap \sigma_2$.

Example 2.61. Below you can find an example of two such cones, in grey we can see their respective duals and the hatched regions correspond to the additions of m for σ_1 and $-m$ for σ_2 . We see that they define the same region τ^{\vee} . This illustrates the inclusions $\mathbf{S}_{\sigma_1} \subseteq \mathbf{S}_{\tau} \supseteq \mathbf{S}_{\sigma_2}$ thus $(U_{\sigma_1})_{\chi^m} = (U_{\sigma_2})_{\chi^{-m}}$



The notion that allows us to extend this reasoning is the notion of fan.

Definition 2.62. Let Σ in $N_{\mathbb{R}}$ be a finite collection of strongly convex rational polyhedral cones $\sigma \subseteq N_{\mathbb{R}}$ such that:

- For all $\sigma \in \Sigma$, all faces $\tau \preceq \sigma$ are elements of Σ .
- For all $\sigma_1, \sigma_2 \in \Sigma$, $\sigma_1 \cap \sigma_2$ is a face of each (and therefore an element of Σ).

Then we say that Σ is a **fan**.

The **support** of Σ is $|\Sigma| := \bigcup_{\sigma \in \Sigma} \sigma$ and we denote $\Sigma(r)$ the set of r -dimensional cones of Σ .

Let us see now the second part of the separation lemma (Lemma 2.38). (The reader may note that the separation lemma is separated in two pieces, which is funny.)

Lemma 2.63. (Separation lemma) Let σ_1, σ_2 be two cones of a fan Σ , then:

$$\mathbf{S}_{\sigma_1 \cap \sigma_2} = \mathbf{S}_{\sigma_1} + \mathbf{S}_{\sigma_2}.$$

Before defining the variety associated to a fan, one must check that the isomorphisms described in (2.1) are compatible with the condition about triplets for abstract varieties (coming from the second bullet of Definition 1.23).

Let us transcript this condition in this case: let $\sigma_i, i \in \{1, 2, 3\}$ be three cones of the same fan, let $\tau = \sigma_1 \cap \sigma_2 \cap \sigma_3$ be the common face of these three cones, let $m_1 \in \sigma_1^{\vee} \cap (-\sigma_2)^{\vee}$ be such that $\sigma_1 \cap \sigma_2 = H_{m_1} \cap \sigma_1 = H_{m_1} \cap \sigma_2$, let $m_2 \in \sigma_1^{\vee} \cap (-\sigma_3)^{\vee}$ be such that $\sigma_1 \cap \sigma_3 =$

$H_{m_2} \cap \sigma_1 = H_{m_2} \cap \sigma_3$ and let $m_3 \in \sigma_2^\vee \cap (-\sigma_3)^\vee$ be such that $\sigma_2 \cap \sigma_3 = H_{m_3} \cap \sigma_2 = H_{m_3} \cap \sigma_3$. The condition rewrites as $(U_{\sigma_1})_{\chi^{m_1}} \cap (U_{\sigma_1})_{\chi^{m_2}} = (U_{\sigma_2})_{\chi^{-m_1}} \cap (U_{\sigma_2})_{\chi^{m_3}}$.

We need to remember that in the proof of Proposition 2.59, we proved that $\mathbf{S}_{\sigma_1 \cap \sigma_2} = \mathbf{S}_{\sigma_1} + \mathbb{Z}(-m_1)$. Therefore, $\mathbf{S}_\tau = \mathbf{S}_{\sigma_1} + \mathbb{Z}(-m_1) + \mathbb{Z}(-m_2) = \mathbf{S}_{\sigma_2} + \mathbb{Z}(m_1) + \mathbb{Z}(-m_3)$. Therefore

$$(U_{\sigma_1})_{\chi^{m_1}} \cap (U_{\sigma_1})_{\chi^{m_2}} = U_\tau = (U_{\sigma_2})_{\chi^{-m_1}} \cap (U_{\sigma_2})_{\chi^{m_3}}.$$

Thus, we can now define the variety associated to a fan.

