

LOUVAIN
School of Management

UNIVERSITE CATHOLIQUE DE LOUVAIN

LOUVAIN SCHOOL OF MANAGEMENT

Behavioral Finance applied to Portfolio theory:

An empirical challenge to Markowitz' framework

Supervisor: Sophie **BÉREAU**

Research Master Thesis submitted by

Marie-Odile **OGER**

Henri **VANHOMWEGEN**

With a view of getting the degree
Master in Management

ACADEMIC YEAR 2015-2016

« *Traditional finance assumes that we are rational, while behavioral finance simply assumes we are normal* »

– Meir Statman

This research master thesis has been conducted through the supervision of Sophie Béreau, Professor at the *Louvain School of Management* in the *Finance department* and member of the *Center for Operations Research and Econometrics* (CORE).

We would like to express our deep gratitude to the people who motivated, helped and supported us during the creation of this Master Thesis by their advices, experience and words of encouragement.

Thank you,

To our supervisor, Sophie Béreau, for her wisdom, advices and unwavering trust in us and our subject;

To the Université Catholique de Louvain, and particularly to the teachers and assistants that led us to where we are today;

To the community at Stackoverflow.com, whose help during our empirical computation was amazing and very much appreciated;

To our families, friends and partner who supported us and motivated us at every step of this experience, which puts an end to our academic journey.

ABSTRACT

Over the past decades, assumptions related to modern portfolio theory such as investors' rationality, market efficiency and Gaussian returns have been challenged, namely through the emergence of behavioral finance. This paper offers a literature review of these shortcomings and presents the alternatives given by behavioral portfolio theory.

The main contribution of this paper relies on an empirical study, largely based on Bourachnikova *et al*. research (2016), that aims at presenting the comparison between Markowitz' framework and both the behavioral model (Shefrin and Statman, 2000) and its mental account extension (Das *et al*., 2010). Our objective is to determine whether or not the optimal behavioral portfolio is located on the efficient frontier (i.e. as understood by Markowitz) and, in the positive case, to discuss the relevant implications of such location in terms of risk.

Based on 2500 simulations, defined by the daily historical prices of 387 stocks stemming from the S&P500 from 2005 to 2015 (Bloomberg), we find that in the case where an optimal solution exists (i.e. 60%), 60% are located on the efficient frontier. More importantly, this solution is almost always (i.e. 95%) located on the upper right side of this frontier, meaning that it is characterized by a high risk profile. We state that while typical standard investors would see this investment opportunity as too risky, a behavioral portfolio investor would rather see it as optimal. We argue that it is the consequence of the different approaches used to assess risk in the respective optimization program of the two models (i.e. standard deviation and safety first constraint).

TABLE OF CONTENTS

INTRODUCTION **1**

PART I: MODERN PORTFOLIO THEORY **5**

1. Modern Portfolio Theory main assumptions 5

1.1. Efficient Market Hypothesis: fundamental and informational efficiency 5

1.2. Investors rationality 7

1.3. Normally distributed returns 8

2. Markowitz' Mean-Variance Framework 9

2.1. Mean and variance as inputs 9

2.2. Effects of diversification 10

2.3. Portfolio selection and efficient frontier creation 12

3. Limitations 13

PART II. BEHAVIORAL PORTFOLIO THEORY **15**

1. Origins of behavioral portfolio theory 15

1.1. Investors lack of rationality 16

1.2. Behavioral biases affecting investors. 18

1.2.1. Myopic loss aversion 18

1.2.2. Social validation/ Herding 19

1.2.3. Overconfidence and self-attribution bias 20

1.3. Efficiency disruptions 21

1.3.1. Equity premium Puzzle 22

1.3.2. Excess volatility 23

1.3.3. Limits of arbitrage 25

2. Behavioral Portfolio Theory: from Shefrin & Statman to CPT 27

2.1. Safety first constraint 27

2.2. Prospect theory 29

2.2.1. Limits of the expected utility framework 29

2.2.1. The mental account structure 31

2.3. SP/A & CPT: Transformation Function of probabilities 32
2.3.1. SP/A 32
2.3.2. Cumulative Prospect Theory 33
2.4. Further development on the CPT 34
2.5. Efficient portfolio selection 36

PART III: METHODOLOGY OF THE EMPIRICAL COMPUTATION 37

1. Data and sample creation 37
2. States of nature matrix creation 37
2.1. Historical return matrix 37
2.2. States of nature matrix 38
3. Portfolio weights matrix generation 39
4. Mean-variance portfolios 40
4.1. Mean-variance portfolio generation 41
4.2. Efficient portfolio selection and frontier creation 42
5. Behavioral portfolios 42
5.1. Safety first constraint 43
5.2. Behavioral portfolios generation 45
5.3. Optimal behavioral portfolio selection 47
5.4. Localization on the mean-variance efficient frontier 47
6. Behavioral Mental accounts 48

PART IV: EMPIRICAL ANALYSIS 51

1. Mean-Variance Portfolios 51
1.1. Useful Statistics 51
1.2. Efficient Frontiers 52
2. Behavioral Portfolios 53
2.1. Safety first Constraint 53
2.2. Optimal Behavioral Portfolios 54
2.2.1. Mean-Variance Efficient Behavioral Portfolios 54
2.2.2. Optimal Portfolios Specifications 55

3. Behavioral Mental Account Portfolios 59

3.1. Specifications 60

3.2. Limits 62

4. Conclusions 63

CONCLUSION 65

BIBLIOGRAPHY 69

APPENDIXES 75

1. Appendix 1: From the expected utility framework to the "mean-variance trade-off". 75

2. Appendix 2: Violation of the Expected Utility framework: examples 77

3. Appendix 3: Cumulative Prospect Theory in details 79

4. Appendix 4: Sample of Mean-Variance framework plots 81

5. Appendix 5: Sample of Optimal Behavioral Portfolios plots 83

6. Appendix 6: Sample of aggregate mental accounts portfolios plots 95

7. Appendix 7: Unsmoothed graph versions 97

8. Appendix 8: 'R' code 99

LIST OF ABREVIATIONS

MPT: Modern Portfolio Theory

GMVP: Global Minimum Variance Portfolio

BPT: Behavioral Portfolio Theory

CPT: Cumulative Prospect Theory

VaR: Value at Risk

MRP: Maximum Return Portfolio

OBP: Optimal Behavioral Portfolio

INTRODUCTION

Through the course of our academic studies, we have been largely familiarized to standard finance and especially to Von-Neumann and Morgenstern's Expected Utility Framework and Markowitz' Modern Portfolio Theory, which both relate to numerous assumptions such as investors' full rationality, market efficiency and, on the empirical side, Gaussian returns. However, through several of our master lectures and through our international experiences, we were able to question those grounds of standard finance. More specifically, we found that many concerns had been raised against the formerly developed theory, namely through the emergence of behavioral finance.

Applied to portfolio theory, this led us to conduct an empirical comparison between Markowitz' framework and both the behavioral model (i.e. Shefrin and Statman, 2000) and its mental account extension (Das *et al*., 2010), our goal being to discuss the similarities and dissimilarities between the different optimal allocations. Towards that end, we will follow the methodology developed by Bourachnikova *et al*. (2016) to locate the optimal behavioral portfolio in the mean-variance space and verify whether or not it will belong to the efficient frontier. Our contribution will lie on the original analysis of the results, namely the consequences stemming from the localization of the optimal behavioral portfolio on the investors' risk profile.

Through the literature review, we learnt that according to standard finance, investors were considered as being *homo œconomicus*, implying that they are consistently rational and make optimal choices based on all available information. The Expected Utility Framework is required to describe investors' rational preferences, one of the most important properties pertaining to the concavity of the utility function, which imply both decreasing marginal utility and risk adverse attitude. Going further, the efficient market hypothesis describes a market where prices and volatility are fully characterized by their underlying fundamentals (i.e. fundamental efficiency) and where historical information is fully and directly integrated in current contemporaneous assets' prices (i.e. weak form of informational efficiency). According to this hypothesis, price changes are thus supposed to follow a random walk, making them impossible to predict and preventing any investor to consistently beat the market. Returns are further assumed to follow a Gaussian distribution, characterized by its

first-two moments (i.e. mean and standard deviation), which allows to move smoothly from expected utility on wealth to the standard mean-variance framework at the core of the portfolio theory problem.

However, we found that many concerns had been raised against the formerly developed theory, mainly through the emergence of behavioral finance. More precisely, regarding the investor rationality and the expected utility framework, we point out that investors are susceptible to commit mistakes, have limitations in their knowledge and skills, and are biased by their emotions. Hence, they make choices that often violate the axioms underlying the expected utility framework, as suggested by Kahneman, Tversky (1973, 1979, 1981, 1989, 1992) and Allais (1953). More interestingly, we soon realized that those biases had an impact on the overall market: while markets are not chaotic there are typically not efficient. Multiple behavioral explanations were developed to explain financial puzzles (i.e. financial anomaly failed to be explained by standard finance), such as myopic loss aversion, over-confidence and social validation. Regarding the distribution of returns, we recall Mandelbrot's findings (1963) to highlight the fact that they are typically not Gaussian.

Going forward with our empirical development, we defined 2500 simulations based on the daily historical prices (Bloomberg) of 387 stocks stemming from the S&P500 from 2005 to 2015. Our analysis consists in three steps. Firstly, we create the mean-variance space for each simulations and their respective efficient frontier, as understood by modern portfolio theory. Secondly, we derive the optimal behavioral portfolios and locate them in the mean-variance space, allowing us to compare both outputs on the same scale and to further analyze the consequence of their position. We find that in the case where an optimal allocation exists (i.e. 60%, contrasting with Markowitz' always-existing optimal solution), 60% are located on the efficient frontier. More importantly, this solution is almost always (i.e. 95%) located on the upper right side of this frontier, meaning that it is characterized by a high risk profile. Our original contribution lies in the analysis of these results. We state that behavioral investors are not to be considered as taking more risk in absolute terms than standard investors, there rather do not value the risk the same way. As we will see, in the behavioral framework, the standard deviation is substituted by the safety first constraint (i.e. an analogous measure to the Value-at-Risk), which justifies our statement. Thirdly, we apply the same methodology to the mental account model, in which an aggregated portfolio consisting of a combination between different subportfolios is created. While the same conclusion can be draw, some discrepancies

arise, namely, the percentage of selected simulations and the proportion of optimal portfolios lying on the mean-variance efficient frontier are both lower.

The present paper is organized as follows: part I is devoted to the review of modern portfolio theory and especially to the underlying assumptions of the mean-variance framework. In part II, using a complete literature review, we describe, on the one hand, the behavioral explanations of Markowitz' framework limitations, and, on the other hand, the founding concepts lying at the core of the alternative behavioral portfolio model. Part III is dedicated to the complete and detailed version of the empirical methodology, which was based on Bourachnikova *et al.* research (2016). Part IV addresses the results of the empirical development as well as the original analysis of their intrinsic consequences. In the final part, we offer the conclusions of our work and suggest directions for further studies.

PART I: MODERN PORTFOLIO THEORY

In 1952, when Harry Markowitz introduced his work relative to portfolio theory, he revolutionized financial practices and created the foundations of Modern portfolio theory (MPT, hereafter). By considering that rational investors only cared about risk and reward, he established what we now refer as being the mean-variance optimization and stated its consequences for efficient portfolio selection. Reward was now associated to the expected return and risk to its standard deviation. Furthermore, he postulated that, thanks to the covariance factor included in the standard deviation mathematical definition, what mattered was not the individual securities composing the portfolio anymore, but rather the portfolio as a whole, i.e. the combination of those securities and how they were related to each other. He further developed this idea by asserting that rational investors should hold diversified portfolios instead of single assets: thanks to diversification, for a same level of expected return, it is possible to combine assets in a portfolio such as to obtain lower risk levels than with isolated securities.

In the first part of this paper, we will introduce the reader with the main assumptions that are at the core of MPT. We will then, broadly remind Markowitz' work on portfolio theory and efficient portfolio selection. Finally, we will draw the readers' attention on the criticisms and limitations this theory has faced over the past decades.

1. Modern Portfolio Theory main assumptions

In order to further understand the practical application of MPT and efficient portfolio selection, one has to make sure that she understands the premises of the theory and not only the mathematical optimization programs that are derived from it. In this section, we will briefly recall the main assumptions that underpin MPT and Markowitz' work: efficient market, investors' rationality and Gaussian distribution of assets' returns.

1.1. Efficient Market Hypothesis: fundamental and informational efficiency

When it comes to efficiency, Markowitz' work is based on the ideas that markets are able to quickly and perfectly take new information into account and that investors can freely have access to accurate information about risk and returns (Hiro, Sofat, 2011, p. 105).

The existence of a fundamental efficient market implies that the price of a security reflects its fundamental value, this is, the discounted value of its future cash flows given the available information. When it comes to MPT, this means that information relative to assets' prices (i.e. mean and variance of returns) is available and accessible to every investor and reflects pertinent ground on which investment decisions can be based. Under this situation, it results that it is not possible for an investor to systematically do better than the market by using the information at her disposal. Indeed, as the prices are supposed to incorporate all available information, it is not possible to further determine their evolution: they therefore represent the best evaluation of the related securities (Alphonse *et al*., 2013, p. 133).

In 1970, when Eugene Fama introduced his article « Efficient Capital Markets: A Review of Theory and Empirical Work », he gave birth to the efficient market hypothesis and to the proper definition of the 'informationally efficient markets'. According to this initial definition, a market is efficient when the prices fully incorporate the available information. He went further in his work by distinguishing three different levels of efficiency: the weak, semi-strong, and strong forms (Fama, 1970). Those forms are introduced hereafter.

Firstly, a market is considered to be weak-efficient when the prices of the securities at any point in time reflect all their past trading information, making it impossible for investors to beat the market using past prices. This form of efficiency is directly derived from the Random walk model, in which it is assumed that successive asset prices changes are independently and identically distributed (Brealy *et al.*, 2005); making the prediction of future returns from past returns impossible. Secondly, a market is considered to be semi-strong efficient when the prices take all the past information as well as the publicly available information into account. It is then not possible to make, for example, additional gains when a company announces its benefits, as any investor can use it as soon as it is publicly communicated. Finally, a market is considered to be strong efficient when the prices reveal all the information: past, public and private. Under this form of efficiency, it is then not possible for people having monopolistically access to private information to make excess gains compared to the market (Alphonse *et al.*, 2013, p. 144).

In 1984, J. Tobin went further than Fama's work by making the distinction between two types of market efficiency. The first one is what he refers to as the 'informational efficiency' and is based on Fama's definition. The second one is more demanding than the preceding one and is called the 'fundamental efficiency'. He postulates that, in such situation, prices not only

rapidly reflect all the available information, but also do it correctly: prices are said to be the best estimate of the security's intrinsic value (Lobão, 2015a).

1.2. Investors rationality

According to Markowitz, the portfolio choice is the result of rational agents' decisions, who are willing to maximize their expected return while minimizing their level of risk. Those agents are thus characterized by two main objectives. On the one hand, they want their returns on investment to be high, and thus prefer more wealth than less. On the other hand, they want their returns to be secured, and thus prefer certainty to uncertainty. The efficient portfolio choice is then the result of the rational investor's risk profile, this is, her disposition and capability to take risk (Markowitz, 1959, p. 6).

Agents under Markowitz' MPT can be described as being *homo œconomicus*. This definition implies that they are consistently rational and defined by three properties. Firstly, they are able to clearly establish their preferences, which can be represented by the following Von Neumann and Morgenstern axioms: comparability, transitivity, independency, continuity and monotonicity (1947). Secondly, their objectives are defined as clearly as their preferences: they behave in a way that allows them to reach those goals without being distracted by any other consideration (i.e. self-interest axiom). They are endlessly driven by a utility maximization goal. Finally, they are able to review their anticipations and modify their behaviors so that they can directly take new information into account without any bias (Alphonse *et al.*, 2013, pp. 131-132).

It should be added that the expected utility theory has often been the source for the definition of rational behavior and choice under uncertainty. This theory relies on three main principles when applied to uncertain outcomes (Kahneman, Tversky, 1979, p.263). The first one states that the utility of any payoffs, with their associated probabilities (i.e. prospect), can be represented as the expected utility of its outcomes, such that:

$$U(x_1p_1; \dots;\ x_np_n) = p_1u(x_1) + \cdots +\ p_nu(x_n)$$

where p_i stands for the probability to achieve the outcome x_i, and where *u()* is a utility function as defined by Von Neumann and Morgenstern. The second one states that the utility of integrating an asset, *a,* to the prospects is superior to the utility of this isolated asset (i.e. non-staiability). We have that:

$$U(a + x_1 p_1; \ldots;\ a + x_n p_n) > u(a)$$

The third one is about risk aversion under uncertain situations. According to the risk aversion principle, an agent will always prefer something for sure than a gambling leading to the same expected gain (Béreau, 2013). Under expected utility theory, this statement can only hold if the utility function is strictly concave. In this case, we face a decreasing marginal utility, with $u" < 0$.

Note that Herbert Simon goes further and defines a rational agent as follow:

> "He is assumed […] to have a well-organized and stable system of preferences, and a skill in computation that enables him to calculate, for the alternative courses of action that are available to him, which of these will permit him to reach the highest attainable point on his preference scale." (Simon, 1955, p. 99)

Considering this definition and Markowitz' mean-variance portfolio theory, one can conclude that there is no rational investor that would choose a portfolio characterized by a lower expected return and a higher level of risk than another, as it is not the 'highest attainable point on his preference scale'.

1.3. Normally distributed returns

One of the main assumption underlying Markowitz' work is that returns are characterized by their first-two moments. This can be explained through the development of Sargent (1987), which stated that the expected utility framework could be considered as equivalent to a "mean-variance trade-off".

Considering the exponential utility function defined over wealth, $\mathcal{U}(c) = -e^{-\lambda c}, \lambda > 0$ which is increasing and concave with wealth and assuming a normal distribution of wealth, we can show that expected utility of wealth is given by (appendix 1):

$$\mathbb{E}\mathcal{U}(\tilde{c}) = -e^{\lambda\left(\mu - \frac{\lambda\sigma^2}{2}\right)}$$

where μ stands for the expected wealth, σ^2 for its variance and λ for the risk aversion coefficient. Considering that an agent whishes to maximize its wealth and thus $\mu - \frac{\lambda\sigma^2}{2}$ there is a trade-off between μ and σ^2.

> "The more the agent is risk-adverse, the more he will penalize the increase in risk (σ^2), along with the increase in expected gain (μ)" (Béreau, 2013).

Expressing the expected utility on returns rather than on wealth and assuming the log-normality of returns, one can switch from the expected utility framework to portfolio theory. Let's recall the Gaussian distribution is symmetric (i.e. skewness equal zero) and fully characterized by those first two moments (Mills, 1956, p.151).

2. Markowitz' Mean-Variance Framework

In this section, we will briefly recall the important steps that Markowitz went through when he introduced its brand new theory in the 1950s. To do so, we will firstly introduce the reader with the elements Markowitz chose to define as the inputs of his theory: the mean and the variance of random returns. Secondly, we will discuss how an investor can build a diversified portfolio and the benefits of doing so. Finally, we will define what Markowitz refers to as an 'efficient portfolio' and the way an investor can be in an optimal situation where she maximizes her expected return and minimizes risk.

2.1. Mean and variance as inputs

In his revolutionary article of 1952, Markowitz rejected the idea of relying solely on the maximization of discounted expected return as a guiding strategy to portfolio selection: he mathematically proved that it did not imply diversification. Indeed, when following this logic, the investor systematically chooses to allocate all her wealth in the security giving her the highest discounted value. He also rejected the idea according to which a portfolio should be diversified by allocating wealth to securities associated with a maximized expected return (Markowitz, 1952, pp. 77-78). By studying the statistical properties of the first two moments of the returns (i.e. mean and variance), he postulated that the variance of returns should be integrated into the portfolio choice process and should be understood as a measure of the risk taken by the investor.

Consider a situation in which R_i is the random return of security i (with $i = 1 \dots N$) and X_i is the weight (i.e. percentage of wealth) allocated to this security (with $\sum X_i = 1$, meaning that all the wealth is invested). We then have that the return of the portfolio is $\sum R_i X_i$.

Consider now that μ_i is the expected value of this random variable, R_i. Recalling that the expectation of a sum is the sum of the expectations, we get that the expected return of the portfolio (E) is the weighted sum of the expected returns of all the *i* securities composing the portfolio and is equal to:

$$\boldsymbol{E = \sum_{i=1}^{N} X_i \mu_i}$$

This relation can be verified for any random variable, no matter their correlation coefficient (i.e. the way they move up or down together). The same conclusion cannot be drawn when it comes to the variance: the variance of a sum is not equal to the sum of the variances. Indeed, a covariance factor has to be taken into account. As a result, in order to derive the variance of a portfolio, one needs to know the weights to be allocated between the random return, the individual standard deviations of those returns as well as the covariances between them. We get the following equation:

$$\boldsymbol{V = \sum_{i=1}^{N} \sum_{j=1}^{N} X_i X_j \sigma_{ij}}$$

The covariance factor is given by $\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$, where ρ_{ij} represents the correlation between the random returns R_i and R_j; and where σ_i and σ_j stand for their respective standard deviation. As the variance of a portfolio depends on the correlation ρ_{ij} between its random returns, and that this correlation factor always takes values between -1 and 1, we can then broadly conclude that the variance of a portfolio is generally smaller than the average of the variances of the individual random returns corresponding to this portfolio (Markowitz, 1952). This leads us to the next part of Markowitz' work, related to diversification.

2.2. Effects of diversification

According to the findings developed in the last section of this paper, the correlation coefficient allows an investor to gain from diversification. This statement is true as long as this coefficient is different from 1. In such case, the returns are said to be perfectly positively correlated and the portfolio standard deviation stays constant over time. Markowitz highly relied on the existence of this factor to convince rational agents that they should think about their portfolio as a whole, rather than in terms of its individual assets (Markowitz, 1959, p. 114).

Markowitz went further in his development and stated that when increasing the number of securities in a portfolio, agents were able to make it less volatile without reducing the level of expected return: some part of the global risk could be diversified away. This the reason why one distinguishes two kinds of risk while mentioning the risk associated to an asset. On the one hand, we have the non-systematic, or idiosyncratic risk, which can be decreased through diversification. On the other hand, we have the systematic risk, or market risk, that diversification cannot eliminate.

Consider a *N* asset situation, for which *1/N* is the fraction of asset *i* invested in the portfolio (with *i* = 1…N); the variance of the portfolio is:

$$\boldsymbol{V = \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{N^2} \sigma_{ij}}$$

This can be re-written as:

$$\boldsymbol{V = \sum_{i=1}^{N} \frac{1}{N^2} \sigma_{ii}^2 + V = \sum_{i=1}^{N} \sum_{i \neq j}^{N} \frac{1}{N^2} \sigma_{ij}}$$

Which is equivalent to:

$$\boldsymbol{V = \frac{1}{N}\left(\frac{1}{N} \sum_{i=1}^{N} \sigma_{ii}^2\right) + \frac{N^2 - N}{N^2}\left(\frac{1}{N^2 - N} \sum_{i=1}^{N} \sum_{i \neq j}^{N} \sigma_{ij}\right)}$$

The first term into brackets corresponds to the average variance of the returns, while the second term gives their average covariance. As *N* increases and tends to infinity, the contribution of individual variances decreases and tends to 0 (as $1/N$ tends to 0), while the contribution of covariance terms approaches the average covariance (as N^2-N/N^2 tends to 1). This last term corresponds to the part of the global risk that cannot be removed when diversifying (i.e. market risk).

Although this rationale is important, Markowitz argues that it is not sufficient for an investor to hold a large number of securities and that what really matters is to choose securities and weights so that the average covariance is small. To this purpose, he recommends agents to hold securities from different industry and with different economic features (Markowitz, 1959, p. 89).

2.3. Portfolio selection and efficient frontier creation

According to Markowitz, a rational investor should select, among the set of feasible portfolios, what he refers to as being an « efficient portfolio ». He defines it as follow:

> "If a portfolio is efficient, it is impossible to obtain a greater average return without incurring greater standard deviation; [or similarly,] it is impossible to obtain a smaller standard deviation without giving up return on the average." (Markowitz, 1959, p.22)

In order to find a set of feasible and efficient portfolios for the investor, one needs to identify the optimal weights that minimize the risk for a given level of expected return (or analogically that maximize the expected return for a given level of risk).

Consider a situation in which one minimizes the risk for a level μ_k of expected return. By restricting the set of portfolios to the one that are feasible (i.e. $\sum_{i=1}^{N} X_i = 1$), we get the following optimization program:

$$min_{X_1 \dots X_N} V = \sum_{i=1}^{N} \sum_{j=1}^{N} X_i X_j \sigma_{ij}$$

$$\text{s.t.} \quad \sum_{i=1}^{N} X_i = 1$$

$$\sum_{i=1}^{N} X_i \mu_i = \mu_k$$

By solving this program with different values for the expected return and plotting the results, we get a parabola made of the feasible portfolios. At this point, one still needs to identify the part of this envelope curve made of the efficient portfolios. It is the global minimum variance portfolio (GMVP), located at the bottom of the parabola, that sets the limit between feasible-efficient portfolios and feasible-inefficient portfolios. The GMVP has the smallest variance among all the feasible portfolios. The efficient frontier is thus made of the upper part of the parabola (i.e. the one located above the GMVP).

Once those portfolios have been identified, it is up to the rationally acting investor to choose the one that suits best her needs according to her risk profile this is, her disposition and capability to take risk (Markowitz, 1959, p 6.).

3. Limitations

When it comes to market efficiency, numerous examples have proven that this assumption was failing: from speculative bubbles to excess volatility of asset prices relative to fundamentals (Lobão, 2015a). Many economists started to develop theories according to which security prices could be predictable based on past information. Studies were conducted to describe market anomalies and gave evidences for the existence of dynamic return occurrences (Slezak, 2003, p. 523). In 1999, Lo and MacKinlay, stated, for example, that stock prices in the short run exhibited some correlation and could thus not follow a random walk, challenging the so-called weak form of efficiency. Short-term momentums, defined as a tendency for price to keep following the same pattern, were used to illustrate this postulate. Similarly, Fama and French (1988) found that some of the long-term return variation could be predicted from past returns and proved the existence of return reversals (i.e. negative serial correlation between returns). Another market anomaly that can be mentioned and that challenges the semi-strong form of efficiency is the post-earnings announcement drift. According to Ball and Brown, this can be defined as the tendency of returns to flow in the same direction for a substantial period of time after an earnings announcement (1968).

This first criticism is directly linked to the second one, according to which investors do not adopt a complete rational behavior. Many times, theorists have justified anomalies by pointing investors' behaviors. The rationality hypothesis has proven to be too restrictive to properly understand financial phenomena (Alphonse *et al*., 2013). On the contrary, agents are susceptible to commit mistakes, have limitations in their knowledge and skills and are influenced by their emotions (Lobão, 2015a). Indeed, investors have often proven to exhibit biased behaviors when making decisions. A typical example is the overconfidence bias, according to which an investor is too confident in her knowledge and skills and would typically buy more shares after a stock price appreciation (Montier, 2010). Note also that, as previously mentioned, rationality has often been defined through the expected utility function. However, agents have often made choices that violated the principles lying at the core of this function. As we will see, Kahneman and Tversky defended that the expected utility theory was not *an adequate descriptive model* for situations of choices under uncertainty (1979, p. 263). Famous examples, such as the Allais or Condorcet Paradoxes have proven the existence of deficiencies in the standard preference theory (appendix 2), as defined by Von Neumann and Morgenstern's axioms and utility function.

A third criticism to Markowitz' work relies on the fact that returns are supposed to follow a Gaussian distribution. However, financial series have proven to exhibit some asymmetries (i.e. positive or negative skewness) and leptokurtosis (i.e. peak in the distribution), which are inconsistent with normality. Indeed, as uncovered by Mandelbrot, some observations go against normal distribution of returns. First, significant price variations are much more recurrent than the one predicted by this distribution. Second, instantaneous significant price changes occur often, contrary to prediction (Mandelbrot, 1963, p.372). Finally, there seems to exist recognizable patterns in price changes (e.g. momentum), challenging the Random walk model as previously stated. The normal distribution is then not able to properly describe the size and the frequency of extreme events. Theories suggest that higher moments than the mean and the standard deviation such as skewness or kurtosis should be considered (Urbani, n.d.). Brimberg and Hurley (2015), Roncalli (2014) raised another criticism concerning the sensitivity of optimal weights (i.e. allocations) to parameters such as the variance and especially the mean return vector. Brimber and Hurley argue that:

> "the first-order conditions of the Markowitz portfolio optimization model result in a linear system that is ill-conditioned and it is this poor conditioning that leads to the extreme sensitivity of the portfolio weights" (Brimberg & Hurley, 2015, p)

Consequently, according to Roncalli (2014), the resulting portfolio might not be well diversified.

While Markowitz' approach has been generalized and is now used to some extent by portfolio managers and financial analysts, the assumptions that were at the core of his work have been challenged over the past decades, considered by many as not predictive of the real world. The first two criticisms according to which market are inefficient and agents are not completely rational lie at the foundation of behavioral finance, a theory that has seen the light thirty years ago and that is at the core of the second part of this paper.

PART II. BEHAVIORAL PORTFOLIO THEORY

In this section we will explore the origins of Behavioral portfolio theory (BPT, hereafter) and elaborate upon the theories on which it stands. Our goal is to clearly establish the reasons that led Shefrin and Statman (2000) to create a new model of portfolio investment. To do so, we will explore the main weaknesses of MPT and analyze the alternatives provided by behavioral finance. We will detail the Prospect theory and the Cumulative prospect theory (Kahneman and Tversky, 1979;1992). Then applying those theories to BPT, we will analyze their effect on both the weighting and value functions, and the new efficient portfolio selection resulting from them. Finally, we will go further by introducing the concept of mental accounts (Kahneman and Tversky, 1979) applied to BPT, its assumptions and limitations.

1. Origins of behavioral portfolio theory

BPT was developed in reaction to multiple criticisms targeting the MPT framework since the seventies. As each pillars on which the MPT was build (investor rationality and expected utility, market efficiency, Gaussian distribution of returns) had been weakened by new discoveries and studies, Shefrin and Statman proposed a new framework to portfolio theory in an attempt to generate a more realistic model.

Investors' rationality was first dismissed by Simon (1957) who defined a new concept named 'bounded rationality'. It states that investors simplify the world surrounding them (through the use of models) because of their cognitive limitations, and make decision based on limited information, which does not lead to optimization.

Nevertheless, the decisive blow was inflicted when Daniel Kahneman and Amos Tversky (1981, 1989) who uncovered different biases (heuristics) afflicting agents making decisions under uncertainty, thus contradicting the theory of rational choice.

Market efficiency, has been one of, if not the most, discussed topic in economics during the last few decades. According to Howard (2014, p.2) the first shots fired at the efficiency paradigm were two studies published by Basu (1977) and Banz (1981) respectively, which showed disruptions in the market efficiency hypothesis. The following decades, a colossal amount of studies were pursued both in an effort to comfort or dismiss the hypothesis (Mehra

& Prescott, 1985; Shiller, 2003; Shleifer &Vishny, 1997). The principal efficiency disruptions will be further discussed in this chapter.

Finally, Kahneman and Tversky (1979,1992), departing from the expected utility framework and its shortcomings, created the prospect theory; followed thirteen years later by the Cumulative Prospect Theory (CPT, hereafter). Prospect Theory departs from the traditional mean-variance framework and is partly integrated in the BPT framework, allowing to leave the variance out of the risk measurement and to substitute it by the safety first constraint (Roy,1952; Tesler,1955) or the SP/A theory (Lopes, 1987). Bourachnikova *et al.* (2016) went further and fully integrated the CPT in BPT.

1.1. Investors lack of rationality

As stated previously, according to MPT investors can be seen as *homo æconomicus*: they are rational, self-interested and make decisions to maximize their expected utility. This assertion is quite strong and unrealistic, as it supposes that agents would be comparable to perfect 'machines' who would, without failure, make an optimal choice based on all available information. This is the reason why Simon introduced the concept of bounded rationality (Simon,1957;1972), which answered some flaws in the rational choice theory, namely choice under uncertainty, limited information and complexity.

> "The first consequence of the principle of bounded rationality is that the intended rationality of an actor requires him to construct a simplified model of the real situation in order to deal with it. He behaves rationally with respect to this model, and such behavior is not even approximately optimal with respect to the real world" (Simon, 1957, p.198)

A bounded rational investor does not use all the available information to make an investment decision but rather behaves rationally within her simplifications of the world, which does not necessarily lead to a rational aggregate market.

Later on, Kahneman and Tversky (1973) published an article that described behavioral biases and heuristics used by people when facing choice under uncertainty: the two psychologists had set, without knowing it, the corner stone of behavioral finance. In this aforementioned article and several others following (1979; 1981; 1989), the authors identified the most common errors (with respect to rationality) that individuals would perpetrate while facing choices under uncertainty and proved that individual investors were not typically rational.

Howard (2014) described investors who succumb to these biases and heuristics as the 'emotional crowds'.

> "Emotional Crowds are made up of those investors who base decisions on emotional and intuitive reactions to unfolding events and anecdotal evidence" (Howards, 2014, p.8)

They are therefore not fully rational, as their actions are driven by various psychological biases and their goals are to reach a satisfactory level (Simon, 1956, p. 129), which generally does not 'optimize'. Applied to portfolio theory, this means that agents would not necessarily choose an efficient portfolio (efficient as understood by MPT), as their choice could be misguided by emotions and heuristics, leading them to faulty allocations of wealth to reach a satisfactory state.

Kahneman (2011, p.41) defined this behavioral pattern as 'System 1 thinking', which is *automatic and quick, requires barely no efforts nor any sense of voluntary control*. System 1 is based on heuristics, this is, short cuts of the mind that people use on a daily basis to facilitate their decision making processes. It has been hard-wire in our brain since the first human being in order to survive (Howard, 2014, p.8). System 1 allows us to be much more efficient and quick in our life and to free our brain from unnecessary and fastidious decisional tasks. While heuristics are extremely useful in our every day life, they can prove to be counter-effective when applied to investment decisions. Such decisions demand in-depth analysis and focused attention to be successful. Indeed, investing is by nature complex, as one needs to juggle between numerous variables in an attempt to predict correctly the evolution of a particular stock or set of securities.

Another example can be retrieved in a book written by Barber & Odean (2011), where the authors stated that the investors' behavior is not truthfully predicted by the model of rational choice. According to them, investors typically hold undiversified portfolios, trade securities too often and are influenced by circumstantial factors.

Howards (2014, p.14) summarized this investors' lack of rationality in his first principle of behavioral portfolio management: *Emotional crowds dominate market pricing and volatility*, meaning that, usually, irrationalities steaming from investor will drive the price of securities in the stock market and influence volatility. Interestingly, this does not mean that markets are completely chaotic, as stated by Lester & Yang (2007, p.1918) and corroborated by a

substantial literature (Mehra & Prescott, 1985; Shiller, 2003; Shleifer &Vishny, 1997). Those theorists demonstrated that markets as whole do not show signs of consistent rationality nor complete irrationality. We will further discuss this topic in section 1.3.

In behavioral finance, investors are rather viewed as 'normal' (Statman, 1999, p.20): they are subject to behavioral biases. These behavioral biases are the subject of our next point.

1.2. Behavioral biases affecting investors.

Normal investors are subject to many behavioral biases that will affect their choices, and thus, their final level of wealth. In order to understand Shefrin & Statman's framework and Kahneman & Tversky prospect and cumulative prospect theories, we will highlight and explain some of the most recurrent biases from which investors might be subject to.

1.2.1. Myopic loss aversion

> "Loss aversion is the property of the prospect theory value function (Kahneman and Tversky 1979) that reductions in wealth, relative to the current reference point, are weighted much more heavily than increases in wealth." (Benartzi & Thaler, 1999, p.365)

As quoted above, the ideas of loss aversion relies on the fact that the pain associated to a loss of amount x is stronger than the joy associated to a gain of the same amount x. This 'irrationality' prevents investors from investing in a perfectly acceptable transaction, and therefore keeps them from maximizing their wealth. A very simple example, and yet enlightening, of loss aversion can be showed by a roll of dice. Let's pretend that one faces a game that pays 110€ if the number is even and makes her pay 100€ if the number is odd (one can repeat the game as much as she desires, given that she owns 100€). It has been empirically proved (Thaler *et al.*, 1997) that one would typically not participate, even if the potential gain (i.e. expected return of 5%) was higher than the respective loss. In order to have a full participation, the game should award 200€ instead of 110€ with the same loss of 100€, which means that investors weight their loss twice as much as their gain (Howard, 2014).

