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Actuarial analysis of epidemiological models

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Preface.

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Chapter 1

Introduction

In 2020, an unprecedented global pandemic has emerged: the COVID-19 pandemic. This outbreak has had many impacts on all economic and financial sectors, including the insurance world. Indeed, a few major insurance companies announced in the press that claims caused by an epidemic such as the COVID-19 cannot be covered. Therefore, for insurance companies the need to develop insurance plans to cover risks related to an epidemic is growing.
In this master thesis, we show how the problem can be tackled and we investigate the pricing of insurances providing hospitalization and death benefits caused by an epidemic.

First, we present different deterministic models to model an epidemic wave, starting from the standard SIR (susceptible-infected-removed) model proposed by Kermack and Mc Kendrick (1927) and improving it by considering time-varying parameters. The time-varying parameters better reflect the effects of preventive measures taken to slow the progression of the virus. Then, based on these models and inspired by Feng and Garrido (2011)[6], we present a first insurance plan.
Next, we add a Brownian noise to our deterministic epidemic models in order to better explain the dynamics of pandemics. Based on these stochastic models, we define insurance plans including different aspects: pricing, reserving, hedging and risk management. Indeed, we determine fair premium rates and present different ways to hedge, either with a reinsurance treaty or by determining a solvency capital requirement.
However, models presented can only model one wave of epidemic, yet in practice, we observe multiwave pattern, this is why at the end of the theoretical part, we present a way to improve the SIR models in order to be able to explain multiwave dynamics.

Finally, in the practical part of this thesis, we apply all the theoretical elements presented to the COVID-19 epidemic in Belgium. We mainly focus on the first wave of COVID 19 by calibrating the models to the first wave data and then on this basis setting up insurance plans to cover hospitalization and death benefits. In a second step, we examine if based on the first wave, we could have predicted the intensity of the second wave and thus set up an insurance plan.

Chapter 2

Deterministic epidemic models

In this chapter, we introduce deterministic models that are suitable to explain an epidemic wave. First, we present the standard SIR (susceptible-infected-recovered) model, an epidemic model proposed by Kermack and Mc Kendrick (1927) and observe its different weaknesses. Therefore, in a second time, in order to improve the standard SIR model, we slightly modify the dynamic and allow parameters to be time-varying in order to take into account the lockdown effects during an epidemic wave. We also present a time-varying parameters model that gives a closed form for the number of infected which is useful to obtain more accurate results.

2.1 SIR model

The SIR (susceptible-infected-recovered) is a standard deterministic model that explains an epidemic wave.
This model is a 3 states model where:

- Susceptibles denote people that are likely to be infected. In the beginning of the wave, we consider that everybody is susceptible to be infected
- Infected denote people that are infected i.e. contagious and thus spread the virus
- Removed denote people that are not longer infected

Figure 2.1.1 illustrates the states evolution of the SIR model.

Figure 2.1.1: SIR scheme

Mathematically speaking, the SIR model is based on a 3 dimensional system of differential equations:

$$\begin{aligned} \frac{dS(t)}{dt} &= \frac{-\beta S(t)I(t)}{N} \ , t \geq t_0 \\ \frac{dI(t)}{dt} &= \frac{\beta S(t)I(t)}{N} - \gamma I(t) \ , t \geq t_0 \\ \frac{dR(t)}{dt} &= \gamma I(t) \ , t \geq t_0 \end{aligned} \tag{2.1.1}$$

with initial conditions: $S(t_0) = S_0, I(t_0) = I_0, R(t_0) = R_0$ and $N = S_0 + I_0 + R_0$ where N is the size of the population.
The parameters β represents the infection rate of the virus and γ stands for the recovery/mortality rate. As this two constants represent rates, they should be positive so we have: $\beta > 0$ and $\gamma > 0$. Note that in this model, we ignore births and fatality deaths so we assume: $S(t) + I(t) + R(t) = N \ \forall t$.

This model does not make the distinction between recovery people and deaths so $R(t)$ stands for both the recovered and the deceased persons. As we will use this model for actuarial purposes, it can be interesting to make this distinction. Consider η the recovery rate and μ the mortality rate, we deduce that: $\gamma = \eta + \mu$ and instead of having a system of 3 equations we get a 4 dimensional system such that:

$$\begin{aligned} \frac{dS(t)}{dt} &= \frac{-\beta S(t)I(t)}{N} \ , t \geq t_0 \\ \frac{dI(t)}{dt} &= \frac{\beta S(t)I(t)}{N} - (\eta + \mu)I(t) \ , t \geq t_0 \\ \frac{dR(t)}{dt} &= \eta I(t) \ , t \geq t_0 \\ \frac{dD(t)}{dt} &= \mu I(t) \ , t \geq t_0 \end{aligned} \tag{2.1.2}$$

with initial conditions: $S(t_0) = S_0, I(t_0) = I_0, R(t_0) = R_0, D(t_0) = D_0$ and $N = S_0 + I_0 + R_0 + D_0$.

To analyse the evolution of an epidemic, we can define the reproduction number R_0, this is the number of susceptibles that are infected by one infected person and at most R_0 is small at least the virus spreads. With the standard SIR model, the reproduction number is defined by:

$$R_0 = \frac{\beta}{\gamma} = \frac{\beta}{\eta + \mu}$$

Thus, the SIR model considers that this reproduction is constant over time and does not consider the effect of preventive measures that are taken when an epidemic wave occurs to slow down the transmission of the virus.

Although the standard model is a simple model for modelling an epidemic wave, it presents three main drawbacks:

- This model fails to reproduce epidemic wave with right fat tails without considering N also as parameter that is not acceptable [1].

[1] We will see it in the practical part

- It considers constant parameters and thus does not assume that the reproduction number decreases with time due to lockdown effects.
- This model does not give a closed formula for $I(t)$, this means that we have to use numerical approximations to get the evolution of the epidemic that is not accurate and optimal

Therefore, in the next section, we present models that can rectify these drawbacks.

2.2 Time-varying parameters SIR models

In order to consider models that can reproduce epidemic waves with fat tails but also taken into account preventive measures of lockdown, we are inspired by Xinzhi Liu (2012)[15] and consider time-varying parameters.
In other words, β and γ become time-dependent:

$$\beta \longrightarrow \beta(t) \text{ and } \gamma = \eta + \mu \longrightarrow \gamma(t) = \eta(t) + \mu$$

Note that we consider a constant mortality rate μ.
In this case, the equation system becomes:

$$\begin{aligned} \frac{dS(t)}{dt} &= \frac{-\beta(t)S(t)I(t)}{N} \ , t \geq t_0 \\ \frac{dI(t)}{dt} &= \frac{\beta(t)S(t)I(t)}{N} - (\eta(t)+\mu)I(t) \ , t \geq t_0 \\ \frac{dR(t)}{dt} &= \eta(t)I(t) \ , t \geq t_0 \\ \frac{dD(t)}{dt} &= \mu I(t) \ , t \geq t_0 \end{aligned} \tag{2.2.1}$$

Typically, we will observe in the practical part of the thesis that $\beta(t)$ and $\gamma(t)$ decrease with time but $\beta(t)$ faster than $\gamma(t)$ due to lockdown effects.

We can also define the reproduction number $R_0(t)$ considering time-varying parameters:

$$R_0(t) = \frac{\beta(t)}{\gamma(t)} = \frac{\beta(t)}{\eta(t)+\mu}$$

Since the infection rate decreases faster than the recovery rate, we observe that the reproduction number decreases with time to take into account the lockdown effects.

However, even considering time-varying parameters, we do not obtain a closed formula for $I(t)$, therefore we also present a model that modifies the dynamic of the SIR in order to obtain an explicit solution for $I(t)$.

2.2.1 Piecewise constant parameters

The piecewise constant parameters SIR model follows the idea that the virus does not spread in the same way depending on the measures taken by the government, particularly if a lockdown is decreed. So we divide the time interval into three sub-intervals:

- First interval: the spread of the virus continues and the measures are not yet felt
- Second interval: the effects of the lockdown are beginning to be felt and the spread is starting to diminish
- Third interval: the effects are really felt and the spread of the virus is greatly reduced

Thus in this approach, $\beta(t)$ and $\gamma(t) = \eta(t) + \mu$ in (2.2.1) are defined by:

$$\begin{aligned} \beta(t) &= \beta_1 1_{t_0 \leq t \leq t_1} + \beta_2 1_{t_1 < t \leq t_2} + \beta_3 1_{t_2 < t \leq t_3} \\ \gamma(t) &= \gamma_1 1_{t_0 \leq t \leq t_1} + \gamma_2 1_{t_1 < t \leq t_2} + \gamma_3 1_{t_2 < t \leq t_3} \end{aligned} \tag{2.2.2}$$

2.2.2 Exponential parameters

The exponential parameters SIR model is a time-varying model that consider decreasing parameters and suppose that $\beta(t)$ and $\gamma(t) = \eta(t) + \mu$ in (2.2.1) are defined by negative exponential function.
We define:

$$\begin{aligned} \beta(t) &= \beta_0 \ exp(-\lambda_1 \ (t - t_0)) \\ \gamma(t) &= \gamma_0 \ exp(-\lambda_2 \ (t - t_0)) \end{aligned} \tag{2.2.3}$$

λ_1 and λ_2 correspond to the speed at which $\beta(t)$ and $\gamma(t)$ decrease and as generally $\beta(t)$ declines faster than $\gamma(t)$, we observe $\lambda_1 > \lambda_2$.

Note that this approach reduces the number of parameters to estimate. In fact, we have only 5 parameters (β_0, γ_0, λ_1, λ_2 and μ) to estimate while for the piecewise constant parameters approach we have 9 parameters. So this approach leads to a more parsimonious model.

2.2.3 Simplified SIR model with exponential parameters

The two time-varying parameters SIR models introduced in the previous subsections take into account the effect of preventive measures and are also able to reproduce epidemic wave with right fat tails[2] but their main drawback is that they do not provide explicit solution for $I(t)$ and thus we always need to use numerical approximations.

Therefore, in order to obtain a closed form for $I(t)$, we slightly modify the dynamic of the SIR model.
We rely on the fact that $S(t) \approx N$ and for this reason we decide to replace $S(t)$ by N in the dynamic of the time-varying SIR model (with exponential parameters).

[2] As we will see in the practical part

It leads to a system of 4 differential equations:

$$\begin{aligned}
\frac{dS(t)}{dt} &= -\beta_0 \ exp(-\lambda_1(t-t_0)) \ I(t) \ , t \geq t_0 \\
\frac{dI(t)}{dt} &= \beta_0 \ exp(-\lambda_1(t-t_0)) \ I(t) - \gamma_0 \ exp(-\lambda_2(t-t_0)) \ I(t) \ , t \geq t_0 \\
\frac{dD(t)}{dt} &= \mu \ I(t) \ , t \geq t_0 \\
\frac{dR(t)}{dt} &= (\gamma_0 \ exp(-\lambda_2(t-t_0)) - \mu) \ I(t) \ , t \geq t_0
\end{aligned} \tag{2.2.4}$$

Note that $\eta(t) = \gamma(t) - \mu$ and the condition: $\forall t, \ N = S(t) + I(t) + D(t) + R(t)$ is still checked since $\forall t, \ dN(t) = dS(t) + dI(t) + dD(t) + dR(t) = 0$.

Since in (2.2.4), the differential of $I(t)$ only depends on $I(t)$, we can deduce its explicit solution:

$$\begin{aligned}
& dI(t) &&= \Big(\beta_0 \ exp(-\lambda_1(t-t_0)) \ I(t) - \gamma_0 \ exp(-\lambda_2(t-t_0)) \ I(t)\Big) \ dt \\
\Longleftrightarrow \quad & \frac{dI(t)}{I(t)} &&= \Big(\beta_0 \ exp(-\lambda_1(t-t_0)) - \gamma_0 \ exp(-\lambda_2(t-t_0))\Big) \ dt \\
\Longleftrightarrow \quad & \int_{t_0}^{t} \frac{dI(s)}{I(s)} &&= \int_{t_0}^{t} \Big(\beta_0 \ exp(-\lambda_1(s-t_0)) - \gamma_0 \ exp(-\lambda_2(s-t_0))\Big) \ ds \\
\Longleftrightarrow \quad & ln\Big(\frac{I(t)}{I(t_0)}\Big) &&= \beta_0 \frac{1 - exp(-\lambda_1(t-t_0))}{\lambda_1} - \gamma_0 \frac{1 - exp(-\lambda_2(t-t_0))}{\lambda_2} \\
\Longleftrightarrow \quad & I(t) &&= I(t_0) \ \Big(\beta_0 \frac{1 - exp(-\lambda_1(t-t_0))}{\lambda_1} - \gamma_0 \frac{1 - exp(-\lambda_2(t-t_0))}{\lambda_2}\Big)
\end{aligned}$$

The fact of having a closed solution for $I(t)$ allow us to no longer use numerical approximations which should lead to more accurate results in our actuarial analysis.

Chapter 3

Insurance plan based on deterministic models

In this chapter, based on deterministic models presented in the chapter 1, we define insurance plans. The aim of such insurance plans is to cover hospitalization and death benefits due to an epidemic wave. Based on the actuarial equilibrium, we deduce a fair premium rate and we also analyse the reserves of the plans.

3.1 Definition of the insurance plans

Based on Feng and Garrido (2011)[6] and Hainaut (2020)[10], we define an insurance plan considering that the whole population of a country takes part of it. In this case, we assume that susceptibles pay premiums as long as they are healthy, infected are compensated for their medical expenses and in case of death a lump sum benefit is paid.
Solidarity is the key of such plan since susceptibles pay for infected but from a certain point of view this kind of plan isn't fair as people who do not receive benefits pay more than people who are infected and thus receive benefits.
Therefore we also present another plan where the payment of premiums is made before the wave such that everyone pays the same premium which is fairer.

3.2 Fair premium and reserves

In this section, we determine the fair premium for the insurance plans just defined.
For that, we make the assumption that all payments are made in a continuous time. Susceptibles pay premiums in form of continuous annuities to a premium rate p, infected receive compensations in form of continuous annuities to a benefit rate b and in case of death a lump sum capital c is paid.
Considering that payments are made continuously is a conservative approach. Since we consider that premiums are received later as they are usually paid at the beginning of the period and conversely, for compensation annuities, we assume that we pay them earlier, as these expenses are often paid at the end of the period.

As the plan has to cover the whole population the fair premium rate, benefit rate and lump sum capital should ensure the financial equilibrium of the plan.

Actual value of premiums = Actual value of benefits

On this basis, we are able to deduce the fair premium rate.
However, we know that this premium rate ensures the actuarial equilibrium taking into account the whole duration of the plan but there is no guarantee that at any time the accumulated premiums are higher or equal to the accumulated benefits, therefore we need to analyse the reserves.

We can define the reserves $Res(t)$ associated to an insurance plan from a retrospective or prospective approach.

Retrospective approach:

$$Res(t) = Accumulated\ value\ of\ premiums\ at\ time\ t - Accumulated\ value\ of\ benefits\ at\ time\ t$$

Prospective approach:

$$Res(t) = \ Value\ at\ time\ t\ of\ future\ benefits\ -\ Value\ at\ time\ t\ of\ future\ premiums$$

To ensure that our deterministic plan is solvent at any time, we want that these reserves remain positive during the whole duration of the contract.
Condition:

$$\forall t\ Res(t) \geq 0$$

If $Res(t) < 0$ that indicates that we don't accumulate enough premiums to meet previous benefits or from a prospective point of view that susceptibles pay at time t too much premiums compared to the futures benefits

Now that the principles to apply to determine the fair premium rate and the reserves are defined, we develop formulas to compute p and $Res(t)$ considering first that premiums are paid during the wave and secondly that those are paid before the wave.

3.2.1 Premium payment during the wave

In this section, we make the assumption that premiums are paid continuously during the wave. The plan begins at t_0, finishes at t_f and if we suppose that premiums are paid from t_0 to t_f, we can deduce the fair premium rate from the equilibrium defined in section 3.2:

$$\begin{aligned} p \int_{t_0}^{t_f} e^{-r(s-t_0)} S(s)\ ds &= b \int_{t_0}^{t_f} e^{-r(s-t_0)} I(s)\ ds + c \int_{t_0}^{t_f} e^{-r(s-t_0)} dD(s) \\ \implies p &= \frac{b \int_{t_0}^{t_f} e^{-r(s-t_0)} I(s)\ ds + c \int_{t_0}^{t_f} e^{-r(s-t_0)} dD(s)}{\int_{t_0}^{t_f} e^{-r(s-t_0)} S(s)\ ds} \end{aligned} \tag{3.2.1}$$

where r is the risk-free interest rate.

Moreover, we know that in the SIR framework, $dD(t) = \mu I(t)\ dt$, then the fair premium rate can be written as:

$$p = \frac{(b+\mu c)\int_{t_0}^{t_f} e^{-r(s-t_0)} I(s)\ ds}{\int_{t_0}^{t_f} e^{-r(s-t_0)} S(s)\ ds} \tag{3.2.2}$$

However, we cannot develop the integrals explicitly (even for the simplified SIR model) and we need to approximate the integrals numerically to obtain our numerical results.

As we know the fair premium rate, we can now deduce the form of the reserves for this plan:

Retrospective approach:

$$Res(t) = p\int_{t_0}^{t} e^{r(t-s)} S(s)\ ds - (b+\mu c)\int_{t_0}^{t} e^{r(t-s)} I(s)\ ds,\ t_0 \leq t \leq t_f \tag{3.2.3}$$

Prospective approach:

$$Res(t) = (b+\mu c)\int_{t}^{tf} e^{-r(s-t)} I(s)\ ds - p\int_{t}^{tf} e^{-r(s-t)} S(s)\ ds,\ t_0 \leq t \leq t_f \tag{3.2.4}$$

Unfortunately, as we will see in the practical part of this thesis, if we consider that premiums are paid during the whole duration of the contract, reserves become negative during the insurance plan. This can be explained by the nature of the event covered, as we are considering an epidemic wave, if we consider a plan with continuous payments until the end of the epidemic, we don't accumulate enough premiums before the peak of the epidemic to cope it.

One solution to get positive reserves is to reduce the time of premium payment in order to accumulate enough premiums before the peak of the wave.
Consider t^* corresponds to the end of the premium payment then the premium rate is given by:

$$p = \frac{(b+\mu c)\int_{t_0}^{t_f} e^{-r(s-t_0)} I(s)\ ds}{\int_{t_0}^{t^*} e^{-r(s-t_0)} S(s)\ ds} \tag{3.2.5}$$

and reserves from a retrospective approach becomes:

$$\begin{aligned} Res(t) &= p\int_{t_0}^{t} e^{r(t-s)} S(s)\ ds - (b+\mu c)\int_{t_0}^{t} e^{r(t-s)} I(s)\ ds,\ t \leq t^* \\ &= \left(p\int_{t_0}^{t^*} e^{r(t^*-s)} S(s)\ ds\right) e^{r(t-t^*)} - (b+\mu c)\int_{t_0}^{t} e^{r(t-s)} I(s)\ ds,\ t > t^* \end{aligned} \tag{3.2.6}$$

In such way, we define t^* to get positive reserves.

In this section, we define the fair premium rate of an insurance plan that considers premium payment during the wave. However this fair premium rate is fair according the the actuarial equilibrium but it is not entirely fair. In fact, in a such plan, as we suppose continuous payments, the total amount of premium paid by a policyholder is equal to:

$$P = p \times (min(t_I, t_f) - t_0)),\ t_I \text{ is the time of infection} \tag{3.2.7}$$

We clearly deduce from (3.2.7) that someone who is not infected during the wave ($t_I > t_f$), pays more than someone who is infected and then receives benefits so it's not entirely fair.

3.2.2 Premium payment before the wave

We see in the previous section that the insurance plan with premium payment during the wave is not totally fair as not everyone pays the same premium.
Therefore, in order to obtain fairer premiums, we consider that in place of paying premiums during the epidemic wave, susceptibles pay premiums before the wave in anticipation of it and if the wave does not occur then premiums are reimbursed.

If we make the assumption that before the wave, the number of susceptibles is equal to N and if we define t_p as the beginning of the premium payment, t_0 the beginning of the wave and t_f the end, the actuarial equilibrium is given by:

$$p\int_{t_p}^{t_0} e^{-r(s-t_p)}N\,ds = \mathbb{P}(\text{wave})\,(b+\mu c)\,e^{-r(t_0-t_p)}\int_0^{t_f-t_0} e^{-rs}I(t_0+s)\,ds+(1-\mathbb{P}(\text{wave}))\,p\,e^{-r(t_0-t_p)}\int_{t_p}^{t_0} e^{r(t_0-s)}N ds \tag{3.2.8}$$

from which we can deduce the fair premium rate:

$$p = \frac{(b+\mu c)\ e^{-r(t_0-t_p)}\int_0^{t_f-t_0} e^{-rs}I(t_0+s)\ ds}{N\ \frac{(1-e^{-r(t_0-t_p)})}{r}} \tag{3.2.9}$$

If we now compute the total amount of premium paid by each policyholder, we obtain:

$$P = p\times(t_0-t_p) \tag{3.2.10}$$

So P does not depend on the time of infection and everyone pays the same amount.

Moreover, considering the premium payment before the wave ensures positive reserves.
In fact, reserves for $t_p \le t < t_0$ can be defined from a retrospective approach as:

$$Res(t) = p\int_{t_p}^{t} e^{r(t-s)}N\ ds \tag{3.2.11}$$

and reserves for $t_0 \le t \le t_f$ from a prospective approach as:

$$Res(t) = (b+\mu c)\int_t^{tf} e^{-r(s-t)}I(s)\ ds \tag{3.2.12}$$

We deduce from (3.2.11) and (3.2.12) that for $t_p \le t \le t_f,\ Res(t)\ge 0$.

Chapter 4

Stochastic epidemic models

All SIR models developed in the first chapter are deterministic, they reflect on mean the reality but they cannot capture all nuances. In reality, dynamics oscillate about deterministic model. In fact, population dynamics are inevitably subjected to environmental noise.
In order to have models that are able to explain this noise, we consider stochastic SIR models. We add randomness to our SIR models by considering a Brownian motion.
As for the deterministic approach, the piecewise constant and exponential parameters SIR models do not give an explicit solution for $\big(I(t)\big)_{t\geq t_0}$ and the simplified model provides an explicit solution by assuming that $\big(I(t)\big)_{t\geq t_0}$ follows a Log-normal distribution.

4.1 Stochastic time-varying parameters SIR models

In the same way as Gray et al. (2011)[9], we decide to randomize our models by adding a Brownian noise to β (the infectious rate). So we assume:

$$\beta(t)\ dt \longrightarrow \beta(t)\ dt + \sigma\ dW(t) \tag{4.1.1}$$

With $W(t)$ a Brownian motion defined on $\big(\Omega, \mathcal{F}, P\big)$ (see appendix 9.1.3), σ a constant.