Definition 2.64. Let Σ be a fan in $N_{\mathbb{R}}$. The variety X_Σ is the collection of affine varieties $\{U_\sigma\}_{\sigma \in \Sigma}$ with Zariski open subsets $U_{\sigma_2 \sigma_1} := (U_{\sigma_1})_{\chi^m}$, where m is such that $H_m \cap \sigma_1 = \sigma_1 \cap \sigma_2 = H_m \cap \sigma_2$, and isomorphisms

$$g_{\sigma_1 \sigma_2} : (U_{\sigma_1})_{\chi^m} \simeq (U_{\sigma_2})_{\chi^{-m}}.$$

We call this variety X_Σ the variety associated to Σ . As for affine varieties, if we need to precise the lattice N , we will write $X_{\Sigma, N}$.

Theorem 2.65. Let Σ be a fan in $N_{\mathbb{R}}$, then the variety X_Σ is a normal toric variety which is Hausdorff under classical topology.

Proof. It is obvious that X_Σ is normal, because of Proposition 2.53. Since all affine varieties are irreducible, X_Σ is irreducible.

Each affine variety defining X_Σ contains a torus of the same dimension and the gluing isomorphisms are the identities. Hence the action of the torus defined on each affine variety is compatible with the gluing of each pair, making X_Σ a toric variety.

We won't see that the variety is Hausdorff under Euclidean topology, □

There is a reciprocal of last theorem. This theorem is often expressed as a corollary of Theorem 3 of Sumigiro [12], as one can see from [8, Thm. 6, p.24].

Theorem 2.66. Let X be a normal separated toric variety with torus T_N , then there exists a fan Σ such that $X = X_\Sigma$.

Example 2.67. Let us see a fan that generates the variety $\mathbb{P}^2(\mathbb{C})$. Let $\sigma_0 = \text{Cone}(e_1, e_2)$, $\sigma_1 = \text{Cone}(e_2, -e_1 - e_2)$ and $\sigma_2 = \text{Cone}(e_1, -e_1 - e_2)$. Let $\Sigma = \{\sigma_0, \sigma_1, \sigma_2, \sigma_1 \cap \sigma_2, \sigma_0 \cap \sigma_2, \sigma_0 \cap \sigma_1, \{0\}\}$.

We can see that:

$$\begin{aligned} U_{\sigma_0} &= \text{Spec}(\mathbb{C}[\mathbf{S}_{\sigma_0}]) = \text{Spec}(\mathbb{C}([x, y])) = \mathbb{C}^2 \\ U_{\sigma_1} &= \text{Spec}(\mathbb{C}[\mathbf{S}_{\sigma_1}]) = \text{Spec}(\mathbb{C}([x^{-1}, x^{-1}y])) = \mathbb{C}^2 \\ U_{\sigma_2} &= \text{Spec}(\mathbb{C}[\mathbf{S}_{\sigma_2}]) = \text{Spec}(\mathbb{C}([xy^{-1}, y^{-1}])) = \mathbb{C}^2. \end{aligned}$$

The intersection of σ_0 and σ_1 is $\text{Cone}(e_2)$, hence $e_1 \in \sigma_0^\vee \cap (-\sigma_1)^\vee$ is such that $\sigma_0 \cap H_{e_2} = \sigma_1 \cap H_{e_2} = \text{Cone}(e_1)$. We have the isomorphism $g_{\sigma_1\sigma_0}^* : \mathbb{C}([x, y])_x \simeq \mathbb{C}([x^{-1}, x^{-1}y])_{x^{-1}}$. Therefore, the isomorphism $g_{\sigma_1\sigma_0} : (\mathbb{C}^2)_{x^{-1}} \rightarrow (\mathbb{C}^2)_x$ is:

$$g_{\sigma_1\sigma_0} : (\mathbb{C}^2)_{x^{-1}} \rightarrow (\mathbb{C}^2)_x, (a, b) \mapsto (1/a, b/a).$$

In the same way, we can show that:

$$\begin{aligned} g_{\sigma_2\sigma_0} &: (\mathbb{C}^2)_{y^{-1}} \rightarrow (\mathbb{C}^2)_y, (a, b) \mapsto (a/b, 1/b) \\ g_{\sigma_2\sigma_1} &: (\mathbb{C}^2)_{x^{-1}y} \rightarrow (\mathbb{C}^2)_{xy^{-1}}, (a, b) \mapsto (1/b, a/b). \end{aligned}$$

One can see that it corresponds to Example 1.24 in the case of $n = 2$, hence $\mathbb{P}^2(\mathbb{C})$ is a toric variety.