One of the main differences between BPT and MPT comes from the distinction between the notions of loss aversion and risk aversion. According to Markowitz' framework (1952), investors are risk averse, which means that they choose investments that minimize variance

(risk) with respect to a target expected return whereas in BPT, investors are loss averse. The main consequence is that the value function threats gain and losses differently, i.e. there is an asymmetry between gains and losses to reflect investors tendency to be risk seeking while facing losses (i.e. they are unwilling to admit a loss) and risk averse while facing gain. This last point will be further analyzed in section 2.

'Myopic', in the introductory definition, refers to the propensity of investors to overly focus their attention on short-term results. Investors with long-term investment horizon tend to evaluate their portfolio over short periods, multiple times, and are affected by losses (loss aversion bias) occurring over theses periods.

> "Rather than viewing such situations as 20, 30 or 40-year time periods, we view them as a series of days, months, quarters, or years. We evaluate performance in each short time period and apply 2-to-1 loss aversion to our gains and losses in each period" (Howard, 2014, p.31)

More precisely, Pound and Shiller (1989) discovered that individual investors could use up to thirty minutes a day solely to evaluate the most recent addition to their portfolio.

Myopic loss aversion is at the center of the 'equity risk premium puzzle', which is up today (as you will understand in point 1.3) one of the most striking example of a market anomalies that cannot be explained by standard finance.

1.2.2. Social validation/ Herding

In the context of MPT, individual investors are self-interested. To put it in another way, they only care about their investment possibilities and do not, in any case, take interest in other's investment opportunities. Nonetheless, this axiom is constantly violated in the real world: humans are social and therefore cannot escape the influence that each individual has on one another (Howard, 2014). They tend to follow trends or to react disproportionally to anecdotal information, which could result in a radical shift of the market. Investors typically do not like to move in opposite direction from the crowd.

Studies such as Sias (2004) found that not only individual investors were subject to herding, but that institutional investors were concerned as well.

> "The growing institutional presence has led to a common perception that institutional herding (i.e., institutional investors following each other into and out of the same securities) impacts security prices and leads to excess volatility and market fragility." (Sias, 2004)

Graham (1999) went further into the subject and empirically proved that institutional analysts could dismiss private information if this information turned out to be in contradiction to a strong public information. From a different perspective, Nofsinger & Sias (1999) used herding to explain disruptions in market efficiency and provided many detailed examples on how strong social validation was and the impact it had on the market (see section 1.3.).

Consequently, believing that investors act in completely independent fashion is not realistic: they interact with one another and form a complex net of connexions. This process highly influences individual investors and their investment strategy. Sometimes, when a systemic event occurs (Lehman' Brothers bankruptcy for example), a 'social consensus' arises and the market as a whole can shift extremely rapidly and drastically. This phenomenon is the main cause of "bubbles" (Montier, 2010). Indeed, during the internet bubble, investors were charmed by the high market value (and zero book value) of internet stocks and herd by investing gigantic amount of money in overprice stocks. Yet, when doubt arose over those particulars stocks, the market flipped (herding in the opposite direction) and the bubble burst. This shows the global impact that this bias can have on markets.

To conclude, Montier (2010, p.96) found empirical evidences demonstrating that, while investors think they are 'independent thinkers', it is typically not the case. They tend to under-valuate the effects of other people's opinions on their decision-making and to be overconfident of their own capacity, which is the subject of our next point.

1.2.3. Overconfidence and self-attribution bias

> "The proposition that investors are overconfident about their valuation and trading skills can explain high observed trading volume." (Statman, Thorley & Vorkink, 2006, p.1531)

Multiple empirical studies over the last decades found that individual investors were overconfident about their own trading abilities: between 60 to 80 % of them believe that they are above average when it comes to, between other skills, trading (Svenson, 1981).

Then, what are the consequences of this bias on investment strategies? Investors who are overconfident tend to trade too frequently, as pointed out by Barber and Odean (2011), which leads to a lower excess return because of the transactions cost faced at each trading steps. Intuitively, this means that this bias prevents individual investor to reach an optimal level. In terms of risks, overconfident investors tend to see their own investments less risky than they are in reality, making more optimistic predictions about future returns (Barber & Odean, 2001).

Nofsinger (2001) stated that when investors are gaining experience, and when more information is available, they feel more secure about about trading, even though dozens of unpredictable factors are still embedded in financial decisions. The illusion of control is also highlighted by Nofsinger (2001) as being one of the main drivers for overconfidence: investor's involvement in the financial decision processes is proportional to the illusion of control she will sense. This means that when an investor is solicited during the entire process of trading, (i.e. without going through an institutional investor), her sense of control will become more intense.

Note that the self-attribution bias, is a different, yet linked, bias. It describes a tenacious tendency for investor to impute their success on their own capabilities, while blaming their losses on bad luck or unpredictable events, rather than to acknowledge their own mistakes (Hastorf, Schneider & Polefka, 1970). Self-attribution can be linked to the overconfidence bias. Indeed, when traders are facing a streak of positive results, they tend to credit their own superior capabilities as investors: consecutives positive results comfort them in their belief and tend to exacerbate their overconfidence (Daniel, Hirshleifer & Subrahmanyan, 1998). This explains that overall overconfidence can fluctuate in the market. The consequences will be explored in the next section.

1.3. Efficiency disruptions

> "Financial markets have been subject to speculative fads, from Dutch tulip mania to junk bonds, and to occasional dramatic losses in value, such as occurred in October1987, that are hard to interprets as rational" (Patel, Zeckhauser, Hendrick, 1991, p.232).

In the previous section, we gave some examples of behavioral biases that could affect individual investors. We will now go further and analyze the consequences of those 'irrationalities' on the global market and will derive how it can explain modern 'financial

puzzles', which find no answer in the market efficiency hypothesis. In the first part of this paper, we distinguished between 'informational efficiency' and 'fundamental efficiency'. We will now examine how each one of these axioms hold when investors are no longer considered as rational. Indeed, according to Julio Lobão (2015a), differentiating the two notions is essential to behavioral finance, as bounded rationality of investors might not affect efficiency in its informational sense (information create rapid fluctuation in price). However, the resulting market will not be fundamentally efficient as information will not be interpreted correctly and prices will not reflect the intrinsic value of securities.

1.3.1. Equity premium Puzzle

The first mention of the equity risk premium puzzle came from an article written by Mehra and Prescott (1985).

> "Historically the average return on equity has far exceeded the average return on short-term virtually default-free debt. Over the ninety-year period 1889-1978 the average real annual yield on the Standard and Poor 500 Index was seven percent, while the average yield on short-term debt was less than one percent." (Mehra & Prescott,1985, p.145)

Their paper aimed to find the reason behind this excessive risk-premium and determine whether or not it could be explained by standard model (based on market efficiency). Their results were unambiguous: it could not. Indeed, although equities are riskier than bonds (in the short run), the magnitude of the equity risk premium is way too high to be solely explained by risk aversion.

In 1997, Siegel & Thaler extended Mehra & Prescott's work in an attempt to find a plausible explanation to this particular issue; they found it in the previously explained concept of 'myopic loss aversion'. According to the authors, when investors have loss averse preferences (such as the typical 'normal' investor assumed by the BPT), then the time horizon on which they evaluate their returns is crucial. Let's take a simple example in which an investor holds a portfolio with a twenty-year investment horizon but evaluates it at the end of each day. Consequently, she will face a considerably larger amount of smaller losses than if she was to evaluate it every year, or better yet, at the end of the period. Because her utility is driven by the overall change of value in his portfolio (i.e. the return) (Siegel & Thaler, 1997, p.197) and because she is much more sensitive to loss than gain, it is obvious that she will suffer a

greater loss of utility if she is to evaluate the portfolio too frequently. Certainly, she will ache from the short-term losses and will try to offset them with new trading positions, which will increase the level of trading and transaction cost thus reducing her final excess return. On the other hand, if she is able to restrain herself and only evaluate her portfolio at the end of the period, not only will she suffer less transaction costs, but she will also decrease her loss, as for periods over twenty years, stock markets have never faced a decrease in real value (Siegel & Thaler, 1997, p.197).

This lead to the final conclusion of the article, according to which, if investors were not affected by 'myopia' and focused more on long-term investment, they would invest much more in stocks than in securities, as they dominate bonds over the long term (Lobão, 2015b). Siegel (2008) found that the safest long-term investment (in order to preserve their purchasing power (Lobão, 2015b)) is a diversified portfolio made of equities.

However, because stocks are riskier than bonds on short-term period and that investors evaluate their portfolio too frequently, they perceive the stock market as riskier than it actually is and therefore ask for a higher risk premium. This leads to fundamental inefficiency but does not affect informational efficiency.

> "If investors focus on the long-term returns of stocks they would recognize how little risk there is, relative to bonds, and would be happy to hold stocks at a smaller equity premium. Instead, they consider short-term volatility, with frequent mental accounting losses, and demand a substantial equity premium as compensation." (Rabin & Thaler, 2001, p.227)

Myopic loss aversion therefore gives a plausible explanation of the equity risk premium puzzle and leads to question the efficiency of using short-term standard deviation (i.e. volatility) to assess the portfolio risk, which is the subject of our next point.

1.3.2. Excess volatility

The level of volatility encountered in the market represents the second persistent anomaly in financial market that standard finance failed to explain. This anomaly was first questioned by Shiller in the beginning of the eighties and summarized in his book 'Market Volatility' (1992), where he empirically disproved the efficient market theory and, especially, the fundamental efficiency axiom.

According to the efficient market theory, prices and volatility in stock prices should be explained by their underlying fundamentals (data that affect the stock price beside the stock trading itself). However, through extensive empirical analysis, Shiller proved that in the real world, volatility was typically not explained by change in fundamentals, but rather by mass psychology, thus violating the fundamental efficiency axiom. This led him to become one of the most influential author of behavioral finance and one of the loudest speaker against the market efficiency hypothesis.

> "The evidence regarding excess volatility seems, to some observers at least, to imply that changes in prices occur for no fundamental reason at all […]." (Shiller, 2003, p.84)

But what could be the cause behind this so-called excess volatility? In his book 'behavioral portfolio management', Howard (2014, p.11) stated that Emotional Crowds, in general, dominate pricing and volatility. In other words, 'normal investors' tend to overwhelm arbitrageurs (which the author refers as Behavioral Data Investors). As a result, prices do not reflect the underlying value of the asset (even at the overall market level). He further argues that prices are driven by emotions (results of behavioral biases) rather than fundamentals; emotions, as understood by Howard, are similar to what Black (1986) described as market noise.

> "Noise in the sense of a large number of small events is often a causal factor much more powerful than a small number of large events can be. Noise makes trading in financial markets possible, and thus allows us to observe prices for financial assets." (Black, 1986, p.529)

According to the author, noise is to be distinguished from information: *noise is the arbitrary element in expectation*, which leads to faulty assessment of expected return and price. Because of noise, markets are fairly inefficient; yet at the same time prevents investors from easily taking advantages of this efficiency, as we will see in the next point over the limits of arbitrage. Shiller confirms both authors' findings by stating that, while markets are not completely chaotic, there are filled with noise, so much so that it dominates the overall movement of the aggregated stock market (Shiller, 2003, p.90).

These researches have considerable consequences on the understanding of volatility. In MPT, risk is measured through the standard deviation (or variance) of portfolios; standard deviation

being a measure of assets/portfolios volatility on the market. Because emotions/noise dominate more often than not the market and the price discovery process, it has a direct impact on volatility. Indeed, excess volatility can be explained by the overreaction of crowds to some market information (Howard, 2014, p.18). This overreaction can be caused by the overconfidence bias, which as stated before, can fluctuate and thus increase or decrease excess volatility. For instance, Scheinkman & Xiong (2003) found that when a bubble is forming in the market, volatility increases substantially because of the differences between overconfident investors.

> "Fluctuations in the value of the bubble contribute an extra component to price volatility. We emphasize that the bubble in our model is a consequence of the divergence of opinions generated by the overestimation of informativeness of the distinct signals". (Scheinkman & Xiong, 2003, p.1186)

Therefore, Howards (2014) made the powerful statement that volatility is mostly a measure of emotions embedded in the market rather than a risk measure. While short-term volatility does impact risk in the short-term, it plays a much smaller role in the long term. This discards the mean-variance optimization used in the MPT, as standard deviation is no longer deemed as an unbiased estimate of risk.

Several new measures of risk have seen the light in the last decades. For example, the semi-variance (Markowitz, 1959), which only cares about the down-side variance or the Value-at-Risk (VaR, hereafter) to cite a few. However in the scope of behavioral finance, we will see in section two that the measure of risk will be defined as the safety first constraint, which can be assimilated to the VaR.

1.3.3. Limits of arbitrage

> "Textbook arbitrage […] requires no capital and entails no risk. In reality, almost all arbitrage requires capital, and is typically risky. Moreover, professional arbitrage is conducted by a relatively small number of highly specialized investors using other people's capital." (Shleifer & Vishny, 1997, p.35)

Shleifer & Vishny (1997) analyzed the assertion that arbitrage, in efficient market, allows driving price back to their fundamental value and ensure that assets are correctly priced. However, we discussed studies in the previous section that demonstrated that it is most

generally not the case: neither price nor volatility are (fully) explained by fundamentals. We argued that this was caused by general overconfidence in the market and by myopic loss aversion, afflicting both individual and institutional investors. We will now give some hindsight about the last anomaly we wanted to discuss, namely arbitrage and its limitations.

In an efficient market, although investors might sometimes behave irrationally, the overall market will always be efficient. This means that when investors behave irrationally, arbitrageurs will exploit the resulting irrationality and will drive the price back to its value, leading to an overall efficient market, fundamentally speaking.

> "Arbitrage is the transaction where a rational agent tries to profit from mispricing" (Pontiff, 2006, p.36)

According to standard finance, this mechanism does not have any cost nor risk embedded in it. Nonetheless, researches have proven this statement to be too simplistic and incorrect (Shleifer, Vishny, 1997; Pontiff, 2006). Indeed, when transaction and holding costs are taken into account in the equation, arbitrage becomes risky and costly, which prevents it from fully driving the irrationalities out of the market. Pontiff (2006, p.36) points out that mispricing is still present in equilibrium, as arbitrageurs are not able to completely offset the consequences of investors' irrationalities on price, because of the cost link to arbitrage.

Howard (2014, p.14) refers to this phenomenon by stating that *emotions trumps arbitrage*, meaning that arbitrageurs (Behavioral Data Investor) do not dominate the price discovery process. As a result, irrationality will prevail on arbitrage and price distortions will be possible and potentially usable.

Another cause to this anomaly comes from the fact that professional arbitrage is executed by a small number of specialized professionals (Shleifer, Vishny, 1997, p.35). Standard finance views arbitrageurs as a large amount of small investors who, alone, have no impact on price distortion but whose collective action will drive irrationalities out of the market. Though, Shleifer & Vishny (1997, p.36) argue that those investors typically do not have the knowledge or information to uncover arbitrage opportunities and engage in them, professional arbitrageurs, however, are highly specialized and possess both the knowledge and information to take large position (with the funds of their client) and engage in arbitrage. We stated previously that institutional investors could also suffer from behavioral biases and deviate from rationality (i.e. herd). Yet, arbitrage, by definition, requires to take an opposing position

from other investors deemed as irrational. This can prove to be emotionally difficult as it requires to go against the crowds. Because professional investors use the resources of external investors (clients) to engage in arbitrage, they need to convince their clients of the necessity to do so. According to Howard (2014), this relation can explain, to some extent, why arbitrage fails to drive price back to their fundamentals. Institutional investors are then prompt to *emotional catering*, meaning that they sometimes choose to herd to satisfy their clients decisions, instead of making them realize their mistakes.

> "The path of least resistance, of course, is to cater to client emotions, since trying to modify client behavior is difficult. Be that as it may, one of the most important things an advisor can do is help clients avoid emotional investing mistakes and thus mitigate the resulting losses." (Howards, 2014, p.117)

This concludes our analysis of the main criticisms regarding the market efficiency hypothesis and their consequences. In the following section we will introduce a new model for portfolio valuation, namely the Behavioral Portfolio Model introduced by Shefrin & Statman in 2000, discuss its fundamentals, and go further in its development by implementing CPT.

2. Behavioral Portfolio Theory: from Shefrin & Statman to CPT

In 2000, when Shefrin & Statman introduced a new positive behavioral portfolio theory, they based their work on previous studies such as the safety first constraint (Roy,1952; Tesler,1955), the SP/A theory (security, potential and aspiration theory) (Lopes, 1987) and parts of the prospect theory (Kahneman & Tversky, 1979). In this section we will give a brief overview of each of these founding researches and go further by implementing CPT (Kahneman & Tversky, 1992) in BPT (BPT_{CPT}) following the methodology pursued by Bourachnikova *et al.* (2016).

2.1. Safety first constraint

The safety first constraint is a theory of choice under uncertainty developed by Roy (1952), which is characterized by a level of subsistence 's' and a level of wealth 'W'. The optimization under Roy's principle consists in minimizing the probability of 'W' to be inferior to level 's', namely the probability of ruin (P {W< s}). Roy supposes normally distributed returns and argues that investors minimize the number of standard deviation where

‘s’ is lower than μ_p (supposing that ‘s’ is always inferior to μ_p, μ_p being the mean return) leading to the following objective function (Shefrin & Statman, 2000, p.130):

$$\text{Min } (s - \mu_p)/\sigma_p$$

Roy implemented his strategy to overcome the limitations of the expected utility framework (Von Neumann & Morgenstern, 1944; 1947), which is implemented in MPT. Indeed, in 1948, Friedman & Savage discovered that investors could combine the purchase of insurance policies and lottery tickets, meaning that they typically do not have a uniform attitude towards risk as supposed by the excepted utility frameworks (Shefrin & Statman, 2000, p.127-128). While expected utility function is concave over all assets, Friedman & Savage’s function is concave over insurance policies and convex over lottery ticket, reflecting an investor different attitude towards risk. We will see in our following point that Prospect theory and CPT use a similar function but defined over losses and gains instead. Roy’s safety first theory is consistent with Friedman & Savage’s observation; yet, his theory was further developed by Tesler in 1955.

> “Telser (1955) developed a model that features both a fixed subsistence level s and a ruin probability α. In Telser's model, a portfolio is considered safe if the probability of ruin does not exceed α.” (Shefrin & Statman, 2000, p.130)

Therefore, the objective function developed by Telser and applied to portfolio theory has the form:

$$\text{Max } E(R) \text{ s.t. } P\{W < s\} \leq \alpha$$

where W stands for the final wealth distribution and R for the portfolio return. Using the safety first constraint allows to leave standard deviation out of the equation and avoid its previously developed limitations. As one can observe, this constraint is based on an extreme quantile α of the final wealth probability distribution. It can be assimilated to a statistical concept labeled ‘value at risk’ (VaR, hereafter), which corresponds to a measure of risk.

> “VaR measures the potential loss in value of a risky asset or portfolio over a defined period for a given confidence interval” (Damodaran, 2011, p. 431).

Indeed, Charpentier and Oulidi (2009, p. 492) postulate that:

> "Roy (1952) relates the disaster to the idea of ruin in insurance, which can also be related, with a modern perspective of financial problems, to Value-at-Risk. In this "safety first principle", investors are looking for portfolio which minimize disaster probabilities, with some expected return constraint"

Therefore this aforementioned objective function is, much like the one introduced by Markowitz, defined over a tradeoff between risk and return. Nevertheless the maximization of the expected returns is, in this case, made under the constraint that the risk (i.e. probability of falling below the subsistence level) lays below a threshold α (i.e. ruin probability), defined ex-ante. This diverges from Markowitz framework where the optimization of the expected returns is constraint by the requirement to reach (or be below) a targeted level of risk (i.e. variance).

We will see in the following point and in the empirical analysis that the objective function E(R) will be replace by E_h(W), thanks to a transformation function of probabilities (Lopes, 1987; Kahneman and Tversky, 1992), and that the subsistence level will be generalized by the aspiration level (Lopes, 1987).

2.2. Prospect theory

As we presented previously, Kahneman & Tversky found that investors were typically not rational while facing choice under uncertainty because of behavioral biases affecting their judgment when investing. Additionally, we will now show that the authors disproved the expected utility framework, which led them to introduce the Prospect theory in 1971. Because BPT only focuses on some part of the theory, namely the mental account structure of the prospect theory, we will not detail the theory entirely but rather focus on the parts of interest to us.

2.2.1. Limits of the expected utility framework

> "[…] people underweight outcomes that are merely probable in comparison with outcomes that are obtained with certainty. This tendency, called the certainty effect, contributes to risk aversion in choices involving sure gains and to risk seeking in choices involving sure losses." (Kahneman & Tversky, 1971, p. 263)

The certainty effect defined by the authors was first discovered by Allais (1953) (as the Allais paradox) and is the best known violation of expected utility. He proposed multiple games

where participants had to choose between two alternatives. These tests not only demonstrated that investors have a tendency to overweight certain opportunities compare to uncertain opportunities, but most importantly that the substitution axiom of expected utility was violated in these examples (appendix 2.1.).

> "The substitution axiom of utility theory asserts that if B is preferred to A, then any (probability) mixture (B, p) must be preferred to the mixture (A, p). Our subjects did not obey this axiom." (Kahneman & Tversky, 1971, p. 266).

The other effect uncovered by the two psychologists through Allais' paradox is called 'the reflection effect'. It reveals that, when prospects are positives, investors are typically risk averse but that this effect reverts when prospects are negatives: indeed, investors become risk seeking. This actually is the definition of loss aversion, and a proof that investors are not risk averse but rather loss averse.

Finally, they defined a last effect labeled 'isolation effect', which leads to inconsistent preferences (appendix 2.2.). It describes the tendency of investors to base their choice on the differences between two prospects while disregarding their similarities (Kahneman & Tversky, 1971, p. 266). However, prospects can be set apart in many different ways and investors can change their preferences depending on the way the two prospects are decomposed and compared (i.e. framing effect) (Kahneman & Tversky, 1971, p. 271). The authors proposed two different formulations of an identical problem, one in a standard form and the other as a sequential formulation and found that investors' preferences would reverse from one formulation to the other. This is explained by the introduction of a dependency between prospects in the sequential formulation (keeping the probabilities and outcomes untouched).

> "The reversal of preferences due to the dependency among events is particularly significant because it violates the basic supposition of a decision-theoretical analysis, that choices between prospects are determined solely by the probabilities of final states." (Kahneman & Tversky, 1971, p. 272).

To conclude, we can state that people underweight outcomes that are least probable with respect to the one obtained with certainty, and tend to dismiss characteristics that are shared by all prospects and focus solely on their differences. Lastly, investors do not have consistent risk aversion but rather express risk aversion over gain and risk seeking behavior over losses.

This is the reason why the value function, as defined in prospect theory, is dependent on gains and losses rather than on a final level of wealth and that decision weights, which artificially modify the probabilities of outcomes, were introduced (see point 2.3.)

2.2.1. The mental account structure

> "Mental accounting is a feature of prospect theory, a framework Kahneman and Tversky (1979) developed to model the tendency to make decisions based on gains and losses relative to a reference point." (Shefrin & Statman, 2000, p.142)

The value in prospect theory is driven by gain and losses, and investors can have either low level, high level or mixed level of expectation, meaning that while they try to avoid ruin or poverty, they also and sometimes simultaneously want to become rich (Shefrin & Statman, 2000, p.141). When investors have mixed expectation levels, they usually hold a portfolio that can be seen as a pyramid, with different layers and different aspiration attached to them. In a simplified vision, one could say that the foundation layer as the sole goal to avoid poverty while the top layer aims at being wealthy (Shefrin & Statman, 2000, p.141). According to the authors, the main difference between the BPT single account (where investor compute their portfolio as a whole) and the BPT multiple accounts (where they divide their wealth between different subportfolios) is that in the former, investors act like they take into account the covariance between prospect, while in the later they are overlooked. (Shefrin & Statman, 2000, p.142). Kroll, Levy & Rapoport (1988) empirically proved that investors overlook the covariance in their portfolio construction, which validates the existence of the mental accounting structures. In 2010, Das, Markowitz, Scheid & Statman further developed and integrated the mental account structure to the MPT. They attached to each account a threshold level, that can be associated to an aspiration (which will be further described in the next point) and they defined the respective risk as the probability of failing fail to reach this aspiration level (Das et al., 2010). Therefore, each account can be viewed as a subportfolio with an aspiration level and a safety first constraint.

> "[…] investors are better at stating their goal thresholds and probabilities of reaching thresholds in MA (the consumption view) than their risk-aversion coefficients in MVT (the production view). Second, that investors are better able to state thresholds and probabilities for sub-portfolios (e.g., retirement, bequest, education, etc.) than for an aggregate portfolio."(Das, Markowitz, Scheid & Statman, p.315)

Furthermore, the authors found that there was no efficiency loss by applying the mental account structure with regards to the MPT. This motivated us to apply this mental account structure combined with the safety first & CPT to the BPT framework. Results will be further developed in the empirical section of this paper.

2.3. SP/A & CPT: Transformation Function of probabilities

2.3.1. SP/A

In 1987, Lopes developed a new theory (based on psychology) that was, much like Roy's theory, consistent with Friedman & Savage's paradox and allowed to leave the expected utility framework as suggested by the findings of Kahneman and Tversky (1972) (see point 2.3.1.). In many ways, it is an extension of the safety first constraint.

SP/A is defined over three poles: Security ('S') which denotes a general will to avoid low level of wealth, Potential ('P') which represents the will to be wealthy and finally Aspiration ('A') that crystallizes the safety first essence by defining a goal, a target to reach (like 's') (Shefrin & Statman, 2000, p.130). The main differences between the SP/A theory and the safety first theory comes from the interaction of emotions (hope and fear) on the cumulative probability function, and the substitution of the subsistence level 's' by a more general aspiration level.

In Telser's model, the probability of ruin is computed as $P(W< s)$. Therefore the safety probability could be modeled as $P(W \geq s) = 1 - P(W<s)$, the decumulative probability function can be rewritten as $D(s) = P(W \geq s)$ (Shefrin & Statman, 2000, p.131). In Lopes' (1982) model, levels of final wealth 'W_i' are ranked from the worst outcomes to the best outcomes, each having an attached probability P_i. Therefore, the expected wealth can be computed as follow:

$$E(W) = \sum P_i W_i, \text{ or alternatively } \sum D_i (W_i - W_{i-1}).$$

> "In this expression for E(W), the individual receives W_1 with certainty (note that $D_1 = 1$), receives the increment $W_2 - W_1$ (that is, an amount over W_1) with probability D_2, receives the further increment $W_3 - W_2$ with probability D_3, and so on." (Shefrin & Statman, 2000, p.131)

Fear and hope, according to Lopes (1982), will affect the relative probability weights of the decumulative function. More precisely, fear will overweight the probabilities corresponding to the worst outcomes relatively to the best outcomes, inversely hope will overweight the probabilities attached to the best outcomes relatively to the worst one. Therefore, fear will make individual act as if they were computing E(W) using a value excessively high for P_1 (and low for P_n, i.e. the last probability of the last outcomes) and hope, as stated before, will do the exact opposite. Hence, the decumulative function will undergo a reweighting of its probabilities. Lopes defined two decumulative probability function, one reflecting hope ($h_p(D) = 1 - (1 - D)^{1+q_p}$), the other fear ($h_s(D) = D^{1+q_s}$) and defined the decumulative transformation function as a convex combination of those two functions: $h(D) = \delta h_s(D) + (1-\delta)h_p(D)$. This transformation function will be used to create a new objective function $E_h(W)$, where each probabilities P_i's will be replaced by : $\pi_i = h(D_{i+1}) - h(D_i)$ and by this mean will depart from the expected utility framework.

2.3.2. Cumulative Prospect Theory

In 1992, Kahneman and Tversky enriched the Prospect theory and created the CPT. We will, for now, only focus our attention on its underlying transformation function of probabilities.

The first observation made in CPT states that investors use decisions weights rather than probabilities, much like Lopes (1982) implemented in her framework, and overweight probabilities of extreme losses and gains. The main difference between Prospect theory and CPT steam from different decision weights functions: while in Prospect theory they are separable, in CPT, as in Lopes's work, they are cumulative (Kahneman & Tversky, 1992, p. 297). The authors took into account developments made by Quiggin (1982) and Schmeidler (1989) with their new model of utilities: rank-dependent utilities and cumulative functional utility respectively, and implemented it to the CPT.

> "Instead of transforming each probability separately, this model transforms the entire cumulative distribution function." (Kahneman & Tversky, 1992, p. 299).

However, the weighting function used in CPT is not exactly similar to the one used by Lopes (1988). Indeed, we will see in our methodology, that the decision weight π_i will be defined, as done by Bourachnikova *et al.* (2016), by:

$$\pi = \begin{bmatrix} \pi_1 \\ \vdots \\ \pi_k \\ \vdots \\ \pi_{1000} \end{bmatrix} = \begin{bmatrix} \pi_1 = w\left(\sum_{j=1}^{1000} p_j\right) - w\left(\sum_{j=2}^{1000} p_j\right) \\ \vdots \\ \pi_k = w\left(\sum_{j=i}^{1000} p_j\right) - w\left(\sum_{j=k+1}^{1000} p_j\right) \\ \vdots \\ \pi_{1000} = w\,(p_{1000}) \end{bmatrix}$$

rather than:

$$\pi_i = h(\mathrm{D}_{i+1}) - h(\mathrm{D}_i)$$

as previously used in Lopes' framework. However, note that they are analogous.

Hence, the weighting function labeled $w(p) = \frac{p^\gamma}{[p^\gamma + (1-p)^\gamma]^{1/\gamma}}$ and integrated in the formula of π, where $\gamma = 0{,}61$, will, much like Lopes function, overweight tails probabilities and reflect the sensitivity of investors for changes in highest and lowest probabilities (Bourachnikova *et al*., 2016, p.430). Be aware that this function does not yet distinguishes between gains and losses; this distinction will be established in our methodology.

As pointed by Bourachnikova *et al.* (2016) we notice that $w(p_i)$ is applied to the decumulative distributive function y:

$$w(\textstyle\sum_{i=1}^{n} p_i) = w(1 - F(y_{i-1}))$$

where F stands for the cumulative function of Y, which is the corresponding to D in lopes' theory. The characteristics of the cumulative prospect theory will be detailed in appendix 3.

2.4. Further development on the CPT

> "Cumulative prospect theory is a modified version of "prospect theory" (Kahneman and Tversky 1979). Under cumulative prospect theory, people evaluate risk using a value function that is defined over gains and losses, that is concave over gains and convex over losses, and that is kinked at the origin." (Barberis & Huang, 2008, p.2066)

In their work *et al.* (2016) underlined that BPT did not integrate the entirety of Prospect theory nor Cumulative Prospect theory. Indeed, four main characteristics lie at the core of this last theory (Bourachnikova *et al.*, 2016, pp. 420; 430). First, agents utilize decision weights instead of probabilities, and put more weights on probabilities of extreme events. Second, the value function that agents maximize is built on gains and losses. Third, a reference point allows making the distinction between those gains and losses. Fourth, the value function reveals decreasing sensitivity and loss aversion. BPT, as defined by Shefrin and Statman (2000), only relies on the first aspect of this theory. Bourachnikova *et al.* criticized this restricted definition and proposed the BPT_{CPT} model that took all these aspects into consideration. We will, in this point, analyze the modified value function proposed by Kahneman and Tversky (1979; 1992).

As we specified before, the transformation function of probabilities, $w(p) = \frac{p^{\gamma}}{[p^{\gamma}+(1-p)^{\gamma}]^{1/\gamma}}$, used to compute behavioral portfolios according to the BPT by Bourachnikova *et al.* (2016) was not yet distinguishing between losses and gains. Yet, Kahneman and Tversky introduced a new value function in their work (1979;1992) which reflected the loss aversion of agent.

$$V(x) = \begin{cases} (y-k)^{0,88} & if\ y \geq k \\ -2.25(-(y-k))^{0,88} & if\ y < k \end{cases}$$

where k represents the reference point previously mentioned on which investors will determine whether they are facing a loss or a gain. This led to a tilted 'S-shaped' value function that is concave over losses and convex over gains, reflecting the risk preferences of each (Fig. 1).

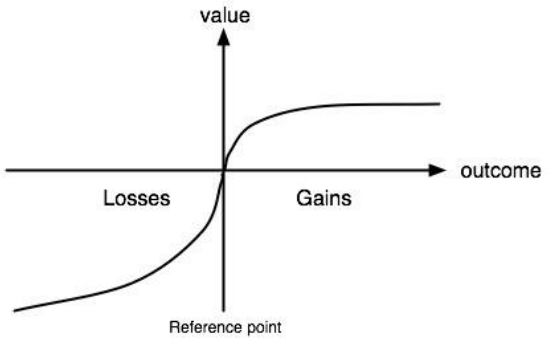


Fig. 1. Value function in Prospect Theory (Drawing by Marc Oliver Rieger)

Because losses and gains are threated differently, the authors refined their transformation function of probabilities by mixing two different functions: one defined over negatives

prospects ($w^-(p)$), the other over positive ones ($w^+(p)$), which definition can be found in appendix 3.

The main difference between the two is embedded in a change in value of the parameter γ, its value being estimated to 0,61 and 0,69 for $w^-(p)$) and ($w^+(p)$) respectively (Kahneman & Tversky, 1992). Bourachnikova *et al.* fully integrated the CPT in BPT and labeled the resulting portfolios as BPT_{CPT}.

2.5. Efficient portfolio selection

Until now, we have described the theory on which Shefrin and Statman based their BPT and described further research which fully incorporated the BPT in CPT. We will now give a brief overview of the portfolio selection process in BPT.

Previously, we saw two different decision weighting functions, one introduced by Lopes in 1988 and one introduced by Kahneman and Tversky in 1992, which both overweight the tails of the cumulative distribution it is applied on. According to Barberis and Huang (2008, p.2066): *The overweighting of tails does not represent a bias in beliefs; it is simply a modeling device that captures the common preference for a lottery-like, or positively skewed, wealth distribution.*

In BPT, Shefrin & Statman use Lopes' transformative functions of probabilities and the safety first constraint defined by Telser (1955). This led to a value function defined as:

$$\text{Max} : E_h(W) = \sum r_i W_i, \text{ subject to : Prob } (W \leq A) \leq \alpha \quad \text{for i : 1...n}$$

$$\text{Where } r_i = h(D_{i+1}) - h(D_i)$$

This characterizes an efficient solution within the meaning of BPT (if the portfolio is view as a single account). However it is crucial to highlight, that for high values of *A* and low values of α, it will become nearly impossible to satisfy the constraint and therefore no optimal portfolio will exist. We will empirically demonstrate this effect in our empirical analysis. In the upcoming empirical analysis, we will use Kahneman and Tversky decision weighting function (see Part III) and their value function, which will be further developed in the following section describing our methodology to conduct an empirical analysis on Behavioral Portfolio Theory.

PART III: METHODOLOGY OF THE EMPIRICAL COMPUTATION

The third part of this paper aims at providing the reader with a complete and detailed version of the empirical methodology that was used to obtain the results further explained in part four. It is largely based on the work of O. Bourachnikova, M. Pfiffelmann and T. Roger in their article "When Behavioral Portfolio Theory meets Markowitz theory" (2016), which was the first attempt to empirically compute the BPT optimal portfolio without restrictions. We have intended to further reproduce their methodology by interpreting and applying it to our own sample. We have also used its premises to go further in the development of the behavioral mental account analysis. This methodology has been computerized and applied to the chosen data set using the R software. A complete version of the code can be found in appendix 8.

1. Data and sample creation

The empirical analysis that lies at the core of this paper will be based on the study of 387 stocks stemming from the S&P500 index, whose daily historical prices have been extracted from the Bloomberg platform. The period of study has been fixed to eleven years, from 2005 to 2015. Although we were aware of the pitfalls related to the use of past information to predict likely returns in the future, we decided to use this complete price history for simplicity's sake in the empirical development. The S&P500 choice comes from the fact that we wanted to illustrate H. Markowitz' recommendation, according to which investors should diversify across different industries (Markowitz, 1952, p.89).

2. States of nature matrix creation

The portfolio choice, in both modern portfolio theory and behavioral finance, is based on expected returns. In order to determine them based on past information, we have followed the method recommended by Bourachnikova *et al.*, further detailed in this section (2016, pp. 421-422).

2.1. Historical return matrix

By means of rolling windows, we computed the 387 stocks monthly returns on a daily basis. Historical returns were then calculated based on their related prices history, using the following definition:

$$R_{t,i} = \log(P_{t,i}) - \log(P_{t-20,i})$$

where $R_{t,i}$ stands for the monthly return of stock *i* for day *t* and $P_{t,i}$ for its price (which incorporates the eventual dividends).