With this perturbation (4.1.1), we obtain stochastic time-varying parameters SIR models with the following differential equations system:

$$\begin{aligned}
dS(t) &= \frac{-\beta(t)S(t)I(t)}{N}\ dt - \frac{\sigma S(t)I(t)}{N}\ dW(t)\ , \sigma \in \mathbb{R} \text{ et } t \geq t_0 \\
dI(t) &= \left(\frac{\beta(t)S(t)I(t)}{N} - \gamma(t)I(t)\right)\ dt + \frac{\sigma S(t)I(t)}{N}\ dW(t)\ , \sigma \in \mathbb{R} \text{ et } t \geq t_0 \\
dD(t) &= \mu\ I(t)\ dt,\ \ t \geq t_0 \\
dR(t) &= (\gamma(t) - \mu)I(t)\ dt,\ \ t \geq t_0
\end{aligned} \tag{4.1.2}$$

However, as for the deterministic case, we can only obtain a closed form for $\big(I(t)\big)_{t\geq t_0}$ with the simplified SIR.

4.2 Stochastic simplified SIR

In this section, we show that we can deduce a closed form for $\big(I(t)\big)_{t\geq t_0}$ from the simplified stochastic SIR model.

Under simplified SIR, we have the following differential equations system:

$$\begin{aligned} dS(t) &= -\beta_0\ exp(-\lambda_1(t-t_0))\ I(t)\ dt - \sigma I(t)\ dW(t) \\ dI(t) &= (\beta_0\ exp(-\lambda_1(t-t_0))\ I(t) - \gamma_0\ exp(-\lambda_2(t-t_0))\ I(t))\ dt + \sigma I(t)\ dW(t) \\ dD(t) &= \mu\ I(t)\ dt \\ dR(t) &= (\gamma_0\ exp(-\lambda_2(t-t_0)) - \mu)\ I(t)\ dt \end{aligned} \tag{4.2.1}$$

As the dynamic of $\big(I(t)\big)_{t\geq t_0}$ only depends on $(t, I(t))$, we can deduce a solution for $\big(I(t)\big)_{t\geq t_0}$.

Proposition 4.2.1. *The closed form for $\big(I(t)\big)_{t\geq t_0}$ using simplified SIR model is given by:*

$$I(t) = I(t_0)\ exp\left(\frac{\beta_0}{\lambda_1}\big(1-exp\ (-\lambda_1(t-t_0))\big)-\frac{\gamma_0}{\lambda_2}\big(1-exp\ (-\lambda_2(t-t_0))\big)-\frac{1}{2}\sigma^2(t-t_0)+\sigma W(t-t_0)\right),\ t \geq t_0$$

Proof. First, we rewrite the differential equation as:

$$\frac{dI(t)}{I(t)} = \big(\beta_0\ exp(-\lambda_1(t-t_0)) - \gamma_0\ exp(-\lambda_2(t-t_0))\big)\ dt + \sigma\ dW(t) \tag{4.2.2}$$

Then, we define $f(t, I(t)) = ln(I(t))$ and we apply the Itô's lemma that we recall in appendix (9.1.4).
Thus we can compute $df(t, I(t))$ using Ito's formula:

$$\begin{aligned} & dln(I(t)) = \left(\frac{1}{I(t)}I(t)\big(\beta_0\ exp(-\lambda_1(t-t_0)) - \gamma_0\ exp(-\lambda_2(t-t_0))\big) + \frac{1}{2}\frac{-1}{I(t)^2}\sigma^2 I(t)^2\right)\ dt + \frac{1}{I(t)}\sigma I(t)\ dW(t) \\ \iff & dln(I(t)) = \left(\big(\beta_0\ exp(-\lambda_1(t-t_0)) - \gamma_0\ exp(-\lambda_2(t-t_0))\big) - \frac{1}{2}\sigma^2\right)\ dt + \sigma\ dW(t) \\ \implies & \int_{t_0}^{t} dln(I(s)) = \int_{t_0}^{t}\left(\beta_0\ exp(-\lambda_1(s-t_0)) - \gamma_0\ exp(-\lambda_2(s-t_0)) - \frac{1}{2}\sigma^2\right)\ ds + \int_{t_0}^{t}\sigma\ dW(s) \\ \iff & ln\left(\frac{I(t)}{I(t_0)}\right) = \frac{\beta_0}{\lambda_1}\big(1 - exp(-\lambda_1(t-t_0))\big) - \frac{\gamma_0}{\lambda_2}\big(1 - exp(-\lambda_2(t-t_0))\big) - \frac{1}{2}\sigma^2(t-t_0) + \sigma\big(W(t) - W(t_0)\big) \\ \iff & I(t) = I(t_0)\ exp\left(\frac{\beta_0}{\lambda_1}\ (1 - exp\ (-\lambda_1(t-t_0))) - \frac{\gamma_0}{\lambda_2}(1 - exp\ (-\lambda_2(t-t_0))) - \frac{1}{2}\sigma^2(t-t_0) + \sigma W(t-t_0)\right) \end{aligned}$$

And that ends the proof. □

Moreover, we know that as $W(t-t_0)$ is a Brownian motion, $W(t-t_0) \sim N(0, \sqrt{t-t_0})$. Then from proposition 4.2.1 we can deduce that:

$$\big(I(t)\big)_{t\geq t_0} \sim Log-normal(\mu(t,t_0), \sigma(t,t_0)) \tag{4.2.3}$$

with $\mu(t,t_0) = log(I(t_0)) + \frac{\beta_0}{\lambda_1}\big(1 - exp(-\lambda_1(t-t_0))\big) - \frac{\gamma_0}{\lambda_2}\big(1 - exp(-\lambda_2(t-t_0))\big) - \frac{1}{2}\sigma^2(t-t_0)$ and $\sigma(t,t_0) = \sigma\sqrt{t-t_0}$.

Proposition 4.2.2. $\mathbb{E}(I(t)|\mathcal{F}_{t_0}) = I(t)^{Det}$, *where $I(t)^{Det}$ is the explicit solution of I(t) considering the deterministic simplified SIR with exponential parameters.*

Proof. Since $I(t)$ follows a Log-normal distribution, we just have to apply the formula of the expectation of a Log-normal random variable that we recall:

$$Z \sim Log - normal(\mu, \sigma) \Longrightarrow \mathbb{E}(Z) = exp\left(\mu + \frac{\sigma^2}{2}\right) \tag{4.2.4}$$

So if we apply this formula to $I(t)$, we obtain:

$$\begin{aligned}
\mathbb{E}(I(t)|\mathcal{F}_{t_0}) &= I(t_0)\ exp\left(\frac{\beta_0}{\lambda_1}\left(1 - exp\left(-\lambda_1(t-t_0)\right)\right) - \frac{\gamma_0}{\lambda_2}\left(1 - exp\left(-\lambda_2(t-t_0)\right)\right) - \frac{1}{2}\sigma^2(t-t_0) + \frac{\sigma^2(t-t_0)}{2}\right) \\
&= I(t_0)\ exp\left(\frac{\beta_0}{\lambda_1}\left(1 - exp\left(-\lambda_1(t-t_0)\right)\right) - \frac{\gamma_0}{\lambda_2}\left(1 - exp\left(-\lambda_2(t-t_0)\right)\right)\right) \\
&= I(t)^{Det}
\end{aligned}$$

□

Note that $\mathcal{F}_{t_0}$ is the natural filtration associated to $\big(I(t)\big)_{t\geq t_0}$. It contains the information about the sample path of $\big(I(t)\big)_{t\geq t_0}$, up to time t_0.

The fact of having an explicit solution for $I(t)$, allows us to obtain more accurate formulas for our actuarial analyses by avoiding Monte Carlo methods.

Chapter 5

Insurance plan based on stochastic models including hedging and risk management

In this chapter, we consider infectious disease insurance plans that cover the costs of an epidemic wave as defined in chapter 3 expect that now we rely on stochastic models.
We first determine the fair premium rates for the insurance plans. However, as we consider stochastic models, the reserves now become stochastic and can be negative, which means that the plans are not solvent in all cases. Therefore, in order to hedge against this type of events, we first consider a reinsurance treaty and secondly we determine a minimum capital to be held in order to ensure the solvency of the plan. We also present a financial product that can be used to hedge against financial impacts of unexpected events due to an epidemic.

5.1 Fair premium rate

In this section, we determine the fair premium rates of the insurance plans defined in section 7.1.2.
As we are no longer in a deterministic world, fair premium rates should ensure the actuarial equilibrium between expected cash-flows:

Actual value of expected premiums = Actual value of expected benefits

5.1.1 Premium payment during the wave

Suppose that premiums are continuously paid till time t^*, that the plan begins at t_0 and finishes at t_f, then the actuarial equilibrium is given by:

$$p_{stoch} \int_{t_0}^{t^*} e^{-r(s-t_0)} \ \mathbb{E}(S(s)|\mathcal{F}_{t_0}) \ ds = (b + \mu c) \int_{t_0}^{t_f} e^{-r(s-t_0)} \ \mathbb{E}(I(s)|\mathcal{F}_{t_0}) \ ds \tag{5.1.1}$$

And then the fair premium rate is defined by:

$$p_{stoch} = \frac{(b+\mu c)\int_{t_0}^{t_f} e^{-r(s-t_0)}\ \mathbb{E}(I(s)|\mathcal{F}_{t_0})\ ds}{\int_{t_0}^{t^*} e^{-r(s-t_0)}\ \mathbb{E}(S(s)|\mathcal{F}_{t_0})\ ds} \tag{5.1.2}$$

Therefore, to obtain the fair premium rate, we have to compute $\mathbb{E}(S(s)|\mathcal{F}_{t_0})$ and $\mathbb{E}(I(s)|\mathcal{F}_{t_0})$. However, except for the simplified model, we don't have closed form for $\mathbb{E}(S(s)|\mathcal{F}_{t_0})$ and $\mathbb{E}(I(s)|\mathcal{F}_{t_0})$ so we need to use Monte Carlo simulations.

Proposition 5.1.1. *For the simplified SIR model,* $\mathbb{E}(S(s)|\mathcal{F}_{t_0}) = S(t)^{Det}$ *and* $\mathbb{E}(I(s)|\mathcal{F}_{t_0}) = I(t)^{Det}$ *where* $S(t)^{Det}$ *and* $I(t)^{Det}$ *are curves* $S(t)$ *and* $I(t)$ *obtained with the deterministic simplified SIR with exponential parameters.*

Proof. The proof is simple, we show in section 4.2 that $\mathbb{E}(I(s)|\mathcal{F}_{t_0}) = I(t)^{Det}$ so we only have to prove $\mathbb{E}(S(s)|\mathcal{F}_{t_0}) = S(t)^{Det}$.

Recalling that under simplified SIR model, $dS(t) = -\beta_0\ exp(-\lambda_1(t-t_0))\ I(t)\ dt - \sigma I(t)\ dW(t)$ and so if we integrate this expression, we obtain:

$$\begin{aligned}\int_{t_0}^{t} dS(s) &= \int_{t_0}^{t} -\beta_0\ exp(-\lambda_1(s-t_0))\ I(s)\ ds - \int_{t_0}^{t} \sigma I(s)\ dW(s)\\ \iff S(t) &= S(t_0) - \int_{t_0}^{t} \beta_0\ exp(-\lambda_1(s-t_0))\ I(s)\ ds - \int_{t_0}^{t} \sigma I(s)\ dW(s)\end{aligned} \tag{5.1.3}$$

By taking the expectation of $S(t)$, we will have:

$$\begin{aligned}\mathbb{E}(S(t)|\mathcal{F}_{t_0}) &= S(t_0) - \int_{t_0}^{t} \beta_0\ exp(-\lambda_1(s-t_0))\ \mathbb{E}(I(s)|\mathcal{F}_{t_0})\ ds\\ &= S(t_0) - \int_{t_0}^{t} \beta_0\ exp(-\lambda_1(s-t_0))\ I(t)^{Det}\ ds\\ &= S(t)^{Det}\end{aligned} \tag{5.1.4}$$

We use the fact that we can switch expectation and deterministic integral but also that the expectation of a stochastic integral is 0.

□

So for the simplified SIR model, using the proposition 5.1.1, we can show that the fair premium rate for the stochastic case coincides with the rate for the deterministic one.

$$\begin{aligned}p_{stoch} &= \frac{(b+\mu c)\int_{t_0}^{t_f} e^{-r(s-t_0)}\ \mathbb{E}(I(s)|\mathcal{F}_{t_0})\ ds}{\int_{t_0}^{t^*} e^{-r(s-t_0)}\ \mathbb{E}(S(s)|\mathcal{F}_{t_0})\ ds}\\ &= \frac{(b+\mu c)\int_{t_0}^{t_f} e^{-r(s-t_0)}\ I(s)^{Det}\ ds}{\int_{t_0}^{t^*} e^{-r(s-t_0)}\ S(s)^{Det}\ ds}\\ &= p_{det}\end{aligned} \tag{5.1.5}$$

However, in practice, we see that optimal parameters of simplified deterministic SIR model do not coincide with parameters for the stochastic model and then we observe different premium

rates.
Concerning the reserves, they are still defined by:

$$
\begin{aligned}
Res(t) &= p_{stoch} \int_{t_0}^{t} e^{r(t-s)} S(s)\ ds - (b+\mu c) \int_{t_0}^{t} e^{r(t-s)} I(s)\ ds,\ t \le t^* \\
&= \left(p_{stoch} \int_{t_0}^{t^*} e^{r(t^*-s)} S(s)\ ds \right) e^{r(t-t^*)} - (b+\mu c) \int_{t_0}^{t} e^{r(t-s)} I(s)\ ds,\ t > t^*
\end{aligned} \tag{5.1.6}
$$

But now we are in a stochastic world, reserves ($Res(t)$) are stochastic and then can become negative in case of bad events. However, we require them to be positive on average, so once again, t^* is defined in order to have in mean positive reserves $\big(\mathbb{E}(Res(t)|\mathcal{F}_{t_0}) \ge 0$ for $t_0 \le t \le t_f\big)$.

5.1.2 Premium payment before the wave

Assume that before the wave, the number of susceptibles is equal to N and if we define t_p as the beginning of the premium payment, t_0 the beginning of the wave and t_f then end, the actuarial equilibrium is given by:

$$
\begin{aligned}
p_{stoch} \int_{t_p}^{t_0} e^{-r(s-t_p)} N\ ds = \mathbb{P}(\text{wave})\ (b+\mu c)\ e^{-r(t_0-t_p)} \int_0^{t_f-t_0} e^{-rs}\ \mathbb{E}(I(t_0+s)|\mathcal{F}_{t_0})\ ds\ + \\
(1-\mathbb{P}(\text{wave}))\ p_{stoch}\ e^{-r(t_0-t_p)} \int_{t_p}^{t_0} e^{r(t_0-s)} N ds
\end{aligned} \tag{5.1.7}
$$

From which we deduce the premium rate:

$$
p_{stoch} = \frac{(b+\mu c)\ e^{-r(t_0-t_p)} \int_0^{t_f-t_0} e^{-rs}\ \mathbb{E}(I(t_0+s)|\mathcal{F}_{t_0})\ ds}{N\ \frac{(1-e^{-r(t_0-t_p)})}{r}} \tag{5.1.8}
$$

To obtain the fair premium rate, we have to compute $\mathbb{E}(I(t)|\mathcal{F}_0)$. Once again, using 5.1.1, we can deduce that for the simplified SIR model, deterministic and stochastic premium rates coincide.

Reserves for this plan are defined by:

$$
\begin{aligned}
Res(t) =& p_{stoch}\ N\ \frac{(e^{r(t-t_p)}-1)}{r},\ t_p \le t \le t_0 \\
=& p_{stoch}\ N \left(\frac{(e^{r(t_0-t_p)}-1)}{r} \right) e^{r(t-t_0)} - (b+\mu c) \int_{t_0}^{t} I(s)\ e^{r(t-s)}\ ds,\ t_0 \le t \le t_f
\end{aligned} \tag{5.1.9}
$$

Reserves can become negative for $t_0 \le t \le t_f$ since now $I(t)$ is stochastic and that p_{stoch} is defined to obtain the equilibrium on average. Note that by (5.1.9) and (5.1.8), we can deduce that $\mathbb{E}(Res(t)|\mathcal{F}_{t_0}) \ge 0\ \forall t$ (see 9.2.1 in appendix).

To conclude this section, it is necessary to mention that an insurance plan with an average based premium is not hedgeable.
Indeed, if we consider a country wanting to set up an insurance plan then obviously an average based premium is not hedgeable as the reserves can become negative.
If we consider an insurance company that decides to cover the risk of epidemic for different countries by charging premiums based on an average, the company is not able to hedge because the risk of epidemic is a systematic risk, undiversifiable that generally affects the whole world, so there can be no compensation between countries.
One solution to achieve a good level of hedging is of course the reinsurance treaty, but in this case the problem is simply transferred and it is the reinsurance company that has to face a hedging problem.
Another solution may be to transfer this epidemic risk to investors willing to take a risky position on this event. This can be done through Catbonds, which are defaultable bonds, so that if a specific event occurs, part or all of the principal is retained by the company.
In our case, in the event of an epidemic, part or all of the Catbond principal is kept by the insurance company to deal with unexpected costs.

5.2 Reinsurance

In the section 5.1, we define a premium rate that cover on average expenses due to an epidemic wave but not at 100%. This means that if bad events occur, there is a high probability that reserves become negative, which can lead to an insolvency of our plan.
In this section, in the same way as Hainaut (2020)[10], we present a first way to reduce the insolvency risk: the reinsurance contract. First we define the reinsurance contract and deduce the fair reinsurance premium. In a second time we investigate if it provides a 100% hedging against the risk of insolvency.

5.2.1 Reinsurance premium

All developments made in the section are based on the simplified SIR model since it provides closed formula for $I(t)$ but note that, in the practical part of this thesis, in order to benchmark our numerical results, we also compute reinsurance premiums for the two other time-varying parameters SIR models using Monte Carlo method.

First, we price a reinsurance contract that gives, at time t, an amount $C\ \left(I(t) - \mathbb{E}(I(t)|\mathcal{F}_{t_0})\right)$ if the number of actual infected exceeds the expectation of $I(t)$. It means that if we observe, at time t, more infected than expected, we receive a capital that can cover unexpected costs.

So the payoff at time t is given by:

$$\text{Payoff} = C\ (I(t) - K)_+ \tag{5.2.1}$$

With $K = \mathbb{E}(I(t)|\mathcal{F}_{t_0})$.

Often, reinsurers price under a pricing measure (P^*) that is different from real measure (P). Since we know that under P, we are covered in mean, reinsurers want to reach a higher level of coverage and then price under P^*. In principle, the dynamic of $I(t)$ under P^* should be different from those under P (more conservative parameters than the estimated one in order to include a safety margin) but here, we assume that dynamics under P^* and P are the same.

So the price of the stop loss contract at time t_0 is given by:

$$\begin{aligned} p^t_{reins} &= C\ e^{-r(t-t_0)}\ \mathbb{E}^*((I(t)-K)_+|\mathcal{F}_{t_0}) \\ &= C\ \ e^{-r(t-t_0)}\ \mathbb{E}((I(t)-K)_+|\mathcal{F}_{t_0}) \end{aligned} \tag{5.2.2}$$

Where $\mathbb{E}^*$ denotes the expectation under P^*, r is the risk-free interest rate and $K = \mathbb{E}(I(t)|\mathcal{F}_0)$.

Proposition 5.2.1.

$$p^t_{reins} = C\ \ e^{-r(t-t_0)}\ \left(I(t_0)\ e^{\mu(t_0,t)+\frac{\nu(t_0,t)^2}{2}}\ \Phi(-d^t_1) - K\ \Phi(-d^t_2) \right) \tag{5.2.3}$$

where:

$$\begin{aligned} \mu(t_0,t) &= \frac{\beta_0}{\lambda_1}\ (1-e^{-\lambda_1(t-t_0)}) - \frac{\gamma_0}{\lambda_2}\ (1-e^{-\lambda_2(t-t_0)}) - \frac{\sigma^2}{2}(t-t_0) \\ \nu(t_0,t) &= \sigma\sqrt{t-t_0} \\ d^t_2 &= \frac{ln(\frac{K}{I(t_0)}) - \mu(t_0,t)}{\nu(t_0,t)} \\ d^t_1 &= d^t_2 - \nu(t_0,t) \end{aligned} \tag{5.2.4}$$

And $\Phi(.)$ is the cumulative distribution function of a standard normal distribution $N(0,1)$

Proof. Recall that the dynamic of $I(t)$ under P (and also under P^* as we suppose the same dynamic) is given by:

$$\frac{dI(t)}{I(t)} = (\beta_0\ exp(-\lambda_1(t-t_0)) - \gamma_0\ exp(-\lambda_2(t-t_0)))\ dt + \sigma\ dW(t) \tag{5.2.5}$$

So we have that $I(t) = I(t_0)e^{\mu(t_0,t)+\nu(t_0,t)\ U}$ with $\mu(t_0,t), \nu(t_0,t)$ defined in (5.2.4) and $U \sim N(0,1)$.
First recall the definition of an expectation of a continuous variable X:

$$\mathbb{E}(g(X)) = \int_{-\infty}^{+\infty} g(x)\ f_X(x)\ dx \tag{5.2.6}$$

For our case, pose $g(u) = (I(t_0)e^{\mu(t_0,t)+\nu(t_0,t)u} - K)_+$ and we have [1]:

$$\begin{aligned} g(u) = (I(t_0)e^{\mu+\nu u} - K)_+ &= 0\ ,\ \text{if}\ u < \frac{1}{\nu}(ln\left(\frac{K}{I(t_0)}\right) - \mu) \\ &= I(t_0)\ e^{\mu+\nu u} - K\ ,\ \text{if}\ u \geq \frac{1}{\nu}(ln\left(\frac{K}{I(t_0)}\right) - \mu) \end{aligned} \tag{5.2.7}$$

[1] To simplify notations, we pose $\mu(t_0,t) = \mu$ and $\nu(t_0,t) = \nu$

So by using that, we have:

$$\begin{aligned}\mathbb{E}((I(t)-K)_+|\mathcal{F}_{t_0}) &= \int_{-\infty}^{+\infty} (I(t_0)e^{\mu+\nu u}-K)_+ \; f_U(u)\; du \\ &= \int_{\frac{1}{\nu}(ln(\frac{K}{I(t_0)})-\mu)}^{+\infty} (I(t_0)e^{\mu+\nu u}-K)\; \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}\; du \\ &= \int_{\frac{1}{\nu}(ln(\frac{K}{I(t_0)})-\mu)}^{+\infty} I(t_0)e^{\mu+\nu u}\; \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}\; du \\ &\quad - K\int_{\frac{1}{\nu}(ln(\frac{K}{I(t_0)})-\mu)}^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}\; du\end{aligned} \tag{5.2.8}$$

<u>The second term is:</u>

$$\begin{aligned}K\int_{\frac{1}{\nu}(ln(\frac{K}{I(t_0)})-\mu)}^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}\; du &= K(1-\int_{-\infty}^{\frac{1}{\nu}(ln(\frac{K}{I(t_0)})-\mu)} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}\; du) \\ &= K(1-\Phi(d_2)) \\ &= K\Phi(-d_2)\end{aligned} \tag{5.2.9}$$

We use that:

$$P(X\leq a) = \int_{-\infty}^{a} f_X(x)\; dx, \text{ if } X \text{ is a continuous variable} \tag{5.2.10}$$

and also that for $X \sim N(0,1)$, we have:

$$(1-\Phi(a)) = (1-P(X\leq a)) = P(X\leq -a) = \Phi(-a) \tag{5.2.11}$$

<u>The first term becomes:</u>

$$\begin{aligned}\int_{\frac{1}{\nu}(ln(\frac{K}{I(t_0)})-\mu)}^{+\infty} I(t_0)e^{\mu+\nu u}\; \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}\; du &= I(t_0)e^{\mu}\int_{\frac{1}{\nu}(ln(\frac{K}{I(t_0)})-\mu)}^{+\infty} \frac{e^{-\frac{u^2-2\nu u}{2}}}{\sqrt{2\pi}}\; du \\ &= I(t_0)e^{\mu}\int_{\frac{1}{\nu}(ln(\frac{K}{I(t_0)})-\mu)}^{+\infty} e^{\frac{\nu^2}{2}}\; \frac{e^{-\frac{(u-\nu)^2}{2}}}{\sqrt{2\pi}}\; du\end{aligned} \tag{5.2.12}$$

Now we make a variable change and pose $s = u-\nu \implies du = ds$ and the condition $u \geq \frac{1}{\nu}(ln(\frac{K}{I(t_0)})-\mu)$ becomes $s \geq \frac{1}{\nu}(ln(\frac{K}{I(t_0)})-\mu)-\nu$. So using this variable change we have:

$$\begin{aligned}\int_{\frac{1}{\nu}(ln(\frac{K}{I(t_0)})-\mu)}^{+\infty} I(t_0)e^{\mu+\nu u}\; \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}\; du &= I(t_0)e^{\mu+\frac{\nu^2}{2}}\int_{\frac{1}{\nu}(ln(\frac{K}{I(t_0)})-\mu)-\nu}^{+\infty} \frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}}\; ds \\ &= I(t_0)e^{\mu+\frac{\nu^2}{2}}\; \Phi(-d_1)\end{aligned} \tag{5.2.13}$$

So if we go back to the expression of $\mathbb{E}((I(t)-K))_+)$, we obtain:

$$\mathbb{E}((I(t)-K))_+) = I(t_0)e^{\mu+\frac{\nu^2}{2}}\; \Phi(-d_1) - K\Phi(-d_2) \tag{5.2.14}$$

And that ends the proof. □

We thus show how to compute a reinsurance premium for a contract that can be used to cover unexpected costs at time t but it's still insufficient to get a prefect hedging so we need to generalize this contract.
If we generalize this reinsurance contract for all t ($t_0 \leq t \leq t_f$), the total payoff at time t_f is given by:

$$Payoff = C \int_{t_0}^{tf} e^{r(t_f - s)}((I(s) - K)_+ \; ds \tag{5.2.15}$$

Where $K = \mathbb{E}(I(s)|\mathcal{F}_{t_0})$ and $C = b + \mu c$ if we want that the payoff at time t_f corresponds to the unexpected costs of the insurance plan.