The other projective planes are also toric varieties, let us fix $n \in \mathbb{N}$, let $e_0 = \sum_{i=1}^n -e_i$, let $\sigma_i = \text{Cone}(e_0, \dots, \hat{e}_i, \dots, e_n)$ for $i \in \{0, \dots, n\}$ and let Σ be the fan containing all faces of the cones σ_i for all $i \in \{0, \dots, n\}$. Similar computations lead to $X_\Sigma = \mathbb{P}^n(\mathbb{C})$.

We now take a look at the product of fans, which is linked to the product of varieties.

Definition 2.68. Let Σ_1, Σ_2 be fans in $(N_1)_{\mathbb{R}}$ and $(N_2)_{\mathbb{R}}$ respectively. The **product** of those fans is the fan

$$\Sigma_1 \times \Sigma_2 := \{\sigma_1 \times \sigma_2 \mid \sigma_1 \in \Sigma_1, \sigma_2 \in \Sigma_2\}$$

in $(N_1)_{\mathbb{R}} \times (N_2)_{\mathbb{R}} = (N_1 \times N_2)_{\mathbb{R}}$.

Proposition 2.69. Let Σ_1, Σ_2 be fans in $(N_1)_{\mathbb{R}}$ and $(N_2)_{\mathbb{R}}$ respectively, then $X_{\Sigma_1 \times \Sigma_2} = X_{\Sigma_1} \times X_{\Sigma_2}$.

Proof. Let us give a sketch of the proof. First, one may observe that for two affine semigroups $\mathbf{S}_1, \mathbf{S}_2$, we have $\mathbb{C}[\mathbf{S}_1] \otimes \mathbb{C}[\mathbf{S}_2] = \mathbb{C}[(\mathbf{S}_1, 0), (0, \mathbf{S}_2)]$ by the morphism mapping $x \otimes y \mapsto xy$. Hence, for two cones σ_1, σ_2 , we have $U_{\sigma_1 \times \sigma_2} = U_{\sigma_1} \times U_{\sigma_2}$.

The gluing maps can therefore be decomposed component per component, which leads to the conclusion. \square

Let us now see two properties that fans can have that have nice links with the corresponding varieties.

Definition 2.70. Let Σ be a fan in $N_{\mathbb{R}}$.

1. If all cones of Σ are smooth (or regular), then we say that Σ is a **smooth** (or **regular**) fan.

2. If $|\Sigma| = N_{\mathbb{R}}$, then we say Σ is **complete**.

The reader may note that in the next theorem, the notion of "smooth" variety appears. We won't see what it means for algebraic varieties, but this notion corresponds to the notion of smoothness for differential manifolds.

Theorem 2.71. *Let Σ be a fan in $N_{\mathbb{R}}$ and X_{Σ} the variety generated by Σ , the following hold:*

1. *The variety X_{Σ} is smooth if and only if Σ is smooth.*
2. *The variety X_{Σ} is compact in the Euclidean topology if and only if Σ is complete.*

Let us remind Proposition 2.48 establishing an equivalence between points of affine toric varieties, maximal ideals of its coordinate ring and semigroup homomorphisms from the affine semigroup of the variety to \mathbb{C} seen with its usual multiplication. The next definition defines some special points of the varieties, which will allow to make links between the orbits of the torus action and the cones of the fan.

Definition 2.72. *Let $\sigma \subseteq N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone and $U_{\sigma} = \text{Spec}(\mathbb{C}[\mathbf{S}_{\sigma}])$ the variety generated by this cone. We define γ_{σ} the point of U_{σ} corresponding to*

$$m \in \sigma^{\vee} \cap M \mapsto \begin{cases} 1 & m \in \sigma^{\perp} \cap M \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.73. *Let U_{σ} be a toric variety generated by a strongly convex rational polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$. Let $u \in N$, then:*

$$u \in \sigma \iff \lim_{t \rightarrow 0} \lambda^u(t) \text{ exists in } U_{\sigma}.$$

Furthermore the limit can be computed: if $u \in \text{Relint}(\sigma)$, then $\lim_{t \rightarrow 0} \lambda^u(t) = \gamma_{\sigma}$.

The reader may note that the point γ_{σ} is a point of an affine toric variety. When we have an abstract toric variety generated by a fan, the point γ_{σ} is seen as a point of the affine variety contained in the abstract variety.