We were then able to establish the following historical return matrix, labeled *R* and made of the 387 monthly returns for the considered time period (i.e. 2005-2015):

$$R = \begin{bmatrix} R_{1,1} & R_{1,2} & \dots & R_{1,387} \\ R_{2,1} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ R_{2749,1} & \dots & \dots & R_{2749,387} \end{bmatrix}$$

where $R_{1,1}$ stands for the monthly return of the first stock for the first period considered (from the 3rd to the 1st of February 2005), where $R_{2,3}$ stands for the monthly return of the third stock for the second period considered (from the 4th of January 2005 to the 2nd of February 2005), and so forth until $R_{2749,387}$.

2.2. States of nature matrix

In order to derive possible states of nature (i.e. scenarios) for each date *t* (with *t* from t=251 to t=2749) Bourachnikova *et al.* applied the Hull & White's bootstrapping method, developed in 1998 and initially used as an historical simulation method for the VaR.

The first step was, for each of the 2500 simulations (*with t = 251…2479*), to randomly select stocks. As Bourachnikova *et al.* did, we randomly selected 80 stocks out of the 387 initially studied, considering that this would provide sufficient diversification to the created portfolios and that it would prevent potential computation issues in terms of both space and time.

The second step was to create 2500 R^* matrices (each of dimensions *250x80*), corresponding to the 2500 considered simulations; a rolling sample approach allowed us to do so. For each simulation, we picked the 250 historical monthly returns prior to date *t* of the 80 randomly selected stocks. We then had that:

$$R_t^* = \begin{bmatrix} R^*_{t-250,1} & R^*_{t-250,2} & \dots & R^*_{t-250,80} \\ R^*_{t-249,1} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ R^*_{t-1,1} & \dots & \dots & R^*_{t-1,80} \end{bmatrix}$$

As an illustration, R^*_{253} corresponds to the $R*$ matrix of the third simulation and is defined as follow:

$$R^*_{253} = \begin{bmatrix} R^*_{3,1} & R^*_{3,2} & \dots & R^*_{3,80} \\ R^*_{4,1} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ R^*_{252,1} & \dots & \dots & R^*_{252,80} \end{bmatrix}$$

The third and last step is dedicated to the states of nature matrix creation. For each of the 2500 simulations, we randomly picked a row of its corresponding R^*_t matrix and repeated this operation 1000 times. By bootstrapping the rows of the matrix and leaving the columns unchanged, the cross-sectional dependence structure of underlying time series remain unchanged. This resulted in 2500 states of nature matrices, labeled θ_t (*with t = 251…2479*), each being made of 80 stocks and their corresponding 1000 states of nature at date *t*. We have:

$$\theta_t = \begin{bmatrix} \theta_{1,1} & \theta_{1,2} & \dots & \theta_{1,80} \\ \theta_{2,1} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \theta_{1000} & \dots & \dots & \theta_{1000,80} \end{bmatrix}$$

where $\theta_{i,j}$ corresponds to *the monthly stock return of stock j if states of nature i occurs* (Bourachnikova *et al.*, 2016, p. 422). Note that each row of θ_t is associated to its corresponding date in R^*_t.

3. Portfolio weights matrix generation

As we previously mentioned, we voluntary reduced the maximum number of stocks a portfolio could be made of to 80. In order to generate those portfolios, we needed to determine the positive weights (i.e. proportion of wealth) invested in those 80 stocks. To that purpose, we followed a two steps process.

First, we limited the number of portfolios to be created to 10,000 for two mains reasons. On the one hand, although an infinite number of portfolios could have been created, we thought that 10,000 could still be a relevant proxy compared to the agent's choice in real world situations. On the other hand, although we tried to create more than 10,000 of them, we soon realized that we were stuck by computing limitations due to the capacity of our computers.

Second, we generated 10,000 different weights combinations. To that end, we followed the procedure recommended by Bourachnikova *et al.* (2016, p. 422) and assumed that the weights associated to each assets were equal to *k/n*, with *k=1,…,n* and *n* being the number of assets composing the portfolio (n_{max} = 80). Then, we considered all the possible integer decompositions of the number *n,* this is, the decompositions of *n* as a sum of positive integers. To that end, we relied on the partition function established by Rademacher in 1937. As an illustration, for n=5, we got seven different integer decompositions:

$$I_1 = \begin{bmatrix}5\\0\\0\\0\\0\end{bmatrix} \quad I_2 = \begin{bmatrix}4\\1\\0\\0\\0\end{bmatrix} \quad I_3 = \begin{bmatrix}3\\2\\0\\0\\0\end{bmatrix} \quad I_4 = \begin{bmatrix}3\\1\\1\\0\\0\end{bmatrix} \quad I_5 = \begin{bmatrix}2\\2\\1\\0\\0\end{bmatrix} \quad I_6 = \begin{bmatrix}2\\1\\1\\1\\0\end{bmatrix} \quad I_7 = \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}$$

Using R, we were able to computerize this calculation, and got 15,796,476 integer decompositions for n=80. We randomly selected 10,000 of those vectors, and then, to ensure a wide diversity in our portfolios, rearranged them randomly. We were left with a matrix, labeled *W,* of dimension *(10000x80)*. To get our 10,000 structures of weights, we divided each entry of this matrix by n=80, allowing us to saturate the feasibility constraint (i.e. sum of weights for each portfolio amounts to 1).

Note that we voluntary chose to restrict the weights to positive values, this is, we made the assumption of a world that did not allow for short-selling. This decision is based on the fact that, considering Bourachnikova's *et al*. study, allowing negative weights structure did not alter the overall results (2016, pp. 425-426). Moreover, this would have required heavy calculations that would have forced us to diminish the size of our sample.

4. Mean-variance portfolios

In this section, we will describe the process that allowed us to get efficient portfolios within the meaning of Harry Markowitz' theory. To that end, we will firstly define how we computed the first two moments of the considered portfolios, this is, their expected returns and standard deviations (i.e. inputs). Then, we will explain how we derived the efficient frontier made of efficient portfolios (i.e. outputs).

4.1. Mean-variance portfolio generation

In this section, we point out the way we derived the expected returns and the standard deviations of the 10,000 portfolios for each 2500 simulations.

In order to get the expected returns of the 10,000 portfolios for the first simulation, we followed a three steps process.

First, we computed the expected return of each stock. To that end, we calculated the average monthly stock return in the states of nature matrix associated to its simulation, resulting in a vector of dimension *(1x80)*. Second, we randomly applied the 10,000 weights (defined in 3.) to this vector, resulting in a matrix made of 10,000 portfolios. Third, by applying the mathematical property of the expectation, according to which the expectation of a sum is the sum of the expectations, we summed up the entries of each row of this matrix, and derived the expected return of the 10,000 portfolios composing the simulation. We repeated this whole process for the 2500 simulations. Note that 80 different stocks were selected in each simulation while keeping the same structure of weights, leaving us with a wide variety of portfolios.

When it comes to the standard deviation of a portfolio, we used a two steps process, which is introduced hereafter. Let us first remind its definition. As we have seen, the variance of a portfolio can be defined as follow:

$$\boldsymbol{V = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij}}$$

In matrices notations, this can be rewritten as:

$$\boldsymbol{V = w' \Sigma w}$$

where $w = (w_1 w_2 \dots w_n)'$ stands for the vector of weights (*with i=1,…,80*) and Σ for the variance-covariance matrix associated with the returns. Σ is of dimension *(80x80)* and takes the following form:

$$\Sigma = \begin{bmatrix} \sigma_{1,1} & \sigma_{1,2} & \cdots & \sigma_{1,80} \\ \sigma_{2,1} & \sigma_{2,2} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \sigma_{80,1} & \cdots & \cdots & \sigma_{80,80} \end{bmatrix}$$

where $\sigma_{i,j}$ stands for the covariance between the returns of stocks *i and j*.

Following this last definition, we firstly computed the variance-covariance matrix, Σ, associated with the monthly stock returns of the first states of nature matrix by using the R pre-installed package labeled 'stats'. Secondly, we applied each vector of weights w and its transposed version to this matrix to compute the variance (and standard deviation) of each generated portfolios composing a simulation. Again, we repeated the whole process for the 2500 simulations.

4.2. Efficient portfolio selection and frontier creation

This section aims at defining the methodology used to determine the set of efficient frontiers and their corresponding plot. As a reminder, Markowitz defines an efficient portfolio as a feasible portfolio for which there is no other feasible alternative with both a higher expected return and a lower variance (Markowitz, 1959, p. 140). Using R, we were able to select and plot them in each of the 2500 simulations using a four step process.

First, we associated, in each of the 2500 simulations, the standard deviation and the expected return of each of the generated portfolios. Second, we ranked those feasible portfolios according to their standard deviation, from the lowest one to the highest one. Third, we dropped all the feasible portfolios that had, when going from the lowest level of standard deviation to the highest one, a lower level of expected return than the previous increment. We were left, for each simulation, with a set of efficient portfolios, labeled S^{ef}.

We finally plotted those sets of efficient portfolios to get their corresponding efficient frontiers. To do so, we defined the horizontal axis as the standard deviation and the vertical axis as the expected return. In each of the 2500 graphs, we added the 'extreme' efficient portfolios, this is, the GMVP (i.e. portfolio with the lowest standard deviation among all the feasible portfolios) and the Maximal return portfolio, MRP hereafter (i.e. portfolio with the highest expected return among all the feasible portfolios).

5. Behavioral portfolios

In this section, we will describe the process that allowed us to get behavioral portfolios within the meaning of Shefrin and Statman's BPT (2000). We will firstly introduce the reader with the way we applied the so-called safety first constraint to restrict the set of feasible portfolios. Secondly, we will explain how we used the CPT's decision weighting function to generate behavioral portfolios. Thirdly, we will briefly explain the way we derived the optimal

behavioral portfolio. Finally, we will establish how we were able to determine whether or not those portfolios were efficient within the meaning of Markowitz' mean-variance theory by locating them on the efficient frontier.

5.1. Safety first constraint

As previously mentioned, the safety first constraint in Shefrin and Statman BPT model (2000) is the result of Roy's safety first approach (1952). According to this principle, investors are willing to minimize the probability of ruin associated with an investment. It goes a bit further by integrating Telser's extension of Roy's work, according to which the probability of ruin has a maximal level of acceptability for the investor, labeled α (1955), and Lopes' theory according to which the subsistence level is replaced by a more general aspiration level. The safety first constraint can thus be written as:

$$p(W < A) < \alpha$$

$$\text{or} \quad p(W \geq A) \geq 1 - \alpha$$

where W stands for the investor's final wealth distribution, A for the Aspiration level and α for the acceptable level of probability of ruin. According to Bourachnikova *et al.* this is equivalent to say that:

> "A portfolio satisfied this constraint if its value at the end of the period is at least equal to aspiration level A in $1 - \alpha\%$ of the states of natures" (Bourachnikova *et al.*, 2016, p.423)

Recall that this constraint is applied on an extreme quantile α of the final wealth (i.e. expected returns) probability distribution and can thus be interpreted as the VaR.

In order to restrict the set of feasible portfolios to the one that satisfied this constraint, we proceeded in four steps. The process hereafter described applies to the first simulation and was repeated 2500 times.

Firstly, we defined values for A and α. As recommended by Bourachnikova *et al.* (2016), we defined 3 different specifications for α: 0,1; 0,2 and 0,3. As they did, we associated A with, on the one hand, a short-term risk free rate (i.e. US 3 months Treasury Bill) and, on the other hand, a long-term risk free rate (i.e. US 10 years Treasury Bond), whose daily historical data

were downloaded from the Federal Reserve Bank of St Louis website. By using the rolling windows principle, we computed, for each aspiration level, their monthly returns on a daily basis. Then, for each of the 2500 simulations (*with t = 251…2479)*, we used the rolling sample approach to pick the 250 historical monthly returns of each aspiration level prior to date *t* (i.e. date of the studied simulation). We were left with 2500 vectors of dimension (*1x250).*

Secondly, we applied the structure of weights to the first states of nature matrix. To do so, we multiplied each row of the weight matrix, *W* (of dimension *10000x80),* by every row of the first θ_t matrix (of dimension *1000x80*). We were left with 10,000 matrices of dimension *(1000x80).* Each of these matrices can be associated to a portfolio, for which there exists 1000 states of nature for the expected return. In our BPT version, we relaxed the assumption according to which returns follow a normal distribution to provide a better representation of the reality. As a result, we did not compute the expected return of the 10,000 portfolios as we did in MVT, but we rather computed the expected return associated to each of the 1000 states of nature for each of the 10,000 portfolios. We were then left with 10,000 vectors of dimension *(1000x1).*

Thirdly, for each of the 10,000 portfolios, we matched every set of 1000 states of nature to their related aspiration level, *A*. As we previously mentioned, each rows of the first states of nature matrix are associated to their corresponding date in *R**. As a result, each row of the 10,000 portfolio vectors could be matched to those dates. Then, for each of the 10,000 portfolios, we compared the values of the 1000 states of nature to their corresponding aspiration level. We only selected those for which the expected return at the end of the period was superior or equivalent.

Finally, we created the set of portfolios that were meeting the safety first constraint. To do so, we calculated, for each of the 10,000 portfolios, the number of states of nature for which the value of the expected return at the end of the period was higher than the aspiration level and compared the result with the acceptability level, α (i.e. 900 when α = 0,1). When the number of states of nature was higher than α, the corresponding portfolio was selected. The set of portfolios *S** was then created and made of all the portfolios, among the 10,000, for which the constraint had been satisfied.

This whole process was repeated six times, considering the specificities for *A* and α that were defined above. We firstly considered a situation where *A* was defined as the 3 Months US Treasury Bill and repeated the constraint for the three different levels of α (0,1; 0,2 and 0,3). We did the same in the second situation, where *A* was now defined as the 10 years US Treasury Bond. Those six iterations were then repeated for the 2500 simulations. Note that we restricted our analysis to the simulations in which there was at least one portfolio that was satisfying the safety first constraint.

5.2. Behavioral portfolios generation

Shefrin and Statman (2000) went further than integrating Roy and Telser's safety first approach to their work and also included Lopes' SP/A theory (1987). As previously mentioned, "Fear" and "Hope" are at the core of this theory and are respectively characterized by the fact that investors overweight the probabilities associated with the poorest and greatest outcomes. In order to take that into account, Shefrin and Statman integrated those emotions into their model by modifying the probability distribution of outcomes. They postulated that the BPT optimal portfolio was then the result of subjective decision weights solving the following maximization program:

$$\max E_{\pi}(W) = \boldsymbol{Y'\pi}$$

$$\text{s.t. } p(W < A) < \alpha$$

where π stands for the transformation function of probabilities. In order to transform probabilities and to determine those subjective weights, we followed Bourachnikova's *et al.* recommended procedure, which substitutes Lopes' transformation function of probabilities by the one developed by Kahneman & Tversky (1992) (keep in mind that they are analogous). This phase is organized in four steps.

We firstly associated a vector *Y* to each portfolio in the *S** set. For a given portfolio, this vector was made of all its expected returns in each of the 1000 states of nature. Secondly, we ranked each entry in those vectors from the lowest to the highest, resulting in:

$$Y_i = \begin{bmatrix} Y_{i,1} \\ Y_{i,2} \\ \vdots \\ Y_{i,1000} \end{bmatrix}$$

where $Y_{i,j}$ is the expected return of the i^{th} portfolio in the j^{th} state of nature.

Thirdly, we assumed that, in each simulation, every state of nature had the same chance of happening. This allowed us to define the following objective probability vector, labeled *p*:

$$p = \begin{bmatrix} p_1 = \dfrac{1}{1000} \\ \vdots \\ p_{1000} = \dfrac{1}{1000} \end{bmatrix}$$

As we previously mentioned, BPT, as defined by Shefrin and Statman (2000), was developed by relying on the first aspect of the CPT only, according to which agents utilize decisions weights instead of probabilities and put more weights on extreme events. In order to reflect this, we fourthly determined the vector of subjective decisions weights, using the following weighting function developed by Kahneman and Tversky in 1992:

$$w(p) = \frac{p^{\gamma}}{[p^{\gamma}+(1-p)^{\gamma}]^{1/\gamma}}$$

where they estimated the γ parameter to 0,61 (1992, p. 312).

By combining it with the probability vector *p,* we were able to derive different ranks of subjective decision weights for our equiprobable objective probabilities. We got:

$$\pi = \begin{bmatrix} \pi_1 \\ \vdots \\ \pi_k \\ \vdots \\ \pi_{1000} \end{bmatrix} = \begin{bmatrix} \pi_1 = w\left(\sum_{j=1}^{1000} p_j\right) - w\left(\sum_{j=2}^{1000} p_j\right) \\ \vdots \\ \pi_k = w\left(\sum_{j=i}^{1000} p_j\right) - w\left(\sum_{j=k+1}^{1000} p_j\right) \\ \vdots \\ \pi_{1000} = w\,(p_{1000}) \end{bmatrix}$$

where π_i is the decision weight associated with the i^{th} state of nature's probability. This vector represents the fact that investors *apply an inverse S-Shape weighting function to the decumulative probability distribution of outcomes* (Bourachnikova *et al.*, 2016, p.421).

Note that the BPT_{CPT} model, introduced by Bourachnikova *et al.*, which integrated all of the four aspects of the CPT, brought two main changes to this development. Namely, they

replaced Y_i by a vector of modified outcomes to reflect the value function as defined under CPT. They also introduced a modified vector of decision weights, which was, as the value function, treating differently gains and losses. In the context of this paper, we did not compute this model for two main reasons. Firstly, we observed that the results derived for the BPT_{CPT} model were nearly equal to the one for the BPT model. Secondly, considering the material at our disposal, this would have required us too heavy calculations in terms of both space and time.

We were finally able to generate feasible behavioral portfolios, in each of the six situations, considering the specificities for A and α previously defined. To do so, we multiplied the subjective decision weights vector π, by each S^* portfolio's expected return vector Y_i (i.e. $\boldsymbol{Y'\pi}$) (for the simulations with at least one selected portfolios).

5.3. Optimal behavioral portfolio selection

We still had to determine optimal behavioral portfolios in each of our six situations. This portfolio can be defined as the one that satisfies the safety first constraint and that guarantees the maximal payoff to the investor. In order to find it, we compared, for each set S^*, all of its feasible behavioral portfolios (generated in *5.3*) and selected the one that maximized $\boldsymbol{Y'\pi}$.

5.4. Localization on the mean-variance efficient frontier

The final objective of this empirical part was to determine, whether or not, the generated behavioral portfolios were belonging to the mean-variance space, and could thus be considered, or not, as mean-variance efficient portfolios in the meaning of Markowitz' theory.

The subjective weights used in the behavioral portfolio generation process served as decisions weights. There were not considered as fraction of wealth allocated between assets, by opposition to the 10,000 weight structures that had initially been used to derive the 10,000 portfolios from which the set S^* followed. As a result, in order to locate those optimal behavioral portfolios in the mean-variance space, we had to firstly search back for the weights structure that had initially be associated with them.

Each BPT optimal portfolio was associated to a specific simulation. In this specific simulation, we searched for the weight combination that had allowed deriving the 1000 states of nature's expected return associated with this optimal portfolio. When looking back in the

mean-variance efficient portfolio set, S^{ef}, for the corresponding simulation, we tracked back the optimal behavioral portfolio weights structure. We were then able to determine, whether or not, a portfolio with the same weight structure existed in S^{ef}. In the positive case, we derived the corresponding portfolio and its first-two moments in the mean-variance space, and located it on its corresponding plot. We repeated this operation for each simulation in which there was an optimal behavioral portfolio, for all the specificities of A and α previously defined.

6. Behavioral Mental accounts

As previously mentioned, the mental account theory postulates that investors do not consider their portfolio as a whole but rather as

> "collection of mental accounting subportfolios where each portfolio is associated with a goal and each goal has a threshold" (Das *et al.*, 2010, p.312).

In the specific case of this paper, we decided to apply this theory to behavioral portfolios and considered the most famous subportfolio division, known as: retirement, education and bequest. We postulated that the aggregate portfolio would be the result of a combination between 60% of the retirement portfolio, 20% of the education portfolio and 20% of the bequest portfolio. In this section, we will briefly recall how we were able to identify each of the optimal subportfolios under BPT and how we aggregated them into a single aggregated portfolio.

In order to derive the optimal subportfolios under BPT, we used the same procedure as the one developed in the previous section of this paper. Firstly, we needed to define the feasible subportfolios, this is, the one satisfying the safety first constraint. To do so, we determined, for each subportfolio, the specificities for the aspiration level, A and for the acceptable level of probability of ruin, α. As those specificities are associated to a risk aversion level, we chose them so that they could best reflect the subportfolios' own characteristics. As a result, we decided to associate the retirement portfolio, for which agents are supposed to exhibit a strong risk averse behavior, with A defined as the short-term risk free rate (i.e. US 3 month Treasury Bill) and α being equal to 0,1. When it comes to the education portfolio, agents are supposed to be less risk averse, and so we defined A as the long-term risk free rate (i.e. US 10 years Treasury Bond) and α as 0,2. Finally, for the bequest portfolio, we assumed that agents were almost risk neutral and decided to define A as 10% percent annualized and α as 0,3. As

the specificities for the first-two subportfolios had already been studied, we already had the set of feasible and optimal behavioral subportfolios. We just needed to repeat the procedure for the third subportfolio, following the same steps as before.

In order to compute the aggregate portfolios and to locate them in the mean-variance space, we decided to restrict our analysis to simulations in which the three optimal subportfolios existed (i.e. satisfied the safety first constraint) and were efficient within the meaning of Markowitz' definition. As for the behavioral portfolio case, we identified the mean-variance efficient subportfolios throughout their simulation number and their weights vector. We were then able to associate a mean and a variance to all of them, and to derive the aggregate portfolios. Indeed, by applying the different weights combination (i.e. 0,6; 0,2 and 0,2) to the three optimal behavioral subportfolios' expected return and standard deviation, we were able to derive the expected return of the corresponding aggregate portfolio as well as its standard deviation, in each simulation. We finally plotted the results on the previously obtained graphs.

PART IV: EMPIRICAL ANALYSIS

1. Mean-Variance Portfolios

In this section, we will analyze the main results obtained by applying the mean-variance framework to our sample of data. As explained previously, by computing the standard deviations and returns of each portfolio composing the simulation (i.e. 10,000) we were able to graph the resulting efficient frontiers. Following is a brief overview of some useful statistics that we were able to collect using the software 'R' and a description of our efficient frontiers.

1.1. Useful Statistics

By using the portfolios resulting from our methodology, we were able to extract some useful statistics such as, the average number of stocks composing our portfolio (of max 80 stocks) as well as their Herfindahl index (i.e. concentration measure that we use to assess the diversification of the portfolio; the higher the index the less diversified). Following is a table summarizing our main findings.

Indicators	Results
Average Number of Stocks	19
Minimal Number of Stocks	4
Median Number of Stocks	18
Maximal Number of Stocks	66
Mean Herfindahl Index	0,1269

Table 1: Useful statistics derived from R

The computation of the mean Herfindahl Index, allows us to certify that our portfolios are well diversified as the observed value (i.e. 0,1269) is close to zero (Larry & Stulz, 1993, p. 13). All these results are consistent with the ones obtained by Bourachnikova *et al* (2016).

1.2. Efficient Frontiers

We will now discuss the way we decided to plot our set of efficient frontiers and will provide an explanatory graph. Given the number of simulations and corresponding graphs, we decided to set a small sample of them only in appendix 4. The complete set can be computed with our 'R' code available in appendix 8.

To plot our frontiers, we set the x-axis as the standard deviation of the portfolios (i.e. σ_p) and the y-axis as the returns (i.e. μ_p). We highlighted each efficient portfolios (of each simulation) and connected them by a non-smoothed line. We decided to use this type of line for clarity's sake. Indeed, as we will see in the next section, the optimal behavioral portfolio will sometimes correspond to an MVT efficient portfolio: we therefore needed to use a line connecting each of them. The frontiers are always bounded by two specific portfolios: the GMVP (on the left) and the MRP (on the right) as explained previously. The resulting graphs lead to the upper part of a parabolic figure turned on its side as one can see in the following figure (Fig. 2.).

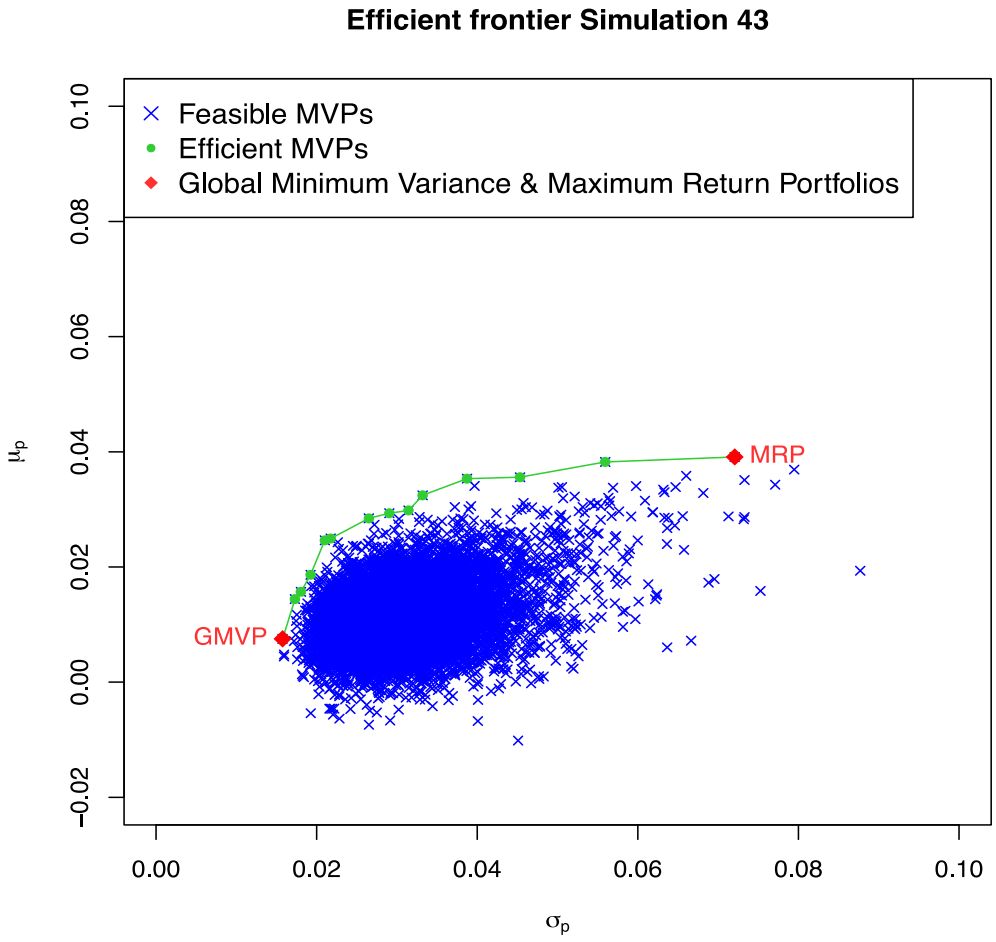


Fig. 2. Efficient frontier of simulation 43. Data: Bloomberg

The entire set of generated MVT portfolio graphs will be used in the following sections to locate the Optimal behavioral portfolio (OBP, hereafter) or the Mental Account aggregated portfolios. As we will see, more often than not, those portfolios will coincide with an MVT efficient portfolio. Explaining the reasons leading to such phenomenon is one of the main goal lying at the core of this paper.

2. Behavioral Portfolios

In the following points, we will discuss our main findings regarding the behavioral side of our empirical analysis. We will examine the statistics that we believed were relevant to this paper and explore their implications. Going further, we will justify those results using the previously introduced theoretical concepts and draw our resulting conclusions.

2.1. Safety first Constraint

Our goal here is to select the simulations in which at least one portfolio satisfies the safety first constraint. Considering the previously defined specifications for A (i.e. aspiration level) and α (i.e. acceptability level), different numbers of simulations were selected in each case. A summary table of those results (Table 2) follows bellow.

	A= ST risk free rate (3-Month T-Bill)	**A= LT risk free rate (10-Year T-Bond)**
$\alpha = 0,1$	16,32%	16,84%
$\alpha = 0,2$	71,32%	79,92%
$\alpha = 0,3$	89,28%	89,64%
Overall Average	59,37%	

Table 2: Eligible simulations

Given the definition of α, it is quite intuitive to understand that the lower it gets, the more selective investors become as they desire to secure safer portfolios, which restrict their choice.

Following the same logic, the number of selected simulations is supposed to move in the opposite direction from the aspiration level (Bourachnikova *et al.*, 2016). However, in our analysis we encountered the opposite results that we propose to document as follow. To begin with, looking at historical average yearly returns for Treasury-Bill and Treasury-Bond, we

observed that long-term rates did not always outperform the short-term ones, i.e. that the yield curve was inverted. This situation is generally encountered in times of future economic recession. As such, the concentration of our data around the crisis could very well be an explanatory factor for this phenomenon. Readers should keep in mind that the effect associated with different parameterizations of α is much stronger than the one resulting from change in aspiration levels.

When defining Lopes' framework, we stated that for high values of aspiration levels and low values of α it would become increasingly difficult to satisfy the safety first constraint, this was confirmed by our empirical results.

2.2. Optimal Behavioral Portfolios

Given the previously selected portfolios, we were able to derive the OBP thanks to Shefrin & Statman (2010) and Kahneman & Tversky (1992) theories. The following points aim at locating the aforementioned portfolios as well as describing their specifications.

2.2.1. Mean-Variance Efficient Behavioral Portfolios

After finding the portfolios that were maximizing the following optimization program:

$$\max E_{\pi}(W) = \boldsymbol{Y'\pi}$$

$$\text{s.t. } p(W < A) < \alpha$$

our next step was to locate the resulting portfolios on the efficient frontiers of the corresponding MPT simulations. This led us to the main findings of our research: in 60% of the eligible simulations, the OBP, lay on the Mean-Variance efficient frontier. All results are given in Table 3.

	A= ST risk free rate (3-Month T-Bill)	**A= LT risk free rate (10-Year T-Bond)**
α = 0,1	61,27%	63,18%
α = 0,2	40,27%	45,16%
α = 0,3	75,64%	75,94%
Overall Average	60,55%	

Table 3: Percentage of OBP located on the efficient frontier

Overall, our results are consistent with Bourachnikova *et al.* (2016) findings. In terms of magnitude however, we found a difference of 10 %; hence, their overall average was equal to 70 %. This can be explained by two main elements: the numbers of stock considered in our sample was much smaller than theirs (387 compared to 1492 respectively) and their time period much larger (11 years compared to 17 years respectively). Keeping in mind these differences, we believe that our results are consistent with their findings.

On the contrary, these results are inconsistent with the findings of Shefrin & Statman (2000), which found that, the OBP would typically not coincide with the efficient frontier. However according to Levy & Levy (2004, p.1015) *when diversification between assets is allowed, the MV and PT-efficient sets* [i.e. prospect theory] *almost coincide.* Das *et al.* (2010) found similar results with the introduction of mental accounts. Therefore, we took a closer look at those coinciding portfolios to analyze their specificities.

2.2.2. Optimal Portfolios Specifications

As previously mentioned, given the efficient frontiers graphs obtained following the mean-variance theory, we were able to select those for which an OBP had been identified. The following point aims at introducing the reader with the graph of a specific simulation to further describe the characteristics of this OBP in terms of both risk and return.

Considering, in simulation 500, an aspiration level equal to the 10-Year Treasury Bond rate and a value for α of 0,3, following is a graph representing the location of the Optimal Behavioral Portfolio (Fig. 3). Note that several graphs have been created for all the specifications of A and α; sample of them can be found in appendix 5.

The first striking characteristic stemming from this graph is the location of the OBP. Indeed, it coincides with the MRP. After analyzing our entire set of simulations, we came to the conclusion that the OBP was almost always located on the upper right side of the efficient frontier (roughly 95% of the cases). Bourachnikova *et al.* (2016, pp. 426-427) reflected these results by demonstrating that investors using the BPT framework were typically less risk averse than their MPT counterparts. More importantly, they proved that a typical mean-variance investor would not hold a BPT optimal portfolio, even if located on the efficient frontier.

We further argue, that the difference of risk aversion characterizing those two types of investors comes from the different approaches used to define risk. Indeed, as previously stated, while MPT investors use the standard deviation as a measure for risk, their BPT counterparts use the acceptability level for the probability of ruin (i.e. α) relative to their aspiration level (i.e. A).

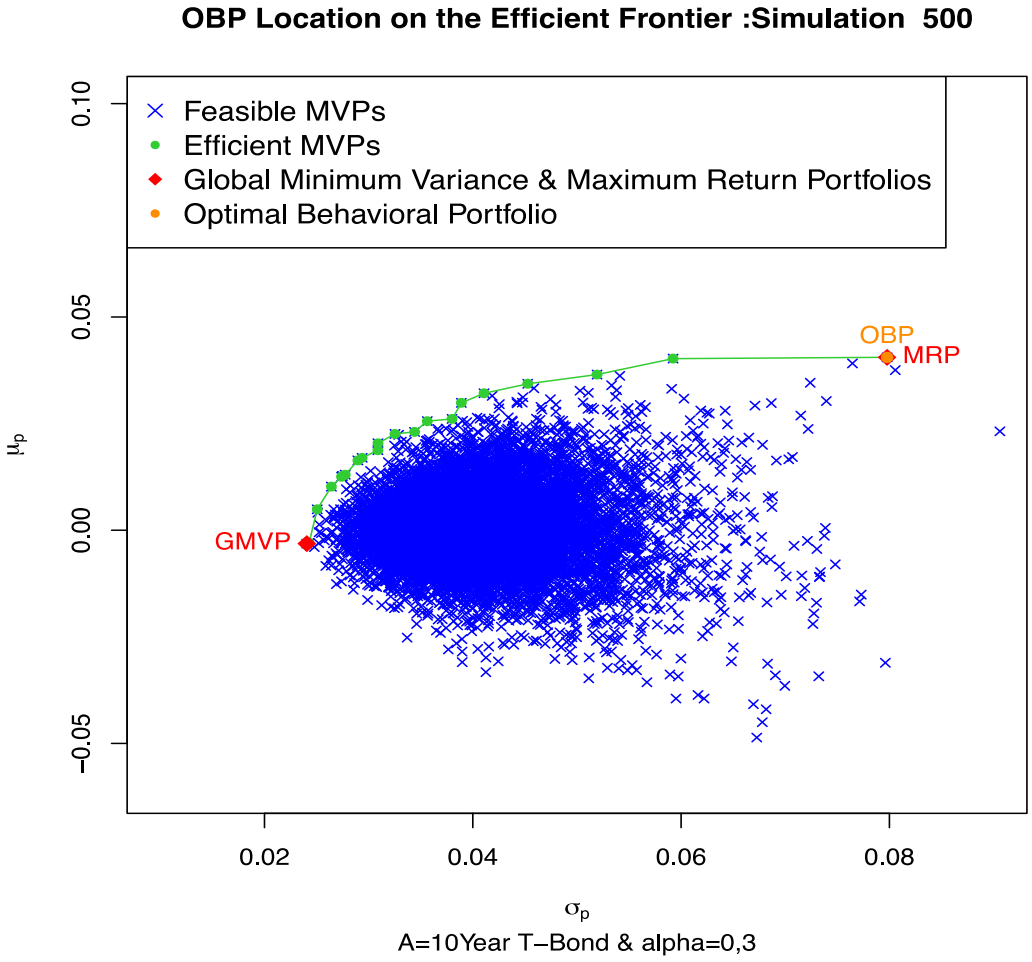


Fig. 3. OBP location in the Mean-Variance space simulation 500. Data: Bloomberg.

Throughout our theoretical analysis, we pointed out multiple shortcomings afflicting standard deviation as a proper measure for risk. Indeed, recalling Howard's statement (2014), standard deviation captures mostly emotions: even if in the short-term it is characterized to some extent by risk, this is no longer the case in the long run. Hence, we previously discussed that multiple biases were at the core of the excess volatility puzzle (Shiller, 1992; Black, 1986): the overconfidence of investors and their disagreements, the lack of rationality characterizing the investors, their propensity to create noise and finally the shortcomings of arbitrage on the financial markets. This led us to believe that standard deviation (being a measure of volatility)

was no longer a relevant measure of risk. Finally, recalling that investors are myopic loss averse, they evaluate their portfolio too frequently. Therefore, their evaluation of risk, through the short-term standard deviation, will be affected by emotions and their constant portfolio updates will decrease their level of utility.