Using (5.2.2) and (5.2.3), the unique reinsurance premium at time t_0 that covers all that exceeds what we expect is thus given by:

$$\begin{aligned} P_{reins} &= e^{-r(t_f - t_0)} \; \mathbb{E}\Big(C \int_{t_0}^{tf} e^{r(t_f - s)}((I(s) - K)_+ \; ds|\mathcal{F}_{t_0} \Big) \\ &= \mathbb{E}\Big(C \int_{t_0}^{tf} e^{-r(s-t_0)} \; ((I(s) - K)_+ \; ds \; |\mathcal{F}_{t_0} \Big) \\ &= C \int_{t_0}^{tf} e^{-r(s-t_0)} \; \mathbb{E}((I(s) - K)_+|\mathcal{F}_{t_0}) \; ds \\ &= \int_{t_0}^{tf} p^s_{reins,\; C=\beta+c\mu} \; ds \\ &= (b + c\mu) \int_{t_0}^{tf} e^{-r(s-t_0)} \; \Big(I(t_0) \; e^{\mu(t_0,s) + \frac{\nu(t_0,s)^2}{2}} \; \Phi(-d^s_1) - K \; \Phi(-d^s_2) \Big) \; ds \end{aligned} \tag{5.2.16}$$

Where $K = \mathbb{E}(I(s)|\mathcal{F}_{t_0})$

So we see that based on the simplified SIR model, we can avoid simulations to get the reinsurance premium that leads to more accurate results.
Unfortunately, we cannot get an explicit value of the integral in (5.2.16) so we need to use numerical approximations to compute the unique reinsurance premium.

Note that as for the insurance company, the risk of epidemic as such is not hedgeable for the reinsurance company because of the systematic nature of the event covered. However, as the reinsurance company has a higher level of diversification, the risk of epidemic could be considered diversifiable with other natural disasters such as earthquakes or cyclones so that there could be compensations between the catastrophic events. Thus a reinsurance premium based on an average could be accepted even if to have a safety margin the reinsurance company would have to charge a risk premium on the pure reinsurance premium.

The aim of the reinsurance treaty is to deal with unexpected costs but does it really lead to a perfect hedging for the insurance plan? We answer this question in the next section.

5.2.2 Does an insurance plan with reinsurance contract lead to a perfect hedging?

We present a way to cover unexpected costs and in this section we analyse if a reinsurance contract in addition to an insurance plan can lead to a perfect hedging.
We see that the theoretical answer depends on the insurance plan considered. First we consider the insurance plan with premium payment before the wave and secondly the plan with premium payment during the wave.

5.2.2.1 Premium payment before the wave

In this section, we show that if we consider this type of insurance plan with in addition an reinsurance contract, we get a prefect cover.

To prove it, we compute the net stochastic reserves that are equal to the accumulated value of the premiums (AP_t^{net}) minus the accumulated net value of the benefits (AB_t^{net}):

$$\begin{aligned} Res^{net}(t) &= AP_t^{net} - AB_t^{net} \\ &= p\ N\ \frac{(e^{r(t-t_p)}-1)}{r},\ t_p \le t < t_0 \\ &\left(p\ N\ \frac{(e^{r(t_0-t_p)}-1)}{r}\right) e^{(t-t_0)r} - (b+c\mu)\int_{t_0}^{t} e^{r(t-s)}\ min\big(\mathbb{E}(I(t)|\mathcal{F}_{t_0}), I(t)\big)\ ds\ ,\ t_0 \le t \le t_f \end{aligned} \tag{5.2.17}$$

Proposition 5.2.2. *For* $t_p \le t \le t_f,\ Res^{net}(t) \ge 0$

Proof. By definition, $R^{net}(t)$ is positive for $t_p \le t < t_0$, let's prove that it's also the case for $t_0 \le t \le t_f$.
For this, we can rewrite $min\big(\mathbb{E}(I(t)|\mathcal{F}_{t_0}), I(t)\big)$:

$$min\big(\mathbb{E}(I(t)|\mathcal{F}_{t_0}), I(t)\big) = \mathbb{E}(I(t)|\mathcal{F}_{t_0}) - \big(\mathbb{E}(I(t)|\mathcal{F}_{t_0}) - I(t)\big)_+ \tag{5.2.18}$$

Using (5.2.18), $Res^{net}(t)$ for $t_0 \le t \le t_f$ is defined by:

$$\begin{aligned} Res^{net}(t) &= \left(p\ N\ \frac{(e^{r(t_0-t_p)}-1)}{r}\right) e^{(t-t_0)r} \\ &\quad - (b+c\mu)\int_{t_0}^{t} e^{r(t-s)}\left(\mathbb{E}(I(s)|\mathcal{F}_{t_0}) - \big(\mathbb{E}(I(s)|\mathcal{F}_{t_0}) - I(s)\big)_+\right) ds \\ &= \left(p\ N\ \frac{(e^{r(t_0-t_p)}-1)}{r}\right) e^{(t-t_0)r} - (b+c\mu)\int_{t_0}^{t} e^{r(t-s)}\mathbb{E}(I(s)|\mathcal{F}_{t_0})\ ds \\ &\quad + (b+c\mu)\int_{t_0}^{t}\big(\mathbb{E}(I(s)|\mathcal{F}_{t_0}) - I(s)\big)_+\ ds \\ &= \mathbb{E}(Res(t)|\mathcal{F}_{t_0}) + (b+c\mu)\int_{t_0}^{t}\big(\mathbb{E}(I(s)|\mathcal{F}_{t_0}) - I(s)\big)_+\ ds \end{aligned} \tag{5.2.19}$$

As we know that by definition of the fair premium rate, the expectation of stochastic reserves are positive for $t_p \le t \le t_f$ (see 9.2.1 in appendix) and as $\big(\mathbb{E}(I(s)|\mathcal{F}_{t_0}) - I(s)\big)_+ \ge 0$ then we prove:

$$\text{For } t_p \le t \le t_f,\ Res^{net}(t) \ge 0 \tag{5.2.20}$$

And that ends the proof. □

By the proposition 5.2.2, we know that $Res^{net}(t_f) \geq 0$, so there is chance that $Res^{net}(t_f) > 0$ and that an amount is paid back to the population.

We can quantify this amount by computing the expectation of $Res^{net}(t_f)$:

$$\begin{aligned} \mathbb{E}(Res^{net}(t_f)|\mathcal{F}_{t_0}) &= \mathbb{E}\Bigg(\mathbb{E}(Res(t_f)|\mathcal{F}_{t_0}) + (b+c\mu)\int_{t_0}^{t_f} \big(\mathbb{E}(I(s)|\mathcal{F}_{t_0}) - I(s)\big)_+ \, ds \, |\mathcal{F}_{t_0}\Bigg) \\ &= \mathbb{E}(Res(t_f)|\mathcal{F}_{t_0}) + (b+c\mu)\int_{t_0}^{t_f} \mathbb{E}\Bigg(\big(\mathbb{E}(I(s)|\mathcal{F}_{t_0}) - I(s)\big)_+|\mathcal{F}_{t_0}\Bigg) \, ds \\ &= (b+c\mu)\int_{t_0}^{t_f} \mathbb{E}\big(\big(K - I(s)\big)_+|\mathcal{F}_{t_0}\big) \, ds \end{aligned} \tag{5.2.21}$$

We define $K = \mathbb{E}(I(s)|\mathcal{F}_{t_0})$ and we use the fact that by definition $\mathbb{E}(Res(t_f)|\mathcal{F}_{t_0}) = 0$.

For the simplified SIR model, we can determine the value of $\mathbb{E}((K - I(s))_+|\mathcal{F}_{t_0})$ using a similar technique as for the computation of $\mathbb{E}((I(s) - K)_+|\mathcal{F}_{t_0})$ in section 5.2.1.

Proposition 5.2.3. *For the simplified SIR model,*

$$\mathbb{E}((K - I(t))_+|\mathcal{F}_{t_0}) = K \ \Phi(d_2^t) - I(t_0) \ e^{\mu(t_0,t)+\frac{\nu^2(t_0,t)}{2}} \ \Phi(d_1^t). \tag{5.2.22}$$

Where parameters are defined by (5.2.4).

As we use similar development than for the pricing of reinsurance contract, we decide to put this proof in appendix (see 9.2.2).

Using (5.2.21), the expected cash back at the end of the insurance plan is given by:

$$\text{Expected cash back} = \frac{\mathbb{E}(Res^{net}(t_f)|\mathcal{F}_{t_0})}{N} \tag{5.2.23}$$

In this section, we prove that this insurance plan with a reinsurance contract, leads to a 100% coverage against the insolvency risk. Moreover, from a marketing point of view it can also be interesting, we can ask the population to pay for the plan by telling them that there is a strong chance that a part of the premium paid is reimbursed at the end of the wave.

5.2.2.2 Premium payment during the wave

Contrary to the plan where premiums are totally paid before the wave, this insurance plan with a reinsurance treaty does, in theory, not cover all the insolvency risk at 100%, but we observe that in practice it can lead to a 100% coverage.

For this insurance plan, the net stochastic reserves are given by:

$$\begin{aligned} Res^{net}(t) &= p\int_{t_0}^{t} e^{r(t-s)} S(s) \, ds - (b+c\mu)\int_{t_0}^{t} e^{r(t-s)} \, min(\mathbb{E}(I(s)|\mathcal{F}_{t_0}), I(s)) \, ds, \ \text{if } t \leq t^* \\ &= p\left(\int_{t_0}^{t^*} e^{r(t^*-s)} S(s) \, ds\right) e^{r(t-t^*)} - (b+c\mu)\int_{t_0}^{t} e^{r(t-s)} \, min\big(\mathbb{E}(I(s)|\mathcal{F}_{t_0}), I(s)\big) \, ds, \ \text{if } t > t^* \end{aligned} \tag{5.2.24}$$

With t^* time until which premiums are paid. Can we say that for $t_0 \leq t \leq t_f,\ Res^{net}(t) \geq 0$? Not necessary.

Proposition 5.2.4. *For $t_0 \leq t \leq t_f,\ Res^{net}(t) \geq 0$ if for $t_0 \leq t \leq t^*,\ S(t) \geq \mathbb{E}(S(t)|\mathcal{F}_{t_0})$*

Proof. Suppose that for $t_0 \leq t \leq t^*,\ S(t) \geq \mathbb{E}(S(t)|\mathcal{F}_{t_0})$ and recall that $\mathbb{E}(Res(t)|\mathcal{F}_{t_0}) \geq 0$ for $t_0 \leq t \leq t_f$[2].

1) If $t_0 \leq t \leq t^*$, we have:

$$\begin{aligned} p \int_{t_0}^{t} e^{r(t-s)}\ S(s)\ ds &\geq p \int_{t_0}^{t} e^{r(t-s)}\ \mathbb{E}(S(s)|\mathcal{F}_{t_0})\ ds \\ &\geq (b+c\mu) \int_{t_0}^{t} e^{r(t-s)}\ \mathbb{E}(I(s)|\mathcal{F}_{t_0})\ ds \\ &\geq (b+c\mu) \int_{t_0}^{t} e^{r(t-s)}\ min\Big(\mathbb{E}(I(s)|\mathcal{F}_{t_0}), I(s)\Big)\ ds \end{aligned} \tag{5.2.25}$$

So for $t_0 \leq t \leq t^*$, $Res^{net}(t) = p \int_{t_0}^{t} e^{r(t-s)} S(s)\ ds - (b+c\mu) \int_{t_0}^{t} e^{r(t-s)}\ min\big(\mathbb{E}(I(s)|\mathcal{F}_{t_0}), I(s)\big)\ ds$ is positive.

2) If $t^* < t \leq t_f$, we have:

$$\begin{aligned} p \left(\int_{t_0}^{t^*} e^{r(t^*-s)}\ S(s)\ ds \right) e^{r(t-t^*)} &\geq p \left(\int_{t_0}^{t^*} e^{r(t^*-s)}\ \mathbb{E}(S(s)|\mathcal{F}_{t_0})\ ds \right) e^{r(t-t^*)} \\ &\geq (b+c\mu) \int_{t_0}^{t} e^{r(t-s)}\ \mathbb{E}(I(s)|\mathcal{F}_{t_0})\ ds \\ &\geq (b+c\mu) \int_{t_0}^{t} e^{r(t-s)}\ min\big(\mathbb{E}(I(s)|\mathcal{F}_{t_0}), I(s)\big)\ ds \end{aligned} \tag{5.2.26}$$

So for $t^* < t \leq t_f$, $Res^{net}(t) = p \left(\int_{t_0}^{t^*} e^{r(t^*-s)}\ S(s)\ ds \right) e^{r(t-t^*)} - (b+c\mu) \int_{t_0}^{t} e^{r(t-s)}\ min\big(\mathbb{E}(I(s)|\mathcal{F}_{t_0}), I(s)\big)\ ds$ is positive.

Hence, by 1) and 2), we prove that $Res^{net}(t) \geq 0$ for $t_0 \leq t \leq t_f$ and that ends the proof. $\square$

Question we can now ask is: can $S(t)$ be smaller than its expectation for $t_0 \leq t \leq t^*$? The theoretical answer is yes, but in practice, normally, $S^*(t)$ (the real number of susceptibles) is always higher than $\mathbb{E}(S(t)|\mathcal{F}_{t_0})$.

Assume that, at time t, we have a higher infection rate than expected then it can appear that $I(t) > \mathbb{E}(I(t)|\mathcal{F}_{t_0})$ and than since we know that in the model we suppose: $N = S(t) + I(t) + R(t)$, we have $S(t) < \mathbb{E}(S(t)|\mathcal{F}_{t_0})$.

However, in practice, $\mathbb{E}(S(t)|\mathcal{F}_{t_0})$ strongly underestimates the real number of susceptibles ($S^*(t)$). For example, we can take the simplified SIR model calibrated with data from the COVID-19 epidemic first wave in Belgium. Figure 5.2.1 shows $\mathbb{E}(S(t)|\mathcal{F}_{t_0})$, we can observe that $\mathbb{E}(S(t)|\mathcal{F}_{t_0})$ underestimates the real number of susceptibles. In fact, it means that more than 800 000 people were infected by COVID-19 in Belgium during the first wave which is far away from the reality.

[2] Remember that t^* is defined such that $\mathbb{E}(Res(t)|\mathcal{F}_{t_0}) \geq 0$ for $t_0 \leq t \leq t_f$

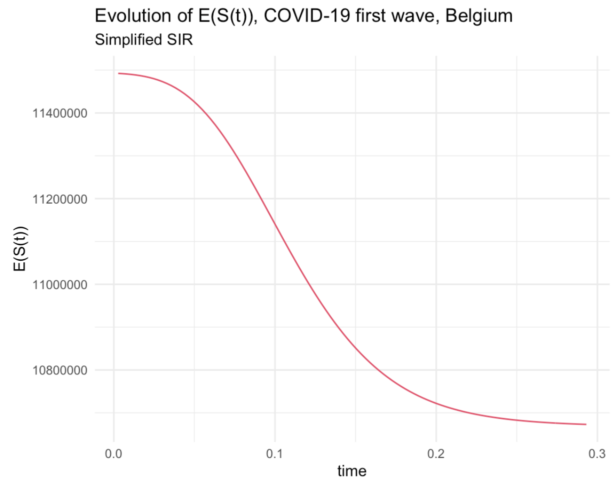


Figure 5.2.1: Evolution of $E(S(t)|\mathcal{F}_{t_0})$

5.3 Capital requirement

In this section, we present another way to hedge against unfavorable events: the solvency capital requirement (SCR). To this end, we consider that in addition to an insurance plan, we require a sufficient capital C that should ensure a probability of solvency at any time of 99.5% to be in line with the Solvency II regulation. First, we show a value-at-risk (Var) approach but this approach does not provide an explicit formula and then we have to use Monte Carlo simulations in order to obtain our capital requirement. Therefore, to obtain an explicit formula, we present in a second time a tail value-at-risk (Tvar) approach that leads to a conservative capital requirement and a formula that does not require simulations.

5.3.1 SCR using a value-at-risk approach

In this section, we determine the value of capital that we must have in addition to the premiums to achieve a solvency probability of 99.5% at any time.

Remember that stochastic reserves are denoted by $Res(t)$ and if we add at time t_0 (the beginning of the wave) a capital $C = SCR$, reserves become:

$$\tilde{Res}(t) = SCR\ e^{r(t-t_0)} + Res(t),\ t_0 \leq t \leq t_f \tag{5.3.1}$$

Our problem is now to find suitable $C = SCR$ such that at any time, stochastic reserves with a capital $C = SCR$ remain positive to a probability level of 99.5%. So we want $C = SCR$ such that:

$$\mathbb{P}\big(\tilde{Res}(t) \geq 0\big) \geq 99.5\%,\ \text{for } t_0 \leq t \leq t_f \tag{5.3.2}$$

Using the value-at-risk theory, we can give a condition on SCR. Before that, we recall the definition of a value-at-risk.

Definition 5.3.1. The value-at-risk at a confidence level α of a random variable X ($Var(X, \alpha)$) is defined by:

$$Var(X, \alpha) = F_X^{-1}(X, \alpha) \tag{5.3.3}$$

Where F_X is the cumulative probability function of X.

Proposition 5.3.2. *If for* $t_0 \leq t \leq t_f,\ SCR\ e^{r(t-t_0)} \geq Var(-Res(t), 99.5\%) \Longrightarrow \mathbb{P}\big(\tilde{Res}(t) \geq 0\big) \geq 99.5\%$

Proof. The proof is simple, we develop expression (5.3.2) and use the value-at-risk theory. Suppose $SCR\ e^{r(t-t_0)} \geq Var(-Res(t), 99.5\%)$, then we have:

$$\begin{aligned} P(\tilde{Res}(t) \geq 0) &= P(SCR\ e^{r(t-t_0)} + Res(t) \geq 0) \\ &= P(SCR\ e^{r(t-t_0)} \geq -Res(t)) \\ &\geq P(Var(-Res(t), 99.5\%) \geq -Res(t)) \\ &= 99.5\% \end{aligned} \tag{5.3.4}$$

So using (5.3.4) we prove that $P(\tilde{Res}(t) \geq 0) \geq 99.5\%$. □

With proposition 5.3.2, we have a condition that SCR should respect.
In the next two sub-sections, we consider the two insurance plans presented previously and determine the value of SCR for each case.

5.3.1.1 Premium payment before the wave

In this section, we consider the insurance plan with premium payment before the wave.
Recall that for $t_0 \leq t \leq t_f$, stochastic reserves for this insurance plan look like:

$$Res(t) = \left(p\ N\frac{(e^{r(t_0-t_p)} - 1)}{r}\right)\ e^{r(t-t_0)} - (b + c\mu)\int_{t_0}^{t} I(t)\ e^{r(t-s)}\ ds, \quad \text{for } t_0 \leq t \leq t_f \tag{5.3.5}$$

If we now add at time t_0 a capital $C = SCR$ and pose $P = p\ N\frac{(e^{r(t_0-t_p)}-1)}{r}$, we obtain:

$$\begin{aligned} \tilde{Res}(t) &= \left(SCR + p\ N\frac{(e^{r(t_0-t_p)} - 1)}{r}\right)\ e^{r(t-t_0)} - (b + c\mu)\int_{t_0}^{t} I(t)\ e^{r(t-s)}\ ds, \quad \text{for } t_0 \leq t \leq t_f \\ &= \Big(SCR + P\Big)\ e^{r(t-t_0)} - (b + c\mu)\int_{t_0}^{t} I(t)\ e^{r(t-s)}\ ds, \quad \text{for } t_0 \leq t \leq t_f \end{aligned} \tag{5.3.6}$$

After defining the stochastic reserves, we can now determine the value of SCR using the proposition 5.3.2.