Let us see how last proposition can be analyzed on a toric variety.

Example 2.74. Let us analyze this proposition on $\mathbb{P}^2(\mathbb{C})$ with the fan of Example 2.67. We will use homogeneous coordinates (x_0, x_1, x_2) , where $x_i \neq 0$ corresponds to being in the affine variety corresponding to σ_i . Points of the torus have coordinates $(1, s, t)$, $s \neq 0 \neq t$ and for $u = (a, b) \in \mathbb{Z}^2$, we have

$$\lambda^u(t) = (1, t^a, t^b).$$

If $a, b > 0$, then $u := (a, b) \in \text{Relint}(\sigma_0)$ and $\lim_{t \rightarrow 0} \lambda^u(t) = \lim_{t \rightarrow 0} (1, t^a, t^b) = (1, 0, 0)$. In the same way, for all $u \in \text{Relint}(\sigma_1)$, we have $\lim_{t \rightarrow 0} \lambda^u(t) = (0, 1, 0)$ and for all $u \in \text{Relint}(\sigma_2)$, we have $\lim_{t \rightarrow 0} \lambda^u(t) = (0, 0, 1)$.

Now let us assume $u = (a, 0)$ for $a > 0$, then $u \in \text{Relint}(\sigma_0 \cap \sigma_1)$ and $\lim_{t \rightarrow 0} \lambda^u(t) = (1, 1, 0)$. For $u = (0, b)$ with $b > 0$, we have $u \in \text{Relint}(\sigma_0 \cap \sigma_2)$ and $\lim_{t \rightarrow 0} \lambda^u(t) = (1, 0, 1)$. For $u = (a, a)$ for $a < 0$, we have $u \in \text{Relint}(\sigma_1 \cap \sigma_2)$ and $\lim_{t \rightarrow 0} \lambda^u(t) = (0, 1, 1)$.

Finally, for $u = (0, 0)$, we have $\lambda^u(t) = (1, 1, 1)$ hence the limit is $(1, 1, 1)$ and $u \in \text{Relint}(\{0\})$.

Let us now take a look at the orbits of our different special points: for a point in the torus $(1, s, t)$, $s \neq 0 \neq t$ and a point of the variety (a, b, c) , $abc \neq 0$, the action is given by

$$(1, s, t) \cdot (a, b, c) = (a, bs, ct).$$

Hence, it is clear that each of our distinguished points is on a different orbit, because the action cannot transform a 0 at a coordinate into a nonzero and vice-versa. Let us check the description of these different orbits:

$$\begin{aligned} (1, 1, 1) &\in \{(x_0, x_1, x_2) | x_i \neq 0 \text{ for all } i\} =: O_{(1,1,1)} \\ (1, 1, 0) &\in \{(x_0, x_1, 0) | x_i \neq 0 \text{ for all } i\} =: O_{(1,1,0)} \\ (1, 0, 1) &\in \{(x_0, 0, x_2) | x_i \neq 0 \text{ for all } i\} =: O_{(1,0,1)} \\ (0, 1, 1) &\in \{(0, x_1, x_2) | x_i \neq 0 \text{ for all } i\} =: O_{(0,1,1)} \\ (0, 0, 1) &\in \{(0, 0, x_2) | x_2 \neq 0\} =: O_{(0,0,1)} \\ (0, 1, 0) &\in \{(0, x_1, 0) | x_1 \neq 0\} =: O_{(0,1,0)} \\ (1, 0, 0) &\in \{(x_0, 0, 0) | x_0 \neq 0\} =: O_{(1,0,0)}. \end{aligned}$$

Since an orbit is characterized by the special point in this orbit, we can establish a correspondence between orbits in the variety and cones of the fan. The orbit O_i corresponds to the cone σ if and only if for all $u \in \text{Relint}(\sigma)$, we have $\lim_{t \rightarrow 0} \lambda^u(t) \in O_i$.

This correspondence is valid in the general case, leading to the following definition.

Definition 2.75. *Let Σ be a fan, X_Σ the variety associated to Σ and T_N be the torus of X_Σ . We define $O(\sigma) := T_N \cdot \gamma_\sigma$.*

We now need to see a nice way to associate a lattice to a cone, which will allow us to describe the orbit corresponding to a cone.