We came to the conclusion that, although, through the mean-variance framework, the OBPs seemed to be characterized by high level of risk, through the BPT framework this was no longer the case. Indeed, while investors in the former model rely on the standard deviation to define risk, investors in the later model use the safety first constraint. As a result, portfolios are no longer deemed as excessively risky by the BPT investors but rather as optimal.

Another striking characteristic of the OBP is that its return and standard deviation are in general extremely high and close or equal the one of the maximum return portfolio. The following graphs (Fig. 4 & 5) aim at illustrating the standard deviation and the expected return respectively of such portfolio for the entirety of the considered time period. Note that, for clarity's sake, we voluntary decided to smooth the line corresponding to the returns and standard deviation of the MRP and GMVP. This explains why, at some point, the OBP returns or standard deviation seem to exceed the ones of the MRP, which is in reality never the case. The unsmoothed graphs are available in appendices 7.1 and 7.2 to prove our point.

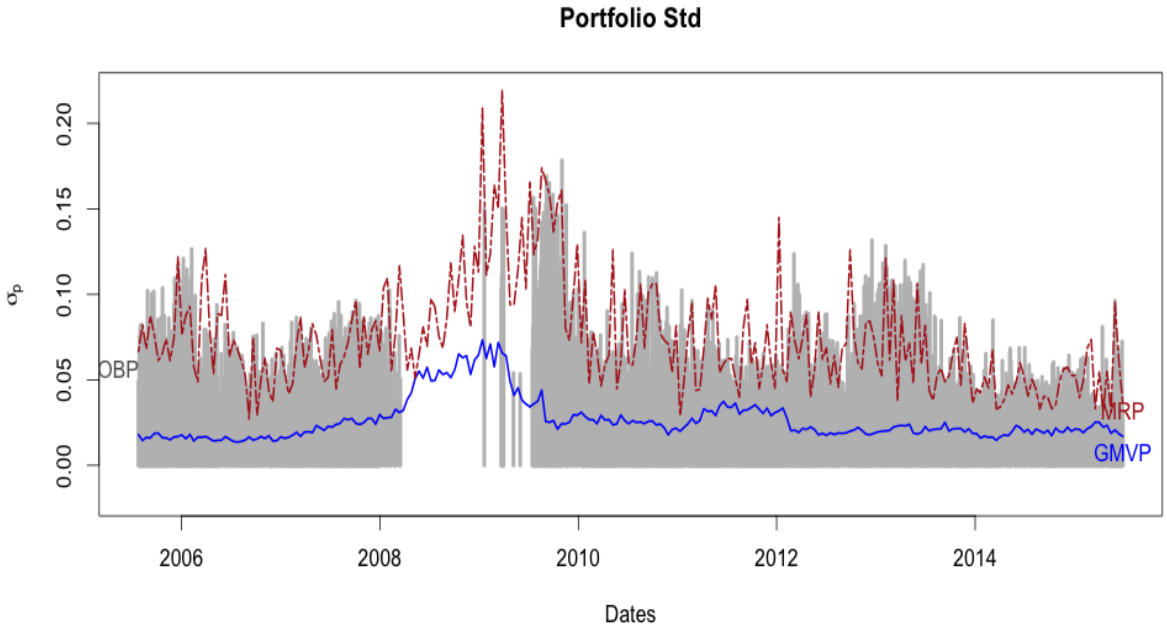


Fig. 4. Standard deviation of the OBP compared to GMVP and MRP

As stated above, we have the graphical confirmation that OBP are characterized by a high level of risk, which is quite similar to the one of the MRP. Keep in mind that, once again,

these results are given by the mean-variance framework. However, as stated above, a typical BPT investor will not perceive this particular portfolio as excessively risky but rather as optimal. The following figure aims at describing the return of the OBP comparatively to the ones of the MRP and GMVP.

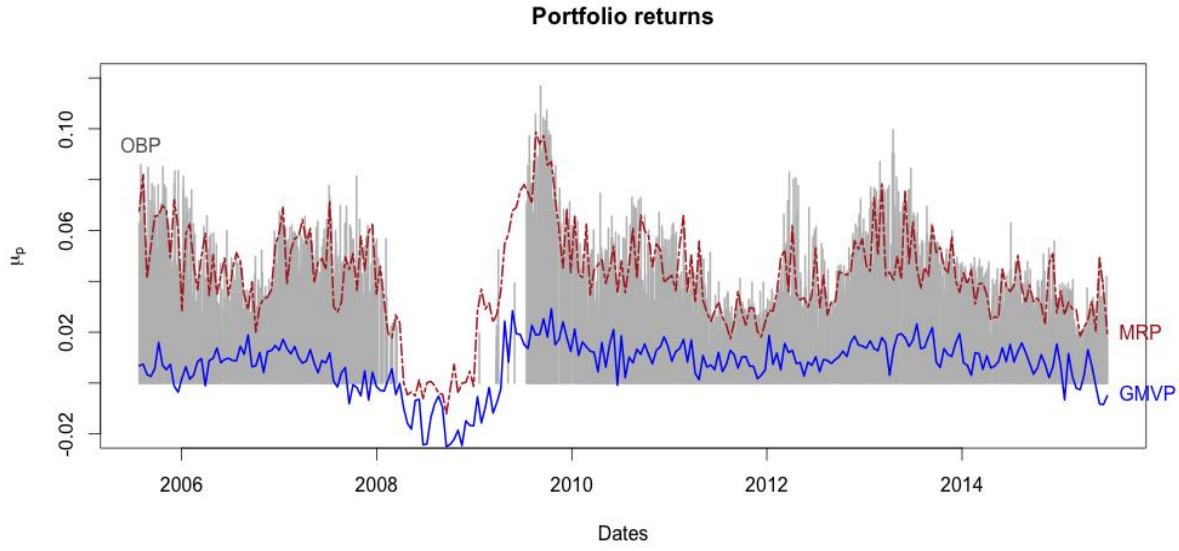


Fig. 5. Return of the OBP compared to GMVP and MR

Given the proximity of returns between the OBP and the MRP, we can conclude that the OBP always lies on the upper right side of the efficient frontier. This confirms all our previous findings.

The second learning stemming from this graph comes from the gap observed between 2008-2009. Indeed, during this period, no optimal solutions for the behavioral portfolios existed. The period of 2008-2009 is not a coincidence: it actually matches the sub-prime crisis, which had disastrous effects on the stock market. As one can see in Figure 5, even the maximum return portfolio during these periods featured negative average returns. Recalling the safety first constraint, we can easily understand that such levels of returns on the stock market prevented behavioral investors from selecting a portfolio that could satisfy it. This led those investors to quit the stock market during this period, as they were unable to find an optimal solution to their wealth allocation.

This also resonates with the theories of Lopes (1982) and Kahneman & Tversky (1992). Clearly, the phenomenon characterized by investors exiting and reentering the stock market can be associated to the concepts of ‘fear’ and ‘hope’, as defined by Lopes, and the

transformative probability function used by Kahneman & Tversky in the cumulative prospect theory.

Behavioral portfolio investors behave in two steps when it comes to their wealth allocation. Firstly, they decide on the amount of capital they are willing to secure. Secondly, they gamble and hope receiving a large amount of money (Bourachnikova, *et al.,* 2016). In a financial crisis situation, 'fear' and thus desire for 'security' are predominant, meaning that investors want to avoid low levels of wealth. As a result, they overweight the probabilities associated to the occurrence of the worst outcomes, relatively to the one of the best outcomes. As they are firstly willing to secure their capital, they set increasingly high specifications for A and α. Hence, the safety first constraint becomes harder, or even impossible, to satisfy; leaving them with no gambling possibility. This explains more deeply the 2008-2009 optimal portfolio gap on figures 4 & 5.

On the contrary, in a 'normal' world, investors are typically characterized by 'fear' and 'hope' and by the 'security' and 'potential' factors. This means that while they try to avoid low levels of wealth they are simultaneously driven by an enrichment purpose. This is, they overweight the probabilities associated to the occurrence of the best outcomes, relatively to the one of the worst outcomes. As they are seeking for high returns, they are more willing to gamble and therefore more attracted to lottery like prospect (i.e. positively skewed). As a result, they do not hesitate to put low specifications for A and α: the safety first constraint can be satisfied more easily.

3. Behavioral Mental Account Portfolios

As previously mentioned, the mental account theory postulates that investors do not consider their portfolio as a whole but rather as a collection of subportfolios (i.e. overlooking the covariance) (Das *et al.*, 2010, p. 232). In this paper, we purposely chose to define the aggregate portfolio as combination between the retirement, education and bequest portfolios, each of them being linked to a specific level of risk aversion (i.e. strong, medium, low respectively) expressed through the safety first constraint. We postulated that the aggregate portfolio would be the result of a combination between 60% of the retirement portfolio, 20% of the education portfolio and 20% of the bequest portfolio. We specially chose to use the same combination as the one defined by Das *et al.* (2010, p. 317).

We empirically derived the aggregated mental account portfolios for which all of the three subportfolios were optimal in the sense of the BPT and thus initially satisfying the safety first constraint. Among them, we identified those that were lying on the mean-variance efficient frontier. Table 4 illustrates the obtained results.

% of optimal MA portfolios	15%
% of optimal MA portfolios lying on the efficient frontier	31,40%

Table 4: overall results for behavioral mental accounts portfolios

3.1. Specifications

As previously, we will analyze graphically the resulting optimal portfolios and try to draw some relevant conclusions. First, we graphed the location of a typical mental account portfolio on the mean-variance efficient frontier (Fig. 6.). A small sample of the entire set of plot can be found in appendix 6.

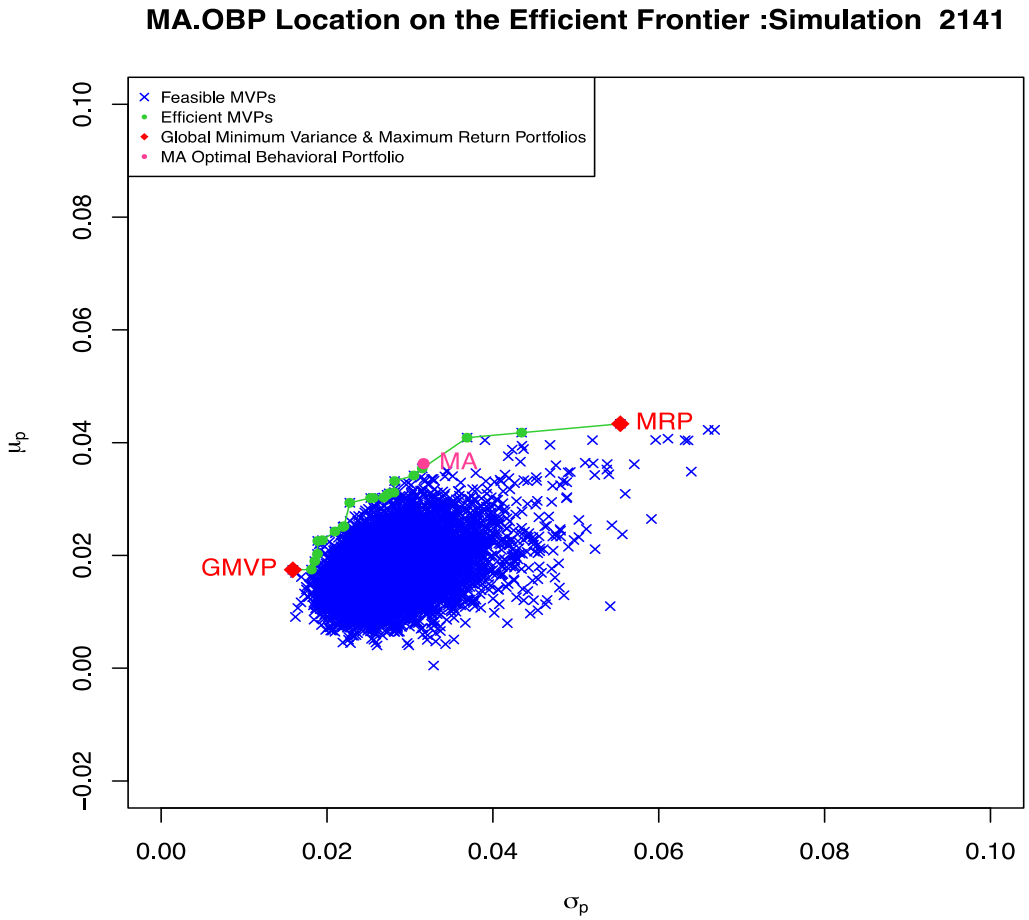


Fig. 6. MA location in the Mean-Variance space simulation 2141. Data: Bloomberg.

As one can see in this case, the efficient mental account portfolio is slightly above the efficient frontier, meaning that the mental account structure allowed having an efficient portfolio that was unachievable otherwise. However, this was typically not the case in other simulations: as seen previously, only 31,40% of mental account portfolios were above or on the efficient frontier. These results are plausible given our assumption according to which all of the three subportfolios should be feasible. Indeed, investors are now facing three distinct safety first constraints, which limits their choices even more. We will see on the next point some of the limitations we faced while computing those portfolios.

After further analysis, we observed that much like OBPs, the mental account portfolios were characterized by high levels of risk and standard deviation and thus were lying on the upper-right side of the efficient frontier (Fig.6). We then decided to analyze their returns along the complete studied time period, as well as their standard deviation. We set them in perspective with the returns and standard deviation of the GMVP and MRP. Following are the graphical restitutions of our findings (Fig. 7., Fig. 8.).

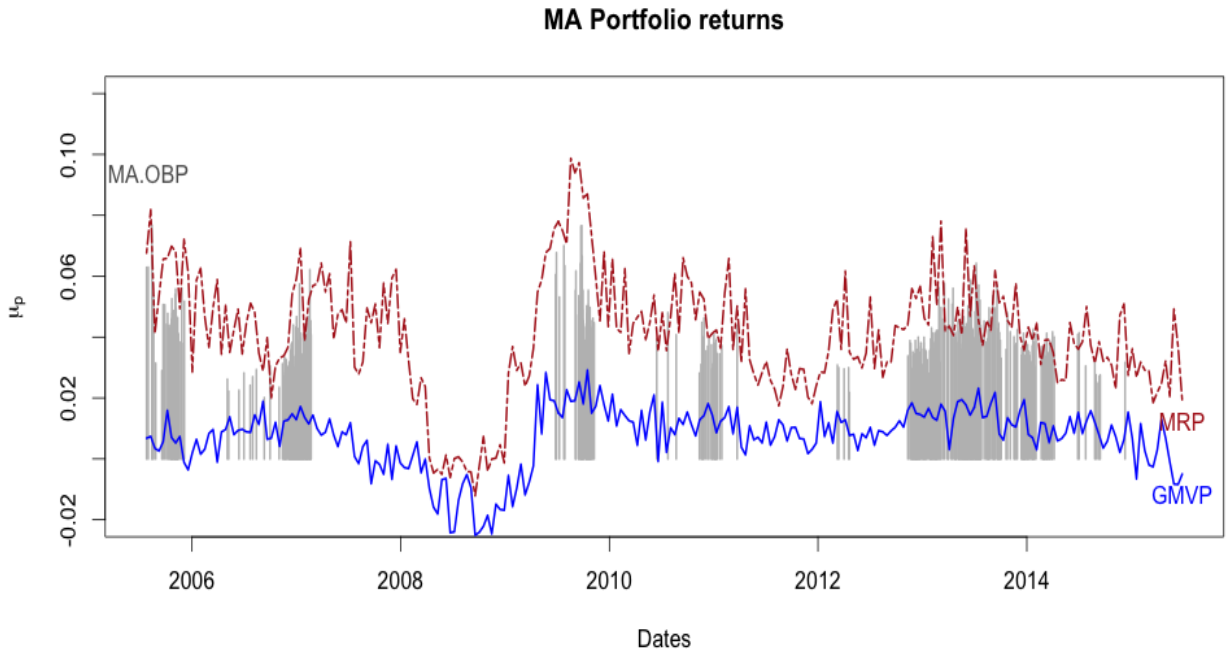


Fig. 7. MA Portfolio returns compared to MRP and GMVP

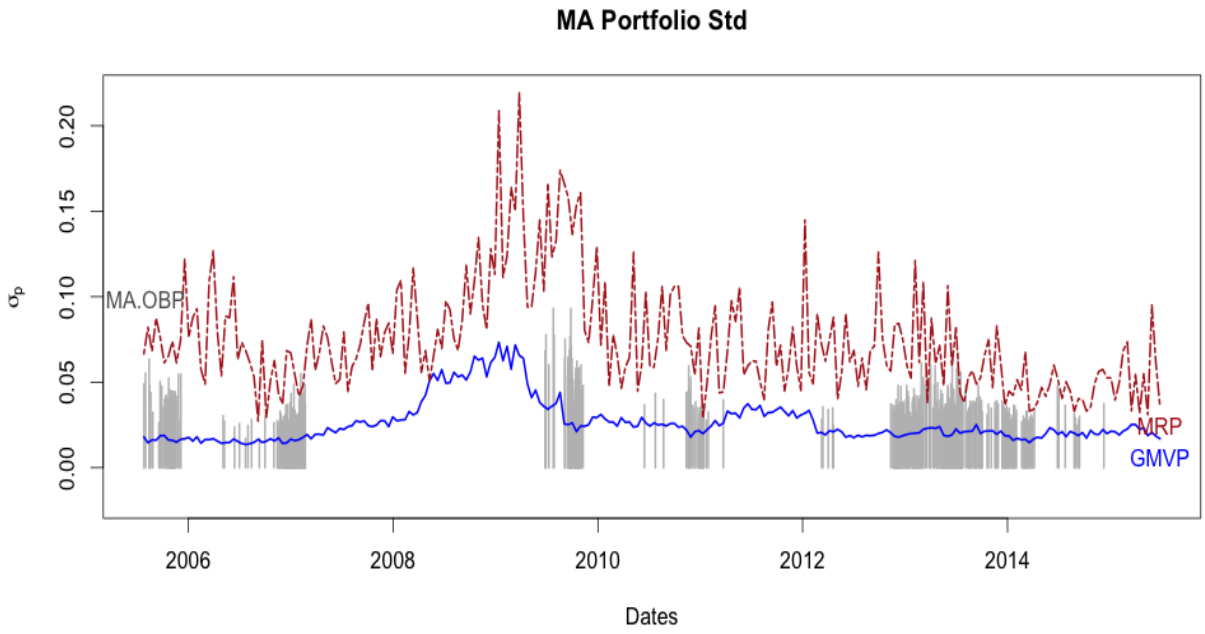


Fig. 8. MA Portfolio Std compared to MRP and GMVP

Given the similarities between this set of graph and the preceding ones, the same conclusions can be drawn.

3.2. Limits

As stated before, we faced some limitations during the computations of the mental account optimal portfolios. The main limitation we encountered is that the number of mental account optimal portfolios lying on the efficient frontier is significantly lower than the one obtained for the behavioral portfolios. According to us, this is mainly due to the fact that we voluntary chose to restrict our analysis to the aggregated portfolios for which all of the three subportfolios were meeting the safety first constraint. Indeed, when one of the subportfolio was non-existent, we could not compare the resulting aggregated portfolio to the MVP portfolios anymore as only part of the investor wealth was invested in the stock market.

Let us illustrate this situation with two examples. On the one hand, think about a situation where the Retirement portfolio has no optimal solution. It is then not reasonable to assume that the investor will decide to allocate the retirement proportion of wealth to a riskier portfolio such as the education or bequest one. Therefore, we believe that this investor will invest in opportunities outside of the stock market to compensate, which its outside of the scope of our paper. On the other hand, think about an investor who does not find an optimal solution for her bequest portfolio. It is then reasonable to assume than she will transfer the

subportfolio wealth allocation to the other two portfolios, the question is then: in which proportions? These are the reasons why we only focused our attention on the mental account aggregated portfolios that were characterized by three subportfolios with optimal solutions.

4. Conclusions

After computing the set of efficient frontiers in the mean-variance framework, we were able to start making various comparisons between three different frameworks: Markowitz' MPT, Shefrin & Statman's BPT, and behavioral mental account.

The first conclusion we were able to draw from our empirical analysis was that in the behavioral framework an optimal solution was not always defined. In our set, only 60% of the simulations had an optimal solution. However, keep in mind that our studied time period was centered around the 2008 crisis and that therefore results could be affected by it, explaining the 10% gap between our study and Bourachnikova's *et al.* (2016) results.

Secondly, we discovered that 60% of the optimal behavioral portfolios would lie on the efficient frontier, as defined in the mean-variance framework. Whereas this result is consistent with Bourachnikova *et al.* research, it contradicts Shefrin and Statman's (2000, p.128) statement, according to which *portfolios on the BPT efficient frontier are generally not on the Mean Variance frontier*. However multiple recent empirical studies tended to prove our point and showed that OBP could coincide with the efficient frontier (Das *et.al*, 2010; Levy & Levy, 2004).

Thirdly, we analyzed the specificities of the OBP that coincided with the mean-variance efficient frontier. We found out that they were characterized by extremely high levels of standard deviation and expected return. Namely, they were located on the upper right side of this frontier. Bourachnikova *et al.* (2016) reflected these results by stating that a risk averse mean-variance investor would not hold a BPT optimal portfolio, even if located on the efficient frontier. However, we went further and stated that this phenomenon was due to the different approaches used to define risk by those two types of investors. As such, behavioral investors cannot be considered as taking more risks in absolute terms. They rather do not value risk the same way as standard investors holding efficient portfolios from the Markowitz' bullet. To illustrate this, we recalled Howards' doubts regarding standard deviation as a measure of risk, as well as the excess volatility puzzle. It is important to notice that a massive

literature about risk and its possible definition exists. However, in the scope of this paper, we decided to limit our analysis to standard deviation and the safety first constraint.

Furthermore, we recalled Lopes' and Kahneman and Tversky theories to justify this location. We also used their findings and namely the concepts of 'fear' and 'hope' to justify the lack of optimal behavioral portfolios around the sub-primes crisis period.

Finally, we implemented the mental account structure to the BPT framework and found, to some extent, similar results. This is, the optimal mental account portfolios were characterized by high levels of risk and standard deviation. However, we also encountered significant dissimilarities. Namely, the number of simulations for which the mental account optimal portfolios were satisfying the constraint, and the number of the number of mental account optimal portfolios lying on the mean-variance efficient frontier were much lower. We argued that they were the consequences of the model's limitations and out of the scope of the present paper.

CONCLUSION

Numerous concepts lying at the core of modern portfolio theory have been criticized by several studies over the past decades. More precisely, investors' irrationality, fundamental inefficiency in the market and non-Gaussian distribution of returns were discussed. Bearing in mind these challenges to Markowitz' framework and considering his initial assumptions as too strong and oversimplifying of the world, we decided to analyze the alternatives given by behavioral finance. To achieve our goal, we grounded our theoretical analysis on several researches challenging Markowitz' framework (Mehra & Prescott, 1985; Shiller, 1992; Howard, 2014; Thaler, 1997;…) and more broadly defined Shefrin and Statman's behavioral portfolio theory (2000) and Tversky and Kahneman cumulative prospect theory's premises (1992). This allowed us to have a broader perspective on portfolio theory and to empirically apply the newly acquired notions. Hence, following Bourachnikova *et al*.'s methodology (2016), we provided an empirical comparison between various concepts of optimal portfolios stemming from both standard and behavioral finance. More specifically, we discussed the characteristics of various behavioral specifications built on the concepts of prospect theory and mental accounts.

Based on 2500 simulation exercises, we were able to define the optimal behavioral portfolios, as understood by Shefrin and Statman (2000) and to compare them with the portfolios generated through the mean-variance framework. The simulations were defined using the daily historical prices (Bloomberg) of 387 stocks stemming from the S&P500 from 2005 to 2015.

In order to move from Markowitz' framework to behavioral portfolio theory, the definition of risk has to be adjusted. While Markowitz defines it as the standard deviation of the returns distribution, Shefrin and Statman rather use the safety first constraint (i.e. analogous to the Value-at-Risk). Doing so allows departing from the shortcomings of the standard deviation and being more predictive of the real world.

Applying the safety first constraint, we demonstrate that, in contrast to Markowitz' framework, an optimal solution to the portfolio allocation problem does not always exist in the behavioral framework (i.e. 60% of the cases). Note that the lack of optimal portfolio is particularly significant during the 2008-2009 financial crisis, which we explain by using the

concepts of ‘fear’ and ‘hope’ developed by Lopes and their consequences on the constraint satisfaction.

Yet, the present study highlights that, in the cases where an optimal solution exists, 60% of them are efficient in the sense of Markowitz. More specifically, they are consistently located on the upper right side of the efficient frontier and thus characterized by a highly risky profile. Consistently with the founding of Bourachnikova *et al*. (2016) a MPT investor would typically not choose to hold such portfolio since an extremely low level of risk aversion characterizes this position.

The new contribution of our paper lays in the explanation of those results: we argue that it lies in the differences of risk measurements implemented to solve the underlying optimization programs. Indeed, while standard investors use the standard deviation of the return distribution as a measure of risk, behavioral investors use the safety first constraint (i.e. assimilated to the VaR). As such, behavioral investors cannot be considered as taking more risks in absolute terms. They rather do not value risk the same way as standard investors holding efficient portfolios from the Markowitz' bullet. In that respect, an extension of that study could be to provide a more general assessment of risk bearing attitude of all types of investors by providing some reverse exercise where safety first constraints would be derived for standard investors and analyzed with respect to their discrepancy to zero, i.e. positive (resp. negative) constraints implying greater (lower) risk taking than optimal behavioral portfolios. This opens the doors to a broader debate regarding the most relevant risk measure to adopt, which would not be bounded to behavioral finance. Knowing that an investor behavior can change over time and depends on the context she faces, one could wonder which risk measure would be most adapted to fulfill her needs at a specific point in time or even consider a model combining multiple risk measures.

The present study also extends Bourachnikova *et al*.’s work (2016) by implementing the mental account structure to the behavioral portfolio theory framework. The mental account aggregated portfolios are made of a combination of three different subportfolios, each being associated to a specific safety first constraint. The empirically derived results are, to some extent, consistent with the ones previously uncovered. This is, high levels of risk and standard deviation characterize the optimal mental account portfolios. Nevertheless, significant dissimilarities should be mentioned: both the percentage of simulations in which at least one

portfolio satisfies the constraint and the proportion of optimal portfolio lying on the mean-variance efficient frontier are much lower. Results are explained by the restriction of the analysis to the case in which all three subportfolios are satisfying their relative constraint. When one of the subportfolio does not have an optimal solution, we cannot compare the resulting aggregated portfolio to the MVP portfolios anymore, as only part of the investor wealth is invested in the stock market.

This leads us to mention a broader limitation affecting both the behavioral portfolio model and its mental account declination. We believe that there is a general lack of explanation regarding the behavior of investors facing the specific case where there is no optimal solution to their portfolio allocation. Do they really exit the stock market? In the positive case, what are their investment alternatives? Further studies should be conducted to analyze the actual behavior of investors in such situation and take into account the external alternative for which they would substitute their portfolio allocation.

BIBLIOGRAPHY

Allais, M.. (1953). Le Comportement de l'Homme Rationnel devant le Risque: Critique des Postulats et Axiomes de l'Ecole Americaine. *Econometrica*, 21(4), 503–546.

Alphonse, P., Desmuliers, G., Grandin, P., & Levasseur, M. (2013). *Gestion de portefeuille et marchés financier.* Montreuil: Pearson France.

Ball, R., & Brown, P. (1968). An Empirical Evaluation of Accounting Income Numbers. *Journal of Accounting Research*, 6(2), 159-178.

Banz, R. W. (1981). The relationship between return and market value of common stocks. *Journal of financial economics*, 9(1), 3-18.

Barber, B.M., Odean, T. (2011). The behavior of individual investors. Available at SSRN: /http://ssrn.com/abstract1⁄41872211S. Retrieved on April/ 28/ 2016.

Barberis, N., & Huang, M. (2008). Stocks as Lotteries: The Implications of Probability Weighting for Security Price. *The American Economic Review*, 98(5), 2066–2100.

Basu, S. (1977). Investment performance of common stocks in relation to their price-earnings ratios: A test of the efficient market hypothesis. *The journal of Finance*, 32(3), 663-682.

Benartzi, S., & Thaler, R. H. (1995). Myopic Loss Aversion and the Equity Premium Puzzle. *The Quarterly Journal of Economics*, 110(1), 73–92.

Benartzi, S., & Thaler, R. H. (1999). Risk Aversion or Myopia? Choices in Repeated Gambles and Retirement Investments. *Management Science*, 45(3), 364–381.

Béreau, S. (2013). LECGE1218 - *Portfolio Theory*. Faculté des Sciences économiques, sociales, politiques et de communication, Université Catholique de Louvain.

Black, F. (1986). Noise. *The Journal of Finance*, 41(3), 529–543.

Bloomberg L.P. (2016) Historical Stock price of S&P 500 from 03/01/2005 to 31/12/2015. Retrieved Feb. 12, 2016 from Bloomberg database.

Bourachnikova, O., Pfiffelmann, M., & Roger, T. (2016). When behavioral portfolio theory meets Markowitz theory. *Economic Modelling*, 53(1), 419–435.

Brealy, R. A., Myers, S. C., & Allen, F. (2005). *Corporate Finance :8th Ediition.* New York: McGraw-Hill Irwin.

Charpentier, A., & Oulidi, A. (2009). Estimating allocations for Value-at-Risk portfolio optimization. *Mathematical Methods of Operations Research*, 69(3), 395-410.

Das, S., Markowitz, H., Scheid, J., & Statman, M. (2010). Portfolio Optimization with Mental Accounts. *The Journal of Financial and Quantitative Analysis*, 45(2), 311–334.

Hurley, W. J., & Brimberg, J. (2015). A note on the sensitivity of the strategic asset allocation problem. Operations Research Perspectives, 2, 133-136.

Fama, E. F. (1970). Efficient Capital Markets: A Review of Theory and Empirical Work. *The Journal of Finance*, 25(2), 383-417.

Fama, E. F., & French, K. R. (1988). Dividend Yield and Expected Stock Returns. Journal of Financial Economics, 22, 3-25.

Federal Reserve Bank of St. Louis. (2016) Historical rate of 3-Month Treasury-Bill and 10-Year Treasury-Bond from 03/01/2005 to 31/12/2015. Retrieved Feb. 23, 2016 from FRED database.

Friedman, M., & Savage, L. J. (1948). The Utility Analysis of Choices Involving Risk. *Journal of Political Economy*, 56(4), 279–304.

Graham, J. R., (1999). Herding among Investment Newsletters: Theory and Evidence. *The Journal of Finance*, 54(1), 237–268.

Hankin, R. K. S. (2006). Additive integer partitions in R. *Journal of Statistical Software*, Code Snippets, 16(1).

Hastorf, A.H., Schneider, D.J., & Polefka, J. (1970). *Person Perception*. Boston: Addison-Wesley.

Hendricks, D., Patel, J., & Zeckhauser, R.(1991). The Rationality Struggle: Illustrations from Financial Markets. *The American Economic Review*, 81(2), 232–236.

Howard, C. T. (2014). *Behavioral Portfolio Management : How succesful investors master their emotions and build superior portfolios*. Petersfield: Harriman House LTD.

Hiro, P. & Sofat, R. (2011). *Strategic Financial Management.* New Delhi: PHI Learning Private Limited.

Hull, J., & White, A. (1988). Incorporating volatility updating into the historical simulation method for value-at-risk. *Journal of Risk*, 5-19.

Kahneman, D. (2011). *Thinking, fast and slow*. New York: Macmillan.

Kahneman, D., & Tversky, A. (1973). Availability: A heuristic for judging frequency and probability. *Cognitive psychology*, 5(2), 207-232.

Kahneman, D., & Tversky, A. (1979). Prospect theory: An analysis of decision under risk. Econometrica: *Journal of the Econometric Society*, 47(2), 263-291.

Kahneman, D., & Tversky, A. (1981). The framing of decisions and the psychology of choice. *Science*, 211(4481), 453-458.

Kahneman, D., & Tversky, A. (1989). *Rational choice and the framing of decision*. Berlin: Springer Heidelberg.

Kahneman, D., & Tversky, A. (1992). Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and uncertainty*, 5(4), 297-323.

Kahneman, D., Schwartz, A., Thaler, R. H., & Tversky, A. (1997). The Effect of Myopia and Loss Aversion on Risk Taking: An Experimental Test. *The Quarterly Journal of Economics*, 112(2), 647–661.

Kroll, Y., Levy, H., & Rapoport, A. (1988). Experimental Tests of the Separation Theorem and the Capital Asset Pricing Model. *The American Economic Review*, 78(3), 500–519.

Larry H. P. Lang, & Stulz, R. M.. (1994). Tobin's q, Corporate Diversification, and Firm Performance. *Journal of Political Economy*, 102(6), 1248–1280.

Levy, H., & Levy, M.. (2004). Prospect Theory and Mean-Variance Analysis. The Review of Financial Studies, 17(4), 1015–1041.

Lobão, J. (2015a). 2FI14 - *Behavioral finance*. Faculdade de Economia da Universidade do Porto.

Lobão, J. (2015b). 2FI02 - *Markets and financial investements*. Faculdade de Economia da Universidade do Porto.

Lo, A. W., & MacKinlay, A. C. *A Non-Random Walk Down Wall Street.* Princeton: Princeton University Press.

Lopes, L. L. (1987). Between hope and fear: The psychology of risk. *Advances in experimental social psychology*, 20(3), 255-295.

Mandelbrot, B. (1963). The variation of certain speculative prices. *Journal of Business*, 36, 394–419.

Markowitz, H. M. (1952). Portfolio Selection. *The Journal of Finance*, 7(1), 77–91.

Markowitz, H. M. (1959). *Portfolio Selection : Efficient diversification of investements*. United States of America: Cowles Foundation for Research in Economics at Yale University.

Mehra, R., Prescott, E. C. (1985). The equity premium: A puzzle. *Journal of Monetary Economics*,15(2), 145-161.

Mills, F. C. (1956). *Introduction to Statistics*. New-York: Henry Holt & co.

Montier, J. (2010). *The little book of behavioral investing: how not to be your own worst enemy*. New-York: Wiley.

Morgenstern, O., & Von Neumann, J. (1944): *Theory of Games and Economic Behavior*, Second edition, 1947; third edition, 1953. Princeton, New Jersey: Princeton University Press.

Nofsinger, J. R. (2001). *Investment madness – How psychology affects your investing... and what to do about it*. New-York: Pearson Education.

Nofsinger, J. R., & Sias, R. W. (1999). *Herding and Feedback Trading by Institutional and Individual Investors*.New-York: Wiley.

Pontiff, J. (2006). Costly arbitrage and the myth of idiosyncratic risk. *Journal of Accounting and Economics*, 42(2), 35-52.

Pound, J., & Shiller, R. (1989). Survey evidence on diffusion of interest and information among investors, *Journal of Economic Behavior & Organization*,12(1), 47-66.

Quiggin, J. (1982). A theory of anticipated utility. *Journal of Economic Behavior & Organization*, 3(4), 323-343.

Rademacher, H. (1937). On the partition function p(n). *Proceedings London Mathematical Society*, 43, 241-254.

R Core Team. (2015). R: A language and environment for statistical computing. R Foundation for Statistical Computing. Vienna: Austria. https://www.R-project.org/.

Roncalli, T. (2014). *Introduction to risk parity and budgeting*. New York: CRC Press.

Roy, A. D. (1952). Safety First and the Holding of Assets. *Econometrica*, 20(3), 431–449.

Sargent, T. J. (1987). *Macroeconomic Theory*. New York: Academic Press.

Schmeidler, D. (1989). Subjective Probability and Expected Utility without Additivity. *Econometrica*, 57(3), 571–587.

Scheinkman, J. A., & Xiong, W. (2003). Overconfidence and Speculative Bubbles. *Journal of Political Economy*, 111(6), 1183–1220.

Shefrin, H., & Statman, M. (2000). Behavioral Portfolio Theory. *The Journal of Financial and Quantitative Analysis*, 35(2), 127–151.

Shiller, R. J. (1992). *Market volatility*. Cambridge (US): MIT press.

Shiller, R. J. (2003). From Efficient Markets Theory to Behavioral Finance. *The Journal of Economic Perspectives*, 17(1), 83–104.

Shleifer, A., & Vishny, R. W. (1997). The Limits of Arbitrage. *The Journal of Finance*, 52(1), 35–55.

Sias, R. W. (2004). Institutional Herding. *The Review of Financial Studies*, 17(1), 165–206

Siegel, J. J. (2005). Perspectives on the Equity Risk Premium. *Financial Analysts Journal*, 61(6), 61–73.