Proposition 5.3.3. *For the insurance plan with premium payment before the wave, if* $SCR = Var(-Res(t_f)\ e^{-r(t_f-t_0)}, 99.5\%) = Var\big((b + c\mu)\int_{t_0}^{tf} e^{-r(s-t_0)} I(s)\ ds, 99.5\%\big) - P$ $\Longrightarrow \mathbb{P}\big(\tilde{Res}(t) \geq 0\big) \geq 99.5\%$, *for* $t_0 \leq t \leq t_f$ *(with* $Res(t)$ *and* $\tilde{Res}(t)$ *defined by (5.3.5) and (5.3.6))*

Proof. Suppose $SCR = Var\big((b+c\mu)\int_{t_0}^{tf} e^{-r(s-t_0)} I(s)\ ds, 99.5\%\big) - P$, our goal is to show that for $t_0 \leq t \leq t_f$, $\mathbb{P}\big(\tilde{Res}(t) \geq 0\big) \geq 99.5\%$.

For that, we use the condition exposed in proposition 5.3.2. Therefore, if we prove that for $t_0 \leq t \leq t_f$, $SCR\ e^{r(t-t_0)} \geq Var\big((b + c\mu)\int_{t_0}^{t} e^{r(t-s)} I(s)\ ds - P\ e^{r(t-t_0)}, 99.5\%\big)$ then we prove that $\mathbb{P}\big(\tilde{Res}(t) \geq 0\big) \geq 99.5\%$.

Before that, we recall properties about integration calculus and value-at-risk.

- $\int_0^t f(s)\ ds \leq \int_0^{t+1} f(s)\ ds$
- If $X \leq Y \Longrightarrow Var(X,\epsilon) \leq Var(Y,\epsilon),\ X$ and Y two random variables
- $Var(a+C(t)X,\epsilon) = a+\big(C(t)\,Var(X,\epsilon)\big)$, where $C(t)$ is a deterministic function and X random variable

Now using these properties we can show that the condition on $C = SCR$ is verified:

$$\begin{aligned} SCR\ e^{r(t-t_0)} &= \left(Var\big((b+c\mu)\int_{t_0}^{tf} e^{-r(s-t_0)}I(s)\ ds, 99.5\%\big) - P\right)\ e^{r(t-t_0)} \\ &= Var\big((b+c\mu)\int_{t_0}^{tf} e^{r(t-s)}I(s)\ ds - P\ e^{r(t-t_0)}, 99.5\%\big) \\ &\geq Var((b+c\mu)\int_{t_0}^{t} e^{r(t-s)}I(s)\ ds - P\ e^{r(t-t_0)}, 99.5\%),\ \text{for } t_0 \leq t \leq tf \end{aligned} \tag{5.3.7}$$

As the condition on SCR is verified, we can conclude that $P(\tilde{Res}(t) \geq 0) \geq 99.5\%$ and that ends the proof. □

Note that the capital requirement defined in proposition 5.3.3 is also applicable in the case where there is no premium payment taking $P = 0$.

Unfortunately, we can know the distribution of $I(t)$ (if we consider the simplified SIR model) but not that of $\left((b+c\mu)\int_{t_0}^{tf} e^{-r(s-t_0)}I(s)\ ds\right)$ and then we cannot obtain a closed formula for the SCR. The only way to determine the SCR using the value-at-risk approach is to use Monte Carlo simulations that is not optimal. Therefore in the next section, we propose a tail value-at-risk approach which leads, in the case of the simplified SIR model, to avoid simulations for computing the SCR.

5.3.1.2 Premium payment during the wave

We consider the insurance plan with premium payment during the wave, then stochastic reserves are defined by:

$$\begin{aligned} Res(t) &= p\int_{t_0}^{t} e^{r(t-s)}\ S(s)\ ds - (b+\mu c)\int_{t_0}^{t} e^{r(t-s)}\ I(s)\ ds,\ \text{if } t_0 \leq t \leq t^* \\ &= p\left(\int_{t_0}^{t^*} e^{r(t^*-s)}\ S(s)\ ds\right)\ e^{r(t-t^*)} - (b+\mu c)\int_{t_0}^{t} e^{r(t-s)}\ I(s)\ ds,\ \text{if } t^* < t \leq t_f \end{aligned} \tag{5.3.8}$$

If we add a capital $C = SCR$ to the stochastic reserves at time t_0, we obtain:

$$\begin{aligned} \tilde{Res}(t) &= SCR\ e^{r(t-t_0)} + \left(p\int_{t_0}^{t} e^{r(t-s)}\ S(s)\ ds - (b+\mu c)\int_{t_0}^{t} e^{r(t-s)}\ I(s)\ ds\right),\ \text{if } t \leq t^* \\ &= SCR\ e^{r(t-t_0)} + \left(p\left(\int_{t_0}^{t^*} e^{r(t^*-s)}\ S(s)\ ds\right)\ e^{r(t-t^*)} - (b+\mu c)\int_{t_0}^{t} e^{r(t-s)}\ I(s)\ ds\right),\ \text{if } t > t^* \end{aligned} \tag{5.3.9}$$

And then by doing a same reasoning than in section 5.3.1.1 and using proposition 5.3.2, we can quantify the minimum capital requirement.

Proposition 5.3.4. *For the insurance plan with premium payment during the wave, if* $SCR = Var(-Res(t_f)\ e^{-r(t_f-t_0)}, 99.5\%)$ *and if for* $t_0 \leq t \leq t^*$, $SCR\ e^{r(t-t_0)} = Var(-Res(t_f)\ e^{-r(t_f-t_0)}, 99.5\%)\ e^{r(t-t_0)} \geq Var(-Res(t), 99.5\%)$ $\Longrightarrow \mathbb{P}\big(\tilde{Res}(t) \geq 0\big) \geq 99.5\%$, *for* $t_0 \leq t \leq t_f$ *(with* $Res(t)$ *and* $\tilde{Res}(t)$ *defined by (5.3.8) and (5.3.9))*

Proof. As the proof is similar to 5.3.1.1, we decide to put it in appendix (see 9.2.3). □

Note that here we have an additional condition on SCR that can be explained by the fact that there is an additional source of randomness in the reserves ($S(t)$) but we observe that in practice this condition is verified.

Unfortunately, as for the case with payment before the wave, we cannot obtain a closed formula for SCR.

5.3.2 SCR using a tail value-at-risk approach

In the section 5.3.1, we show that based on the value-at-risk theory, we can determine a minimal capital requirement but the main drawback of the method is that it does not lead to an explicit formula. In this section, we consider a tail value-at-risk approach that can lead to a better formula to compute the capital requirement.
Note that for this approach, we only consider the simplified SIR model and the insurance plan with premium payment before the wave.

First we can recall the definition of a tail value-at-risk.

Definition 5.3.5. The tail value-at-risk at a confidence level α of a random variable X is defined by:

$$TVar(X, \alpha) = \frac{1}{(1-\alpha)} \int_{\alpha}^{1} Var(X, \epsilon)\ d\epsilon \tag{5.3.10}$$

Obviously, by definition 5.3.5, we can see that:

$$Var(X, \alpha) \leq TVar(X, \alpha) \tag{5.3.11}$$

So by proposition (5.3.3) and (5.3.11), we can say that if we take:

$$SCR = TVar\Big((b + c\mu) \int_{t_0}^{tf} e^{-r(s-t_0)} I(s)\ ds, 99.5\%\Big) - P \tag{5.3.12}$$

Then the probability of solvency is higher than 99.5%.

Moreover, we know that the tail value-at-risk has the subadditivity property, thus we observe for random variables X and Y:

$$TVar(X + Y, \alpha) \leq TVar(X, \alpha) + TVar(Y, \alpha) \tag{5.3.13}$$

If now we approximate the integral value in (5.3.10) by a sum[3] such that:

$$\int_{t_0}^{t_f} e^{-r(s-t_0)} I(s)\, ds \approx \sum_{i=0}^{n-1} e^{-r\ (t_{i+1}-t_0)}\ I(t_{i+1})\ \delta_i \tag{5.3.14}$$

where $\delta_i = 1/365$, $t_i = i\ \delta_i$ and $t_0 < t_1 < ... < t_n = t_f$

Then, we can apply the subadditivity property of the tail value-at-risk in order to get a conservative capital $C_{Tvar}^{Discrete}$ such that $P(\tilde{Res}(t) \geq 0) \geq 99.5\%$ defined by:

$$SCR_{Tvar}^{Discrete} := (b + c\mu) \sum_{i=0}^{n-1} e^{-r\ (t_{i+1}-t_0)}\ TVar(I(t_{i+1}), 99.5\%)\ \delta_i - P \tag{5.3.15}$$

This formula is great because as we know the distribution of $I(t)$, we can deduce the value of $TVar(I(t))$ and then obtained a formula that does not require Monte Carlo simulations.

For that, we first compute the value-at-risk of $I(t)$.

Proposition 5.3.6.

$$Var(I(t), \alpha) = I(t_0)\ exp\ \left(\mu(t, t_0) + \Phi^{-1}(\alpha)\ \sigma\sqrt{t - t_0}\right) \tag{5.3.16}$$

Where $\Phi^{-1}(.)$ *is the quantile of a standard normal random variable and* $\mu(t, t_0) = \frac{\beta_0}{\lambda_1}(1 - e^{-\lambda_1\ (t-t_0)})) - \frac{\gamma_0}{\lambda_2}(1 - e^{-\lambda_2\ (t-t_0)})) - \frac{\sigma^2\ (t-t_0)}{2}$.

Proof. By definition 5.3.1, we know that $Var(I(t), \alpha)$ is such that:

$$P(I(t) \leq Var(I(t), \alpha)) = \alpha \tag{5.3.17}$$

As $I(t)$ follows a Log-normal distribution, we have[4]:

$$\begin{aligned}
& P\big(I(t) \leq Var\big) = \alpha \\
\iff & P\big(ln(I(t)) \leq ln(Var)\big) = \alpha \\
\iff & P\Bigg(ln(I(t_0)) + \mu(t, t_0) + \sigma W(t - t_0) \leq ln(Var)\Bigg) = \alpha \\
\iff & P\Bigg(Z \leq \frac{ln(Var) - ln(I(t_0)) + \mu(t, t_0)}{\sigma\sqrt{t - t_0}}\Bigg) = \alpha,\ Z \sim N(0, 1) \\
\iff & \frac{ln(Var) - ln(I(t_0)) - \mu(t, t_0)}{\sigma\sqrt{t - t_0}} = \Phi^{-1}(\alpha) \\
\Longrightarrow & Var = I(t_0)\ exp\ \left(\mu(t, t_0) + \Phi^{-1}(\alpha)\ \sigma\sqrt{t - t_0}\right)
\end{aligned} \tag{5.3.18}$$

□

As we know the explicit form of $Var(I(t), \alpha)$, we can now determine the expression of the $TVar(I(t), \alpha)$

[3]For our numerical results, we have to approximate numerically the integral since we don't have a closed form of the integral

[4]To simplify the notations $Var(I(t), \alpha)$ is denoted by Var

Proposition 5.3.7.

$$TVar(I(t),\alpha) = \frac{I(t_0)\ exp(\mu(t,t_0) + \frac{\sigma(t,t_0)^2}{2})\ \Phi(\sigma(t,t_0) - \Phi^{-1}(\alpha))}{(1-\alpha)} \tag{5.3.19}$$

with $\mu(t,t_0) = \frac{\beta_0}{\lambda_1}(1-e^{-\lambda_1\ (t-t_0)})) - \frac{\gamma_0}{\lambda_2}(1-e^{-\lambda_2\ (t-t_0)})) - \frac{\sigma^2\ (t-t_0)}{2}$ *and* $\sigma(t,t_0) = \sigma\sqrt{t-t_0}$.

Proof. To prove it, we first recall the definition of the expected shortfall to a level α of a random variable X ($ES(X,p)$):

$$ES(X,\alpha) = \mathbb{E}((X - Var(X,\alpha))_+) \tag{5.3.20}$$

Moreover, we know that using expected shortfall and value-at-risk, we can deduce the expression of the tail value-at-risk. In fact Denuit and Charpentier (2005)[5] show that:

$$TVar(X,\alpha) = Var(X,\alpha) + \frac{ES(X,\alpha)}{(1-\alpha)} \tag{5.3.21}$$

Therefore, we use (5.3.21) to deduce the expression of $TVar(I(t),\alpha)$.

As we already know the value of $Var(I(t),\alpha)$, we develop the expression of the expected shortfall.

The expected shortfall defined by (5.3.20) has the same expression than the reinsurance premium computed in section 5.2.1 (see proposition 5.2.1).
So by defining $K = Var(I(t),\alpha)$, $\mu(t,t_0) = \frac{\beta_0}{\lambda_1}(1-e^{-\lambda_1\ (t-t_0)})) - \frac{\gamma_0}{\lambda_2}(1-e^{-\lambda_2\ (t-t_0)})) - \frac{\sigma^2\ (t-t_0)}{2}$ and $\sigma(t,t_0) = \sigma\sqrt{t-t_0}$, we have:

$$\begin{aligned} ES(I(t),\alpha) = I(t_0)\ e^{\mu(t,t_0)+\frac{\sigma^2(t,t_0)}{2}}\Phi\Bigg(-\frac{ln\big(\frac{Var(I(t),\alpha)}{I(t_0)}\big) - \mu(t,t_0)}{\sigma(t,t_0)} + \sigma(t,t_0)\Bigg) \\ - Var(I(t),\alpha)\Phi\Bigg(-\frac{ln\big(\frac{Var(I(t),\alpha)}{I(t_0)}\big) - \mu(t,t_0)}{\sigma(t,t_0)}\Bigg) \end{aligned} \tag{5.3.22}$$

But since we know that $Var(I(t),\alpha) = I(t_0)\ exp\ (\mu(t,t_0) + \Phi^{-1}(\alpha)\ \sigma\sqrt{t-t_0})$, we can simplify the expression.

The first part becomes:

$$I(t_0)\ e^{\mu(t,t_0)+\frac{\sigma^2(t,t_0)}{2}}\Phi\Bigg(-\frac{ln\big(\frac{Var(I(t),\alpha)}{I(t_0)}\big) - \mu(t,t_0)}{\sigma(t,t_0)} + \sigma(t,t_0)\Bigg) = I(t_0)\ e^{\mu(t,t_0)+\frac{\sigma^2(t,t_0)}{2}}\Phi\big(\sigma(t,t_0) - \Phi^{-1}(\alpha)\big) \tag{5.3.23}$$

And the second,

$$\begin{aligned} Var(I(t),\alpha)\Phi\Bigg(-\frac{ln\big(\frac{Var(I(t),\alpha)}{I(t_0)}\big) - \mu(t,t_0)}{\sigma(t,t_0)}\Bigg) &= Var(I(t),\alpha)\ \Phi(-\Phi^{-1}(\alpha)) \\ &= Var(I(t),\alpha)\ (1-\Phi(\Phi^{-1}(\alpha))) \\ &= Var(I(t),\alpha)\ (1-\alpha) \end{aligned} \tag{5.3.24}$$

So finally, we obtain:

$$ES(I(t),\alpha) = I(t_0)\ e^{\mu(t,t_0)+\frac{\sigma^2(t,t_0)}{2}}\Phi(\sigma(t,t_0)-\Phi^{-1}(\alpha)) - Var(I(t),\alpha)\ (1-\alpha) \quad (5.3.25)$$

Now $Var(I(t),\alpha)$ and $ES(I(t),\alpha)$ are known, we can determine the expression of $TVar(I(t),\alpha)$

$$\begin{aligned} TVar(I(t),\alpha) &= Var(I(t),\alpha) + \frac{ES(I(t),\alpha)}{(1-\alpha)} \\ &= Var(I(t),\alpha) + \frac{I(t_0)\ e^{\mu(t,t_0)+\frac{\sigma^2(t,t_0)}{2}}\Phi\big(\sigma(t,t_0)-\Phi^{-1}(\alpha)\big)}{(1-\alpha)} - Var(I(t),\alpha) \\ &= \frac{I(t_0)\ e^{\mu(t,t_0)+\frac{\sigma^2(t,t_0)}{2}}\Phi\big(\sigma(t,t_0)-\Phi^{-1}(\alpha)\big)}{(1-\alpha)} \end{aligned} \quad (5.3.26)$$

And that ends the proof. □

As we have an explicit expression of $TVar(I(t),\alpha)$, $SCR_{Tvar}^{Discrete}$ is defined by:

$$\begin{aligned} SCR_{Tvar}^{Discrete} &= (b+c\mu)\sum_{i=0}^{n-1} e^{-r\ (t_{i+1}-t_0)}\ TVar(I(t_{i+1}), 99.5\%)\ \delta_i - P \\ &= (b+c\mu)\sum_{i=0}^{n-1} e^{-r\ (t_{i+1}-t_0)}\frac{I(0)\ e^{\mu(t_{i+1},t_0)+\frac{\sigma^2(t_{i+1},t_0)}{2}}\Phi\big(\sigma(t_{i+1},t_0)-\Phi^{-1}(99.5\%)\big)}{0.005}\ \delta_i - P \end{aligned} \quad (5.3.27)$$

To conclude this section, we see that we obtain a capital requirement with a closed formula which no longer requires Monte Carlo simulations that leads to more accurate and conservative results.

5.4 Exotic option based on an epidemic infected index

In this section, we present a financial instrument that allows market participants to hedge against economic impacts of an epidemic.
If we take the case of a country that sets up an insurance plan to cover the costs of an epidemic, this exotic option can enable to hedge against unexpected events (in the event that the premiums are not sufficient to cover the expenses).

5.4.1 Payoff at maturity

This financial product looks like an exotic product that could typically be found on a financial market but here the underlying is not a financial asset but an index of infected. If we consider a 1-year contract, the payoff of the product is defined by:

$$P_1 = C\ \min\left(\frac{\big(\max_{0\le t\le 1} I(t) - K_1\big)_+}{K_2-K_1}, 1\right) \quad (5.4.1)$$

Where C is the nominal amount of the contract, K_1 and K_2 are two strikes defined in advance.

We can analyse the different possible payoffs according to the infected index:

- $\max_{0\leq t\leq 1} I(t) < K_1 \Longrightarrow P_1 = 0$
- $K_1 \leq \max_{0\leq t\leq 1} I(t) < K_2 \Longrightarrow P_1 = C \times R$ where $R = \frac{\max_{0\leq t\leq 1} I(t) - K_1}{K_2 - K_1} < 1$
- $\max_{0\leq t\leq 1} I(t) \geq K_2 \Longrightarrow P_1 = C$

In the case of an insurance plan, if we consider $C = SCR$ or $C = SCR\ e^r$ to take into account the time value of money[5] and K_1, K_2 appropriately chosen[6], then this product allows to reimburse part or all of the unexpected costs related to a wave not covered by the premiums.

5.4.2 Fair price

Now that the product is defined, we can determine its price at time $t = 0$.
Generally financial products (vanilla options, exotic options, futures,...) are priced under the risk neutral measure Q but here, we present an exotic product with an underlying (infected index) that is not a financial asset. That is why our product is much closer from weather options or real options than traditional equity options. Consequently, there is no natural risk neutral measure and thus we decide to price our product under the real measure P possibly adding a risk premium.

In order to use the SIR framework, we make some assumptions:

- There is a maximum of 1 wave per year
- The maximum of infected is reached during a wave
- τ defined the stochastic wave start time.
- Epidemic wave is modelled by a SIR model independent from τ with duration T and $I^{SIR}(0) = I_0 \neq I(\tau)$ and we assume the number of infected using SIR model is denoted by $I^{SIR}(t)$

We are aware that these assumptions are not all true. Indeed, for example, epidemics such as COVID-19 present multiwave pattern. Moreover, we could also consider that the initial number of infected of the wave depends on τ by taking $I^{SIR}(0) = I(\tau)$ but here in order to have independence between τ and the stochastic process $I^{SIR}(t)$, we assume: $I^{SIR}(0) = I_0 \neq I(\tau)$.

[5] The SCR is the amount to be held at the beginning of the wave to have a 99.5% probability of solvency so if we want this value at the end of the year we have to capitalise it

[6] K_1 and K_2 should be chosen such that in more than 99.5% the amount received by the option is sufficient to meet the unexpected costs, in practice we determine K_1 and K_2 by simulations

Under those assumptions, we can rewrite the payoff at time $t = 1$ by:

$$
\begin{aligned}
P_1 =& 1_{\{\tau>1\}} \times 0 + 1_{\{\tau\leq 1\}}\ C\ \min\left(\frac{\left(\max\limits_{\tau\leq t\leq \min(\tau+T,1)} I(t) - K_1\right)_+}{K_2 - K_1}, 1\right) \\
=& 1_{\{\tau\leq 1\}}\ C\ \min\left(\frac{\left(\max\limits_{\tau\leq t\leq \min(\tau+T,1)} I(t) - K_1\right)_+}{K_2 - K_1}, 1\right) \\
=& 1_{\{\tau\leq 1\}}\ C\ \min\left(\frac{\left(\max\limits_{0\leq t\leq \min(1-\tau,T)} I^{SIR}(t) - K_1\right)_+}{K_2 - K_1}, 1\right)
\end{aligned}
$$

If we keep this payoff to price our product, it is not easy to compute the expectation knowing that: τ and $\max\limits_{0\leq t\leq \min(1-\tau,T)} I^{SIR}(t)$ are not independent.

This is why we make a conservative pricing assumption and suppose that no matter when a wave starts, the maximum is determined over the total duration of a wave i.e T:

$$
\max_{0\leq t\leq \min(1-\tau,T)} I^{SIR}(t) \longrightarrow \max_{0\leq t\leq T} I^{SIR}(t)
$$

And the pricing payoff becomes:

$$
P_1^* = 1_{\{\tau\leq 1\}}\ C\ \min\left(\frac{\left(\max\limits_{0\leq t\leq T} I(t)^{SIR} - K_1\right)_+}{K_2 - K_1}, 1\right)
$$

The assumption we make is conservative since $P_1^* \geq P_1$ and moreover since we assume independence between τ and $\max\limits_{0\leq t\leq T} I(t)^{SIR}$, we can easily price the product at time $t = 0$ considering a deterministic risk-free interest rate r.

The fair price at $t = 0$ is given by:

$$
\begin{aligned}
P_0 &= \mathbb{E}(e^{-r}\ P_1^*) \\
&= e^{-r}\mathbb{E}(P_1^*) \\
&= e^{-r}\ C\ \mathbb{E}(1_{\tau<1})\ \mathbb{E}\left(\min\left(\frac{\left(\max\limits_{0\leq t\leq T} I(t)^{SIR} - K_1\right)_+}{K_2 - K_1}, 1\right)\right) \\
&= e^{-r}\ C\ P(\tau<1)\ \mathbb{E}\left(\min\left(\frac{\left(\max\limits_{0\leq t\leq T} I(t)^{SIR} - K_1\right)_+}{K_2 - K_1}, 1\right)\right)
\end{aligned}
$$

However, we cannot determine an analytical formula for $\mathbb{E}\left(\min\left(\frac{\left(\max\limits_{0\leq t\leq T} I(t)^{SIR} - K_1\right)_+}{K_2 - K_1}, 1\right)\right)$ so we have to use Monte Carlo simulations to deduce its value. If we can quantify $P(\tau < 1)$ then we can give the price of our product at time $t = 0$.
Note that τ can be modelled as the first jump of a Poisson process and that we can add a risk premium but since we make a conservative assumption for pricing, we decide to not add this premium.