Definition 2.76. *Let σ be a strongly convex rational polyhedral cone and N_σ the sublattice spanned by σ . We define $N(\sigma) := N/N_\sigma$.*

One may want to adapt the pairing of M, N to a pairing involving $N(\sigma)$, since to an element of $N(\sigma)$ we can freely add or subtract an element of σ , in the dual lattice we need to be orthogonal to σ , therefore there is a perfect pairing between $\sigma^\perp \cap M$ and $N(\sigma)$.

Lemma 2.77. *Let σ be a strongly convex rational polyhedral cone in $N_{\mathbb{R}}$, then the orbit $O(\sigma)$ is isomorphic to the torus with character lattice $N(\sigma)$:*

$$\begin{aligned} O(\sigma) &= \left\{ \gamma : \mathbf{S}_\sigma \rightarrow \mathbb{C} \mid \gamma(m) \neq 0 \iff m \in \sigma^\perp \cap M \right\} \\ &\simeq \text{hom}_{\mathbb{Z}}(\sigma^\perp \cap M, \mathbb{C}^*) \simeq T_{N(\sigma)}. \end{aligned}$$

The next theorem first summarizes our previous discussion and then describes the variety corresponding to each cone in terms of orbits and finishes by translating the notion of a cone being a face of another cone in terms of orbits.

Theorem 2.78. (*Orbit-cone correspondence*) *Let Σ be a cone and X_Σ the variety associated to Σ . Then:*

1. *There is a bijective correspondence*

$$\begin{aligned} \{\text{cones } \sigma \in \Sigma\} &\longleftrightarrow \{T_N - \text{orbits in } X_\Sigma\} \\ \sigma &\longleftrightarrow O(\sigma) \simeq \text{hom}_{\mathbb{Z}}(\sigma^\perp \cap M, \mathbb{C}^*). \end{aligned}$$

2. *The dimension of $O(\sigma)$ is the codimension of σ .*
3. *The affine variety $\text{Spec}(\mathbb{C}[\mathbf{S}_\sigma])$ is the union of orbits*

$$\text{Spec}(\mathbb{C}[\mathbf{S}_\sigma]) = \bigcup_{\tau \preceq \sigma} O(\tau).$$

4. *Let $\sigma, \tau \in \Sigma$ be two cones, then τ is a face of σ , $\tau \preceq \sigma$ if and only if $O(\sigma) \subseteq \overline{O(\tau)}$, moreover*

$$\overline{\text{Spec}(\mathbb{C}[\mathbf{S}_\sigma])} = \bigcup_{\tau \preceq \sigma} O(\tau)$$

where the topological closure is in both Zariski and Euclidean topology.

Example 2.79. We already took a look the first point for $\mathbb{P}^2(\mathbb{C})$. The second point illustrates in the fact that the orbit of $(1, 1, 1)$ has dimension 2, while the corresponding cone has dimension 0. In the same way, the three one dimensional cones corresponds to the three orbits of dimension 1 and the two dimensional cones corresponds to the three fixed points of the torus action.

The third and fourth points corresponds to the affine components we have described in Example 2.67.

Let us now take a look at the notion of toric morphism for abstract varieties.

Definition 2.80. *Let Σ_1 and Σ_2 be two fans in $(N_1)_{\mathbb{R}}$ and $(N_2)_{\mathbb{R}}$ respectively. Let $X_{\Sigma_1}, X_{\Sigma_2}$ be the varieties associated to those fans. A morphism $\phi : X_{\Sigma_1} \rightarrow X_{\Sigma_2}$ is **toric** if $\phi(T_{N_1}) \subseteq T_{N_2}$ and $\phi|_{T_{N_1}} : T_{N_1} \rightarrow T_{N_2}$ is a group homomorphism.*

One may remember that this definition was equivalent to being a toric morphism in the affine case. Here it is more obvious to start from this definition and then to find the corresponding notion in terms of \mathbb{Z} -linear mappings.

Let us remember that in the affine case, a morphism is toric if it corresponds to a semigroup homomorphism between the semigroups generating the varieties. If we think of the semigroups as the semigroup generated by the cones, that we denote \mathbf{S}_{σ_1} and \mathbf{S}_{σ_2} , we have that the image of \mathbf{S}_{σ_2} must be in \mathbf{S}_{σ_1} . Taking the dual of this map, we get that the image of σ_1 should be in σ_2 , which is a motivation for the next definition.