Siegel, J. J. (2008). *Stocks for the Long Run*: 4th Edition. New-York: McGraw-Hill Companies.

Siegel, J. J., & Thaler, R. H. (1997). Anomalies: The Equity Premium Puzzle. *The Journal of Economic Perspectives*, 11(1), 191–200.

Simon, H. A. (1955). A Behavioral Model of Rational Choice. *The Quarterly Journal of Economics*, 69(1), 99–118.

Simon, H. A. (1957). *Models of Man, Social and Rational: Mathematical Essays on Rational Human Behavior in a Social Setting*. New York: Wiley.

Simon, H. A. (1972). Theories of bounded rationality. *Decision and organization*, 1(1), 161-176.

Slezak, S. L. (2003). On the Impossibility of Weak-Form Efficient Markets. *The Journal of Finance and Quantitative Analysis*, 38(3), 523-554.

Statman, M. (1999). Behaviorial finance: Past battles and future engagements. *Financial Analysts Journal*, 55(6), 18-27.

Statman, M., Thorley, S., & Vorkink, K. (2006). Investor Overconfidence and Trading Volume. The Review of Financial Studies, 19(4), 1531–1565.

Svenson, O. (1981). Are we all less risky and more skillful than our fellow drivers?. *Acta Psychologica*, 47(2), 143-148.

Tobin, J. (1984). On the efficiency of the financial system. *Lloyds Bank Review*, 153, 1-15.

Telser, L. G. (1955). Safety First and Hedging. *The Review of Economic Studies*, 23(1), 1–16.

Urbani, P. (n.d.). When Normality is not enough. Available at Edge-fund.com: http://www.edge-fund.com/UrbaniXX.pdf. Retreived on April/ 28/ 2016.

Warnes. G. R., Bolker. B., Gorjanc. G, Grothendieck G., *et al*. (2015). gdata: Various R Programming Tools for Data Manipulation. R package version 2.17.0. https://CRAN.R-project.org/package=gdata.

APPENDIXES

1. Appendix 1: From the expected utility framework to the "mean-variance trade-off"

In this note[1], we demonstrate under which assumptions the expected utility framework may be considered as equivalent to a "mean-variance trade-off". The analysis is taken from pp. 154-155 in Sargent (1987).

Consider that the utility function defined over wealth, c, is exponential and given by:

$$\mathcal{U}(c) = -\mathrm{e}^{-\lambda c}, \lambda > 0$$

This utility function is increasing and concave with wealth, as:

$$\mathcal{U}'(c) = -\lambda \mathrm{e}^{-\lambda c} > 0$$
$$\mathcal{U}''(c) = -\lambda^2 \mathrm{e}^{-\lambda c} < 0$$

Since the utility function is concave, it reflects risk aversion. A more specific measure of it is given by, among others, the Arrow-Pratt index of absolute risk aversion, defined as:

$$R_A(c) \equiv -\frac{\mathcal{U}'(c)}{\mathcal{U}''(c)} = \lambda$$

This means that if an individual is characterized by the utility function given in the first equation, its risk aversion is captured by the λ parameter. The greater λ, the more the agent will be risk adverse.

We assume now that $\tilde{c}$ is distributed normally with mean μ and standard deviation σ. The density of $\tilde{c}$ is given by:

$$p(\tilde{c}) = \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-\frac{(\tilde{c}-\mu)^2}{2\sigma^2}}$$

Hence, expected $\mathbb{E}\mathcal{U}(\tilde{c})$ is given by:

$$\mathbb{E}\mathcal{U}(\tilde{c}) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} (-1) \cdot \mathrm{e}^{-\lambda c} \cdot \mathrm{e}^{-\frac{(c-\mu)^2}{2\sigma^2}} dc$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} -\mathrm{e}^{-\left(\lambda c + \frac{(c-\mu)^2}{2\sigma^2}\right)} dc$$

[1] Taken from Sophie Béreau's 5th lecture complement in LECGE 1218 - Portfolio Theory, initially inspired from Yossi Spiegel

Noticing that:

$$\lambda c + \frac{(c-\mu)^2}{2\sigma^2} = \frac{(c-\mu\ +\lambda\,\sigma^2)^2}{2\sigma^2} + \lambda \cdot \left(\mu - \frac{\lambda\sigma^2}{2}\right)$$

And substituting this expression into that of $\mathbb{E}\mathcal{U}(\tilde{c})$, we have:

$$\mathbb{E}\mathcal{U}(\tilde{c}) = -\frac{\mathrm{e}^{-\lambda\left(\mu-\frac{\lambda\sigma^2}{2}\right)}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{(c-\mu+\lambda\,\sigma^2)^2}{2\sigma^2}} dc$$

Since $\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{(c-\mu')^2}{2\sigma^2}} dc$ for all μ' including $\mu' = \mu - \lambda\sigma^2$, we finally get

$$\mathbb{E}\mathcal{U}(\tilde{c}) = -\mathrm{e}^{\lambda\left(\mu-\frac{\lambda\sigma^2}{2}\right)}$$

From this result, it comes that it is equivalent for the agent to maximize his wealth $\tilde{c}$ or the expression $\mu - \frac{\lambda\sigma^2}{2}$.

From previous result, it comes that the agent is interested in maximizing his expected wealth μ minus the variance multiplied by the parameter capture his risk-aversion. There is thus a trade-off between μ and σ^2, the more the agent is risk adverse, the more he will penalize the increase in risk (σ^2) along with the increase in expected gain (μ).

2. Appendix 2: Violation of the Expected Utility framework: examples

2.1. Appendix 2.1: Certainty effect and the violation of the substitution axiom

Tversky and Kahneman (1979, pp. 266-267) selected 100 participants to whom they proposed different situation in which they had the choice between to solutions. Following is the results of their studies and their implications on the expected utility framework.

Problem 1

Choice A: make a gain of 4000 with a probability of 0,80, 0 otherwise – A = (4000;0,80)
Choice B: make a gain of 3000 with certainty – B = (3000)
Results: 20% for A and 80% for B

Problem 2

Choice C: make a gain of 4000 with a probability of 20%, 0 otherwise – C = (4000;0,20)
Choice D: make a gain of 3000 with a probability of 25%, 0 otherwise – D = (3000;0,25)
Results: 65% for C and 35% for D

Analysis

In the first problem, choice B is preferred to choice A (80% vs. 20%). However, in the second one, choice C is preferred to choice D (65% vs. 35%). C can be expressed in terms of choice A, such that: C = (A;0,25). Similarly, D = (B;0,25). According to problem 1, D should be preferred to A. However, this is not the case, proving that the substitution axiom of the expected utility theory is violated. To prove the certainty effect, see that the reduction of probability of gain from 1 to 0,25 has more effect than the reduction of this probability from 0,8 to 0,2.

2.2. Appendix 2.2: example of the isolation effect

Problem 3

Consider the following two-stages game where a player has 75% chance of winning nothing and 25% chance to move to the second stage (choice is made at time 0). If the second stage is reached, the investor is facing problem 1, as defined above (i.e. A = (4000;0,80), B = (3000)).

Note that in terms of final outcomes, the investor faces a choice between (4000;0,20) and (3000;0,25), namely problem 2. As a result, problem 3 can be considered as a sequential formulation of problem 2.

Out of the considered subjects, 78% chosen to play in problem 3 rather than in problem 2, even though both are strictly equivalent. Proving the following point:

> "The reversal of preferences due to the dependency among events is particularly significant because it violates the basic supposition of a decision-theoretical analysis, that choices between prospects are determined solely by the probabilities of final states" (Kahneman, Tversky, 1979, p.272).

Indeed, investors overlook the first stage of problem 3 and are lured to choose it because the second stage seems more attractive than problem 2.

3. Appendix 3: Cumulative Prospect Theory in details

Those details were retrieved from Tversky and Kahneman (1992) and Bourachnikova *et al.* (2016, appendix A).

Under CPT, a prospect F can be denoted as a series of pairs $(x_i, a_i, i = -m, \dots n)$, which have been ranked in increasing order and that yield outcome x_i if a_i occurs. This prospect is evaluated through a valuation function, denoted $V(F)$, in which positive subscripts are used to denote positive outcomes, and vice-versa. We have:

$$V(F) = V(F^+) + V(F^-)$$

$V(F^+)$ and $V(F^-)$ can then be respectively rewritten as:

$$V(F^+) = \sum_{i=1}^{n-1} \left(w^+ \left(\sum_{j=i}^{n} p_j \right) - w^+ \left(\sum_{j=i+1}^{n} p_j \right) v(x_i) + w^+(p_n) v(x_n) \right)$$

$$V(F^-) = \sum_{i=m+1}^{0} \left(w^- \left(\sum_{j=-m}^{i} p_j \right) - w^- \left(\sum_{j=-m}^{i-1} p_j \right) v(x_i) + w^-(p_n) v(x_n) \right)$$

In those formulas, *v* can interpreted as a strictly increasing value function defined relatively to a reference point, x_0, satisfying $v(x_0) = v(0) = 0$. w^+ and w^- are strictly increasing functions satisfying $w^+(0) = w^-(0) = 0$ and $w^+(1) = w^-(1) = 1$.

Tversky and Kahneman (1992, p. 309) derived a two-part power function form for *v*, which is given as follow:

$$v(x) = \begin{Bmatrix} x^\alpha & \text{if x} > 0 \\ -\lambda(-x)^\beta & \text{if x} < 0 \end{Bmatrix}$$

The median values for the α, β, λ parameters where respectively estimated to 0,88; 0,88 and 2,25. α and β allow to express the resulting shape of the function : for $0 < \alpha < 1$, and $0 < \beta < 1$ *v* is concave over gains and convex over losses. λ allows to express the degree of loss aversion.

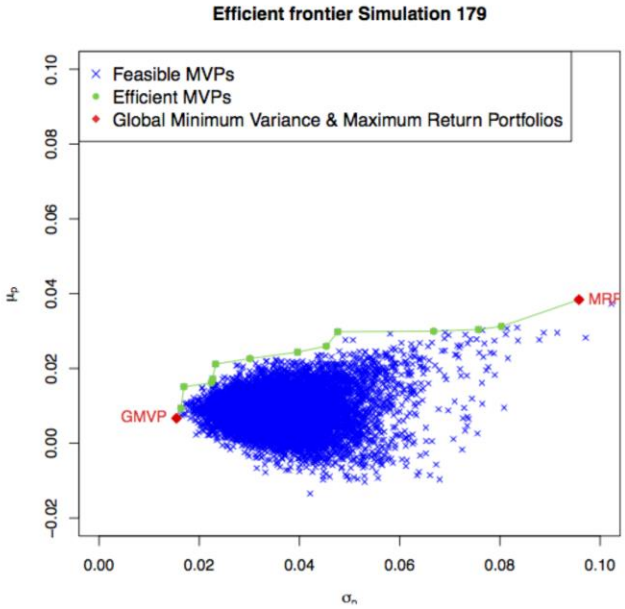
Finally, the weighting function, w, was also defined by Tversky and Kahneman for positive and negative prospects:

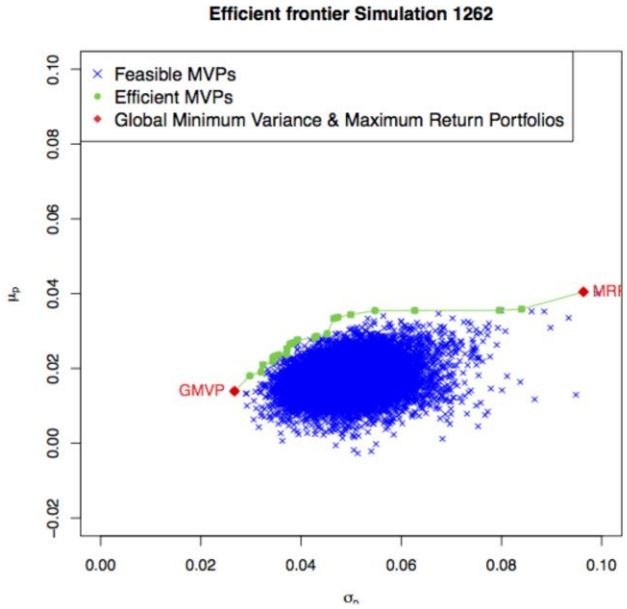
$$w^{+}(p) = \frac{p^{\gamma^{+}}}{\left[p^{\gamma^{+}} + (1-p)^{\gamma^{+}}\right]^{1/\gamma^{+}}}$$

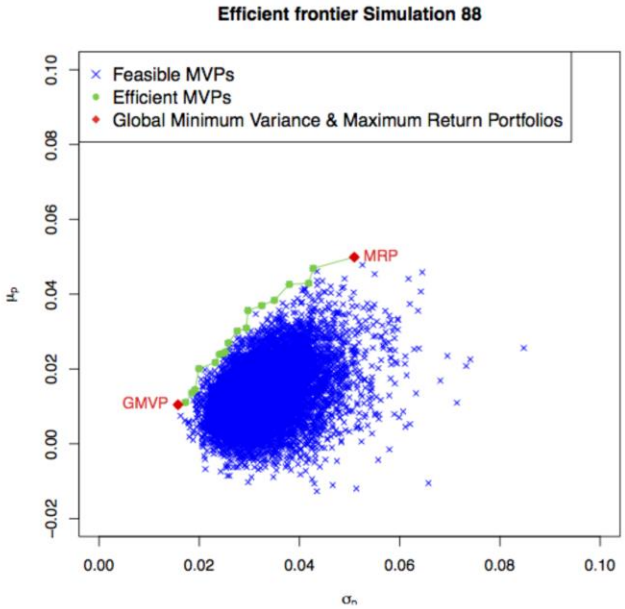
$$w^{-}(p) = \frac{p^{\gamma^{-}}}{[p^{\gamma^{-}} + (1-p)^{\gamma^{-}}]^{1/\gamma^{-}}}$$

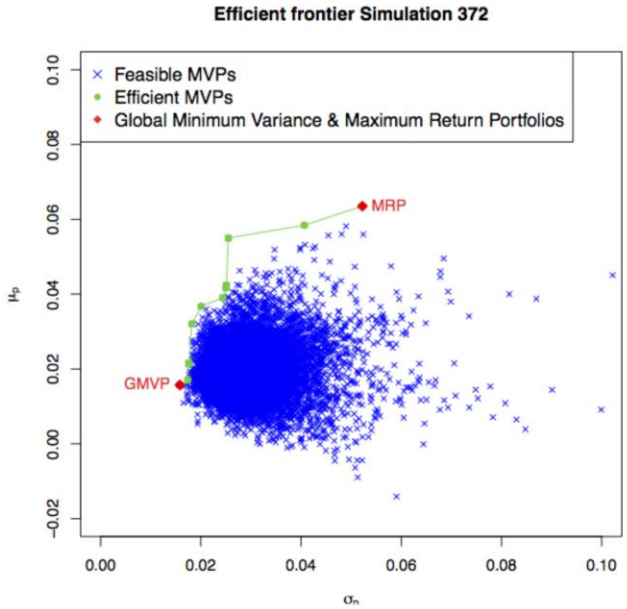
Similarly, $\lambda < 1$ shows the overweighting of low probabilities and the greater sensitivity to variations in probabilities for extremely low and high probabilities. As defined by Tversky and Kahneman (1992), this function is concave near 0 and convex near 1 and the parameters $\gamma^{+}$and γ^{-} where respectively estimated to 0.61 and 0.69.

4. Appendix 4: Sample of Mean-Variance framework plots

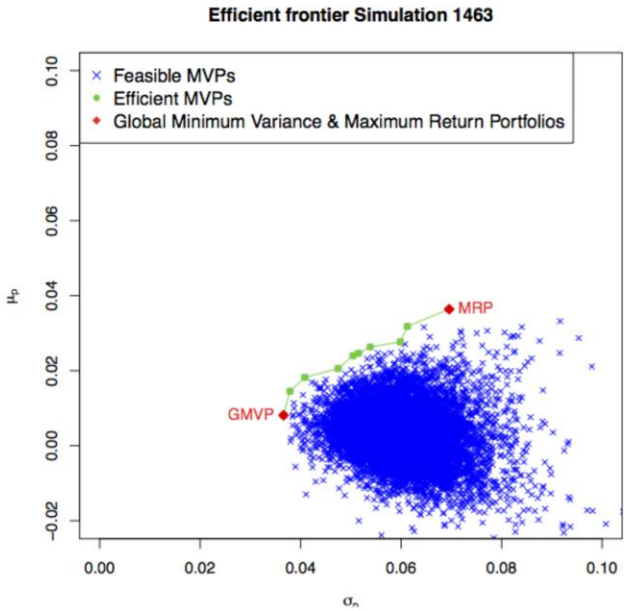


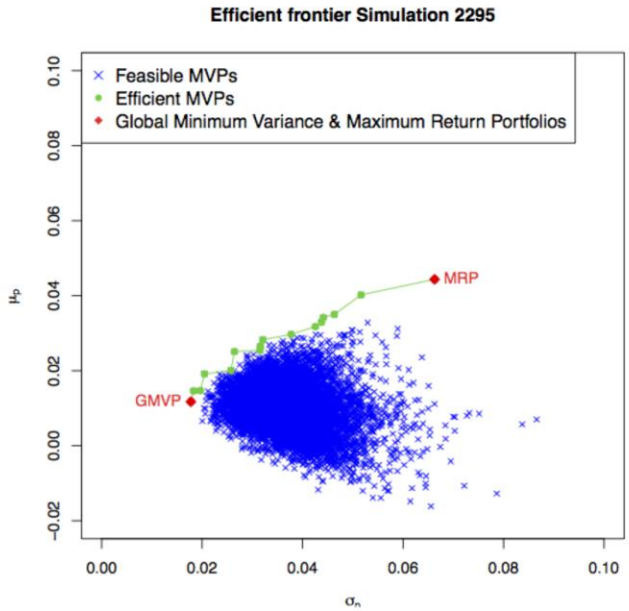


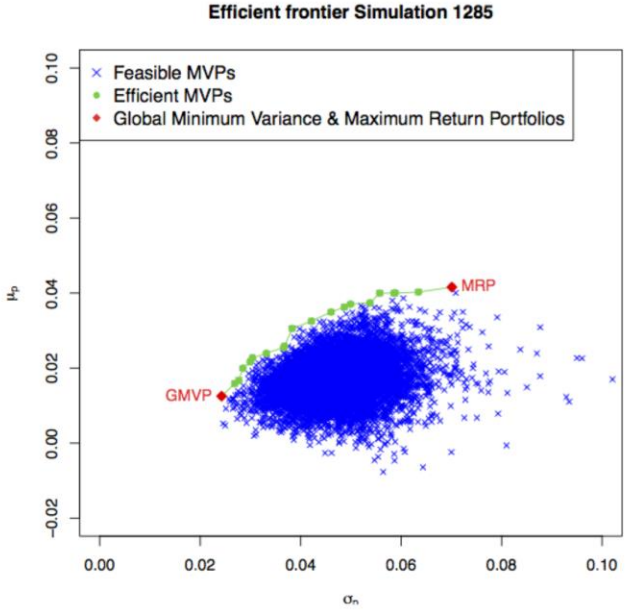


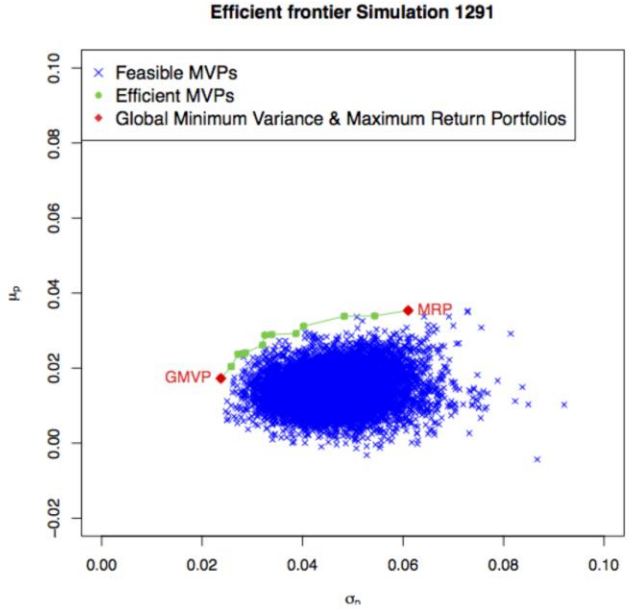


Appendix 4 (continued)



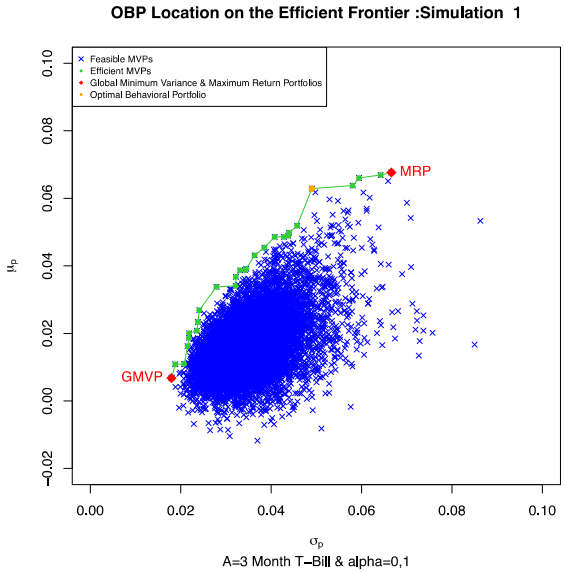


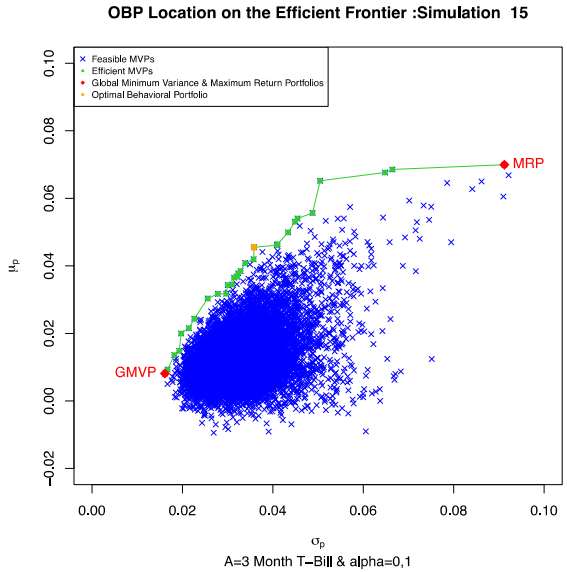


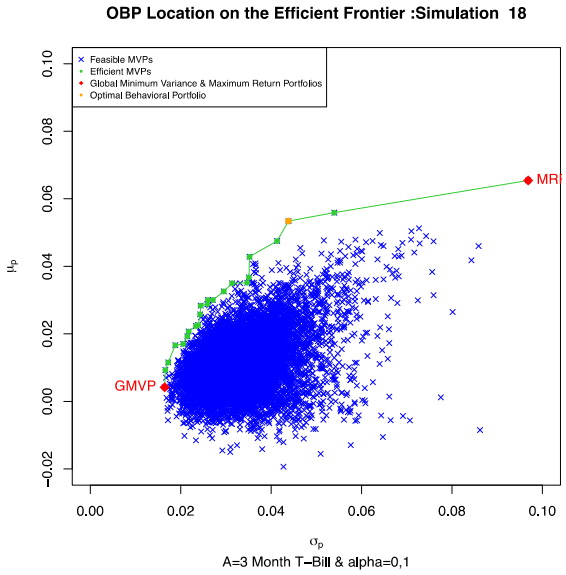


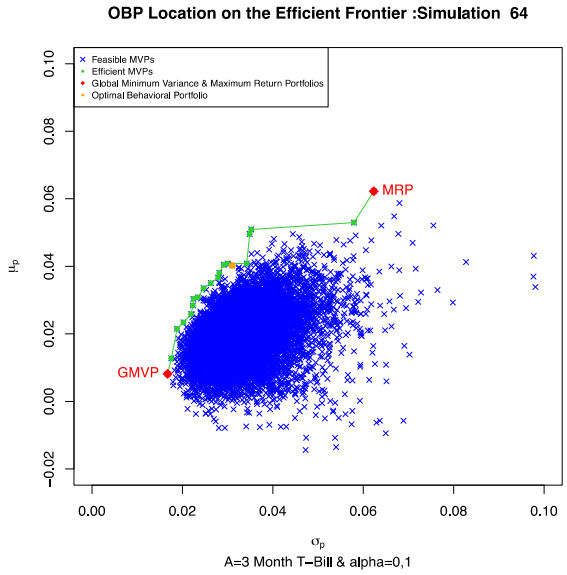
5. Appendix 5: Sample of Optimal Behavioral Portfolios plots

5.1. Appendix 5.1: A = Short-term risk free rate and $\alpha = 0,1$

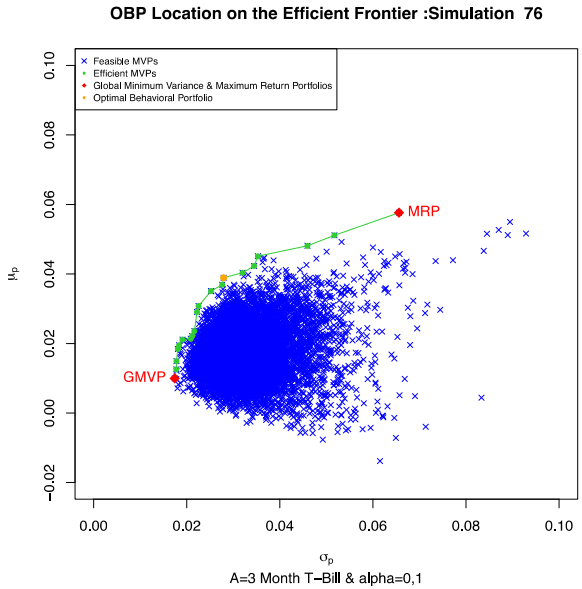


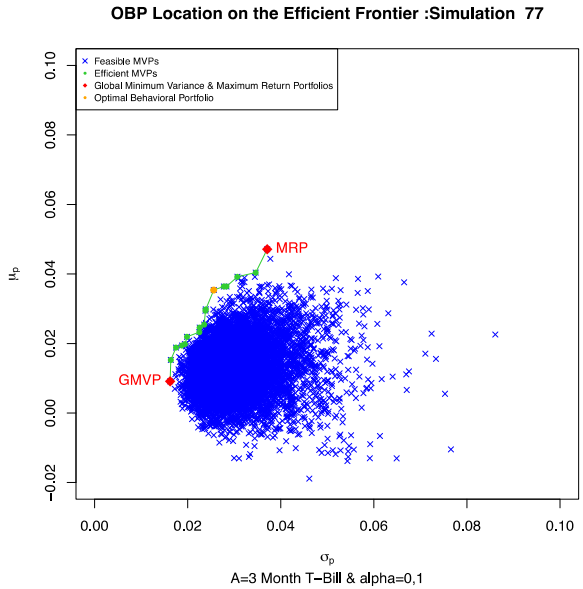


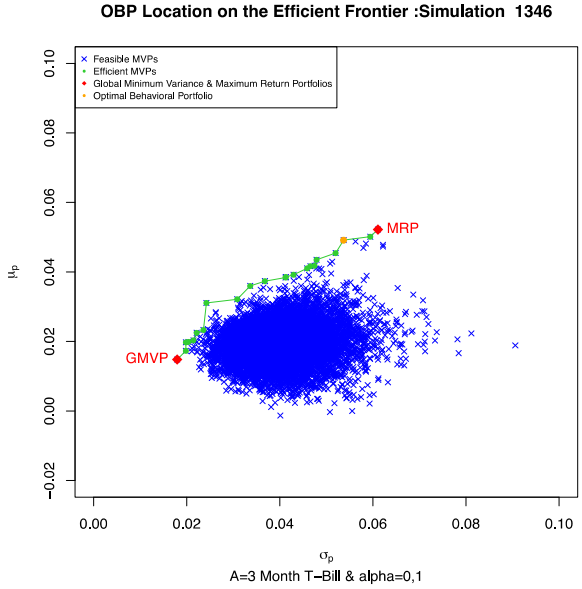


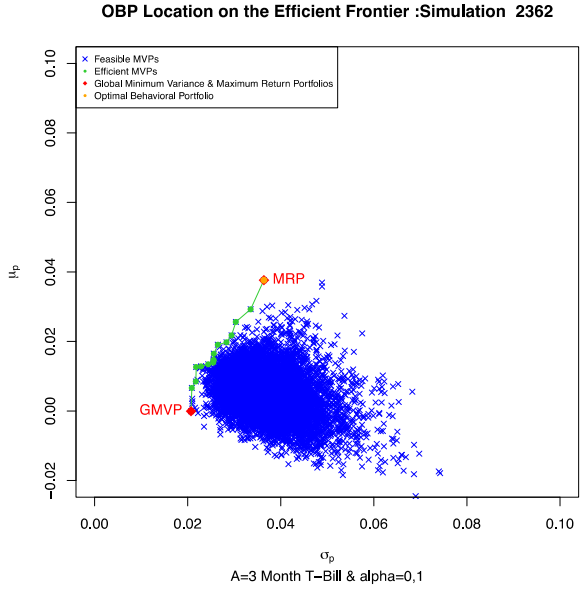


Appendix 5.1 (continued)

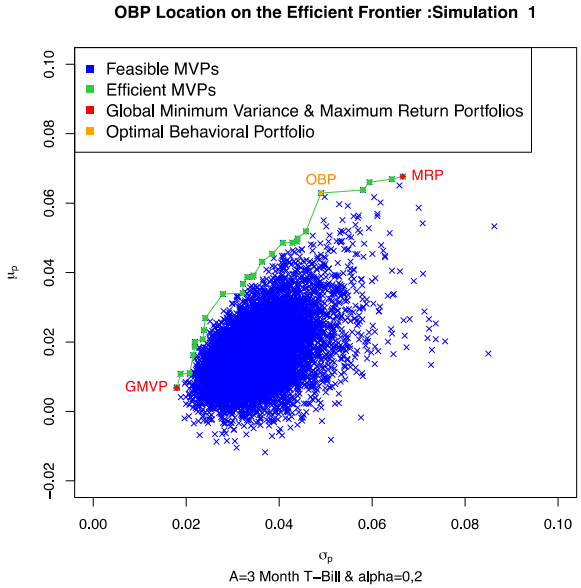


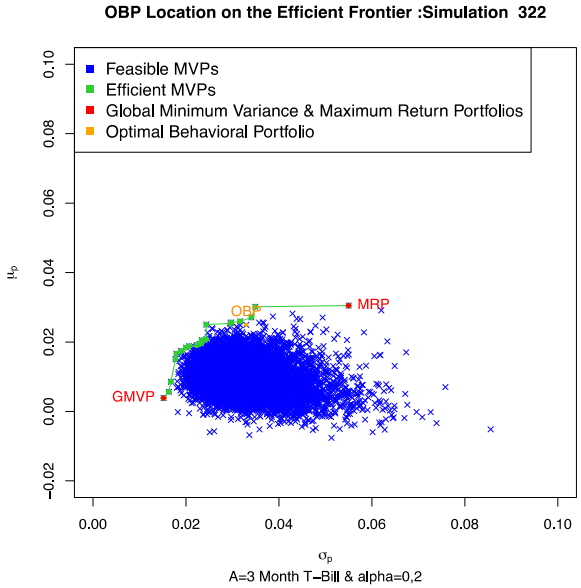


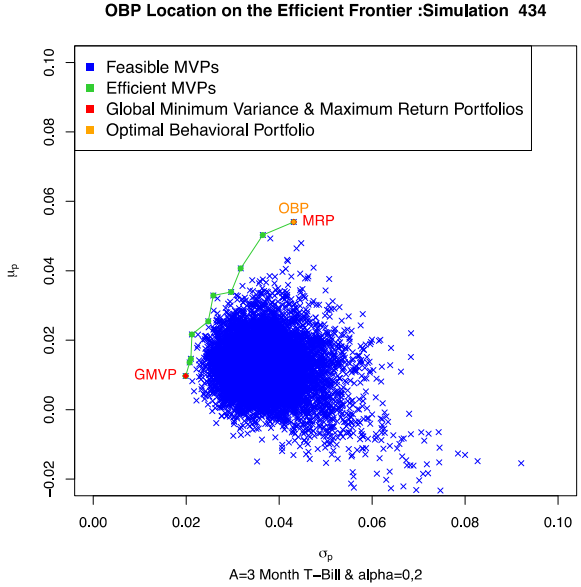


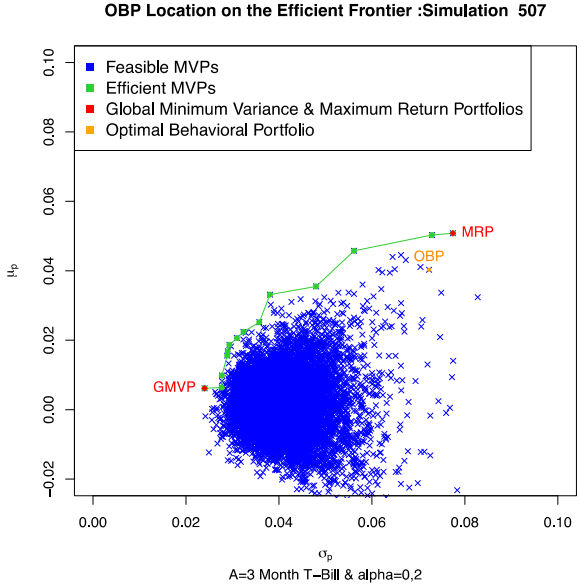


5.2. Appendix 5.2: A = Short-term risk free rate and $\alpha = 0,2$

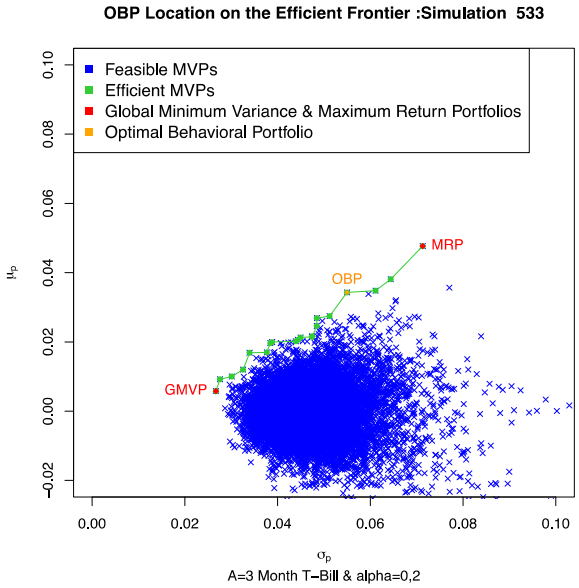


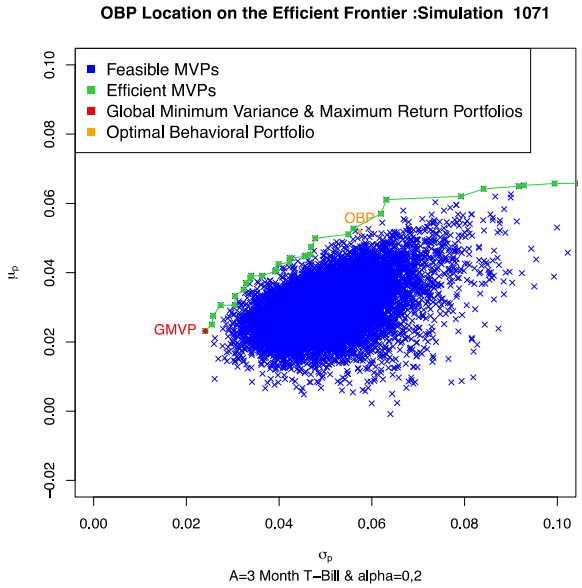


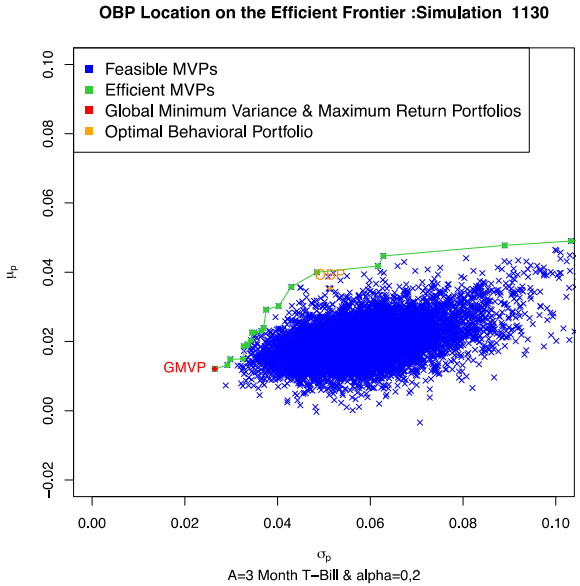


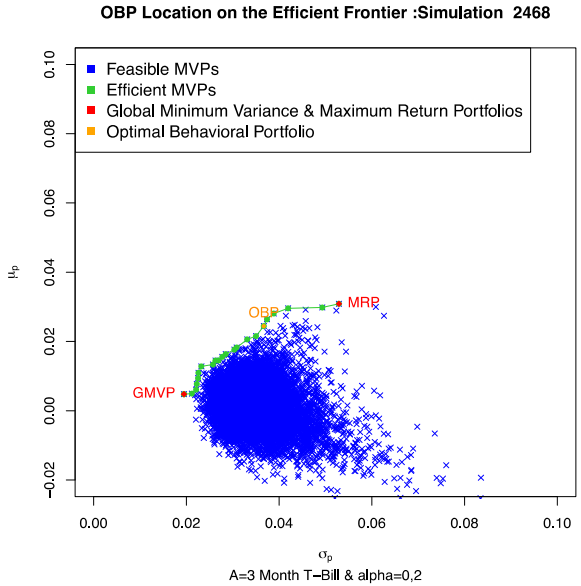


Appendix 5.2 (continued)