Chapter 6

Stochastic multiwave epidemic models

In the previous sections, we define models that are suitable to model an epidemic wave. However, in general, we observe that epidemics produce not only one wave but multiwave pattern. In this section, we give an idea of how we can modify time-varying parameters SIR models in order to take into account several waves.

Time-varying parameters SIR models presented in the previous sections are models that can explain well an epidemic wave. Therefore, we slightly modify the dynamic of these models in order to model the multiwave effects of an epidemic.
Remember that the dynamic of $I(t)$ under the stochastic SIR model is defined by:

$$dI(t) = \left(\frac{\beta(t)S(t)I(t)}{N} - \gamma(t)I(t)\right) dt + \frac{\sigma S(t)I(t)}{N} \, dW(t), \; t \geq t_0 \tag{6.0.1}$$

We consider the case where $\beta(t)$ and $\gamma(t)$ are negative exponential function:

$$\begin{aligned} \beta(t) = \beta_0 \; exp(-\lambda_1 \; (t - t_0)) &\longrightarrow d\beta(t) = -\lambda_1 \; \beta(t) \; dt, \; \text{with } \beta(t_0) = \beta_0 \\ \gamma(t) = \gamma_0 \; exp(-\lambda_2 \; (t - t_0)) &\longrightarrow d\gamma(t) = -\lambda_2 \; \gamma(t) \; dt, \; \text{with } \gamma(t_0) = \gamma_0 \end{aligned} \tag{6.0.2}$$

Typically, as we see in the practical part of this thesis, we observe $\lambda_1 > \lambda_2$ due to lockdown effects.

The SIR model with these parameters can only model one wave because as beta decreases faster than gamma, we observe a constantly decreasing reproduction rate ($R_0(t) = \frac{\beta(t)}{\gamma(t)}$) which means that we are heading towards an extinction of the epidemic.
However, this model does not take into account a possible relaxation of the population in the face of social distancing measures when the end of the lockdown is decreed. To consider the relaxation of the population, we will modify the dynamic of parameters $\beta(t)$ and $\gamma(t)$ to allow these parameters to increase again after a wave.

We define a_β and a_γ two constants that quantify the relaxation of the population and if we suppose that social distancing measures are taken when the number of infected exceeds a certain threshold K and are lifted when the number of infected falls below this threshold, dynamics of parameters become:

$$\begin{aligned} \frac{d\beta(t)}{\beta(t)} &= \Big(-\lambda_1 \, 1_{I(t)\geq K} + a_\beta \, 1_{I(t)<K} \Big) \, dt \\ \frac{d\gamma(t)}{\gamma(t)} &= \Big(-\lambda_2 \, 1_{I(t)\geq K} + a_\gamma \, 1_{I(t)<K} \Big) \, dt \end{aligned} \tag{6.0.3}$$

Typically if we consider that after a wave, the reproduction number slowly increases then we should have that $a_\beta > a_\gamma$.

Nevertheless, if we consider the SIR model with parameters dynamics described in (6.0.3), we obtain a much more complicated model since parameters depend now on $I(t)$ and thus are stochastic.
Therefore, this multiwave model does not give a closed formula for $I(t)$ even if we make the simplification $S(t) = N$ in the dynamics of $I(t)$. This means that the actuarial results presented in the previous sections can still be applied but have to be computed using each time Monte Carlo simulations.

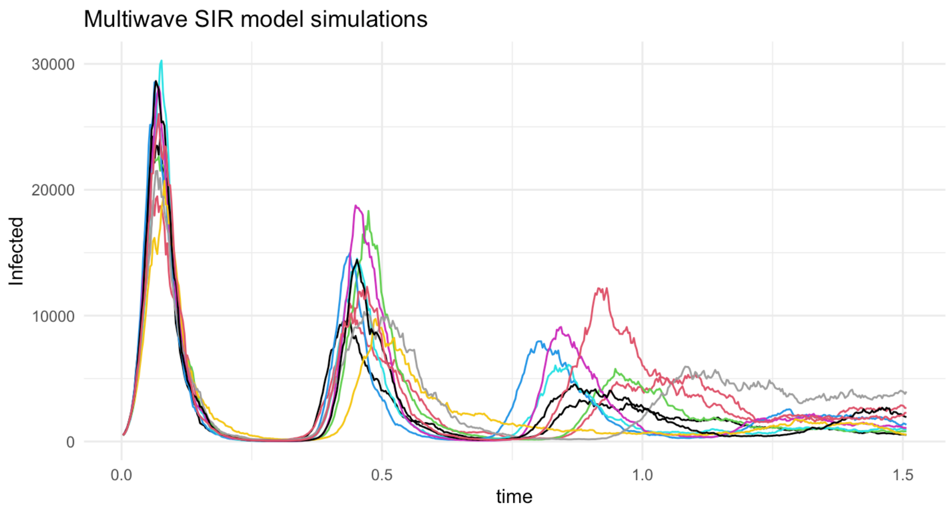


Figure 6.0.1: Multiwave epidemic model based on exponential parameters SIR model with $\beta_0 = 1250$, $\gamma_0 = 1150$, $\lambda_1 = 9$, $\lambda_2 = 8.5$, $a_\beta = 8$, $a_\gamma = 7$, $\sigma = 1$, $K = 800$, $I(0) = 500$ and $N = 10\ 000\ 000$

Figure 6.0.1 shows 10 simulated paths of the multiwave model based on the exponential parameters SIR model with arbitrarily chosen parameters. We can notice that over time the waves become smaller and smaller.
Unfortunately, this kind of scheme considering less and less intense waves is not able to generate paths similar to the evolution of the COVID-19 epidemic in Belgium since the second wave was much stronger than the first one.

Chapter 7

Practical application

In this chapter, we apply all theoretical elements presented in the previous chapters to a concrete case: the COVID-19 epidemic in Belgium.
As the COVID-19 outbreak in Belgium has presented multiwave pattern and as the SIR models presented can only model one wave, we mainly concentrate on the first wave of COVID-19.
First, we calibrate our models to the first wave data and then on this basis, we make actuarial analysis: insurance plan, reserving, reinsurance treaty, capital requirement.
In a second part, we examine if based on the information from the first wave, we could have predicted the intensity of the second wave and thus set up an insurance plan to cover second wave expenses.

7.1 First wave of COVID-19 in Belgium

We consider that the first wave of COVID-19 in Belgium occurred from the 1^{st} March to the 15^{th} June. To obtain the evolution of the number of deaths and infected during the first wave, we rely on the data from the library "coronavirus" in R[1]. This dataset provides the daily number of deaths but also the detected cases of COVID-19.

However, in order to use the SIR models, we are interested to get the dynamic of the number of infected at time t, i.e the number of infectious people which is not provided. Therefore, we need to make some assumptions in order to create the dataset of the number of infected:

- COVID-19 tests are on average done two days before having results
- The incubation period of COVID-19 is on average 5 days and for asymptomatic people, we advice to do a test 5 days after having a contact with an infected person.
- Infected remain contagious 7 days after the first symptoms

With those hypothesis, we postulate that when someone is positive, it is contagious 7 days before, and remains contagious during 5 days (12 days of infection in total).

Figure 7.1.1 illustrates the number of infected at time t considering the three hypothesis and we can already notice the presence of a right fat tail.

[1]See, https://github.com/RamiKrispin/coronavirus, package developed by Rami Krispin.

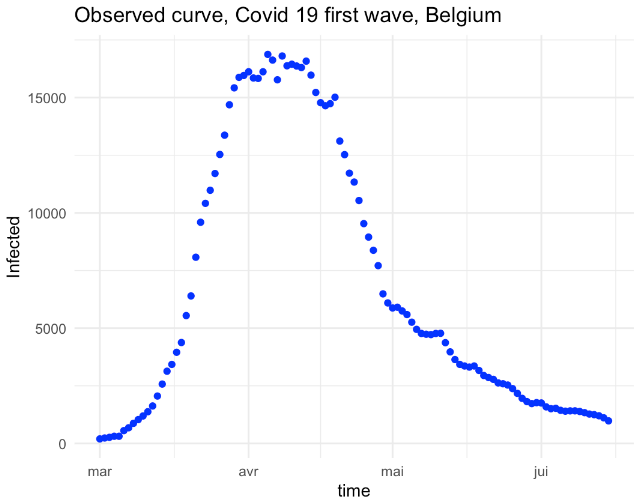


Figure 7.1.1: Observed number of infected of COVID-19 from the 1^{st} March to the 15^{th} June in Belgium

Now that we have the dataset of the number of infected but also the number of deaths, we can fit our SIR models to the data.

7.1.1 Calibration of deterministic models

In this section, our aim is to calibrate the SIR models to the first wave of COVID-19. To this end, we use a least square approach on the number of infected, in other words, we look for the set of parameters that gives an infected curve ($I(t)$) that is as close as possible to the one observed ($I^*(t)$).
Note that we explain the calibration of the classical SIR model but we can easily deduce the calibration of the time-varying parameters SIR models.

Mathematically speaking, we look for β and $\gamma = \eta + \mu$[2] that minimize the quadratic error between $I(t)$ and $I^*(t)$.
We can formalize this as follows:

$$(\hat{\beta}, \hat{\gamma}) = \arg\min \left(\sum_t (I(t) - I^*(t))^2 \right) \tag{7.1.1}$$

However, we know that SIR models, expect the simplified model, do not provide explicit form for $I(t)$. Therefore, to get the dynamics of the number of infected, we have to use a numerical approximations. The numerical approach chosen is the implicit Euler scheme that we recall in appendix (see 9.1.1).

Nevertheless, with the least square method, we estimate β and γ, but it is still two parameters to fit: the mortality rate μ and the recovery rate η. We show now how to calibrate these parameters.

[2] As the effect of η and μ are inseparable for $I(t)$, we estimate their sum (γ), in the following we show how we estimate η and μ separately

From SIR models, we postulate that:

$$dD(t) = \mu\ I(t)\ dt$$

So by integrating both side of the equation, we can find $D(t)$ depending on $I(t)$:

$$\int_{t_0}^{t} dD(s) = \mu \int_{t_0}^{t} I(s)\ ds$$
$$\iff D(t) = \mu \int_{t_0}^{t} I(s)\ ds,\ D(t_0) = 0$$

Then, we can easily isolate μ and we obtain:

$$\mu = \frac{D(t)}{\int_{t_0}^{t} I(t)\ dt}$$

Unfortunately, we don't have a closed form of the integral expression, so we need to use a numerical approximation of the integral (see 9.1.2). Using this approximation, we obtain:

$$\int_{t_0}^{t} I(t)\ dt = \sum_{i=1}^{n} I(t_i)\ \delta_{t_i}$$

Note that for our results, we take $\delta_{t_i} = 1/365$.
As the total number of deaths till the end of the wave $D^*(t_f)$ is easily available and as we know $I(t)$ for optimal β and γ, we can estimate μ by the following formula:

$$\hat{\mu} = \frac{D^*(t_f)}{\sum_{i=1}^{n} I(t_i)}\ 365$$

As we estimate γ and μ, we can deduce the estimation of η ($\eta = \gamma - \mu$) and in such way we calibrate all parameters of the model.

To get our numerical results, we use the R software and obtained results are reported in the table 7.1.1.
Note that according to Statbel[3], the size of Belgium's population the 1^{st} January 2020 was equal to 11 492 641 and $I(t_0) = 199$ which corresponds to the number of infected the 1^{st} March.

N	$\hat{\beta}$	$\hat{\eta}$	$\hat{\mu}$	SSE
11 492 641	937.5872	876.8159	6.531	783 413 684
38 203	90.00415	13.9238	4.915	59 221 265

Table 7.1.1: Parameters estimated from standard SIR model on yearly basis

However, the classical SIR model does not replicate well the infected curve as we can see in the figure 7.1.2 which can be explained by the nature of the observed curve that has a right fat tail.
So to improve our result, we can consider N also as a parameter to calibrate and doing that,

[3]https://statbel.fgov.be/fr/themes/population/structure-de-la-population

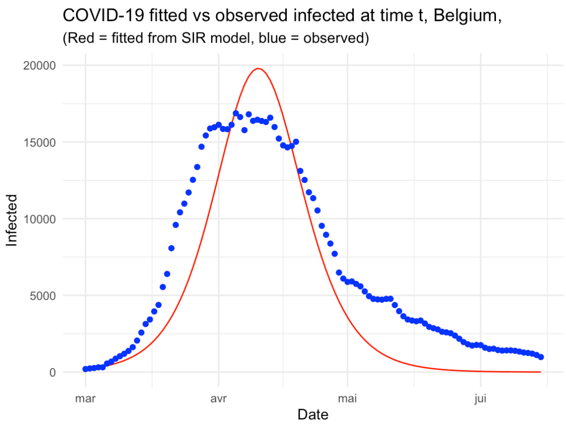


Figure 7.1.2: fitted vs observed infected with initial N

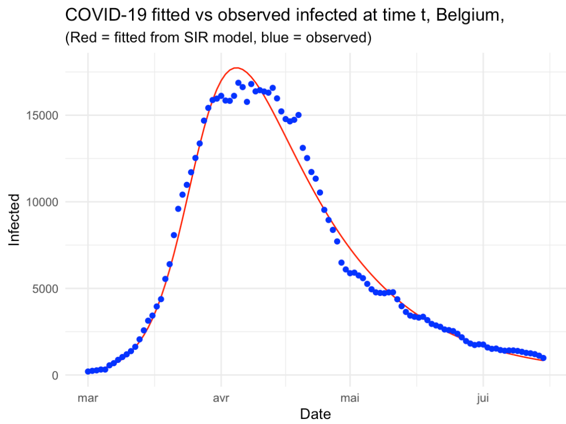


Figure 7.1.3: fitted vs observed infected with adjusted N

the fitting is better (see figure 7.1.3). Nevertheless the adjusted $\tilde{N}$ is significantly smaller than N so in order to obtain more coherent results, we consider time-varying parameters SIR models.

$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\eta}_1$	$\hat{\eta}_2$	$\hat{\eta}_3$	$\hat{\mu}$	t_1	t_2	SSE
1162.89	615.66	303.4687	1081.19	581.78	284.3041	4.88	22/365	62/365	16 803 745

Table 7.1.2: Piecewise constant parameters estimated on yearly basis

$\hat{\beta}_0$	$\hat{\gamma}_0$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\mu}$	SSE
1300.248925	1187.245388	8.924704	8.440068	4.86	40 572 756

Table 7.1.3: Exponential parameters estimated on yearly basis

The three time-varying parameters SIR models presented in section (2.2) can be calibrated by the same way.

Tables 7.1.2, 7.1.3 and 7.1.4 report the fitted parameters. We see, in the parameters, the impact of lockdown effects. In fact, for each model, we obverse that $\beta(t)$ and $\gamma(t)$ decrease but $\beta(t)$ faster than $\gamma(t)$ which implies that $R_0(t)$ decreases with time.

If we look at the SSE, we see that piecewise constant parameters SIR model replicates the observed curve much better than the two others time-varying parameters SIR models. However, this model contains almost two times more parameters than the two others that are more parsimonious and as we can see at Figure 7.1.4 they also replicate quite well the first wave curve.

As the classical SIR model does not replicate well the first wave curve, we decide, for the following, to only consider time-varying parameters SIR models.

$\hat{\beta}_0$	$\hat{\gamma}_0$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\mu}$	SSE
1056.900136	943.105819	8.636480	7.500669	4.873649	63 224 280

Table 7.1.4: Simplified SIR, exponential parameters estimated on yearly basis

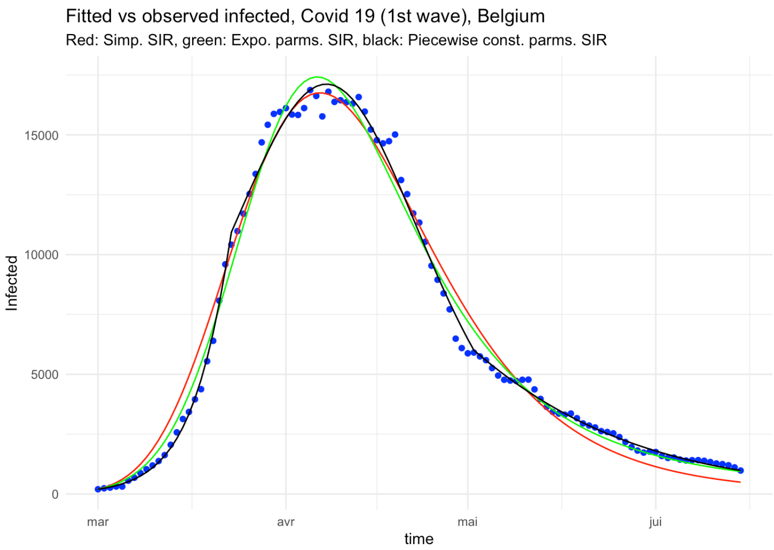


Figure 7.1.4: Comparison of time-varying parameters SIR models

7.1.2 Fair premium rate based on deterministic models

In chapter 3, we present insurance plans and show how to determine the fair premium rates and reserves. Based on fitted SIR models and assuming an epidemic wave of duration T equals to 3 and a half months ($T = t_f - t_0 = 3.5/12$), we now compute those rates and reserves.

To simplify notations, we define:

- Insurance plan 1 with fair premium rate p_1 and reserves $Res_1(t)$: the insurance plan with premium payment during the whole duration of the wave
- Insurance plan 2 with fair premium rate p_2 and reserves $Res_2(t)$: the insurance plan with premium payment during the first 2 months of the wave (we take $t^* = t_0 + 2/12$)
- Insurance plan 3 with fair premium rate p_3 and reserves $Res_3(t)$: the insurance plan with premium payment 2 months before the wave ($t_0 - t_p = 2/12$)

We consider two cases of benefits: in the first case, we consider a benefit rate b equals to 365 000 which represents 1000€ per day and no lump sum capital and in the second, we consider a lump sum capital c equals to 200 000€ in case of death but no other benefit.
Note that for all plan, we will use a risk-free interest rate r equals to 2%.

Now that the assumptions are defined, we can work out numerically the premium rates. First, we approximate $I(t)$ using the Euler's scheme and then approximate numerically the integrals. Note that for the simplified SIR, we just approximate the integrals since we have a closed form for $I(t)$.

Numerical results in function of SIR models and insurance plans are reported in table 7.1.5.

Model	b	c	fair p_1	fair p_2	fair p_3
Piecewise constant parms	365 000	0	230.7406	396.3492	380.5511
	0	200 000	617.4272	1060.571	1018.298
Exponential parms	365 000	0	229.083	396.1307	381.9985
	0	200 000	612.9916	1055.995	1018.322
Simplified SIR	365 000	0	227.3342	394.0092	381.3486
	0	200 000	607.0942	1052.199	1018.389

Table 7.1.5: Fair premium rates, $r = 2\%$, $T = 3.5/12$

We observe in table 7.1.5 that there are no significant differences between premium rates depending on the SIR models, the three SIR models lead to similar premium rates. We also observe that fair premium rate p_1 is always lower than p_2 and p_3, this is simply due to the fact that the duration of premium payment is not the same.

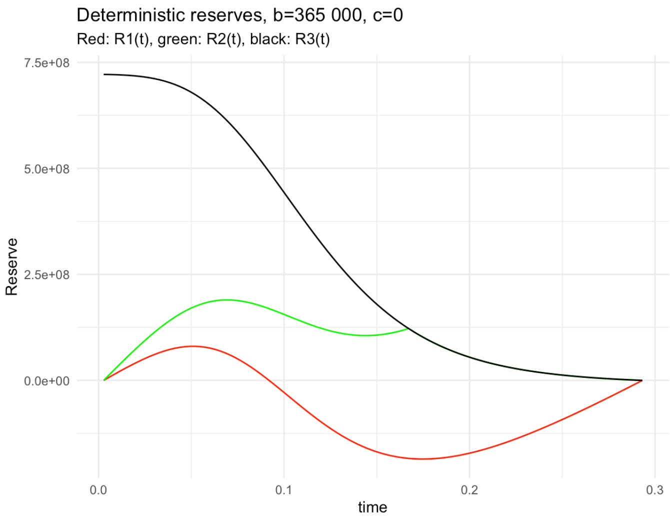


Figure 7.1.5: Evolution of reserves $(Res_i(t))$, $b = 365\ 000$, $c = 0$, using simplified SIR model

Figure 7.1.5 shows the evolution of reserves $(Res_i(t))$ for the case $b = 365\ 000$ and $c = 0$ and for the simplified SIR model [4].

We observe that reserves for plan 1 become negative while the reserves for the two other plans stay continuously above zero. As we can't accept negative reserves for the reasons exposed in section 3.2, we decide for the following to keep only insurance plan 2 and 3.

We can also analyse the amounts of premiums paid by each policyholder, table 7.1.6 reports those amounts in function of the insurance plan and time of infection using the simplified SIR model.
As mentioned in section 3.2, we see that for the insurance plan 3, everybody pays the same

[4] We have the same conclusions if we consider the case $b = 0$, $c = 200\ 000$ and the other SIR models

Time of infection (t_I)	P_2	P_3
t_0	0	63.56€
$t_0 + 1/12$	32.83€	63.56€
$> t_0 + 2/12$	65.67€	63.56€

Table 7.1.6: Amounts of premiums P_i, $b = 365\ 000$ and $c = 0$, $r = 2\%$, $T = 3.5/12$, simplified SIR

amount of premium while in the second plan, it depends on the time of infection and people who receive benefits pay less than people who receive nothing. So plan 3 is fairer.

Moreover, considering the insurance plan 3, as we can see at figure 7.1.5, we already constitute reserves in the beginning of the wave (t_0) while in plan 2, the reserves at time t_0 are null and then the insurer is exposed to a higher risk.

7.1.3 Calibration of stochastic models

In this section, we calibrate stochastic SIR models presented in chapter 4 to the case of the first wave of COVID-19 in Belgium.

First, we calibrate the simplified SIR model, as it gives a closed formula for $I(t)$ it is easier to calibrate this model.
For that, we use the Markov property of $I(t)$.

Denote $\{I(t_i)\}_{i=0}^{n}$ an observed path of $I(t)$ and pose:

$$l(\theta) = p_\theta(I(t_0), I(t_1), ..., I(t_n)), \text{ the likelihood depending on } \theta \text{ (parameters to estimate)} \tag{7.1.2}$$

$p_\theta(I(t_1), I(t_2), ..., I(t_n))$ is the probability of observing the path.

Our goal is to maximize that probability.