Definition 2.81. *Let Σ_1 and Σ_2 be two fans in $(N_1)_{\mathbb{R}}$ and $(N_2)_{\mathbb{R}}$ respectively. A \mathbb{Z} -linear mapping $\bar{\phi} : N_1 \rightarrow N_2$ is **compatible** with the fans Σ_1 and Σ_2 if for all cone $\sigma_1 \in \Sigma_1$, there exists $\sigma_2 \in \Sigma_2$ such that $\bar{\phi}_{\mathbb{R}}(\sigma_1) \subseteq \sigma_2$.*

Let us now see a theorem that states that we have chosen the right definition.

Theorem 2.82. *Let Σ_1 and Σ_2 be two fans in $(N_1)_{\mathbb{R}}$ and $(N_2)_{\mathbb{R}}$ respectively and X_{Σ_1} and X_{Σ_2} the varieties associated to those fans.*

Then there is a bijective correspondence between the \mathbb{Z} -linear maps $\bar{\phi} : N_1 \rightarrow N_2$ compatible with Σ_1 and Σ_2 and the toric morphisms $\phi : X_{\Sigma_1} \rightarrow X_{\Sigma_2}$.

Moreover, if $\bar{\phi}$ is such linear mapping then the map ϕ associated to it is such that $\phi|_{T_{N_1}} = \bar{\phi} \otimes 1 : N_1 \otimes_{\mathbb{Z}} \mathbb{C}^ \rightarrow N_2 \otimes_{\mathbb{Z}} \mathbb{C}^*$.*

Proposition 2.83. *Let N, N' be two lattices such that N' is a subgroup of finite index of N , let Σ be a fan in $N_{\mathbb{R}}$ and $N'_{\mathbb{R}}$ and let $G = N/N'$.*

Then the toric morphism $\phi : X_{\Sigma, N'} \rightarrow X_{\Sigma, N}$ induced by the inclusion $N' \hookrightarrow N$ presents $X_{\Sigma, N}$ as the quotient $X_{\Sigma, N'}/G$.

In other words, $X_{\Sigma, N}$ is the geometric quotient of the action of G on $X_{\Sigma, N'}$, written $X_{\Sigma, N} = X_{\Sigma, N'}/G$.

Let us now see some sort of primality for toric varieties, which will take an important role when looking at toric varieties as quotients of the affine space.

Definition 2.84. Let X be a toric variety, X has a **torus factor** if there is an equivariant isomorphism between X and the product of a nontrivial torus with a toric variety of smaller dimension.

There is a nice characterization of varieties having torus factors. The first criterion will be easier to explicit when looking at the characterization of varieties having no torus factors. The second criterion is a criteria about the one-dimensional cones of the fan of the variety. Since it is a spanning criteria in a lattice, it is quite easy to verify.

Proposition 2.85. Let Σ be a fan in $N_{\mathbb{R}}$ and X_{Σ} the variety associated to this fan. Then the following are equivalent:

- The variety X_{Σ} has a torus factor.
- There exists a nonconstant morphism $X_{\Sigma} \rightarrow \mathbb{C}^*$.
- The ray generators $u_{\rho}, \rho \in \Sigma(1)$, do not span $N_{\mathbb{R}}$.

To get the corollary, we just take the negation of each bullet of the previous theorem. The first criterion tells us that the only way to have a nontrivial morphism from a toric variety to \mathbb{C}^* is to have all the nontrivial part of the morphism in a torus factor.

Corollary 2.86. Let Σ be a fan in $N_{\mathbb{R}}$ and X_{Σ} the variety associated to this fan. Then the following are equivalent:

- X_{Σ} has no torus factors.
- All morphisms $X_{\Sigma} \rightarrow \mathbb{C}^*$ are constant.
- The ray generators $u_{\rho}, \rho \in \Sigma(1)$, span $N_{\mathbb{R}}$.

We may now take a look at a proposition that describes the torus factor of a variety. One important remark is that the fan need to be of smaller dimension than the lattice it is embedded in to be able to use this proposition to describe a torus factor. (Adding a torus factor to a variety is trivial.)