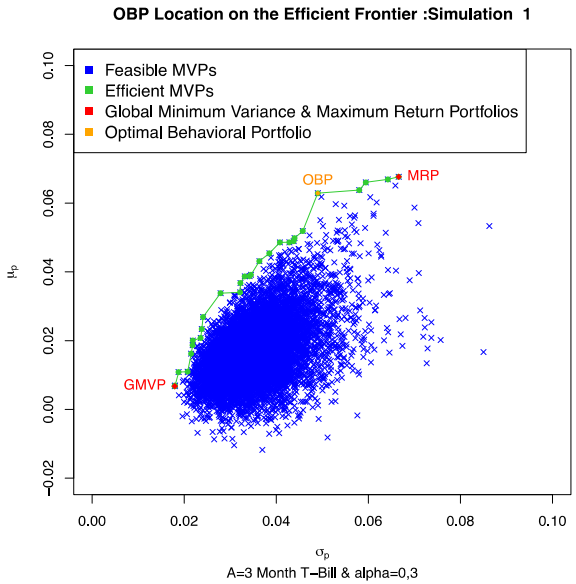


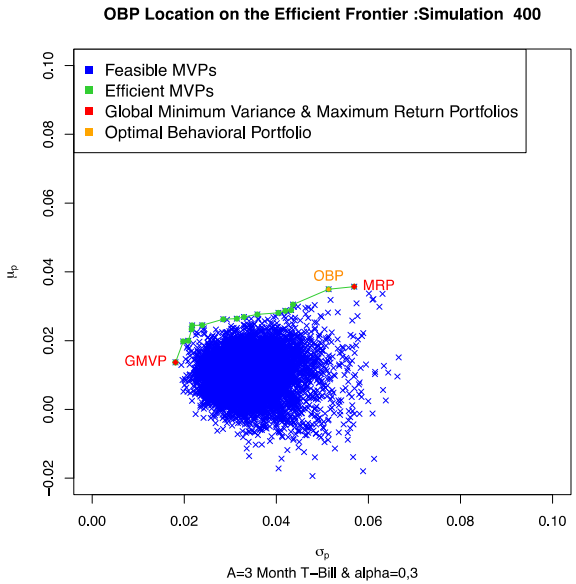


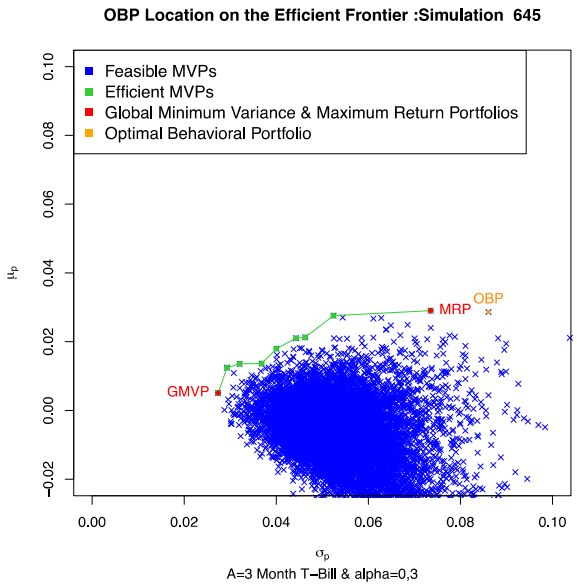


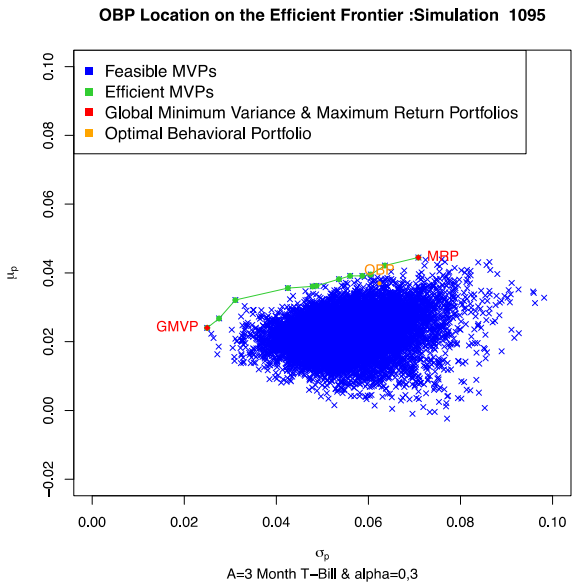


5.3. Appendix 5.3: A = Short-term risk free rate and $\alpha = 0,3$

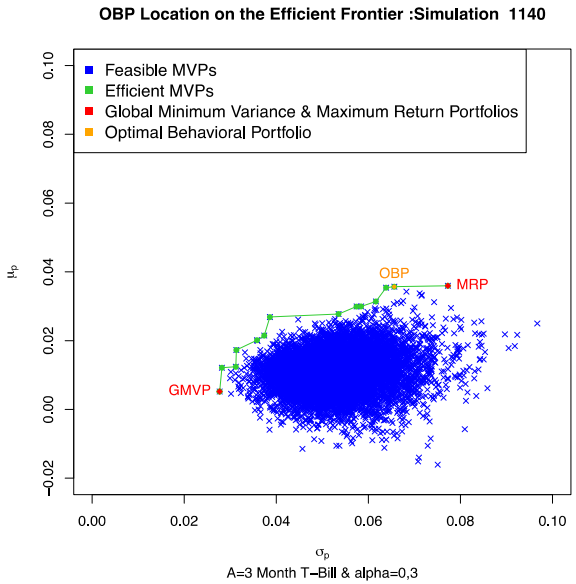


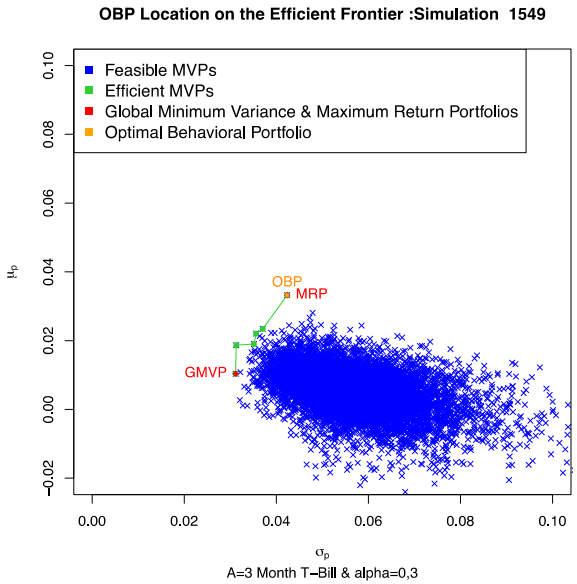


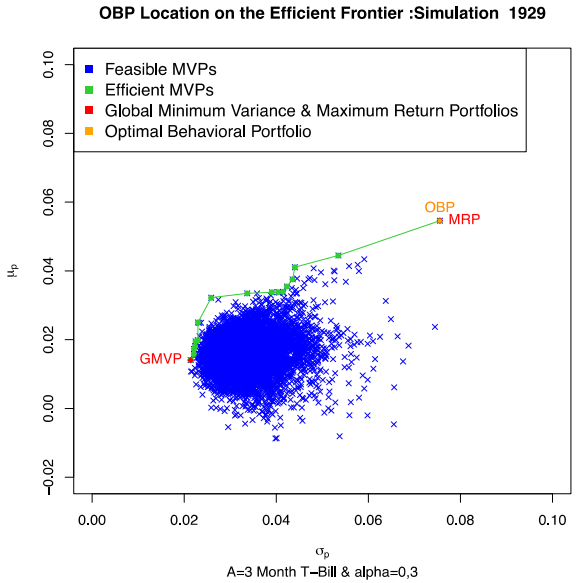


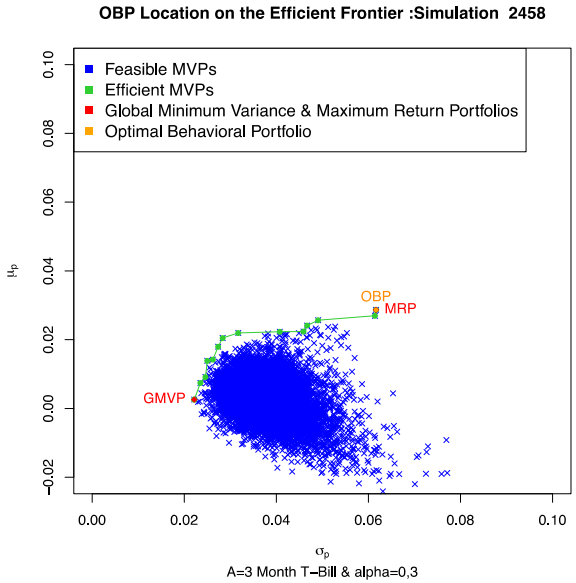


Appendix 5.3 (continued)

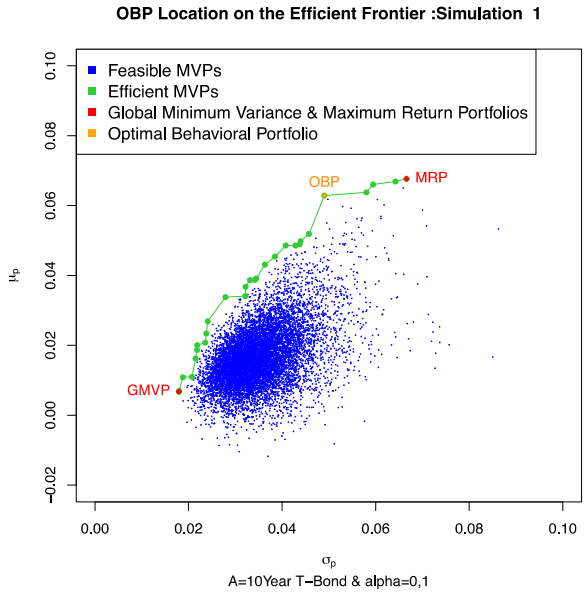


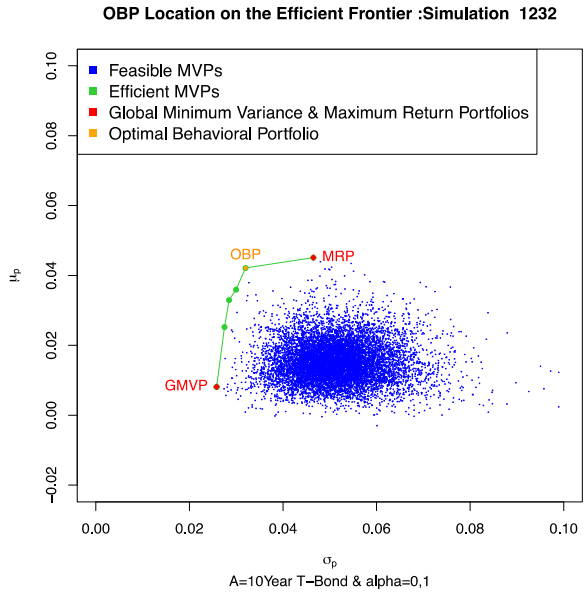


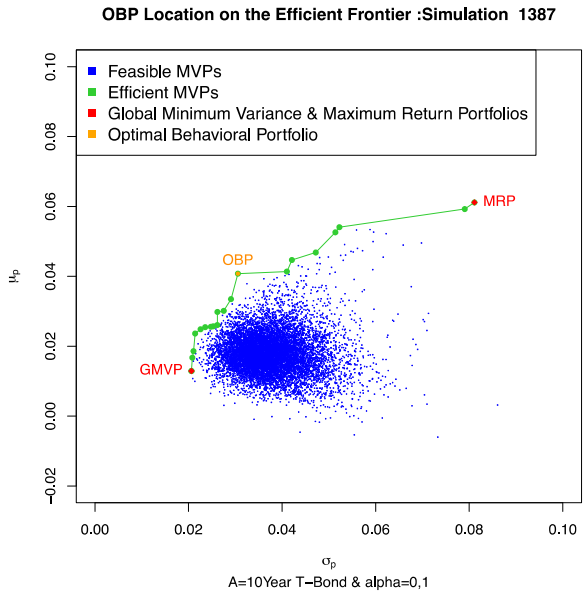


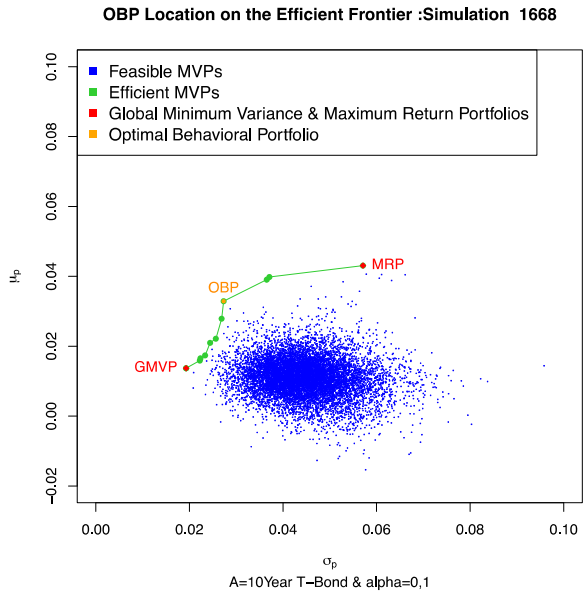


5.4. Appendix 5.4: A = Long-term risk free rate and $\alpha = 0,1$

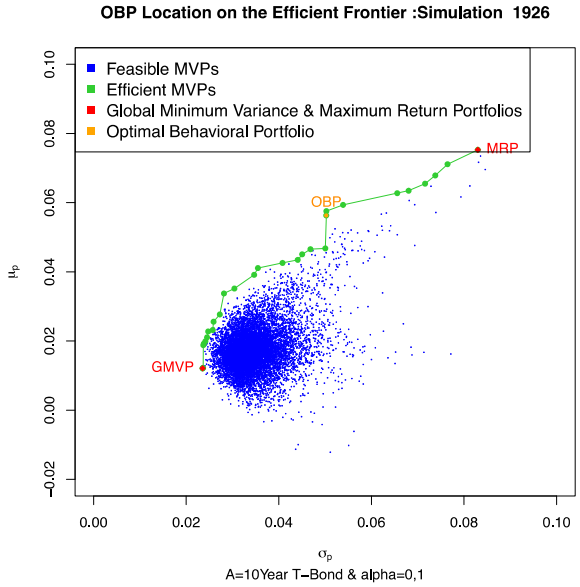


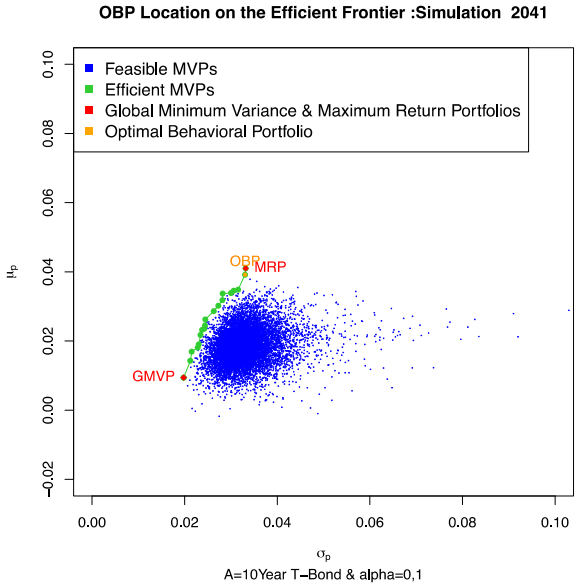


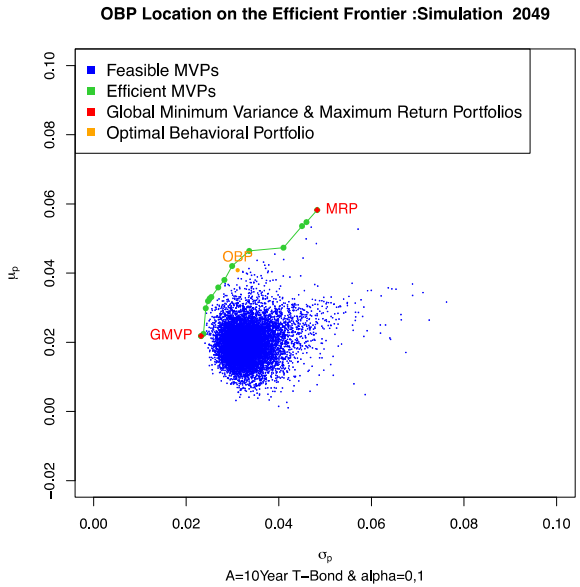


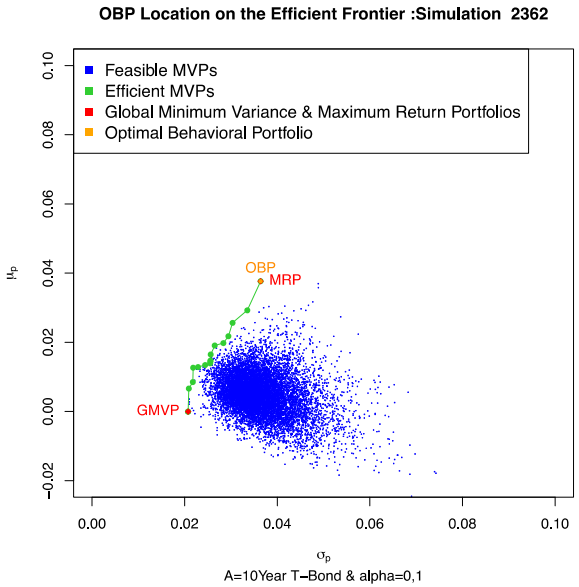


Appendix 5.4 (continued)

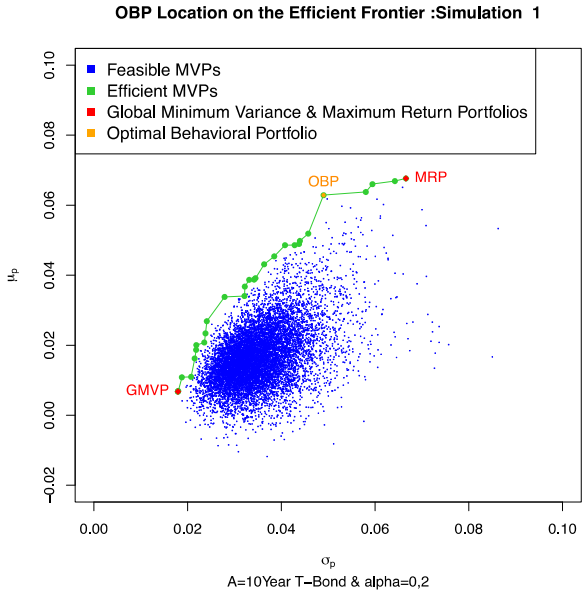


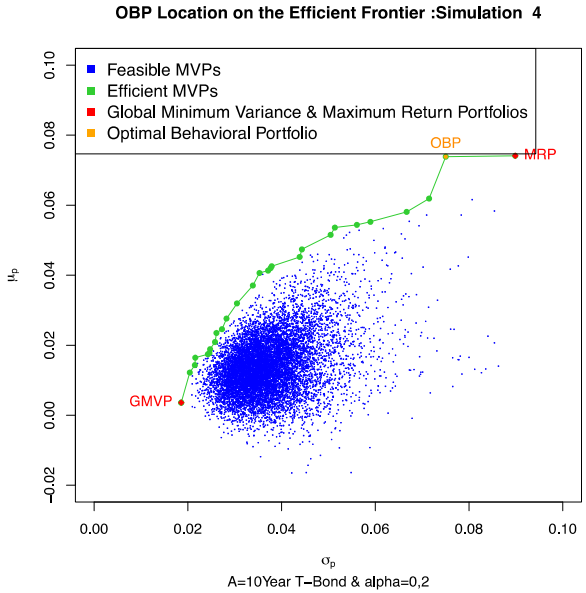


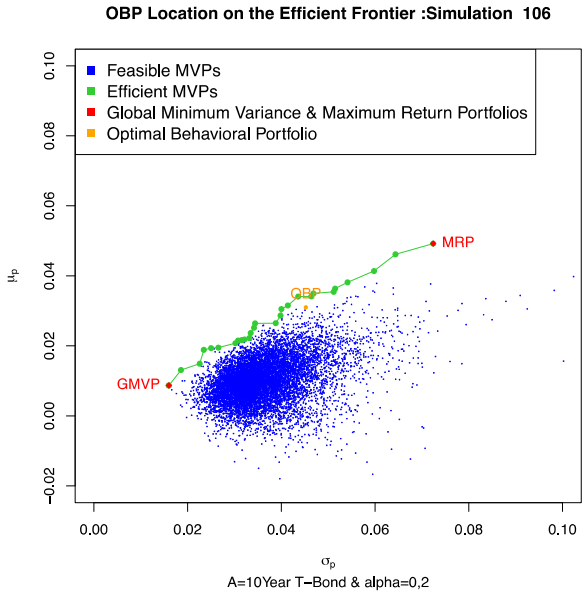


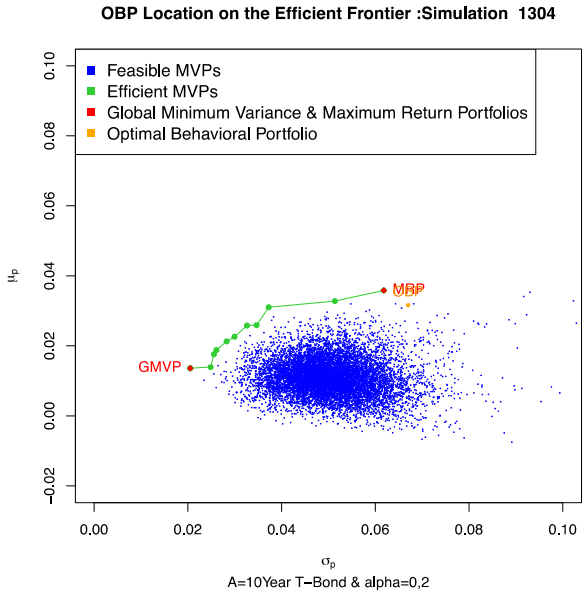


5.5. Appendix 5.5: A = Long-term risk free rate and α = 0,2

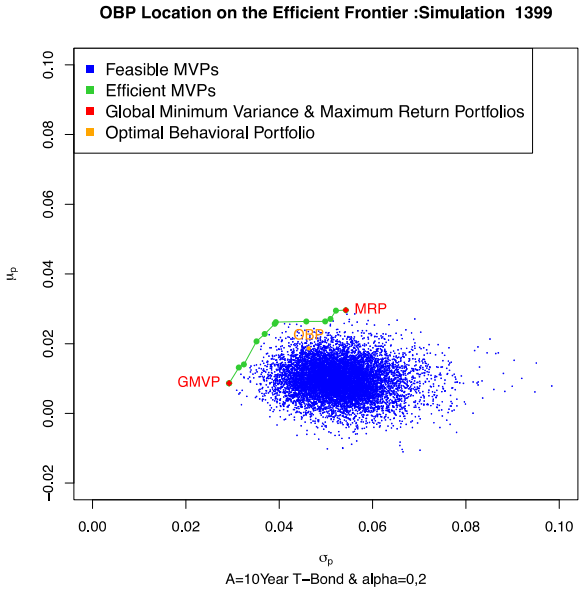


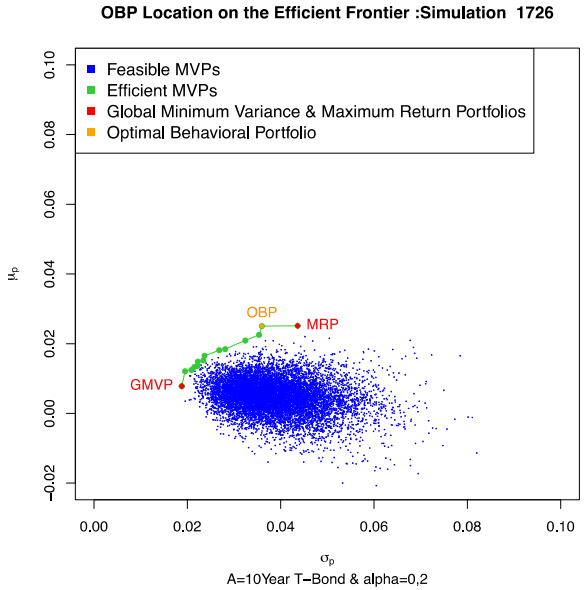


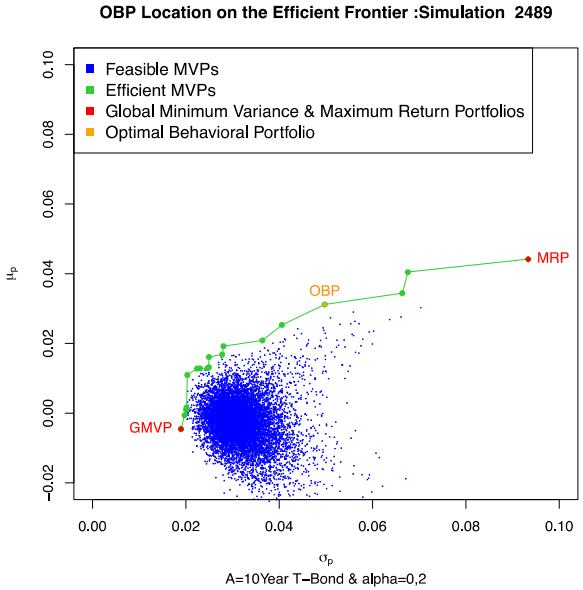


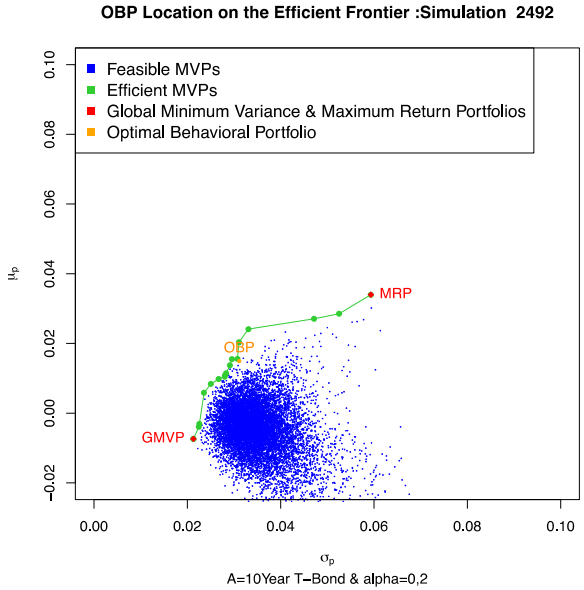


Appendix 5.5 (continued)

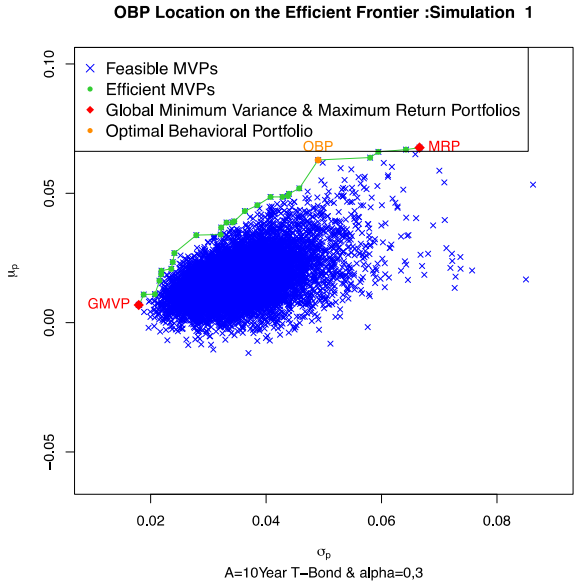


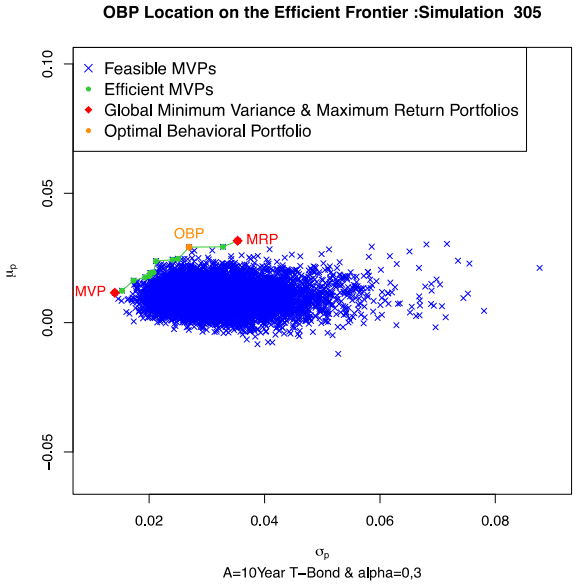


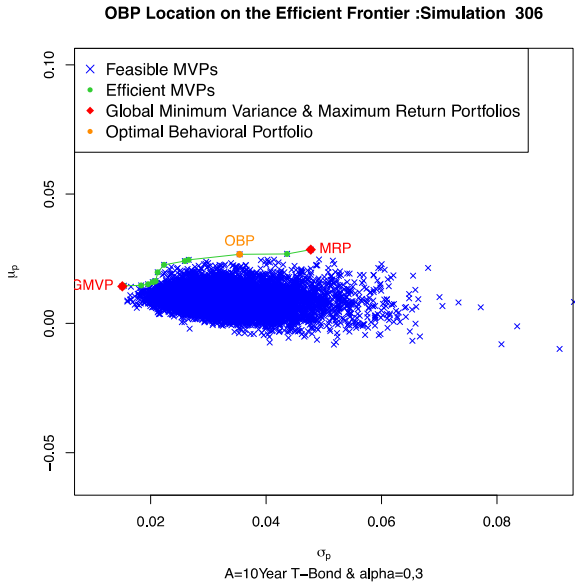


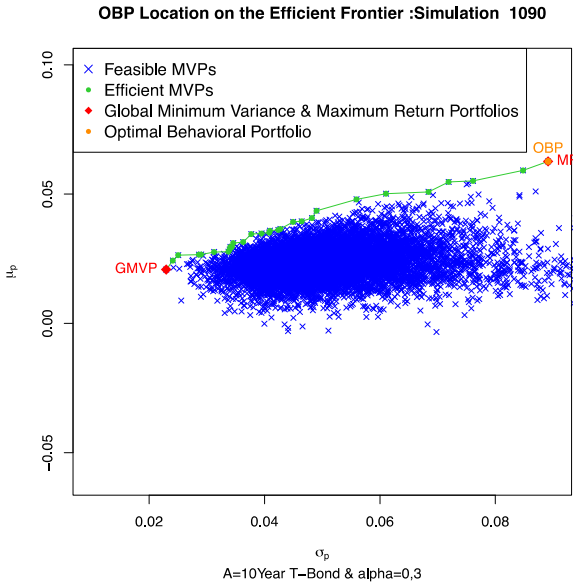


5.6. Appendix 5.6: A = Long-term risk free rate and α = 0,3

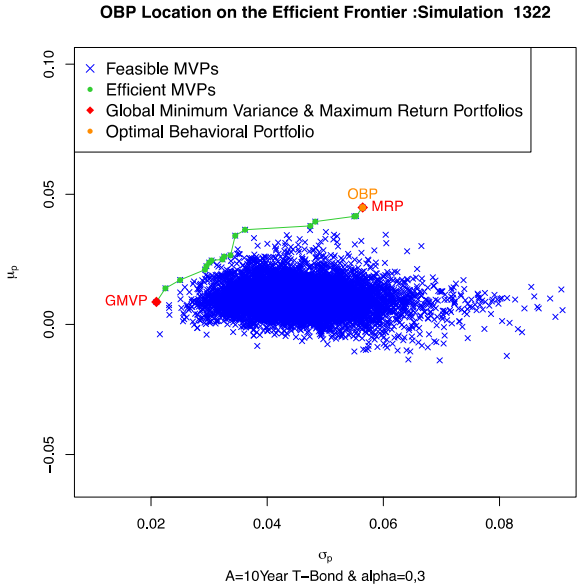


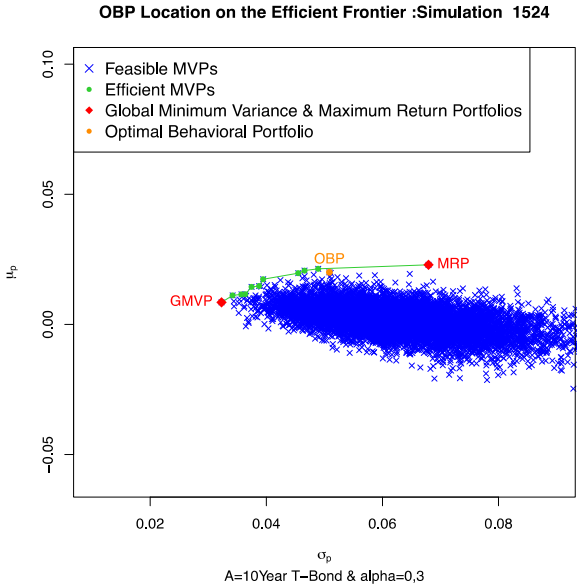


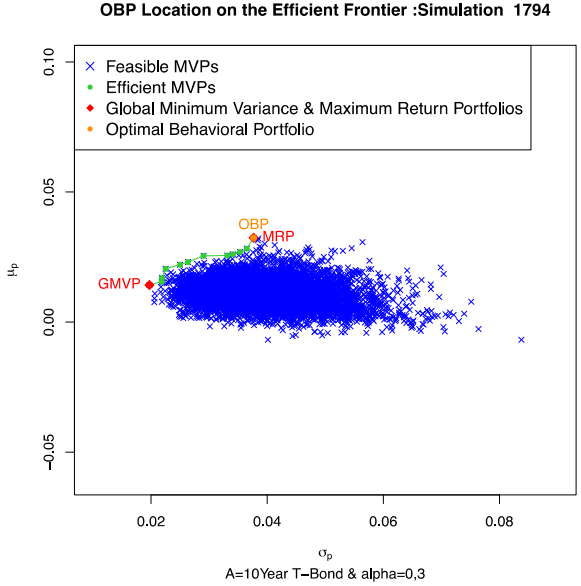


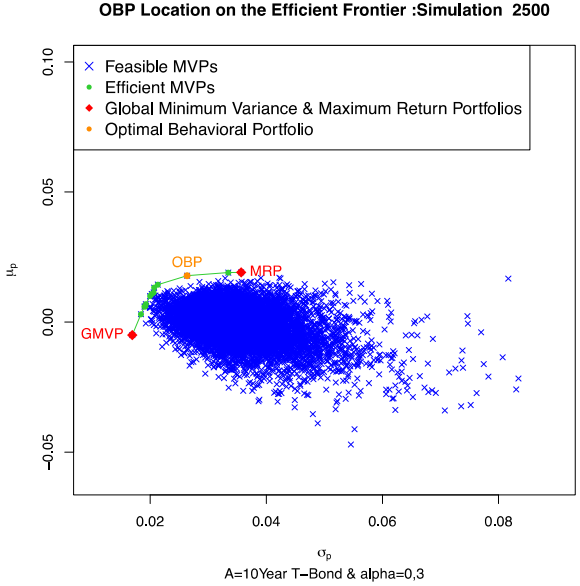


Appendix 5.6 (continued)