Since $I(t)$ is a Markov process, we can develop $l(\theta)$ such that:

$$\begin{aligned} l(\theta) = p_\theta(I(t_0), I(t_1), ..., I(t_n)) &= p_\theta(I(t_n)|I(t_{n-1}), ..., I(t_0))\ p(I(t_{n-1}), ..., I(t_0)) \\ &= p_\theta(I(t_n)|I(t_{n-1}))\ p(I(t_{n-1}), ..., I(t_0)) \end{aligned} \tag{7.1.3}$$

And by repeating this n times that leads to:

$$l(\theta) = \prod_{i=1}^{n} p_\theta(I(t_i)|I(t_{i-1})) \tag{7.1.4}$$

Taking now the log-likelihood leads to:

$$L(\theta) = \sum_{i=1}^{n} log(p_\theta(I(t_i)|I(t_{i-1}))) \tag{7.1.5}$$

Then using a maximum log-likelihood approach, the optimal set of parameters θ is given by:

$$\hat{\theta} = arg\ max\ \sum_{i=1}^{n} log(p_\theta(I(t_i)|I(t_{i-1}))) \tag{7.1.6}$$

For the simplified SIR, we can evaluate $p_\theta(I(t_i)|I(t_{i-1}))$.

In fact, we can show that:

$$I(t_i) = I(t_{i-1})\ exp\left(\frac{\gamma_0}{\lambda_2}\left(e^{-\lambda_2(t_i-t_0)}-e^{-\lambda_2(t_{i-1}-t_0)}\right)-\frac{\beta_0}{\lambda_1}\left(e^{-\lambda_1(t_i-t_0)}-e^{-\lambda_1(t_{i-1}-t_0)}\right)-\frac{1}{2}\sigma^2\delta_t+\sigma Z\right) \tag{7.1.7}$$

Where $Z \sim N(0, \sqrt{\delta_t})$
So, we have that $\big(I(t_i)|I(t_{i-1})\big) \sim LogNormal(\mu_i, \sigma_i)$, with $\mu_i = ln(I(t_{i-1}) + \frac{\gamma_0}{\lambda_2}\left(e^{-\lambda_2(t_i-t_0)} - e^{-\lambda_2(t_{i-1}-t_0)}\right) - \frac{\beta_0}{\lambda_1}\left(e^{-\lambda_1(t_i-t_0)} - e^{-\lambda_1(t_{i-1}-t_0)}\right) - \frac{1}{2}\sigma^2\delta_t$ and $\sigma_i = \sigma\delta_t$

And then we get that:

$$p_\theta(I(t_i)|I(t_{i-1})) = \frac{1}{I(t_i)\sigma_i\ \sqrt{2\pi}}\ e^{-\frac{(ln(I(t_i)-\mu_i)^2}{2\ \sigma_i}} \tag{7.1.8}$$

So using (7.1.6) and (7.1.8), we obtain the optimal parameters.
Note that in the parameter estimation process, we estimate $\beta(t)$, $\gamma(t) = \eta(t) + \mu$ and σ, so in order to obtain the optimal μ (the mortality rate) and $\eta(t)$, we use a same approach than for the deterministic case and we estimate μ by:

$$\hat{\mu} = \frac{D^*(t_f)}{\sum_t \mathbb{E}(I(t)|\mathcal{F}_0)}\ 365 \tag{7.1.9}$$

$\eta(t)$ is deduce by $\eta(t) = \gamma(t) - \mu$.

For the two other SIR models (piecewise constant and exponential parameters) it is more difficult to calibrate the models since we don't have closed form for $I(t)$.
In order, to calibrate those models, we use the implicit Euler scheme (see appendix 9.3.2) to discretize the dynamic of $I(t)$ and we obtain:

$$I(t_i) \approx I(t_{i-1}) + \left(\frac{\beta(t_{i-1})S(t_{i-1})I(t_{i-1})}{N} - \gamma(t_{i-1})I(t_{i-1})\right)\delta_t + \frac{\sigma S(t_{i-1})I(t_{i-1})}{N}\ \sqrt{\delta_t}\ \varepsilon_i \tag{7.1.10}$$

Where $\varepsilon_i \sim N(0,1)$ and $\delta_t = \frac{1}{365}$

Using this approximation, we can conclude that:

$$\big(I(t_i)|I(t_{i-1}), S(t_{i-1})\big) \sim N(\mu_{i-1}, \sigma_{i-1})$$

With $\mu_{i-1} = I(t_{i-1}) + \left(\frac{\beta(t_{i-1})S(t_{i-1})I(t_{i-1})}{N} - \gamma(t_{i-1})I(t_{i-1})\right)\delta_t$ and $\sigma_{i-1} = \frac{\sigma S(t_{i-1})I(t_{i-1})}{N}\ \sqrt{\delta_t}$

Then we estimate $p_\theta(I(t_i)|I(t_{i-1}), S(t_{i-1}))$ by:

$$\tilde{p}_\theta(I(t_i)|I(t_{i-1}), S(t_{i-1})) = \frac{1}{\sigma_{i-1}\ \sqrt{2\pi}}\ e^{-\left(\frac{I(t_i)-\mu_{i-1}}{2\ \sigma_{i-1}}\right)^2}$$

Finally, using a conditional log-likelihood approach, we obtain optimal set of parameters θ such that:

$$\hat{\theta} = arg\ max\ \sum_{i=1}^{n} log(\tilde{p}_\theta(I(t_i)|I(t_{i-1}, S(t_{i-1})))$$
$$= arg\ max\ \sum_{i=1}^{n} log\Big(\frac{1}{\sigma_{i-1}\ \sqrt{2\pi}}\ e^{-\left(\frac{I(t_i)-\mu_{i-1}}{2\ \sigma_{i-1}}\right)^2}\Big)$$

Tables 7.1.7, 7.1.8 and 7.1.9 report the calibrated parameters of our processes and note that once again we consider $N = 11\ 492\ 641$ and $I(t_0) = 199$.

$\hat{\beta}_0$	$\hat{\gamma}_0$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\mu}$	$\hat{\sigma}$
1049.434103	950.571400	7.866149	6.995303	4.73529	1.192667

Table 7.1.7: Calibrated parameters for simplified SIR model

$\hat{\beta}_0$	$\hat{\gamma}_0$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\mu}$	$\hat{\sigma}$
1300.397583	1187.096907	8.178057	7.751199	4.691202	1.445309

Table 7.1.8: Calibrated parameters for exponential parameters SIR model

$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\eta}_1$	$\hat{\eta}_2$	$\hat{\eta}_3$	$\hat{\mu}$	t_1	t_2	$\hat{\sigma}$
1153.366	616.7189	301.3734	1072.805	580.612	285.4767	4.735038	22/365	62/365	1.402324

Table 7.1.9: Calibrated parameters for piecewise constant parameters SIR model

To see what our stochastic processes look like, we can simulate paths of $I(t)$, schemes used are presented in appendix (9.3.1 and 9.3.2).

Figure 7.1.6 shows simulation of 1000 sample paths of $I(t)$ with our three SIR models. We see that the three models lead to realistic paths of epidemic wave.

However, we note that the simplified model, due to the log-normality of $I(t)$, allows the generation of paths which present right fat tails that can deviate strongly from the expectation, thus allowing for larger extreme values and stronger epidemic wave. The simplified model nevertheless, on average, underestimates the peak of the number of infected.

It is also interesting to look at the cumulative number of deaths. Table 7.1.10 presents the expectation and quantiles for $D(t)$ at the end of the wave and since the simplified SIR model allows for stronger epidemic waves, quantiles of $D(t_f)$ are higher than the two others models. Note that empirical histograms of $D(t_f)$ are presented in appendix 9.4.1 and we observe that for the simplified SIR model, $D(t_f)$ presents right fat tails due to the log-normality of $I(t)$ while for the two others, the distribution of $D(t_f)$ is more close to a normal distribution.

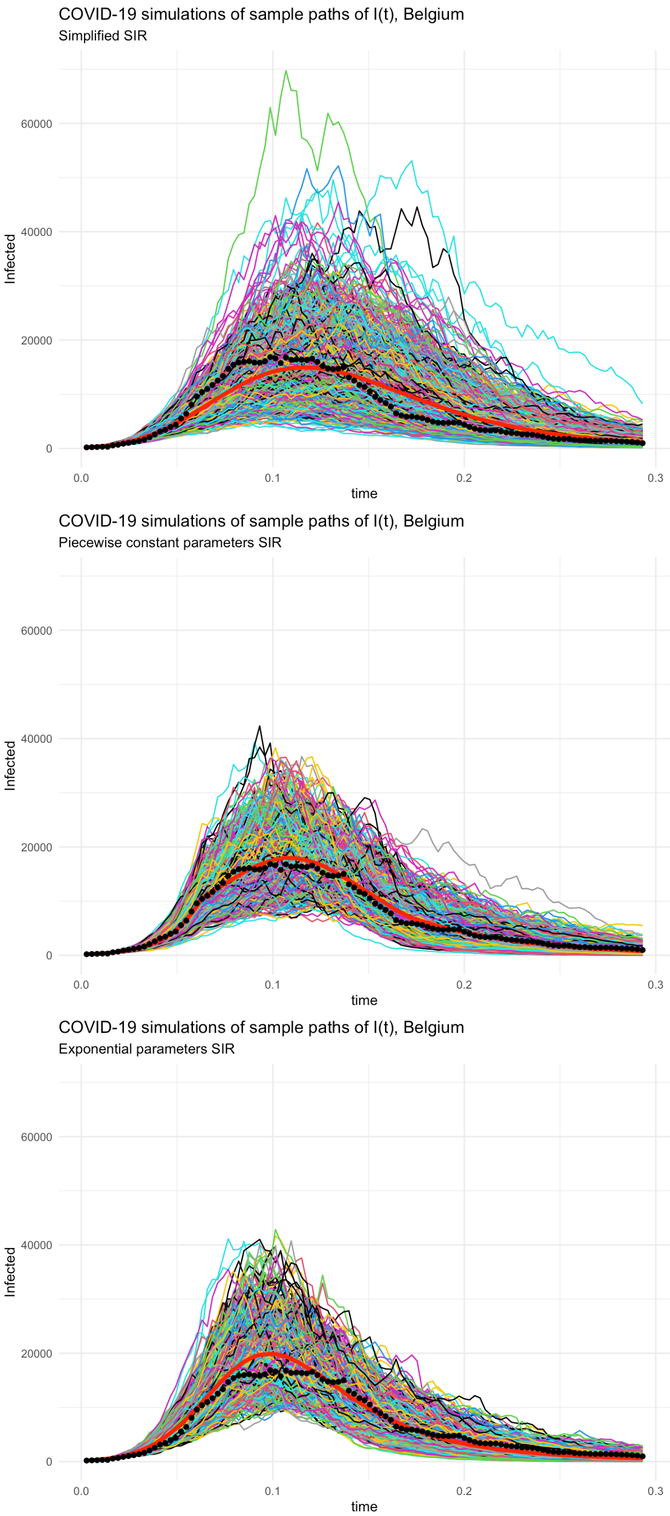


Figure 7.1.6: Simulations of stochastic SIR models, in red the expectation and in black dots the observed infected first wave curve

Model	$\mathbb{E}(D(t_f))$	$\mathrm{Var}(D(t_f), 95\%)$	$\mathrm{Var}(D(t_f), 99\%)$
Piecewise constant parms	9661.206	11 432.19	12 245.14
Exponential parms	9659.814	11 953.91	13 016.02
Simplified SIR	9641.356	16 810.79	21 903.96

Table 7.1.10: Expectation and extreme quantiles of $D(t_f)$

7.1.4 Fair premium rate based on stochastic models

In this section, we consider insurance plans described in section 7.1.2 and we determine fair premium rates but this time based on stochastic SIR models.
Note that we only consider insurance plan 2 and 3 since insurance plan 1 leads to negative reserves for the deterministic case and in the stochastic case, we require that stochastic reserves remain positive on average.

Model	b	c	fair p_2	fair p_3
Piecewise constant parms	365 000	0	408.7691	392.4775
	0	200 000	1060.5684	1018.299
Exponential parms	365 000	0	413.4687	396.2039
	0	200 000	1062.83	1018.451
Simplified SIR	365 000	0	403.1341	391.8873
	0	200 000	1046.004	1016.822

Table 7.1.11: Fair premium rates with, r=2%, T=3.5/12

Table 7.1.11 reports the fair premium rates for the three different SIR models and for the two different plans considered. Note that for the piecewise constant parameters and exponential parameters SIR models, we use Monte Carlo simulations in order to compute the expectation of $I(t)$ and we take 100.000 simulations.

We observe that the fair premium rate for the simplified SIR model is always lower than the two others, it's coming from the fact that as seen in the previous section, the simplified SIR model underestimates in mean the number of infected and thus the benefits.
If we compare those premium rates with those determined by the deterministic models 7.1.5, we see that there are small differences coming from the fact that parameters are not the same and also that for the piecewise constant parameters and exponential parameters SIR models the expectation of $I(t)$ is not equal to the deterministic model. Note that the differences between insurance plan 2 and 3 are the same that ones explained in section 7.1.2.

The main difference with the deterministic plan is that here, reserves are stochastic. We know that on average they are positive, but now there is no guarantee that in all situations they remain positive as shown in figure 7.1.7. Therefore, it is important to know how we can hedge against events that could lead to insolvency.

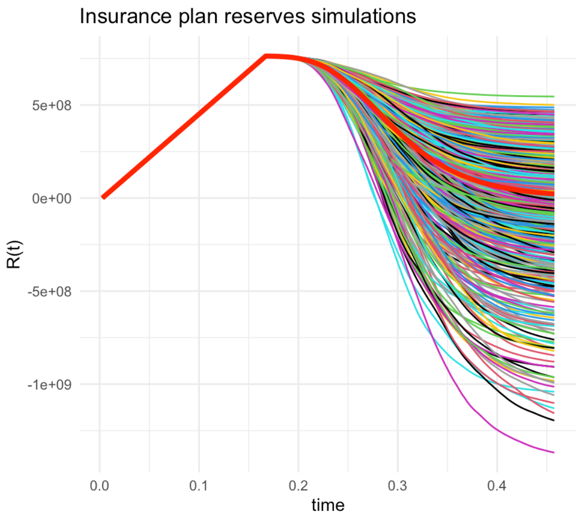


Figure 7.1.7: Insurance plan 3 reserves simulations using simplified SIR model, in red the expectation, b=365 000

7.1.5 Reinsurance

In the previous section, we see that since we consider stochastic models, our numerical reserves can become negative. In order to hedge our position against bad events, we can use stop loss reinsurance defined in section 5.2. Thus we compute the reinsurance premium for a stop loss reinsurance on top of the insurance plans described in section 7.1.2. For the simplified SIR model, we can use closed formula developed in section 5.2 but for the two others models, we have to use Monte Carlo simulations.

Model	b	c	P_{reins}
Simplified SIR	365 000	0	124 729 018€
	0	200 000	323 631 821€
Piecewise constant parms SIR	365 000	0	92 723 958€
	0	200 000	240 576 146€
Exponantial parms SIR	365 000	0	91 152 602€
	0	200 000	234 309 737€

Table 7.1.12: Reinsurance premium, $r = 2\%$, T= 3.5/12

Table 7.1.12 reports the reinsurance premiums for the three different models. We can observe that simplified SIR model leads to higher premiums than the two SIR models, that can be explained by the fact that, as we can see at figure 7.1.6, the simplified model can generate larger waves with right tails that can deviate significantly from the expectation due to the log-normality distribution of $I(t)$.
Moreover, we show in section 5.2 that if we consider the insurance plan 3 with a reinsurance treaty then we are hedged at 100%. In this case, there are a lot of chance that reserves are positive at the end of the wave, that means that a cash back needs to be paid to policyholder.

Figure 9.4.2 in appendix shows simulations of net reserves for the insurance plan 3 and we see that they remain positive.

Using formula (5.2.21) and (5.2.23), we can compute the expected amount policyholder can expect to receive at the end of the wave. Table 7.1.13 reports those amounts by using the simplified SIR model but we can do the same thing for the two other SIR models. The reinsurance

b	c	Cash back(€)
365 000	0	10.92
0	200 000	28.33

Table 7.1.13: Cash back amount for the insurance plan 3 using the simplified SIR model

treaty is also great from a marketing point a view since we can ask premium to population promising a possible cash back at the end. For the case $c = 365\ 000$ and $b = 0$, we ask a total premium of 65.31€ ($391.8873 \times 2/12$) and we say that on average, policyholders receive at the end 10.92€.

7.1.6 Capital requirement

In this section, we consider insurance plans presented in the previous sections but we suppose now that there is no reinsurance treaty and we want to determine the minimum capital to require such that our insurance plans stay solvent at a 99.5% level.

To determine the capital using the Var approach, we need to use Monte Carlo simulations in order to simulate a quantile (see 9.3.4 in appendix). For the $Tvar$ approach, simulations are not necessary but this approach is only available for the simplified SIR model and for the insurance plan 3.

Table 7.1.14 reports our results using the simplified SIR model and figure 7.1.8 shows the evolution of $SCR \times e^{rt}$ vs $Var(-Res(t), 99.5\%)$. We observe that the condition $SCR \times e^{rt} \geq Var(-Res(t), 99.5\%)$ required on SCR is verified[5](see proposition 5.3.2).
Note that results using piecewise constant parameters and exponential parameters are available in appendix 9.4.1.

Model	Insurance plan	b	c	SCR_{Var} (€)	SCR_{TVar} (€)
Simp. SIR	No premium	365 000	0	1 865 962 380	2 391 982 139
		0	200 000	4 841 574 246	6 206 426 906
	Premium payment before	365 000	0	1 124 390 044	1 650 409 803
		0	200 000	2 917 431 601	4 282 284 261
	Premium payment during	365 000	0	1 140 969 690	/
		0	200 000	2 960 449 711	/

Table 7.1.14: Capital requirement using simplified SIR model, $r = 2\%$, $T = 3.5/12$

[5]We take the case $b = 365\ 000$ and $c = 0$ but we can do the same conclusion for the case $b = 0$ and $c = 200\ 000$

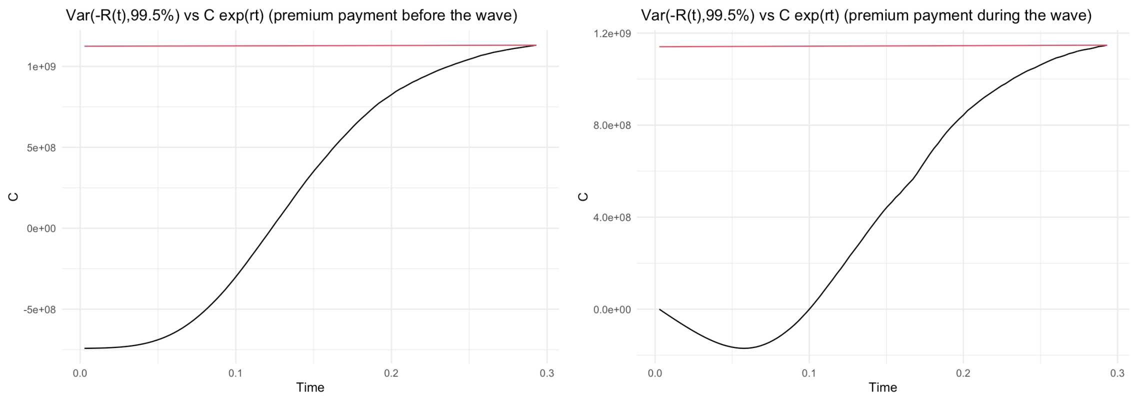


Figure 7.1.8: $SCR \times e^{rt}$ vs $Var(-Res(t), 99.5\%)$, in red $SCR \times e^{rt}$ and in black $Var(-Res(t), 99.5\%)$, $b = 365\ 000$, $c = 0$

First notice that the simplified SIR model leads to far more conservative capital requirement than the two others models which is explained by the log-normality of the number of infected using the simplified model which is felt in the distribution of $-Res(t_f)$.
Indeed, the histograms of $-Res(t_f)$ for insurance plan 3 according to the different SIR models are presented in appendix (see figure 9.4.3). We observe a distribution including a right fat tail for the simplified model while for the two other models, we observe more of a normal distribution. A right fat tail distribution allows for larger extreme quantiles, which is why we observe much more conservative capital under the simplified model.

Moreover, those results confirm that the $Tvar$ approach is more conservative as we observe an additional safety margin compared to the SCR_{Var}. If we compare the capital considering premium payment before the wave with those supposing the payment during the wave, we observe that these capitals are almost the same:

$$SCR_{Var}^{P\ before} \approx SCR_{Var}^{P\ during}$$

The fact that can explain why $SCR_{Var}^{P\ during}$ is slightly higher is that in this plan, the total amount of premiums received is random contrary to the other plan where the total of premiums is known that means that we can receive lower premiums than expected and to face that, we need more capital. Moreover, as seen in section (7.1.2), if we suppose premium payment before the wave, we already constitute reserves at the beginning of the wave, so we are less exposed to an insolvency risk at the beginning and then need less capital.

Finally, if we compare these capitals, with the reinsurance premiums computed in section 7.1.5, we observe that the capital requirements are always higher than the reinsurance premiums and that's logical because reinsurance premiums are based on an average of unexpected costs while the capital requirements are based on a extreme quantile. Note that the difference is much more pronounced for the simplified SIR model due to the log-normality of $I(t)$.

7.1.7 Price of an exotic option based on an epidemic infected index

In the theoretical part of the thesis, we present a financial product that allows investors to hedge against to economic consequence of an epidemic.

This financial product can also be used to hedge our insurance plan against bad events.

For that, we can take $C = SCR^{TVar} \times e^r$ to take account of the time value of money.
K_1 and K_2 should be chosen suitable such that:

- If $\max_{0 \leq t \leq 1} I(t) < K_1$ then premiums are sufficient to cover epidemic expenses
- If $K_1 \leq \max_{0 \leq t \leq 1} I(t) < K_2$ then the exotic option is sufficient to reimburse all the unexpected costs
- If $\max_{0 \leq t \leq 1} I(t) \geq K_2$ the exotic option is sufficient to reimburse all the unexpected costs expect in ε % of the case where $\varepsilon < 0.05$ because we keep in mind that with the SCR we are not hedged at 100%

To find suitable K_1 and K_2, we can do wave simulations and see if with this K_1, K_2 the conditions described above are fulfilled. Note that in this section, we only consider the insurance plan 3 with $b = 365\ 000$, $c = 0$ and use the simplified SIR model with $I^{SIR}(0) = 199$ and optimal parameters calibrated to the first wave of COVID-19 in Belgium (table 7.1.7).

Using 100 000 simulations of epidemic waves based on simplified SIR model, we find that $K_1 = 12\ 500$ and $K_2 = 30\ 000$ are suitable since there are only 8 simulations where the compensations are not sufficient to reimburse all the unexpected costs which represents only 0.008% that is less than 0.5%.

Once C, K_1 and K_2 are defined, we can compute the price of such exotic option using Monte Carlo simulations since we don't have a closed form for the price.
Table 7.1.15 reports our results as a function of the probability of having a wave.