Proposition 2.87. Let $N' \subseteq N$ be two lattices and Σ be a fan in $N'_{\mathbb{R}}$ and in $N_{\mathbb{R}}$. Let $\dim N_{\mathbb{R}} = n$ and $\dim N'_{\mathbb{R}} = k$, then if N' is spanned by a subset of a basis of N , $X_{\Sigma, N}$ has a torus factor given by an isomorphism:

$$\phi : X_{\Sigma, N} \simeq X_{\Sigma, N'} \times (\mathbb{C}^*)^{n-k}.$$

In general, we can find a sublattice of finite index $N'' \subseteq N$ such that N' is spanned by a subset of a basis of N'' , writing $G = N/N''$, we can extend last proposition to:

$$X_{\Sigma, N} \simeq X_{\Sigma, N''}/G \simeq X_{\Sigma, N'} \times (\mathbb{C}^*)^{n-k}/G.$$

2.3 Quotients of toric varieties

We investigate on the way to see toric varieties as quotients of the affine space. One usual way to define the projective plane $\mathbb{P}^n(\mathbb{C})$ is to consider $\mathbb{C}^{n+1} \setminus \{0\} / \sim$, where $x \sim y$ if there is a scalar λ such that $\lambda x = y$. Let us try to achieve a similar result for all toric varieties, by seeing X_Σ as the almost geometric quotient $(\mathbb{C}^r \setminus Z) // G$, where we know r, Z and G .

Let us work on that for a toric variety X_Σ with torus T and having no torus factors, we need to start from Theorem 4.1.3 of [4] which can be stated as the following exact sequence:

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathbb{Z}^{\Sigma(1)} \longrightarrow \text{Cl}(X_\Sigma) \longrightarrow 0.$$

The element $\text{Cl}(X_\Sigma)$ is called the class group of X_Σ , class group is an invariant of algebraic varieties which takes the form of an abelian group.

We can apply the left exact functor $\text{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ to all elements of the exact sequence, leading to the following exact sequence:

$$0 \longrightarrow \text{Hom}_{\mathbb{Z}}(\text{Cl}(X_\Sigma), \mathbb{C}^*) \longrightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\Sigma(1)}, \mathbb{C}^*) \longrightarrow \text{Hom}_{\mathbb{Z}}(\mathcal{M}, \mathbb{C}^*) \longrightarrow 0.$$

We will take $\text{Hom}_{\mathbb{Z}}(\text{Cl}(X_\Sigma), \mathbb{C}^*)$ as a definition for the group G , while $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\Sigma(1)}, \mathbb{C}^*) \simeq (\mathbb{C}^*)^{\Sigma(1)}$ and $\text{Hom}_{\mathbb{Z}}(\mathcal{M}, \mathbb{C}^*) \simeq T$. This can be summarized as:

$$1 \longrightarrow G \longrightarrow (\mathbb{C}^*)^{\Sigma(1)} \longrightarrow T \longrightarrow 1.$$

Let us now construct the set Z , that we will call the **exceptionnal set**.

We start by fixing a convention for the coordinate ring of $\mathbb{C}^{\Sigma(1)}$:

$$S := \mathbb{C}[x_\rho | \rho \in \Sigma(1)].$$

Then $\text{Spec}(S) = \mathbb{C}^{\Sigma(1)}$. We call S the **total coordinate ring** of X_Σ .

We now define some special monomials, which will be used to describe the irrelevant ideal, which in turns is helpful to describe the exceptionnal set.

Definition 2.88. *Let $\sigma \in \Sigma$, we define*

$$x^{\hat{\sigma}} := \prod_{\rho \notin \sigma(1)} x_\rho.$$

We can see that $x^{\hat{\sigma}}$ is the product of the variables corresponding to rays that are not rays of σ .

We can now define the **irrelevant ideal**:

$$B(\Sigma) := \langle x^{\hat{\sigma}} \mid \sigma \in \Sigma \rangle \subseteq S.$$

Now we define the affine variety corresponding to this ideal:

$$Z(\Sigma) := \mathcal{V}(B(\Sigma)) \subseteq \mathbb{C}^{\Sigma(1)}.$$

The set $Z(\Sigma)$ is the **exceptional set**.

To see X_{Σ} as a quotient, we first need a morphism $\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma) \rightarrow X_{\Sigma}$ and then we need to see that this morphism makes an almost geometric quotient.