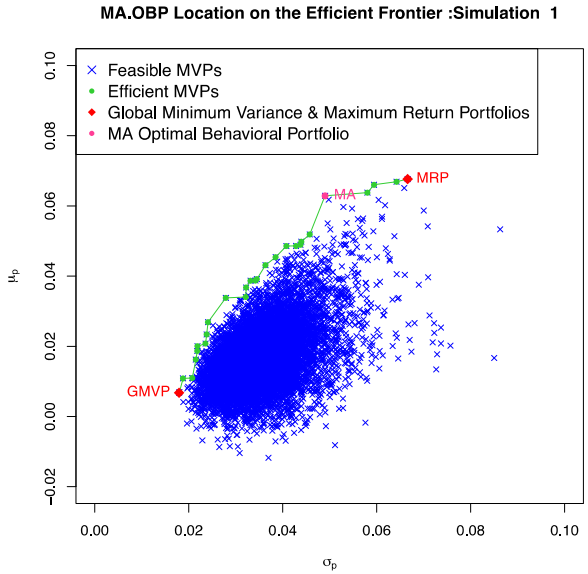


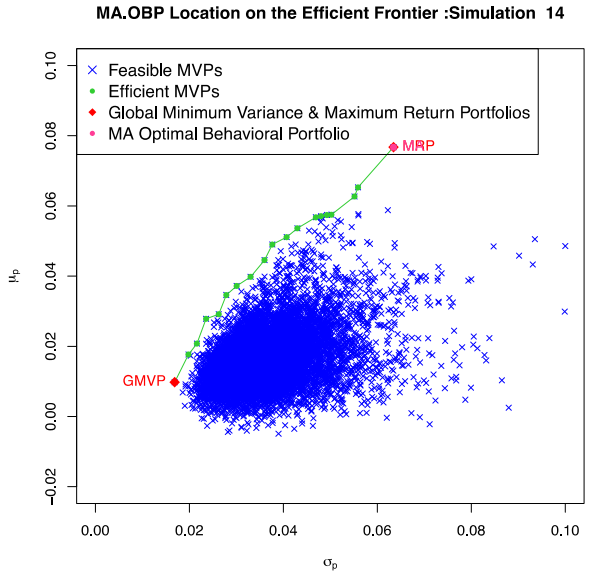


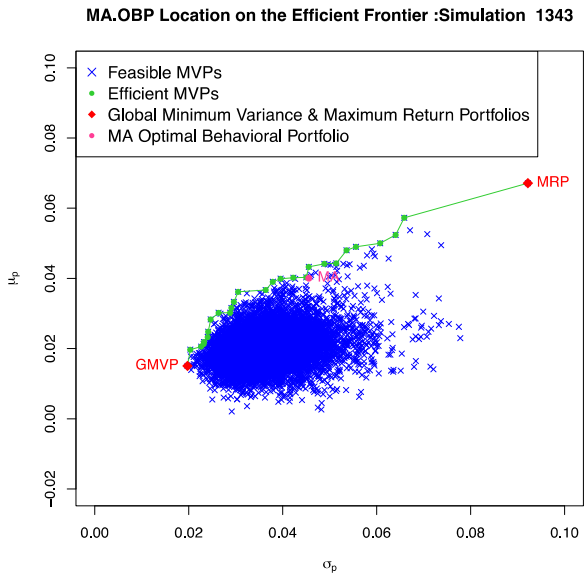


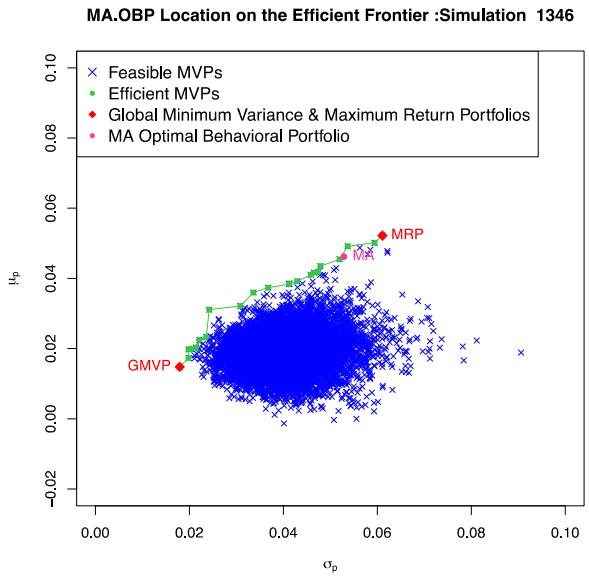


6. Appendix 6: Sample of aggregate mental accounts portfolios plots

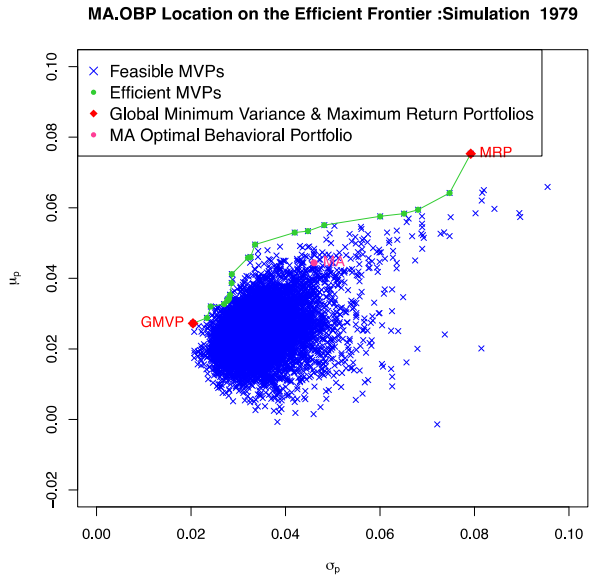


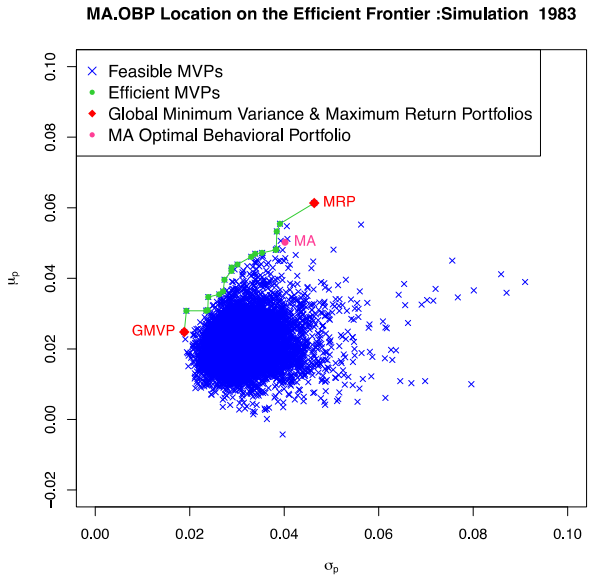


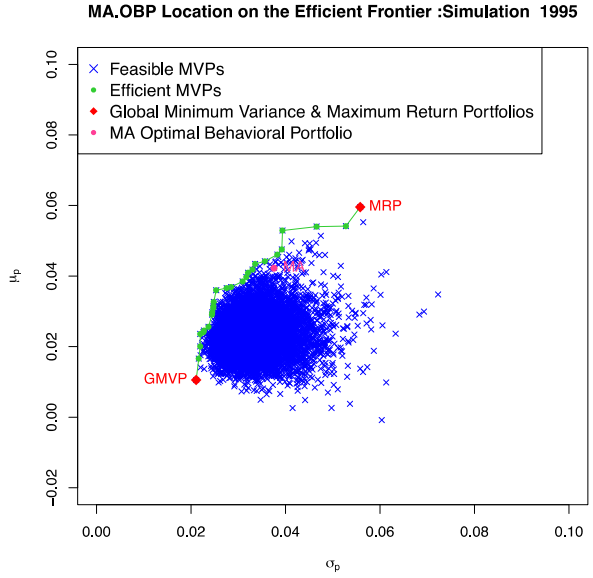


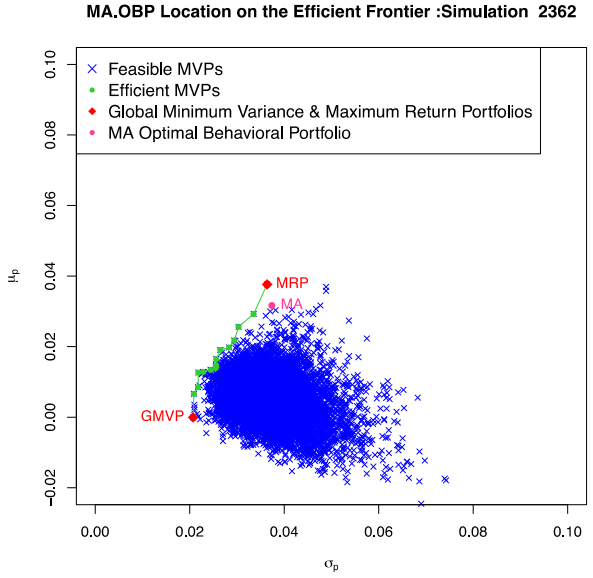


Appendix 6 (continued)