$P(\tau < 1)$	Price
0.01	12 649 700€
0.1	126 496 996€
0.5	632 484 981€
1	1 264 969 962€

Table 7.1.15: Price of the exotic option at time $t = 0$ for $C = C^{TVar,\ b=365\ 000,\ c=0} \times e^r$, $K_1 = 12\ 500$ and $K_2 = 30\ 000$

The advantage of this option is of course to buy it when the probability of occurrence of an epidemic is low and that we are aware of the impacts of an epidemic.
For the case $P(\tau < 1) = 1$, we observe that the price of this exotic option is much larger than the reinsurance premium. It explains by the fact that the reinsurance premium is based on an average of unexpected expenses while the price of this exotic option depends on SCR^{TVar} which is based on a tail value-at-risk measure.
Moreover, we know that with a reinsurance treaty, the risk of insolvency is transferred from the insurance company to the reinsurance company and that is it that must cover against bad events, not the insurance company. We clearly see that if the reinsurance company wants to hedge its position with this product, it could ask a much higher reinsurance premium.

Note that unlike the reinsurance treaty, this exotic option does not lead to a perfect cover there can be cases where compensations are not sufficient. The compensations received are not directly related to the number of infected people exceeding the expected number, which implies that we also may receive a much higher compensation than the actual unexpected costs and make profit.

7.2 Second wave of COVID-19 in Belgium

We have seen that in Belgium a second and stronger wave of COVID-19 has taken place few months after the first one. In this chapter, we try, on the basis of simplified SIR model with parameters calibrated to the first wave, to set up an insurance plan that could have covered expenses due to the second wave.

First we can have a look on the second wave infected curve. Figure 7.2.1 shows the curve making the same assumptions than for the first wave.

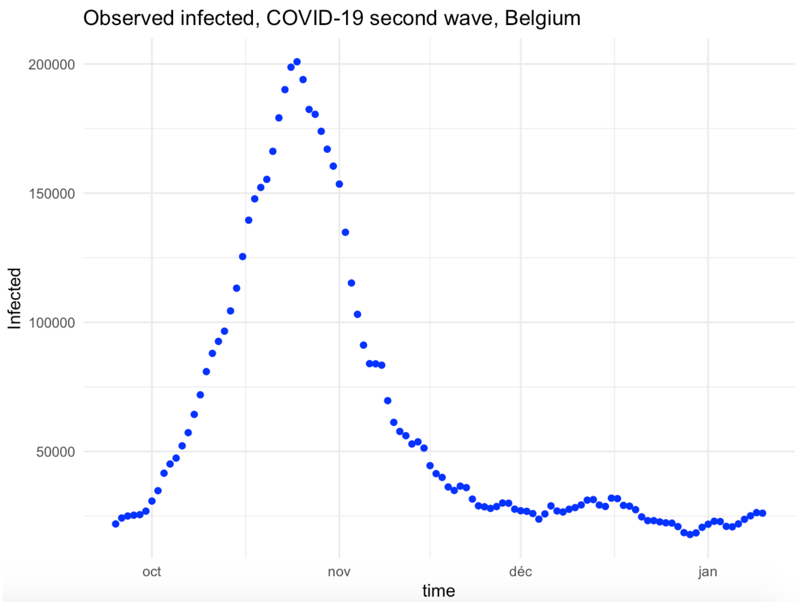


Figure 7.2.1: Observed infected curve for the second wave

What can we notice?
First of all, we can clearly see that the second wave is stronger than the first one, figure 7.2.1 shows a peak close to 200 000 whereas for first wave, the peak is around 16 000 infected (see figure 7.1.1).

The fact that the number of infected in the second wave is higher can be explained by the fact that COVID-19 testings became more widespread, more people were being tested and inevitably more people were positive, which suggests that the observed number of infected in the first wave strongly underestimates the actual number of infected.
We also see that $I(t_0)$ (the number of infected at the beginning) is between 15 000 and 20 000 whereas for the first wave $I(t_0)$ was equal to 199.

Moreover, the total number of deaths during the second wave is equal to 10 113 while for the first wave we observe 9661 deaths. The difference between the total number of deaths is not very significant, whereas the number of infected between the two waves is very different. This confirms the idea that in the first wave, there were much more infected than the figures indicated.

7.2.1 Can the simplified SIR model with first wave parameters predict waves similar to the second one?

In this section, we examine if the simplified SIR model with parameters from the first wave can predict wave paths like the second wave.

Since we observe that the number of infected at the beginning of the second wave is around 15 000, we make the assumption that $I(t_0) = 15\ 000$.
So using the simplified SIR model, with $I(t_0) = 15\ 000$, we can simulate paths of $I(t)$ for the second wave. Figure 7.2.2 presents the second wave and the expectation of three simplified SIR models.

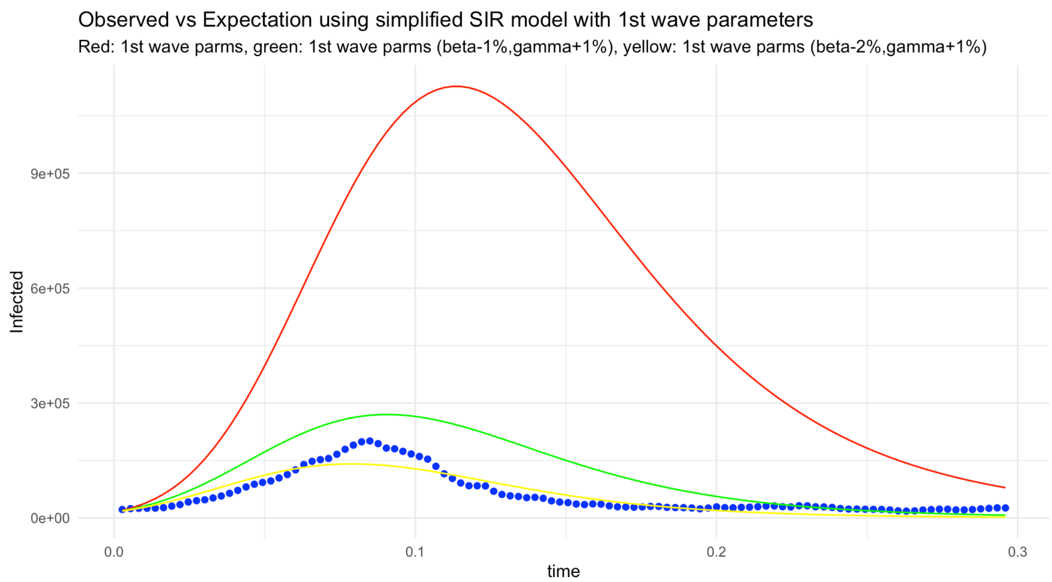
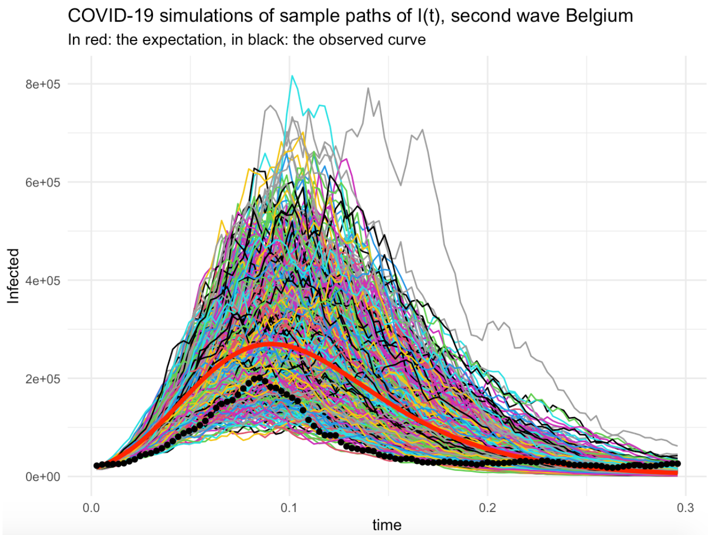


Figure 7.2.2: Observed curve vs simplified SIR expectation using 1st wave parameters

We see that if we keep same optimal parameters than for the first wave, then we strongly overestimates the second wave and moreover the observed curve isn't a likely path. However, we notice that by slightly modifying β_0 and γ_0, we obtain more coherent results and find models that can lead to similar paths to the one observed for the second wave as we can see at figure 7.2.3.

Since we show that simplified SIR model with similar parameters than the first wave can be suitable to predict second wave path, we can price an insurance plan on those basis.



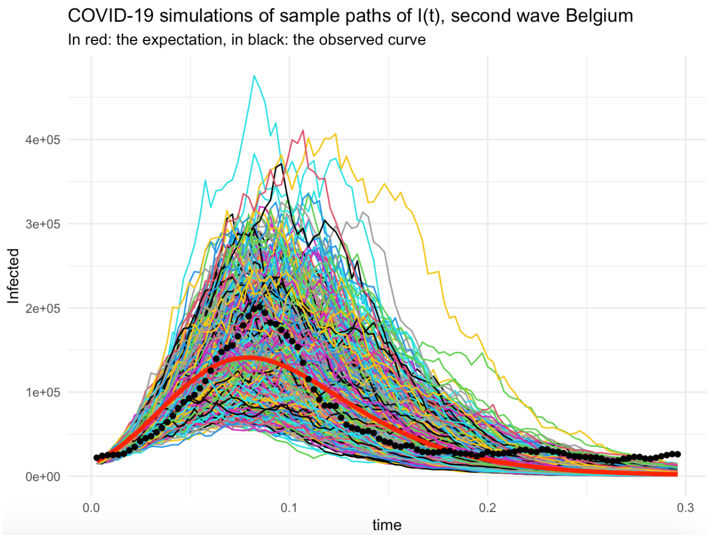


Figure 7.2.3: 1000 paths simulation of simplified SIR model for the second wave; top: $\beta_0 - 1\%$ and $\gamma_0 + 1\%$, bottom: $\beta_0 - 2\%$ and $\gamma_0 + 1\%$. In red $\mathbb{E}(I(t)|\mathcal{F}_0)$, in black dots the observed curve

However, the simplified model is not be able to correctly predict the number of deaths due to the second wave. Indeed, as we can see in table 7.2.1, the simplified model, by considering the same mortality rate as the 1st wave, strongly overestimates the number of total deaths, as the real number of deaths during the second wave (10 113) is not even included in the 1% quantile. This can be explained by the fact that the first wave mortality rate is determined on the basis of the number of infected in the first wave that is considerably lower than the number of infected in the second wave.
Therefore, in the following we don't consider a plan to cover death benefits.

Model	$\mathbb{E}(D(t_f))$	$D(t_f)$, 1% percentile
Simp. SIR	729 372	287 695
Simp SIR with $\beta_0 - 1\%$ and $\gamma_0 + 1\%$	160 179	71 270
Simp SIR with $\beta_0 - 2\%$ and $\gamma_0 + 1\%$	79 877	37 793

Table 7.2.1: Expectation and 1% percentile of $D(t_f)$ for the second wave

7.2.2 Insurance plan to cover second wave costs

Based on the predictions made using simplified SIR model for the second wave, we can set up an insurance plan.

Suppose that the 25th July 2020, Belgian government decides to establish an insurance plan based on the first wave simplified SIR model with premium payment from the 15th July to the 25th September, benefits payment from the 25th September to the 10th January and project the number of infected at the beginning of the wave equals to 15 000. As it's far away from $I(t_0)$ for the first wave, we need to compute the fair premium rate.
Note that since models can't get likely number of deaths for the second wave, we decide to only cover hospitalized benefits.

Using the actuarial equilibrium between the value of premiums and the expected value of future benefits, we can obtain premium rates like defined in section 3.2. Note that to simplify our hypothesis, we consider that the size of the population that takes part in the financing of the plan during the first 2 months is equal to N, that's maybe a non conservative hypothesis since we don't take into account the number of deaths caused by the first wave but also the number of infected during the premium payment.

Model	b	Premium rate	Total amount paid
Simp. SIR	365 000	29 581.549	4 930.26€
	36 500	2 958.1549	493.026€
Simp SIR with $\beta_0 - 1\%$ and $\gamma_0 + 1\%$	365 000	6 514.006	1 085.67€
	36 500	651.4006	108.57€
Simp SIR with $\beta_0 - 2\%$ and $\gamma_0 + 1\%$	365 000	3 242.716	540.45€
	36 500	324.2716	54.04€

Table 7.2.2: Premium rates for the second wave insurance plan

However, we observe at table 7.2.2 that an insurance plan that gives 1000€ per day of infection isn't sustainable, we can't accept a total amount paid of at least 540.45€ . To still set up an insurance plan, we can reduce the benefit amount by 10 and give 100€ per day. In this case, if we exclude the simplified SIR model considering same parameters than the first wave, we obtain more reasonable premiums: 108.57€ or 54.04€ for the plan that gives 100€ per day of infection.

We can now look at whether these insurance plans would have been solvent to cover the second wave, to do this we look at the evolution of the reserves.
Figure 7.2.4 shows the evolution of reserves considering the 2 preferred models and we notice

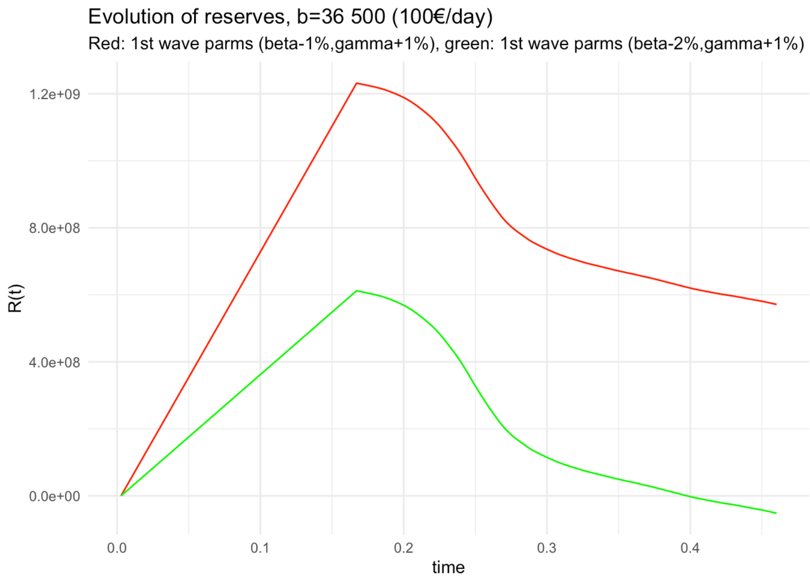


Figure 7.2.4: Reserves of insurance plans for the second wave

that for the case $\beta_0 - 2\%$ and $\gamma_0 + 1\%$ reserves become negative, which means that this plan would haven't been solvent (without reinsurance contract or solvency capital requirement). However, the simplified model with $\beta_0 - 1\%$ and $\gamma_0 + 1\%$ leads to a solvent insurance plan with premiums.
Since, for the case $\beta_0 - 1\%$ and $\gamma_0 + 1\%$, reserves at the end of the plan are positive, an amount should be paid back to the population.

$$Cash\ back = \frac{Res(t_f)}{N}$$

This amount is reported in the table 7.2.3.

Model	b	Cash back
Simp SIR with $\beta_0 - 1\%$ and $\gamma_0 + 1\%$	36 500	49.80€

Table 7.2.3: Cash back for insurance plan, $r = 2\%$

Thus we see that basis on the first wave an insurance plan could have been put in place, if now we consider the case where the government does not require premium payment, we must then determine a capital (SCR) to be blocked in order to cover the expenses with a probability of 99.5% at any time. If we rely on the Tvar approach developed in section 5.3.2, we can quantify these amounts.
Table 7.2.4 reports those amounts, once again we notice that simplified SIR model with first wave parameters leads to huge amounts compared to the two others models and also that it's more reasonable to consider $b = 36\ 500$ (100€ /day) because it leads to more acceptable capital requirement.

Model	b	SCR
Simp SIR	365 000	180 741 975 731€
	36 500	18 074 197 573€
Simp SIR with $\beta_0 - 1\%$ and $\gamma_0 + 1\%$	365 000	35 862 791 317€
	36 500	3 586 279 132€
Simp SIR with $\beta_0 - 2\%$ and $\gamma_0 + 1\%$	365 000	16 838 465 837€
	36 500	1 683 846 584€

Table 7.2.4: SCR for the second wave insurance plan

To conclude this section, we notice that basis of a previous COVID-19 wave model, a coherent and solvent insurance plan can be set up. Maybe that we can find a better model to fit the second wave curve but since the aim here is to price an insurance plan based on our past knowledge, the results obtained can be considered as good and likely.

Chapter 8

Conclusion

This master thesis presents models for actuarial valuation of insurance plans covering claims caused by a pandemic.

First, we show how to improve the standard SIR model, in order to obtain models able to reproduce epidemic waves such as COVID-19 but also an epidemic model (the simplified SIR model) giving an explicit solution for the number of infected $I(t)$.
Next, we extend our deterministic models by adding a Brownian noise to the infection rate to better reflect the evolution of an epidemic wave. We show that, as for the deterministic model, the simplified model provides a closed solution for the number of infected and assume that $I(t)$ follows a Log-normal distribution.

On these basis, we establish national insurance plans providing hospitalization and death benefits based on solidarity between the susceptibles and infected. The results obtained in the practical part show that such a plan could have be set up for the first wave of COVID-19 in Belgium by asking for a premium amount that remains reasonable (60-65€). We also cover different aspects of hedging: reinsurance treaty and solvency capital requirement. For the simplified SIR model, we show that we can obtain formulas that do not require Monte Carlo simulations, which allows to obtain more accurate results. For the solvency capital requirement, we show that we can obtain a conservative capital that does not require simulations based on a tail-var approach. We also develop a financial product to hedge against the economic impacts of an epidemic wave and we explain how this product can be used in the case of an insurance plan.

Epidemics, such as COVID-19, generally present multiwave pattern. We prove in the practical part that based on the first wave of COVID-19, we could have predicted the second wave and thus implemented an insurance plan. However, multiwave epidemic model does not yet exist in the literature, therefore at the end of the theoretical part, we indicate how the problem can be tackled and present a model that is able to reproduce this multiwave pattern of an epidemic. Unfortunately, in practice, this model can only be applied to epidemics with increasingly smaller waves, which is not the case for the COVID-19 epidemic in Belgium.

This thesis opens the way to further research. Instead of only randomize the infection rate, we may think to also randomize the mortality rate, in this case, we can consider a stochastic

model with two correlated Brownian motions.
We could also think of an epidemic model that takes into account the number of hospitalized people by considering a hospitalization rate. This would allow to define an insurance plan that would be more in line with the real costs of hospitalization benefits.
Finally, in the same idea as the proposed multiwave model, we could think of a model allowing to reproduce the multiwave pattern that could provide a closed formula for the number of infected and not necessarily assuming waves of decreasing intensity.

Chapter 9

Appendix

9.1 Concepts

9.1.1 Implicit Euler scheme

Define:

$$dX_t = f(t, X_t),\ \ X \text{ a vector of n dimensions} \tag{9.1.1}$$

We divide the interval $[0, t]$ on which we want the solution into (m-1) sub-intervals:

$$0 = t_0 < t_1 < t_2 \ ... < t_{m-1} < t_m = t$$

where the discretization step $h = t_{i+1} - t_i$
We define $X_i = X_{t_i}$ and $f_i = f(t_i, X_{t_i})$, implicit Euler scheme:

$$\begin{aligned} &X_0 = X_{t_0} \\ &X_{i+1} = X_i + hf_{i+1} \quad\ i = 1, 2, ..., m \end{aligned} \tag{9.1.2}$$

9.1.2 Numerical integration

The numerical approximation of an integral is given by:

$$\int_{t_0}^{t} f(s)ds = \sum_{i=0}^{m-1} h_i\ f(t_{i+1}) \tag{9.1.3}$$

with:

- f a continuous function
- $0 = t_0 < t_1 < t_2 \ ... < t_{m-1} < t_m = t$
- $h_i = t_{i+1} - t_i$

If we apply this concept to the valuation of the fair premium using deterministic model, we obtain:

$$\int_{t_0}^{t} e^{-r(s-t_0)} S(s)ds = \sum_{i=0}^{m-1} h_i\ e^{-r(t_{i+1}-t_0)} S_{i+1}$$

$$\int_{t_0}^{t} e^{-r(s-t_0)} I(s)ds = \sum_{i=0}^{m-1} h_i\ e^{-r(t_{i+1}-t_0)} I_{i+1}$$

Where S_i and I_i are worked out with the implicit Euler's scheme.

9.1.3 Brownian motion

Let $\big(\Omega, \mathcal{F}, P\big)$ be a probability set. A standard Brownian motion $(W_t)_t$ is a Gaussian stochastic process with independent and stationary increments.
Then the Brownian motion $(W_t)_t$ is such that:

- $W(0) = 0$
- For all $0 = t_0 < t_1 < ... < t_{m-1} < t_m$, the increments $(W_{t_{i+1}} - W_{t_i})_{i=1,...,m}$ are independent and distributed as normal with mean and variance:

$$\begin{aligned} \mathbb{E}\Big((W_{t_{i+1}} - W_{t_i})\Big) &= 0 \\ \mathbb{V}\Big((W_{t_{i+1}} - W_{t_i})\Big) &= t_{i+1} - t_i \end{aligned} \tag{9.1.4}$$

9.1.4 Itô's lemma

Let $\big(\Omega, \mathcal{F}, P\big)$ and $(W_t)_{t\geq 0}$ a $\mathcal{F}_t$-Brownian motion. Let $(X_t)_{0\leq t\leq T}$ an Itô process such that:

$$dX_t = \mu_t dt + \sigma_t dW_t, \quad 0 \leq t \leq T$$

where $(\mu_t)_{0\leq t\leq T}$ and $(\sigma_t)_{0\leq t\leq T}$ are $\mathcal{F}_t$-adapted processes, such that:

$$\int_0^T |\mu_s| ds < +\infty \text{ and } \int_0^T |\sigma_s| ds < +\infty$$

If $f(t,x)$ is a function $\mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$, C^1 with respect to time and C^2 with respect to x, then for all $t \in [0,T]$:

$$d(f(t,X_t)) = \left(\frac{\partial f}{\partial t}(t,X_t) + \mu_t \frac{\partial f}{\partial x}(t,X_t) + \frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial x^2}(t,X_t)\right) dt + \sigma_t \frac{\partial f}{\partial x}(t,X_t)\ dW_t$$

9.1.5 Markov property

Define $(X_t)_t$ a stochastic process defined in $\big(\Omega, \mathcal{F}, P\big)$, $(X_t)_t$ is markovian if:

$$\forall\ n,\ \forall\ 0 = t_0 < t_1 < ... < t_n < t$$

$$P(X_t \leq y | X_{t_0}, X_{t_1}, ..., X_{t_n}) = P(X_t \leq y | X_{t_n})$$

So a markovian process is a memoryless process, the conditional probability of X_t depends only on the last state known.