Let us note that we use $e_{\rho}, \rho \in \Sigma(1)$ as a notation for the canonical basis of $\mathbb{Z}^{\Sigma(1)}$.

Definition 2.89. Let $\sigma \in \Sigma$, we define the cone

$$\tilde{\sigma} := \text{Cone}(e_{\rho} \mid \rho \in \sigma(1)) \subseteq \mathbb{R}^{\Sigma(1)}.$$

For a fan Σ , the collection of cones $\tilde{\sigma}$ forms a fan in $(\mathbb{Z}^{\Sigma(1)})_{\mathbb{R}} = \mathbb{R}^{\Sigma(1)}$ that we denote:

$$\tilde{\Sigma} := \{\tau \mid \tau \preceq \tilde{\sigma} \text{ for some } \sigma \in \Sigma\}.$$

Here are some properties of this fan.

Proposition 2.90. 1. The affine variety $\mathbb{C}^{Z(\Sigma)}$ is the toric variety of the fan $\tilde{\Sigma}$.

2. The map sending $e_{\rho} \rightarrow u_{\rho}$ defines a map of lattices $\mathbb{Z}^{Z(\Sigma)} \rightarrow N$ that is compatible with the fans $\tilde{\Sigma}$ in $\mathbb{R}^{\Sigma(1)}$ and Σ in $N_{\mathbb{R}}$.
3. The toric morphism corresponding to the map of the previous bullet is constant on G -orbits. We denote this map π .

Let us now finish with the main result of this section.

Theorem 2.91. Let X_{Σ} be a toric variety without torus factors, then:

1. The toric morphism π is an almost geometric quotient for the action of G on $\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma) // G$.
2. Moreover, π is a geometric quotient if and only if Σ is simplicial.

Conclusion

In this thesis we learned multiple concepts of general algebraic geometry, ranging from affine geometry, abstract geometry and quotients in the framework of algebraic geometry. We then took a look at these concepts for toric varieties.

During this journey, we learned about the geometry of polyhedral cones, we have made deep links between combinatorics and toric varieties, we analyzed the orbits of the torus actions on toric varieties. All these notions allowed to state powerful theorems.

There are many paths that personally I would have like to investigate more, but had no time to, such as the notion of prime divisors, the applications of toric varieties to coding theory and the link between toric ideals and integer programming. Nonetheless, it has been an enriching experience.

Bibliography

- [1] A. Borel. *Linear Algebraic Groups*. Springer, 1991.
- [2] A. Brøndsted. *An Introduction to Convex Polytopes*. Springer, 198.
- [3] D. Cox, J. Little, and D. O’Shea. *Ideals, Varieties, and Algorithms*. Graduate studies in mathematics. Springer, Cham, 2007.
- [4] D.A. Cox, J.B. Little, and H.K. Schenck. *Toric Varieties*. Graduate studies in mathematics. American Mathematical Soc., 2011.
- [5] M. Demazure. Sous-groupes algébriques de rang maximum du groupe de cremona. *Annales scientifiques de l’Ecole Normale Supérieure*, 4e série, 3(4):507–588, 1970.
- [6] I. Dolgachev. *Lectures on Invariant Theory*. London Mathematical Society Lecture Note Series. Cambridge University Press, 2003.
- [7] W. Fulton and R. Weiss. *Algebraic Curves: An Introduction to Algebraic Geometry*. Math Lecture Notes Series. Benjamin, 1969.
- [8] G. Kempf, F. Knudsen, D. Mumford, and B. Saint-Donat. *Toroidal Embeddings 1*. Lecture Notes in Mathematics. Springer Berlin Heidelberg, 1973.
- [9] S. Lang. *Algebra*. Springer, New York, NY, 2002.
- [10] M. Nagata. On the 14-th problem of hilbert. *American Journal of Mathematics*, 81(3):766–772, 1959.
- [11] D. Perrin. *Géométrie algébrique. Une introduction*. InterEditions-CNRS, 1995.
- [12] H. Sumihiro. Equivariant completion. *J. Math. Kyoto Univ.*, 14:1–28, 1974.
- [13] J-P. Tignol. Lectures: LMAT1331 : Algèbre commutative, 2019.
- [14] Š. Špenko. Lectures: MATH-F525 : Algebraic geometry, 2020.

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