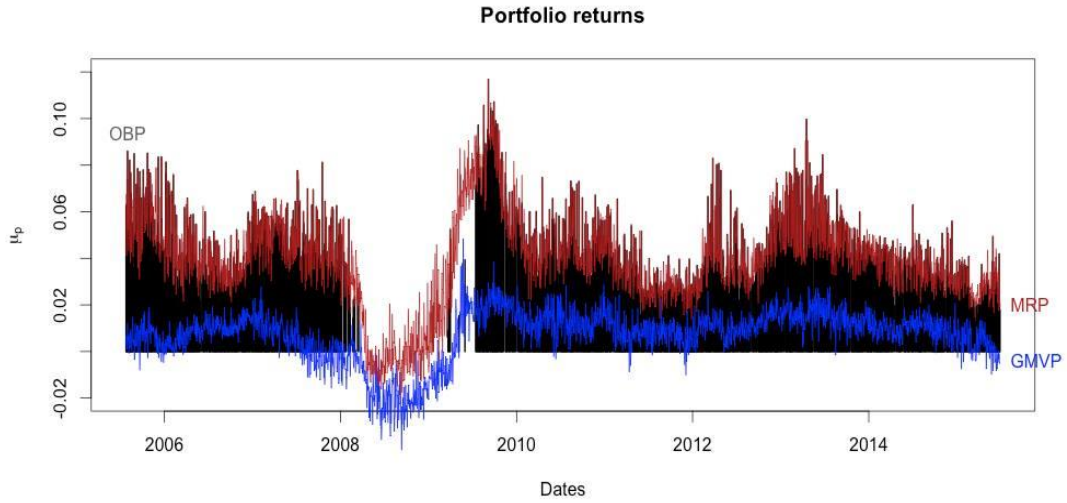




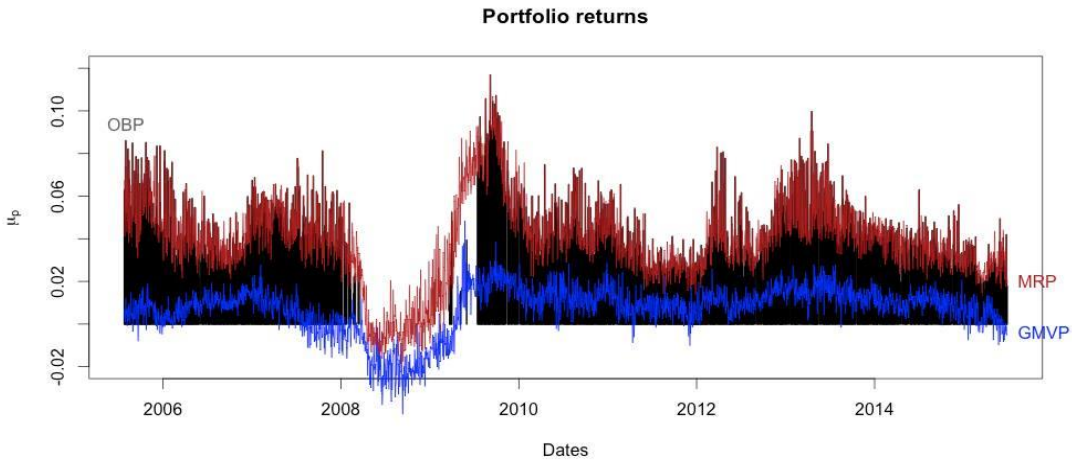


7. Appendix 7: Unsmoothed graph versions

7.1. Appendix 7.1: Standard deviation of the OBP compared to GMVP and MRP (unsmoothed)



7.2. Appendix 7.2: Return of the OBP compared to GMVP and MRP (unsmoothed)



8. Appendix 8: 'R' code

```
##### 1. Mean-Variance Portfolio #####

install.packages("gdata") #allows us to import excell files into R
library(gdata)

#### 1.1. Importation of the historical price database (Bloomberg) ####

data.price = read.xls("~/Desktop/DATABASE HENRI MARIE_O MASTER
THESIS.xlsm",sheet=1)
View(data.price)

Mat.p = as.matrix(data.price, rownames.force= NA) # transform the
data.frame into a matrix
Mat.price=t(Mat.p[,-1])
k=Mat.p[,1]
colnames(Mat.price)=k
M=rownames(Mat.price)
View(Mat.price)

#### 1.2. Importation of the Return spreadsheet and creation of the
corresponding matrix ####

data.return= read.xls("~/Desktop/DATABASE HENRI MARIE_O MASTER
THESIS.xlsm",sheet=3)
Mat.r=as.matrix(data.return, rowname.force=TRUE)
Mat.return= Mat.r[,-1]
l=Mat.r[,1]
View(Mat.return) #matrice of daily computed monthly log returns
rownames(Mat.return)=l

#### 1.3. Creation of 80 stock portfolios for each date starting at 2015-
02-01 ####

Mat.return.t <- matrix(t(Mat.return), nrow = 387, ncol = 2749)
colnames(Mat.return.t)=l
rownames(Mat.return.t)=t(M)
R.mat <- lapply(1:2500, function(i){
  Mat.return.t[sample(nrow(Mat.return.t),size=80,replace=FALSE),][,c(i:(i +
249))] #list of all matrix R* (250 monthly returns for each t : (250:2749)
and 80 ,â† stock for each simulations)
})
A1=as.data.frame(t((R.mat[[1]]))) #example R* simulation 1
View(A1)

#### 1.4. Creation of the states of nature matrix ####

Data.SON <- lapply(1:2500, function(i){

t(as.data.frame(R.mat[[i]]))[sample(nrow(t(as.data.frame(R.mat[[i]]))),size
=1000,replace=TRUE),]
})  #list of all states of nature'
B1=as.data.frame(Data.SON[[1]]) #example
View(B1)

#### 1.5. Possible weights combinations MVP ####

install.packages("partitions")
library(partitions)
```

```
P(80) #possible integer decomposition of n=80
Integer.part=t(parts(80))

W1=as.data.frame(Integer.part[sample(nrow(Integer.part),size=10000,replace=
FALSE),]) #100K possible Integer decompositions
View(W1)
W2=t(apply(W1, 1, sample)) #randomly reorder the values in each vector
View(W2)
W=as.data.frame(W2)/80 # weights to use in each situations

Weights=replicate(length(Data.SON), W, simplify=FALSE)
for(i in 1:2500){colnames(Weights[[i]])=t(colnames(Data.SON[[i]]))}
#Respective Weights matrix for each simulation
View(Weights[[69]]) #test

average=sum(rowSums(W!=0))/10000 #Average number of stocks in a portfolio
median=median(rowSums(W!=0)) #Median of the number of stocks in a portfolio
max=max(rowSums(W!=0)) #Max amount stock
min=min(rowSums(W!=0)) #Min amount stock
HI=sum(W*W)/10000 #Mean Herfindahl index
HI

#### 1.6. MV Portfolios ####

### 1.6.1 Expected returns of each stock ###

ER=replicate(length(Data.SON),matrix(nrow=1,ncol=80),simplify=FALSE)
ER=lapply(1:2500,function(j){as.data.frame(lapply(1:80,function(i){mean(as.
numeric(Data.SON[[j]][,i]))}))})
for(i in 1:2500){colnames(ER[[i]])=t(colnames(Data.SON[[i]]))}
View(ER[[675]]) #example
View(ER)

### 1.6.2. Apply stock weights to corresponding returns ###

W.ret=lapply(1:2500,function(i){data.frame((mapply(`*`,Weights[[i]],t(as.nu
meric(ER[[i]])),SIMPLIFY=FALSE)))})
View(W.ret[[1]]) #test --> verified manually :  ok

### 1.6.3. Return of the MV efficient portfolios ###

Returns.P=lapply(1:2500,function(j){rowSums(W.ret[[j]],dims=1)})
View(Returns.P[[67]]) #test --> to verify

### 1.6.4. Variance/covariance matrix ###

library(stats)
VAR=replicate(length(Data.SON),matrix(nrow=1,ncol=80),simplify=FALSE)
VAR=lapply(1:2500,function(i){var(Data.SON[[i]])}) #variance-covariance
matrix
View(VAR[[1]]) #test
VAR.STOCK= lapply(1:2500,function(i){diag(VAR[[i]])}) #variance of each
stock
STD.STOCK= lapply(1:2500,function(i){sqrt(VAR.STOCK[[i]])}) #standard
deviation of each stock
View(VAR.STOCK[[67]]) #test --> verified
View(STD.STOCK[[67]]) #test --> verified

### 1.6.5 Variance of the MV efficient portfolios ###
```

```
VAR.PORT=lapply(1:2500,function(j){diag(as.matrix(W)%*%as.matrix(VAR[[j]])%
*%t(as.matrix(W)))})
View(VAR.PORT[[1]]) #test --> verified
STD.P=lapply(1:2500,function(i){sqrt(as.data.frame(VAR.PORT[[i]]))})
for(i in 1:2500){STD.P[[i]]=as.data.frame(STD.P[[i]])
                            colnames(STD.P[[i]])='std'}
View(STD.P[[5]]) # test verified

### 1.6.6. Mean-Variance Portfolios ###

Portfolios=lapply(1:2500,function(i){as.data.frame(cbind(as.data.frame(STD.
P[[i]]),Returns.P[[i]]))}) # all MV portfolios with respective returns and
std
for (i in 1:2500){colnames(Portfolios[[i]])=c('std','Returns')
rownames(Portfolios[[i]])=NULL}
MVPortfolios=lapply(1:2500,function(i){Portfolios[[i]][order(Portfolios[[i]
]$std),]})  ## feasible Portfolios sorted by std (lowest to highest)
View(Portfolios[[1]]) # test --> verified ok

### 1.6.7. Global minimum variance portfolio ###

GMVP=lapply(1:2500,
function(i){as.data.frame(MVPortfolios[[i]])[which.min(as.data.frame(MVPort
folios[[i]])$std),]})

### 1.6.8. Max return Portfolio ###

MRP=lapply(1:2500,
function(i){as.data.frame(MVPortfolios[[i]])[which.max(as.data.frame(MVPort
folios[[i]])$Returns),]})

#### 1.7. Efficent frontier ####

### 1.7.1. Portfolio belonging to the efficient frontier ###

S.ef <- lapply( 1:2500,function(i){MVPortfolios[[i]][c(1,
which(MVPortfolios[[i]]$Returns == cummax(MVPortfolios[[i]]$Returns))),]})
# efficient portfolios among feasible portfolios
View(S.ef[[6]])

### 1.7.2. Plor of the Efficient frontier ###

Title= paste("Efficient frontier Simulation ",1:2500,sep="")
PDFPath = "~/Desktop/MVP1.plots.pdf"
pdf(file=PDFPath)
for (i in
1:2500){plot(MVPortfolios[[i]]$std,MVPortfolios[[i]]$Returns,main=Title[i],
type="p",ylim=c(-
0.02,0.1),xlim=c(0,0.1),pch=4,col="blue",ylab=expression(mu[p]),xlab=expres
sion(sigma[p]), cex=0.8)
points( S.ef[[i]],col="limegreen",pch=20,type="o",cex=1.4)
points(GMVP[[i]]$std,GMVP[[i]]$Returns,pch=18,col="red",cex=1.6)
text(GMVP[[i]],labels="GMVP", col="firebrick1", pos=2)
points(MRP[[i]]$std,MRP[[i]]$Returns,pch=18,col="red",cex=1.6)
text(MRP[[i]],labels="MRP", col="firebrick1", pos=4)
legend("topleft",legend=c("Feasible MVPs","Efficient MVPs ", "Global
Minimum Variance & Maximum Return Portfolios"),
col=c("blue","limegreen","firebrick1"),pch=c(4,20,18),bty="o",cex=1.2)
}
dev.off()
```

```
##### 2. Behavioral Portfolios #####

#### 2.1. Weighting function of behavioral portfolios ####

data.pi=read.xls("~/Desktop/DATABASE HENRI MARIE_O MASTER
THESIS.xlsm",sheet=4) # weighting function of behavioral portfolios
data.pi=read.xls("~/Desktop/Vecteur-Pi+.xlsm",sheet=1)
Weights.pi=as.matrix(data.pi,rownames.force=NA)
W.pi=as.numeric(Weights.pi[,-1])
View(W.pi)

#### 2.2. Safety first constraint ####

dates=lapply(1:2500,function(j){rownames(Data.SON[[j]])}) #store dates of
matrix Theta (Data.SON)
stocks=lapply(1:2500,function(j){colnames(Data.SON[[j]])})#store stocks
name of matrix Theta (Data.SON)
no.label=lapply(1:2500,function(i){matrix(data = as.numeric(Data.SON[[i]]),
nrow = 1000, ncol = 80, byrow = FALSE,
dimnames = NULL)}) # only keep numerical values

Theta=lapply(1:2500,function(i){data.frame(no.label[[i]], row.names = NULL,
check.rows = FALSE,
check.names = TRUE,  stringsAsFactors = FALSE)}) #create numerical
data.frame of Theta
View(Theta[[1]]) #test -> verified ok

### 2.2.1. Decomposition of all feasible portfolio in each state of nature
###

WBP=lapply(1:2500,function(j){as.data.frame(tcrossprod(as.matrix(W),as.matr
ix(Theta[[j]])))})
for(i in 1:2500){colnames(WBP[[i]])=dates[[i]]}
View(WBP[[1]]) #test -->verified

### 2.2.2. Aspiration Levels ###

## 2.2.2.1. A =rf.ST ##

r.st=read.xls("~/Desktop/DATABASE HENRI MARIE_O MASTER
THESIS.xlsm",sheet=5)
r.st=read.xls("~/Desktop/STBILL.xlsm",sheet=1)
r.st= as.data.frame(r.st)
r.st.sim <-lapply(1:2500,function(i){r.st[c(i:(i + 249)),c(1:2)]}) # create
set of 250 dates
View(t(r.st.sim[[1]]))

BF.port.st=lapply(1:2500,function(i){merge(x=r.st.sim[[i]], y=t(WBP[[i]]),
by.x="DATE", by.y="row.names")}) # match feasible portfolio and
corresponding aspiration level    ##20 min
BP.st=lapply(1:2500,function(i){as.data.frame(lapply(3:10002,function(j){as
.data.frame(sum(ifelse(BF.port.st[[i]]$ST.T.Bill.Rate<=BF.port.st[[i]][,j],
1,0)))}))}) # check which portfolios are higher than A ## (36-46) min
for(i in 1:2500){BP.st[[i]]=t(BP.st[[i]])
rownames(BP.st[[i]])=c(1:10000)}
View(BP.st[[1]])

  # Alpha = 0,1
```

```
      # Safety first behavioral portfolio A=st.bill alpha=0,1 #

SFB.alpha1.st=lapply(1:2500,function(i){as.data.frame(BP.st[[i]][BP.st[[i]]
>= 900,rownames=TRUE])}) #S*
for(i in 1:2500 ){colnames(SFB.alpha1.st[[i]])=i}
SFB.alpha1.st=Filter(function(x) dim(x)[1]>0,SFB.alpha1.st)
View(SFB.alpha1.st[[1]])
Statistic.eligible.simulation.a1st=Filter(function(x)
dim(x)[1]>0,SFB.alpha1.st)

S.star.Y=lapply(1:length(SFB.alpha1.st),function(i){as.data.frame(t(WBP[[i]
][as.numeric(rownames(SFB.alpha1.st[[i]])),]))})  #  Y matrices
corresponding to S* portfolios
View(S.star.Y[[1]])

Y.sorted=lapply(1:length(SFB.alpha1.st),function(i){lapply(1:nrow(SFB.alpha
1.st[[i]]),function(j){as.data.frame(sort(S.star.Y[[i]][,j],decreasing=FALS
E))})}) #Y* matrices sorted from lowest outcomes to highest
for(i in
1:length(SFB.alpha1.st)){Y.sorted[[i]]=as.data.frame(Y.sorted[[i]])
colnames(Y.sorted[[i]])=rownames(SFB.alpha1.st[[i]])}

      # Optimal behavioral portfolios #

OBP=lapply(1:length(SFB.alpha1.st),function(i){as.data.frame(lapply(1:nrow(
SFB.alpha1.st[[i]]),function(j){as.data.frame(t(as.matrix(Y.sorted[[i]][[j]
]))%*%as.matrix(as.numeric(W.pi)))}))})
for(i in
1:length(SFB.alpha1.st)){colnames(OBP[[i]])=rownames(SFB.alpha1.st[[i]])
rownames(OBP[[i]])=colnames(SFB.alpha1.st[[i]])}
View(OBP[[1]])

      # Optimum Behavioral portfolio #

Max.OBP.a1st= lapply(1:length(SFB.alpha1.st),function(i){t(apply(OBP[[i]],
1, function(x) as.numeric(c(names(which.max(x)), max(x)))))})
for(i in
1:length(Statistic.eligible.simulation.a1st)){rownames(Max.OBP.a1st[[i]])=c
olnames(Statistic.eligible.simulation.a1st[[i]])}
Points.OBP.a1st=do.call(rbind.data.frame, Max.OBP.a1st)

# Optimal Behavioral portfolio belonging to the efficient frontier #

OBP.Sef.a1st=lapply(1:length(Statistic.eligible.simulation.a1st),function(i
){as.data.frame(merge(Max.OBP.a1st[[i]],S.ef[[as.numeric(colnames(Statistic
.eligible.simulation.a1st[[i]]))]],by.x="V1",by.y="row.names"))})
Match.OBP.Sef.a1st=(as.numeric(length(Filter(function(x)
dim(x)[1]>0,OBP.Sef.a1st)))/as.numeric(length(Max.OBP.a1st)))


# Plot

Title= paste("OBP Location on the Efficient Frontier :Simulation
",as.numeric(rownames(Points.OBP.a1st)),sep="")
PDFPath = "~/Desktop/plots.a1.st.pdf"
pdf(file=PDFPath)
for(i in
1:nrow(Points.OBP.a1st)){plot(MVPortfolios[[as.numeric(rownames(Max.OBP.a1s
t[[i]]))]]$std,MVPortfolios[[as.numeric(rownames(Max.OBP.a1st[[i]]))]]$Retu
rns,main=Title[i],sub="A=3 Month T-Bill & alpha=0,1",  type="p",ylim=c(-
```

```
0.02,0.1),xlim=c(0,0.1),pch=4,col="blue",ylab=expression(mu[p]),xlab=expres
sion(sigma[p]), cex=0.8)
points(
S.ef[[as.numeric(rownames(Max.OBP.a1st[[i]]))]],col="limegreen",pch=20,type
="o",cex=1.4)
points(GMVP[[as.numeric(rownames(Max.OBP.a1st[[i]]))]]$std,GMVP[[as.numeric
(rownames(Max.OBP.a1st[[i]]))]]$Returns,pch=18,col="red",cex=1.6)
text(GMVP[[as.numeric(rownames(Max.OBP.a1st[[i]]))]],labels="GMVP",
col="red", pos=2)
points(MRP[[as.numeric(rownames(Max.OBP.a1st[[i]]))]]$std,MRP[[as.numeric(r
ownames(Max.OBP.a1st[[i]]))]]$Returns,pch=18,col="red",cex=1.6)
text(MRP[[as.numeric(rownames(Max.OBP.a1st[[i]]))]],labels="MRP",
col="red", pos=4)
points(Portfolios[[as.numeric(rownames(Max.OBP.a1st[[i]]))]][as.numeric(Max
.OBP.a1st[[i]][,1]),],col="orange",pch=20,cex=1.6)
text(Portfolios[[as.numeric(rownames(Max.OBP.a1st[[i]]))]][as.numeric(Max.O
BP.a1st[[i]][,1]),], labels="OBP",col='darkorange',pos=3)
legend("topleft",legend=c("Feasible MVPs","Efficient MVPs ", "Global
Minimum Variance & Maximum Return Portfolios","Optimal Behavioral
Portfolio"),
col=c("blue","limegreen","red","orange"),pch=c(4,20,18,20),bty="o",cex=1.2)
}
dev.off()

  #Alpha = 0,2

      # Safety first behavioral portfolio A=st.bill alpha=0,2 #

SFB.alpha1.st=lapply(1:2500,function(i){as.data.frame(BP.st[[i]][BP.st[[i]]
>= 800,rownames=TRUE])}) #S*
for(i in 1:2500 ){colnames(SFB.alpha1.st[[i]])=i}
SFB.alpha1.st=Filter(function(x) dim(x)[1]>0,SFB.alpha1.st)
Statistic.eligible.simulation.a2st=Filter(function(x)
dim(x)[1]>0,SFB.alpha1.st)
View(Statistic.eligible.simulation.a2st)

S.star.Y=lapply(1:length(SFB.alpha1.st),function(i){as.data.frame(t(WBP[[i]
][as.numeric(rownames(SFB.alpha1.st[[i]])),]))})  #  Y matrices
corresponding to S* portfolios
View(S.star.Y[[1]])

Y.sorted=lapply(1:length(SFB.alpha1.st),function(i){lapply(1:nrow(SFB.alpha
1.st[[i]]),function(j){as.data.frame(sort(S.star.Y[[i]][,j],decreasing=FALS
E))})}) #Y* matrices sorted from lowest outcomes to highest
for(i in
1:length(SFB.alpha1.st)){Y.sorted[[i]]=as.data.frame(Y.sorted[[i]])
colnames(Y.sorted[[i]])=rownames(SFB.alpha1.st[[i]])}
View(Y.sorted[[1]])

      # Optimal behavioral portfolios #

OBP=lapply(1:length(SFB.alpha1.st),function(i){as.data.frame(lapply(1:nrow(
SFB.alpha1.st[[i]]),function(j){as.data.frame(t(as.matrix(Y.sorted[[i]][[j]
]))%*%as.matrix(as.numeric(W.pi)))}))})
for(i in
1:length(SFB.alpha1.st)){colnames(OBP[[i]])=rownames(SFB.alpha1.st[[i]])
rownames(OBP[[i]])=colnames(SFB.alpha1.st[[i]])}
View(OBP[[1]])

      # Optimum Behavioral portfolio #
```

```
Max.OBP.a2st=
lapply(1:length(Statistic.eligible.simulation.a2st),function(i){t(apply(OBP
[[i]], 1, function(x) as.numeric(c(names(which.max(x)), max(x))))[])})
for(i in
1:length(Statistic.eligible.simulation.a2st)){rownames(Max.OBP.a2st[[i]])=c
olnames(Statistic.eligible.simulation.a2st[[i]])}
View(Max.OBP.a2st) #test --> ok
Points.OBP.a2st=do.call(rbind.data.frame, Max.OBP.a2st)

      # Optimal Behavioral portfolio belonging to the efficient frontier #

OBP.Sef.a2st=lapply(1:length(Statistic.eligible.simulation.a2st),function(i
){as.data.frame(merge(Max.OBP.a2st[[i]],S.ef[[as.numeric(colnames(Statistic
.eligible.simulation.a2st[[i]]))]],by.x="V1",by.y="row.names"))})
Match.OBP.Sef.a2st=(as.numeric(length(Filter(function(x)
dim(x)[1]>0,OBP.Sef.a2st)))/as.numeric(length(Max.OBP.a2st)))

      # Plot

Title= paste("OBP Location on the Efficient Frontier :Simulation
",as.numeric(rownames(Points.OBP.a2st)),sep="")
PDFPath = "~/Desktop/plots.a2.st.pdf"
pdf(file=PDFPath)
for(i in
1:nrow(Points.OBP.a2st)){plot(MVPortfolios[[as.numeric(rownames(Max.OBP.a2s
t[[i]]))]]$std,MVPortfolios[[as.numeric(rownames(Max.OBP.a2st[[i]]))]]$Retu
rns,main=Title[i],sub="A=3 Month T-Bill & alpha=0,2",  type="p",ylim=c(-
0.02,0.1),xlim=c(0,0.1),pch=4,col="blue",ylab=expression(mu[p]),xlab=expres
sion(sigma[p]), cex=0.8)
points(S.ef[[as.numeric(rownames(Max.OBP.a2st[[i]]))]],col="limegreen",pch=
20,type='o',cex=1.4)
points(GMVP[[as.numeric(rownames(Max.OBP.a2st[[i]]))]]$std,GMVP[[as.numeric
(rownames(Max.OBP.a2st[[i]]))]]$Returns,pch=18,col="red")
text(GMVP[[as.numeric(rownames(Max.OBP.a2st[[i]]))]],labels="GMVP",
col="red", pos=2)
points(MRP[[as.numeric(rownames(Max.OBP.a2st[[i]]))]]$std,MRP[[as.numeric(r
ownames(Max.OBP.a2st[[i]]))]]$Returns,pch=18,col="red")
text(MRP[[as.numeric(rownames(Max.OBP.a2st[[i]]))]],labels="MRP",
col="red", pos=4)
points(Portfolios[[as.numeric(rownames(Max.OBP.a2st[[i]]))]][as.numeric(Max
.OBP.a2st[[i]][,1]),],pch=20,col="orange",)
text(Portfolios[[as.numeric(rownames(Max.OBP.a2st[[i]]))]][as.numeric(Max.O
BP.a2st[[i]][,1]),], labels="OBP",col='darkorange',pos=3)
legend("topleft",legend=c("Feasible MVPs","Efficient MVPs ", "Global
Minimum Variance & Maximum Return Portfolios","Optimal Behavioral
Portfolio"),
col=c("blue","limegreen","red","orange"),pch=15,bty="o",cex=1.2)
}
dev.off()

  #Alpha = 0,3

      # Safety first behavioral portfolio A=st.bill alpha=0,3 #

SFB.alpha1.st=lapply(1:2500,function(i){as.data.frame(BP.st[[i]][BP.st[[i]]
>= 700,rownames=TRUE])}) #S*
for(i in 1:2500 ){colnames(SFB.alpha1.st[[i]])=i}
SFB.alpha1.st=Filter(function(x) dim(x)[1]>0,SFB.alpha1.st)
Statistic.eligible.simulation.a3st=Filter(function(x)
dim(x)[1]>0,SFB.alpha1.st)
```

```
S.star.Y=lapply(1:length(SFB.alpha1.st),function(i){as.data.frame(t(WBP[[i]
][as.numeric(rownames(SFB.alpha1.st[[i]])),]))})  #  Y matrices
corresponding to S* portfolios
View(S.star.Y[[1]])

Y.sorted=lapply(1:length(SFB.alpha1.st),function(i){lapply(1:nrow(SFB.alpha
1.st[[i]]),function(j){as.data.frame(sort(S.star.Y[[i]][,j],decreasing=FALS
E))})}) #Y* matrices sorted from lowest outcomes to highest
for(i in
1:length(SFB.alpha1.st)){Y.sorted[[i]]=as.data.frame(Y.sorted[[i]])
colnames(Y.sorted[[i]])=rownames(SFB.alpha1.st[[i]])}

      # Optimal behavioral portfolios #

OBP=lapply(1:length(SFB.alpha1.st),function(i){as.data.frame(lapply(1:nrow(
SFB.alpha1.st[[i]]),function(j){as.data.frame(t(as.matrix(Y.sorted[[i]][[j]
]))%*%as.matrix(as.numeric(W.pi)))}))})
for(i in
1:length(SFB.alpha1.st)){colnames(OBP[[i]])=rownames(SFB.alpha1.st[[i]])
rownames(OBP[[i]])=colnames(SFB.alpha1.st[[i]])}
View(OBP[[1]])

      # Optimum Behavioral portfolio #

Max.OBP.a3st=
lapply(1:length(Statistic.eligible.simulation.a3st),function(i){t(apply(OBP
[[i]], 1, function(x) as.numeric(c(names(which.max(x)), max(x)))))})
for(i in
1:length(Statistic.eligible.simulation.a2st)){rownames(Max.OBP.a3st[[i]])=c
olnames(Statistic.eligible.simulation.a3st[[i]])}
View(Max.OBP.a3st[[20]]) #test --> ok
Points.OBP.a3st=do.call(rbind.data.frame, Max.OBP.a3st)

      # Optimal Behavioral portfolio belonging to the efficient frontier #

OBP.Sef.a3st=lapply(1:length(Statistic.eligible.simulation.a3st),function(i
){as.data.frame(merge(Max.OBP.a3st[[i]],S.ef[[as.numeric(colnames(Statistic
.eligible.simulation.a3st[[i]]))]],by.x="V1",by.y="row.names"))})
Match.OBP.Sef.a3st=(as.numeric(length(Filter(function(x)
dim(x)[1]>0,OBP.Sef.a3st)))/as.numeric(length(Max.OBP.a3st)))

      # Plot

Title= paste("OBP Location on the Efficient Frontier :Simulation
",as.numeric(rownames(Points.OBP.a3st)),sep="")
PDFPath = "~/Desktop/plots.a3.st.pdf"
pdf(file=PDFPath)
for(i in
1:nrow(Points.OBP.a3st)){plot(MVPortfolios[[as.numeric(rownames(Max.OBP.a3s
t[[i]]))]]$std,MVPortfolios[[as.numeric(rownames(Max.OBP.a3st[[i]]))]]$Retu
rns,main=Title[i],sub="A=3 Month T-Bill & alpha=0,3",  type="p",ylim=c(-
0.02,0.1),xlim=c(0,0.1),pch=4,col="blue",ylab=expression(mu[p]),xlab=expres
sion(sigma[p]), cex=0.8)
points(S.ef[[as.numeric(rownames(Max.OBP.a3st[[i]]))]],col="limegreen",pch=
20,type='o',cex=1.4)
points(GMVP[[as.numeric(rownames(Max.OBP.a3st[[i]]))]]$std,GMVP[[as.numeric
(rownames(Max.OBP.a3st[[i]]))]]$Returns,pch=18,col="red")
text(GMVP[[as.numeric(rownames(Max.OBP.a3st[[i]]))]],labels="GMVP",
col="red", pos=2)
points(MRP[[as.numeric(rownames(Max.OBP.a3st[[i]]))]]$std,MRP[[as.numeric(r
ownames(Max.OBP.a3st[[i]]))]]$Returns,pch=18,col="red")
```

```
text(MRP[[as.numeric(rownames(Max.OBP.a3st[[i]]))]],labels="MRP",
col="red", pos=4)
points(Portfolios[[as.numeric(rownames(Max.OBP.a3st[[i]]))]][as.numeric(Max
.OBP.a3st[[i]][,1]),],pch=20,col="orange",)
text(Portfolios[[as.numeric(rownames(Max.OBP.a3st[[i]]))]][as.numeric(Max.O
BP.a3st[[i]][,1]),], labels="OBP",col='darkorange',pos=3)
legend("topleft",legend=c("Feasible MVPs","Efficient MVPs ", "Global
Minimum Variance & Maximum Return Portfolios","Optimal Behavioral
Portfolio"),
col=c("blue","limegreen","red","orange"),pch=15,bty="o",cex=1.2)
}
dev.off()

## 2.2.2.2. A= rf.LT Aspiration Levels ##

r.lt=read.xls("~/Desktop/DATABASE HENRI MARIE_O MASTER
THESIS.xlsm",sheet=6)
r.lt=read.xls("~/Desktop/LTBOND.xlsm",sheet=1)
r.lt= as.data.frame(r.lt)
r.lt.sim <-lapply(1:2500,function(i){r.lt[c(i:(i + 249)),c(1:2)]})
View(r.lt.sim[[1]])

BF.port.lt=lapply(1:2500,function(i){merge(x=r.lt.sim[[i]], y=t(WBP[[i]]),
by.x="DATE", by.y="row.names")}) # match feasible portfolio and
corresponding aspiration level    ##20 min
View(BF.port.lt[[1]])
BP.lt=lapply(1:2500,function(i){as.data.frame(lapply(3:10002,function(j){as
.data.frame(sum(ifelse(BF.port.lt[[i]]$LT.T.Bond.Rate.<=BF.port.lt[[i]][,j]
,1,0)))}))}) # check which portfolios are higher than A ## (36-46) min
for(i in 1:2500){BP.lt[[i]]=t(BP.lt[[i]])
rownames(BP.lt[[i]])=c(1:10000)}
View(BP.lt[[1]])

  # Alpha = 0,1 #

      # Safety first behavioral portfolio A=lt.bond alpha=0,1 #

SFB.alpha1.lt=lapply(1:2500,function(i){as.data.frame(BP.lt[[i]][BP.lt[[i]]
>= 900,rownames=TRUE])}) #S*
for(i in 1:2500 ){colnames(SFB.alpha1.lt[[i]])=i}
SFB.alpha1.lt=Filter(function(x) dim(x)[1]>0,SFB.alpha1.lt)
Statistic.eligible.simulation.a1lt=Filter(function(x)
dim(x)[1]>0,SFB.alpha1.lt)
View(SFB.alpha1.lt[[1]])
S.star.Y=lapply(1:length(SFB.alpha1.lt),function(i){as.data.frame(t(WBP[[i]
][as.numeric(rownames(SFB.alpha1.lt[[i]])),]))})  #  Y matrices
corresponding to S* portfolios
View(S.star.Y[[1]])

Y.sorted=lapply(1:length(SFB.alpha1.lt),function(i){lapply(1:nrow(SFB.alpha
1.lt[[i]]),function(j){as.data.frame(sort(S.star.Y[[i]][,j],decreasing=FALS
E))})}) #Y* matrices sorted from lowest outcomes to highest
for(i in
1:length(SFB.alpha1.lt)){Y.sorted[[i]]=as.data.frame(Y.sorted[[i]])
colnames(Y.sorted[[i]])=rownames(SFB.alpha1.lt[[i]])}

      # Optimal behavioral portfolios #

OBP=lapply(1:length(SFB.alpha1.lt),function(i){as.data.frame(lapply(1:nrow(
SFB.alpha1.lt[[i]]),function(j){as.data.frame(t(as.matrix(Y.sorted[[i]][[j]
]))%*%as.matrix(as.numeric(W.pi)))}))})
```

```
for(i in
1:length(SFB.alpha1.lt)){colnames(OBP[[i]])=rownames(SFB.alpha1.lt[[i]])
rownames(OBP[[i]])=colnames(SFB.alpha1.lt[[i]])}
View(OBP[[1]])

      # Optimum Behavioral portfolio #

Max.OBP.a1lt= lapply(1:length(SFB.alpha1.lt),function(i){t(apply(OBP[[i]],
1, function(x) as.numeric(c(names(which.max(x)), max(x)))))})
for(i in
1:length(Statistic.eligible.simulation.a1lt)){rownames(Max.OBP.a1lt[[i]])=c
olnames(Statistic.eligible.simulation.a1lt[[i]])}
View(Max.OBP.a1lt[[2]]) #test --> ok
Points.OBP.a1lt=do.call(rbind.data.frame, Max.OBP.a1lt)

      # Optimal Behavioral portfolio belonging to the efficient frontier #

OBP.Sef.a1lt=lapply(1:length(Statistic.eligible.simulation.a1lt),function(i
){as.data.frame(merge(Max.OBP.a1lt[[i]],S.ef[[as.numeric(colnames(Statistic
.eligible.simulation.a1lt[[i]]))]],by.x="V1",by.y="row.names"))})
Match.OBP.Sef.a1lt=(as.numeric(length(Filter(function(x)
dim(x)[1]>0,OBP.Sef.a1lt)))/as.numeric(length(Max.OBP.a1lt)))

      # Plot

Title= paste("OBP Location on the Efficient Frontier :Simulation
",as.numeric(rownames(Points.OBP.a1lt)),sep="")
PDFPath = "~/Desktop/plots.a1.lt.pdf"
pdf(file=PDFPath)
for(i in
1:nrow(Points.OBP.a1lt)){plot(MVPortfolios[[as.numeric(rownames(Max.OBP.a1l
t[[i]]))]]$std,MVPortfolios[[as.numeric(rownames(Max.OBP.a1lt[[i]]))]]$Retu
rns,main=Title[i],sub="A=10Year T-Bond & alpha=0,1",  type="p",ylim=c(-
0.02,0.1),xlim=c(0,0.1),pch=4,col="blue",ylab=expression(mu[p]),xlab=expres
sion(sigma[p]), cex=0.8)
points(S.ef[[as.numeric(rownames(Max.OBP.a1lt[[i]]))]],col="limegreen",pch=
20,type='o',cex=1.4)
points(GMVP[[as.numeric(rownames(Max.OBP.a1lt[[i]]))]]$std,GMVP[[as.numeric
(rownames(Max.OBP.a1lt[[i]]))]]$Returns,pch=18,col="red")
text(GMVP[[as.numeric(rownames(Max.OBP.a1lt[[i]]))]],labels="GMVP",
col="red", pos=2)
points(MRP[[as.numeric(rownames(Max.OBP.a1lt[[i]]))]]$std,MRP[[as.numeric(r
ownames(Max.OBP.a1lt[[i]]))]]$Returns,pch=18,col="red")
text(MRP[[as.numeric(rownames(Max.OBP.a1lt[[i]]))]],labels="MRP",
col="red", pos=4)
points(Portfolios[[as.numeric(rownames(Max.OBP.a1lt[[i]]))]][as.numeric(Max
.OBP.a1lt[[i]][,1]),],pch=20,col="orange",)
text(Portfolios[[as.numeric(rownames(Max.OBP.a1lt[[i]]))]][as.numeric(Max.O
BP.a1lt[[i]][,1]),], labels="OBP",col='darkorange',pos=3)
legend("topleft",legend=c("Feasible MVPs","Efficient MVPs ", "Global
Minimum Variance & Maximum Return Portfolios","Optimal Behavioral
Portfolio"),
col=c("blue","limegreen","red","orange"),pch=15,bty="o",cex=1.2)
}
dev.off()

  # Alpha = 0,2 #

      # Safety first behavioral portfolio A=lt.bond alpha=0,2 #
```

```
SFB.alpha1.lt=lapply(1:2500,function(i){as.data.frame(BP.lt[[i]][BP.lt[[i]]
>= 800,rownames=TRUE])}) #S*
for(i in 1:2500 ){colnames(SFB.alpha1.lt[[i]])=i}
SFB.alpha1.lt=Filter(function(x) dim(x)[1]>0,SFB.alpha1.lt)
Statistic.eligible.simulation.a2lt=Filter(function(x)
dim(x)[1]>0,SFB.alpha1.lt)

S.star.Y=lapply(1:length(SFB.alpha1.lt),function(i){as.data.frame(t(WBP[[i]
][as.numeric(rownames(SFB.alpha1.lt[[i]])),]))})  #  Y matrices
corresponding to S* portfolios
View(S.star.Y[[1]])

Y.sorted=lapply(1:length(SFB.alpha1.lt),function(i){lapply(1:nrow(SFB.alpha
1.lt[[i]]),function(j){as.data.frame(sort(S.star.Y[[i]][,j],decreasing=FALS
E))})}) #Y* matrices sorted from lowest outcomes to highest
for(i in
1:length(SFB.alpha1.lt)){Y.sorted[[i]]=as.data.frame(Y.sorted[[i]])
colnames(Y.sorted[[i]])=rownames(SFB.alpha1.lt[[i]])}

      # Optimal behavioral portfolios #

OBP=lapply(1:length(SFB.alpha1.lt),function(i){as.data.frame(lapply(1:nrow(
SFB.alpha1.lt[[i]]),function(j){as.data.frame(t(as.matrix(Y.sorted[[i]][[j]
]))%*%as.matrix(as.numeric(W.pi)))}))})
for(i in
1:length(SFB.alpha1.lt)){colnames(OBP[[i]])=rownames(SFB.alpha1.lt[[i]])
rownames(OBP[[i]])=colnames(SFB.alpha1.lt[[i]])}
View(OBP[[1]])

      # Optimum Behavioral portfolio #

Max.OBP.a2lt=
lapply(1:length(Statistic.eligible.simulation.a2lt),function(i){t(apply(OBP
[[i]], 1, function(x) as.numeric(c(names(which.max(x)), max(x)))))})
for(i in
1:length(Statistic.eligible.simulation.a2lt)){rownames(Max.OBP.a2lt[[i]])=c
olnames(Statistic.eligible.simulation.a2lt[[i]])}
View(Max.OBP.a2lt) #test --> ok
Points.OBP.a2lt=do.call(rbind.data.frame, Max.OBP.a2lt)

      # Optimal Behavioral portfolio belonging to the efficient frontier #

OBP.Sef.a2lt=lapply(1:length(Statistic.eligible.simulation.a2lt),function(i
){as.data.frame(merge(Max.OBP.a2lt[[i]],S.ef[[as.numeric(colnames(Statistic
.eligible.simulation.a2lt[[i]]))]],by.x="V1",by.y="row.names"))})
Match.OBP.Sef.a2lt=(as.numeric(length(Filter(function(x)
dim(x)[1]>0,OBP.Sef.a2lt)))/as.numeric(length(Max.OBP.a2lt)))

      # Plot

Title= paste("OBP Location on the Efficient Frontier :Simulation
",as.numeric(rownames(Points.OBP.a2lt)),sep="")
PDFPath = "~/Desktop/plots.a2.lt.pdf"
pdf(file=PDFPath)
for(i in
1:nrow(Points.OBP.a2lt)){plot(MVPortfolios[[as.numeric(rownames(Max.OBP.a2l
t[[i]]))]]$std,MVPortfolios[[as.numeric(rownames(Max.OBP.a2lt[[i]]))]]$Retu
rns,main=Title[i],sub="A=10Year T-Bond & alpha=0,2",  type="p",ylim=c(-
0.02,0.1),xlim=c(0,0.1),pch=4,col="blue",ylab=expression(mu[p]),xlab=expres
sion(sigma[p]), cex=.1)
```

```
points(S.ef[[as.numeric(rownames(Max.OBP.a2lt[[i]]))]],col="limegreen",pch=
20,type='o',cex=1.4)
points(GMVP[[as.numeric(rownames(Max.OBP.a2lt[[i]]))]]$std,GMVP[[as.numeric
(rownames(Max.OBP.a2lt[[i]]))]]$Returns,pch=18,col="red")
text(GMVP[[as.numeric(rownames(Max.OBP.a2lt[[i]]))]],labels="GMVP",
col="red", pos=2)
points(MRP[[as.numeric(rownames(Max.OBP.a2lt[[i]]))]]$std,MRP[[as.numeric(r
ownames(Max.OBP.a2lt[[i]]))]]$Returns,pch=18,col="red")
text(MRP[[as.numeric(rownames(Max.OBP.a2lt[[i]]))]],labels="MRP",
col="red", pos=4)
points(Portfolios[[as.numeric(rownames(Max.OBP.a2lt[[i]]))]][as.numeric(Max
.OBP.a2lt[[i]][,1]),],pch=20,col="orange",)
text(Portfolios[[as.numeric(rownames(Max.OBP.a2lt[[i]]))]][as.numeric(Max.O
BP.a2lt[[i]][,1]),], labels="OBP",col='darkorange',pos=3)
legend("topleft",legend=c("Feasible MVPs","Efficient MVPs ", "Global
Minimum Variance & Maximum Return Portfolios","Optimal Behavioral
Portfolio"),
col=c("blue","limegreen","red","orange"),pch=15,bty="o",cex=1.2)
}
dev.off()

  # Alpha = 0,3

      # Safety first behavioral portfolio A=lt.bond alpha=0,3 #

SFB.alpha1.lt=lapply(1:2500,function(i){as.data.frame(BP.lt[[i]][BP.lt[[i]]
>= 700,rownames=TRUE])}) #S*
for(i in 1:2500 ){colnames(SFB.alpha1.lt[[i]])=i}
View(SFB.alpha1.lt[[1]])
SFB.alpha1.lt=Filter(function(x) dim(x)[1]>0,SFB.alpha1.lt)
Statistic.eligible.simulation.a3lt=Filter(function(x)
dim(x)[1]>0,SFB.alpha1.lt)

S.star.Y=lapply(1:length(SFB.alpha1.lt),function(i){as.data.frame(t(WBP[[i]
][as.numeric(rownames(SFB.alpha1.lt[[i]])),]))})  #  Y matrices
corresponding to S* portfolios
View(S.star.Y[[1]])

Y.sorted=lapply(1:length(SFB.alpha1.lt),function(i){lapply(1:nrow(SFB.alpha
1.lt[[i]]),function(j){as.data.frame(sort(S.star.Y[[i]][,j],decreasing=FALS
E))})}) #Y* matrices sorted from lowest outcomes to highest
for(i in
1:length(SFB.alpha1.lt)){Y.sorted[[i]]=as.data.frame(Y.sorted[[i]])
colnames(Y.sorted[[i]])=rownames(SFB.alpha1.lt[[i]])}

      # Optimal behavioral portfolios#

OBP=lapply(1:length(SFB.alpha1.lt),function(i){as.data.frame(lapply(1:nrow(
SFB.alpha1.lt[[i]]),function(j){as.data.frame(t(as.matrix(Y.sorted[[i]][[j]
]))%*%as.matrix(as.numeric(W.pi)))}))})
for(i in
1:length(SFB.alpha1.lt)){colnames(OBP[[i]])=rownames(SFB.alpha1.lt[[i]])
rownames(OBP[[i]])=colnames(SFB.alpha1.lt[[i]])}
View(OBP[[1]])

      # Optimum Behavioral portfolio #

Max.OBP.a3lt=
lapply(1:length(Statistic.eligible.simulation.a3lt),function(i){t(apply(OBP
[[i]], 1, function(x) as.numeric(c(names(which.max(x)), max(x))))[])})
```

```
for(i in
1:length(Statistic.eligible.simulation.a3lt)){rownames(Max.OBP.a3lt[[i]])=c
olnames(Statistic.eligible.simulation.a3lt[[i]])}
View(Max.OBP.a3lt[[20]]) #test --> ok
Points.OBP.a3lt=do.call(rbind.data.frame, Max.OBP.a3lt)

      # Optimal Behavioral portfolio belonging to the efficient frontier #

OBP.Sef.a3lt=lapply(1:length(Statistic.eligible.simulation.a3lt),function(i
){as.data.frame(merge(Max.OBP.a3lt[[i]],S.ef[[as.numeric(colnames(Statistic
.eligible.simulation.a3lt[[i]]))]],by.x="V1",by.y="row.names"))})
Match.OBP.Sef.a3lt=(as.numeric(length(Filter(function(x)
dim(x)[1]>0,OBP.Sef.a3lt)))/as.numeric(length(Max.OBP.a3lt)))

      # Plot

Title= paste("OBP Location on the Efficient Frontier :Simulation
",as.numeric(rownames(Points.OBP.a3lt)),sep="")
PDFPath = "~/Desktop/a3.lt.pdf"
pdf(file=PDFPath)
for(i in
1:2500){plot(MVPortfolios[[as.numeric(rownames(Max.OBP.a3lt[[i]]))]]$std,MV
Portfolios[[as.numeric(rownames(Max.OBP.a3lt[[i]]))]]$Returns,main=Title[i]
,sub="A=10Year T-Bond & alpha=0,3",  type="p",ylim=c(-
0.06,0.1),xlim=c(0.01,0.09),pch=4,col="blue",ylab=expression(mu[p]),xlab=ex
pression(sigma[p]), cex=0.8)
points(
S.ef[[as.numeric(rownames(Max.OBP.a3lt[[i]]))]],col="limegreen",pch=20,type
='o',cex=1.4)
points(GMVP[[as.numeric(rownames(Max.OBP.a3lt[[i]]))]]$std,GMVP[[as.numeric
(rownames(Max.OBP.a3lt[[i]]))]]$Returns,pch=18,col="red",cex=1.6)
text(GMVP[[as.numeric(rownames(Max.OBP.a3lt[[i]]))]],labels="GMVP",
col="red", pos=2)
points(MRP[[as.numeric(rownames(Max.OBP.a3lt[[i]]))]]$std,MRP[[as.numeric(r
ownames(Max.OBP.a3lt[[i]]))]]$Returns,pch=18,col="red",cex=1.6)
text(MRP[[as.numeric(rownames(Max.OBP.a3lt[[i]]))]],labels="MRP",
col="red", pos=4)
text(Portfolios[[as.numeric(rownames(Max.OBP.a3lt[[i]]))]][as.numeric(Max.O
BP.a3lt[[i]][,1]),], labels="OBP",col='darkorange',pos=3)
points(Portfolios[[as.numeric(rownames(Max.OBP.a3lt[[i]]))]][as.numeric(Max
.OBP.a3lt[[i]][,1]),],pch=20,col="darkorange",cex=1.6)
legend("topleft",legend=c("Feasible MVPs","Efficient MVPs ", "Global
Minimum Variance & Maximum Return Portfolios","Optimal Behavioral
Portfolio"),
col=c("blue","limegreen","red","darkorange"),pch=c(4,20,18,20),bty="o",cex=
1.2)
}
dev.off()

##### 3. Mental Account #####

#### 3.1. Behavioral Mental Account ####

### 3.1.1. Retirement portfolio A=ST-Bill alpha=0,1 ###

Retirement=Max.OBP.a1st

### 3.1.2. Education Portfolio A=LT-Bond alpha=0.2 ###

Education=Max.OBP.a2lt
```

```
### 3.1.3. Bequest Portfolio A=annualised 10% alpha=0.3 ###

BP.lt=lapply(1:2500,function(i){as.data.frame(lapply(3:10002,function(j){as
.data.frame(sum(ifelse(0.00797<=BF.port.lt[[i]][,j],1,0)))}))}) # check
which portfolios are higher than A ## (36-46) min
for(i in 1:2500){BP.lt[[i]]=t(BP.lt[[i]])
rownames(BP.lt[[i]])=c(1:10000)}

    ## Safety first constraint ##

SFB.alpha1.lt=lapply(1:2500,function(i){as.data.frame(BP.lt[[i]][BP.lt[[i]]
>= 700,rownames=TRUE])}) #S*
for(i in 1:2500 ){colnames(SFB.alpha1.lt[[i]])=i}
View(SFB.alpha1.lt[[1]])
SFB.alpha1.lt=Filter(function(x) dim(x)[1]>0,SFB.alpha1.lt)
Bequest=Filter(function(x) dim(x)[1]>0,SFB.alpha1.lt)
S.star.Y=lapply(1:length(SFB.alpha1.lt),function(i){as.data.frame(t(WBP[[i]
][as.numeric(rownames(SFB.alpha1.lt[[i]])),]))})  #  Y matrices
corresponding to S* portfolios
View(S.star.Y[[1]])
Y.sorted=lapply(1:length(SFB.alpha1.lt),function(i){lapply(1:nrow(SFB.alpha
1.lt[[i]]),function(j){as.data.frame(sort(S.star.Y[[i]][,j],decreasing=FALS
E))})}) #Y* matrices sorted from lowest outcomes to highest
for(i in
1:length(SFB.alpha1.lt)){Y.sorted[[i]]=as.data.frame(Y.sorted[[i]])
colnames(Y.sorted[[i]])=rownames(SFB.alpha1.lt[[i]])}

    ## Optimal Bequest behavioral portfolios ##

OBP.B=lapply(1:length(SFB.alpha1.lt),function(i){as.data.frame(lapply(1:nro
w(SFB.alpha1.lt[[i]]),function(j){as.data.frame(t(as.matrix(Y.sorted[[i]][[
j]]))%*%as.matrix(as.numeric(W.pi)))}))})
for(i in
1:length(SFB.alpha1.lt)){colnames(OBP.B[[i]])=rownames(SFB.alpha1.lt[[i]])
rownames(OBP.B[[i]])=colnames(SFB.alpha1.lt[[i]])}
View(OBP.B)

    ## Optimum Bequest Behavioral portfolio ##

Max.OBP.a3.10.B=
lapply(1:length(SFB.alpha1.lt),function(i){t(apply(OBP.B[[i]], 1,
function(x) as.numeric(c(names(which.max(x)), max(x)))))})
for(i in
1:length(SFB.alpha1.lt)){rownames(Max.OBP.a3.10.B[[i]])=colnames(Bequest[[i
]])}
View(Max.OBP.a3.10.B) #test --> ok

### 3.1.4. Agregate Behavioral Portfolio ###

Optimal.retirement= do.call(rbind.data.frame, Retirement)
Optimal.Education= do.call(rbind.data.frame, Education)
Optimal.Bequest= do.call(rbind.data.frame, Max.OBP.a3.10.B)
Aggregate.B.portfolio=merge(merge(Optimal.retirement,Optimal.Education,by="
row.names"),Optimal.Bequest,by.x="Row.names",by.y="row.names")
Aggregate.B.portfolio<- Aggregate.B.portfolio[ -c(3, 5, 7) ]
colnames(Aggregate.B.portfolio)=c('Row.names','Portfolio.R','Portfolio.E','
Portfolio.B')
Return.Aggregate.B.portfolio=lapply(1:nrow(Aggregate.B.portfolio),function(
i){(sum(c(0.6*(as.numeric((Portfolios[[as.numeric((Aggregate.B.portfolio[i,
]$Row.names))]][as.numeric(Aggregate.B.portfolio[i,]$Portfolio.R),2]))),
0.2*(as.numeric((Portfolios[[as.numeric((Aggregate.B.portfolio[i,]$Row.name
```

```
s))]][as.numeric(Aggregate.B.portfolio[i,]$Portfolio.E),2])),
0.2*(as.numeric((Portfolios[[as.numeric((Aggregate.B.portfolio[i,]$Row.name
s))]][as.numeric(Aggregate.B.portfolio[i,]$Portfolio.B),2])))))})
Return.Aggregate.B.portfolio=do.call(rbind.data.frame,Return.Aggregate.B.po
rtfolio)
rownames(Return.Aggregate.B.portfolio)=c(Aggregate.B.portfolio$Row.names)
colnames(Return.Aggregate.B.portfolio)=c('Aggregate.Return')
std.Aggregate.B.portfolio=lapply(1:nrow(Aggregate.B.portfolio),function(i){
(sum(c(0.6*(as.numeric((Portfolios[[as.numeric((Aggregate.B.portfolio[i,]$R
ow.names))]][as.numeric(Aggregate.B.portfolio[i,]$Portfolio.R),1])),
0.2*(as.numeric((Portfolios[[as.numeric((Aggregate.B.portfolio[i,]$Row.name
s))]][as.numeric(Aggregate.B.portfolio[i,]$Portfolio.E),1])),
0.2*(as.numeric((Portfolios[[as.numeric((Aggregate.B.portfolio[i,]$Row.name
s))]][as.numeric(Aggregate.B.portfolio[i,]$Portfolio.B),1])))))})
std.Aggregate.B.portfolio=do.call(rbind.data.frame,std.Aggregate.B.portfoli
o)
rownames(std.Aggregate.B.portfolio)=c(Aggregate.B.portfolio$Row.names)
colnames(std.Aggregate.B.portfolio)=c('Aggregate.Std')
MA.OBP=as.data.frame(cbind(Return.Aggregate.B.portfolio,std.Aggregate.B.por
tfolio))

### 3.1.4. Plot of Agregate Behavioral Portfolio ###

Title= paste("MA.OBP Location on the Efficient Frontier :Simulation
",as.numeric(rownames(MA.OBP)),sep="")
PDFPath = "~/Desktop/MA.OBP.pdf"
pdf(file=PDFPath)
for(i in
1:nrow(MA.OBP)){plot(MVPortfolios[[as.numeric(rownames(MA.OBP[i,]))]]$std,M
VPortfolios[[as.numeric(rownames(MA.OBP[i,]))]]$Returns,main=Title[i],
type="p",ylim=c(-
0.02,0.1),xlim=c(0,0.1),pch=4,col="blue",ylab=expression(mu[p]),xlab=expres
sion(sigma[p]), cex=0.8)
points( S.ef[[as.numeric(rownames(MA.OBP[i,]))]],col="limegreen",pch=20,
type='o',cex=1.4)
points(GMVP[[as.numeric(rownames(MA.OBP[i,]))]]$std,GMVP[[as.numeric(rownam
es(MA.OBP[i,]))]]$Returns,pch=18,col="red",cex=1.6)
text(GMVP[[as.numeric(rownames(MA.OBP[i,]))]],labels="GMVP", col="red",
pos=2)
points(MRP[[as.numeric(rownames(MA.OBP[i,]))]]$std,MRP[[as.numeric(rownames
(MA.OBP[i,]))]]$Returns,pch=18,col="red",cex=1.6)
text(MRP[[as.numeric(rownames(MA.OBP[i,]))]],labels="MRP", col="red",
pos=4)
points(MA.OBP[i,]$Aggregate.Std,MA.OBP[i,]$Aggregate.Return,pch=20,col="vio
letred1",cex=1.6)
text(MA.OBP[i,]$Aggregate.Std,MA.OBP[i,]$Aggregate.Return,labels="MA",
col="violetred1", pos=4)

legend("topleft",legend=c("Feasible MVPs","Efficient MVPs ", "Global
Minimum Variance & Maximum Return Portfolios","MA Optimal Behavioral
Portfolio"),
col=c("blue","limegreen","red","violetred1"),pch=c(4,20,18,20),bty="o",cex=
1.2)
}
dev.off()

##### 4. Usefull Graphs for the empiricla analysis #####

Date.plot=lapply(1:2500, function(i){rownames(Data.SON[[i]])})
Mean.date=as.data.frame(lapply(1:2500,
function(i){mean.Date(as.Date(Date.plot[[i]]))}))
```

```
Mean.date=as.data.frame(as.vector(t(Mean.date)))
Mean.date=as.data.frame(cbind(as.numeric(c(1:2500)),Mean.date))
colnames(Mean.date)=c('row.names','Date')

Max.return.point=do.call(rbind.data.frame, MRP)
Max.return.point=cbind(Max.return.point,Mean.date)
colnames(Max.return.point)=c("stdMRP",'ReturnsMRP','Date')
rownames(Max.return.point)=c(1:2500)
GMVP.points=do.call(rbind.data.frame, GMVP)
GMVP.points=cbind(GMVP.points,Mean.date)
colnames(GMVP.points)=c("stdGMVP",'ReturnsGMVP','Date')
rownames(GMVP.points)=c(1:2500)
OBP.S=Filter(function(x) dim(x)[1]>0,OBP.Sef.a3lt,all=FALSE)
OBP.S=do.call(rbind.data.frame,OBP.S)
a=rownames(Points.OBP.a3lt)

OBP.Points=cbind(Points.OBP.a3lt,a)

OBP.plot=merge(OBP.S,OBP.Points,by.x='V2',by.y='V2',all=FALSE,rownames=TRUE
,sort=FALSE)
OBP.plot.date=merge(Mean.date,OBP.plot,by.x="Date",by.y="a",all.x=TRUE)
colnames(OBP.plot.date)=c('Row.names','row.names','Date','V2','V1','std','R
eturns','Simulations')
Data.table=cbind(OBP.plot.date,GMVP.points,Max.return.point)

Standard=Data.table$std
Returns.graph=Data.table$Returns
Dates=seq(as.Date("2005/07/26"),as.Date("2015/06/29"),along.with=
OBP.plot.date$Date)

#return smooth
plot(Dates,Returns.graph,main="Portfolio returns", col='grey',ylim=c(-
0.02,0.12),type='h', lwd=2,ylab=expression(mu[p]),)
lines(spline(Dates,Data.table$ReturnsGMVP, method='n',
n=250),col='blue',lwd=2)
lines(spline(Dates,Data.table$ReturnsMRP, method='n',
n=250),col='firebrick',lwd=2,lty=6)
text(Dates[5],Returns.graph[5],col='grey37',labels="OBP", pos=3,cex=1)
text(Dates[2500],Data.table$ReturnsMRP[2500],col='firebrick',labels="MRP",
pos=4,cex=1)
text(Dates[2500],Data.table$ReturnsGMVP[2500],col='blue',labels="GMVP",
pos=4,cex=1)

#return non-smooth
plot(Dates,Returns.graph,main="Portfolio returns", col='black',ylim=c(-
0.02,0.12),type='h', lwd=2,ylab=expression(mu[p]),)
lines(spline(Dates,Data.table$ReturnsGMVP, method='n',
n=2500),col='blue',lwd=1)
lines(spline(Dates,Data.table$ReturnsMRP, method='n',
n=2500),col='firebrick',lwd=1)
text(Dates[5],Returns.graph[5],col='grey37',labels="OBP", pos=3,cex=1)
text(Dates[2500],Data.table$ReturnsMRP[2500],col='firebrick',labels="MRP",
pos=4,cex=1)
text(Dates[2500],Data.table$ReturnsGMVP[2500],col='blue',labels="GMVP",
pos=4,cex=1)

#std smoothed

plot(Dates,Standard,col='grey',ylim=c(-0.02,0.22),main="Portfolio
Std",type='h', lwd=4,ylab=expression(sigma[p]))
lines(spline(Dates,Data.table$stdGMVP, method='n', n=250),col='blue',lwd=2)
```

```
lines(spline(Dates,Data.table$stdMRP, method='n',
n=250),col='firebrick',lwd=2,lty=6)
text(Dates[33],Returns.graph[200],col='black',labels="OBP", pos=2,cex=1,2)
text(Dates[2500],Data.table$ReturnsMRP[2500],col='firebrick',labels="MRP",
pos=3,cex=1)
text(Dates[2500],Data.table$ReturnsGMVP[2500],col='blue',labels="GMVP",
pos=3,cex=1)

#std non-smooth

plot(Dates,Standard,col='black',ylim=c(-0.02,0.22),main="Portfolio
Std",type='h', lwd=4,ylab=expression(sigma[p]))
lines(spline(Dates,Data.table$stdGMVP, method='n',
n=2500),col='blue',lwd=1)
lines(spline(Dates,Data.table$stdMRP, method='n',
n=2500),col='firebrick',lwd=1)
text(Dates[33],Returns.graph[200],col='black',labels="OBP", pos=2,cex=1,2)
text(Dates[2500],Data.table$ReturnsMRP[2500],col='firebrick',labels="MRP",
pos=3,cex=1)
text(Dates[2500],Data.table$ReturnsGMVP[2500],col='blue',labels="GMVP",
pos=3,cex=1)

d=as.numeric(rownames(MA.OBP))
MA.OBP.graph=cbind(d,MA.OBP)
MA.data.table=merge(Data.table,MA.OBP.graph,by.x="Row.names",by.y="d",all=T
RUE)
Standard.MA=MA.data.table$Aggregate.Std
Returns.graph.MA=MA.data.table$Aggregate.Return
Dates=seq(as.Date("2005/07/26"),as.Date("2015/06/29"),along.with=
OBP.plot.date$Date)
Std.graph.MA=MA.data.table$Aggregate.Std

#return MA smooth
plot(Dates,Returns.graph.MA,main="MA Portfolio Returns",
col='grey',ylim=c(-0.02,0.22),type='h', lwd=2,ylab=expression(sigma[p]))
lines(spline(Dates,MA.data.table$ReturnsGMVP, method='n',
n=250),col='blue',lwd=2)
lines(spline(Dates,MA.data.table$ReturnsMRP, method='n',
n=250),col='firebrick',lwd=2,lty=6)
text(Dates[5],Returns.graph[5],col='grey37',labels="MA.OBP", pos=3,cex=1)
text(Dates[2500],MA.data.table$ReturnsMRP[2500],col='firebrick',labels="MRP
", pos=1,cex=1)
text(Dates[2500],MA.data.table$ReturnsGMVP[2500],col='blue',labels="GMVP",
pos=1,cex=1)

#std MA smooth

plot(Dates,Std.graph.MA,main="MA Portfolio Std", col='grey',ylim=c(-
0.02,0.22),type='h', lwd=2,ylab=expression(sigma[p]))
lines(spline(Dates,MA.data.table$stdGMVP, method='n',
n=250),col='blue',lwd=2)
lines(spline(Dates,MA.data.table$stdMRP, method='n',
n=250),col='firebrick',lwd=2,lty=6)
text(Dates[5],Returns.graph[5],col='grey37',labels="MA.OBP", pos=3,cex=1)
text(Dates[2500],MA.data.table$stdMRP[2500],col='firebrick',labels="MRP",
pos=1,cex=1)
text(Dates[2500],MA.data.table$stdGMVP[2500],col='blue',labels="GMVP",
pos=1,cex=1)
```