9.2 Demonstrations

9.2.1 Demo. 1

Let's prove that for the insurance plan with premium payment before the wave, we have:

$$\mathbb{E}(Res(t)|\mathcal{F}_{t_0}) \geq 0, \; t_0 \leq t \leq t_f \tag{9.2.1}$$

Proof. For $t \geq t_0$, stochastic reserves are defined by:

$$Res(t) = p_{stoch}\left(\int_{t_p}^{t_0} e^{r(t_0-s)} \; N \; ds\right) e^{r(t-t_0)} - (b + \mu \; c) \; \int_{t_0}^{t} I(s) \; e^{r(t-s)} \; ds$$

So if we prove that:

$$p_{stoch}\left(\int_{t_p}^{t_0} e^{r(t_0-s)} \; N \; ds\right) e^{r(t-t_0)} \geq (b + \mu \; c) \; \int_{t_0}^{t} \mathbb{E}(I(s)|\mathcal{F}_{t_0}) \; e^{r(t-s)} \; ds, \quad \forall t \geq t_0 \tag{9.2.2}$$

Then we prove (9.2.1).
Let's show it for $t_0 \leq t \leq t_f$.

$$\begin{aligned}
p_{stoch}\left(\int_{t_p}^{t_0} e^{r(t_0-s)} \; N \; ds\right) e^{r(t-t_0)} &= p_{stoch} \; e^{rt} \; N \; \left(\frac{e^{-rt_p} - e^{-rt_0}}{r}\right) \\
&= \frac{(b + \mu \; c) \; \int_{t_0}^{t_f} \; e^{-r(s-t)} \; \mathbb{E}(I(s)|\mathcal{F}_{t_0}) \; ds \; e^{rt_p} \; \left(e^{-rt_p} - e^{-rt_0}\right)}{1 - e^{-r(t_0-t_p)}} \\
&= \left((b + \mu \; c) \int_{t_0}^{t_f} \; e^{r(t-s)} \; \mathbb{E}(I(s)|\mathcal{F}_{t_0}) \; ds\right) \frac{(1 - e^{-r(t_0-t_p)})}{(1 - e^{-r(t_0-t_p)})} \\
&= (b + \mu \; c) \int_{t_0}^{t_f} \; e^{r(t-s)} \; \mathbb{E}(I(s)|\mathcal{F}_{t_0}) \; ds \\
&\geq (b + \mu \; c) \int_{t_0}^{t} \; e^{r(t-s)} \; \mathbb{E}(I(s)|\mathcal{F}_{t_0}) \; ds
\end{aligned}$$

□

And that ends the proof.

9.2.2 Demo. 2

Proof of proposition 5.2.3.

Proof. To prove this proposition, we use the fact that[1]:

$$I(t) - K = (I(t) - K)_{+} - (K - I(t))_{+}$$

Thus, we have

$$(K - I(t))_{+} = (I(t) - K)_{+} - (I(t) - K)$$

[1]What is called the call/put parity in finance

So we have that:

$$\mathbb{E}((K - I(t))_+ | \mathcal{F}_{t_0}) = \mathbb{E}((I(t) - K)_+ | \mathcal{F}_{t_0}) - \mathbb{E}(I(t) | \mathcal{F}_{t_0}) + K$$

But from proposition 5.2.1, we know the expression of $\mathbb{E}((I(t)-K)_+|\mathcal{F}_{t_0})$ and from proposition 5.1.1 the expression $\mathbb{E}(I(t)|\mathcal{F}_{t_0})$.
If we define:

$$\begin{aligned}
\mu(t_0, t) &= \frac{\beta_0}{\lambda_1} \; (1 - e^{-\lambda_1 (t - t_0)})) - \frac{\gamma_0}{\lambda_2} \; (1 - e^{-\lambda_2 (t - t_0)})) - \frac{\sigma^2}{2} (t - t_0) \\
\nu(t_0, t) &= \sigma \sqrt{t - t_0} \\
d_2^t &= \frac{ln(\frac{K}{I(t_0)}) - \mu(t_0, t)}{\nu(t_0, t)} \\
d_1^t &= d_2^t - \nu(t_0, t)
\end{aligned}$$

We obtain:

$$\begin{aligned}
\mathbb{E}((K - I(t))_+ | \mathcal{F}_{t_0}) &= I(t_0) \; e^{\mu(t_0,t) + \frac{\nu(t_0,t)^2}{2}} \; \Phi(-d_1^t) - K \; \Phi(-d_2^t) - I(t_0) \; e^{\mu(t_0,t) + \frac{\nu(t_0,t)^2}{2}} + K \\
&= I(t_0) \; e^{\mu(t_0,t) + \frac{\nu(t_0,t)^2}{2}} \; \Big(\Phi(-d_1^t) - 1 \Big) + K (1 - \Phi(-d_2^t)) \\
&= I(t_0) \; e^{\mu(t_0,t) + \frac{\nu(t_0,t)^2}{2}} \; \big(- \Phi(d_1^t) \big) + K \; \Phi(d_2^t) \\
&= K \; \Phi(d_2^t) - I(t_0) \; e^{\mu(t_0,t) + \frac{\nu(t_0,t)^2}{2}} \; \Phi(d_1^t)
\end{aligned}$$

And that ends the proof. □

9.2.3 Demo. 3

Proof of proposition 5.3.4.

Proof. We use proposition 5.3.2 and equations (5.3.8) and (5.3.9).
Suppose:

- $SCR = Var(-Res(t_f) \; e^{-r(t_f - t_0)}, 99.5\%) = Var\Big((b+\mu c) \int_{t_0}^{t_f} \; e^{r(t_0 - s)} \; I(s) \; ds - p \int_{t_0}^{t^*} \; e^{r(t_0 - s)} \; S(s) \; ds, 99.5\% \Big)$
- For $t \leq t^*$, $SCR \; e^{r(t-t_0)} = Var(-Res(t_f) \; e^{-r(t_f - t_0)}, 99.5\%) \; e^{r(t-t_0)} \geq Var(-Res(t), 99.5\%)$,

The proof is made considering two cases: $t > t^*$ and $t \leq t^*$.
Case 1: $t > t^*$
In this case, we have:

$$\begin{aligned}
SCR \; e^{r(t-t_0)} &= e^{r(t-t_0)} \; Var\Big((b + \mu c) \int_{t_0}^{tf} e^{r(t_0 - s)} \; I(s) \; ds - p \int_{t_0}^{t^*} e^{r(t_0 - s)} \; S(s) \; ds, 99.5\% \Big) \\
&= Var\Big((b + \mu c) \int_{t_0}^{tf} e^{r(t - s)} \; I(s) \; ds - p \int_{t_0}^{t^*} e^{r(t - s)} \; S(s) \; ds, 99.5\% \Big) \\
&\geq Var\Big((b + \mu c) \int_{t_0}^{t} e^{r(t - s)} \; I(s) \; ds - p \int_{t_0}^{t^*} e^{r(t - s)} \; S(s) \; ds, 99.5\% \Big) \\
&= Var(-Res(t), 99.5\%)
\end{aligned}$$

So for $t > t^*$ we prove that $SCR\ e^{r(t-t_0)} \geq Var(-Res(t), 99.5\%) \Longrightarrow P(\tilde{Res}(t) \geq 0) \geq 99.5\%$

Case 2: $t \leq t^*$
In this case, using the assumption made, we obtain:

$$\begin{aligned} SCR\ e^{r(t-t_0)} &= e^{r(t-t_0)}\ Var\Big((b+\mu c)\int_{t_0}^{tf} e^{r(t_0-s)}\ I(s)\ ds - p\int_{t_0}^{t^*} e^{r(t_0-s)}\ S(s)\ ds, 99.5\%\Big) \\ &= Var\Big((b+\mu c)\int_{t_0}^{tf} e^{r(t-s)}\ I(s)\ ds - p\int_{t_0}^{t^*} e^{r(t-s)}\ S(s)\ ds, 99.5\%\Big) \\ &\geq Var(-Res(t), 99.5\%) \end{aligned}$$

So for $t \leq t^*$ we have that $SCR\ e^{r(t-t_0)} \geq Var(-Res(t), 99.5\%) \Longrightarrow P(\tilde{Res}(t) \geq 0) \geq 99.5\%$

By case 1 and case 2 we show that the necessary condition (proposition 5.3.2) is validated, we can conclude that $\forall t_0 \leq t \leq t_f,\ P(\tilde{Res}(t) \geq 0) \geq 99.5\%$ and that ends the proof. □

9.3 Monte Carlo methods and simulation schemes

9.3.1 Simplified SIR simulation algorithm

Remember that with the simplified SIR model, $I(t)$ for $t \geq t_0$ is given by:

$$I(t) = I(t_0) e^{\frac{\beta_0}{\lambda_1}\ (1-e^{-\lambda_1(t-t_0)}) - \frac{\gamma_0}{\lambda_2}(1-e^{-\lambda_2(t-t_0)}) - \frac{1}{2}\sigma^2}\ e^{\sigma W(t-t_0)}$$

So for s<t, we can rewrite $I(t)$ as:

$$I(t) = I(s)\ e^{\frac{\gamma_0}{\lambda_2}\ (e^{-\lambda_2(t-t_0)} - e^{-\lambda_2(s-t_0)}) - \frac{\beta_0}{\lambda_1}\ (e^{-\lambda_1(t-t_0)} - e^{-\lambda_1(s-t_0)})} e^{-\frac{1}{2}\sigma^2(t-s)}\ e^{\sigma(W(t)-W(s))}$$

We discretize [0,T] into sub-interval such that: $0 = t_0 < t_1 < t_2 < ... < t_m = T$ and $t_i = i\ \delta_t$ with $\delta_t = \frac{1}{365}$
We can simulate paths of $I(t)$ by applying the following algorithm:

$$\begin{aligned} I(0) =& I_0 \\ I(t_i) =& I(t_{i-1})\ e^{\frac{\gamma_0}{\lambda_2}\ (e^{-\lambda_2(t_i-t_0)} - e^{-\lambda_2(t_{i-1}-t_0)}) - \frac{\beta_0}{\lambda_1}\ (e^{-\lambda_1(t_i-t_0)} - e^{-\lambda_1(t_{i-1}-t_0)})} e^{-\frac{1}{2}\sigma^2(\delta_t)}\ e^{\sigma Z} \end{aligned}$$

Where $Z \sim N(0, \sqrt{\delta_t})$.

9.3.2 Euler scheme for simulation

Suppose $(X_t)_{0 \leq t \leq T}$ a stochastic process defined by:

$$dX_t = b(X_t, t)\ dt + \sigma(X_t, t)\ dW_t$$

Divide the interval $[0, t]$ into (m-1) sub-intervals:

$$0 = t_0 < t_1 < t_2\ ... < t_{m-1} < t_m = t$$

where the discretization step $h = t_{i+1} - t_i$ Supppose $X_i = X_{t_i}$, $b_i = b(X_{t_i}, t_i)$ and $\sigma_i = \sigma(X_{t_i}, t_i)$, then we can simulate X_t such that:

$$X_0 = X_{t_0}$$
$$X_{i+1} = X_i + h\ b_i + \sigma_i\ Z$$

Where $Z \sim N(0, \sqrt{h})$.

9.3.3 Monte Carlo expectation

Suppose $(X_t)_{t\geq t_0}$ is a stochastic process, the Monte Carlo expectation of $(X_t)_{t\geq t_0}$ is given by:

$$\forall t \geq t_0,\ \mathbb{E}^{MC}(X_t|\mathcal{F}_{t_0}) = \frac{1}{n}\sum_{i=1}^{n} X_t^{(i)}$$

Where $\big((X_t)^i_{t\geq t_0}\big)_{i=1,\dots n}$ are n simulations of the process $(X_t)_{t\geq t_0}$.

9.3.4 Monte Carlo quantile

Suppose X is a random variable.
The Monte Carlo algorithm of $Var(X, \epsilon)$ is given by:
1. Simulate n independent random observations X_1, X_2, ... , X_n of X
2. Compute the empirical distribution function:

$$Fn(x) = \frac{1}{n}\sum_{i=1}^{n} 1_{\{X_i \leq x\}}$$

3. Estimate $Var(X, \epsilon)$ by $\hat{Var}(X, \epsilon)_{MC} = Fn^{-1}(\epsilon)$

9.4 Figures and numerical results

Model	Insurance plan	b	c	SCR_{Var} (€)
Piecewise constant SIR	No premium	365 000	0	965 150 684
		0	200 000	2 504 123 378
	Premium payment before	365 000	0	222 461 506
		0	200 000	577 185 790
	Premium payment during	365 000	0	224 696 945
		0	200 000	582 985 412
Expo. SIR	No premium	365 000	0	1 041 122 430
		0	200 000	2 676 227 741
	Premium payment before	365 000	0	291 381 748
		0	200 000	749 002 523
	Premium payment during	365 000	0	298 480 924
		0	200 000	767 252 117

Table 9.4.1: Capital requirement using piecewise constant and exponential parameters SIR, $r = 2\%$, $T = 3.5/12$

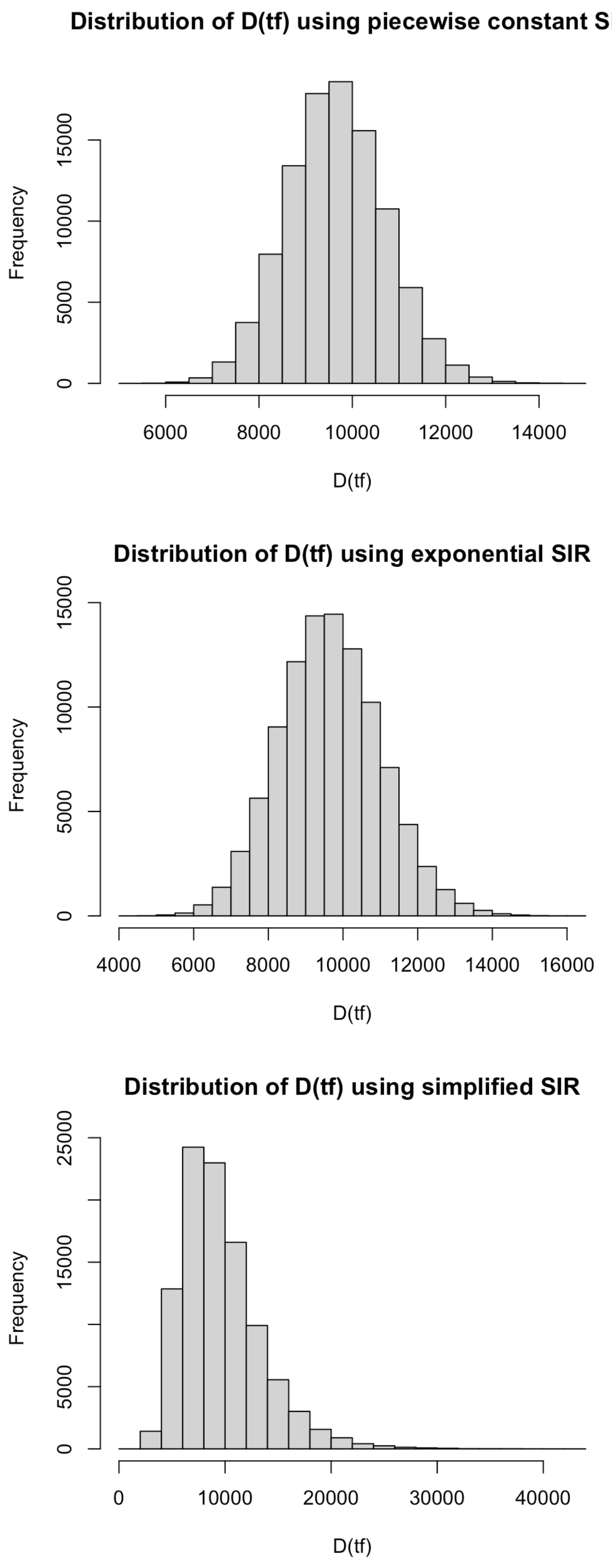


Figure 9.4.1: Histograms of $D(t_f)$ for 100 000 simulations

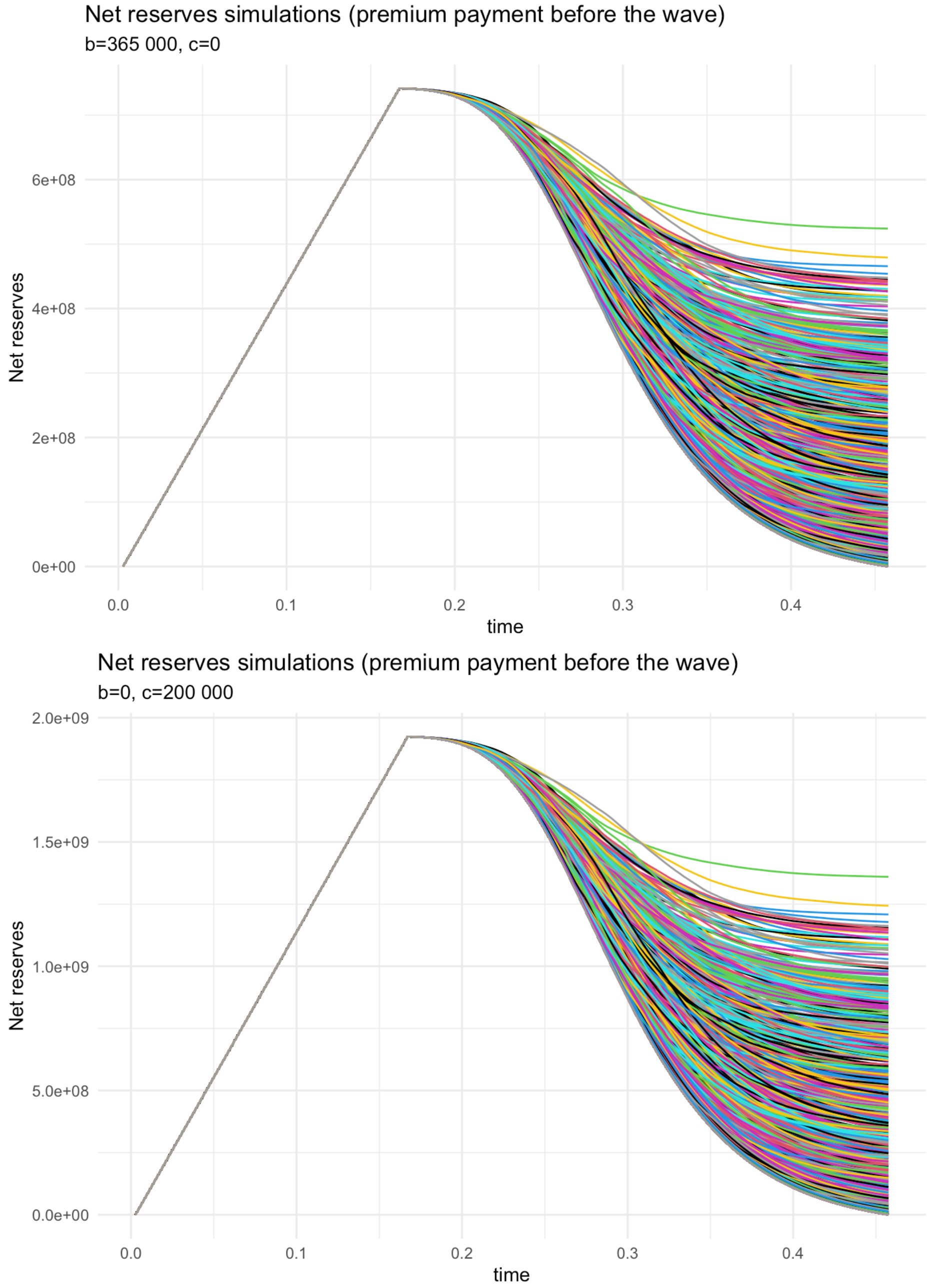


Figure 9.4.2: Net reserves simulations for insurance plan with premium payment before the wave

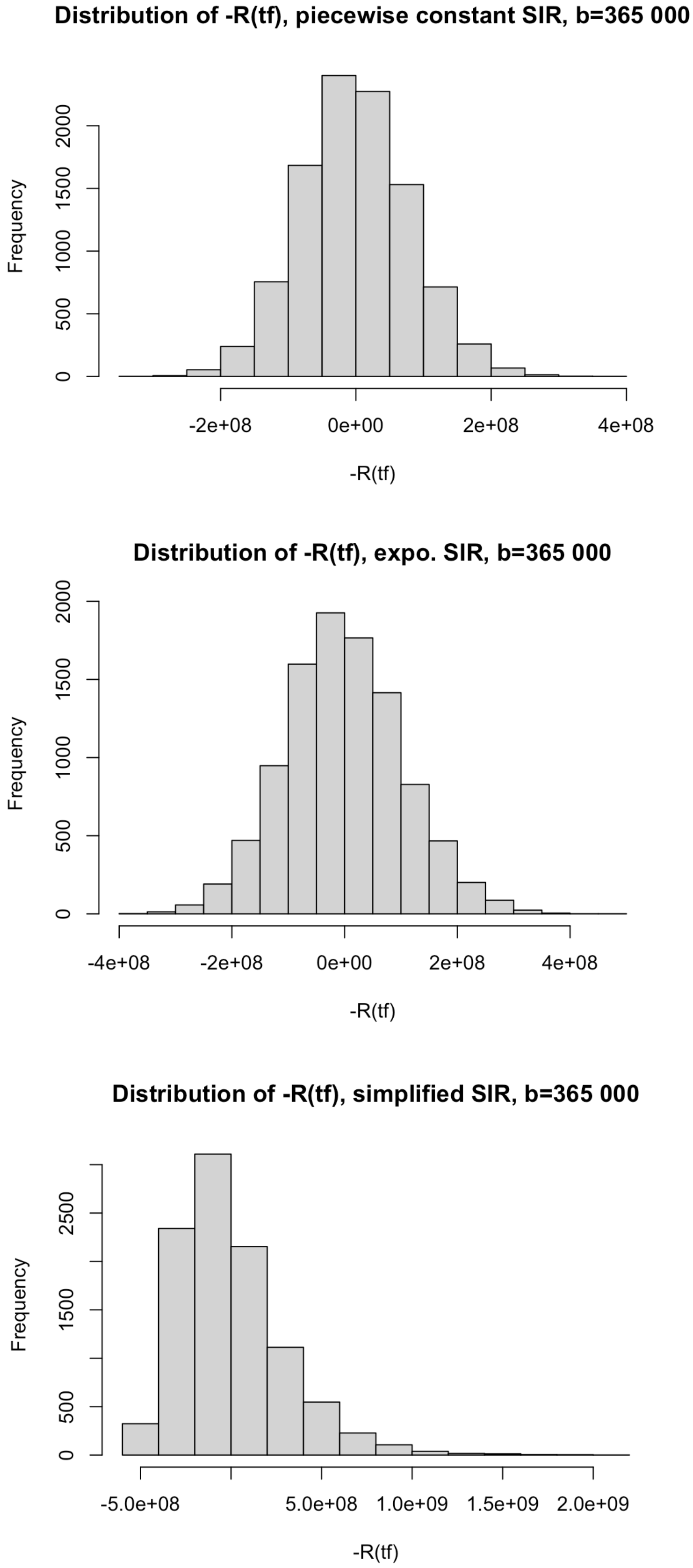


Figure 9.4.3: Histograms of $-Res(t_f)$ for 100 000 simulations and the insurance plan with premium payment before the wave

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